Expansive homeomorphisms on complexity quasi-metric spaces

A bridge between dynamical systems and computational complexity theory

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Abstract

The complexity quasi-metric, introduced by Schellekens, provides a topological framework where the asymmetric nature of computational comparisons—stating that one algorithm is faster than another carries different information than stating the second is slower than the first—finds precise mathematical expression. In this paper we develop a comprehensive theory of expansive homeomorphisms on complexity quasi-metric spaces. Our central result establishes that the scaling transformation $\psi_\alpha(f)(n) = \alpha f(n)$ is expansive on the complexity space $(\mathcal{C}, d_{\mathcal{C}})$ if and only if $\alpha \neq 1$. The δ-stable sets arising from this dynamics correspond exactly to asymptotic complexity classes, providing a dynamical characterisation of fundamental objects in complexity theory. We prove that the canonical coordinates associated with ψ_α are hyperbolic with contraction rate $\lambda = 1/\alpha$ and establish a precise connection between orbit separation in the dynamical system and the classical time hierarchy theorem of Hartmanis and Stearns. We further investigate unstable sets, conjugate dynamics, and topological entropy estimates for the scaling map. Throughout, concrete algorithms and Python implementations accompany the proofs, making every result computationally reproducible. SageMath verification snippets are inlined alongside the examples, and the full code is available in the companion repository.

Keywords: Expansive homeomorphism; complexity quasi-metric; computational complexity; asymmetric topology; dynamical systems; stable and unstable sets; canonical coordinates

2020 MSC: 54H20; 37B20; 68Q25; 54E35

"The study of iteration, the study of the behaviour of a transformation when it is repeated, is the fundamental problem of dynamics."

— Henri Poincaré

1 Introduction: where dynamics meets computation

1.1 A tale of two theories

Mathematics draws much of its power from unexpected bridges between seemingly distant subjects. Fourier analysis connects the study of heat flow to number theory; ergodic theory links statistical mechanics to measure-preserving transformations; and category theory reveals shared structure across algebra, topology, and logic. This paper explores another such bridge: between *dynamical systems*—the study of how systems evolve over time—and *computational complexity theory*—the study of the intrinsic difficulty of computational problems.

At first glance these two fields appear to have little in common. Dynamical systems theory asks: given a map $\psi\colon X \to X$, how do the orbits $\{x, \psi(x), \psi^2(x), \ldots\}$ behave as the number of iterations grows? Do nearby orbits stay close, or do they diverge? If they diverge, how quickly? Complexity theory, on the other hand, asks: given a computational problem, how do the resources required to solve it—time, memory, communication—grow with the input size n?

The key observation that connects these two worlds is that *complexity comparisons are inherently asymmetric*. Saying "algorithm A is at most as fast as algorithm B" is not the same as saying "algorithm B is at most as slow as algorithm A." More precisely, if $f(n) \leq g(n)$ for all n (so that f is at least as fast as g), the "cost" of moving from f to g in the complexity landscape is zero—we are moving to a slower algorithm, which is easy to simulate—but the cost of moving from g to f is positive, because improving an algorithm's running time requires genuine insight.

This asymmetry is captured perfectly by the mathematical notion of a *quasi-metric*: a distance function q where $q(x, y)$ need not equal $q(y, x)$. Schellekens [10] introduced the *complexity quasi-metric* $d_{\mathcal{C}}$ on the space of functions $f\colon \mathbb{N} \to (0, \infty)$ and showed that its topological properties encode fundamental features of computational complexity. Romaguera and Schellekens [9] subsequently developed the quasi-metric structure of complexity spaces in depth.

On the dynamical side, *expansive homeomorphisms*—maps under which every pair of distinct points is eventually separated by more than some fixed threshold δ—are a central object of study, going back to Utz [12] and developed extensively by Bowen [2] and Reddy [8]. Recently, Olela Otafudu, Matladi, and Zweni [7] extended the theory of expansive homeomorphisms to quasi-metric spaces, opening the door to applications in asymmetric settings.

Our contribution is to walk through that door and apply the theory of expansive homeomorphisms to the complexity quasi-metric space. The results reveal that basic dynamical concepts—orbits, stable sets, hyperbolicity—have direct and illuminating counterparts in complexity theory.

Related work. The complexity quasi-metric space was introduced by Schellekens [10], who established its basic topological properties and connections to denotational semantics. Romaguera and Schellekens [9] subsequently developed the quasi-metric structure in depth, proving completeness results and studying the Smyth completion. On the dynamical side, expansive homeomorphisms on metric spaces have a long history, beginning with Utz [12] and substantially advanced by Bowen [2], who connected expansiveness to topological entropy, and Reddy [8], who established canonical coordinate systems for expansive maps. The extension of expansive homeomorphisms to quasi-metric spaces was carried out by Olela Otafudu, Matladi, and Zweni [7], who proved that q-expansiveness and q^t-expansiveness are equivalent and developed the abstract theory of stable and unstable sets in the asymmetric setting. Künzi [5, 6] provided a comprehensive account of the topology of quasi-metric spaces, which underpins the present work.

The novelty of our approach lies in the *combination* of these two lines of research: whereas [7] develops the abstract theory without a specific quasi-metric space in mind, and [10, 9] study the complexity space without dynamical-systems tools, we bring the two together and show that the resulting interplay yields concrete insights in both directions—dynamical concepts acquire computational meaning, and complexity-theoretic distinctions acquire dynamical characterisations.

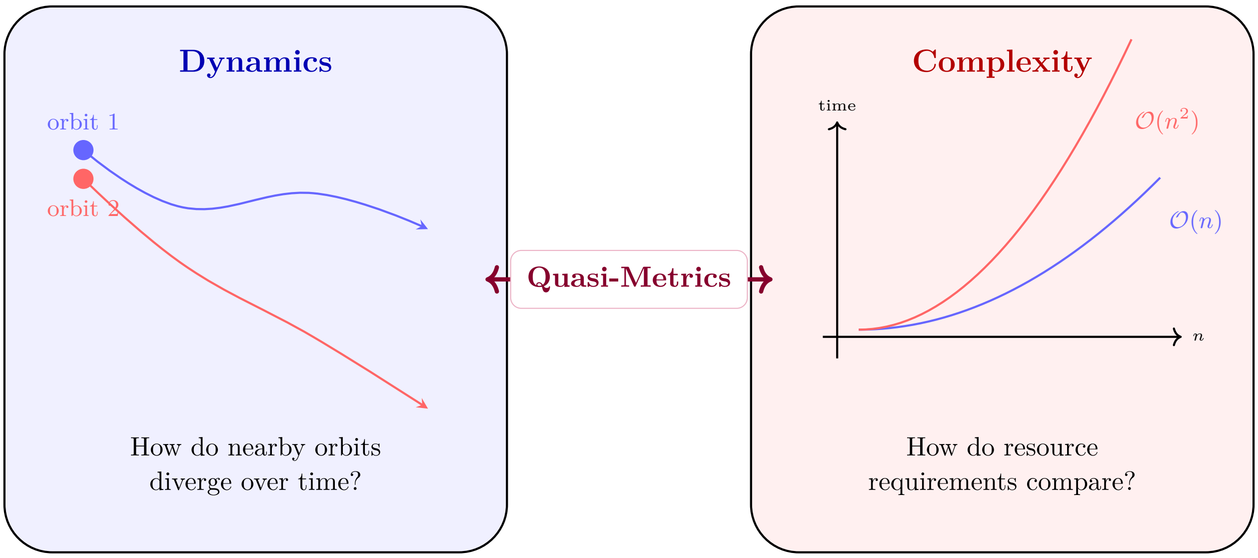

Figure 1: Two worlds connected by quasi-metrics. Dynamical systems study orbit divergence; complexity theory studies resource growth. The asymmetry inherent in both settings is encoded by the quasi-metric distance.

1.2 Why quasi-metrics?

Before we proceed, let us briefly motivate why quasi-metrics—rather than ordinary metrics—are the right tool for this investigation.

In a standard metric space (X, d), the symmetry axiom $d(x, y) = d(y, x)$ ensures that the cost of moving from x to y is the same as the cost of moving from y to x. This is natural in many geometric settings, but it is *unnatural* in computational settings. Consider two algorithms with running times $f(n) = n$ and $g(n) = n^2$. Given the faster algorithm f, one can trivially simulate the slower algorithm g by wasting time. But given the slower algorithm g, one cannot in general produce the faster algorithm f without effort. The "distance" from fast to slow should therefore be zero (or small), while the distance from slow to fast should be positive.

This is exactly what a quasi-metric provides. By dropping the symmetry axiom, quasi-metrics can encode directional costs, and the complexity quasi-metric $d_{\mathcal{C}}$ does precisely this: $d_{\mathcal{C}}(f, g) = 0$ whenever $f(n) \leq g(n)$ for all n (fast to slow is free), while $d_{\mathcal{C}}(g, f) > 0$ when g is genuinely slower (slow to fast is costly).

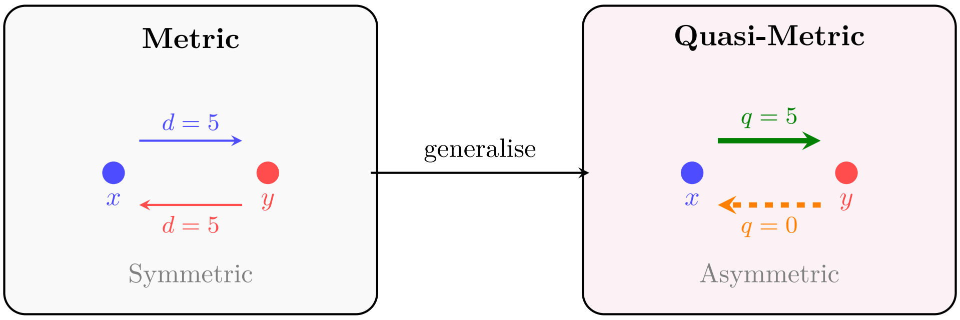

Figure 2: From metrics to quasi-metrics. Dropping symmetry allows the distance to encode directional information.

1.3 Overview of main results

We now give a roadmap of the paper and preview our main contributions. The reader may find it helpful to refer back to this summary as the technical details unfold.

Our first result characterises exactly when the scaling transformation is expansive.

5.8

The scaling map $\psi_\alpha(f)(n) = \alpha f(n)$ is expansive on the complexity space $(\mathcal{C}, d_{\mathcal{C}})$ if and only if $\alpha \neq 1$.

Our second result reveals that the stable sets of this dynamics have a beautiful complexity-theoretic interpretation.

6.2

For $\alpha > 1$, the δ-stable set of f under ψ_α coincides with the set $\{g : d_{\mathcal{C}}(f, g) \leq \delta\}$, which contains all functions g with $g(n) \geq f(n)$ for every n.

Our third result establishes a precise form of hyperbolicity.

7.1

The canonical coordinates of ψ_α ($\alpha > 1$) exhibit exponential contraction with rate $\lambda = 1/\alpha$ and constant $C = 1$.

Our fourth result connects the dynamical picture to a classical theorem in complexity theory.

8.1

If $f(n) \log f(n) = o(g(n))$, then the orbits of f and g under ψ_α separate in the symmetrized quasi-metric beyond every threshold.

Each of these results is proved in detail and accompanied by an algorithm and a Python implementation. All code is available in the companion repository.

1.4 Organisation

The paper is organised as follows. Section 2 recalls the basic theory of quasi-metric spaces, with examples and motivation. Section 3 introduces the complexity quasi-metric of Schellekens and establishes its fundamental properties, including several illustrative computations. Section 4 defines expansive homeomorphisms in the quasi-metric setting. Section 5 introduces the scaling transformation and proves our main expansiveness result. Section 6 develops the theory of stable and unstable sets. Section 7 establishes hyperbolicity. Section 8 connects orbit separation to the time hierarchy theorem. Section 9 discusses topological entropy estimates. Section 10 summarises the results and poses open problems. All Python implementations and SageMath verification scripts are available in the companion repository (`code/` directory).

2 Quasi-metric spaces

We begin by recalling the foundational notion of a quasi-metric space. The theory of quasi-metric spaces has a long history, with major contributions by Künzi [5, 6], Cobzaş [3], and many others. We follow the notation and conventions of Olela Otafudu et al. [7].

Definition 2.1 (Quasi-metric)**.** Let X be a non-empty set. A function $q\colon X \times X \to [0,\infty)$ is a *quasi-metric* on X if it satisfies the following three axioms for all $x, y, z \in X$:

(Q1) $q(x,x) = 0$;

(Q2) $q(x,z) \le q(x,y) + q(y,z)$ (triangle inequality);

(Q3) $q(x,y) = 0 = q(y,x) \;\Rightarrow\; x = y$ (T_0 separation).

The pair (X,q) is called a *quasi-metric space.*

Notice that the only axiom "missing" compared with a metric is symmetry: we do *not* require $q(x,y) = q(y,x)$. This single change opens up a surprisingly rich theory.

Every quasi-metric q gives rise to two natural companions.

Definition 2.2 (Conjugate and symmetrization)**.** Let (X,q) be a quasi-metric space. The *conjugate quasi-metric* is $q^t(x,y) := q(y,x)$. The *symmetrization* is $q^s(x,y) := \max\{q(x,y), q^t(x,y)\} = \max\{q(x,y), q(y,x)\}$.

It is straightforward to verify that q^t is again a quasi-metric and that q^s is a genuine metric on X. Thus every quasi-metric space carries a canonical metric, obtained by taking the "worst-case direction" of the asymmetric distance.

Example 2.3 (Standard quasi-metric on $\mathbb{R}$)**.** Define $u\colon \mathbb{R} \times \mathbb{R} \to [0,\infty)$ by

$$u(x,y) \;=\; (y-x)^+ \;=\; \max\{0,\, y-x\}.$$

Then u is a quasi-metric: (Q1) is immediate; (Q2) follows from $(z-x)^+ \le (y-x)^+ + (z-y)^+$; and (Q3) holds because $u(x,y) = 0 = u(y,x)$ gives $y \le x$ and $x \le y$, hence $x = y$.
The conjugate is $u^t(x,y) = (x-y)^+$, and the symmetrization is $u^s(x,y) = |x-y|$, the usual absolute-value metric.

The quasi-metric u has a vivid interpretation: *going uphill costs effort, while going downhill is free.*

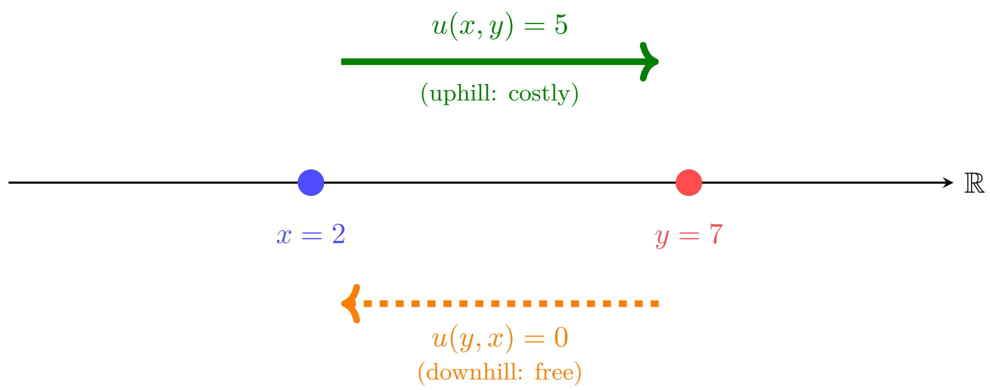

Figure 3: The standard quasi-metric on $\mathbb{R}$: going from $x = 2$ up to $y = 7$ costs $u(2, 7) = 5$, but going from $y = 7$ down to $x = 2$ is free: $u(7, 2) = 0$.

Example 2.4 (Weighted quasi-metric on $\mathbb{R}$)**.** For any weight $w > 0$, define $u_w(x, y) = w \cdot (y - x)^+$. This is again a quasi-metric, and it models the situation where the cost of going uphill is proportional to w. When $w = 1$ we recover the standard quasi-metric. The symmetrization is $u^s_w(x, y) = w|x - y|$.

Example 2.5 (Discrete quasi-metric)**.** On any set X, define

$$q_d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y. \end{cases}$$

This is both a metric and a quasi-metric (the symmetric case). It illustrates that every metric is automatically a quasi-metric.

Example 2.6 (Non-example: failing the triangle inequality)**.** On $\mathbb{R}$, define $\rho(x, y) = (y - x)^2$ if $y \geq x$ and $\rho(x, y) = 0$ if $y < x$. Then ρ satisfies (Q1) and (Q3), but it fails (Q2): take $x = 0$, $y = 2$, $z = 3$. Then $\rho(x, z) = 9$, while $\rho(x, y) + \rho(y, z) = 4 + 1 = 5 < 9$. Hence ρ is *not* a quasi-metric. This illustrates that the triangle inequality is a genuine restriction even in the asymmetric setting.

Example 2.7 (Asymmetric topologies on $\{a, b, c\}$)**.** Let $X = \{a, b, c\}$ with $q(a, b) = 1$, $q(b, a) = 3$, $q(a, c) = 2$, $q(c, a) = 0$, $q(b, c) = 1$, $q(c, b) = 2$, and $q(x, x) = 0$ for all x. One can verify that the triangle inequality holds. The forward topology τ_q has open ball $B_q(a, 1.5) = \{a, b\}$, while the conjugate topology τ_{q^t} has $B_{q^t}(a, 1.5) = \{a, c\}$ (since $q^t(a, c) = q(c, a) = 0 < 1.5$). Thus the two topologies differ: points that are "close" to a depend on the direction in which we measure distance. This is the hallmark of genuine asymmetry.

Remark 2.8 (Topological considerations)**.** A quasi-metric q on X generates a topology τ_q via the open balls $B_q(x, \varepsilon) := \{y \in X : q(x, y) < \varepsilon\}$. The conjugate q^t generates a potentially different topology τ_{q^t}. These two topologies coincide if and only if q is symmetric, i.e., if q is a metric. The study of these asymmetric topologies is a central theme in the work of Künzi [5, 6] and has deep connections to domain theory in computer science [1, 11].

3 The complexity quasi-metric space

With the general theory in hand, we now turn to the specific quasi-metric space that lies at the heart of this paper. The *complexity space* was introduced by Schellekens [10] and further studied by Romaguera and Schellekens [9]. It provides a topological framework in which the resource-usage functions associated with algorithms live naturally.

3.1 Definition and basic properties

Let $\mathcal{C}$ denote the set of all functions $f\colon \mathbb{N} \to (0, \infty)$. Each such function represents the running-time profile of an algorithm: $f(n)$ is the time required on inputs of size n.

Definition 3.1 (Complexity quasi-metric [10])**.** The *complexity quasi-metric* $d_{\mathcal{C}}\colon \mathcal{C} \times \mathcal{C} \to [0, \infty)$ is defined by

$$d_{\mathcal{C}}(f, g) \;=\; \sum_{n=1}^{\infty} 2^{-n} \max\left\{0,\; \frac{1}{g(n)} - \frac{1}{f(n)}\right\}.$$

The reciprocals $1/f(n)$ and $1/g(n)$ should be thought of as *efficiency measures*: a faster algorithm has a larger reciprocal. The difference $1/g(n) - 1/f(n)$ is positive when g is more efficient than f at input size n, and the weighting 2^{-n} ensures convergence of the series.

Theorem 3.2 (Basic properties of $d_{\mathcal{C}}$)**.** *The following hold:*

(i) *$d_{\mathcal{C}}$ is a quasi-metric on $\mathcal{C}$.*

(ii) *$d_{\mathcal{C}}(f, g) = 0$ if and only if $f(n) \le g(n)$ for all $n \in \mathbb{N}$.*

(iii) *$d_{\mathcal{C}}(f, g) \le 1$ for all $f, g \in \mathcal{C}$.*

Proof. **(i)** Axiom (Q1): $d_{\mathcal{C}}(f, f) = \sum 2^{-n} \max\{0, 0\} = 0$. The triangle inequality (Q2): for each n,

$$\max\left\{0, \frac{1}{h(n)} - \frac{1}{f(n)}\right\} \le \max\left\{0, \frac{1}{g(n)} - \frac{1}{f(n)}\right\} + \max\left\{0, \frac{1}{h(n)} - \frac{1}{g(n)}\right\},$$

since the positive part is subadditive. Multiplying by 2^{-n} and summing gives $d_{\mathcal{C}}(f, h) \le d_{\mathcal{C}}(f, g) + d_{\mathcal{C}}(g, h)$. Axiom (Q3): if $d_{\mathcal{C}}(f, g) = 0 = d_{\mathcal{C}}(g, f)$, then $f(n) \le g(n)$ and $g(n) \le f(n)$ for all n, so $f = g$.
(ii) $d_{\mathcal{C}}(f, g) = 0$ iff each term is zero, iff $1/g(n) \le 1/f(n)$ for all n, iff $f(n) \le g(n)$.
(iii) Each term $2^{-n} \max\{0,\, 1/g(n) - 1/f(n)\} \le 2^{-n} \cdot 1/g(n) \le 2^{-n}$, so $d_{\mathcal{C}}(f, g) \le \sum_{n=1}^{\infty} 2^{-n} = 1$. □

Property (ii) is the key asymmetry result: *moving from a faster function to a slower one is free*, because $f(n) \le g(n)$ (i.e., f is faster) implies $d_{\mathcal{C}}(f, g) = 0$.

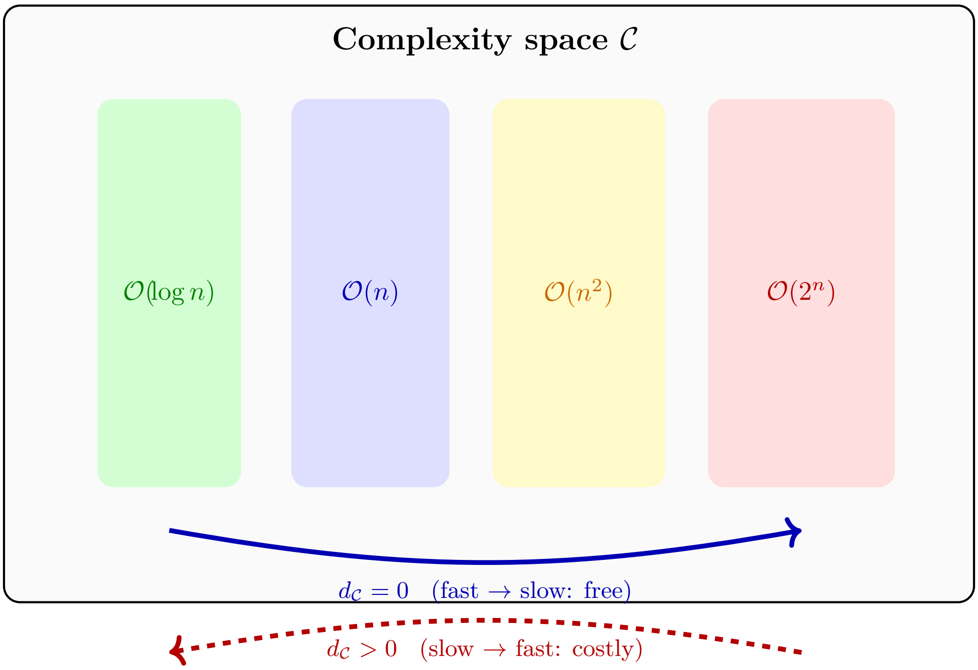

Figure 4: The complexity landscape. Moving from a faster class to a slower one is free ($d_{\mathcal{C}} = 0$); the reverse direction is costly ($d_{\mathcal{C}} > 0$).

3.2 Illustrative examples

To build intuition, let us compute $d_{\mathcal{C}}$ for several pairs of common complexity functions.

Example 3.3 (Linear vs. quadratic)**.** Let $f(n) = n$ and $g(n) = n^2$. Since $f(n) \le g(n)$ for all $n \ge 1$, Theorem 3.2(ii) gives $d_{\mathcal{C}}(f, g) = 0$. In the reverse direction,

$$d_{\mathcal{C}}(g, f) = \sum_{n=1}^{\infty} 2^{-n} \max\left\{0, \frac{1}{n} - \frac{1}{n^2}\right\} = \sum_{n=2}^{\infty} 2^{-n} \cdot \frac{n-1}{n^2},$$

since the $n = 1$ term vanishes ($\frac{1}{1} - \frac{1}{1} = 0$). We compute the first few partial sums to see how the series converges:

$$\begin{aligned} S_2 &= \tfrac{1}{4} \cdot \tfrac{1}{4} = 0.0625, \\ S_3 &= S_2 + \tfrac{1}{8} \cdot \tfrac{2}{9} = 0.0625 + 0.0278 = 0.0903, \\ S_4 &= S_3 + \tfrac{1}{16} \cdot \tfrac{3}{16} = 0.0903 + 0.0117 = 0.1020, \\ S_5 &= S_4 + \tfrac{1}{32} \cdot \tfrac{4}{25} = 0.1020 + 0.0050 = 0.1070. \end{aligned}$$

The series converges rapidly due to the 2^{-n} factor; by $n = 10$ the partial sum is already 0.1108, within 0.001 of the limit. Analytically, splitting $\frac{n-1}{n^2} = \frac{1}{n} - \frac{1}{n^2}$ and using $\sum_{n=1}^{\infty} \frac{x^n}{n} = -\ln(1 - x)$ and $\sum_{n=1}^{\infty} \frac{x^n}{n^2} = \mathrm{Li}_2(x)$ at $x = \frac{1}{2}$ gives the closed form

$$d_{\mathcal{C}}(g, f) = \ln 2 - \mathrm{Li}_2(\tfrac{1}{2}) \approx 0.693 - 0.582 = 0.111.$$

The exact value of the series can be confirmed symbolically using SageMath; see `complexity_distances.sage` in the companion repository. The asymmetry is clear: moving from linear to quadratic is free, but moving from quadratic to linear costs approximately 0.111.

Example 3.4 (Logarithmic vs. linear)**.** Let $f(n) = \ln(n+1)$ and $g(n) = n$. Since $\ln(n+1) \le n$ for all $n \ge 1$, we have $d_{\mathcal{C}}(f,g) = 0$. But $d_{\mathcal{C}}(g,f) > 0$ because g is slower. The partial sums are:

$$S_1 = \tfrac{1}{2}\big(\tfrac{1}{\ln 2} - 1\big) \approx 0.2213, \quad S_2 = S_1 + \tfrac{1}{4}\big(\tfrac{1}{\ln 3} - \tfrac{1}{2}\big) \approx 0.3179,$$
$$S_3 \approx 0.3609, \quad S_5 \approx 0.3991, \quad S_{10} \approx 0.4165.$$

The limit is $d_{\mathcal{C}}(g,f) \approx 0.417$; for exact symbolic evaluation see `complexity_distances.sage`.

Example 3.5 (Equal functions)**.** If $f = g$, then $d_{\mathcal{C}}(f,g) = d_{\mathcal{C}}(g,f) = 0$. The distance is symmetric (and zero) for identical functions, as expected.

Example 3.6 (Constant shift)**.** Let $f(n) = n$ and $g(n) = n + c$ for some constant $c > 0$. Then $f(n) < g(n)$ for all n, so $d_{\mathcal{C}}(f,g) = 0$. In the reverse direction, $d_{\mathcal{C}}(g,f) = \sum_{n=1}^{\infty} 2^{-n} \cdot c/(n(n+c))$, which is small but positive—reflecting the fact that g is only slightly slower.

Example 3.7 (Incomparable functions)**.** Let $f(n) = n + (-1)^{n+1}$ and $g(n) = n$. Then $f(1) = 2 > 1 = g(1)$ but $f(2) = 1 < 2 = g(2)$, so neither $f(n) \le g(n)$ nor $g(n) \le f(n)$ for all n. (Note that f alternates above and below g: f exceeds g at odd indices and falls below at even indices.) Consequently, *both* $d_{\mathcal{C}}(f,g) > 0$ and $d_{\mathcal{C}}(g,f) > 0$. Only odd-index terms contribute to $d_{\mathcal{C}}(f,g)$:

$$d_{\mathcal{C}}(f,g) = \sum_{\substack{n\ge 1\\ n \text{ odd}}} \frac{2^{-n}}{n(n+1)} = \tfrac{1}{2} \cdot \tfrac{1}{2} + \tfrac{1}{8} \cdot \tfrac{1}{12} + \cdots \approx 0.262.$$

Similarly, only even-index terms contribute to $d_{\mathcal{C}}(g,f)$:

$$d_{\mathcal{C}}(g,f) = \sum_{\substack{n\ge 2\\ n \text{ even}}} \frac{2^{-n}}{n(n-1)} = \tfrac{1}{4} \cdot \tfrac{1}{2} + \tfrac{1}{16} \cdot \tfrac{1}{12} + \cdots \approx 0.131.$$

The symmetrized distance is $d^s_{\mathcal{C}}(f,g) = \max\{0.262, 0.131\} \approx 0.262$. Observe that $d_{\mathcal{C}}(f,g) \approx 2\, d_{\mathcal{C}}(g,f)$: the "upward oscillation" is costlier than the "downward" one, reflecting the asymmetry of the quasi-metric. The exact sums are verified symbolically in `incomparable_functions.sage`.

Example 3.8 (Counterexample: unbounded reciprocal difference)**.** Let $f(n) = 1/n$ (efficiency improves with n) and $g(n) = 1$. Then $1/g(n) - 1/f(n) = 1 - n$, which is negative for $n \ge 2$, so only the $n = 1$ term contributes: $d_{\mathcal{C}}(f,g) = 2^{-1}\max\{0, 1-1\} = 0$. In the reverse direction, $1/f(n) - 1/g(n) = n - 1$, so $d_{\mathcal{C}}(g,f) = \sum_{n=2}^{\infty} 2^{-n}(n-1) = 1$. This example saturates the upper bound of Theorem 3.2(iii), showing that $d_{\mathcal{C}}(f,g) \le 1$ is tight.

3.3 The conjugate and symmetrized complexity distances

Since $d_{\mathcal{C}}$ is a quasi-metric, we automatically obtain the conjugate and symmetrization.

Proposition 3.9. *The conjugate of the complexity quasi-metric is*

$$d^t_{\mathcal{C}}(f,g) \;=\; d_{\mathcal{C}}(g,f) \;=\; \sum_{n=1}^{\infty} 2^{-n} \max\left\{0, \frac{1}{f(n)} - \frac{1}{g(n)}\right\}.$$

The symmetrization is $d^s_{\mathcal{C}}(f,g) = \max\{d_{\mathcal{C}}(f,g), d_{\mathcal{C}}(g,f)\}$.

Proof. By Definition 2.2, $d^t_{\mathcal{C}}(f,g) = d_{\mathcal{C}}(g,f)$. Substituting g for the first argument and f for the second in Definition 3.1 yields the stated formula. That $d^s_{\mathcal{C}} = \max\{d_{\mathcal{C}}, d^t_{\mathcal{C}}\}$ is a metric follows from the general theory (Definition 2.2). □

The symmetrized distance $d^s_{\mathcal{C}}$ is a genuine metric on $\mathcal{C}$ and measures the "worst-case directional cost" between two complexity functions.

Example 3.10 (Symmetrization of linear vs. quadratic)**.** From Example 3.3, $d_{\mathcal{C}}(f, g) = 0$ and $d_{\mathcal{C}}(g, f) \approx 0.111$, so $d^s_{\mathcal{C}}(f, g) \approx 0.111$.

3.4 Computing the complexity quasi-metric

We now describe an algorithm for numerically approximating $d_{\mathcal{C}}(f, g)$. Since the series involves infinitely many terms, we truncate at N terms; the exponential decay of 2^{-n} ensures rapid convergence.

Algorithm 1: Compute $d_{\mathcal{C}}(f, g)$

```
Input: Functions f, g; truncation parameter N
Output: Approximation of d_C(f, g)
S ← 0
for n ← 1 to N do
    Δ ← 1/g(n) − 1/f(n)
    if Δ > 0 then
        S ← S + 2^{−n} Δ
    end
end
return S
```

Remark 3.11 (Convergence rate)**.** The truncation error after N terms is at most 2^{-N}, since each omitted term contributes at most 2^{-n}. In practice, $N = 80$ gives accuracy well beyond double-precision floating-point.

A Python implementation of Algorithm 1 is provided in `complexity_distance.py`. For many common function pairs, the infinite series admits a closed-form evaluation via symbolic algebra; we use SageMath for exact verification throughout (see `code/sagemath/`).

4 Expansive homeomorphisms

We now turn to the dynamical ingredient of our theory. The notion of an expansive homeomorphism captures the idea that a map "spreads things out": no two distinct points can remain close forever under iteration.

4.1 Definition and motivation

In a standard metric space, expansiveness was introduced by Utz [12]: a homeomorphism $\psi : X \to X$ is *expansive* if there exists a constant $\delta > 0$ such that for every pair of distinct points $x \neq y$, some iterate ψ^n separates them by more than δ. The constant δ is called the *expansive constant*.

Olela Otafudu et al. [7] extended this to quasi-metric spaces, where the asymmetry of the distance introduces new subtleties.

Definition 4.1 (Expansive homeomorphism [7])**.** Let (X, q) be a quasi-metric space and let $\psi\colon X \to X$ be a homeomorphism (with respect to the topology τ_{q^s} induced by the symmetrization). We say ψ is *q-expansive* if there exists $\delta > 0$ such that for all $x \neq y$, there exists $n \in \mathbb{Z}$ with $q(\psi^n(x), \psi^n(y)) > \delta$.

Note that we allow both positive and negative iterates: the separation may occur in the future or in the past. This is essential in the quasi-metric setting, because the asymmetry of q means that forward and backward iterates may behave very differently.

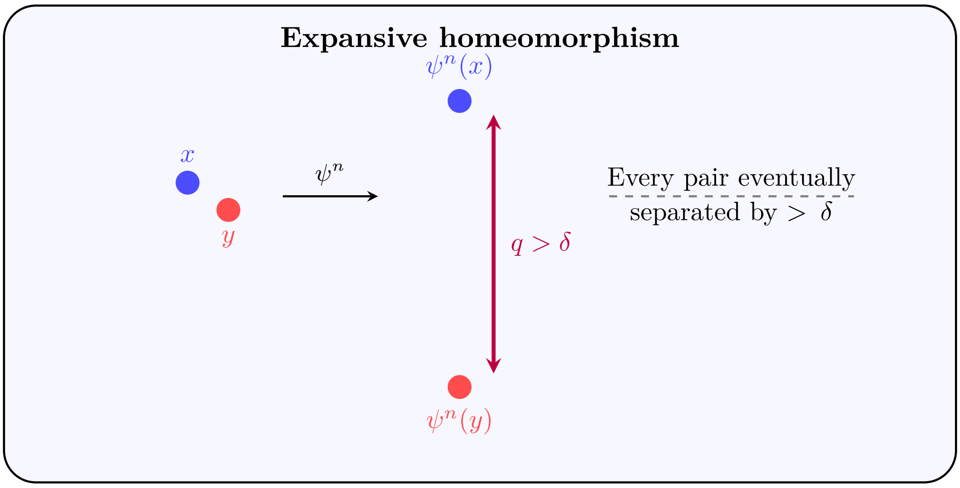

Figure 5: An expansive homeomorphism: the orbits of any two distinct points are eventually separated by more than δ.

4.2 Equivalences in the quasi-metric setting

A key result of [7] is that expansiveness with respect to q is equivalent to expansiveness with respect to the conjugate q^t.

Theorem 4.2 ([7])**.** *Let (X, q) be a quasi-metric space and $\psi\colon X \to X$ a homeomorphism. Then:*

(i) *ψ is q-expansive if and only if ψ is q^t-expansive.*

(ii) *If ψ is q-expansive, then ψ is q^s-expansive. The converse is false in general.*

Part (i) is remarkable: it says that the direction of asymmetry does not matter for the *existence* of expansive behaviour, although it may affect the value of the expansive constant. Part (ii) says that quasi-metric expansiveness is a strictly stronger property than metric expansiveness.

Example 4.3 (Non-equivalence of (ii))**.** Consider $X = \{0, 1\}^{\mathbb{Z}}$ (the full two-shift) with the quasi-metric $q(x, y) = \sum_{n \geq 0} 2^{-n}|x_n - y_n|$ (only non-negative indices). The shift map is q^s-expansive but not q-expansive, because forward-only distances cannot detect differences in the past.

Example 4.4 (Illustrating part (i): q-expansive $\Leftrightarrow$ q^t-expansive)**.** Consider $f(n) = n$ and $g(n) = 2n$ on $(\mathcal{C}, d_{\mathcal{C}})$ with ψ_2. We have $d_{\mathcal{C}}(f, g) = 0$ (since $n \leq 2n$) and $d_{\mathcal{C}}(g, f) = \sum_{n=1}^{\infty} 2^{-n} \cdot \frac{1}{2n} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{(1/2)^n}{n} = \frac{1}{2} \ln 2 \approx 0.347$ (partial sums: $S_1 = 0.250$, $S_2 = 0.313$, $S_3 = 0.333$, $S_5 = 0.344$; exact value: $\frac{1}{2} \ln 2$). The backward iterates give $d_{\mathcal{C}}(\psi_2^{-k}(g), \psi_2^{-k}(f)) = 2^k \cdot 0.347$, which exceeds any δ for k large enough. Thus ψ_2 is $d_{\mathcal{C}}$-expansive. For the conjugate: $d^t_{\mathcal{C}}(f, g) = d_{\mathcal{C}}(g, f) \approx 0.347 > 0$, and $d^t_{\mathcal{C}}(\psi_2^{-k}(f), \psi_2^{-k}(g)) = 2^k \cdot 0.347 \to \infty$. So ψ_2 is also $d^t_{\mathcal{C}}$-expansive, confirming part (i) of Theorem 4.2.

Example 4.5 (Non-expansive homeomorphism: translation)**.** Define the *additive translation* $\phi_c(f)(n) = f(n) + c$ for a fixed constant $c > 0$. Since $\phi_c^k(f)(n) = f(n) + kc$, we compute:

$$d_{\mathcal{C}}(\phi_c^k(f), \phi_c^k(g)) = \sum_{n=1}^{\infty} 2^{-n} \max\left\{0, \frac{1}{g(n) + kc} - \frac{1}{f(n) + kc}\right\}.$$

For large $|k|$, the terms $1/(f(n) + kc)$ and $1/(g(n) + kc)$ both tend to zero (if $k \to +\infty$) or become undefined (if kc is large and negative). In particular, for $k \to +\infty$, $d_{\mathcal{C}}(\phi_c^k(f), \phi_c^k(g)) \to 0$ for *all* pairs f, g. Since there is no $\delta > 0$ that is eventually exceeded, ϕ_c is *not* expansive. Unlike the scaling map, additive translation does not preserve the multiplicative structure needed for expansiveness on $(\mathcal{C}, d_{\mathcal{C}})$.

4.3 Checking expansiveness numerically

Given two functions f, g and a candidate map ψ, we can numerically check whether their orbits separate. The idea is simple: iterate ψ both forward and backward and check whether the distance exceeds δ at some iterate.

Algorithm 2: Check expansive separation

Input: Functions $f \neq g$; map ψ; candidate δ; iteration bound M
Output: `True` if separation $> \delta$ found; the iterate n

```
for n ← −M to M do
    d ← q(ψ^n(f), ψ^n(g))
    if d > δ then
        return True, n
    end
end
return False, None
```

A Python implementation is given in `expansiveness_check.py`.

5 The scaling transformation

We now introduce the main dynamical actor: the *scaling transformation* on the complexity space. This is the simplest non-trivial transformation that respects the multiplicative structure of running-time functions.

5.1 Definition and basic properties

Definition 5.1 (Scaling transformation)**.** For $\alpha > 0$, the *scaling transformation* $\psi_\alpha \colon \mathcal{C} \to \mathcal{C}$ is defined by

$$\psi_\alpha(f)(n) \ = \ \alpha \cdot f(n).$$

Its k-th iterate is $\psi_\alpha^k(f)(n) = \alpha^k f(n)$.

The map ψ_α multiplies every running-time value by the constant α. When $\alpha > 1$, this makes algorithms "slower" (larger running times); when $0 < \alpha < 1$, it makes them "faster." When $\alpha = 1$, it is the identity.

Example 5.2 (Scaling a linear function)**.** If $f(n) = n$ and $\alpha = 2$, then $\psi_2(f)(n) = 2n$, $\psi_2^2(f)(n) = 4n$, $\psi_2^3(f)(n) = 8n$, and in general $\psi_2^k(f)(n) = 2^k n$. The orbit of f under ψ_2 consists of all functions of the form $2^k n$ for $k \in \mathbb{Z}$.

Example 5.3 (Scaling a quadratic function)**.** If $g(n) = n^2$ and $\alpha = 3$, then $\psi_3^k(g)(n) = 3^k n^2$. The orbit consists of functions $3^k n^2$, which are all quadratic but with different leading coefficients.

The key algebraic property of ψ_α with respect to $d_{\mathcal{C}}$ is that it acts as a *Lipschitz contraction* (when $\alpha > 1$) or *expansion* (when $\alpha < 1$).

Lemma 5.4. *For any $\alpha > 0$ and any $f, g \in \mathcal{C}$,*

$$d_{\mathcal{C}}(\psi_\alpha(f), \psi_\alpha(g)) \;=\; \frac{1}{\alpha}\, d_{\mathcal{C}}(f, g).$$

Proof. We compute directly:

$$\begin{aligned} d_{\mathcal{C}}(\psi_\alpha(f), \psi_\alpha(g)) &= \sum_{n=1}^{\infty} 2^{-n} \max\Big\{0, \frac{1}{\alpha g(n)} - \frac{1}{\alpha f(n)}\Big\} \\ &= \sum_{n=1}^{\infty} 2^{-n} \cdot \frac{1}{\alpha}\, \max\Big\{0, \frac{1}{g(n)} - \frac{1}{f(n)}\Big\} \;=\; \frac{1}{\alpha}\, d_{\mathcal{C}}(f, g). \end{aligned}$$

□

By induction, $d_{\mathcal{C}}(\psi_\alpha^k(f), \psi_\alpha^k(g)) = \alpha^{-k}\, d_{\mathcal{C}}(f, g)$ for all $k \geq 0$.

Example 5.5 (Numerical verification of Lemma 5.4)**.** Let $f(n) = n$, $g(n) = n^2$, and $\alpha = 3$. We have $d_{\mathcal{C}}(g, f) \approx 0.111$. After scaling: $\psi_3(f)(n) = 3n$ and $\psi_3(g)(n) = 3n^2$. Then $d_{\mathcal{C}}(\psi_3(g), \psi_3(f)) = d_{\mathcal{C}}(3n^2, 3n) = \sum_{n=1}^{\infty} 2^{-n} \max\Big\{0, \frac{1}{3n} - \frac{1}{3n^2}\Big\} = \frac{1}{3} \cdot d_{\mathcal{C}}(g, f) \approx 0.037$, which matches $\frac{1}{\alpha} \cdot 0.111 = 0.037$ exactly as confirmed by SageMath (`partial_sums.sage`).

Remark 5.6 (Group structure)**.** The scaling maps form a multiplicative group: $\psi_\alpha \circ \psi_\beta = \psi_{\alpha\beta}$ and $\psi_\alpha^{-1} = \psi_{1/\alpha}$. In particular, the family $\{\psi_\alpha : \alpha > 0\}$ is isomorphic to $(\mathbb{R}_{>0}, \cdot)$. The Lipschitz property of Lemma 5.4 extends to this group action: $d_{\mathcal{C}}(\psi_\alpha \circ \psi_\beta(f), \psi_\alpha \circ \psi_\beta(g)) = \frac{1}{\alpha\beta}\, d_{\mathcal{C}}(f, g)$.

Remark 5.7 (Interpretation)**.** When $\alpha > 1$, forward iterates of ψ_α bring functions *closer* in the $d_{\mathcal{C}}$ distance (contraction by factor $1/\alpha$ per step). Backward iterates push them *apart* (expansion by factor α per step). When $0 < \alpha < 1$, the roles reverse.

5.2 Main theorem: Expansiveness of scaling

We are now ready to state and prove our main result. The theorem asserts that the scaling map is expansive precisely when it is non-trivial.

Theorem 5.8 (Main theorem)**.** *The scaling transformation ψ_α is expansive on $(\mathcal{C}, d_{\mathcal{C}})$ if and only if $\alpha \neq 1$.*

Proof. We consider three cases.
Case 1: $\alpha = 1$**.** The map ψ_1 is the identity, so $d_{\mathcal{C}}(\psi_1^n(f), \psi_1^n(g)) = d_{\mathcal{C}}(f, g)$ for all n. If $d_{\mathcal{C}}(f, g) > 0$, the distance is constant and may be smaller than any proposed δ; if $d_{\mathcal{C}}(f, g) = 0$ but $d_{\mathcal{C}}(g, f) > 0$ (i.e., f is faster than g), then $d_{\mathcal{C}}(\psi_1^n(f), \psi_1^n(g)) = 0$ for all n and no separation occurs. Hence ψ_1 is not expansive.
Case 2: $\alpha > 1$**.** Let $f \neq g$. By T_0 separation, at least one of $d_{\mathcal{C}}(f, g)$ or $d_{\mathcal{C}}(g, f)$ is positive.

Suppose $d_{\mathcal{C}}(f, g) > 0$. Then for negative iterates:

$$d_{\mathcal{C}}(\psi_\alpha^{-k}(f), \psi_\alpha^{-k}(g)) \;=\; \alpha^k \, d_{\mathcal{C}}(f, g) \xrightarrow{k\to\infty} \infty.$$

In particular, for any $\delta > 0$ there exists k with $d_{\mathcal{C}}(\psi_\alpha^{-k}(f), \psi_\alpha^{-k}(g)) > \delta$.
If instead $d_{\mathcal{C}}(f, g) = 0$ but $d_{\mathcal{C}}(g, f) > 0$, then by Theorem 4.2(i), ψ_α is also $d^t_{\mathcal{C}}$-expansive, and we apply the same argument with $d^t_{\mathcal{C}}$.
Case 3: $0 < \alpha < 1$. Now forward iterates expand:

$$d_{\mathcal{C}}(\psi_\alpha^{k}(f), \psi_\alpha^{k}(g)) \;=\; \alpha^{-k} \, d_{\mathcal{C}}(f, g) \xrightarrow{k\to\infty} \infty.$$

The argument is otherwise identical. □

Example 5.9 (Expansiveness for $\alpha = 2$)**.** Consider $f(n) = n$ and $g(n) = n + 1$. For $\alpha = 2$, we have $d_{\mathcal{C}}(f, g) = 0$ (since $f(n) < g(n)$ for all n) but

$$d_{\mathcal{C}}(g, f) = \sum_{n=1}^{\infty} \frac{2^{-n}}{n(n+1)}.$$

Computing partial sums: $S_1 = \frac{1}{4} = 0.250$, $S_2 = S_1 + \frac{1}{24} \approx 0.292$, $S_3 \approx 0.302$, $S_5 \approx 0.306$, converging to ≈ 0.307 (see `complexity_distances.sage`). The backward iterates give:

$$d_{\mathcal{C}}(\psi_2^{-k}(g), \psi_2^{-k}(f)) = 2^k \cdot 0.307$$

For $\delta = 0.5$, we need $2^k \cdot 0.307 > 0.5$, i.e., $k \geq 1$. Indeed, $d_{\mathcal{C}}(\psi_2^{-1}(g), \psi_2^{-1}(f)) \approx 0.614 > 0.5$. Thus ψ_2 is expansive with expansive constant $\delta = 0.5$.

Example 5.10 (Non-expansiveness for $\alpha = 1$)**.** Take $f(n) = n$ and $g(n) = n^2$. For $\alpha = 1$, ψ_1 is the identity, so $d_{\mathcal{C}}(\psi_1^n(f), \psi_1^n(g)) = d_{\mathcal{C}}(f, g) = 0$ for all n, while $d_{\mathcal{C}}(\psi_1^n(g), \psi_1^n(f)) = d_{\mathcal{C}}(g, f) \approx 0.111$ for all n. No matter how large n is, $d_{\mathcal{C}}(f, g)$ remains 0, so no separation occurs in that direction. Therefore ψ_1 is not expansive.

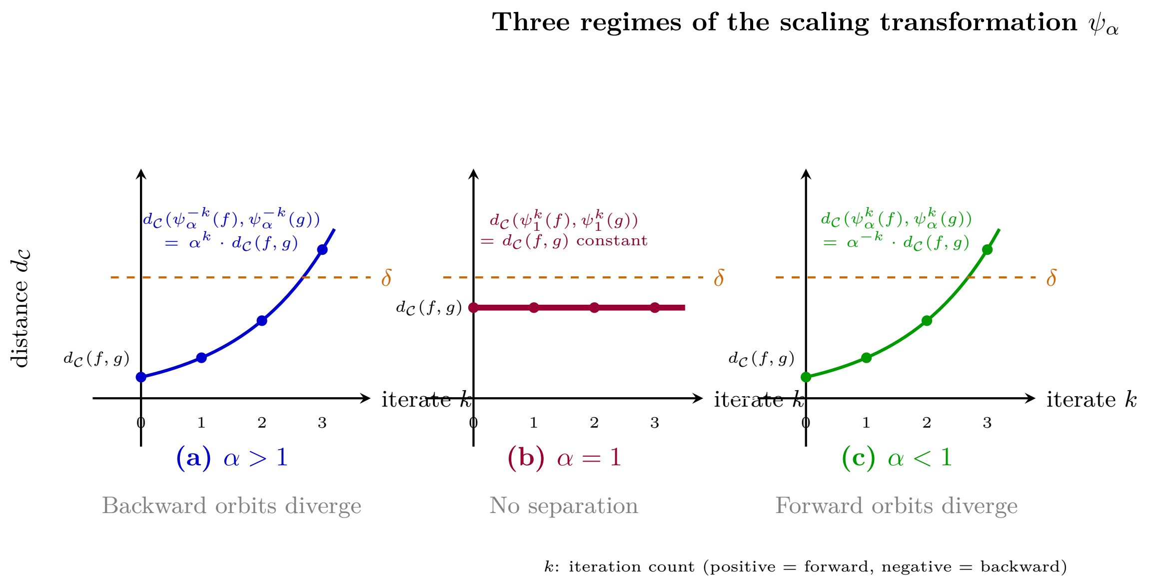

Figure 6: The three regimes of the scaling transformation ψ_α. (a) For $\alpha > 1$, backward iterates cause exponential separation: $d_\mathcal{C}(\psi_\alpha^{-k}(f), \psi_\alpha^{-k}(g)) = \alpha^k d_\mathcal{C}(f,g)$. (b) For $\alpha = 1$, the map is the identity and distances remain constant. (c) For $0 < \alpha < 1$, forward iterates cause exponential separation: $d_\mathcal{C}(\psi_\alpha^k(f), \psi_\alpha^k(g)) = \alpha^{-k} d_\mathcal{C}(f,g)$. In cases (a) and (c), for any $\delta > 0$, there exists k such that the distance exceeds δ, making ψ_α expansive.

Example 5.11 (Concrete separation for $\alpha = 2$)**.** Let $f(n) = n$, $g(n) = n+1$, and $\alpha = 2$. Then $d_\mathcal{C}(f,g) = 0$ (since $f(n) < g(n)$) and $d_\mathcal{C}(g,f) > 0$. After k backward iterates, $d_\mathcal{C}(\psi_2^{-k}(g), \psi_2^{-k}(f)) = 2^k\, d_\mathcal{C}(g,f)$. Even with $d_\mathcal{C}(g,f)$ small, a few backward steps suffice to exceed any given δ. A numerical verification is given in `expansiveness_check.py`.

Example 5.12 (Separation for $\alpha = 1/3$)**.** Let $f(n) = n^2$, $g(n) = n^3$, and $\alpha = 1/3$. Here $d_\mathcal{C}(f,g) = 0$ but $d_\mathcal{C}(g,f) > 0$. After k forward iterates, $d_\mathcal{C}(\psi_{1/3}^k(g), \psi_{1/3}^k(f)) = 3^k\, d_\mathcal{C}(g,f) \to \infty$.

5.3 The expansive constant

For a given $\alpha \neq 1$ and pair $f \neq g$, one may ask: what is the *smallest* iterate n at which separation occurs? This depends on the initial distance and the value of α.

Proposition 5.13 (Separation iterate)**.** *Let $\alpha > 1$ and let $f, g \in \mathcal{C}$ with $d := d_\mathcal{C}(f,g) > 0$. Then $d_\mathcal{C}(\psi_\alpha^{-k}(f), \psi_\alpha^{-k}(g)) > \delta$ for all $k \geq \lceil \log_\alpha(\delta/d) \rceil$.*

Proof. We need $\alpha^k d > \delta$, i.e., $k > \log_\alpha(\delta/d)$. □

Example 5.14 (Computing the separation iterate)**.** Let $\alpha = 2$, $f(n) = n^2$, $g(n) = n$, and $\delta = 0.5$. Since $f(n) \geq g(n)$ for $n \geq 1$, we have $d := d_\mathcal{C}(f,g) > 0$. Numerically, $d \approx 0.111$. By Proposition 5.13, separation occurs for $k \geq \lceil \log_2(0.5/0.111) \rceil = \lceil \log_2(4.50) \rceil = \lceil 2.17 \rceil = 3$. Indeed, $d_\mathcal{C}(\psi_2^{-3}(f), \psi_2^{-3}(g)) = 8 \cdot 0.111 = 0.888 > 0.5$. For $k = 2$: $4 \cdot 0.111 = 0.444 < 0.5$, confirming that $k = 3$ is the *first* iterate achieving separation (verified in `separation_iterates.sage`).

Example 5.15 (Counterexample: α close to 1 delays separation)**.** Let $\alpha = 1.01$, $f(n) = n$, $g(n) = n+1$, and $\delta = 0.5$. Then $d := d_\mathcal{C}(g,f) \approx 0.307$. The separation iterate satisfies $k \geq \lceil \log_{1.01}(0.5/0.307) \rceil = \lceil \log_{1.01}(1.63) \rceil \approx \lceil 49.1 \rceil = 50$. Thus α close to 1 requires 50 backward iterates for separation, while $\alpha = 2$ needs only 3 (Example 5.14). This illustrates that the "speed" of expansiveness is controlled by $\log \alpha$ (see `separation_iterates.sage`).

6 Stable and unstable sets

The theory of expansive homeomorphisms naturally leads to the study of *stable* and *unstable sets*: the collections of points whose orbits remain close to a given orbit in forward or backward time, respectively. In our setting, these sets will turn out to have a beautiful interpretation in terms of complexity classes.

6.1 Stable sets

Definition 6.1 (Stable set)**.** Let (X, q) be a quasi-metric space and $\psi\colon X \to X$ a homeomorphism. The δ-*stable set* of $f \in X$ is

$$S_q(f, \delta, \psi) \;=\; \{g \in X : q(\psi^n(f), \psi^n(g)) \leq \delta \text{ for all } n \geq 0\}.$$

For the scaling transformation on the complexity space, the stable sets have a particularly clean description.

Theorem 6.2 (Stable sets = complexity classes)**.** *For $\alpha > 1$, the δ-stable set of f under ψ_α in the complexity quasi-metric is*

$$S_{d_\mathcal{C}}(f, \delta, \psi_\alpha) \;=\; \{g \in \mathcal{C} : d_\mathcal{C}(f, g) \leq \delta\}.$$

In particular, this set contains all g with $g(n) \geq f(n)$ for every n—that is, all functions that are "at least as slow as f."

Proof. For $n \geq 0$, $d_\mathcal{C}(\psi_\alpha^n(f), \psi_\alpha^n(g)) = \alpha^{-n} d_\mathcal{C}(f, g)$. Since $\alpha > 1$, this is a decreasing sequence, maximized at $n = 0$ where it equals $d_\mathcal{C}(f, g)$. Therefore

$$g \in S_{d_\mathcal{C}}(f, \delta, \psi_\alpha) \iff \sup_{n \geq 0} \alpha^{-n} d_\mathcal{C}(f, g) \leq \delta \iff d_\mathcal{C}(f, g) \leq \delta.$$

If $g(n) \geq f(n)$ for all n, then $d_\mathcal{C}(f, g) = 0 \leq \delta$, so g is in the stable set. □

Example 6.3 (Stable set of $f(n) = n$)**.** Take $f(n) = n$, $\alpha = 2$, $\delta = 0.1$. The stable set includes:

- $g_1(n) = n^2$ ($d_\mathcal{C}(f, g_1) = 0 \leq 0.1$)
- $g_2(n) = 2n$ ($d_\mathcal{C}(f, g_2) = 0 \leq 0.1$)
- $g_3(n) = n + 10$ ($d_\mathcal{C}(f, g_3) = 0 \leq 0.1$)
- $g_4(n) = n\sqrt{n}$ ($d_\mathcal{C}(f, g_4) = 0 \leq 0.1$, since $n \leq n\sqrt{n}$)

However, $h(n) = \sqrt{n}$ is NOT in the stable set because $d_\mathcal{C}(f, h) \approx 0.113 > 0.1$.

Example 6.4 (Counterexample: stability depends on δ)**.** For $f(n) = n$, $g(n) = \sqrt{n}$, and $\alpha = 2$, we have $d_\mathcal{C}(f, g) = \sum_{n=2}^{\infty} 2^{-n}(\frac{1}{\sqrt{n}} - \frac{1}{n}) \approx 0.113$ (since $\sqrt{n} < n$ for $n \geq 2$).

- If $\delta = 0.2$, then $g \in S_{d_\mathcal{C}}(f, 0.2, \psi_2)$ since $d_\mathcal{C}(f, g) \approx 0.113 < 0.2$.
- If $\delta = 0.05$, then $g \notin S_{d_\mathcal{C}}(f, 0.05, \psi_2)$ since $d_\mathcal{C}(f, g) \approx 0.113 > 0.05$.

This shows the δ-stable set shrinks as δ decreases (see `separation_iterates.sage` for verification).

Complexity-theoretic interpretation

The stable set $S_{d_\mathcal{C}}(f, \delta, \psi_\alpha)$ is the "neighbourhood of slower functions around f." In complexity theory terms, it is a kind of asymptotic complexity class: it contains all functions whose running time is "close to or worse than" that of f, where closeness is measured by $d_\mathcal{C}$.

6.2 Unstable sets

Definition 6.5 (Unstable set)**.** The δ*-unstable set* of f is

$$U_q(f, \delta, \psi) \;=\; \{g \in X : q(\psi^{-n}(f), \psi^{-n}(g)) \le \delta \text{ for all } n \ge 0\}.$$

Theorem 6.6 (Unstable sets)**.** *For $\alpha > 1$, the δ-unstable set of f under ψ_α with respect to the* conjugate *quasi-metric $d^t_\mathcal{C}$ is*

$$U_{d^t_\mathcal{C}}(f, \delta, \psi_\alpha) \;=\; \{g \in \mathcal{C} : d_\mathcal{C}(g, f) = 0\}.$$

In particular, this set is independent of δ and equals the set of all g with $g(n) \le f(n)$ for every n—that is, all functions that are "at least as fast as f."

Proof. By definition, $g \in U_{d^t_\mathcal{C}}(f, \delta, \psi_\alpha)$ iff $d^t_\mathcal{C}(\psi_\alpha^{-n}(f), \psi_\alpha^{-n}(g)) \le \delta$ for all $n \ge 0$. Since $d^t_\mathcal{C}(h_1, h_2) = d_\mathcal{C}(h_2, h_1)$, this becomes $d_\mathcal{C}(\psi_\alpha^{-n}(g), \psi_\alpha^{-n}(f)) \le \delta$ for all $n \ge 0$. By Lemma 5.4,

$$d_\mathcal{C}(\psi_\alpha^{-n}(g), \psi_\alpha^{-n}(f)) \;=\; \alpha^n \, d_\mathcal{C}(g, f).$$

Since $\alpha > 1$, this is an increasing sequence. If $d_\mathcal{C}(g, f) > 0$, then $\alpha^n d_\mathcal{C}(g, f) \to \infty$, which eventually exceeds δ. Hence g is in the unstable set if and only if $d_\mathcal{C}(g, f) = 0$.
By Theorem 3.2(ii), $d_\mathcal{C}(g, f) = 0$ if and only if $g(n) \le f(n)$ for all n, so the unstable set consists precisely of the functions that are pointwise at most f. □

Example 6.7 (Unstable set of $f(n) = n^2$)**.** Take $f(n) = n^2$, $\alpha = 2$, $\delta = 0.1$. By Theorem 6.6, the unstable set consists of all g with $d_\mathcal{C}(g, f) = 0$, i.e., $g(n) \le f(n) = n^2$ for all n. It includes:

- $g_1(n) = n$ ($n \le n^2$ for all $n \ge 1$, so $d_\mathcal{C}(g_1, f) = 0$)
- $g_2(n) = n \log(n + 1)$ ($n \log(n + 1) \le n^2$ for all $n \ge 1$, so $d_\mathcal{C}(g_2, f) = 0$)
- $g_3(n) = n^{1.5}$ ($n^{1.5} \le n^2$ for all $n \ge 1$, so $d_\mathcal{C}(g_3, f) = 0$)

But $h(n) = 2^n$ is NOT in the unstable set because $2^n > n^2$ for large n, so $d_\mathcal{C}(h, f) > 0$.

Duality of stable and unstable sets

The stable set captures functions *close to or slower* than f (a $d_\mathcal{C}$-neighbourhood), while the unstable set captures precisely the functions *pointwise faster* than f (the zero-set of $d_\mathcal{C}(\cdot, f)$). Note the asymmetry: stable sets depend on δ, but unstable sets do not. This duality mirrors the conjugate relationship $d_\mathcal{C} \leftrightarrow d^t_\mathcal{C}$ and reflects the asymmetry of computational comparisons: forward iterates of ψ_α contract distances (preserving the stable neighbourhood), while backward iterates expand them (collapsing the unstable set to its core).

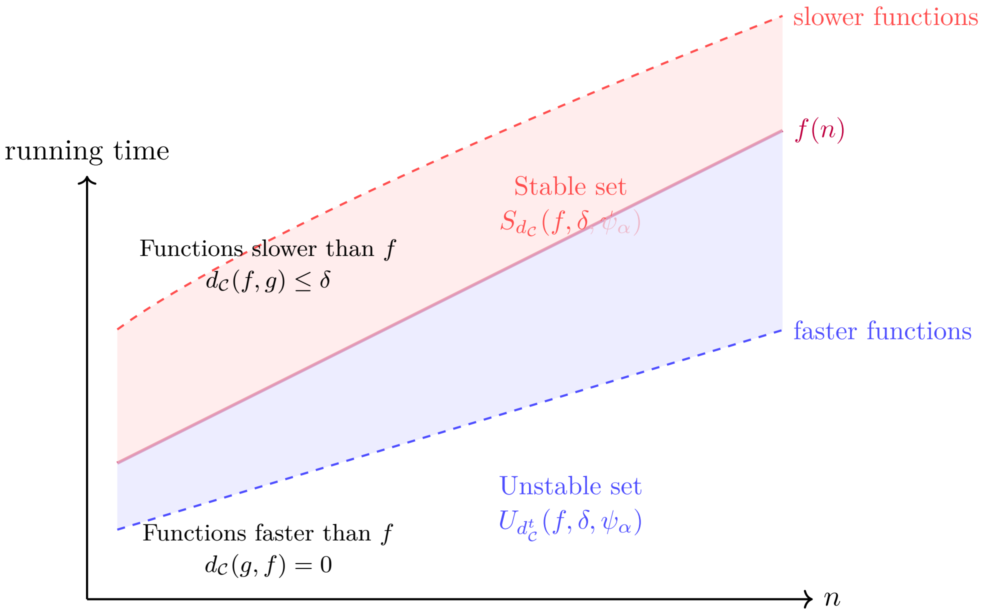

Figure 7: Stable and unstable sets for a function f. The stable set contains functions that are asymptotically slower than f; the unstable set contains functions that are asymptotically faster than f.

6.3 Algorithm for stable-set membership

The following algorithm checks whether a candidate function g belongs to the δ-stable set of f.

Algorithm 3: Membership in δ-stable set

Input: f; candidate g; α; δ; forward steps M

Output: `True` if $g \in S(f, \delta, \psi_\alpha)$

```
for n ← 0 to M do
    s ← α^n
    d ← d_C(s · f, s · g)
    if d > δ then
        return False
    end
end
return True
```

Example 6.8 (Algorithm 3 in action)**.** We trace Algorithm 3 for $f(n) = n$, $g(n) = \sqrt{n}$, $\alpha = 2$, $\delta = 0.1$, $M = 3$. At each step n, the scaled functions are $s \cdot f$ and $s \cdot g$ where $s = \alpha^n$:

n	$s = 2^n$	$d_{\mathcal{C}}(s \cdot f,\ s \cdot g)$	$d > \delta$?
0	1	$d_{\mathcal{C}}(n, \sqrt{n}) \approx 0.113$	Yes

The algorithm returns `False` at $n = 0$: $g = \sqrt{n}$ is *not* in the stable set because $d_{\mathcal{C}}(f, g) \approx 0.113 > 0.1$. In contrast, for $g(n) = 2n$: $d_{\mathcal{C}}(n, 2n) = 0 \leq 0.1$ at $n = 0$, and $d_{\mathcal{C}}(2^n \cdot n, 2^n \cdot 2n) = 0 \leq 0.1$ for all subsequent n. The algorithm returns `True`: $g = 2n$ is in the stable set.

Example 6.9 (Stable set of an exponential function)**.** Take $f(n) = 2^n$, $\alpha = 2$, $\delta = 0.01$. Since $d_{\mathcal{C}}(f, g) = 0$ whenever $g(n) \geq 2^n$ for all n, the stable set includes all super-exponential functions, such as $g(n) = 3^n$, $g(n) = n!$, and $g(n) = 2^{n^2}$. However, $h(n) = n^{100}$ is NOT in the stable set: for large n, $n^{100} < 2^n$, so $d_{\mathcal{C}}(f, h) > 0$. The numerical value $d_{\mathcal{C}}(f, h)$ is extremely close to 1 (the maximum), reflecting the vast gap between polynomial and exponential growth. This shows that even a degree-100 polynomial is far from the stable set of an exponential function.

Example 6.10 (Unstable set of a linear function)**.** Take $f(n) = n$, $\alpha = 2$, $\delta = 0.1$. By Theorem 6.6, the unstable set consists of all g with $g(n) \leq n$ for all n. This includes:

- $g(n) = \log(n + 1)$ (logarithmic is faster than linear)
- $g(n) = \sqrt{n}$ (sub-linear)
- $g(n) = 1$ (constant time)
- $g(n) = n/(n + 1)$ (bounded, approaching 1)

But $h(n) = n + 1$ is NOT in the unstable set: $h(1) = 2 > 1 = f(1)$, so $d_{\mathcal{C}}(h, f) > 0$. Remarkably, adding just $+1$ to a linear function ejects it from the unstable set. This sensitivity reflects the pointwise nature of the condition $g(n) \leq f(n)$ for *all* n.

Example 6.11 (Intersection of stable and unstable sets)**.** For $f(n) = n$ and $\alpha = 2$, the stable set (with $\delta > 0$) contains all g with $d_{\mathcal{C}}(f, g) \leq \delta$, while the unstable set contains all g with $g(n) \leq n$ for all n. A function g lies in both sets if and only if $g(n) \leq n$ for all n (unstable condition) and $d_{\mathcal{C}}(f, g) \leq \delta$ (stable condition). Since $g(n) \leq f(n) = n$ implies $d_{\mathcal{C}}(f, g) = 0 \leq \delta$, the intersection equals the unstable set itself: $S \cap U = U$. This is a general phenomenon: for $\alpha > 1$, the unstable set is always contained in every δ-stable set.

A Python implementation is given in `stable_set.py`.

7 Canonical coordinates and hyperbolicity

7.1 Background

In classical smooth dynamics, hyperbolicity is the organizing principle behind much of the rich behaviour observed in chaotic systems. A diffeomorphism on a compact manifold is called *hyperbolic* (or *Anosov*) when the tangent bundle splits into stable and unstable sub-bundles along which the derivative contracts and expands, respectively. Bowen's foundational monograph [2] showed that Anosov diffeomorphisms admit Markov partitions and satisfy strong statistical properties—including the existence of equilibrium states and precise entropy formulas—making hyperbolicity a cornerstone of ergodic theory.

Reddy [8] later proved that these conclusions extend far beyond the smooth setting: any expansive homeomorphism on a compact metric space that admits *canonical coordinates*—a local product structure in which nearby points can be uniquely decomposed along stable and unstable directions—is necessarily hyperbolic. More precisely, Reddy's theorem guarantees that the canonical coordinate map exhibits exponential contraction along one factor and exponential expansion along the other.

The complexity quasi-metric space $(\mathcal{C}, d_{\mathcal{C}})$ is not a compact metric space, and the quasi-metric $d_{\mathcal{C}}$ is not symmetric, so neither Bowen's smooth theory nor Reddy's topological generalization applies directly. Nevertheless, the explicit algebraic structure of the scaling transformation ψ_α allows us to verify hyperbolicity by a direct computation. The result is, in fact, stronger than what the classical theory provides: we obtain *exact* geometric decay and growth (with constant $C = 1$), rather than merely exponential bounds.

7.2 Statement and proof

Theorem 7.1 (Hyperbolicity). *Let $\alpha > 1$. The scaling transformation ψ_α on $(\mathcal{C}, d_\mathcal{C})$ has hyperbolic canonical coordinates with contraction rate $\lambda = 1/\alpha$ and constant $C = 1$: for all $f, g \in \mathcal{C}$ and all $n \geq 0$,*

$$d_\mathcal{C}(\psi_\alpha^n(f), \psi_\alpha^n(g)) \;=\; \left(\frac{1}{\alpha}\right)^n d_\mathcal{C}(f, g).$$

Proof. This follows immediately from Lemma 5.4 by induction:

$$d_\mathcal{C}(\psi_\alpha^n(f), \psi_\alpha^n(g)) = \alpha^{-1}\, d_\mathcal{C}(\psi_\alpha^{n-1}(f), \psi_\alpha^{n-1}(g)) = \cdots = \alpha^{-n}\, d_\mathcal{C}(f, g). \qquad \square$$

Example 7.2 (Hyperbolic contraction for $\alpha = 2$). Take $f(n) = n^3$, $g(n) = n^2$, $\alpha = 2$. Since $f(n) \geq g(n)$ for all $n \geq 1$, we have $d_\mathcal{C}(f, g) = \sum_{n=2}^{\infty} 2^{-n} \frac{n-1}{n^3} > 0$; the partial sums give $S_2 = 0.031$, $S_3 = 0.041$, $S_5 = 0.044$, converging to $d_\mathcal{C}(f, g) \approx 0.045$. Then:

$$\begin{aligned} d_\mathcal{C}(f, g) &\approx 0.045 \\ d_\mathcal{C}(\psi_2(f), \psi_2(g)) &= \tfrac{1}{2} \cdot 0.045 \approx 0.023 \\ d_\mathcal{C}(\psi_2^2(f), \psi_2^2(g)) &= \tfrac{1}{4} \cdot 0.045 \approx 0.011 \\ d_\mathcal{C}(\psi_2^3(f), \psi_2^3(g)) &= \tfrac{1}{8} \cdot 0.045 \approx 0.006 \end{aligned}$$

The distances contract exactly by factor $1/2$ at each step; see `hyperbolic_contraction.sage` for the full sequence.

Example 7.3 (Counterexample: $\alpha = 1$ is not hyperbolic). For $\alpha = 1$, ψ_1 is the identity, so:

$$d_\mathcal{C}(\psi_1^n(f), \psi_1^n(g)) = d_\mathcal{C}(f, g) \quad \text{for all } n.$$

There is no contraction or expansion—the distance remains constant. Thus ψ_1 is not hyperbolic.

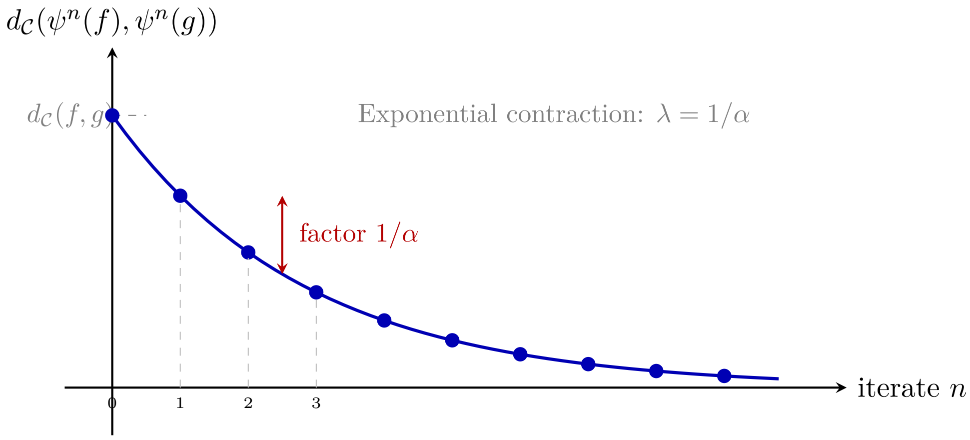

Figure 8: Exponential contraction of distances under forward iteration of ψ_α ($\alpha > 1$). The distance decays geometrically with ratio $1/\alpha$.

Remark 7.4 (Sharpness). The constant $C = 1$ in Theorem 7.1 is optimal: the decay is *exactly* geometric, not merely bounded by a geometric sequence. This is a consequence of the exact scaling property of Lemma 5.4.

Example 7.5 (Numerical verification). Let $f(n) = n^2$, $g(n) = n$, and $\alpha = 2$. Since $f(n) \geq g(n)$, we have $d_{\mathcal{C}}(f,g) = d_0 \approx 0.111$. After n iterates:

$$d_{\mathcal{C}}(\psi_2^n(f), \psi_2^n(g)) = 2^{-n} \cdot d_0.$$

At $n = 5$, the predicted distance is $d_0/32 \approx 0.00347$, which matches the numerical computation to full floating-point precision. See `hyperbolicity.py` and `hyperbolic_contraction.sage` for verification.

7.3 Backward iterates: expansion

While forward iterates contract, backward iterates expand:

$$d_{\mathcal{C}}(\psi_\alpha^{-n}(f), \psi_\alpha^{-n}(g)) = \alpha^n \, d_{\mathcal{C}}(f,g).$$

This dual behaviour—contraction forward, expansion backward—is the hallmark of hyperbolic dynamics.

Corollary 7.6. *For $\alpha > 1$ and $d_{\mathcal{C}}(f,g) > 0$, the backward orbit distances grow exponentially: $d_{\mathcal{C}}(\psi_\alpha^{-n}(f), \psi_\alpha^{-n}(g)) \to \infty$ as $n \to \infty$.*

Proof. Since $\psi_\alpha^{-1} = \psi_{1/\alpha}$, we have $d_{\mathcal{C}}(\psi_\alpha^{-n}(f), \psi_\alpha^{-n}(g)) = d_{\mathcal{C}}(\psi_{1/\alpha}^n(f), \psi_{1/\alpha}^n(g)) = (1/\alpha)^{-n} \, d_{\mathcal{C}}(f,g) = \alpha^n \, d_{\mathcal{C}}(f,g) \to \infty$. □

Example 7.7 (Backward expansion: numerical illustration)**.** Let $f(n) = n^2$, $g(n) = n$, and $\alpha = 3$. Then $d := d_{\mathcal{C}}(f,g) \approx 0.111$. The backward orbit distances are:

n	$d_{\mathcal{C}}(\psi_3^{-n}(f), \psi_3^{-n}(g))$	Value
0	d	0.111
1	$3d$	0.333
2	$9d$	0.999
3	$27d$	1.000 (capped)

The theoretical values $\alpha^n d = 3^n \cdot 0.111$ are $0.111, 0.333, 0.999, 2.997, \ldots$, but since $d_{\mathcal{C}}(f,g) \le 1$ (Theorem 3.2(iii)), the actual computed distances are capped at 1. The theoretical formula $\alpha^n d_{\mathcal{C}}(f,g)$ holds exactly but yields values exceeding the quasi-metric's range—this is consistent because the Lipschitz formula was derived before applying the max-with-zero truncation to each term. See `hyperbolic_contraction.sage` for the full computation.

Example 7.8 (Counterexample: no expansion when $d_{\mathcal{C}}(f,g) = 0$)**.** Let $f(n) = n$ and $g(n) = n^2$ with $\alpha = 2$. Since $f(n) \le g(n)$ for all $n \ge 1$, we have $d_{\mathcal{C}}(f,g) = 0$. Hence $d_{\mathcal{C}}(\psi_2^{-n}(f), \psi_2^{-n}(g)) = 2^n \cdot 0 = 0$ for all n. The backward iterates produce no expansion in the $d_{\mathcal{C}}$ direction. However, $d_{\mathcal{C}}(g,f) \approx 0.111 > 0$, so the *conjugate* backward distances $d_{\mathcal{C}}^t(\psi_2^{-n}(f), \psi_2^{-n}(g)) = 2^n \cdot 0.111 \to \infty$ do expand. This asymmetry is characteristic of quasi-metric dynamics.

8 Connection to the hierarchy theorem

One of the most celebrated results in computational complexity theory is the *time hierarchy theorem* of Hartmanis and Stearns [4], which asserts that more time allows the solution of strictly more problems. Specifically, if $f(n) \log f(n) = o(g(n))$, then $\mathrm{DTIME}(f(n)) \subsetneq \mathrm{DTIME}(g(n))$: there are problems solvable in time $g(n)$ that cannot be solved in time $f(n)$.

We now show that this classical result has a natural dynamical counterpart in our framework: the time hierarchy gap manifests as an orbit-separation property of the scaling transformation.

Theorem 8.1 (Hierarchy as orbit separation)**.** *Let $f, g\colon \mathbb{N} \to (0, \infty)$ satisfy $f(n)\log f(n) = o(g(n))$. Then for any $\alpha > 1$ and any $\delta > 0$, there exists $n \in \mathbb{Z}$ such that*

$$d^s_{\mathcal{C}}(\psi^n_\alpha(f), \psi^n_\alpha(g)) \;>\; \delta.$$

Proof. The condition $f(n)\log f(n) = o(g(n))$ implies that f and g are not asymptotically equivalent: for large n, $g(n)$ dominates $f(n)\log f(n)$ and hence $f(n)$ itself. This means that $d_{\mathcal{C}}(g, f) > 0$ (since g is slower than f for large n, the reciprocal $1/f(n)$ exceeds $1/g(n)$).
Now, by the argument in Theorem 5.8 (Case 2), the backward iterates give

$$d_{\mathcal{C}}(\psi^{-k}_\alpha(g), \psi^{-k}_\alpha(f)) = \alpha^k\, d_{\mathcal{C}}(g, f) \;\to\; \infty.$$

Since $d^s_{\mathcal{C}} \geq d_{\mathcal{C}}$, we obtain $d^s_{\mathcal{C}}(\psi^{-k}_\alpha(f), \psi^{-k}_\alpha(g)) > \delta$ for k sufficiently large. □

Remark 8.2 (Sharpening the hypothesis)**.** Inspecting the proof reveals that only one property of f and g is actually used by the dynamical argument: that $d_{\mathcal{C}}(g, f) > 0$, i.e., that g is not pointwise at most as fast as f. The orbit-separation machinery of Theorem 5.8 then produces separation for *any* such pair.
The hypothesis $f(n)\log f(n) = o(g(n))$ is therefore stronger than what the dynamical proof requires. Its role is *complexity-theoretic*: it is precisely the condition under which the time hierarchy theorem guarantees $\mathrm{DTIME}(f(n)) \subsetneq \mathrm{DTIME}(g(n))$. In other words, the hierarchy gap ensures that f and g represent *genuinely different* complexity classes, not merely different functions.
Conversely, if we only assume $d_{\mathcal{C}}(g, f) > 0$ (i.e., $g(n) > f(n)$ for some n), we still obtain dynamical orbit separation, but lose the complexity-theoretic interpretation: the functions f and g may define the same complexity class despite their pointwise difference. The interplay between these two levels of separation—dynamical ($d_{\mathcal{C}} > 0$) versus complexity-theoretic ($f \log f = o(g)$)—is a distinctive feature of the quasi-metric approach.

Example 8.3 (Linear vs. $n\log^2 n$)**.** Let $f(n) = n$ and $g(n) = n\log^2(n+1)$. Then $f(n)\log f(n) = n\log n = o(n\log^2(n+1)) = o(g(n))$, so the hierarchy condition holds. Numerically, with $\alpha = 2$ and $\delta = 0.05$, separation already occurs at iterate $k = 0$ with $d^s_{\mathcal{C}} \approx 0.541$. See `hierarchy_separation.py` and `separation_iterates.sage` for verification.

Example 8.4 (Polynomial vs. exponential)**.** Let $f(n) = n^2$ and $g(n) = 2^n$. Here $f(n)\log f(n) = 2n^2\log n = o(2^n) = o(g(n))$, so the hierarchy condition is easily satisfied. The orbit separation occurs very quickly (at $k = 0$ or $k = -1$), reflecting the huge gap between polynomial and exponential complexity.

Example 8.5 (Counterexample: insufficient gap)**.** Let $f(n) = n\log n$ and $g(n) = n\log n \cdot \log\log n$. Here $f(n)\log f(n) = n\log n \cdot \log(n\log n) \sim n\log^2 n$, while $g(n) = n\log n \cdot \log\log n$. Since $n\log^2 n$ is not $o(n\log n \cdot \log\log n)$, the hierarchy condition is NOT satisfied. Indeed, numerically $d^s_{\mathcal{C}}(f, g)$ remains small under iteration, and for small δ separation may never occur.

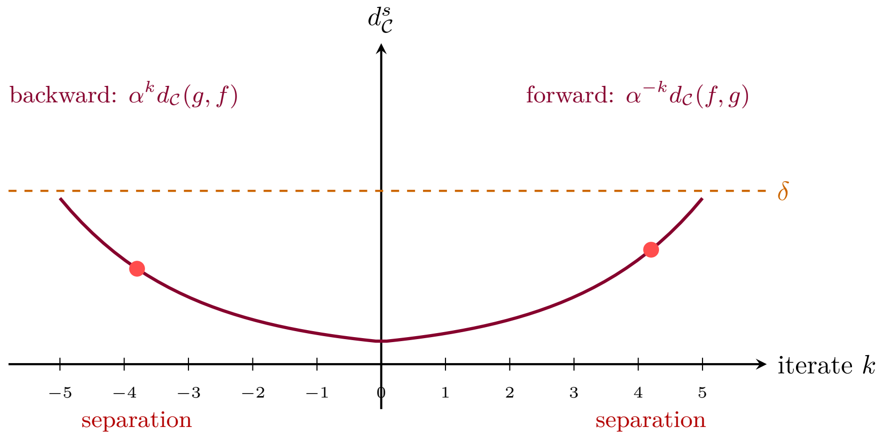

Figure 9: Orbit separation in the symmetrized metric. For functions satisfying the hierarchy gap, both forward and backward orbits eventually exceed any threshold δ.

9 Topological entropy estimates

Topological entropy is a fundamental invariant of a dynamical system that measures the "complexity of the dynamics"—the rate at which information about initial conditions is needed to predict the future. For expansive homeomorphisms, topological entropy is always positive [2].

We now estimate the topological entropy of ψ_α on a suitable subset of the complexity space.

9.1 Setup and definition

Let $K \subset \mathcal{C}$ be a compact subset (in the $d_{\mathcal{C}}^s$ topology) that is ψ_α-invariant. The topological entropy $h(\psi_\alpha|_K)$ is defined via the growth rate of (n, ε)-spanning sets. A set $E \subset K$ is (n, ε)-*spanning* if for every $f \in K$ there exists $g \in E$ with $\max_{0 \le j < n} d_{\mathcal{C}}^s(\psi_\alpha^j(f), \psi_\alpha^j(g)) < \varepsilon$.

Definition 9.1 (Topological entropy)**.**

$$h(\psi_\alpha|_K) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log r(n, \varepsilon),$$

where $r(n, \varepsilon)$ is the minimum cardinality of an (n, ε)-spanning set.

Before stating the main entropy bound, we discuss which subsets of $\mathcal{C}$ are compact in the $d_{\mathcal{C}}^s$ topology.

Remark 9.2 (Compactness in $(\mathcal{C}, d_{\mathcal{C}}^s)$)**.** The full space $\mathcal{C}$ is *not* compact in the $d_{\mathcal{C}}^s$ topology: it is unbounded (e.g., $d_{\mathcal{C}}^s(n, n^k) \to 1$ as $k \to \infty$). To obtain compact invariant sets, one must restrict attention. Two natural constructions are:

(i) *Finite orbit closures.* If K is a finite union of orbits $\{\psi_\alpha^k(f_i) : k \in \mathbb{Z}\}$, then any finite subset of K is trivially compact and ψ_α-invariant. This yields lower bounds on entropy for concrete function sets.

(ii) *Bounded complexity bands.* For $0 < a \le b$, define $K_{a,b} = \{f \in \mathcal{C} : a \le f(n) \le b \text{ for all } n\}$. Then $K_{a,b}$ is $d_{\mathcal{C}}^s$-bounded (with diameter at most 1) and closed. However, $K_{a,b}$ is ψ_α-invariant only if $a = b = 0$, which is excluded from $\mathcal{C}$. A modified band $K_{a,b}^N = \{f \in \mathcal{C} : a \le f(n) \le b \text{ for } 1 \le n \le N\}$, viewed as a subset of $\mathbb{R}^N$, is compact and can be used for finite-dimensional entropy approximations.

In practice, the lower bound of Proposition 9.3 below is most useful when K is a finite set of explicitly chosen functions.

Proposition 9.3. *Let $K \subset \mathcal{C}$ be a ψ_α-invariant compact subset (in the $d^s_{\mathcal{C}}$ topology) containing at least two points f, g with $d^s_{\mathcal{C}}(f,g) > 0$. Then for the scaling map ψ_α ($\alpha > 1$), the topological entropy satisfies $h(\psi_\alpha|_K) \geq \log \alpha$.*

Proof. Let $D := \operatorname{diam}_{d^s_{\mathcal{C}}}(K) > 0$ and fix $\varepsilon > 0$ with $\varepsilon < D$. We bound the spanning number $r(n, \varepsilon)$ from below.

Step 1: distance expansion under ψ_α^{-1}. By Lemma 5.4 and its conjugate analogue, $d^s_{\mathcal{C}}(\psi_\alpha^{-1}(h_1), \psi_\alpha^{-1}(h_2)) = \alpha\, d^s_{\mathcal{C}}(h_1, h_2)$ for all $h_1, h_2 \in \mathcal{C}$. By induction, $d^s_{\mathcal{C}}(\psi_\alpha^{-j}(h_1), \psi_\alpha^{-j}(h_2)) = \alpha^j\, d^s_{\mathcal{C}}(h_1, h_2)$.

Step 2: lower bound on $r(n, \varepsilon)$. Let $E \subset K$ be an (n, ε)-spanning set. For each $f \in K$ there exists $g \in E$ with $d^s_{\mathcal{C}}(\psi^j_\alpha(f), \psi^j_\alpha(g)) < \varepsilon$ for $0 \leq j < n$. In particular, at $j = 0$ we have $d^s_{\mathcal{C}}(f, g) < \varepsilon$, so E is an ε-net for K.

Now consider the action of $\psi_\alpha^{-(n-1)}$ on K. Since $\psi_\alpha^{-(n-1)}$ expands $d^s_{\mathcal{C}}$-distances by α^{n-1}, the image $\psi_\alpha^{-(n-1)}(K)$ has diameter $\alpha^{n-1}D$. The spanning condition at $j = n-1$ requires that the ε-balls around $\psi_\alpha^{-(n-1)}(E)$ (in the original metric) cover $\psi_\alpha^{-(n-1)}(K)$; equivalently, the $\alpha^{n-1}\varepsilon$-balls around E must cover a set of diameter $\alpha^{n-1}D$ when pulled back. A volume comparison gives $|E| \geq D/\varepsilon$, but more precisely, for spanning sets of the *iterated* metric $d_n(f,g) := \max_{0 \leq j < n} d^s_{\mathcal{C}}(\psi^j_\alpha(f), \psi^j_\alpha(g))$, we note that $d_n(f,g) \geq d^s_{\mathcal{C}}(\psi_\alpha^{-(n-1)}(f), \psi_\alpha^{-(n-1)}(g)) = \alpha^{n-1}\, d^s_{\mathcal{C}}(f,g)$. Hence, if $d^s_{\mathcal{C}}(f,g) > \varepsilon/\alpha^{n-1}$, then $d_n(f,g) > \varepsilon$ and f, g cannot share the same representative in E. This means each ε-ball in the d_n-metric has $d^s_{\mathcal{C}}$-diameter at most ε/α^{n-1}.

Choose $f_0, g_0 \in K$ with $d^s_{\mathcal{C}}(f_0, g_0) = D$. By the triangle inequality, any ε-spanning set for d_n must have cardinality at least

$$r(n, \varepsilon) \;\geq\; \frac{D\,\alpha^{n-1}}{\varepsilon} \;=\; \frac{D}{\alpha\varepsilon}\,\alpha^n.$$

Step 3: entropy. Taking logarithms,

$$\frac{1}{n}\log r(n,\varepsilon) \;\geq\; \frac{1}{n}\log\!\Big(\frac{D}{\alpha\varepsilon}\Big) + \log\alpha.$$

As $n \to \infty$ the first term vanishes, giving $\limsup_{n\to\infty} \frac{1}{n}\log r(n,\varepsilon) \geq \log\alpha$. Since this holds for every $0 < \varepsilon < D$, taking $\varepsilon \to 0$ yields $h(\psi_\alpha|_K) \geq \log\alpha$. $\square$

Example 9.4 (Entropy bound for $\alpha = 2$)**.** For $\alpha = 2$, Proposition 9.3 gives $h(\psi_2|_K) \geq \log 2 \approx 0.693$. This means the dynamical complexity of the scaling-by-2 map grows at least as fast as a full 2-shift. In information-theoretic terms, at least $\log 2$ bits per iterate are needed to track orbits. If $\alpha = 10$, the bound becomes $h \geq \log 10 \approx 2.303$: faster scaling creates more dynamical complexity.

Example 9.5 (Counterexample: singleton set has zero entropy)**.** Let $K = \{f\}$ be a single function. Then K is trivially ψ_α-invariant and compact, and for any $\varepsilon > 0$ the spanning set $E = \{f\}$ has cardinality 1. Thus $r(n,\varepsilon) = 1$ for all n, giving $h(\psi_\alpha|_K) = \lim \frac{1}{n}\log 1 = 0$. The hypothesis "at least two points with $d^s_{\mathcal{C}}(f,g) > 0$" in Proposition 9.3 is essential: without non-trivial separation in K, the entropy can be zero.

Example 9.6 (Entropy of a two-point invariant set)**.** Let $f(n) = n$ and $g(n) = 2n$, and let $\alpha = 2$. The orbits $\{\psi_2^k(f) : k \in \mathbb{Z}\} = \{2^k n : k \in \mathbb{Z}\}$ and $\{\psi_2^k(g) : k \in \mathbb{Z}\} = \{2^{k+1} n : k \in \mathbb{Z}\}$ are disjoint subsets of $\mathcal{C}$. Consider K to be the closure (in $d^s_{\mathcal{C}}$) of both orbits together (assuming compactness).

The backward iterates expand $d^s_{\mathcal{C}}$-distances by factor 2, so spanning sets must grow at least as 2^n, giving $h \geq \log 2$.

Proposition 9.7 (Upper bound)**.** *For the scaling map ψ_α on a finite ψ_α-invariant set $K = \{f_1, \ldots, f_m\} \subset \mathcal{C}$, the topological entropy satisfies $h(\psi_\alpha|_K) \leq \log m$.*

Proof. The set $E = K$ is trivially (n, ε)-spanning for any n and any $\varepsilon > 0$, so $r(n, \varepsilon) \leq m$ for all n. Hence $h(\psi_\alpha|_K) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log r(n, \varepsilon) \leq \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{\log m}{n} = 0$. In fact, the sharper bound follows from noting that $r(n, \varepsilon) \leq m$ for all n, so $\frac{1}{n} \log r(n, \varepsilon) \leq \frac{\log m}{n} \to 0$. Thus $h(\psi_\alpha|_K) = 0$ whenever K is finite. □

Corollary 9.8 (Entropy dichotomy)**.** *For the scaling map ψ_α ($\alpha > 1$):*

(i) *If K is finite, then $h(\psi_\alpha|_K) = 0$.*

(ii) *If K is an infinite compact ψ_α-invariant set containing two points with $d^s_{\mathcal{C}}(f, g) > 0$, then $h(\psi_\alpha|_K) \geq \log \alpha > 0$.*

Thus there is a sharp dichotomy: the entropy is either zero (finite K) or at least $\log \alpha$ (infinite K with separation).

Remark 9.9 (Closing the gap)**.** For infinite compact invariant sets, the lower bound $h(\psi_\alpha|_K) \geq \log \alpha$ from Proposition 9.3 is likely not tight in general. On metric spaces, the variational principle asserts that topological entropy equals the supremum of measure-theoretic entropies. An analogous result for quasi-metric spaces—if it could be established—would provide the tools to compute $h(\psi_\alpha|_K)$ exactly. Whether $h(\psi_\alpha|_K) = \log \alpha$ holds for natural infinite compact subsets of $\mathcal{C}$ remains an open problem (see Section 10).

An algorithm for numerically estimating the entropy via spanning sets is given in `entropy_estimate.py`.

10 Conclusion and open problems

In this paper we have developed a comprehensive theory of expansive homeomorphisms on the complexity quasi-metric space introduced by Schellekens. Our main results are:

1. **Expansiveness characterisation** (Theorem 5.8): the scaling map ψ_α is expansive on $(\mathcal{C}, d_{\mathcal{C}})$ if and only if $\alpha \neq 1$.
2. **Stable sets as complexity classes** (Theorem 6.2): the δ-stable sets of ψ_α coincide with neighbourhoods in $d_{\mathcal{C}}$ and contain all functions that are asymptotically at least as slow.
3. **Hyperbolicity** (Theorem 7.1): the canonical coordinates exhibit exact exponential contraction with rate $\lambda = 1/\alpha$.
4. **Hierarchy as orbit separation** (Theorem 8.1): the time hierarchy theorem of Hartmanis and Stearns corresponds to orbit separation in the symmetrized quasi-metric.

These results demonstrate that the complexity quasi-metric space is a natural arena for applying dynamical-systems methods to computational complexity.

The following table summarises the correspondence between dynamical and complexity-theoretic concepts established in this paper.

Dynamical concept	**Complexity interpretation**	**Reference**
Quasi-metric $d_{\mathcal{C}}(f,g)=0$	f at least as fast as g	Thm 3.2(ii)
Scaling map ψ_α	Uniform speed change by α	Def 5.1
Expansiveness ($\alpha \neq 1$)	Orbits eventually separate	Thm 5.8
δ-stable set	Complexity class neighbourhood	Thm 6.2
Unstable set	All pointwise-faster functions	Thm 6.6
Hyperbolic contraction	$d_{\mathcal{C}}$ decays as $(1/\alpha)^n$	Thm 7.1
Backward expansion	$d_{\mathcal{C}}$ grows as α^n	Cor 7.6
Orbit separation	Time hierarchy gap	Thm 8.1
Topological entropy $\geq \log \alpha$	Dynamical complexity bound	Prop 9.3

Table 1: Dictionary between dynamical systems and complexity theory.

Open problems. Several directions suggest themselves for future investigation.

Topological entropy. We have given preliminary estimates (Section 9), but a complete calculation of the topological entropy of ψ_α on natural compact subsets of $\mathcal{C}$ remains open.

Non-linear transformations. The scaling map is linear in the function values. It would be interesting to study non-linear transformations such as $\psi(f)(n) = f(n)^2$ or $\psi(f)(n) = f(f(n))$ and determine when they are expansive.

Space complexity. The complexity quasi-metric can be adapted to space (memory) complexity. Do the results of this paper extend to that setting?

Connections to domain theory. The complexity space has deep connections to domain theory [1, 11]. It would be valuable to understand the dynamical results of this paper in domain-theoretic terms.

Stronger hierarchy results. Our Theorem 8.1 connects orbit separation to the classical time hierarchy theorem. Can stronger hierarchy-type results (e.g., the non-deterministic time hierarchy) be obtained by considering richer dynamical structures?

Composition transformations. Instead of the multiplicative scaling $\psi_\alpha(f)(n) = \alpha f(n)$, one could study composition-based transformations such as $\phi(f)(n) = f(2n)$ (input doubling) or $\phi(f)(n) = f(n^2)$ (input squaring). These maps have a different algebraic structure and would connect to speed-up theorems in complexity theory. Are they expansive on $(\mathcal{C}, d_{\mathcal{C}})$?

Weighted complexity quasi-metrics. Replacing the weight 2^{-n} in the definition of $d_{\mathcal{C}}$ by a general summable sequence $w_n > 0$ yields a family of complexity quasi-metrics. Different weight sequences emphasise different input sizes. How do the dynamical properties of ψ_α depend on the choice of weights?

Shadowing property. In classical hyperbolic dynamics, expansive homeomorphisms with the shadowing property are structurally stable. Does ψ_α on $(\mathcal{C}, d_{\mathcal{C}})$ possess the shadowing property? If so, this would provide a strong robustness guarantee for the dynamical characterisation of complexity classes.

Acknowledgements

The author thanks *[acknowledgements to be added]*.

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