Exercises

Exercise 1

Consider the following one-dimensional boundary-value problem:

$$\begin{cases} -(\alpha u')' + \gamma u = f, & 0 < x < 1, \\ u(0) = 0, \\ \alpha(1)u'(1) + \delta(1)u(1) = 0, \end{cases} \tag{1}$$

where $\alpha = \alpha(x)$, $\gamma = \gamma(x)$, $f = f(x)$ are assigned functions with $0 \leq \gamma(x) \leq \gamma_1$ and $0 < \alpha_0 \leq \alpha(x) \leq \alpha_1$, $\forall x \in [0,1]$, while $\delta \in \mathbb{R}$. Moreover, suppose that $f \in L^2(0,1)$.

1. Write the problem's weak formulation specifying the appropriate functional spaces and hypotheses on the data to guarantee existence and uniqueness of the solution.

Solution: *we seek $u \in V = \{v \in H^1(0,1) : v(0) = 0\}$ such that $a(u,v) = F(v)$ $\forall v \in V$ where*

$$a(u,v) = \int_0^1 \alpha u'v'dx + \int_0^1 \gamma uvdx + \delta u(1)v(1), \qquad F(v) = \int_0^1 fvdx.$$

The existence and uniqueness of the solution of the weak problem are guaranteed if the hypotheses of the Lax-Milgram lemma hold. The form $a(\cdot,\cdot)$ is continuous as we have

$$|a(u,v)| \leq 2\max(\alpha_1, \gamma_1)||u||_V||v||_V + +|\delta||v(1)||u(1)|,$$

from which, considering that $u(1) = \int_0^1 u'dx$, we obtain

$$|a(u,v)| \leq M||u||_V||v||_V \quad \textit{with } M = 3\max(\alpha_1, \gamma_1, |\delta|).$$

We have coercivity if $\delta \geq 0$ as in such case we find

$$|a(u,u)| \geq \alpha_0||u'||^2_{L^2(0,1)} + u^2(1)\delta \geq \alpha_0||u'||^2_{L^2(0,1)}.$$

To find the inequality in $||\cdot||_V$ invoking the Poincaré inequality, i.e., $||u||_{L^2(0,1)} \leq C_p||u'||_{L^2(0,1)}$ $\forall u \in V$, it suffices to prove that

$$\frac{1}{1+C_p^2}||u||^2_V \leq ||u'||^2_{L^2(0,1)},$$

and then to conclude that

$$|a(u,u)| \geq \alpha^*||u||^2_V \quad \textit{with } \alpha^* = \frac{\alpha_0}{1+C_p^2}.$$

The fact that F is a linear and continuous functional can be verified immediately:

$$|F(v)| \leq ||f||_{L^2(0,1)}||v||_V.$$

Exercise 2

Let us consider a thin rod of length L, having temperature t_0 at the extremum $x = 0$ and insulated at the other extremum $x = L$. Let us suppose that the transversal section of the rod has constant surface equal to A and that the perimeter of A be p. The temperature t of the rod at a generic point $x \in (0, L)$ then satisfies the following mixed boundary-value problem

$$\begin{cases} -kAt'' + \sigma pt = 0, & x \in (0, L), \\ t(0) = t_0, \\ t'(L) = 0. \end{cases} \tag{2}$$

having denoted by k the thermal conductivity coefficient and by σ the convective transfer coefficient.

1. Verify that the exact solution of this problem is
$$t(x) = t_0 \frac{\cosh[m(L-x)]}{\cosh(mL)}, \tag{3}$$
with $m = \sigma p/kA$.

2. Write the weak formulation of (2), then its Galerkin linear finite element approximation.

3. Fix $L = 1$, $A = 1/k$, $\sigma = 2/p$ and verify that the hypothesis of the Lax-Milgram lemma are satisfied.

Solution 1.: *Left to the reader.*
Solution 2.: *First we rewrite problem* (2) *as an equivalent mixed boundary value problem with homogeneous Dirichlet boundary conditions in 0 for the unknown* $u = t - t_0$, *obtaining*

$$\begin{cases} -kAu'' + \sigma pu = pt_0, & x \in (0, L), \\ u(0) = 0, \\ u'(L) = 0. \end{cases} \tag{4}$$

The variational formulation reads: find $u \in V = \{v \in H^1(0, L) : v(0) = 0\}$ *such that* $a(u, v) = F(v)\ \forall v \in V$ *where*

$$a(u, v) = \int_0^L kAu'v'dx + \int_0^L \sigma puvdx, \qquad F(v) = \int_0^1 pt_0vdx.$$

To derive the finite element approximation of the above problem we introduce a partition T_h *of* $(0, L)$ *in* $N + 1$ *subintervals* $K_j = (x_{j-1}, x_j)$, *called elements, having width* $h_j = x_j - x_{j-1}$ *with* $0 = x_0 < x_1 < ... < x_N < x_{N+1} = L$, *and set* $h = \max_j h_j$. *Since the functions of* $H^1(0, L)$ *are continuous functions on* $[0, L]$, *we can construct the following space*

$$X_h^1 = \{v \in C^0(0, L) : v_{|K_j} \in \mathbb{P}^1\, \forall K_j \in T_h\},$$

having denoted by $\mathbb{P}^1$ *the space of polynomials with degree lower than or equal to* 1 *in the variable* x. *Then, the Galerkin linear finite element approximation is:*
find $u_h \in V_h = \{v \in X_h^1 : v(0) = 0\} \subset V$ *such that* $a(u_h, v) = F(v)\ \forall v \in V_h$.
Solution 3.: *The hypotheses of the Lax-Milgram lemma can be easily checked:*

- continuity of $a(\cdot, \cdot) : |a(u, v)| \leq 3||u||_V||v||_V$,
- coercivity of $a(\cdot, \cdot) : a(u, u) \geq ||u||_V^2$,
- continuity of $F(\cdot) : |F(v)| \leq pt_0||v||_V$.

Exercise 3

Consider a viscous fluid located between two horizontal plates, parallel and at a distance of $2H$. Suppose that the upper plate, having temperature t_{sup}, moves at a relative speed of U with respect to the lower one, having temperature t_{inf}. In such case the temperature $t : (0, 2H) \to \mathbb{R}$ of the fluid satisfies the following Dirichlet problem

$$\begin{cases} -\dfrac{d^2t}{dy^2} = \alpha(H-y)^2, \quad y \in (0, 2H), \\ t(0) = t_{inf}, \\ t(2H) = t_{sup}. \end{cases} \tag{5}$$

where $\alpha = \dfrac{4U^2\mu}{H^4k}$, k being the thermal conductivity coefficient and μ the viscosity of the fluid.

1. Find the exact solution $t(y)$.
2. Write the weak formulation.
3. Fix $H = 1$, $\alpha = 10$ and verify that the hypothesis of the Lax-Milgram lemma are satisfied.

Solution 1.: *The exact solution*

$$t(y) = \frac{\alpha}{12}(H-y)^4 + \frac{t_{inf} - t_{sup}}{2H}(H-y) + \frac{t_{inf} + t_{sup}}{2} + \frac{\alpha H^4}{12},$$

is obtained integrating two times with respect to y from 0 to $2H$ and imposing the boundary conditions.

Solution 2.: *We write the weak formulation for the following equivalent problem in the unknown $u = t - t_{inf} - \dfrac{(t_{sup} - t_{inf})}{2H}y$:*

$$\begin{cases} -\dfrac{d^2u}{dy^2} = \alpha(H-y)^2, \quad y \in (0, 2H), \\ u(0) = 0, \\ u(2H) = 0. \end{cases} \tag{6}$$

The variational formulation reads: find $u \in V = \{v \in H^1(0, 2H) : v(0) = v(2H) = 0\}$ such that $a(u, v) = F(v)$ $\forall v \in V$ where

$$a(u, v) = \int_0^{2H} u'v'dy, \qquad F(v) = \int_0^{2H} \alpha(H-y)^2 v dy.$$

Solution 3.: *The hypotheses of the Lax-Milgram lemma can be easily checked:*

- continuity of $a(\cdot, \cdot) : |a(u, v)| \le ||u||_V ||v||_V$,
- coercivity of $a(\cdot, \cdot) : a(u, u) = ||u'||^2_{L^2(0,2)} \ge \dfrac{1}{1 + C_p^2}||u||^2_V$,
- continuity of $F(\cdot) : |F(v)| \le 10||v||_V$.