Directed Acyclic Graphs & Topological Sort

CS16: Introduction to Data Structures & Algorithms

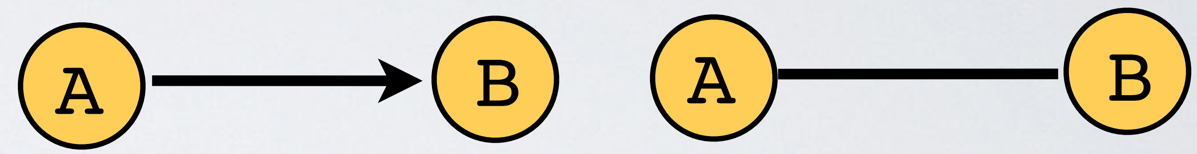
Spring 2019

Outline

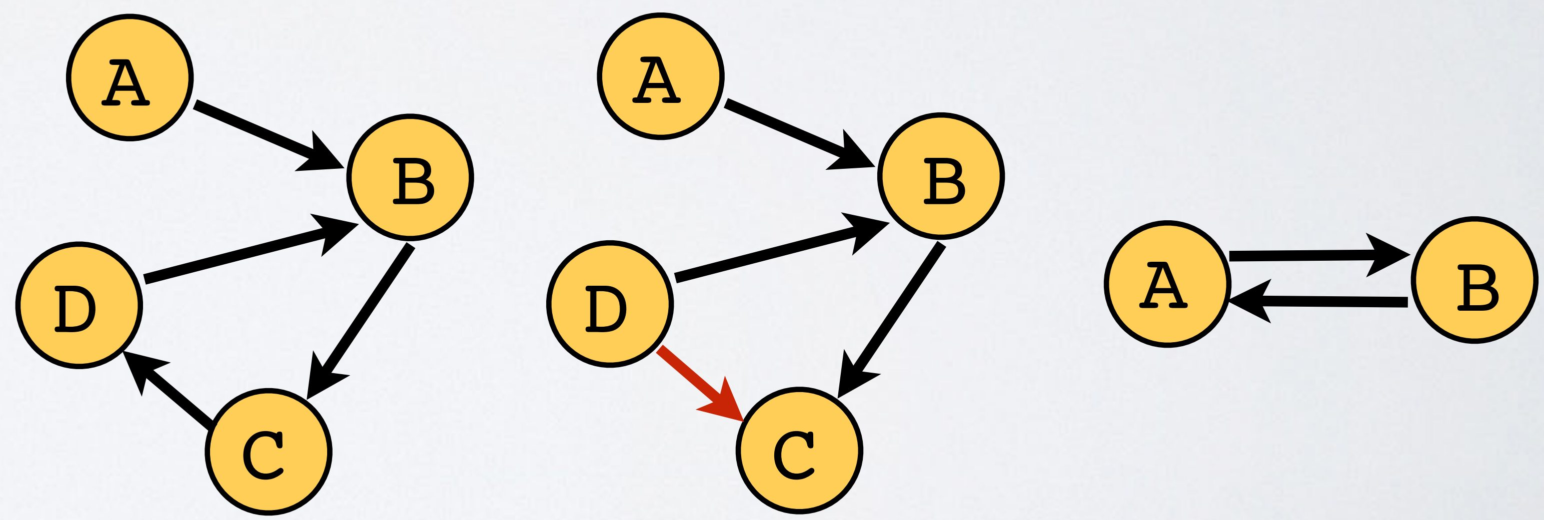
- Directed Acyclic Graphs
- Topological Sort
  - Hand-simulation
  - Pseudo-code
  - Runtime analysis

Directed Acyclic Graphs

- A DAG is **directed** & **acyclic**
- Directed
  - edges have origin & destination…
  - ….represented by a directed arrow
- Acyclic
  - No cycles!
  - Starting from any vertex, there is no path that leads back to the same vertex



Directed Undirected

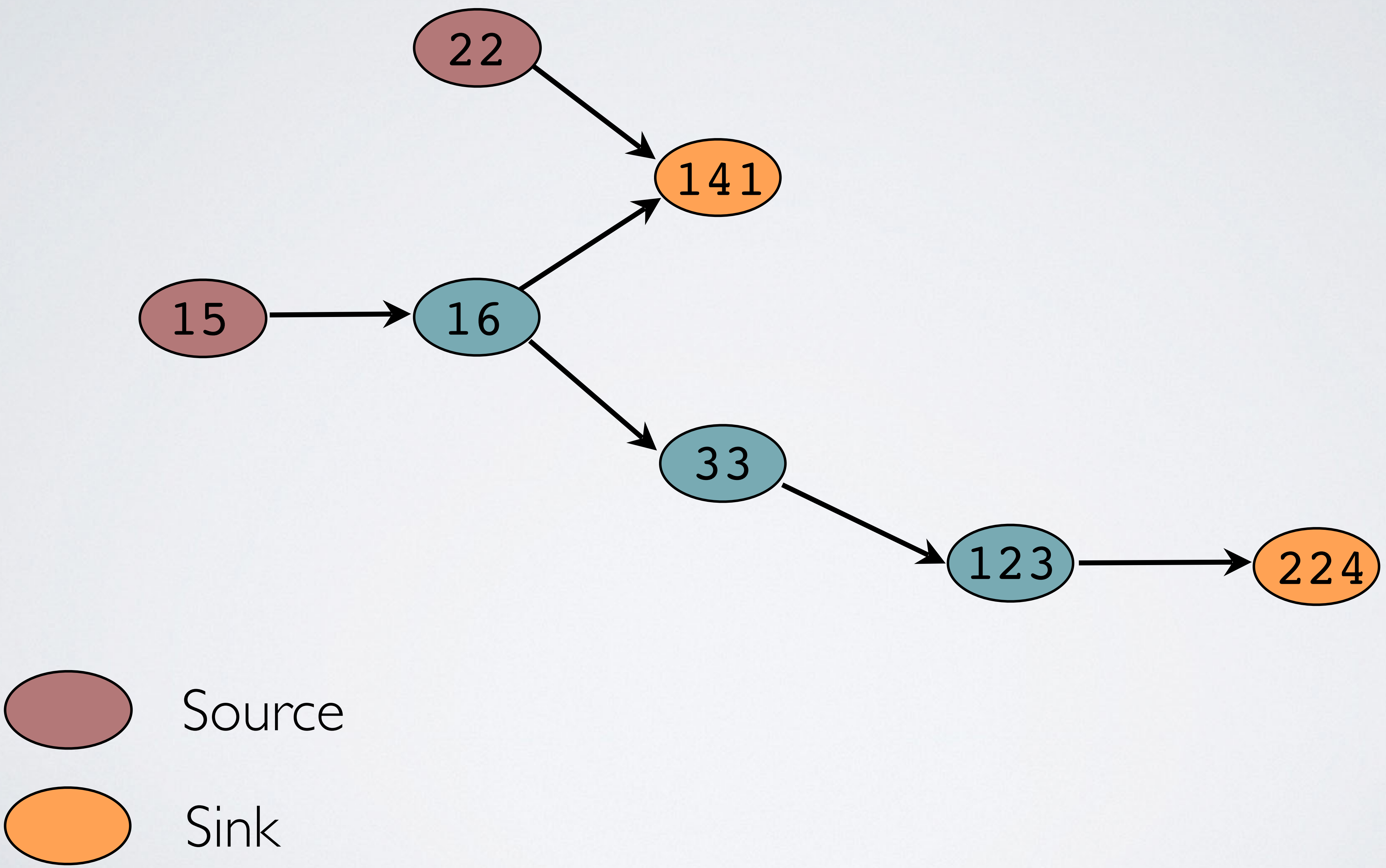


Cyclic Acyclic Cyclic

Directed Acyclic Graphs

- DAGs often used to model situations in which completing certain things depend on completing other things
  - ex: course prerequisites or small tasks in a big project
- Terminology
  - Sources: vertices with no incoming edges (no dependencies)
  - Sinks: vertices with no outgoing edges
  - In-degree of a vertex: number of incoming edges of the vertex
  - Out-degree of a vertex: number of outgoing edges of the vertex

Directed Acyclic Graphs — Example

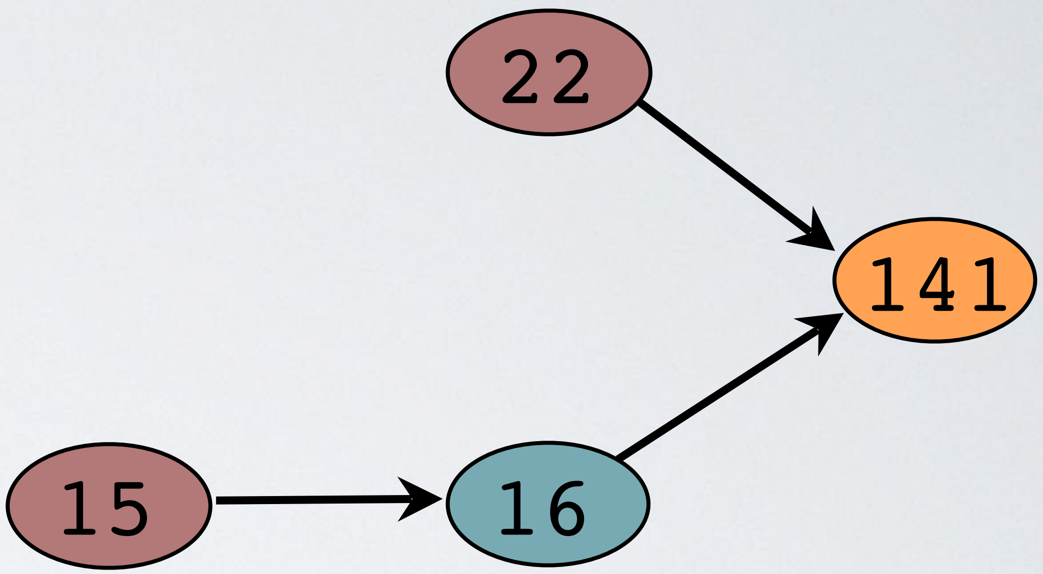


Topological Sort

- Imagine you are a CS concentrator
- You need to plan your courses for next 3 years
- How can you do that taking into account pre-requisites?
  - Represent courses w/ a DAG
  - Use topological sort!
    - Produces topological ordering of a DAG

Topological Sort

- Topological Ordering
  - ordering of vertices in DAG…
  - …such that for each vertex v…
  - …all of v's prereqs come before it in the ordering
- Topological Sort
  - Algorithm that produces topological ordering given a DAG

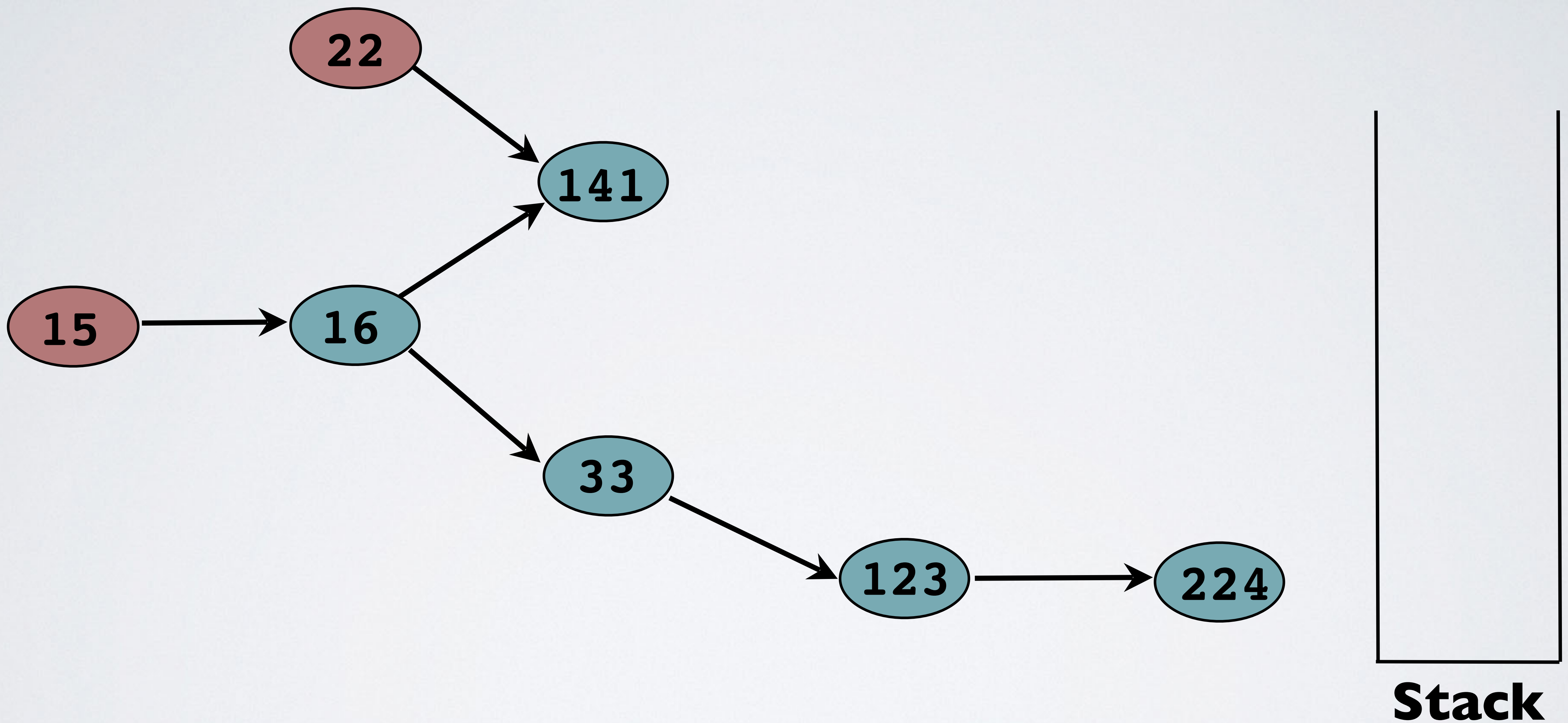


- Valid topological orderings
  - `15,16,22,141`
  - `22,15,16,141`
  - `15,22,16,141`

Topological Sort—General Strategy

- If vertex has no prerequisites (i.e., is a source), we can visit it!
- Once we visit a vertex,
  - all of it's outgoing edges can be deleted
  - because that prerequisite has been satisfied
- Deleting edges might create new sources
  - which we can now visit
- Data Structures needed
  - DAG to top-sort
  - A structure to keep track of sources
  - A list to keep track of the resultant topological ordering

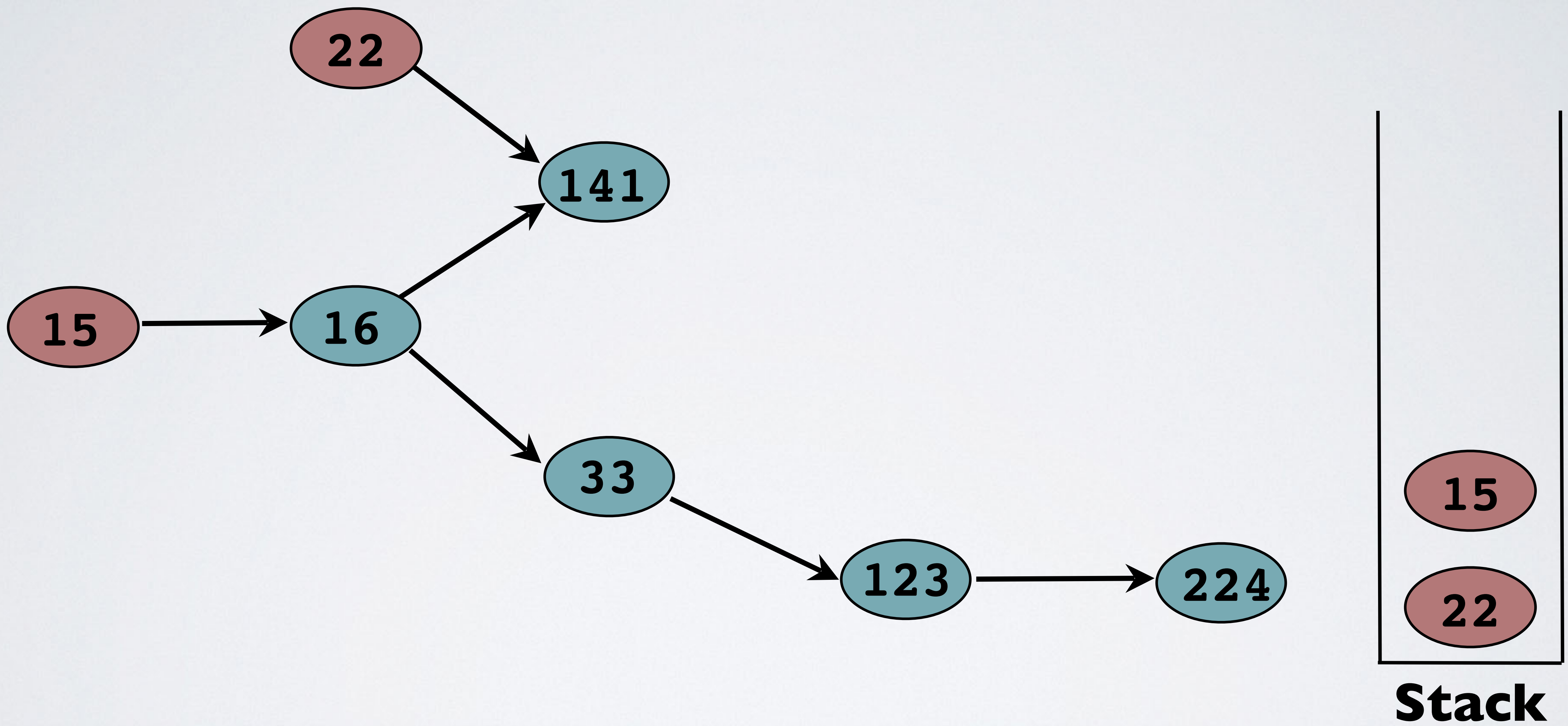
Topological Sort—Simulation



List:

Topological Sort—Simulation

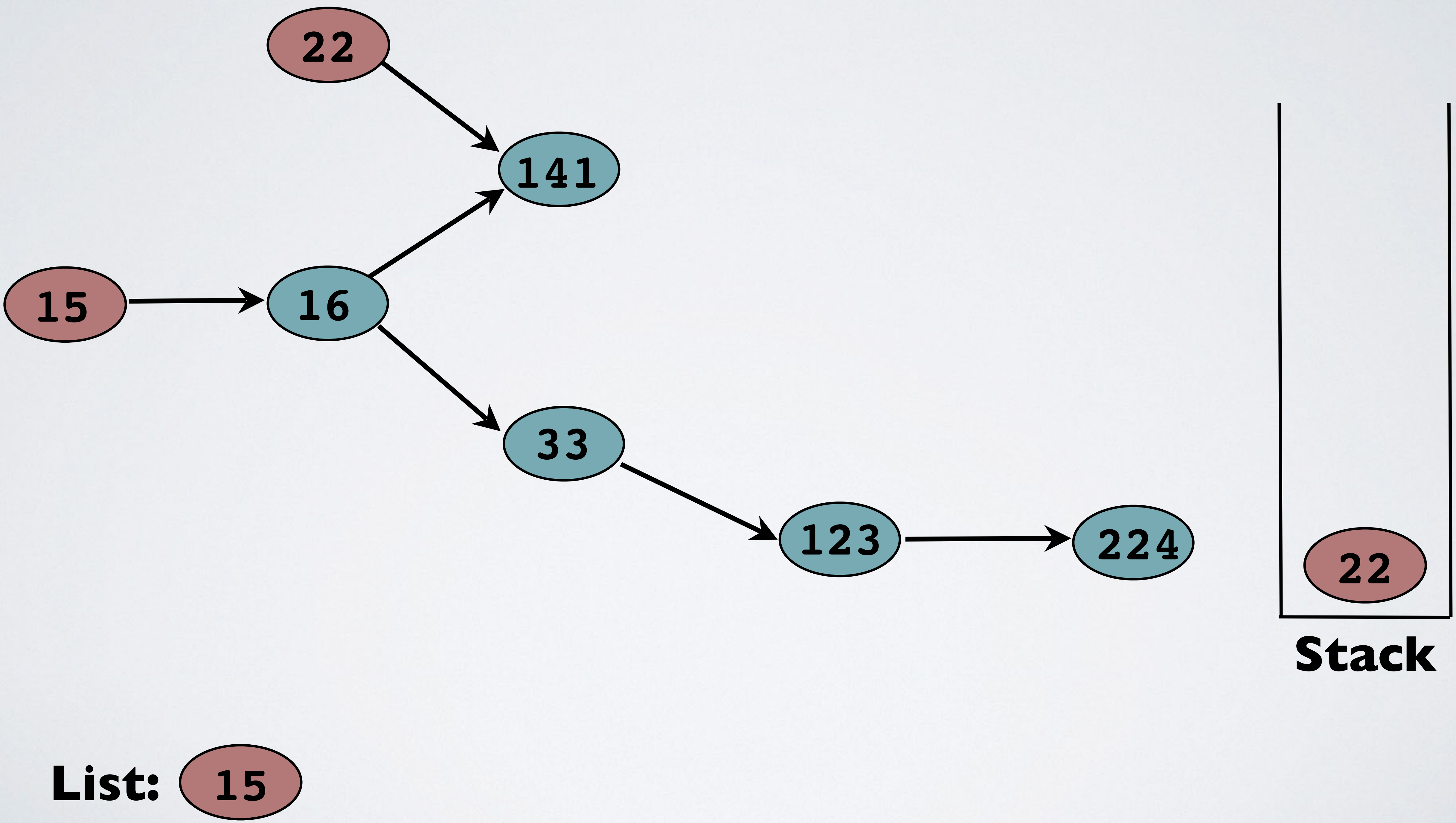
Populate Stack with source vertices



List:

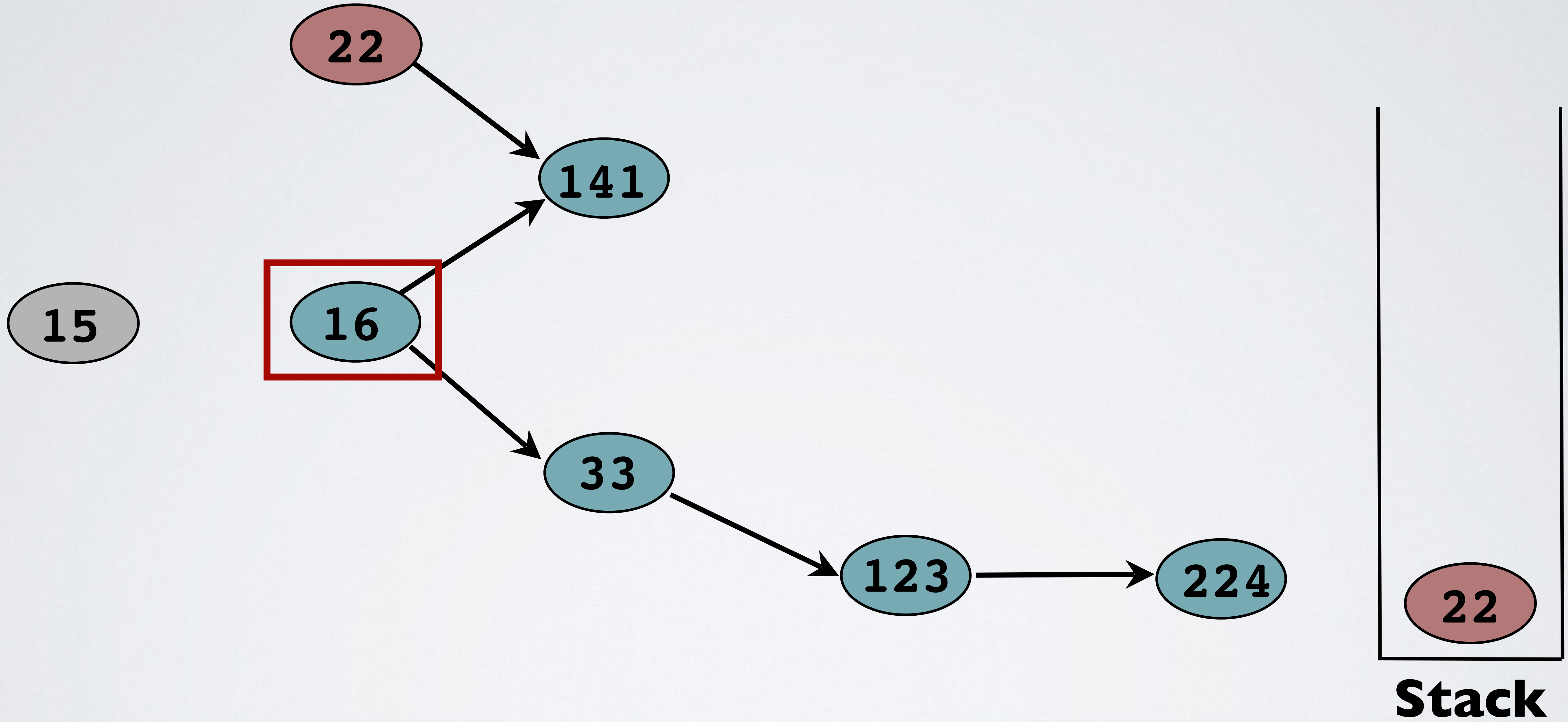
Topological Sort—Simulation

Pop from stack and add to list



Topological Sort—Simulation

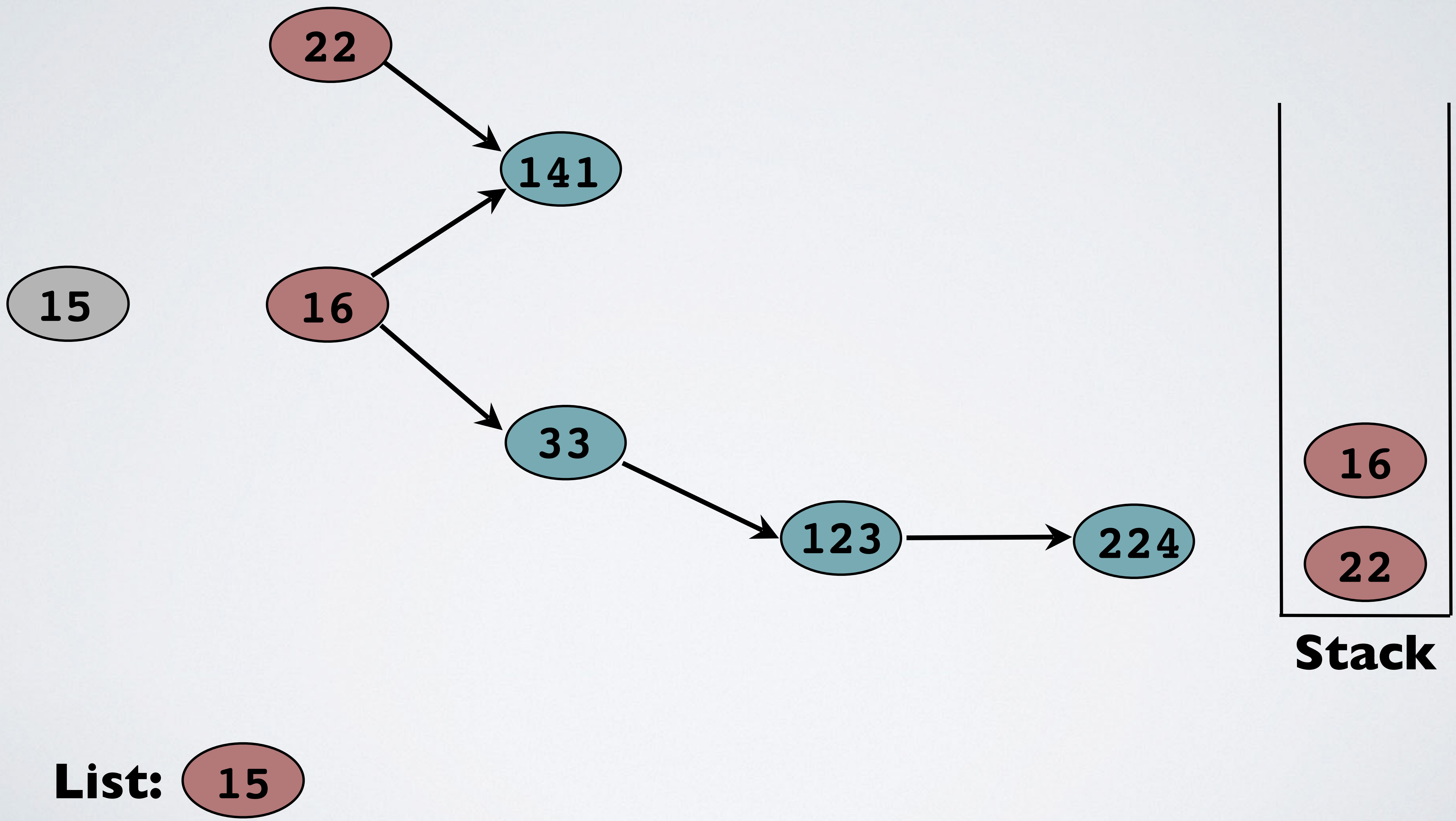
Remove outgoing edges & check corresponding vertices



List: 15

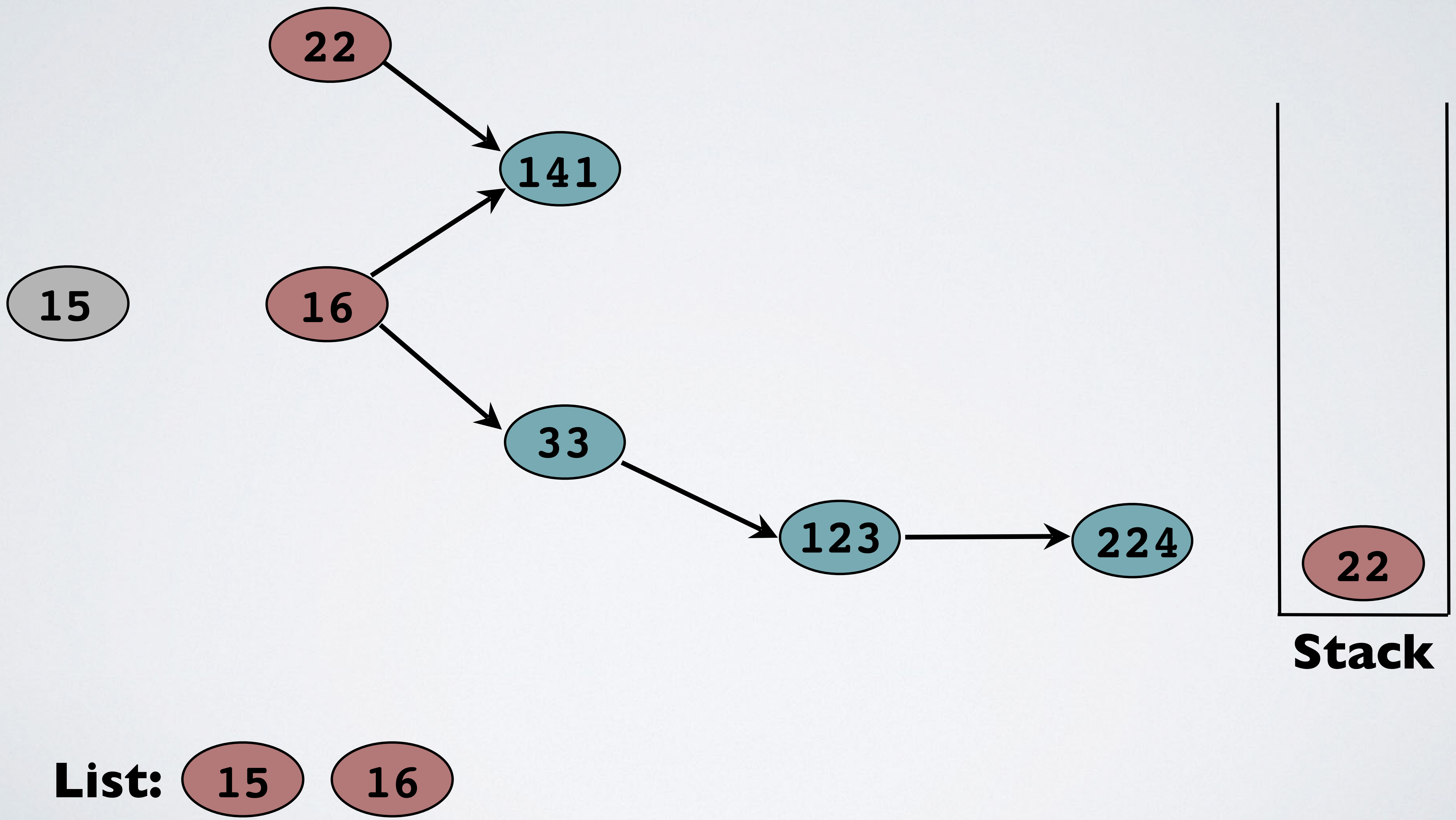
Topological Sort—Simulation

16 has no more incoming edges so push it on the stack



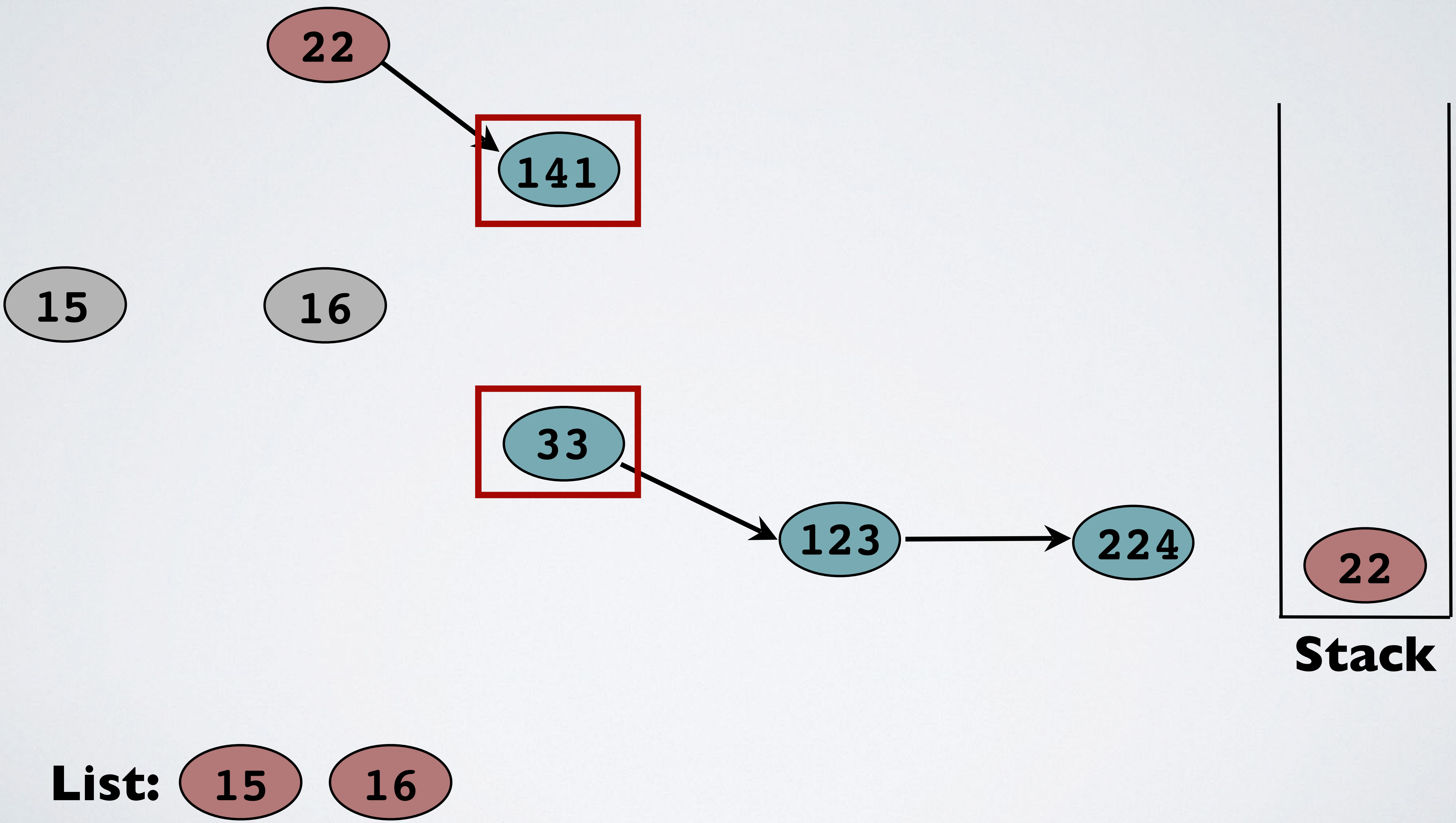
Topological Sort—Simulation

Pop from the stack and add to list



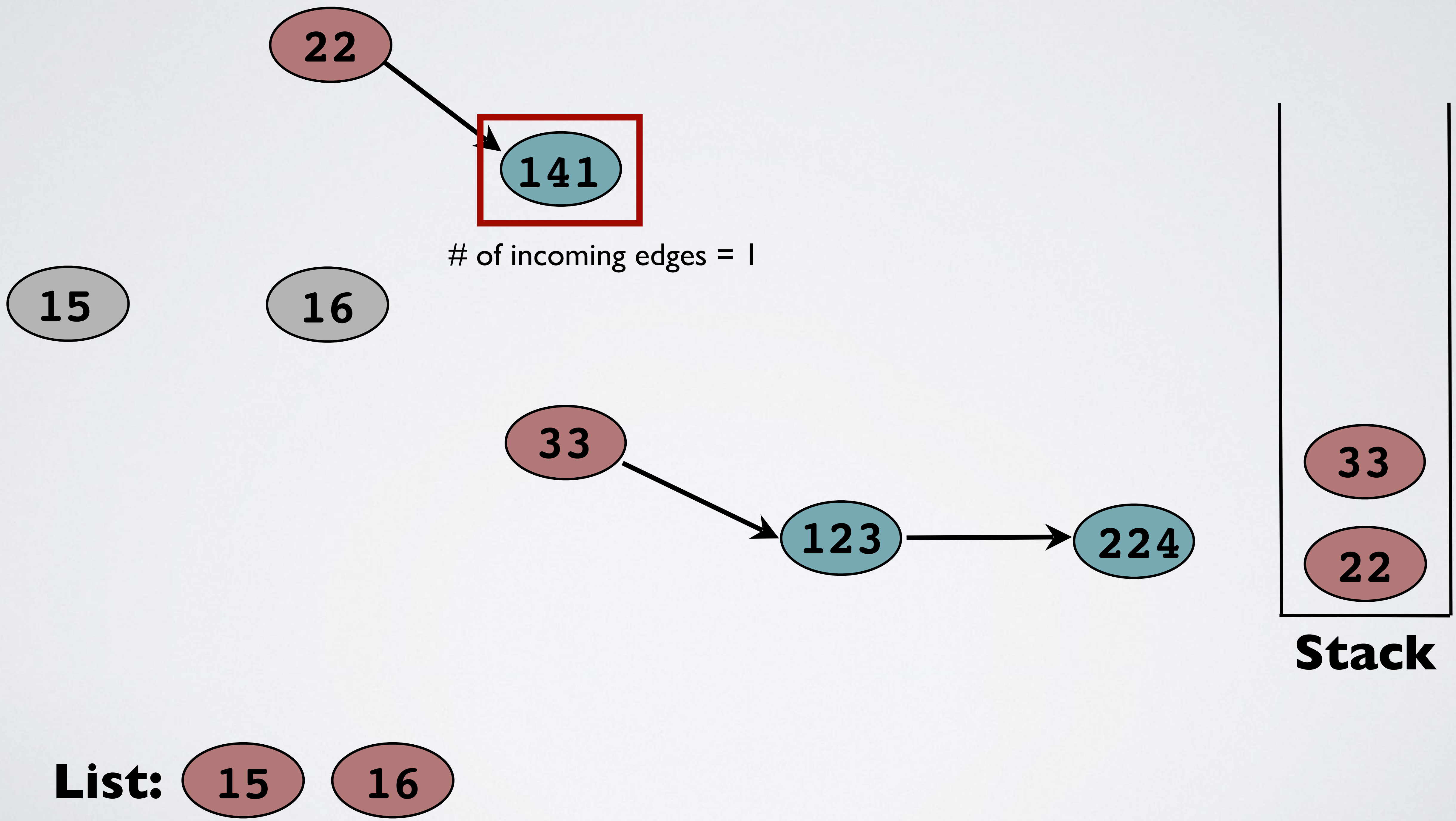
Topological Sort—Simulation

Remove outgoing edges & check the corresponding vertices



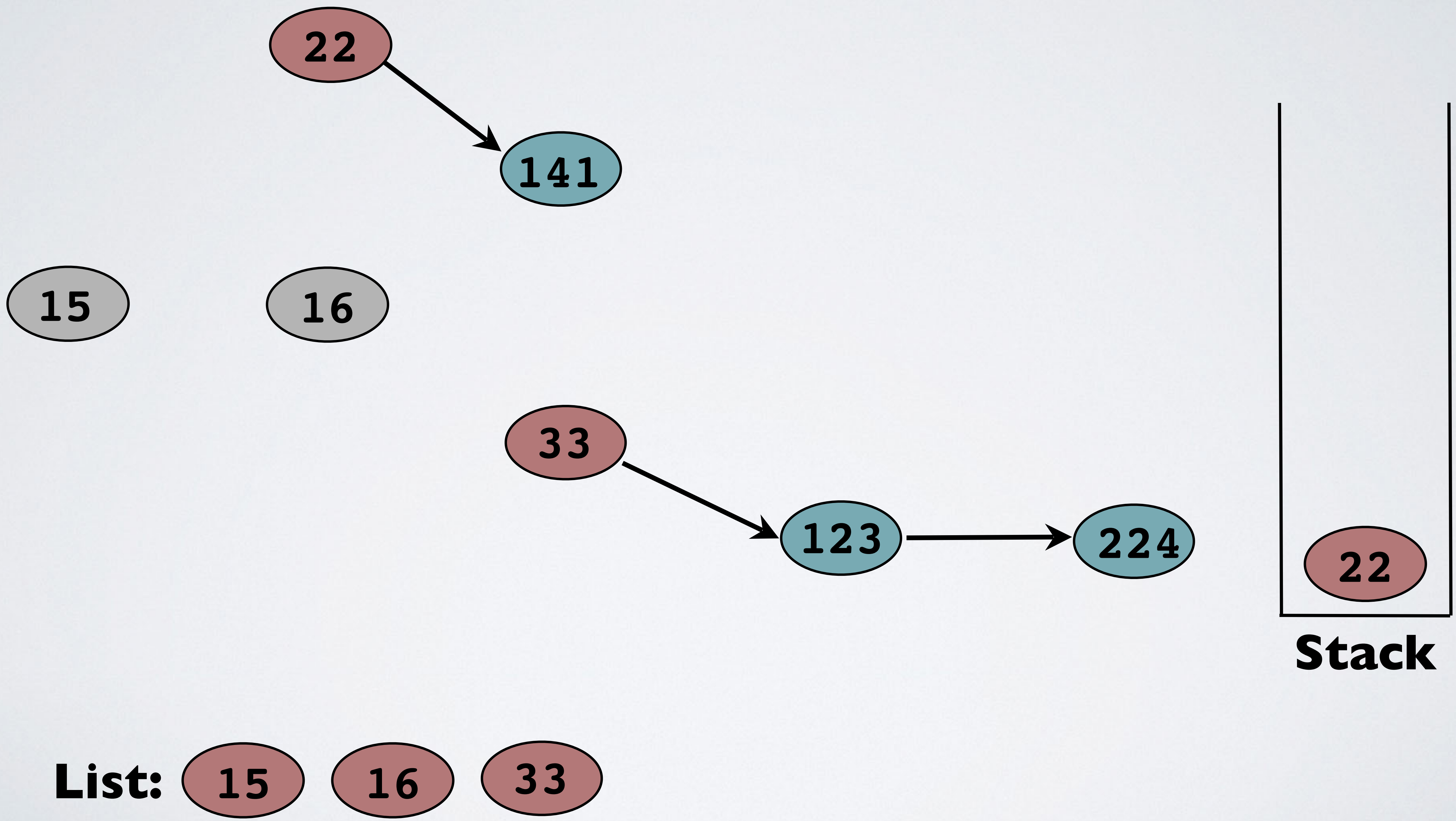
Topological Sort—Simulation

33 has no more incoming edges so push it onto the stack
141 still has an incoming edge

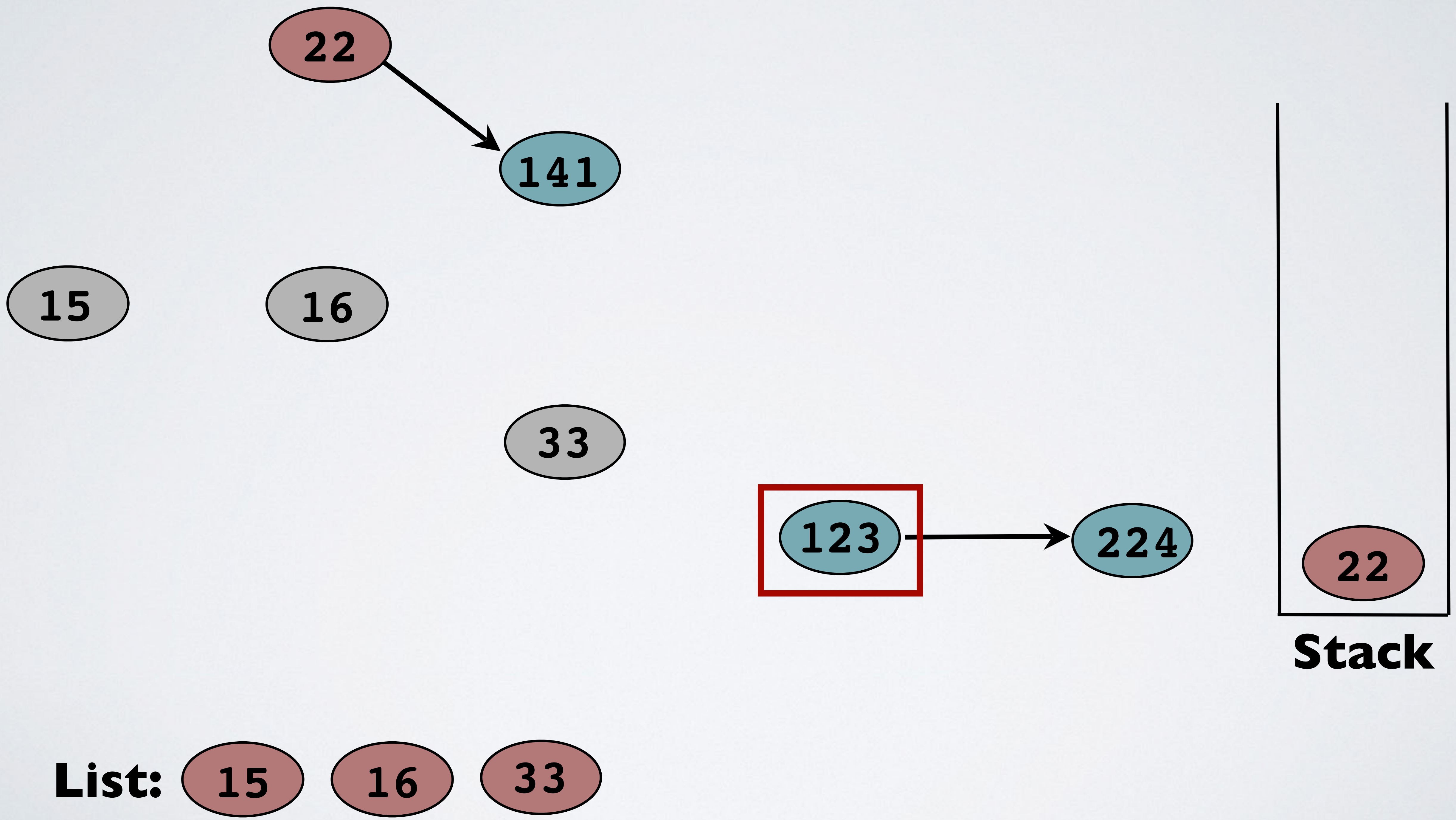


Topological Sort—Simulation

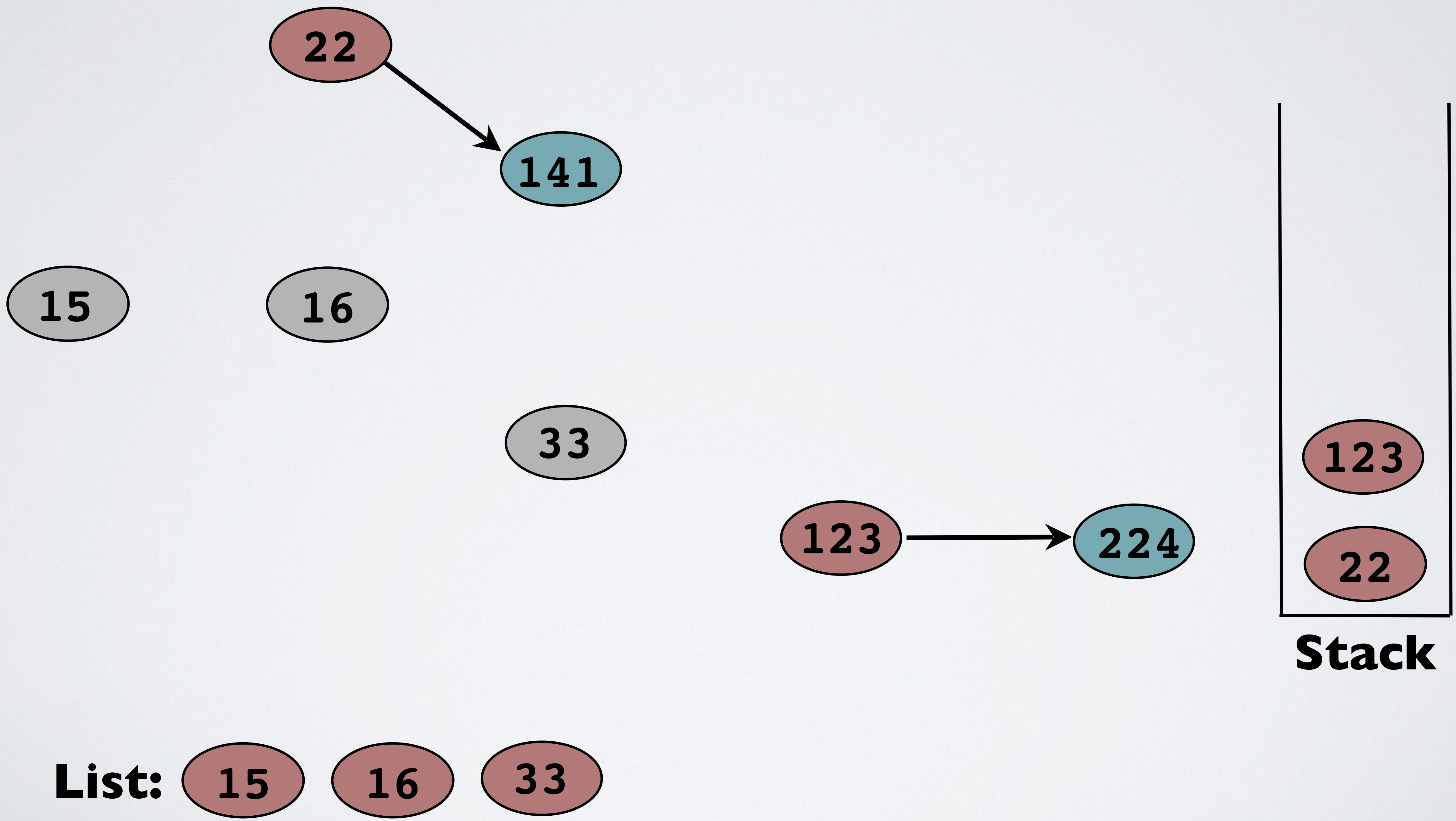
Pop from the stack & repeat!



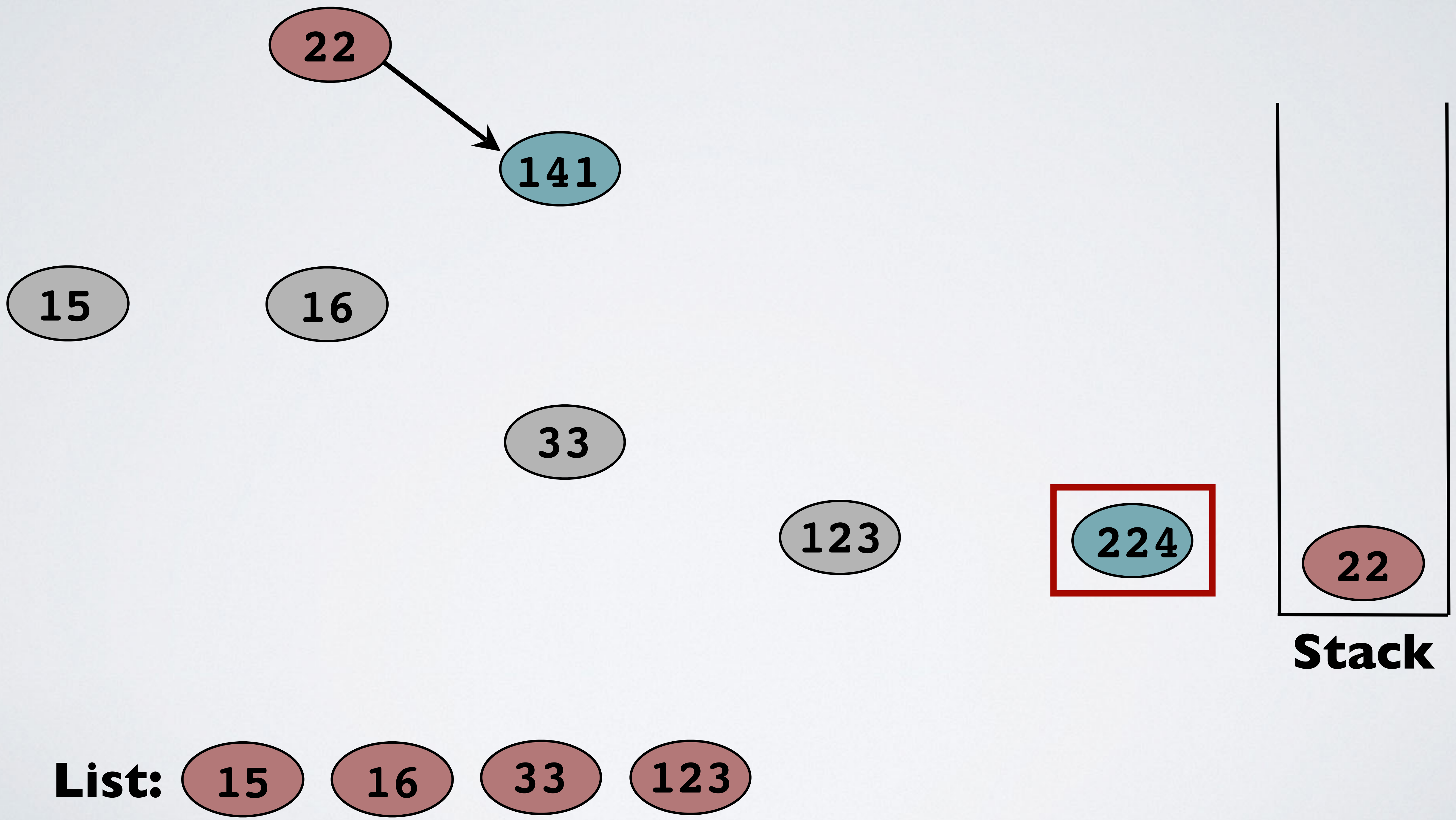
Topological Sort—Simulation



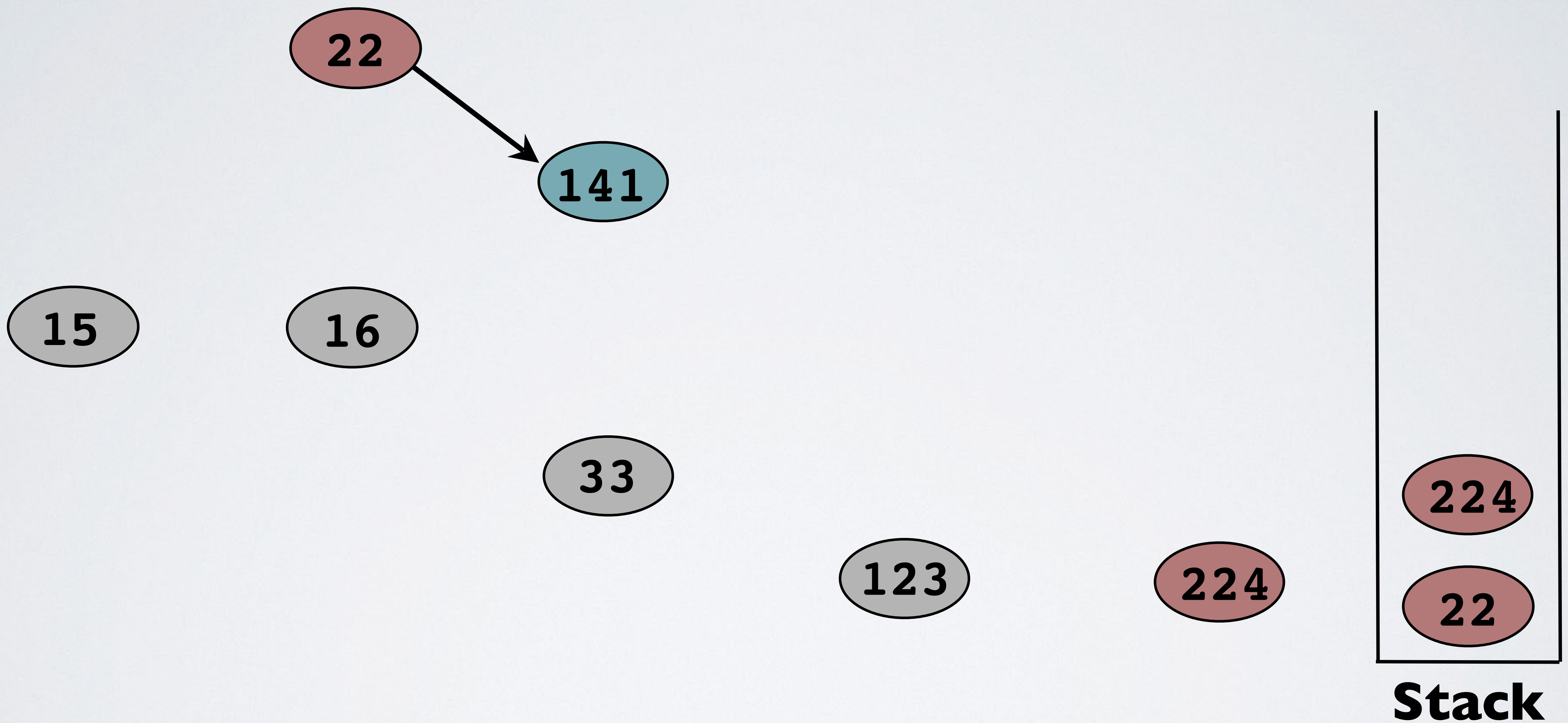
Topological Sort—Simulation



Topological Sort—Simulation

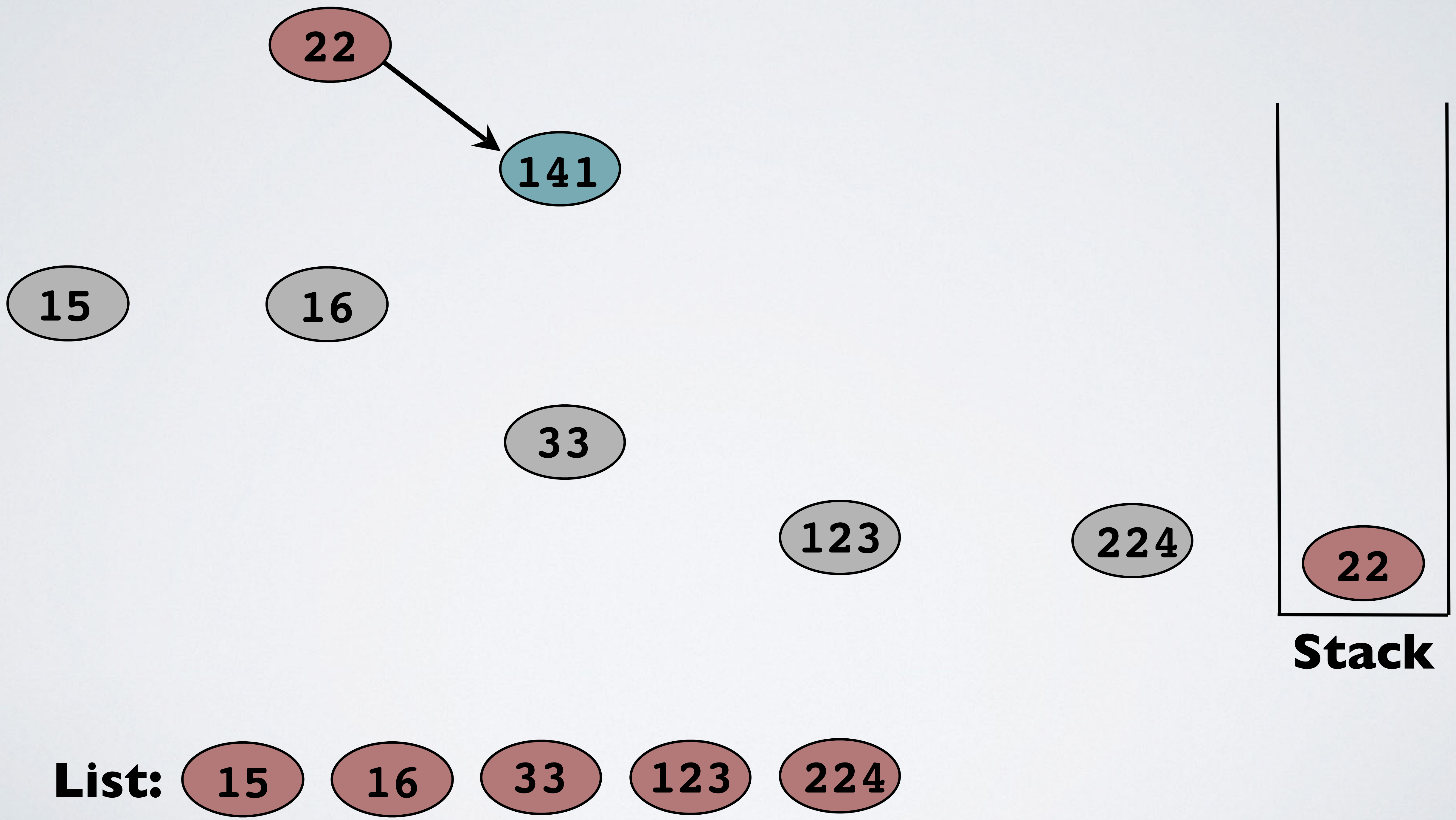


Topological Sort—Simulation

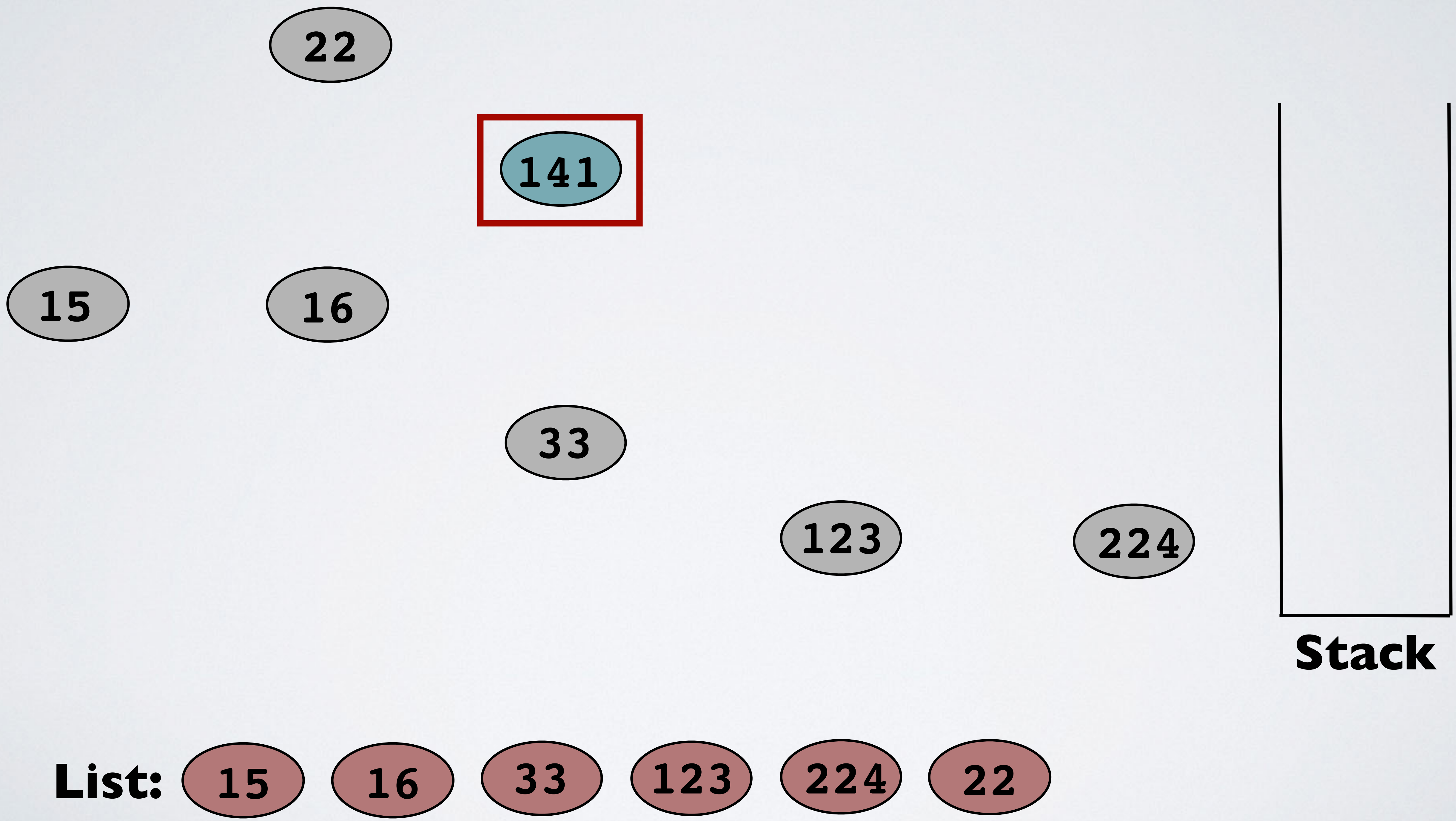


List: 15 16 33 123

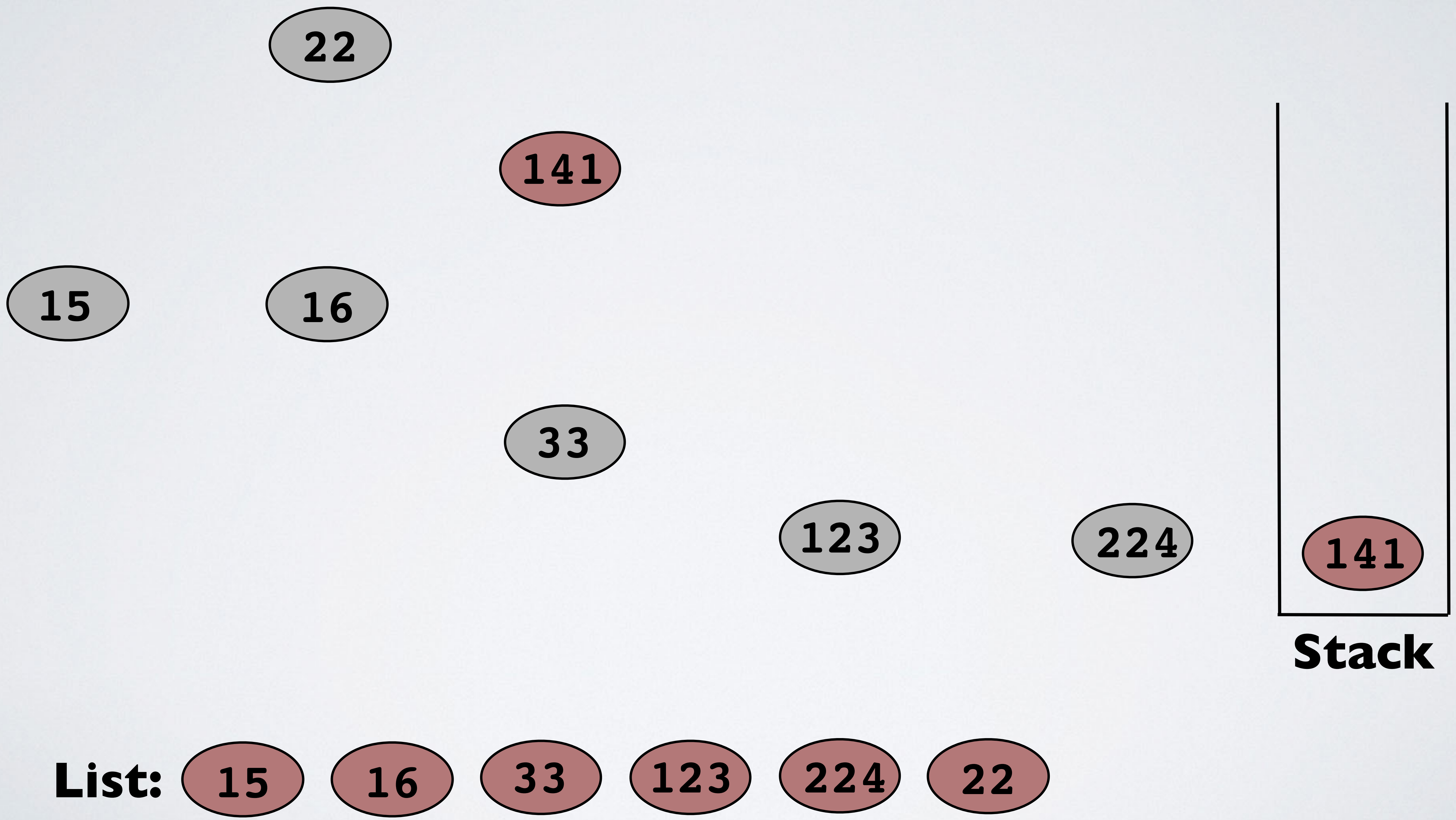
Topological Sort—Simulation



Topological Sort—Simulation

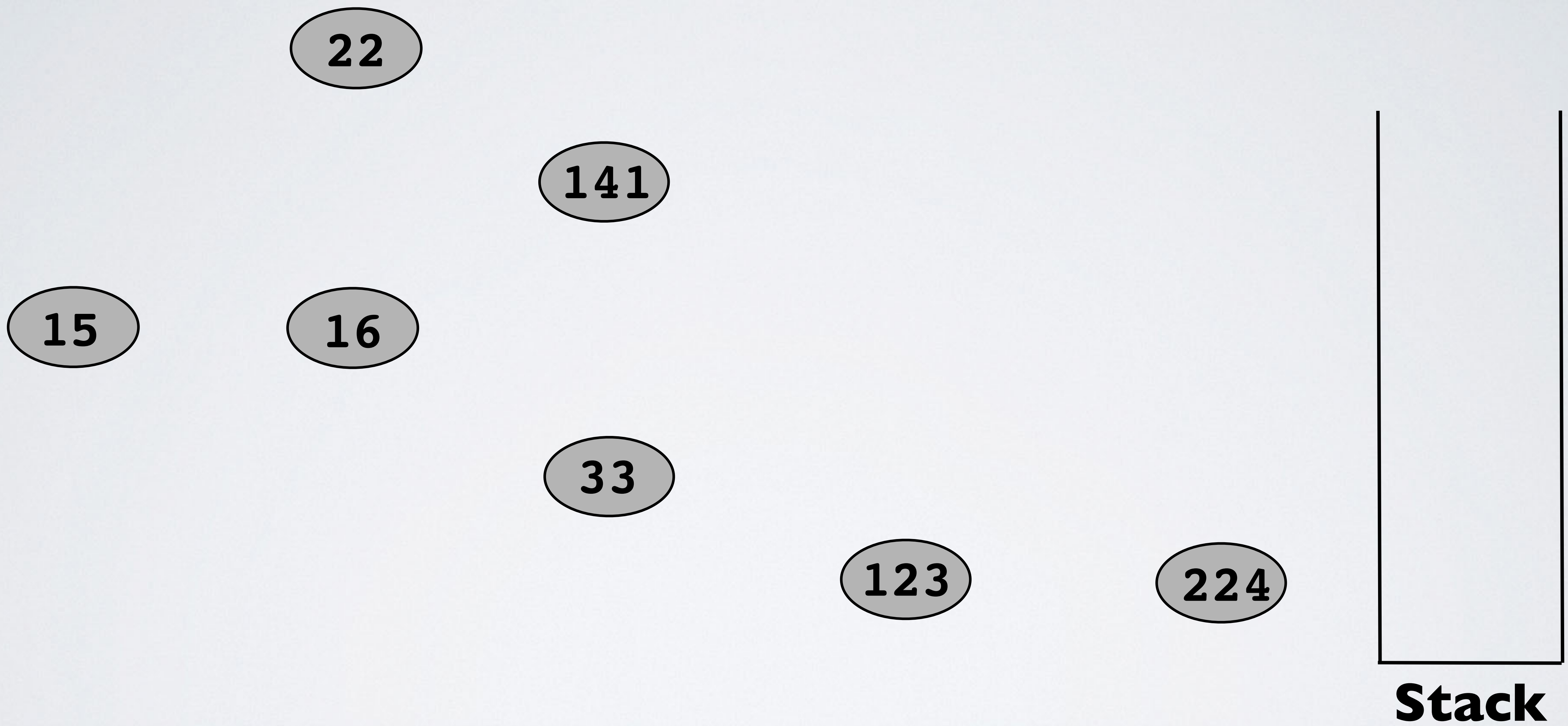


Topological Sort—Simulation



Topological Sort—Simulation

We're done!



List: 15 16 33 123 224 22 141

Topological Sort—Stack

Activity #1

2 min

Topological Sort—Stack

Activity #1

2 min

Topological Sort—Stack

Activity #1

1 min

Topological Sort—Stack

Activity #1

0 min

Topological Sort Pseudo-code

```
function top_sort(graph g):
    // Input: A DAG g
    // Output: A list of vertices of g, in topological order
    s = Stack()
    l = List()
    for each vertex in g:
        if vertex is source:
            s.push(vertex)
    while s is not empty:
        v = s.pop()
        l.append(v)
        for each outgoing edge e from v:
            w = e.destination
            delete e
            if w is a source:
                s.push(w)
    return l
```

Topological Sort Runtime

```
function top_sort(graph g):
    // Input: A DAG g
    // Output: A list of vertices of g, in topological order
    s = Stack()
    l = List()
    for each vertex in g:
        if vertex is source:
            s.push(vertex)
    while s is not empty:
        v = s.pop()
        l.append(v)
        for each outgoing edge e from v:
            w = e.destination
            delete e
            if w is a source:
                s.push(w)
    return l
```

Looping through every vertex to find sources is $O(|V|)$

Topological Sort Runtime

```
function top_sort(graph g):
    // Input: A DAG g
    // Output: A list of vertices of g, in topological order
    s = Stack()
    l = List()
    for each vertex in g:
        if vertex is source:
            s.push(vertex)
    while s is not empty:
        v = s.pop()
        l.append(v)
        for each outgoing edge e from v:
            w = e.destination
            delete e
            if w is a source:
                s.push(w)
    return l
```

Looping through every vertex to find sources is O(|V|)

Stack will hold each vertex once

At each iteration we only visit outgoing edges from popped vertex. So every edge visited once.

Total runtime: O(|V|+|E|)

Topological Sort—Queue

Activity #2

2 min

Topological Sort—Queue

Activity #2

2 min

Topological Sort—Queue

Activity #2

1 min

Topological Sort—Queue

Activity #2

0 min

Topological Sort Variations

- What if we're not allowed to modify original DAG?
  - How do we delete edges?
  - Use decorations!
- Start by decorating each vertex with it's in-degree
  - Instead of deleting edge
  - decrement in-degree of destination vertex by `1`
  - then push vertex on stack when in-degree is `0`!

Topological Sort Variations

- Do we need to use a stack?
  - No! Any data structure like a list or queue would work
  - All we're doing is keeping track of sources
- Different structures might yield different topological orderings
  - Why do they all work ?
  - Vertices are only added to structure when they become a source
    - i.e., when all of it’ s "prerequisites" have been visited
  - This invariant is maintained throughout algorithm…
  - …and guarantees a valid topological ordering!

Topological Sort

Activity #3

2 min

Topological Sort

Activity #3

2 min

Topological Sort

Activity #3

1 min

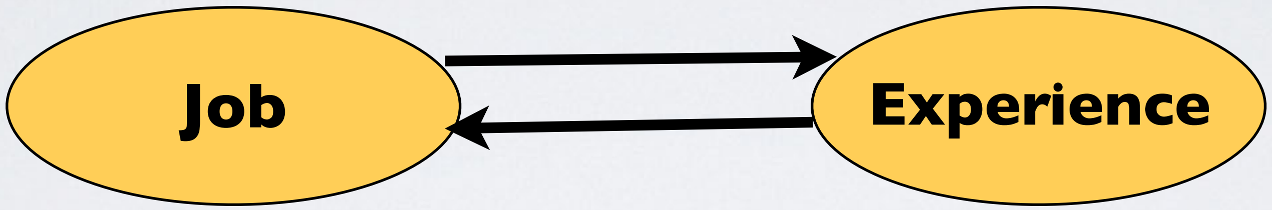
Topological Sort

Activity #3

0 min

Top Sort: Why only on DAGs ?

- If the graph has a cycle…



- …we don't have a valid topological ordering
- We can use top sort to check if a DAG has a cycle
- Run top sort on graph
  - if there are edges left at the end but no more sources
  - then there must be a cycle