EXERCISES

Exercise 1

Using the divergence Theorem[1] verify the following Green's formula:

$$-\int_\Omega \operatorname{div}(\nabla u)\, v\, d\mathbf{x} = \int_\Omega \nabla u \cdot \nabla v\, d\mathbf{x} - \int_{\partial\Omega} \frac{\partial u}{\partial \mathbf{n}} v\, d\mathbf{x}.$$

***Solution**: Applying the divergence Theorem first to the function $\boldsymbol{a} = (\phi\psi, 0)^\top$ and then to $\boldsymbol{a} = (0, \phi\psi)^\top$, we get the relations*

$$\int_\Omega \frac{\partial \phi}{\partial x_i}\psi\, d\Omega = -\int_\Omega \phi \frac{\partial \psi}{\partial x_i}\, d\Omega + \int_{\partial\Omega} \phi\psi n_i\, d\gamma, \quad i = 1, 2. \tag{1}$$

Note also that if we take $\mathbf{a} = \mathbf{b}\phi$, where $\mathbf{b}$ and ϕ are respectively a vector and a scalar field, the the divergence Theorem yields

$$\int_\Omega \phi\, \operatorname{div}\mathbf{b}\, d\Omega = -\int_\Omega \mathbf{b}\cdot\nabla\phi\, d\Omega + \int_{\partial\Omega} \mathbf{b}\cdot\mathbf{n}\phi\, d\gamma,$$

*which is called Green formula for the divergence operator. We exploit (**??**) by keeping into account the fact that $\Delta u = \operatorname{div}(\nabla u) = \sum_{i=1}^2 \frac{\partial}{\partial x_i}\left(\frac{\partial u}{\partial x_i}\right)$. Supposing that all the integral that appear are meaningful, we find*

$$\begin{aligned}
-\int_\Omega \Delta u v\, d\Omega &= -\sum_{i=1}^2 \int_\Omega \frac{\partial}{\partial x_i}\left(\frac{\partial u}{\partial x_i}\right) v\, d\Omega \\
&= \sum_{i=1}^2 \int_\Omega \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}\, d\Omega - \sum_{i=1}^2 \int_{\partial\Omega} \frac{\partial u}{\partial x_i} v n_i\, d\gamma \\
&= \int_\Omega \sum_{i=1}^2 \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}\, d\Omega - \int_{\partial\Omega} \left(\sum_{i=1}^2 \frac{\partial u}{\partial x_i} n_i\right) v\, d\gamma,
\end{aligned}$$

that is

$$-\int_\Omega \operatorname{div}(\nabla u)\, v\, d\Omega = \int_\Omega \nabla u \cdot \nabla v\, d\Omega - \int_{\partial\Omega} \frac{\partial u}{\partial \mathbf{n}} v\, d\gamma.$$

[1] Let $\mathbf{a}(\mathbf{x}) = (a_1(\mathbf{x}), a_2(\mathbf{x}))^\top$ be a regular enough vectorial function, $\mathbf{n}(\mathbf{x}) = (n_1(\mathbf{x}), n_2(\mathbf{x}))^\top$ the outward unit normal to $\partial\Omega$ and $\mathbf{x} = (x_1, x_2)^\top$ the spatial coordinate vector, then

$$\int_\Omega \operatorname{div}(\mathbf{a})\, d\mathbf{x} = \int_{\partial\Omega} \mathbf{a}\cdot\mathbf{n}\, d\mathbf{x}.$$

Exercise 2

For each of the following problem, write the corresponding weak formulation, the Galerkin linear finite element approximation and verify that the hypothesis of the Lax-Milgram lemma are satisfied.

1. **Neumann boundary value problem**

$$\begin{cases} -\Delta u + \sigma u = f, & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \mathbf{n}} = \phi, & \text{on } \partial\Omega, \end{cases}$$

where $\sigma \in L^\infty(\Omega)$ is such that $\sigma(\mathbf{x}) \geq \sigma_0 > 0$ a.e. in Ω, $f \in L^2(\Omega)$ and $\phi \in L^2(\partial\Omega)$.

Solution*: we seek $u \in V := H^1(\Omega)$ such that $a(u, v) = F(v)$ $\forall v \in V$ where*

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v \, d\Omega + \int_\Omega \sigma u v \, d\Omega, \qquad F(v) = \int_\Omega f v \, d\Omega + \int_{\partial\Omega} \phi v \, d\gamma.$$

To derive the finite element approximation of the above problem we introduce a partition T_h of Ω in N triangles K_j, called elements, having diameter h_j, and set $h = \max_{j=1,..,N} h_j$. Then, we can construct the following space

$$X_h^1 = \{v \in C^0(\Omega) : v_{|K_j} \in \mathbb{P}^1 \, \forall K_j \in T_h\},$$

having denoted by $\mathbb{P}^1$ the space of polynomials with degree lower than or equal to 1 in the variables x and *y. Then, the Galerkin linear finite element approximation is:*
find $u_h \in X_h^1 \subset V$ such that $a(u_h, v) = F(v)$ $\forall v \in X_h^1$.

The hypotheses of the Lax-Milgram lemma can be easily checked:

- continuity of $a(\cdot, \cdot)$:

$$\begin{aligned} |a(u, v)| &\leq ||\nabla u||_{L^2(\Omega)} ||\nabla v||_{L^2(\Omega)} + ||\sigma||_{L^\infty(\Omega)} ||u||_{L^2(\Omega)} ||v||_{L^2(\Omega)} \\ &\leq \max\left(1, ||\sigma||_{L^\infty(\Omega)}\right) ||u||_V ||v||_V \end{aligned}$$

- coercivity of $a(\cdot, \cdot)$:

$$a(u, u) = ||\nabla u||^2_{L^2(\Omega)} + \int_\Omega \sigma u^2 \, d\Omega \geq ||\nabla u||^2_{L^2(\Omega)} + \sigma_0 ||u||^2_{L^2(\Omega)} \geq \min(1, \sigma_0) \, ||u||^2_{H^1(\Omega)}$$

- continuity of $F(\cdot)$:

$$|F(v)| \leq ||f||_{L^2(\Omega)} ||v||_V + ||\phi||_{L^2(\partial\Omega)} ||v||_{L^2(\partial\Omega)} \leq \left(||f||_{L^2(\Omega)} + ||\phi||_{L^2(\partial\Omega)}\right) ||v||_V$$

2. **Mixed boundary value problem**

$$\begin{cases} -\Delta u = f, & \text{in } \Omega, \\ u = g & \text{on } \Gamma_D, \\ \dfrac{\partial u}{\partial \mathbf{n}} = \phi, & \text{on } \Gamma_N, \end{cases} \tag{2}$$

where $\Gamma_D \cup \Gamma_N = \partial\Omega$, with $\Gamma_D \cap \Gamma_N = \emptyset$, $f \in L^2(\Omega)$, $g \in L^2(\Gamma_D)$ and $\phi \in L^2(\Gamma_N)$.

***Solution**: We start introducing a function R_g, called lifting of the boundary data, such that*

$$R_g \in H^1(\Omega), \quad R_g = g \, on \, \Gamma_D.$$

Furthermore, we suppose that such lifting be continuous, i.e. that

$$\exists C > 0 : ||R_g||_{H^1(\Omega)} \leq C||g||_{L^2(\Gamma_D)} \quad \forall g \in L^2(\Gamma_D).$$

*We set $w = u - R_g$ and we begin by observing that $w = u - R_g = 0$ on Γ_D, that is $w \in H^1_{\Gamma_D}(\Omega)$. Moreover, since $\nabla u = \nabla w + \nabla R_g$, problem (**??**) becomes: find $w \in V := H^1_{\Gamma_D}(\Omega)$:*

$$a(w, v) = F(v) \qquad \forall v \in V,$$

where

$$a(w, v) = \int_\Omega \nabla w \cdot \nabla v \, d\Omega, \quad and \quad F(v) = \int_\Omega f v \, d\Omega + \int_{\Gamma_N} \phi v \, d\gamma - \int_\Omega \nabla R_g \cdot \nabla v \, d\Omega.$$

Having defined $X^1_{h,\Gamma_D} = \left\{v \in X^1_h : v = 0 \, on \, \Gamma_D\right\}$, the Galerkin linear finite element approximation is: find $w_h \in X^1_{h,\Gamma_D} \subset V$ such that $a(w_h, v) = F(v) \; \forall v \in X^1_{h,\Gamma_D}$.
The hypotheses of the Lax-Milgram lemma can be easily checked:

- continuity of $a(\cdot, \cdot)$:

$$|a(w, v)| \leq ||\nabla w||_V ||\nabla v||_V,$$

- coercivity of $a(\cdot, \cdot)$:

$$a(w, w) = ||\nabla w||^2_{L^2(\Omega)} \geq \frac{1}{1 + C_p^2} ||w||_V,$$

- continuity of $F(\cdot)$:

$$\begin{aligned} |F(v)| &\leq ||f||_{L^2(\Omega)} ||v||_V + ||\phi||_{L^2(\partial\Omega)} ||v||_{L^2(\partial\Omega)} + ||R_g||_{H^1(\Omega)} ||v||_V \\ &\leq \left(||f||_{L^2(\Omega)} + ||\phi||_{L^2(\partial\Omega)} + ||g||_{L^2(\partial\Omega)}\right) ||v||_V. \end{aligned}$$

3. **Linear Elasticity**

$$\begin{cases} -\mathrm{div}(\sigma(\mathbf{u})) = \mathbf{f} & \text{in } \Omega \subset \mathbb{R}^2 \\ \mathbf{u} = \mathbf{0} & \text{on } \Gamma_D, \\ \sigma(\mathbf{u})\mathbf{n} = \mathbf{g}, & \text{on } \Gamma_N, \end{cases}$$

where $\Gamma_D \cup \Gamma_N = \partial\Omega$, with $\Gamma_D \cap \Gamma_N = \emptyset$. As usual $\mathbf{n} = (n_1, n_2)^\top$ is the outward unit normal to $\partial\Omega$, $\mathbf{u} = (u_1, u_2)^\top$ is the unknown displacement vector, $\mathbf{f} \in [L^2(\Omega)]^2$ and $\mathbf{g} \in [L^2(\Gamma_N)]^2$ are two assigned functions. Moreover let

$$\sigma(\mathbf{u}) = \lambda \mathrm{div}(\mathbf{u}) \mathbb{I} + 2\mu\epsilon(\mathbf{u}), \quad \epsilon(\mathbf{u}) = \frac{1}{2}\left(\nabla\mathbf{u} + \nabla\mathbf{u}^\top\right), \tag{3}$$

with λ, μ positive constants and $\mathbb{I}$ the identity tensor. We recall that the above system describes the displacement $\mathbf{u}$ of a homogeneous isotropic elastic body that occupies the region Ω when is excited by an external volume force $\mathbf{f}$ and a surface load $\mathbf{g}$ (on Γ_N).

***Solution**: the weak formulation of can be found by observing that $\sigma_{ij} = \sigma_{ji}$ and by using the following Green formula*

$$\sum_{i,j=1}^{2}\int_{\Omega}\sigma_{ij}(\mathbf{u})\epsilon_{ij}(\mathbf{v})\,d\Omega=\sum_{i,j=1}^{2}\int_{\partial\Omega}\sigma_{ij}(\mathbf{u})n_j v_i\,d\gamma-\sum_{i,j=1}^{2}\int_{\Omega}\frac{\partial\sigma_{ij}(\mathbf{u})}{\partial x_j}v_i\,d\Omega. \tag{4}$$

By assuming $\mathbf{v}\in V=\left(H^1_{\Gamma_D}(\Omega)\right)^2$ *(the space of vectorial functions that have components* $v_i\in H^1_{\Gamma_D}(\Omega)$ *for* $i=1,2$*), the weak formulation reads: find* $\mathbf{u}\in V$ *such that*

$$a(\mathbf{u},\mathbf{v})=F(\mathbf{v})\quad\forall\mathbf{v}\in V,$$

with

$$\begin{aligned}
a(\mathbf{u},\mathbf{v}) &= \int_{\Omega}\lambda\mathrm{div}(\mathbf{u})\mathrm{div}(\mathbf{v})\,d\Omega+2\mu\sum_{i,j=1}^{2}\int_{\Omega}\epsilon_{ij}(\mathbf{u})\epsilon_{ij}(\mathbf{v})\,d\Omega,\\
F(\mathbf{v}) &= \int_{\Omega}\mathbf{f}\cdot\mathbf{v}\,d\Omega+\int_{\Gamma_N}\mathbf{g}\cdot\mathbf{v}\,d\gamma.
\end{aligned}$$

The Galerkin linear finite element approximation is: find $\mathbf{u}\in\left(X^1_{h,\Gamma_D}\right)^2\subset V$ *such that* $a(\mathbf{u},\mathbf{v})=F(\mathbf{v})\ \forall\mathbf{v}\in\left(X^1_{h,\Gamma_D}\right)^2$.

The hypotheses of the Lax-Milgram lemma can be checked knowing that the following Korn inequality holds:

$$\exists C_0>0:\sum_{i,j=1}^{2}\int_{\Omega}\epsilon_{ij}(\mathbf{v})\epsilon_{ij}(\mathbf{v})\,d\Omega\geq C_0||\mathbf{v}||^2_V\quad\forall\mathbf{v}\in V.$$