Fourier from the ground up

1 Prerequisites

We assume basic knowledge of real and complex numbers, calculus, and algebra. None of the following formulas should be any surprise to you:

$$e^{a+bi} = e^a (\cos b + i \sin b)$$

$$f(x) = \sum_{n \geq 0} \frac{1}{n!} f^{(n)}(0) x^n$$

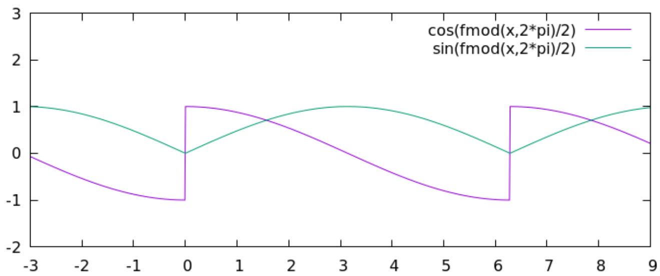
$$\int_a^b f'(x)\mathrm{d}x = f(b) - f(a)$$

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_a^x f(t)\mathrm{d}t = f(x)$$

We denote by $\mathbf{T}$ the quotient of the additive group $\mathbf{R}$ by the subgroup $2\pi\mathbf{Z}$. Thus, functions $f : \mathbf{R} \to \mathbf{C}$ that are 2π-periodic can be understood as functions $f : \mathbf{T} \to \mathbf{C}$, and their integral over one period is denoted indistinctly as

$$\int_{\mathbf{T}} f \quad = \quad \int_{\mathbf{T}} f(\theta)\,\mathrm{d}\theta \quad = \quad \int_0^{2\pi} f(\theta)\,\mathrm{d}\theta \quad = \quad \int_{-\pi}^{\pi} f(\theta)\,\mathrm{d}\theta$$

Notice that the elements of $\mathbf{T}$ can bijectively identified by numbers in $[0, 2\pi)$, but the topological spaces $\mathbf{T}$ and $[0, 2\pi)$ are different. For example, the function $\cos\frac{\theta}{2}$ is continuous on $[0, 2\pi)$ but discontinuous on $\mathbf{T}$, while the function $\sin\frac{\theta}{2}$ is continuous on both spaces, bot only differentiable on $[0, 2\pi)$.



2 The algebra of trigonometric polynomials

2.1 Definition. *A* trigonometric polynomial *is an expression of the form*

$$f(\theta) = \sum_{n\in\mathbf{Z}} F_n e^{in\theta}$$

where all the coefficients $F_n \in \mathbf{C}$ are zero except, maybe, a finite number of them. The set of all trigonometric polynomials is denoted by $\mathscr{P}$.

There are two ways to interpret a trigonometric polynomial: as a function $\mathbf{Z} \to \mathbf{C}$ defined by $n \mapsto F_n$, or as a function $\mathbf{R} \to \mathbf{C}$ defined by $\theta \mapsto f(\theta)$. Most of Fourier analysis deals with the duality between these two interpretations. There is a third interpretation, as finite Laurent series $f(z)$ evaluated on the unit circle $z = e^{i\theta}$, but it is of minor interest for the present course.

Let us introduce some **common language**. A trigonometric polynomial is usually called a *signal.* The indices n are called the *frequencies* and the coefficient F_n is called the *amplitude of f at the frequency n.* The mapping $n \mapsto |F_n|^2$ is called the *power spectrum* of the signal f. Building the signal from its amplitudes is called is called *synthesis*, and extracting the amplitudes from a signal is called *analysis.*

The monomial $e^{in\theta}$ is called a *pure wave of frequency n.* Thus, synthesis consists in creating a signal as a linear combination of pure waves, and analysis consists in recovering the coefficients of this linear combination. Using this language, we say that harmonic analysis consists in studying the duality between signals $f(\theta)$ and their spectra F_n; how do the operations on signals correspond to operations on their spectra, and vice-versa.

2.2 Proposition. *(Elementary properties) The following properties hold:*

1. *If $f \in \mathscr{P}$ then $f(\theta)$ is a function $\mathbf{R} \to \mathbf{C}$ which is 2π-periodic and $\mathscr{C}^\infty$.*
2. *If $f \in \mathscr{P}$ then F_n is a function $\mathbf{Z} \to \mathbf{C}$ of finite support.*
3. *The set $\mathscr{P}$ is a vector space over $\mathbf{C}$.*
4. *If $h = \lambda f + \mu g$ then $H_n = \lambda F_n + \mu G_n$.*
5. *The set $\mathscr{P}$ is an algebra (thus, closed by pointwise product $f(\theta)g(\theta)$)*

Proof. (1) Each monomial $e^{in\theta}$ is $\mathscr{C}^\infty$ and 2π-periodic, and f is a finite linear combination of such monomials, so it is also $\mathscr{C}^\infty$ and 2π-periodic. (2) This is a rewriting of the defintion of $\mathscr{P}$. (3,4) This result is immediate by linearity of finite sums. (5) The product of two finite sums is still a finite sum. □

The point (3) of this proposition is of fundamental importance. In a more general context, it is called the *superposition principle.* Although it is algebraically trivial, it may be non-intuitive when using the language of signal processing: when computing the sum of two signals, their amplitudes at each frequency add up separately. *There can be no destructive interference between different frequencies.*

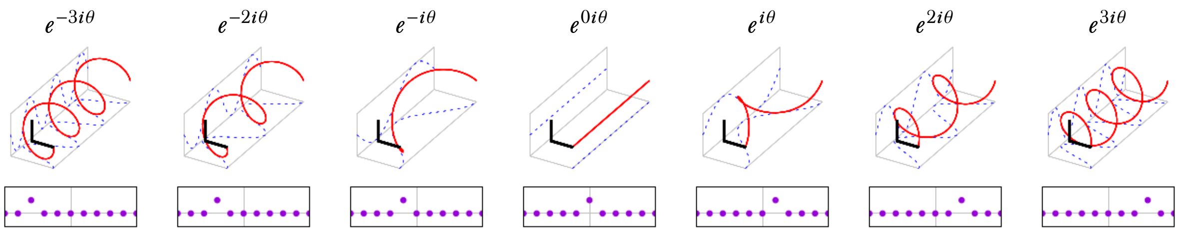
2.3 Proposition. *(Symmetry properties)*

1. f *even* $\iff$ F *even*
2. f *real* $\iff$ F *hermitian*
3. f *real and even* $\iff$ F *real and even*
4. f *real and odd* $\iff$ F *pure imaginary and odd*

2.4 Definition. *(Examples of trigonometric polynomials) Here we show several examples of trigonometric polynomials. They are all very important and will be used later. For each case we specify the values of their non-zero amplitudes.*

1. *The pure waves:* $e_n(\theta) = e^{in\theta}$ $\qquad F_n = 1$
2. *The pure waves with a phase offset:* $e_{n,\psi}(\theta) = e^{in(\theta+\psi)}$ $\qquad F_n = e^{in\psi}$
3. *The cosine waves:* $\cos(n\theta) = \dfrac{e^{in\theta} + e^{in\theta}}{2}$ $\qquad F_n = F_{-n} = \frac{1}{2}$
4. *The sine waves:* $\sin(n\theta) = \dfrac{e^{in\theta} - e^{in\theta}}{2i}$ $\qquad F_n = -F_{-n} = \frac{1}{2i}$
5. *The Dirichlet kernel:* $\mathcal{D}_N(\theta) = \displaystyle\sum_{n=-N}^{N} e^{in\theta}$ $\qquad F_n = 1\,,\ |n| \leq N$
6. *The Fejér kernel:* $\mathcal{K}_N(\theta) = \dfrac{1}{N}\displaystyle\sum_{n=0}^{N-1} \mathcal{D}_n(\theta)$ $\qquad F_n = \frac{N-|n|}{N}\,,\ |n| \leq N$
7. *The Gibbs kernel:* $\mathcal{G}_N(\theta) = \displaystyle\sum_{n=1}^{N} \dfrac{\sin n\theta}{n}$
8. *The odd sampling kernel:* $\mathcal{T}_{2N+1}(\theta) = \frac{1}{2N+1}\mathcal{D}_N(\theta)$
9. *The even sampling kernel:* $\mathcal{T}_{2N}(\theta) = \frac{\mathcal{D}_N(\theta)+\cos\frac{N\theta}{2}}{2N}$

When interpreting a trigonometric polynomial as a 2π-periodic function $\mathbf{R} \to \mathbf{C}$, it helps to plot it as a closed curve in the complex plane. The monomials $e^{in\theta}$ for $n \neq 0$ all correspond to the unit circle traversed n times, clockwise for $n < 0$, anticlockwise for $n > 0$.



Exercice 1. *Prove the following properties of the Dirichlet kernel* $\mathcal{D}_N$*:*

1. *It is a real-valued, even function:* $\mathcal{D}_N(\theta) = 1 + 2\sum_{n=1}^{N} \cos n\theta$
2. *It is a periodic generalization of the sinc function:* $\mathcal{D}_N(\theta) = \dfrac{\sin\left(N+\frac{1}{2}\right)\theta}{\sin\frac{\theta}{2}}$
3. $\mathcal{D}_N(\theta) = \sin N\theta \cot\dfrac{\theta}{2} + \cos N\theta$
4. *Value at zero:* $\mathcal{D}_N(0) = 2N+1$
5. *Zero values:* $\mathcal{D}_N(\theta) = 0 \iff \theta = \dfrac{2\pi k}{2N+1}, \quad k = \pm 1, \ldots, \pm N$
6. *The limit as* $N \to \infty$ *is zero almost everyhere:* $0 < |\theta| \leq \pi \implies \lim_{N\to\infty} \mathcal{D}_N(\theta) = 0$
7. *The integral does not depend on* N*:* $\int_{\mathbf{T}} \mathcal{D}_N = 2\pi$
8. *The* L^2 *norm goes to infinity very fast:* $\int_{\mathbf{T}} |\mathcal{D}_N|^2 =$
9. *The* L^1 *norm goes to infinity slowly* $\int_{\mathbf{T}} |\mathcal{D}_N| = c\log N + O(1) \qquad \text{as } N \to \infty$
10. *Convolution by* $\mathcal{D}_N$ *gives partial sums:* $f \in \mathcal{P} \implies \mathcal{D}_N * f = \mathcal{S}_N f$

Exercice 2. *Prove the following properties of the Féjer kernel* $\mathcal{K}_N$*:*

1. *The amplitudes decay linearly until zero:* $\mathcal{K}_N(\theta) = \sum_{|n|<N} \dfrac{N-|n|}{N} e^{in\theta}$
2. *It generalizes the squared sinc:* $\mathcal{K}_N(\theta) = \dfrac{\sin^2 N\frac{\theta}{2}}{N\sin^2\frac{\theta}{2}}$
3. $\mathcal{K}_N(0) = 2N+1$
4. $\mathcal{K}_N\left(\dfrac{2\pi k}{2N+1}\right) = 0 \qquad k = \pm 1, \ldots, \pm N$
5. *The limit as* $N \to \infty$ *is zero almost everyhere:* $0 < |\theta| \leq \pi \implies \lim_{N\to\infty} \mathcal{K}_N(\theta) = 0$
6. $\int_{\mathbf{T}} \mathcal{D}_N = 2\pi$
7. $\int_{\mathbf{T}} |\mathcal{D}_N|^2 =$
8. $\int_{\mathbf{T}} |\mathcal{D}_N| = c\log N + O(1) \qquad \text{as } N \to \infty$
9. $f \in \mathcal{P} \implies (\mathcal{D}_N * f)(\theta) = \sum_{|n|\leq N} F_n e^{in\theta}$
10. *Convolution by* $\mathcal{D}_N$ *gives Cesàro sums:* $f \in \mathcal{P} \implies \mathcal{K}_N * f = \dfrac{\mathcal{S}_0 f + \cdots + \mathcal{S}_{N-1} f}{N}$

2.1 Convolution theorems

Proposition 2.2(3) explains how to analyze the pointwise sum of two signals. The convolution theorem explains how to analyze their pointwise product:

2.5 Proposition (First convolution theorem). *Let $h(\theta) = f(\theta)g(\theta)$, then*

$$H_n = \sum_{k\in\mathbf{Z}} F_k G_{n-k}.$$

Proof. The proof is a verification, just as for 2.2(3), but we write it down fully because it is very important. By definition we have

$$h(\theta) = f(\theta)g(\theta) = \left(\sum_{n\in\mathbf{Z}} F_n e^{in\theta}\right)\left(\sum_{m\in\mathbf{Z}} G_m e^{in\theta}\right)$$

expanding the product:

$$= \sum_{n\in\mathbf{Z}}\sum_{m\in\mathbf{Z}} F_n G_m e^{i(n+m)\theta}$$

change of variable $m = k - n$ (keeping n constant)

$$= \sum_{n\in\mathbf{Z}}\sum_{k\in\mathbf{Z}} F_n G_{k-n} e^{ik\theta}$$

reordering

$$= \sum_{k\in\mathbf{Z}}\left(\sum_{n\in\mathbf{Z}} F_n G_{k-n}\right) e^{ik\theta}$$

thus, identifying the coefficients we have that $h_k = \sum_{n\in\mathbf{Z}} F_n G_{k-n}$. □

The sequence $h_n = \sum_{k\in\mathbf{Z}} F_k G_{n-k}$ is called the *discrete convolution* of the sequences F_n and G_n. Thus, the first convolution theorem says that the spectrum of a pointwise product is the convolution of the spectra.

The second convolution theorem is the dual version of this statement: the pointwise product of spectra is the spectrum of the convolution. For that, we need to define the convolution of two signals:

2.6 Definition (Convolution of two trigonometric polynomials). *Let $f, g \in \mathcal{P}$. We define the* periodic convolution *of f and g as*

$$(f * g)(\theta) = \int_0^{2\pi} f(t)g(\theta - t)\mathrm{d}t$$

2.7 Proposition. *The function $h = f * g$ is well-defined and it belongs to $\mathcal{P}$. Moreover, $h_n = 2\pi f_n g_n$.*

Proof. The expression that defines $(f * g)(\theta)$ is an integral of a C^∞ function on a compact domain, so it gives a finite, well-defined number. We can check that $f*g \in \mathcal{P}$ by computing the integral explicitly:

$$(f * g)(\theta) = \int_0^{2\pi} f(t)g(\theta - t)\mathrm{d}t =$$

□

2.2 Norm, product and energy conservation

2.3 Degree

2.4 Sampling and the DFT

2.5 Zero-padding and aliasing

2.6 Analytic properties

The set $\mathscr{P}$ of trigonometric polynomials $\mathscr{P}$ is like the set $\mathbf{Q}$ of rational numbers. They are both easy to define and manipulate, and all the results about them can be proved by finite arguments. However, they have the ugly property of not being complete nor closed under elementary operations. For example, the (pointwise) limit of a sequence of elements of $\mathscr{P}$ can be a non-smooth, or even a discontinuous function, that does not belong to $\mathscr{P}$. Even worse, the absolute value $|f|$ of a function $f \in \mathscr{P}$ is not typically an element of $\mathscr{P}$. There are several, different completions of the space $\mathscr{P}$. The most natural completion of $\mathscr{P}$ is the Hilbert space $L^2([0, 2\pi])$, that is obtained using the norm $\|f\|_2$ defined above. We will see that using stricter norms (involving the derivatives of f) we obtain completions that are smaller spaces (Sobolev spaces). And we can even obtain larger completions (periodic distributions) by using topologies that do not come from any norm.

2.7 Higher dimensions

A trigonometric polynomial of N coefficients is an expression of the form

$$f(\theta) = \sum_{n=0}^{N-1} f_n e^{in\theta}$$

An M-sampling is obtained by evaluating f at the M points $\theta_k = \frac{2\pi k}{M}$. Sampling theory studies the relationship between the vector of coefficients f_n and the vector of samples $f(\theta_k)$, depending on the values of M and N.

The ideal case is when $\boxed{N = M}$, or perfect sampling. In that case there are as many coefficients as samples, and there is a linear bijection between them, whose matrix is given by the DFT:

$$f(\theta_k) = \sum_{n=0}^{N-1} f_n e^{\frac{2\pi ikn}{N}} \qquad k = 0, \ldots N$$

This formula says that the IDFT of the vector f_n is the vector $f(\theta_k)$.

When $\boxed{M > N}$ we have more samples than coefficients. This is called oversampling, or redundant sampling, or zero-padding, depending on the point of view:

$$f(\theta_k) = \sum_{n=0}^{N-1} f_n e^{\frac{2\pi ikn}{M}} + \sum_{n=N}^{M-1} 0 \cdot e^{\frac{2\pi ikn}{M}} = \sum_{n=0}^{M-1} zp(f_n) e^{\frac{2\pi ikn}{M}}$$

where $zp(f_n)$ is the vector f_n padded by zeros until length M. This says that the IDFT of the vector $zp(f_n)$ is the vector $f(\theta_k)$.

When $\boxed{M < N}$ we have less samples than coefficients. This is called undersampling or aliasing. We can still write the samples as the IDFT of a vector involving the coefficients:

$$f(\theta_k) = \sum_{n=0}^{N-1} f_n e^{\frac{2\pi ikn}{M}} = \sum_{n=0}^{N-1} f_n e^{\frac{2\pi ik(n\%M)}{M}}$$

where we have performed the euclidean division of n by M, that gives $n = qM + (n\%M)$, and the term containing qM disappears because the exponential is 2π-periodic. Now, several indices n may have the same residue modulo M, and we can group them:

$$f(\theta_k) = \sum_{n=0}^{M-1} \left(\sum_{(n\%M)=m} f_n \right) e^{\frac{2\pi ikm}{M}} = \sum_{m=0}^{M-1} \operatorname{alias}(f_n) e^{\frac{2\pi ikm}{M}}$$

so the samples $f(\theta_k)$ are, as always the IDFT of the coefficients $\operatorname{alias}(f_n)$.

Note: The choice of interval for the indices $n \in \{0, \ldots, N-1\}$ may appear strange, the usual choice being $n \in \left\{-\frac{N}{2}, \ldots, \frac{N}{2} - 1\right\}$. But we can actually take an arbitrary interval $I_N \subset \mathbf{Z}$, as long as our euclidean residue is defined to be inside: $p\%N \in I_N$. With this notation, the formulas above become

This is a trigonometric polynomial of N coefficients:

$$F(\theta) = \sum_{n \in I_N} f_n e^{in\theta}$$

And those are M samples:

$$\theta_k = \frac{2\pi k}{M} \qquad k = 0, \ldots, M$$

Sampling theory studies the relationship between the vectors $(f_n) \in \mathbf{C}^N$ and $(F(\theta_k)) \in \mathbf{C}^M$.

The case $\boxed{M = N}$ is called ideal sampling. The relationship is a linear bijection between samples and coefficients that is, by definition, the `DFT`.

$$F(\theta_k) = \sum_{n \in I_M} f_n e^{\frac{2\pi i n k}{M}}$$

We have $F(\theta_k) = \texttt{IDFT}(f_n)$ (this is our definition of `IDFT`).

$\boxed{M > N}$ oversampling or zero-padding:

$$F(\theta_k) = \sum_{n \in I_N} f_n e^{\frac{2\pi i n k}{M}} + \sum_{n \in I_M \setminus I_N} 0 \cdot e^{\frac{2\pi i n k}{M}} = \sum_{m \in I_M} zp(f)_m e^{\frac{2\pi i m k}{M}}$$

We have $F(\theta_k) = \texttt{IDFT}(zp(f)_m)$, where

$$zp(f)_m = \begin{cases} f_m & \text{if } m \in I_N \\ 0 & \text{if } m \in I_M \setminus I_N \end{cases}$$

$\boxed{M < N}$ undersampling or aliasing:

$$F(\theta_k) = \sum_{n \in I_N} f_n e^{\frac{2\pi i n k}{M}} = \sum_{n \in I_N} f_n e^{\frac{2\pi i (n\%M) k}{M}} = \sum_{m \in I_M} \left(\sum_{(n\%M)=m} f_n \right) e^{\frac{2\pi i m k}{M}}$$

We still have $F(\theta_k) = \texttt{IDFT}(alias(f)_m)$, where

$$alias(f)_m = \sum_{(n \bmod_I M)=m} f_n$$