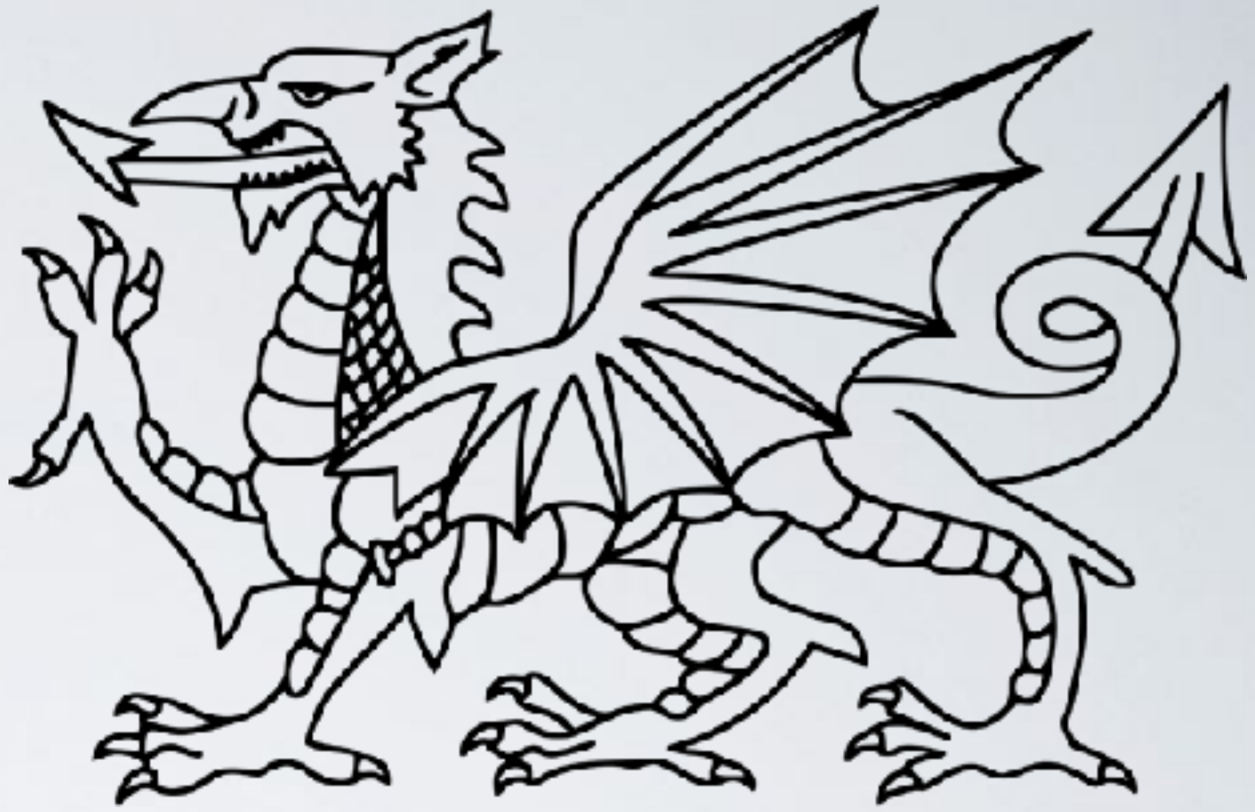
Analysis of Algorithms & Big-O

CS16: Introduction to Algorithms & Data Structures
Spring 2019

Outline

- Running time
- Big-**O**
- Big-Ω and Big-Θ



What is an “Efficient” Algorithm

- Possible efficiency measures
  - Total amount of time on a stopwatch?
  - Low memory usage?
  - Low power consumption?
  - Network usage?
- In CS16 we will focus on *running time*

Q: How should we measure running time?

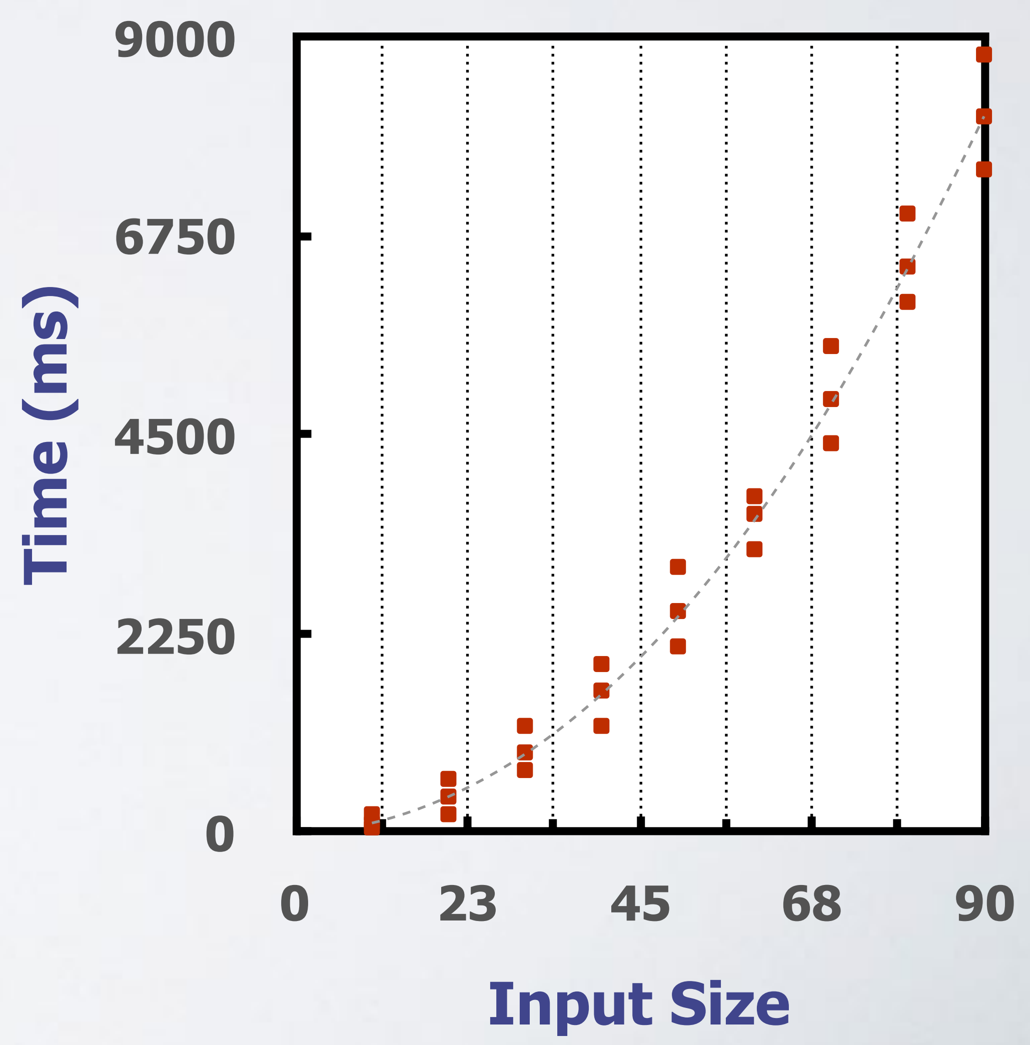
A Simple Algorithm

```
function sum_array(array)
    // Input: an array of 100 integers
    // Output: the sum of the integers
    if array.length = 0
       return error
    sum = 0
    for i in [0, array.length-1]:
      sum = sum + array[i]
     return sum
```

- How do we measure its running time?

Measuring Running Time

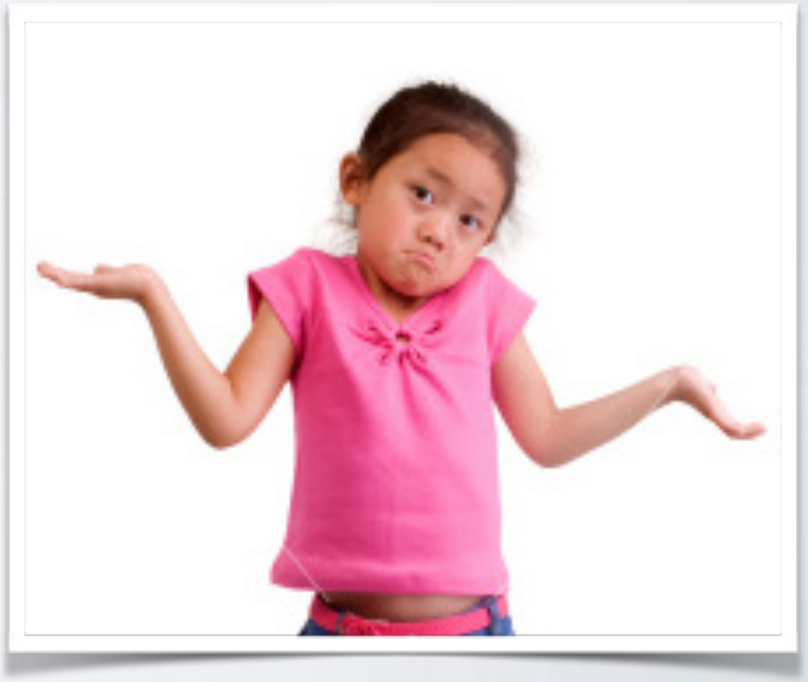
- Experimentally?
  - Implement algorithm
  - Run algorithm on inputs of different size
  - Measure time it takes to finish
  - Plot the results



Q: Was that useful?

Experimental Running Time

- How large should the array be in the experiment?
- Which array should we use (i.e., which ints)?
- Which hardware should we run on?
- Which operating system?
- Which compiler should we use?
- Which compiler flags?
- …



Measuring Running Time

- We need a measure that is
  - independent of hardware
  - independent of OS
  - independent of compiler
  - …
- It should depend only on
  - "intrinsic properties of the algorithm"

Q: What is the *intrinsic* running time of an algorithm?

A Simple Algorithm

```
function sum_array(array)
    // Input: an array of integers
    // Output: the sum of the integers
    if array.length = 0
       return error
    sum = 0
    for i in [0, array.length-1]:
      sum = sum + array[i]
     return sum
```

Knuth's Observation

- Experimental running time can be determined using
  - Time of each operation & frequency of each operation
- Example:
  - run **`sum_array`** on array of size **`100`**

```
time(sum_array) = time(read)·100 + time(add)·99 + time(comp)·1
                = 3ms·100 + 100ms·99 + 10ms·1
                = 10.21s
```

- **`Key insight!`**
  - the time an operation takes depends on hardware but…
  - *the number of times an operation is repeated does not depend on hardware*
  - So let's ignore time and only focus on *number of times* an operation is repeated

Knuth's Observation

- How do we ignore time?
  - we'll assume each operation takes `1` unit of time
- Example:
  - sum_array on array of size `100`

```
time(sum_array) = time(read)·100 + time(add)·99 + time(comp)·1
                = 1·100 + 1·99 + 1·1
                = 100 reads + 99 adds + 1 comp
```

- Let's simplify and just report total number of operations
  - time`(sum_array) = 200 ops`

Elementary Operations

- Most algorithms make use of standard "elementary" operations:
  - Math: `+,-,*,/,max,min,log,sin,cos,abs,...`
  - Comparisons: `==,>,<,≤,≥`
  - Variable assignment
  - Variable increment or decrement
  - Array allocation
  - Creating a new object
  - Function calls and value returns
  - Careful: an object's constructor & function calls may have elementary ops too!
- In practice all these operations take different amounts of time but
  - **we will assume each operation takes 1 unit of time**

What is Running Time?

"Running time"

=

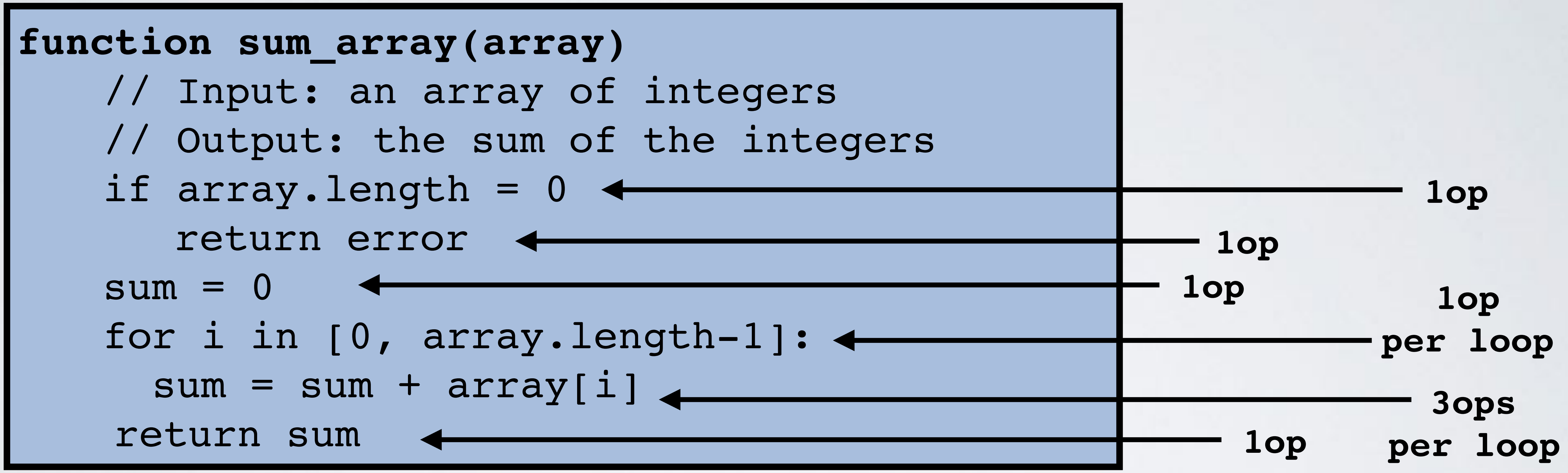
Number of elementary operations

Running time $\neq$ Experimental time

Towards **Algorithmic** Running Time

- Problem #1
  - experimental running time depends on hardware
  - solution: *focus on number of operations*

A Simple Algorithm



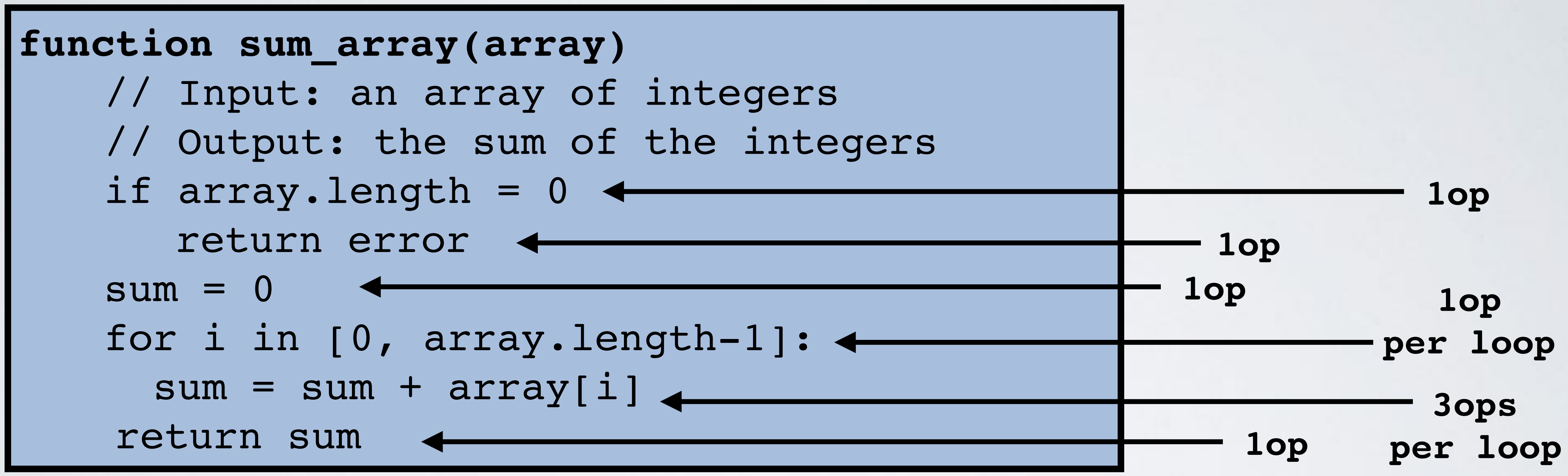
```
function sum_array(array)
    // Input: an array of integers
    // Output: the sum of the integers
    if array.length = 0
       return error
    sum = 0
    for i in [0, array.length-1]:
      sum = sum + array[i]
     return sum
```

- Do we count "**`return error`**"?
  - depends on whether input array is empty
  - if **`array`** is empty then **`sum_array`** takes `2 ops`
  - if **`array`** is not empty then **`sum_array`** takes $3+4\cdot n$ **ops**

Towards **Algorithmic** Running Time

- Problem #1
  - experimental running time depends on hardware
  - solution: *focus on number of operations*
- Problem #2
  - number of operations depends on input
  - solution: *focus on number of operations for worst-case input*

A Simple Algorithm



```
function sum_array(array)
    // Input: an array of integers
    // Output: the sum of the integers
    if array.length = 0
       return error
    sum = 0
    for i in [0, array.length-1]:
      sum = sum + array[i]
     return sum
```

- What is the worst-case input for our algorithm?
  - any array that is non-empty
  - so we'll just ignore "`return error`"

What is Running Time?

Worst-case running time

=

Number of elementary operations on worst-case input

A Simple Algorithm

```
function sum_array(array)
    // Input: an array of integers
    // Output: the sum of the integers
    if array.length = 0           ←  1op
       return error               ←  1op
    sum = 0
    for i in [0, array.length-1]: ←  1op per loop
      sum = sum + array[i]        ←  3ops per loop
    return sum                    ←  1op
```

- How many times does loop execute?
  - depends on *size* of input array

Towards an **Algorithmic** Running Time

- Problem #1
  - experimental running time depends on hardware
  - solution: *focus on **number** of operations (Knuth's observation)*
- Problem #2
  - number of operations depends on input
  - solution: *focus on number of operations on **worst-case** input! Why?*
- Problem #3
  - number of operations depends on input size
  - solution: *focus on number of operations as a function of **input size n**.*

A Simple Algorithm

```
function sum_array(array)
    // Input: an array of integers
    // Output: the sum of the integers
    if array.length = 0          ← 1op
       return error              ← 1op
    sum = 0                      ← 1op
    for i in [0, array.length-1]:  ← n
      sum = sum + array[i]       ← 3ops
     return sum                  ← 1op
```

- How many times does loop execute?
  - depends on *size* of input array
  - `sum_array` takes `3+4·n ops`

What is Running Time?

Worst-case running time

=

T(n): *Number of elementary operations*

on worst-case input

as a function of input size n

Constant Running Time

```
function first(array):
    // Input: an array
    // Output: the first element
    return array[0]         ← 2ops
```

- How many operations are executed?
  - `T(n)=2` ops
  - What if array has `100` elements?
  - What if array has `100,000` elements?
- **key observation:**
  - *running time does not depend on array size!*

```
function argmax(array)
    // Input: an array
    // Output: the index of the maximum value
    index = 0  ←──────────────────────────────────── 1op
    for i in [1, array.length):  ←────────────────── 1op per loop
      if array[i] > array[index]:  ←──────────────── 3ops per loop
        index = i  ←──────────────────────────────── 1op per loop
                                                     (sometimes)
     return index  ←──────────────────────────────── 1op
```

Activity #1

1 min

```
function argmax(array)
    // Input: an array
    // Output: the index of the maximum value
    index = 0  ←──────────────────────────────────── 1op
    for i in [1, array.length):  ←────────────────── 1op per loop
      if array[i] > array[index]:  ←──────────────── 3ops per loop
        index = i  ←──────────────────────────────── 1op per loop
                                                      (sometimes)
     return index  ←───────────────────────────────── 1op
```

Activity #1

1 min

```
function argmax(array)
    // Input: an array
    // Output: the index of the maximum value
    index = 0
    for i in [1, array.length):
      if array[i] > array[index]:
        index = i
    return index
```

1op

1op per loop

3ops per loop

1op per loop
(sometimes)

1op

Activity #1

0 min

Linear Running Time

```
function argmax(array)
    // Input: an array
    // Output: the index of the maximum value
    index = 0                                        1op
    for i in [1, array.length):                      1op per loop
      if array[i] > array[index]:                    3ops per loop
        index = i                                    1op per loop
                                                     (sometimes)
     return index                                    1op
```

- How many operations are executed?
    - `T(n)=5n+2` ops where `n=size(array)`
- **key observation:**
    - *running time depends (mostly) on array size*

```
function possible_products(array):
    // Input: an array
    // Output: a list of all possible products
    //          between any two elements in the list
    products = []                          ← 1op
    for i in [0, array.length):            ← 1op per loop
        for j in [0, array.length):        ← 1op per loop per loop
            products.append(array[i] * array[j])   ← 4ops per loop per loop
    return products                        ← 1op
```

Activity #2

1 min

```
function possible_products(array):
    // Input: an array
    // Output: a list of all possible products
    //          between any two elements in the list
    products = []                                    1op
    for i in [0, array.length):                      1op per loop
        for j in [0, array.length):                  1op per loop per loop
            products.append(array[i] * array[j])     4ops per loop per loop
    return products                                  1op
```

Activity #2

1 min

```
function possible_products(array):
    // Input: an array
    // Output: a list of all possible products
    //          between any two elements in the list
    products = []
    for i in [0, array.length):
        for j in [0, array.length):
            products.append(array[i] * array[j])
    return products
```

1op

1op per loop

1op per loop per loop

4ops per loop per loop

1op

Activity #2

0 min

Quadratic Running Time

```
function possible_products(array):
    // Input: an array
    // Output: a list of all possible products
    //         between any two elements in the list
    products = []
    for i in [0, array.length):
        for j in [0, array.length):
            products.append(array[i] * array[j])
    return products
```

1op

1op per loop

1op per loop per loop

4ops per loop per loop

- How many operations are executed?
  - `T(n)=5n²+n+2` operations where `n=size(array)`
- **key observation:**
  - running time depends (mostly) on the **square** of array size

Running Times

Constant

independent of input size

Linear

depends on input size

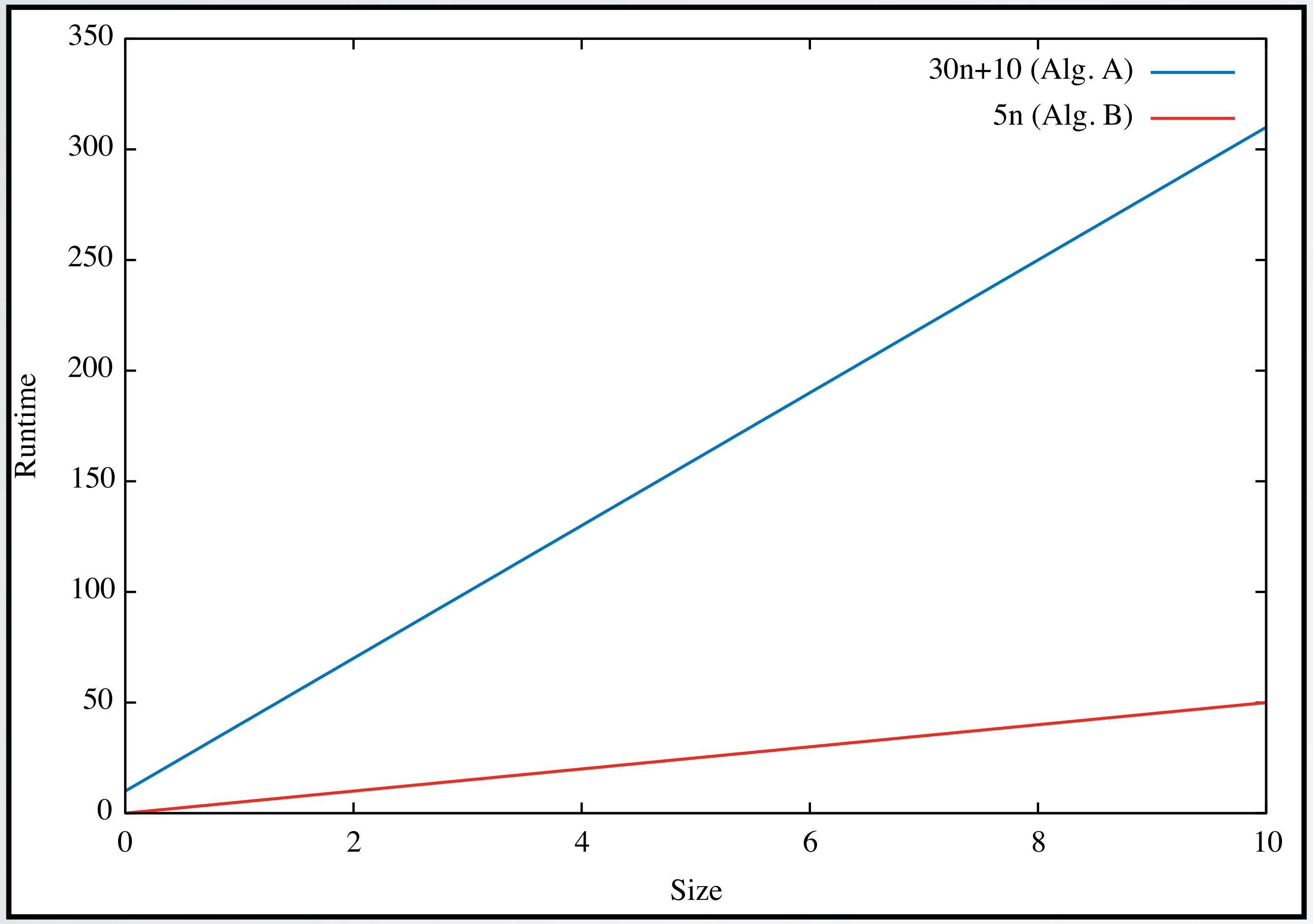
Quadratic

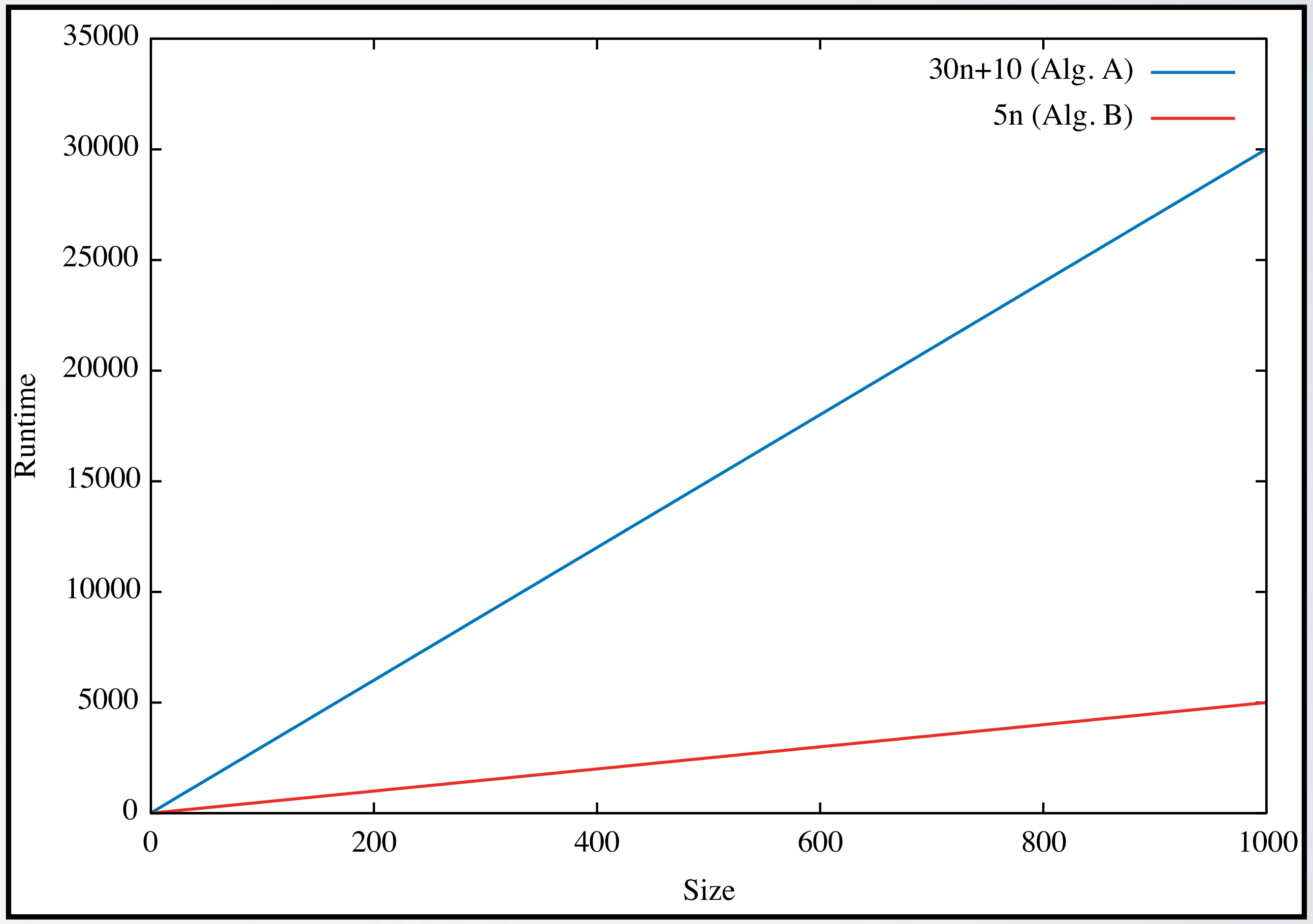
depends on square of input size

Q: how do we compare running times?

Which Algorithm is Better?

- Algorithm A takes $T_A(n)=30n+10$ ops
- Algorithm B takes $T_B(n)=5n$ ops

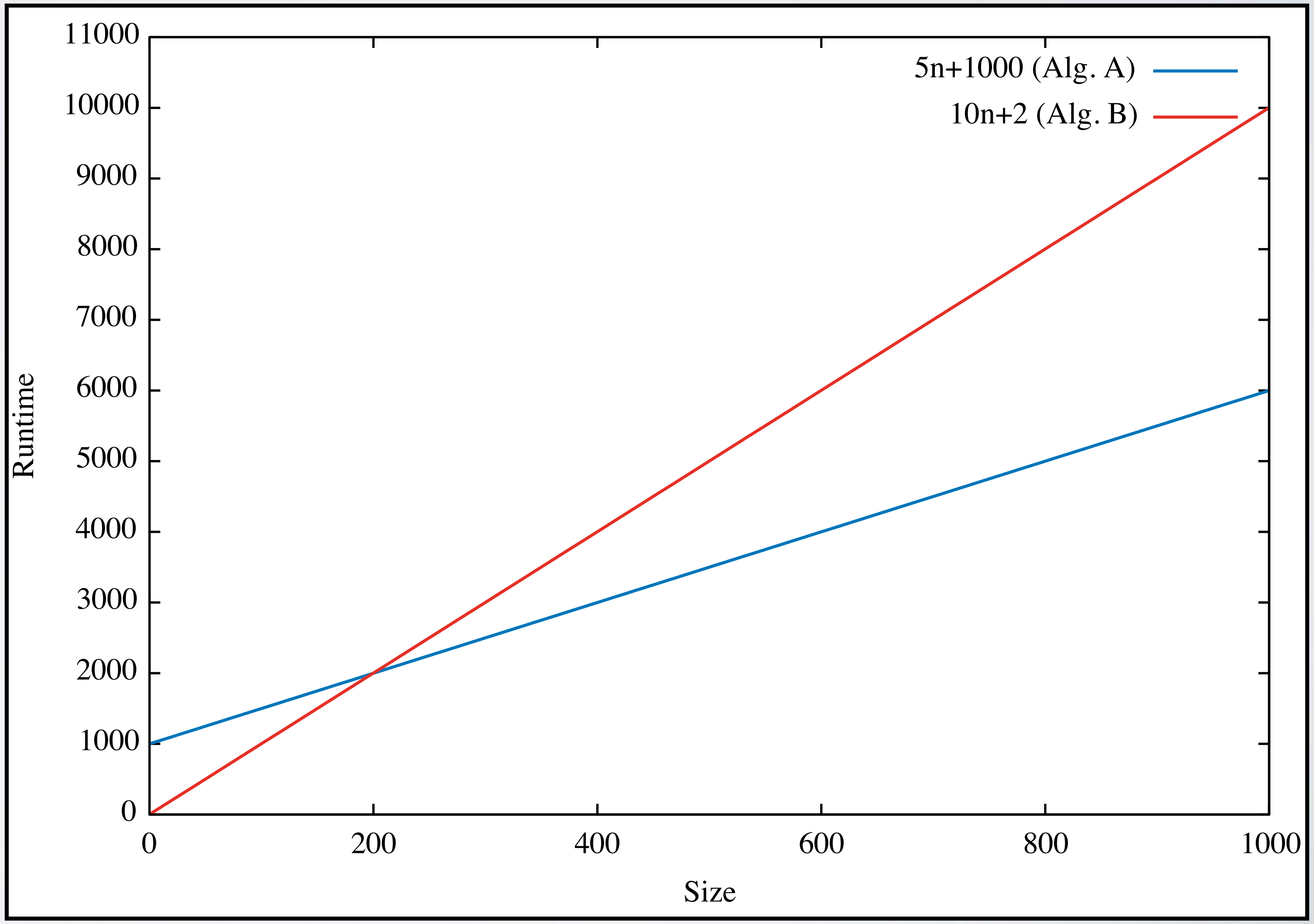




Which Algorithm is Better?

- Alg A takes $T_A(n)=5n+1000$ ops
- Alg B takes $T_B(n)=10n+2$ ops
- It depends on **n**

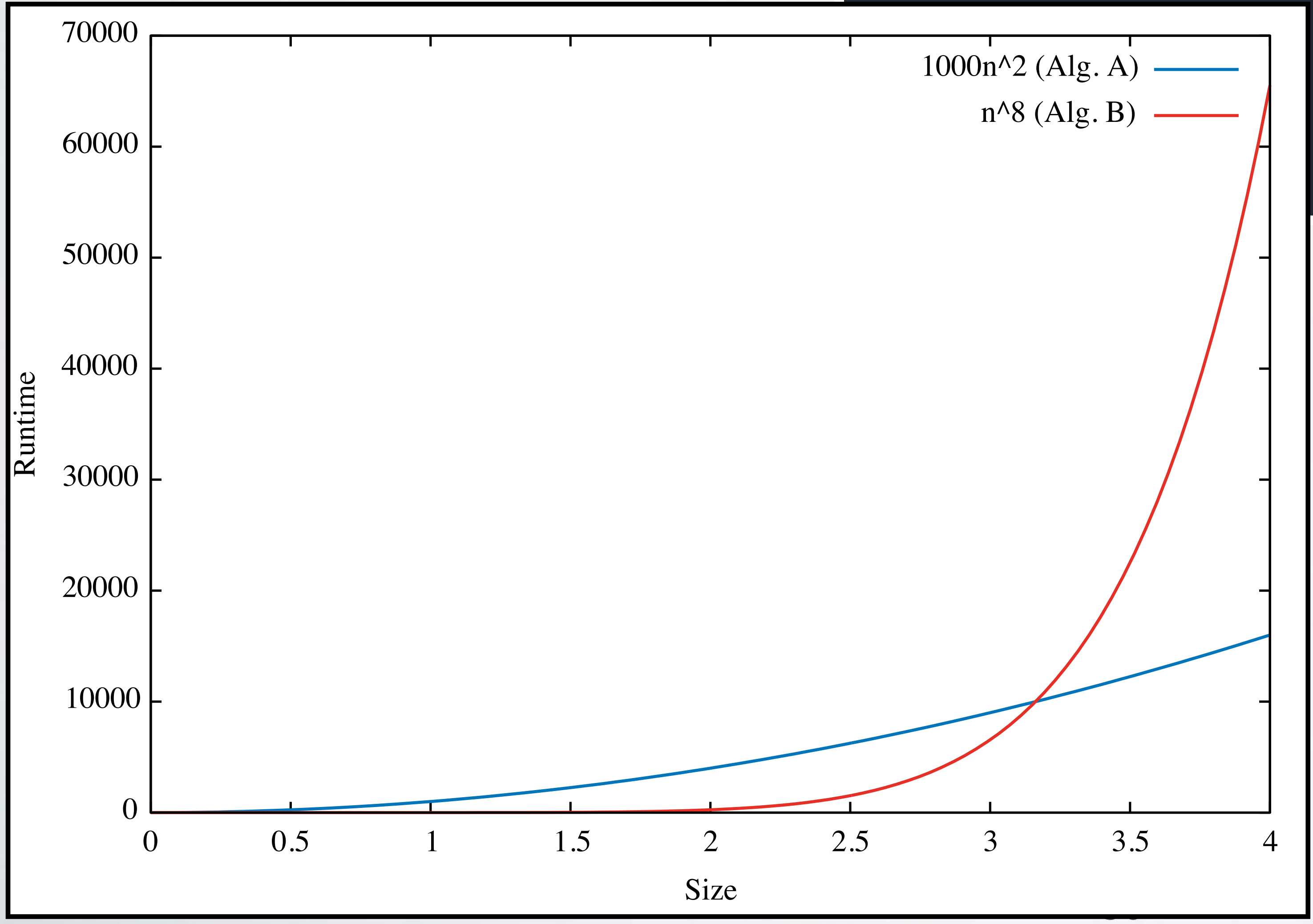
```
rtime(A) < rtime(B) ⟺ 5n+1000 < 10n+2
                    ⟺ 5n > 998
                    ⟺ n > 199.6
```

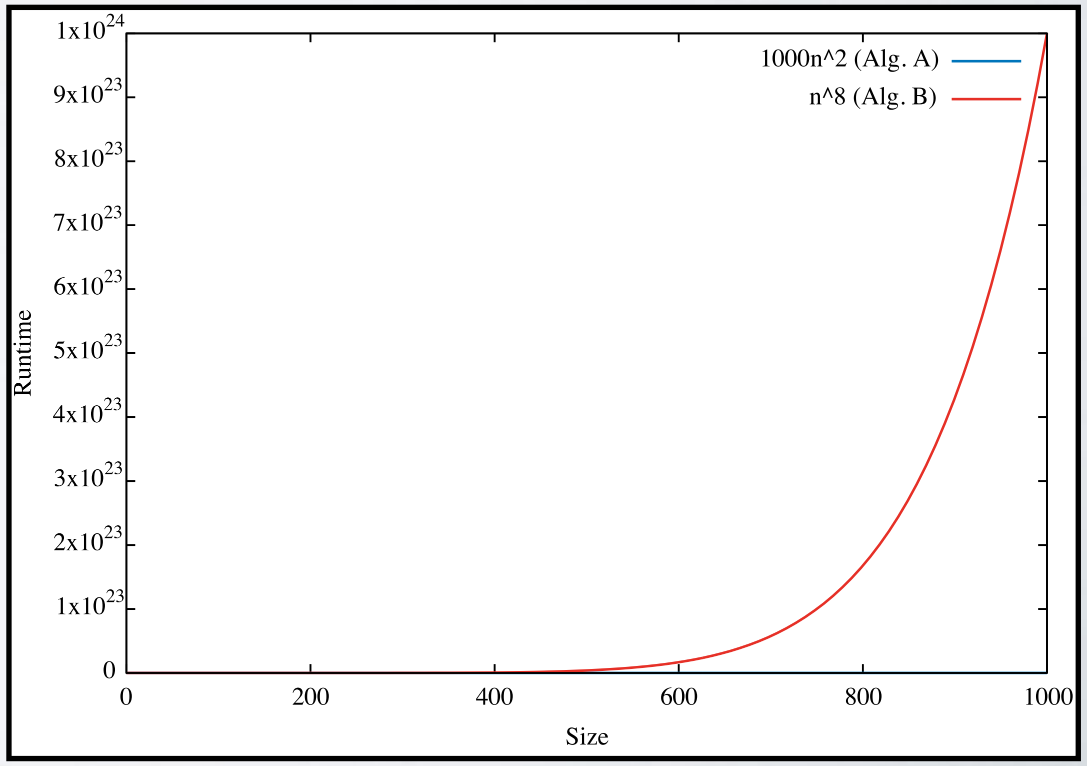


Which Algorithm is Better?

- Alg A takes $T_A(n)=1000n^2$ ops
- Alg B takes $T_B(n)=n^8$ ops
- It depends on `n`

$$
\begin{aligned}
\text{rtime(A)} < \text{rtime(B)} &\iff 1000n^2 < n^8 \\
&\iff 1000n^2 - n^8 < 0 \\
&\iff n^2(1000 - n^6) < 0 \\
&\iff 1000 - n^6 < 0 \\
&\iff n > 1000^{1/6} \\
&\iff n > 3.16\ldots
\end{aligned}
$$





What is Running Time?

Asymptotic worst-case running time

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Number of elementary operations

on worst-case input

as a function of input size n

when n tends to infinity

In CS "running time" usually means asymptotic worst-case running time…but not always!

we will learn about other kinds of running times

Comparing Running Times

Comparing *asymptotic* running times

=

$T_A(n)$ is better than $T_B(n)$ if

for large enough n

$T_A(n)$ grows slower than $T_B(n)$

Q: can we formalize all this mathematically?

Big-O

Definition (Big-O): $T_A(n)$ is $O(T_B(n))$ if there exists positive constants c and n_0 such that:

$$T_A(n) \leq c \cdot T_B(n)$$

for all $n \geq n_0$

- $T_A(n)$'s order of growth is at most $T_B(n)$'s order of growth
- Examples
  - $2n+10$ is $O(n)$
  - $n^{10}+2019$ is $O(n^{10})$ and also $O(n^{50})$

Big-O

- How do we find "the Big-O of something"?
  - Usually you "eyeball" it
  - Then you try to prove it
    - (though most of the time in CS16 it will be simple enough that you don't need to prove it)

Big-O Examples

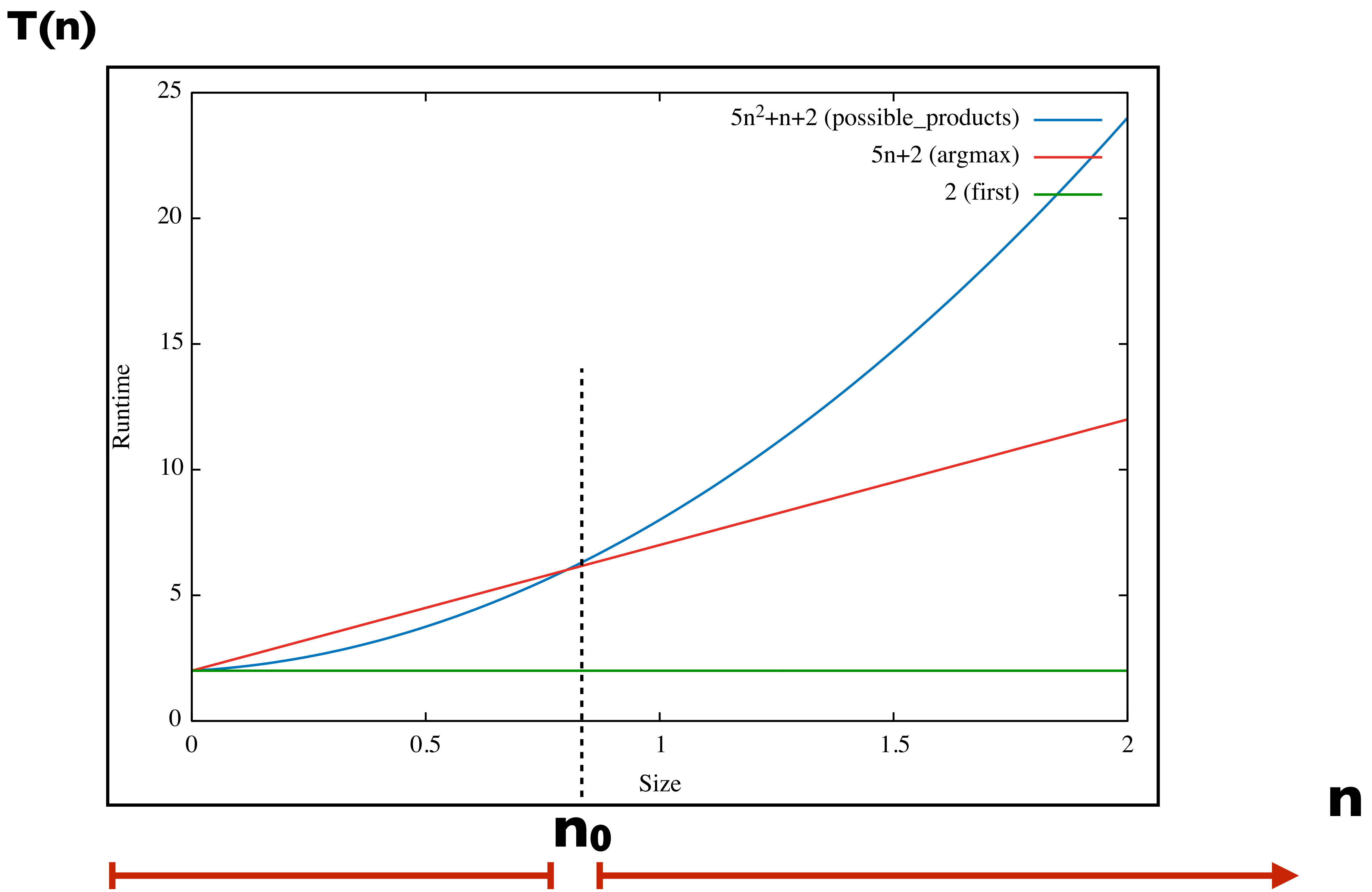
Definition (Big-O): $T_A(n)$ is $O(T_B(n))$ if there exists positive constants c and n_0 such that:

$$T_A(n) \leq c \cdot T_B(n)$$

for all $n \geq n_0$

- $2n+10$ is $O(n)$
  - for example, choose $c=3$ and $n_0=10$
- Why? because
  - $2n+10 \leq 3 \cdot n$ when $n \geq 10$
  - for example, $2 \cdot 10+10 \leq 3 \cdot 10$

Plotting Running Times



We don't care what happens here

We only care what happens here

Experimental measurement

Big-O

More Big-O Examples

- n^2 is not $O(n)$. Why?
  - To prove that n^2 is $O(n)$ we have to find a positive constant c and a positive constant n_0 such that
    - $n^2 \leq c \cdot n$ for all $n > n_0$
  - This is not possible!
    - equivalent to asking that
    - $n \leq c$ for all $n > n_0$

Big-O & Growth Rate

Activity #3

1 min

Big-O & Growth Rate

Activity #3

1 min

Big-O & Growth Rate

Activity #3

0 min

Eyeballing Big-O

- If `T(n)` is a polynomial of degree `d` then `T(n)` is $O(n^d)$
- In other words you can ignore
  - lower-order terms
  - constant factors
- Examples
  - $1000n^2+400n+739$ is $O(n^2)$
  - $n^{80}+43n^{72}+5n+1$ is $O(n^{80})$
- *For the Big-O, use the smallest upper bound*
  - $2n$ is $O(n^{50})$ but that's not really a useful bound
  - instead it is better to say that $2n$ is $O(n)$

Example Big-O Analysis

- Given algorithm, find number of ops as a function of input size
  - first: $T(n)=2$
  - argmax: $T(n)=5n+2$
  - possible_products: $T(n)=5n^2+n+3$
- Replace constants with "**c**" (they are irrelevant as **n** grows)
  - first: $T(n)=c$
  - argmax: $T(n)=c_0n+c_1$
  - possible_products: $T(n)=c_0n^2+n+c_1$

Example Big-O Analysis

- Discard constants & use smallest possible degree
  - first: `T(n)=c` is `O(1)`
  - argmax: $T(n)=c_0 n+c_1$ is `O(n)`
  - possible_products: $T(n)=c_0 n^2+n+c_1$ is $O(n^2)$
- The convention for `T(n)=c` is to write `O(1)`

Big-O

Definition (Big-O): $T_A(n)$ is $O(T_B(n))$ if there exists positive constants c and n_0 such that:

$$T_A(n) \leq c \cdot T_B(n)$$

for all $n \geq n_0$

- $T_A(n)$'s growth rate is upper bounded by $T_B(n)$'s growth rate
- But what if we need to express a lower bound?
    - we use Big-Ω notation!

Big-Omega

Definition (Big-Ω): $T_A(n)$ is $\Omega(T_B(n))$ if there exists positive constants $\mathbf{c}$ and $\mathbf{n_0}$ such that:

$$T_A(n) \geq c \cdot T_B(n)$$

for all $\mathbf{n} \geq \mathbf{n_0}$

- $T_A(n)$'s growth rate is lower bounded by $T_B(n)$'s growth rate
- What about an upper **and** a lower bound?
  - We use Big-**P** notation

Big-Theta

Definition (Big-P): $T_A(n)$ is **P**$(T_B(n))$ if it is **O**$(T_B(n))$ and **Ω**$(T_B(n))$.

- $T_A(n)$'s growth rate is the same as $T_B(n)$'s

More Examples

Activity #4

2 min

More Examples

Activity #4

1 min

More Examples

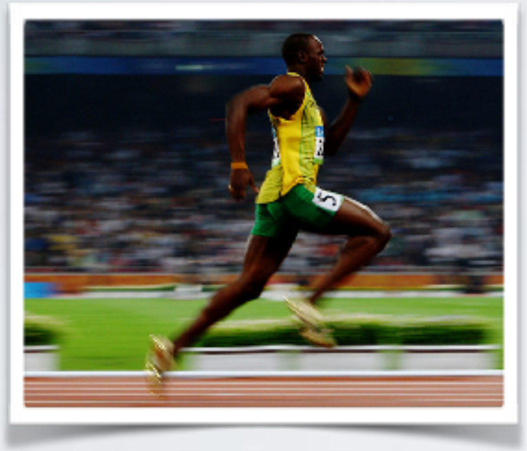
Activity #4

0 min

More Examples

T(n)	Big-O	Big-Ω	Big-**P**
an + b	?	?	**P**(n)
$an^2 + bn + c$	?	?	**P**(n^2)
a	?	?	**P**(1)
$3^n + an^{40}$	?	?	**P**(3^n)
an + b log n	?	?	**P**(n)

Running Times


`O(1)`
independent of input size

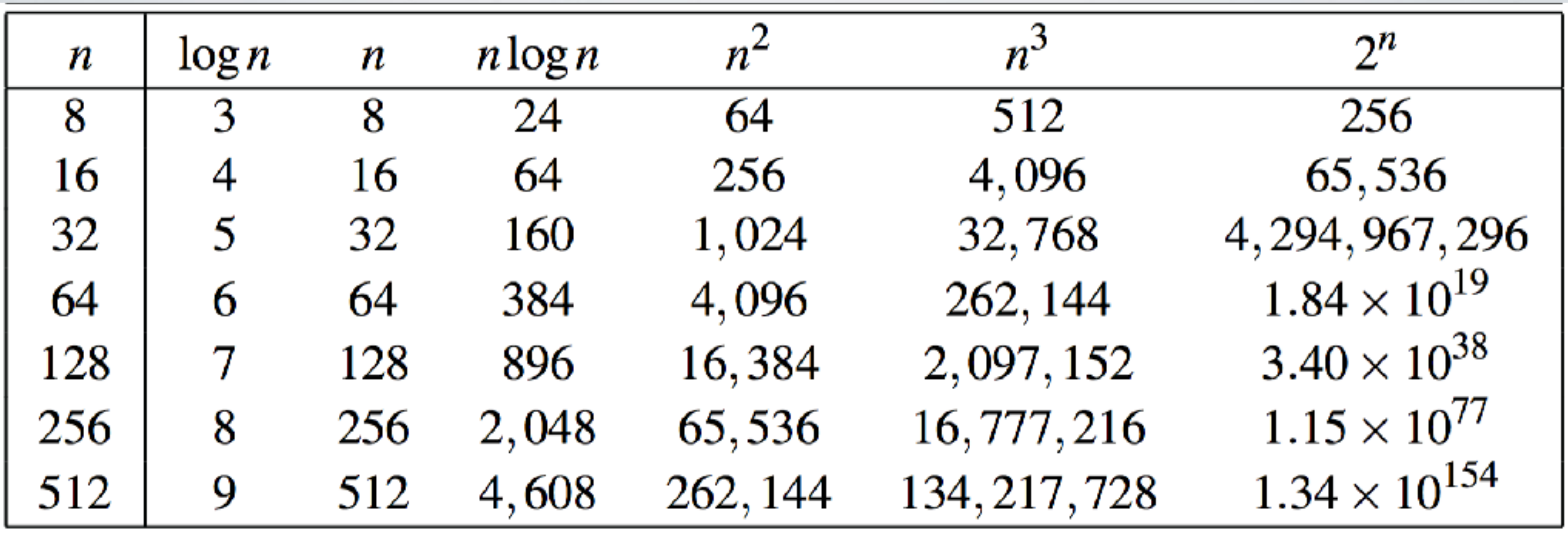
`O(n)`
depends on input size

`O(n²)`
depends on square of input size

`O(n³)`
depends on cube of input size

`O(n⁷⁰)`
depends on 70th power of input size

`O(2ⁿ)`
exponential in input size



n	$\log n$	n	$n \log n$	n^2	n^3	2^n
8	3	8	24	64	512	256
16	4	16	64	256	4,096	65,536
32	5	32	160	1,024	32,768	4,294,967,296
64	6	64	384	4,096	262,144	1.84×10^{19}
128	7	128	896	16,384	2,097,152	3.40×10^{38}
256	8	256	2,048	65,536	16,777,216	1.15×10^{77}
512	9	512	4,608	262,144	134,217,728	1.34×10^{154}

Readings

- Asymptotic runtime and Big-O
  - Dasgupta et al. section 0.3 (pp. 15-17)
  - Roughgarden Part 1, Chap 2

Announcements

- Homework 1 due this Friday at 5pm!
- Thursday is in-class Python lab!
- If you are able to work on your own laptop
  - Go to McMillan 117 (here!)
- Make sure you can log into your CS account before attending lab
- See SunLab consultant if you have any account issues!
- Sections started yesterday
  - if you are not signed up, you could be in trouble!

References

- Slide #10
  - the portrait on the left is a drawing; really!
- Slide #25
  - Usain Bolt (constant): 8-time Olympic gold medalist and greatest sprinter of all time
  - Sally Pearson (linear): 2012 Olympic world champion in 100m hurdles, 2011 and 2017 World Champion
  - Wilson Kipsang (quadratic): former marathon world-record holder, Olympic marathon bronze medalist
  - Eliud Kipchoge (quadratic): 2016 Olympic marathon gold medalist, greatest marathoner of the modern era