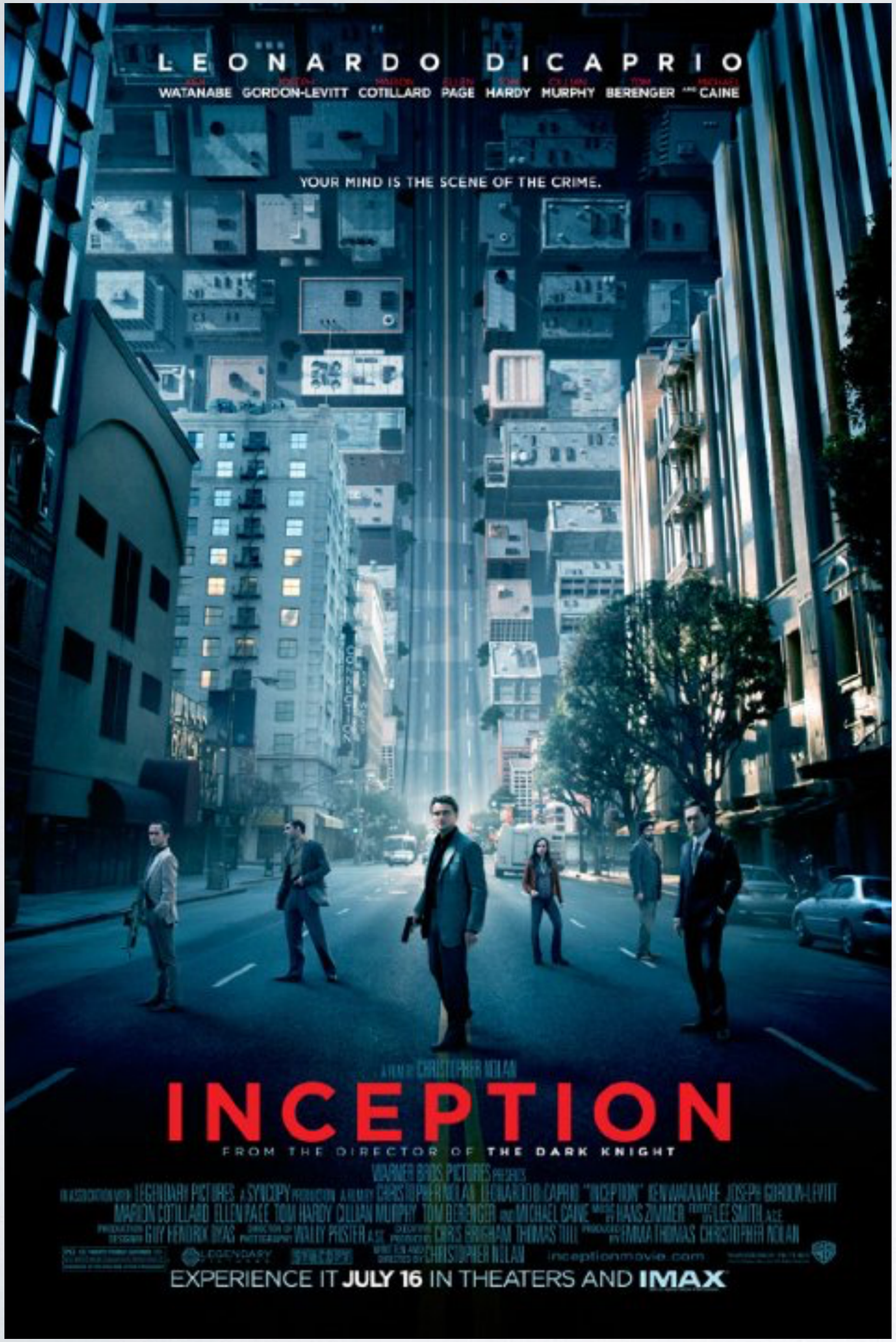
Recursion & Induction

CS16: Introduction to Algorithms & Data Structures
Spring 2019

Outline

- Recursion
- Recurrence relations
- Plug & chug
- Induction
- Strong vs. weak induction

"Something defined in terms of itself"


LEONARDO DICAPRIO
WATANABE GORDON-LEVITT COTILLARD PAGE HARDY MURPHY BERENGER AND CAINE
YOUR MIND IS THE SCENE OF THE CRIME.
INCEPTION
FROM THE DIRECTOR OF THE DARK KNIGHT
EXPERIENCE IT JULY 16 IN THEATERS AND IMAX

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Recursion

- What is a recursive problem?
  - a problem defined in terms of itself
- What is a recursive function?
  - a function defined in terms of itself
  - example: Factorial, Fibonacci
- At each level, the problem/function/pic gets easier/smaller
- How can we solve recursive problems?

Recursive Algorithms

- Algorithms that call themselves
  - Call themselves on smaller inputs (sub-problems)
  - Combine the results to find solution to larger input
- Recursive algorithms
  - Can be very easy to describe & implement :-)
  - Can be hard to think about and to analyze :-(

Factorial

iterative: $n! = \prod_{i=1}^{n} i = n \times (n-1) \times \cdots \times 1$

recursive: $n! = n \times (n-1)!, \text{ with } 1! = 1$

Recursive Factorial — Simulation

```
def factorial(n):
    if n == 1:
        return 1
    else:
        return n * factorial(n-1)
```

- call **factorial**(3)

Recursive Factorial — Simulation

```
def factorial(n):
    if n == 1:
        return 1
    else:
        return n * factorial(n-1)
```

- call **factorial**(3)
  - <u>level #1</u>: 3≠1 so 3 × **factorial**(2)

Recursive Factorial — Simulation

```
def factorial(n):
    if n == 1:
        return 1
    else:
        return n * factorial(n-1)
```

- call **factorial**(3)
  - level #1: 3≠1 so 3 × **factorial**(2)
    - level #2: 2≠1 so 2 × **factorial**(1)

Recursive Factorial — Simulation

```
def factorial(n):
    if n == 1:
        return 1
    else:
        return n * factorial(n-1)
```

- call **factorial**(3)
  - level #1: 3≠1 so 3 x **factorial**(2)
    - level #2: 2≠1 so 2 x **factorial**(1)
      - level #3: 1==1 so return 1

Recursive Factorial — Simulation

```
def factorial(n):
    if n == 1:
        return 1
    else:
        return n * factorial(n-1)
```

- call **factorial**(3)
  - level #1: 3≠1 so 3 x **factorial**(2)
    - level #2: 2≠1 so 2 x **1**
      - ~~level #3: 1==1 so return **1**~~

Recursive Factorial — Simulation

```
def factorial(n):
    if n == 1:
        return 1
    else:
        return n * factorial(n-1)
```

- call **factorial**(3)
  - level #1: 3≠1 so 3 x **2**
    - ~~level #2: 2≠1 so 2 x **1**~~
      - ~~level #3: 1==1 so return **1**~~

Recursive Factorial — Simulation

```
def factorial(n):
    if n == 1:
        return 1
    else:
        return n * factorial(n-1)
```

- call **factorial**(3) = **6**
  - ~~fact(3): 3≠1 so 3 × 2~~
    - ~~level #2: 2≠1 so 2 × 1~~
      - ~~level #3: 1==1 so return 1~~

Wait a minute!!

you keep *calling* factorial but never actually *implemented it*

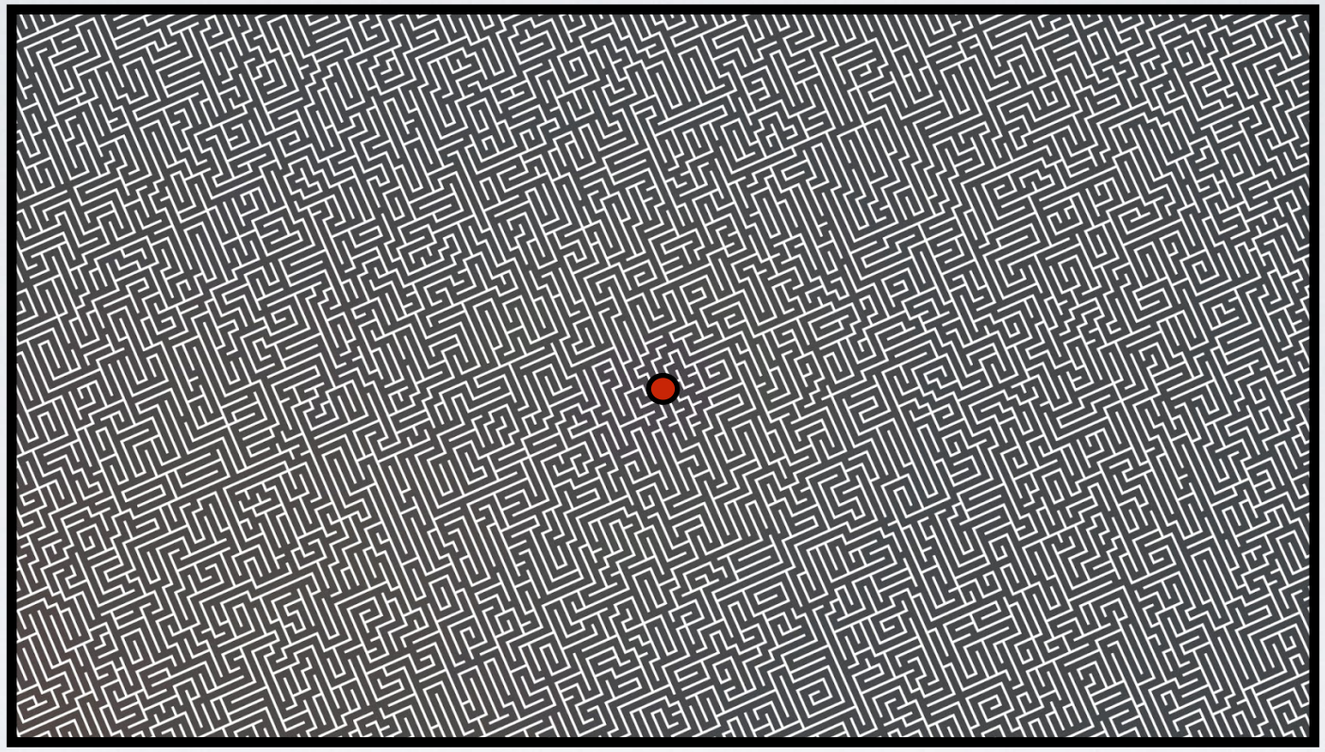
Recursive Factorial — Simulation

```
def factorial(n):
    if n == 1:
        return 1
    else:
        return n * factorial(n-1)
```

Recursive Factorial — Simulation

```
def factorial(n):
    if n == 1:
        return 1
    else:
        return n * factorial(n-1)
```

Recursion & Clones



- At each intersection
  - clone yourself twice and send one Left and one Right
  - wait for clones to report a path to exit (if it exists) and its length
  - pick direction that gets you to exit the fastest

Example: recursive `array_max`

```
def array_max(array, n):
    if n == 1:
        return array[0]
    else:
        return max(array[n-1], array_max(array, n-1))
```

Activity #1

Example: recursive `array_max`

```
def array_max(array, n):
    if n == 1:
        return array[0]
    else:
        return max(array[n-1], array_max(array, n-1))
```

Activity #1

2 min

Example: recursive `array_max`

```
def array_max(array, n):
    if n == 1:
        return array[0]
    else:
        return max(array[n-1], array_max(array, n-1))
```

Activity #1

1 min

Example: recursive `array_max`

```
def array_max(array, n):
    if n == 1:
        return array[0]
    else:
        return max(array[n-1], array_max(array, n-1))
```

Activity #1

0 min

Example: recursive `array_max`

```
def array_max(array, n):
    if n == 1:
        return array[0]
    else:
        return max(array[n-1], array_max(array, n-1))
```

```
array_max([5,1,9,2], 4) = max(2, array_max([5,1,9], 3))
                        = max(2, max(9, array_max([5,1], 2)))
                        = max(2, max(9, max(1, array_max([5], 1))))
                        = max(2, max(9, max(1, 5)))
                        = max(2, max(9, 5))
                        = max(2, 9)
                        = 9
```

***Note**: we keep *entire* array but only *show* relevant items

Running Time of Recursive Algos

- Difficult to analyze :-(
- With iterative algorithms
  - we can count # of ops per loop
- How can we count # ops in a recursive step?
  - We can't…

```
def factorial(n):
    out = 1
    for i in range(1, n+1):
        out = i * out
    return out
```

```
def factorial(n):
 if n == 1:
  return 1
 else:
  return n * factorial(n-1)
```

Recurrence Relations

- Functions that express run time recursively

$$\underbrace{T(n) = 2 \cdot T(n-1) + 10}_{\text{general case}}, \quad \text{with } \underbrace{T(1) = 8}_{\text{base case}}$$

  - part 1: # of operations in *general* case
  - part 2: # of operations in *base* case

Example: recursive `array_max`

```
def array_max(array, n):
    if n == 1:
        return array[0]
    else:
        return max(array[n-1], array_max(array, n-1))
```

$$\underbrace{T(n) = T(n-1) + c_1}_{\text{general case}}, \quad \text{with } \underbrace{T(1) = c_0}_{\text{base case}}$$

What about Big-Oh?

- general: constant # ops for comp & max + cost of recursive call
- base: constant # ops for comp and return

Big-O from Recurrence Relation

- Step #1: Plug & Chug
  - algebraic manipulations to guess a Big-O expression
- Step #2: Induction
  - prove that Big-O expression is correct

Example: recursive `array_max`

$$\underbrace{T(n) = T(n-1) + c_1}_{\text{general case}}, \quad \text{with } \underbrace{T(1) = c_0}_{\text{base case}}$$

Activity #2

Example: recursive `array_max`

$$\underbrace{T(n) = T(n-1) + c_1}_{\text{general case}}, \quad \text{with } \underbrace{T(1) = c_0}_{\text{base case}}$$

Activity #2

3 min

Example: recursive `array_max`

$$\underbrace{T(n) = T(n-1) + c_1}_{\text{general case}}, \quad \text{with } \underbrace{T(1) = c_0}_{\text{base case}}$$

Activity #2

2 min

Example: recursive `array_max`

$$\underbrace{T(n) = T(n-1) + c_1}_{\text{general case}}, \quad \text{with } \underbrace{T(1) = c_0}_{\text{base case}}$$

Activity #2

1 min

Example: recursive `array_max`

$$\underbrace{T(n) = T(n-1) + c_1}_{\text{general case}}, \quad \text{with } \underbrace{T(1) = c_0}_{\text{base case}}$$

Activity #2

0 min

Plug & Chug

$$
\begin{aligned}
T(1) &= c_0 \\
T(2) &= c_1 + T(1) = c_1 + c_0 \\
T(3) &= c_1 + T(2) = c_1 + c_1 + c_0 = 2c_1 + c_0 \\
T(4) &= c_1 + T(3) = c_1 + 2c_1 + c_0 = 3c_1 + c_0 \\
T(5) &= c_1 + T(4) = c_1 + 3c_1 + c_0 = 4c_1 + c_0 \\
&\vdots \\
T(n) &= c_1 + T(n-1) = (n-1)c_1 + c_0
\end{aligned}
$$

- Recurrence *solution*: closed form expression

$$\boxed{T(n) = (n-1) \cdot c_1 + c_0 = O(n)}$$

Are we done?

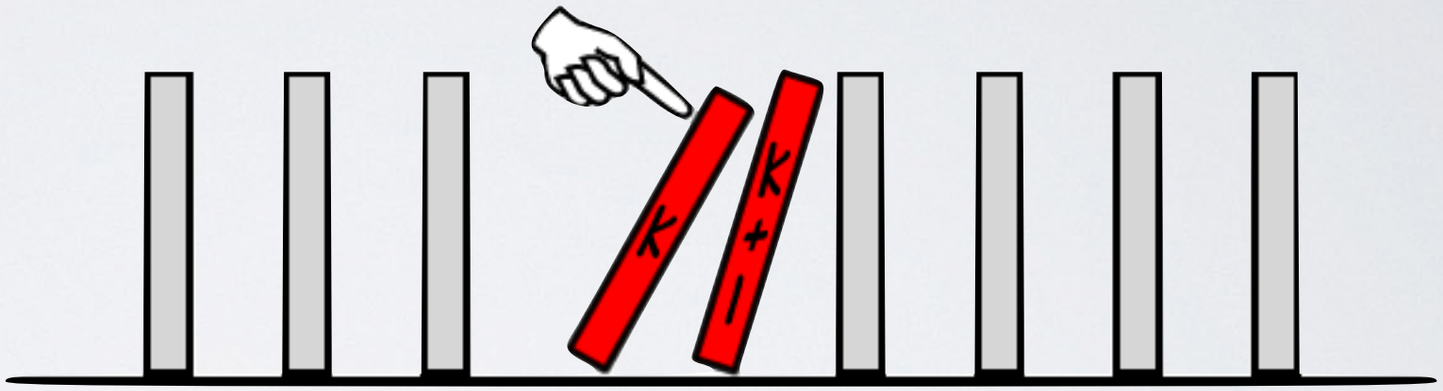
- That was just a guess…not a proof!
  - plugged & chugged to find a pattern
  - and then we guessed at a Big-O
- How can we be sure?
- We prove it using Induction

Induction

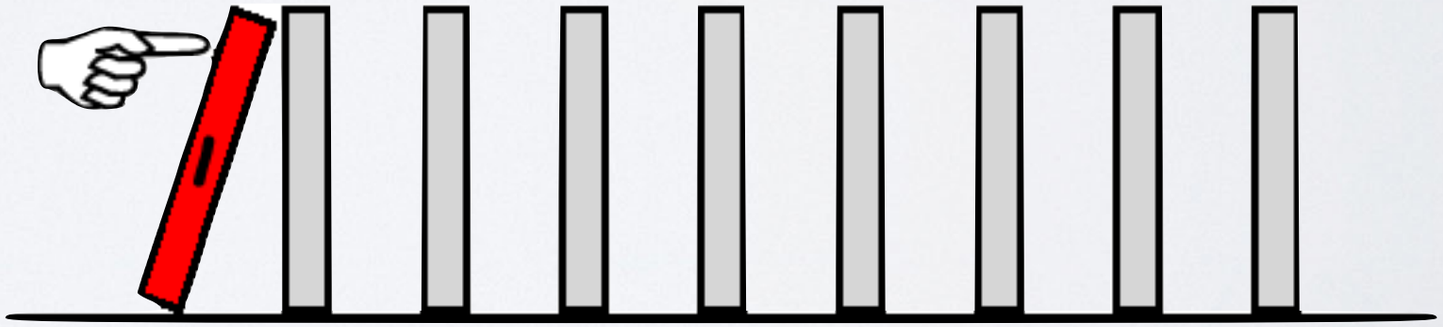
- Proof technique to prove statements about *well-ordered* sets
  - well-ordered: order between elements
  - example: the integers, recurrence relations
- Idea:
  - prove that the statement **P** is true for base case
  - prove that **if P** is true for some case, then **P** is true for the next case
- Example for integers
  - prove that a statement **P** is true for `n=1`
  - prove that **if P** is true for `n=k` then **P** is true for `n=k+1`

Induction

Inductive step:



Base case:



Induction for `array_max`

- **P**(n): solution of $T(n) = T(n-1) + c_1,\ \text{w/}\ T(1) = c_0$ is

$$f(n) = (n-1) \cdot c_1 + c_0$$

base case

- Prove for base case: **`n=1`**
  - $T(1) = c_0$ and $f(1) = (1-1) \cdot c_1 + c_0 = c_0$
- Inductive *assumption*: **`n=k`**
  - $T(k) = f(k)$
- Inductive step: $T(k+1) = T(k) + c_1$

$$= (k-1) \cdot c_1 + c_0 + c_1$$

$$= k \cdot c_1 + c_0$$

Induction Example #2

P(n): $A(n) = \sum_{i=1}^{n} 2i$ is equal to $f(n) = n \cdot (n+1)$

- Base case: `n = 1`
  - $A(1) = 2$ and $f(1) = 1 \cdot (1+1) = 2$
- Inductive assumption: n=k
  - $\sum_{i=1}^{k} 2i = k \cdot (k+1)$
- Inductive step

$$A(k+1) = \sum_{i=1}^{k+1} 2i$$

$$= \sum_{i=1}^{k} 2i + 2 \cdot (k+1)$$

$$\ldots = k \cdot (k+1) + 2 \cdot (k+1)$$

$$= (k+1) \cdot (k+2)$$

$$= f(k+1)$$

Induction Example #3

$$P(n) = \sum_{i=1}^{n} i = \frac{n \cdot (n+1)}{2}$$

Activity #3

Induction Example #3

$$P(n) = \sum_{i=1}^{n} i = \frac{n \cdot (n+1)}{2}$$

Activity #3

4 min

Another Induction Example

$$P(n) = \sum_{i=1}^{n} i = \frac{n \cdot (n+1)}{2}$$

Activity #3

3 min

Another Induction Example

$$P(n) = \sum_{i=1}^{n} i = \frac{n \cdot (n+1)}{2}$$

Activity #3

2 min

Another Induction Example

$$P(n) = \sum_{i=1}^{n} i = \frac{n \cdot (n+1)}{2}$$

Activity #3

1 min

Another Induction Example

$$P(n) = \sum_{i=1}^{n} i = \frac{n \cdot (n+1)}{2}$$

Activity #3

0 min

Another Induction Example

$$P(n) = \sum_{i=1}^{n} i = \frac{n \cdot (n+1)}{2}$$

- Prove base case: **`n=1`**
  - $\sum_{i=1}^{1} i = 1$ and $\frac{1 \cdot (1+1)}{2} = 1$
- Induction assumption: **`n=k`**
  - $\sum_{i=1}^{k} i = \frac{k \cdot (k+1)}{2}$
- Prove induction step!

Another Induction Example

- Prove induction step

$$\sum_{i=1}^{k+1} i = 1 + 2 + \cdots + k + (k+1)$$

$$= \sum_{i=1}^{k} i + (k+1)$$

$$= \frac{k \cdot (k+1)}{2} + (k+1)$$

$$= \frac{k \cdot (k+1)}{2} + \frac{2 \cdot (k+1)}{2} \qquad \times \frac{2}{2}$$

$$= \frac{(k+1) \cdot (k+2)}{2} \qquad \text{factor out } (k+1)$$

Induction assumption

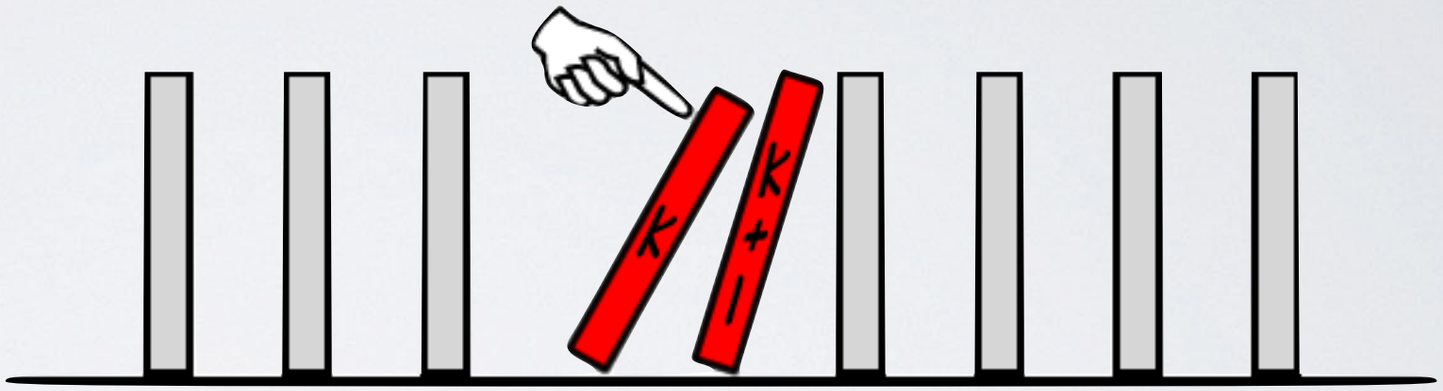
$$\sum_{i=1}^{k} i = \frac{k \cdot (k+1)}{2}$$

Strong vs. Weak Induction

- Weak induction
  - induction step assumes true for **`n=k`** and
  - proves true for **`n=k+1`**
- Strong induction
  - induction step assumes true for **`n=1,2,…,k`** and
  - proves true for **`n=k+1`**
- Strong vs. weak refers to *assumption*
  - not strength of proof

Strong vs. Weak Induction

Weak:



Strong:

?

Readings

- Induction handout on course page
  - http://cs.brown.edu/courses/cs016/static/files/docs/induction.pdf