Segway Model

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1 Parameters and notation

Our notation is similar to [1].

g	$=$	9.80665	:	gravitational acceleration (m/s^2)
m	$=$	0.0165 / 0.0295	:	NXT2/RCX wheel weight (kg)
R	$=$	0.0216 / 0.0408	:	NXT2/RCX wheel radius (m)
w	$=$	0.022 / 0.015	:	NXT2/RCX wheel width (m)
J_w	$=$	$mR^2/2$	:	wheel inertia moment $(kg \cdot m^2)$
M	$=$	0.55	:	body weight (kg)
W	$=$	0.15	:	body width (m)
D	$=$	0.045	:	body depth (m)
H	$=$	0.158	:	body height (m)
L	$=$	$H/2$	:	center of mass distance from the wheel axle (m)
J_ψ	$=$	0.00805	:	body pitch moment of inertia $(kg \cdot m^2)$
J_ϕ	$=$	0.005	:	body yaw moment of inertia $(kg \cdot m^2)$
ψ_0	$=$	-2	:	balanced pitch deviation from upright position (deg)
B	$=$	0.18	:	DC motor constant $(Nm \cdot s/rad)$
K	$=$	20	:	DC motor constant (Nm/V)
ψ	$=$		:	body pitch angle (rad)
$\theta_{l/r}$	$=$	$\theta \mp W\phi/(2R)$	:	left/right wheel roll angles (rad)
θ	$=$	$(\theta_l + \theta_r)/2$	:	axle midpoint roll angle (rad)
ϕ	$=$	$R(\theta_r - \theta_l)/W$	:	body yaw angle (rad)
$\mathbf{p}_{l/r}$	$=$	$[x_{l/r}\ y_{l/r}\ R]^T$	:	left/right wheel positions (m)
$\mathbf{p}_m$	$=$	$[x_m\ y_m\ R]^T$	:	axle midpoint position (m)
$\mathbf{p}_c$	$=$	$[x_c\ y_c\ z_c]^T$	:	body center of mass position (m)
$v_{l/r}$	$=$		:	left/right DC motor control power (V)

1.1 Robot position

The position of the center of mass and its derivative can be expressed as

$$\mathbf{p}_c = \mathbf{p}_m + L \begin{bmatrix} \sin\psi\cos\phi \\ \sin\psi\sin\phi \\ \cos\psi \end{bmatrix} \quad , \quad \dot{\mathbf{p}}_c = \dot{\mathbf{p}}_m + L \begin{bmatrix} \dot\psi\cos\psi\cos\phi - \dot\phi\sin\psi\sin\phi \\ \dot\psi\cos\psi\sin\phi + \dot\phi\sin\psi\cos\phi \\ -\dot\psi\sin\psi \end{bmatrix} , \tag{1}$$

where $\mathbf{p}_m = (\mathbf{p}_l + \mathbf{p}_r)/2$. Furthermore, if the ϕ is constant and the wheels are rotating $\delta\theta_l$ and $\delta\theta_r$ radians in δt seconds, then the wheel positions are changing by

$$\delta\mathbf{p}_l = R\,\delta\theta_l \begin{bmatrix} \cos\phi \\ \sin\phi \\ 0 \end{bmatrix} \quad , \quad \delta\mathbf{p}_r = R\,\delta\theta_r \begin{bmatrix} \cos\phi \\ \sin\phi \\ 0 \end{bmatrix} . \tag{2}$$

As we take $\delta t \to 0$, ϕ becomes constant on this infinitely short δt interval, and so we obtain

$$\dot{\mathbf{p}}_l = R\,\dot\theta_l \begin{bmatrix} \cos\phi \\ \sin\phi \\ 0 \end{bmatrix} \quad , \quad \dot{\mathbf{p}}_r = R\,\dot\theta_r \begin{bmatrix} \cos\phi \\ \sin\phi \\ 0 \end{bmatrix} \quad , \quad \dot{\mathbf{p}}_m = R\,\dot\theta \begin{bmatrix} \cos\phi \\ \sin\phi \\ 0 \end{bmatrix} . \tag{3}$$

1.2 Pitch equilibrium

This model assumes that the center of mass is of the robot body is just above the axle at the middle when the robot is in equilibrium. In real it is not quite true, so one can shift the pitch angle by a constant ψ_0 and use $\psi - \psi_0$ in the model.

But notice that moving the body center of mass also changes the body moment of inertias, J_ψ and J_ϕ, so these quantities should be identified for a fixed ψ_0.

2 Motion equations

The system is divided into two subsystems: the driven segway robot and the driving DC motors.

2.1 Segway robot dynamics

The Lagrangian of the system (without the DC motor) is

$$L = E_t + E_r - E_p\,, \tag{4}$$

where E_t is the translational kinetic energy, E_r is the rotational kinetic energy and E_p is the potential energy defined as

$$\begin{aligned} E_t &= \frac{m}{2}\left(\dot{x}_l^2 + \dot{y}_l^2\right) + \frac{m}{2}\left(\dot{x}_r^2 + \dot{y}_r^2\right) + \frac{M}{2}\left(\dot{x}_c^2 + \dot{y}_c^2 + \dot{z}_c^2\right), \\ E_r &= \frac{J_w}{2}\dot{\theta}_l^2 + \frac{J_w}{2}\dot{\theta}_r^2 + \frac{J_\psi}{2}\dot{\psi}^2 + \frac{J_\phi}{2}\dot{\phi}^2\,, \\ E_p &= mgR + mgR + Mgz_c\,. \end{aligned} \tag{5}$$

Then to find the equations of motion with respect to ψ, ϕ and θ, we write the Euler-Lagrange equations for the F_ψ, F_ϕ, F_θ generalized forces (see appendix A) as

$$\begin{aligned} F_\psi &= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\psi}}\right) - \frac{\partial L}{\partial \psi} = \left(ML^2 + J_\psi\right)\ddot{\psi} + \left(MLR\cos\psi\right)\ddot{\theta} - ML^2\dot{\phi}^2\sin\psi\cos\psi - MgL\sin\psi\,, \\ F_\theta &= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = \left(\frac{MLR}{2}\cos\psi\right)\ddot{\psi} + \left(mR^2 + J_w + \frac{MR^2}{2}\right)\ddot{\theta} - \frac{MLR}{2}\dot{\psi}^2\sin\psi\,, \\ F_\phi &= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} = \left(mR^2 + J_w + \frac{2R^2}{W^2}\left(ML^2\sin^2\psi + J_\phi\right)\right)\ddot{\phi} + \frac{2MR^2L^2}{W^2}\dot{\phi}\dot{\psi}\sin\psi\cos\psi\,. \end{aligned} \tag{6}$$

2.2 DC motor dynamics

A detailed description of a DC motor model can be found in [3]. However, now we will use a simplified model, which captures only that the torque is damped by the shaft's rotational velocity and proportional to the applied power. So

$$F_{\theta_{l/r}} = -B(\dot{\theta}_{l/r} - \dot{\psi}) + Kv_{l/r}\,, \tag{7}$$

where B $(Nm \cdot rad/s)$ and K (Nm/V) are DC motor constants. Then we can express the generalized forces as

$$\begin{aligned} F_\psi &= -(F_{\theta_l} + F_{\theta_r}) = 2B(\dot{\theta} - \dot{\psi}) - K(v_l + v_r)\,, \\ F_\theta &= \frac{1}{2}(F_{\theta_l} + F_{\theta_r}) = -B(\dot{\theta} - \dot{\psi}) + K\frac{v_l + v_r}{2}\,, \\ F_\phi &= \frac{R}{W}(F_{\theta_r} - F_{\theta_l}) = -B\dot{\phi} + \frac{R}{W}(v_r - v_l)\,. \end{aligned} \tag{8}$$

2.3 State space form

By connecting the two subsystems and rearranging the equations, we get

$$\begin{aligned} \left(ML^2 + J_\psi\right)\ddot{\psi} + (MLR\cos\psi)\ddot{\theta} &= ML^2\dot{\phi}^2\sin\psi\cos\psi + MgL\sin\psi + 2B\left(\dot{\theta} - \dot{\psi}\right) - K\left(v_l + v_r\right), \\ (MLR\cos\psi)\ddot{\psi} + \left((2m + M)R^2 + 2J_w\right)\ddot{\theta} &= MLR\dot{\psi}^2\sin\psi - 2B\left(\dot{\theta} - \dot{\psi}\right) + K\left(v_l + v_r\right), \\ \left(mR^2W^2 + W^2J_w + 2R^2\left(ML^2\sin^2\psi + J_\phi\right)\right)\ddot{\phi} &= -2MR^2L^2\dot{\phi}\dot{\psi}\sin\psi\cos\psi - BW^2\dot{\phi} + RWK\left(v_r - v_l\right). \end{aligned} \tag{9}$$

By introducing the $\mathbf{x} = [\psi\ \theta\ \phi\ \dot\psi\ \dot\theta\ \dot\phi\ x_m\ y_m]^T$ state and the $\mathbf{v} = [v_l\ v_r]^T$ control, we can write (9) into

$$H(\mathbf{x})\,\dot{\mathbf{x}} = \widehat{f}(\mathbf{x}) + \widehat{G}\,\mathbf{v}\,, \tag{10}$$

where $H : \mathbb{R}^8 \to \mathbb{R}^{8\times 8}$, $\widehat{f} : \mathbb{R}^8 \to \mathbb{R}^8$, $\widehat{G} \in \mathbb{R}^{8\times 2}$,

$$H(\mathbf{x}) = \begin{bmatrix} \mathbf{I}_{3\times3} & \mathbf{0}_{3\times2} & \mathbf{0}_{3\times1} & \mathbf{0}_{3\times2} \\ \mathbf{0}_{2\times3} & \widehat{H}(\mathbf{x}) & \mathbf{0}_{2\times1} & \mathbf{0}_{1\times2} \\ \mathbf{0}_{1\times3} & \mathbf{0}_{1\times2} & \widehat{h}(\mathbf{x}) & \mathbf{0}_{1\times2} \\ \mathbf{0}_{2\times3} & \mathbf{0}_{2\times2} & \mathbf{0}_{2\times1} & \mathbf{I}_{2\times2} \end{bmatrix} , \quad \widehat{f}(\mathbf{x}) = \begin{bmatrix} \dot\psi \\ \dot\theta \\ \dot\phi \\ \widehat{f}_4(\mathbf{x}) \\ \widehat{f}_5(\mathbf{x}) \\ \widehat{f}_6(\mathbf{x}) \\ R\,\dot\theta\cos\phi \\ R\,\dot\theta\sin\phi \end{bmatrix} , \quad \widehat{G} = K \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ -1 & -1 \\ 1 & 1 \\ -RW & RW \\ 0 & 0 \\ 0 & 0 \end{bmatrix} , \tag{11}$$

with

$$\begin{aligned} &\widehat{H}(\mathbf{x}) = \begin{bmatrix} \widehat{H}_{11} & \widehat{H}_{12}(\mathbf{x}) \\ \widehat{H}_{12}(\mathbf{x}) & \widehat{H}_{22} \end{bmatrix} , \quad \begin{aligned} \widehat{H}_{11} &= ML^2 + J_\psi \\ \widehat{H}_{12}(\mathbf{x}) &= MLR\cos\psi \\ \widehat{H}_{22} &= (2m+M)R^2 + 2J_w \end{aligned} , \\ &\widehat{h}(\mathbf{x}) = mR^2W^2 + W^2J_w + 2R^2\big(ML^2\sin^2\psi + J_\phi\big)\,, \\ &\widehat{f}_4(\mathbf{x}) = ML^2\dot\phi^2\sin\psi\cos\psi + MgL\sin\psi + 2B\big(\dot\theta - \dot\psi\big)\,, \\ &\widehat{f}_5(\mathbf{x}) = MLR\dot\psi^2\sin\psi - 2B\big(\dot\theta - \dot\psi\big)\,, \\ &\widehat{f}_6(\mathbf{x}) = -2MR^2L^2\dot\phi\dot\psi\sin\psi\cos\psi - BW^2\dot\phi\,. \end{aligned} \tag{12}$$

Notice that $\det \widehat{H}(\mathbf{x}) > 0$ for all $\mathbf{x}$ because

$$\begin{aligned} \det \widehat{H}(\mathbf{x}) &= \big(ML^2 + J_\psi\big)\Big((2m+M)R^2 + 2J_w\Big) - \big(MLR\cos\psi\big)^2 \\ &> M^2L^2R^2 - M^2L^2R^2\cos^2\psi = M^2L^2R^2\big(1-\cos^2\psi\big) \ge 0\,. \end{aligned} \tag{13}$$

Furthermore, for all $\mathbf{x}$, we have $\widehat{h}(\mathbf{x}) > 0$ and so $\det H(\mathbf{x}) > 0$. Hence, $\exists H(\mathbf{x})^{-1}$ for all $\mathbf{x}$, and the state space form becomes

$$\dot{\mathbf{x}} = H(\mathbf{x})^{-1}\Big(\widehat{f}(\mathbf{x}) + \widehat{G}\,\mathbf{v}\Big) = H(\mathbf{x})^{-1}\widehat{f}(\mathbf{x}) + H(\mathbf{x})^{-1}\widehat{G}\,\mathbf{v} = f(\mathbf{x}) + G(\mathbf{x})\,\mathbf{v}\,, \tag{14}$$

where

$$f(\mathbf{x}) = \begin{bmatrix} \dot\psi \\ \dot\theta \\ \dot\phi \\ f_4(\mathbf{x}) \\ f_5(\mathbf{x}) \\ f_6(\mathbf{x}) \\ R\dot\theta\cos\phi \\ R\dot\theta\sin\phi \end{bmatrix} , \quad G(\mathbf{x}) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ g_4(\mathbf{x}) & g_4(\mathbf{x}) \\ g_5(\mathbf{x}) & g_5(\mathbf{x}) \\ -g_6(\mathbf{x}) & g_6(\mathbf{x}) \\ 0 & 0 \\ 0 & 0 \end{bmatrix} , \tag{15}$$

with

$$\begin{aligned} f_4(\mathbf{x}) &= \frac{\widehat{H}_{22}\widehat{f}_4(\mathbf{x}) - \widehat{H}_{12}(\mathbf{x})\widehat{f}_5(\mathbf{x})}{\widehat{H}_{11}\widehat{H}_{22} - \widehat{H}_{12}(\mathbf{x})^2} , & g_4(\mathbf{x}) &= \frac{-K\big(\widehat{H}_{22} + \widehat{H}_{12}(\mathbf{x})\big)}{\widehat{H}_{11}\widehat{H}_{22} - \widehat{H}_{12}(\mathbf{x})^2} , \\ f_5(\mathbf{x}) &= \frac{-\widehat{H}_{12}(\mathbf{x})\widehat{f}_4(\mathbf{x}) + \widehat{H}_{11}\widehat{f}_5(\mathbf{x})}{\widehat{H}_{11}\widehat{H}_{22} - \widehat{H}_{12}(\mathbf{x})^2} , & g_5(\mathbf{x}) &= \frac{K\big(\widehat{H}_{12}(\mathbf{x}) + \widehat{H}_{11}\big)}{\widehat{H}_{11}\widehat{H}_{22} - \widehat{H}_{12}(\mathbf{x})^2} , \\ f_6(\mathbf{x}) &= \frac{\widehat{f}_6(\mathbf{x})}{\widehat{h}(\mathbf{x})} , & g_6(\mathbf{x}) &= \frac{KRW}{\widehat{h}(\mathbf{x})}\,. \end{aligned} \tag{16}$$

A Derivation of the Euler-Lagrange equations

Notice that

$$\begin{aligned}
\dot{x}_c \cos\phi + \dot{y}_c \sin\phi &= \dot{x}_m \cos\phi + \dot{y}_m \sin\phi + L\dot{\psi}\cos\psi\big(\cos^2\phi + \sin^2\phi\big) = R\dot{\theta} + L\dot{\psi}\cos\psi\,,\\
\dot{x}_c \sin\phi - \dot{y}_c \cos\phi &= \dot{x}_m \sin\phi - \dot{y}_m \cos\phi - L\dot{\phi}\sin\psi\big(\sin^2\phi + \cos^2\phi\big) = -L\dot{\phi}\sin\psi\,.
\end{aligned} \tag{17}$$

Then

$$\begin{aligned}
F_\psi &= \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\psi}}\right) - \frac{\partial L}{\partial\psi}\\
&= \frac{d}{dt}\Big(ML\big(\dot{x}_c\cos\psi\cos\phi + \dot{y}_c\cos\psi\sin\phi - \dot{z}_c\sin\psi\big) + J_\psi\dot{\psi}\Big)\\
&\quad + ML\Big(\dot{x}_c\big(\dot{\psi}\sin\psi\cos\phi + \dot{\phi}\cos\psi\sin\phi\big) + \dot{y}_c\big(\dot{\psi}\sin\psi\sin\phi - \dot{\phi}\cos\psi\cos\phi\big) + \dot{z}_c\dot{\psi}\cos\psi - g\sin\psi\Big)\\
&= \frac{d}{dt}\Big(ML\cos\psi\big(\dot{x}_c\cos\phi + \dot{y}_c\sin\phi\big) + ML^2\dot{\psi}\sin^2\psi + J_\psi\dot{\psi}\Big)\\
&\quad + ML\Big(\dot{\psi}\sin\psi\big(\dot{x}_c\cos\phi + \dot{y}_c\sin\phi\big) + \dot{\phi}\cos\psi\big(\dot{x}_c\sin\phi - \dot{y}_c\cos\phi\big) - L\,\dot{\psi}^2\sin\psi\cos\psi - g\sin\psi\Big)\\
&\overset{(17)}{=} \frac{d}{dt}\Big(MLR\,\dot{\theta}\cos\psi + ML^2\,\dot{\psi} + J_\psi\dot{\psi}\Big)\\
&\quad + ML\big(R\,\dot{\theta}\,\dot{\psi}\sin\psi + L\,\dot{\psi}^2\sin\psi\cos\psi - L\,\dot{\phi}^2\cos\psi\sin\psi - L\,\dot{\psi}^2\sin\psi\cos\psi - g\sin\psi\big)\\
&= MLR\,\ddot{\theta}\cos\psi - MLR\,\dot{\theta}\,\dot{\psi}\sin\psi + \big(ML^2 + J_\psi\big)\ddot{\psi} + ML\big(R\,\dot{\theta}\,\dot{\psi}\sin\psi - L\,\dot{\phi}^2\sin\psi\cos\psi - g\sin\psi\big)\\
&= \big(ML^2 + J_\psi\big)\ddot{\psi} + \big(MLR\cos\psi\big)\ddot{\theta} - ML^2\,\dot{\phi}^2\sin\psi\cos\psi - MgL\sin\psi\,.
\end{aligned} \tag{18}$$

Furthermore, notice that $\partial\phi/\partial\theta_{l/r} = \mp R/W$ and

$$\begin{aligned}
\dot{x}_{l/r}\cos\phi + \dot{y}_{l/r}\sin\phi &= R\,\dot{\theta}_{l/r}\cos^2\phi + R\,\dot{\theta}_{l/r}\sin^2\phi = R\,\dot{\theta}_{l/r}\,,\\
\dot{x}_{l/r}\sin\phi - \dot{y}_{l/r}\cos\phi &= R\,\dot{\theta}_{l/r}\cos\phi\sin\phi - R\,\dot{\theta}_{l/r}\sin\phi\cos\phi = 0\,,\\
\dot{x}_c\cos\phi + \dot{y}_c\sin\phi &= R\,\dot{\theta}\cos^2\phi + L\big(\dot{\psi}\cos\psi\cos^2\phi - \dot{\phi}\sin\psi\sin\phi\cos\phi\big)\\
&\quad + R\,\dot{\theta}\sin^2\phi + L\big(\dot{\psi}\cos\psi\sin^2\phi + \dot{\phi}\sin\psi\cos\phi\sin\phi\big) = R\,\dot{\theta} + L\dot{\psi}\cos\psi\,,\\
\dot{x}_c\sin\phi - \dot{y}_c\cos\phi &= R\,\dot{\theta}\cos\phi\sin\phi + L\big(\dot{\psi}\cos\psi\cos\phi\sin\phi - \dot{\phi}\sin\psi\sin^2\phi\big)\\
&\quad - R\,\dot{\theta}\sin\phi\cos\phi - L\big(\dot{\psi}\cos\psi\sin\phi\cos\phi + \dot{\phi}\sin\psi\cos^2\phi\big) = -L\dot{\phi}\sin\psi\,.
\end{aligned} \tag{19}$$

Then the forces of the left/right axle endpoints can be expressed as

$$\begin{aligned}
F_{\theta_{l/r}} &= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_{l/r}}\right) - \frac{\partial L}{\partial \theta_{l/r}} \\
&= \frac{d}{dt}\Bigg(mR\big(\dot{x}_{l/r}\cos\phi + \dot{y}_{l/r}\sin\phi\big) \\
&\qquad\qquad + \frac{MR}{2}\Big[\dot{x}_c\Big(\cos\phi \pm \frac{2L}{W}\sin\psi\sin\phi\Big) + \dot{y}_c\Big(\sin\phi \mp \frac{2L}{W}\sin\psi\cos\phi\Big)\Big] + J_w\dot{\theta}_{l/r} \mp \frac{R}{W}J_\phi\dot{\phi}\Bigg) \\
&\quad \mp \frac{mR^2}{W}\big(\dot{x}_l\dot{\theta}_l\sin\phi - \dot{y}_l\dot{\theta}_l\cos\phi + \dot{x}_r\dot{\theta}_r\sin\phi - \dot{x}_r\dot{\theta}_r\cos\phi\big) \\
&\quad \mp \frac{MR}{W}\Big(\dot{x}_c\big[R\dot{\theta}\sin\phi + L\big(\dot{\psi}\cos\psi\sin\phi + \dot{\phi}\sin\psi\cos\phi\big)\big] \\
&\qquad\qquad + \dot{y}_c\big[-R\dot{\theta}\cos\phi + L\big(-\dot{\psi}\cos\psi\cos\phi + \dot{\phi}\sin\psi\sin\phi\big)\big]\Big) \\
&\overset{(19)}{=} \frac{d}{dt}\Big(mR^2\dot{\theta}_l + \frac{MR}{2}\big(R\dot{\theta} + L\dot{\psi}\cos\psi\big) \mp \frac{MRL^2}{W}\dot{\phi}\sin^2\psi + J_w\dot{\theta}_{l/r} \mp \frac{R}{W}J_\phi\dot{\phi}\Big) \\
&\quad \mp 0 \mp \frac{MR}{W}\Big(R\dot{\theta} + L\dot{\psi}\cos\psi\Big)\big(-L\dot{\phi}\sin\psi\big) \pm \frac{MRL}{W}\dot{\phi}\sin\psi\big(R\dot{\theta} + L\dot{\psi}\cos\psi\big) \\
&= mR^2\ddot{\theta}_{l/r} + \frac{MR}{2}\big(R\ddot{\theta} + L\ddot{\psi}\cos\psi - L\dot{\psi}^2\sin\psi\big) \mp \frac{MRL^2}{W}\big(\ddot{\phi}\sin^2\psi + 2\dot{\phi}\dot{\psi}\sin\psi\cos\psi\big) \\
&\qquad\qquad + J_w\ddot{\theta}_{l/r} \mp \frac{R}{W}J_\phi\ddot{\phi} \\
&= \Big(mR^2 + J_w\Big)\ddot{\theta}_{l/r} + \frac{MR^2}{2}\ddot{\theta} + \Big(\frac{MRL}{2}\cos\psi\Big)\ddot{\psi} - \frac{MRL}{2}\dot{\psi}^2\sin\psi \\
&\qquad\qquad \mp \frac{R}{W}\Big(ML^2\sin^2\psi + J_\phi\Big)\ddot{\phi} \mp \frac{2MRL^2}{W}\dot{\phi}\dot{\psi}\sin\psi\cos\psi\,.
\end{aligned} \tag{20}$$

By using F_{θ_l} and F_{θ_r}, we can compute the F_θ, F_ϕ generalized forces as

$$\begin{aligned}
F_\theta &= F_{\theta_l}\frac{\partial\theta}{\partial\theta_l} + F_{\theta_r}\frac{\partial\theta}{\partial\theta_r} = \frac{F_{\theta_l} + F_{\theta_r}}{2} \\
&= \Big(mR^2 + J_w + \frac{MR^2}{2}\Big)\ddot{\theta} + \Big(\frac{MLR}{2}\cos\psi\Big)\ddot{\psi} - \frac{MLR}{2}\dot{\psi}^2\sin\psi\,, \\
F_\phi &= F_{\theta_l}\frac{\partial\phi}{\partial\theta_l} + F_{\theta_r}\frac{\partial\phi}{\partial\theta_r} = \frac{R}{W}\Big(F_{\theta_r} - F_{\theta_r}\Big) \\
&= \Big(mR^2 + J_w + \frac{2R^2}{W^2}\big(ML^2\sin^2\psi + J_\phi\big)\Big)\ddot{\phi} + \frac{2MR^2L^2}{W^2}\dot{\phi}\dot{\psi}\sin\psi\cos\psi\,.
\end{aligned} \tag{21}$$

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