Fourier from the ground up

1 The algebra of trigonometric polynomials

Definition 1. *A* trigonometric polynomial *is an expression of the form*

$$f(\theta) = \sum_{n\in\mathbf{Z}} f_n e^{in\theta}$$

where all the coefficients $f_n \in \mathbf{C}$ are zero except, maybe, a finite number of them. The set of all trigonometric polynomials is denoted by $\mathscr{P}$.

There are two ways to interpret a trigonometric polynomial: as a function $\mathbf{Z} \to \mathbf{C}$ defined by $n \mapsto f_n$, or as a function $\mathbf{R} \to \mathbf{C}$ defined by $\theta \mapsto f(\theta)$. Most of Fourier analysis deals with the duality between these two interpretations.

Let us introduce some **common language**. A trigonometric polynomial is usually called a *signal*. The indices n are called the *frequencies* and the coefficient f_n is called the *amplitude of f at the frequency n*. The mapping $n \mapsto |f_n|^2$ is called the *power spectrum* of the signal f. Building the signal from its amplitudes is called is called *synthesis*, and extracting the amplitudes from a signal is called *analysis*.

The monomial $e^{in\theta}$ is called a *pure wave of frequency n*. Thus, synthesis consists in creating a signal as a linear combination of pure waves, and analysis consists in recovering the coefficients of this linear combination. The whole of harmonic analysis consists in studying the duality between signals $f(\theta)$ and their spectra f_n; how do the operations on signals correspond to operations on their spectra, and vice-versa.

Definition 2. *a b c*

Proposition 3. *(Elementary properties) The following properties hold:*

1. *If $f \in \mathscr{P}$ then $f(\theta)$ is a function $\mathbf{R} \to \mathbf{C}$ which is 2π-periodic and $\mathscr{C}^\infty$.*
2. *If $f \in \mathscr{P}$ then f_n is a function $\mathbf{Z} \to \mathbf{C}$ of finite support.*
3. *The set $\mathscr{P}$ is a vector space over $\mathbf{C}$.*
4. *If $h = \lambda f + \mu g$ then $h_n = \lambda f_n + \mu g_n$.*
5. *The set $\mathscr{P}$ is an algebra (thus, closed by pointwise product $f(\theta)g(\theta)$)*

Proof. (1) Each monomial $e^{in\theta}$ is $\mathscr{C}^\infty$ and 2π-periodic, and f is a finite linear combination of such monomials, so it is also $\mathscr{C}^\infty$ and 2π-periodic. (2) This is a rewriting of the defintion of $\mathscr{P}$. (3,4) This result is immediate by linearity of finite sums. (5) The product of two finite sums is still a finite sum. □

When interpreting a trigonometric polynomial as a 2π-periodic function $\mathbf{R} \to \mathbf{C}$, it helps to plot it as a closed curve in the complex plane. The monomials $e^{in\theta}$ for $n \neq 0$ all correspond to the unit circle traversed n times, clockwise for $n < 0$, anticlockwise for $n > 0$.