Final Review

CS16: Introduction to Data Structures & Algorithms

Spring 2019

Outline

- Dynamic Programming
- PageRank
- Kruskal's Algorithm
- Prim-Jarnik Algorithm
- Functional Programming
- Hardness
- Neural Networks

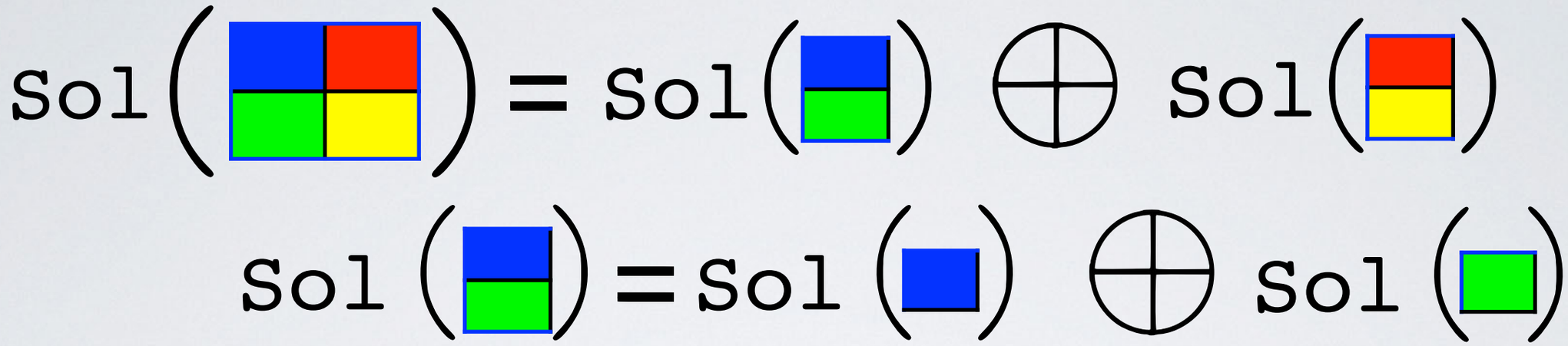
What is Dynamic Programming?

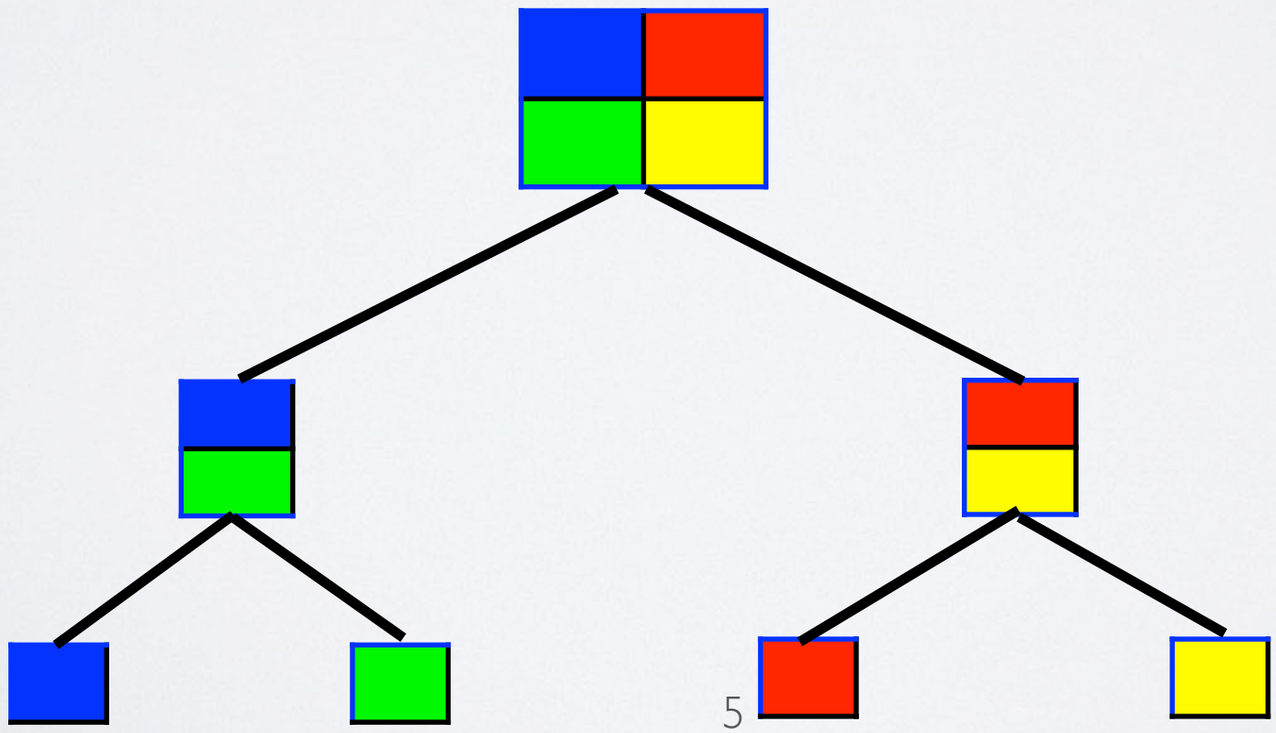
- Algorithm design paradigm/framework
  - Design efficient algorithms for optimization problems
- Optimization problems
  - "find the **best** solution to problem **`X`**"
  - "what is the **shortest** path between **`u`** and **`v`** in **`G`**"
  - "what is the **minimum** spanning tree in **`G`**"
- Can also be used for non-optimization problems

When is Dynamic Programming Applicable?

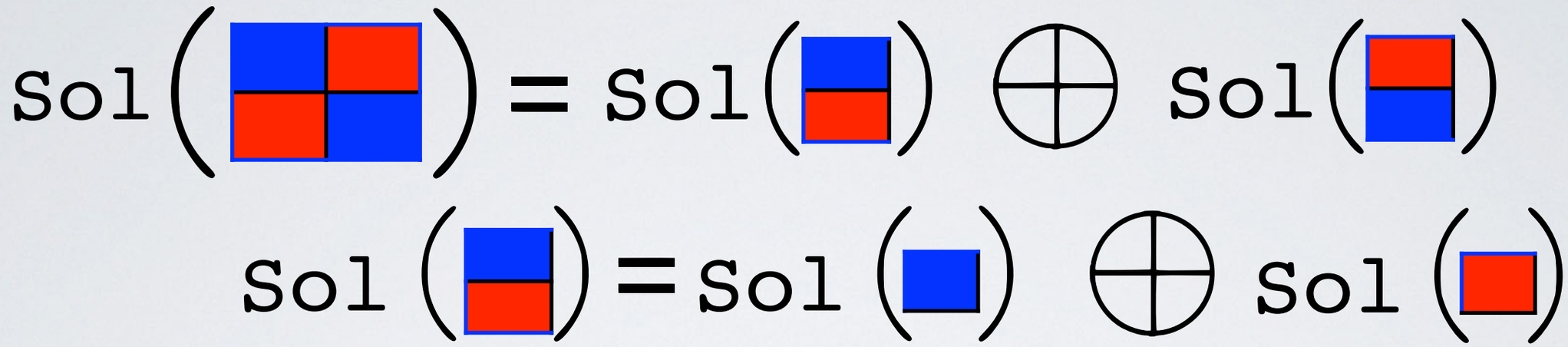
- Condition #1: sub-problems
  - The problem can be solved recursively
  - Can be solved by solving sub-problems
- Condition #2: overlapping sub-problems
  - Same sub-problems need to be solved many times

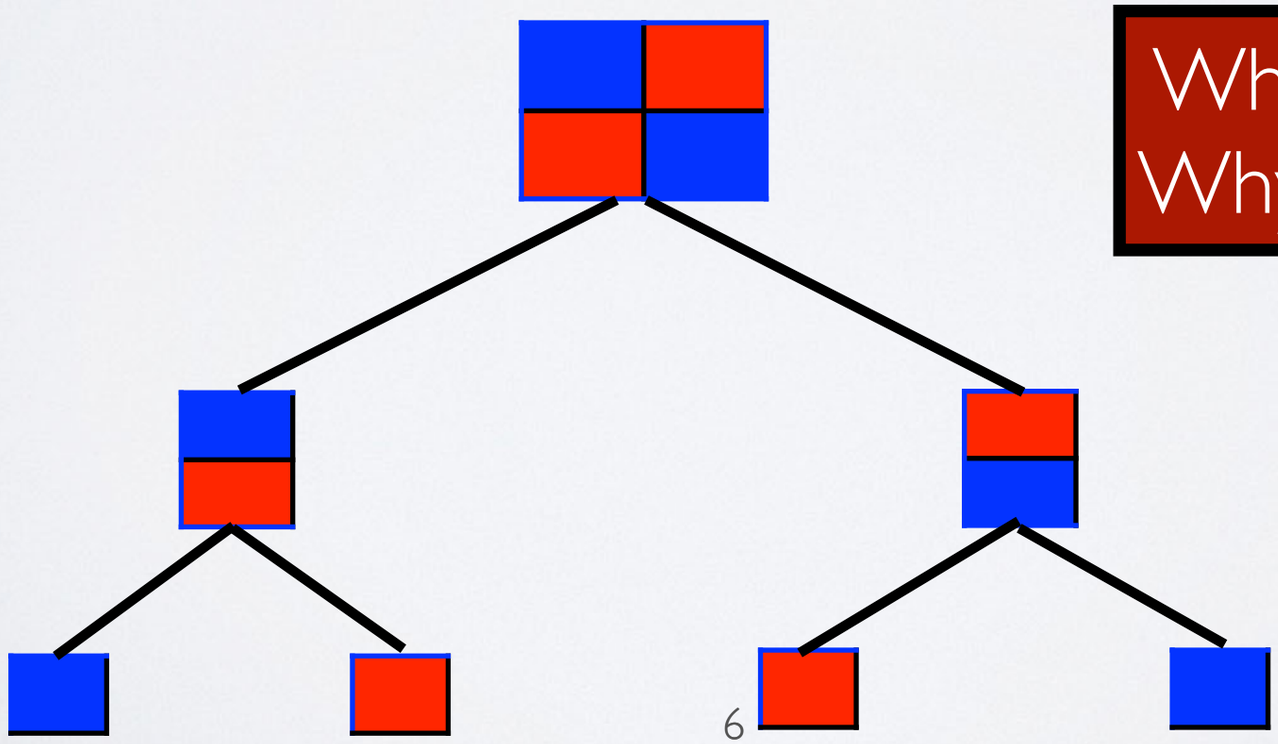
Sub-Problems





Overlapping Sub-Problems





Why solve red twice?
Why solve blue twice?

When is Dynamic Programming Applicable?

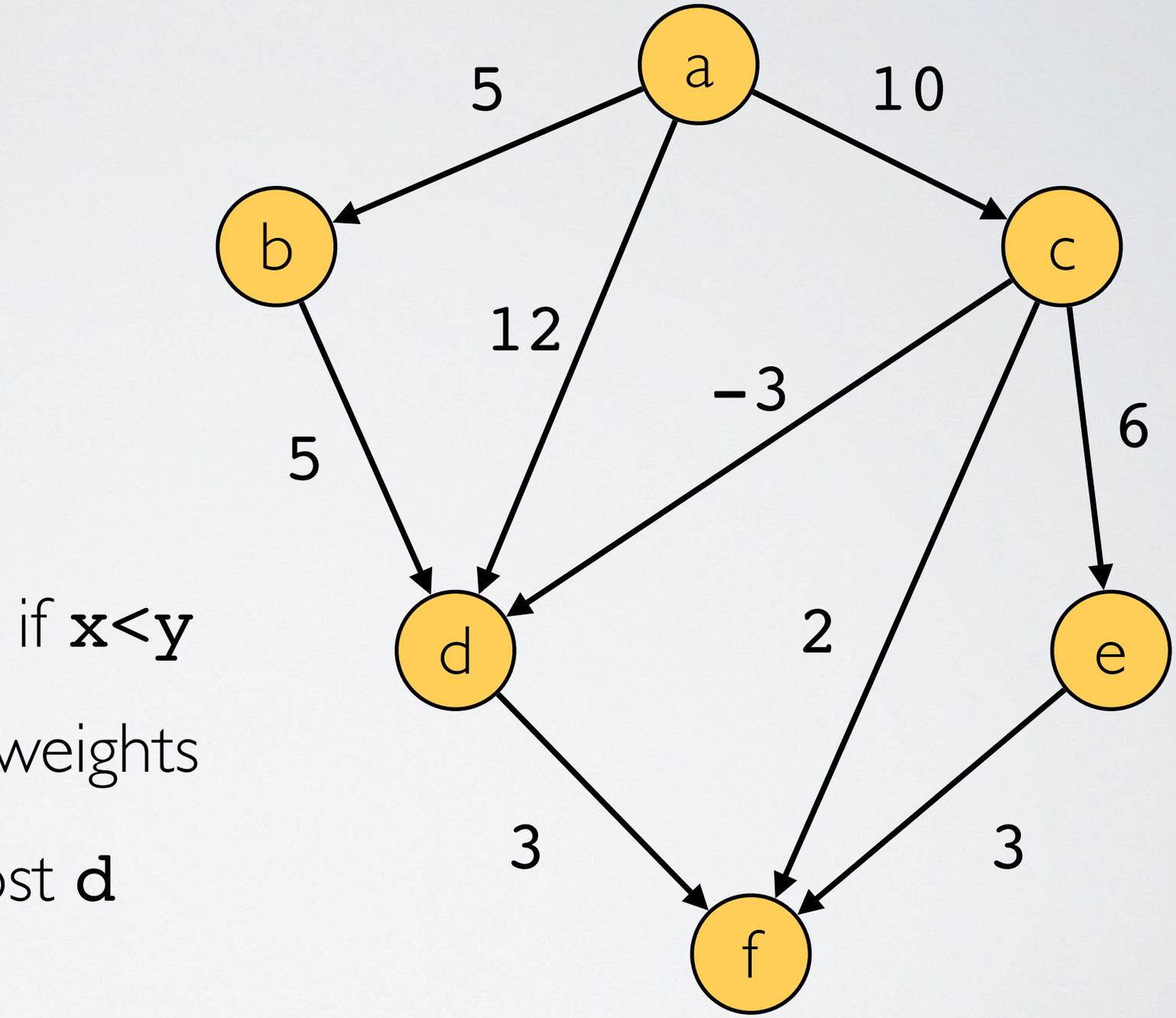
- Core idea
  - Decompose problem into its sub-problems
  - and if sub-problems are overlapping then
  - solve each sub-problem once and store the solution
  - use stored solution when you need to solve sub-problem again

Steps to Solving a Problem w/ DP

- What are the **sub-problems**?
- What is the "**magic**" step?
  - Given solution to a sub-problem…
  - …how do I *combine* them to get solution to the problem?
- Which **(topological) order** on sub-problems can I use?
  - so that solutions to sub-problems available before I need them
- Design iterative **algorithm**
  - that solves sub-problems in order and stores their solution

Shortest Path in Layered Directed Graph

- Layered
  - edge **`(x,y)`** only if **`x<y`**
- Negative & positive weights
- vertex degree at most **`d`**



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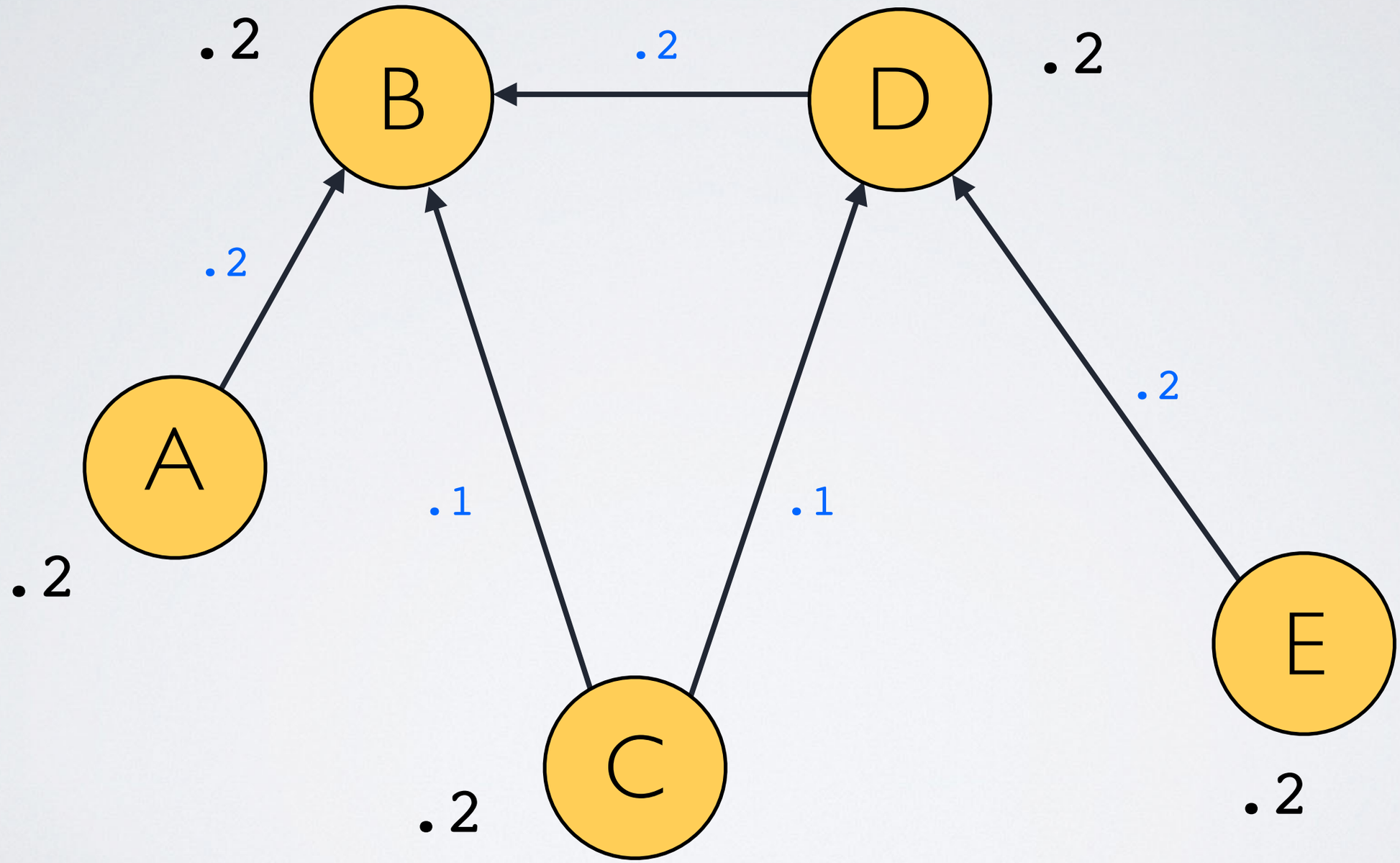
The Basic PageRank Algorithm

- At every round
  - each vertex splits its PR evenly among its outgoing edges
  - each vertex receives PR from all its incoming edges
  - this is done using an *update rule* which is run on every vertex
- The update rule for Basic PageRank is:

$$\mathsf{PR}(v) = \sum_{u \in \mathsf{in}(v)} \frac{\mathsf{PR}(u)}{|\mathsf{out}(u)|}$$

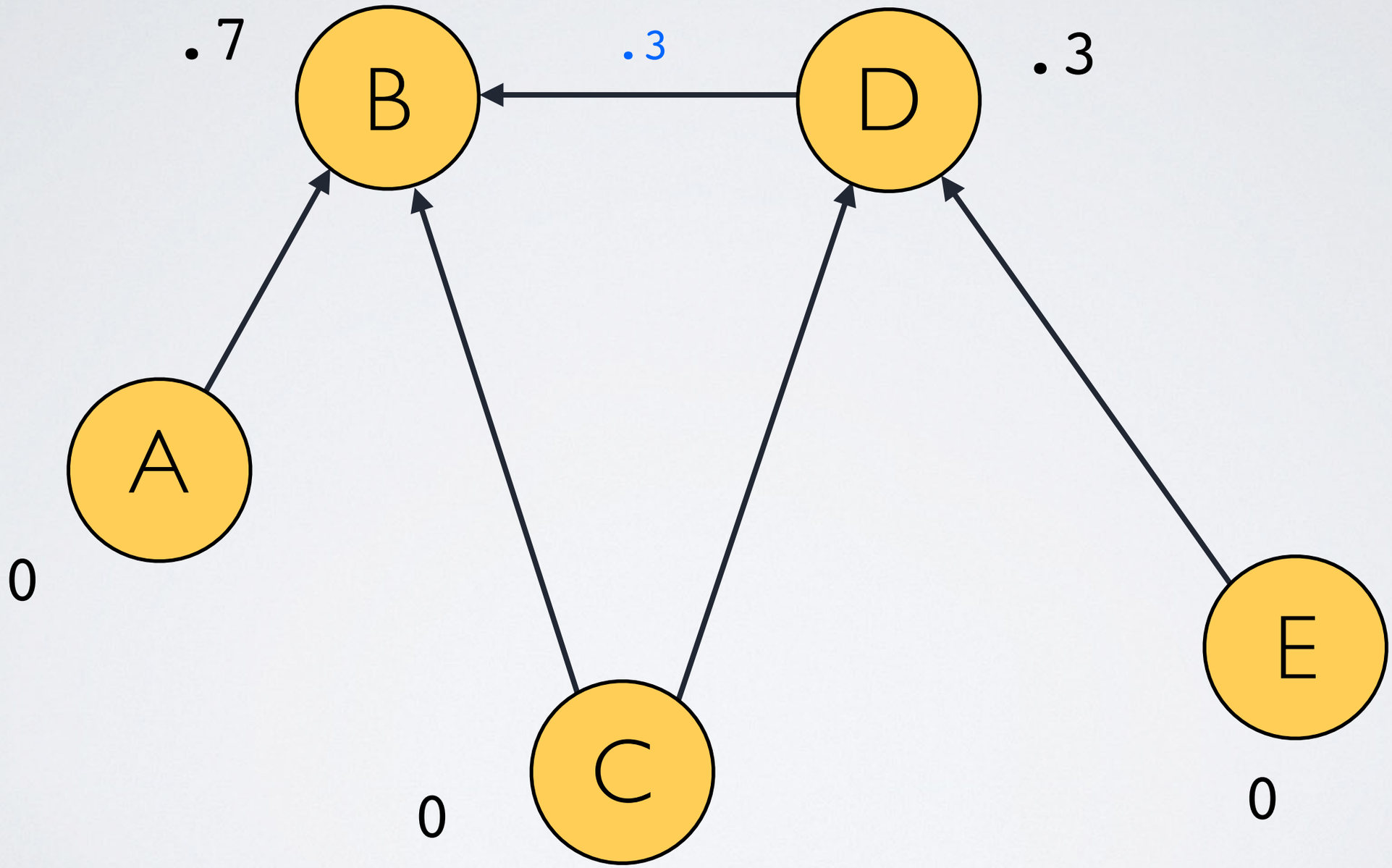
Basic PageRank: Example 2

Round 1



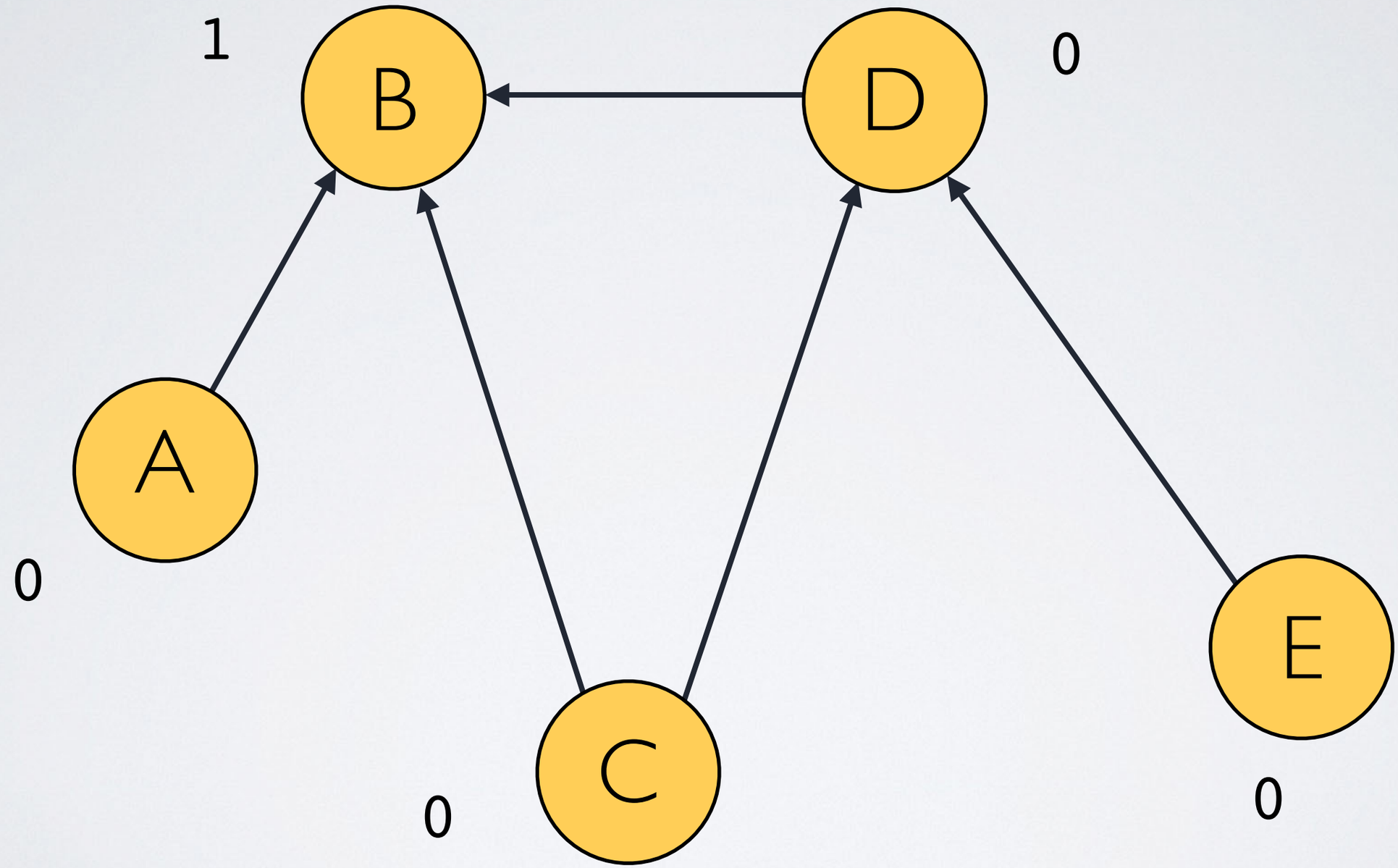
Basic PageRank: Example 2

Round **2**

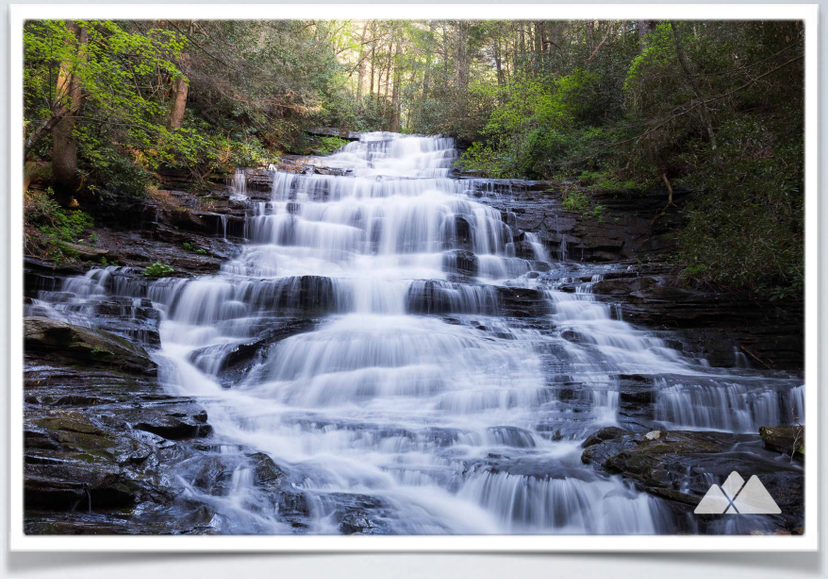


Basic PageRank: Example 2

Round 3



Basic PageRank



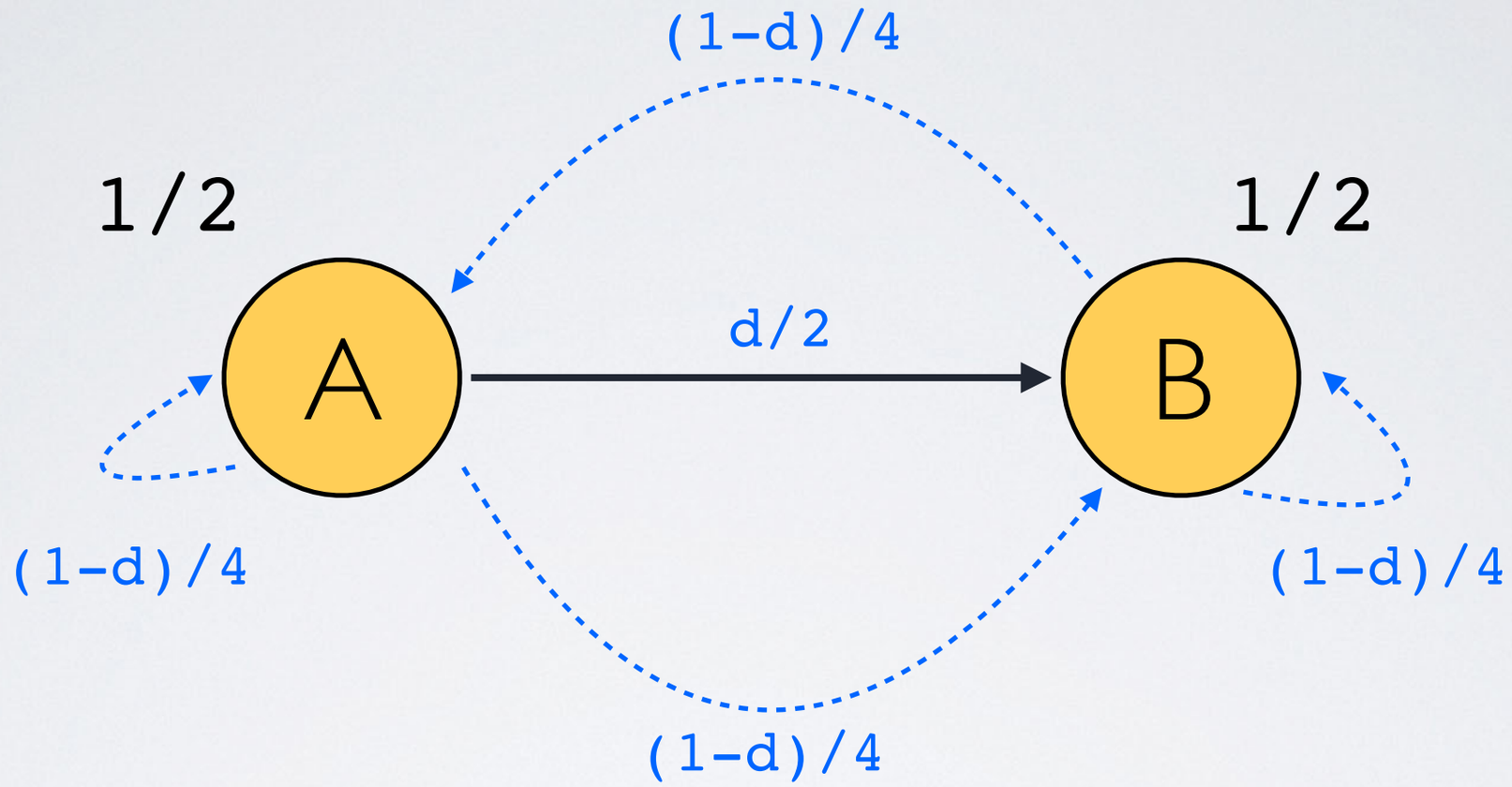
- Basic PageRank doesn't work for certain graphs
  - e.g.: graphs with sinks or with cycles with no outgoing edges
  - all the pagerank gets trapped there
- How do we handle "rank traps"?
- Water flows down from high elevation to low elevation
  - why doesn't all the water end up at the lowest points on Earth?
  - because some of the water evaporates…
  - …and rains back down on the high elevation points

Handling Rank Traps

- Let's make some of the pagerank evaporate!
  - We need a new *update rule*
- In basic update rule, nodes gave all their pagerank to neighbors
- In new update rule, a node will
  - give a **`d`** fraction of its PR to its neighbors (split evenly)
  - give a **`1-d`** fraction of its PR to all other nodes (split evenly)
  - this guarantees that pagerank doesn't accumulate anywhere
  - **`d`** is usually set to **`.85`**

What happens if the node is a sink?

Disappearing PageRank



Disappearing PageRank

- The sum of the pageranks *does not* sum to `1`:

$$\frac{(1-d)}{2} + \frac{1}{2} = 1 - \frac{d}{2} < 1$$

  - since `0 < d < 1`

- We lost `d/2` of B's pagerank when we updated

Handling Sinks

- There are several ways to handle sinks
- The simplest is to modify the graph as follows
  - if $\mathbf{v}$ is a sink, add an edge from $\mathbf{v}$ to every other node in the graph
- Then use the update rule we described on slide #38

The Real PageRank Algorithm

- Add edges connecting every sink to every other node
- At every round, each vertex
  - splits a **`d`** fraction of its PR evenly among its outgoing edges
  - splits a **`(1-d)`** fraction of its PR evenly among all nodes
- **`d`** is called the *damping factor* & is usually set to **`.85`**

The Real PageRank Algorithm

- At every round the PR of each vertex **v** is updated using:

$$\mathsf{PR}(v) = \left(\sum_{u \in \mathsf{in}(v)} \frac{d \cdot \mathsf{PR}(u)}{|\mathsf{out}(u)|} \right) + \sum_{u \in V} \frac{(1-d) \cdot \mathsf{PR}(u)}{|V|}$$

1. nodes with edges pointing to **v**

2. **d** fraction of **u**'s PR

3. number of edges leaving u

4. **(1-d)** fraction of u's PR

$$= \left(d \cdot \sum_{u \in \mathsf{in}(v)} \frac{\mathsf{PR}(u)}{|\mathsf{out}(u)|} \right) + \frac{1-d}{|V|} \cdot \sum_{u \in V} \mathsf{PR}(u)$$

$$= \left(d \cdot \sum_{u \in \mathsf{in}(v)} \frac{\mathsf{PR}(u)}{|\mathsf{out}(u)|} \right) + \frac{1-d}{|V|} \quad \downarrow \; 1$$

$$= \frac{1-d}{|V|} + d \cdot \sum_{u \in \mathsf{in}(v)} \frac{\mathsf{PR}(u)}{|\mathsf{out}(u)|}$$

The Real PageRank

- Runtime of a round **O(|E|)**
  - iterate over every vertex **v** and over all incoming edges to **v**
- How many rounds should we run?
- Until the pageranks "stabilize"
  - pageranks stop changing even though we run more rounds
- We can prove that
  - if we run for large enough number of rounds then pageranks will stabilize
  - that number could be very large for some graphs…
  - …but in practice it's usually reasonable

Alternative Sink Handling

- You can also handle sinks without modifying the graph
  - but you need a slightly different update rule

$$\mathsf{PR}(v) = \frac{1-d}{|V|} + d \cdot \sum_{u \in \mathsf{in}(v)} \frac{\mathsf{PR}(u)}{|\mathsf{out}(u)|} + \sum_{u \in \mathsf{sinks}(G)} \frac{d \cdot \mathsf{PR}(u)}{|V|}$$

$$= \frac{1-d}{|V|} + d \cdot \left(\sum_{u \in \mathsf{in}(v)} \frac{\mathsf{PR}(u)}{|\mathsf{out}(u)|} + \sum_{u \in \mathsf{sinks}(G)} \frac{\mathsf{PR}(u)}{|V|} \right)$$

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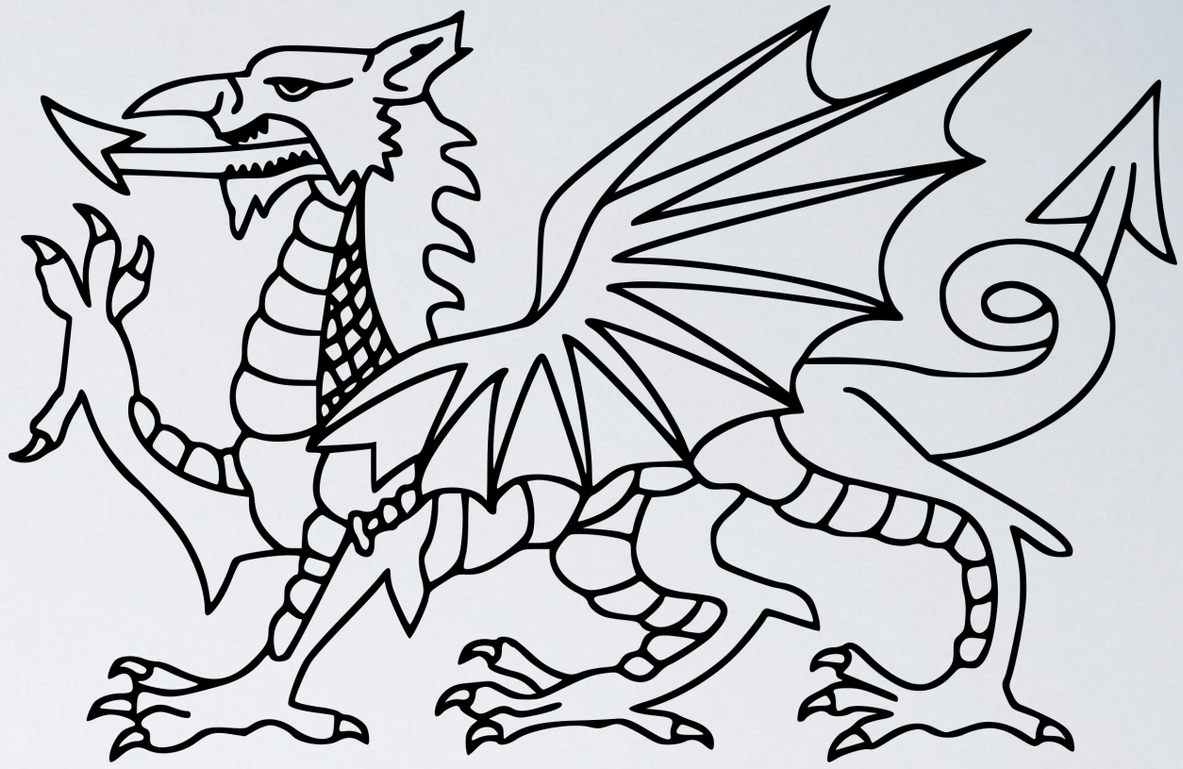


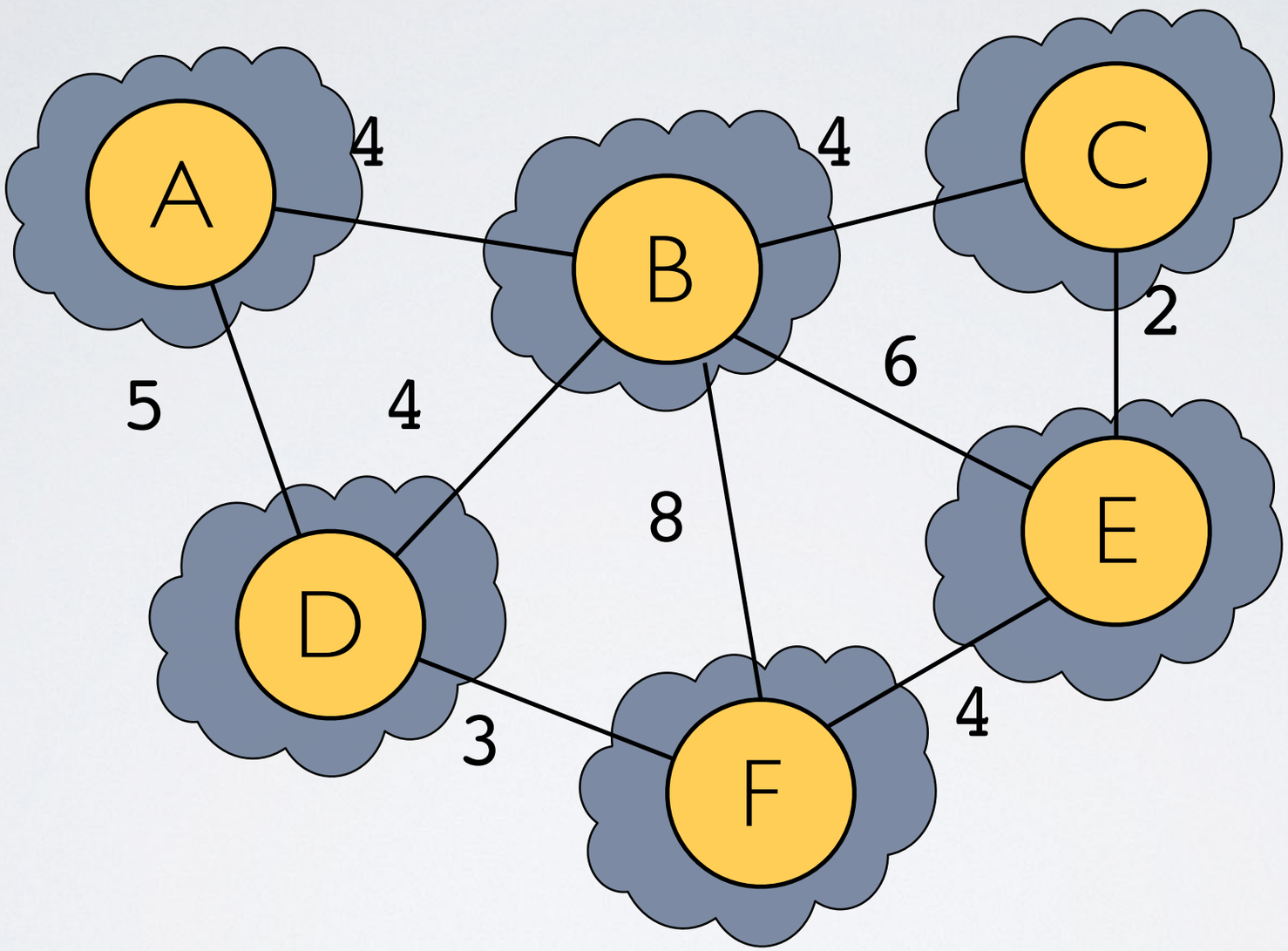
Kruskal's Algorithm

- Sort edges by weight in increasing order
- For each edge in sorted list
  - If adding edge does not create cycle…
    - …add it to MST
- Stop when you have gone through all edges

Kruskal

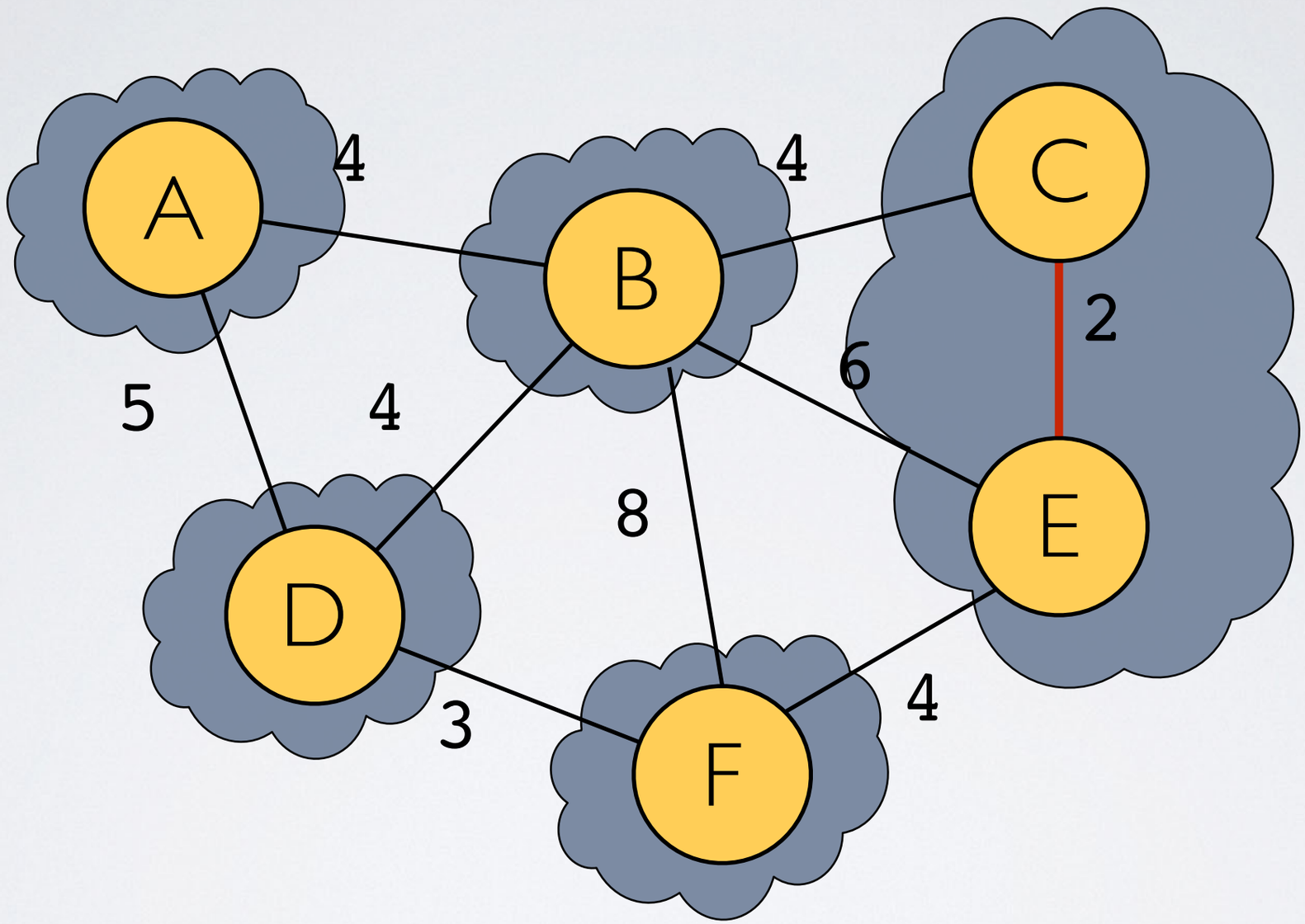
- How can we tell if adding edge will create cycle?
- Start by giving each vertex its own "cloud"
- If both ends of lowest-cost edge are in same cloud
  - we know that adding the edge will create a cycle!
- When edge is added to MST
  - merge clouds of the endpoints

Example




```
edges = [(C,E),(D,F),(B,C),(E,F),(B,D),(A,B),(A,D),(B,E),(B,F)]
```

Example

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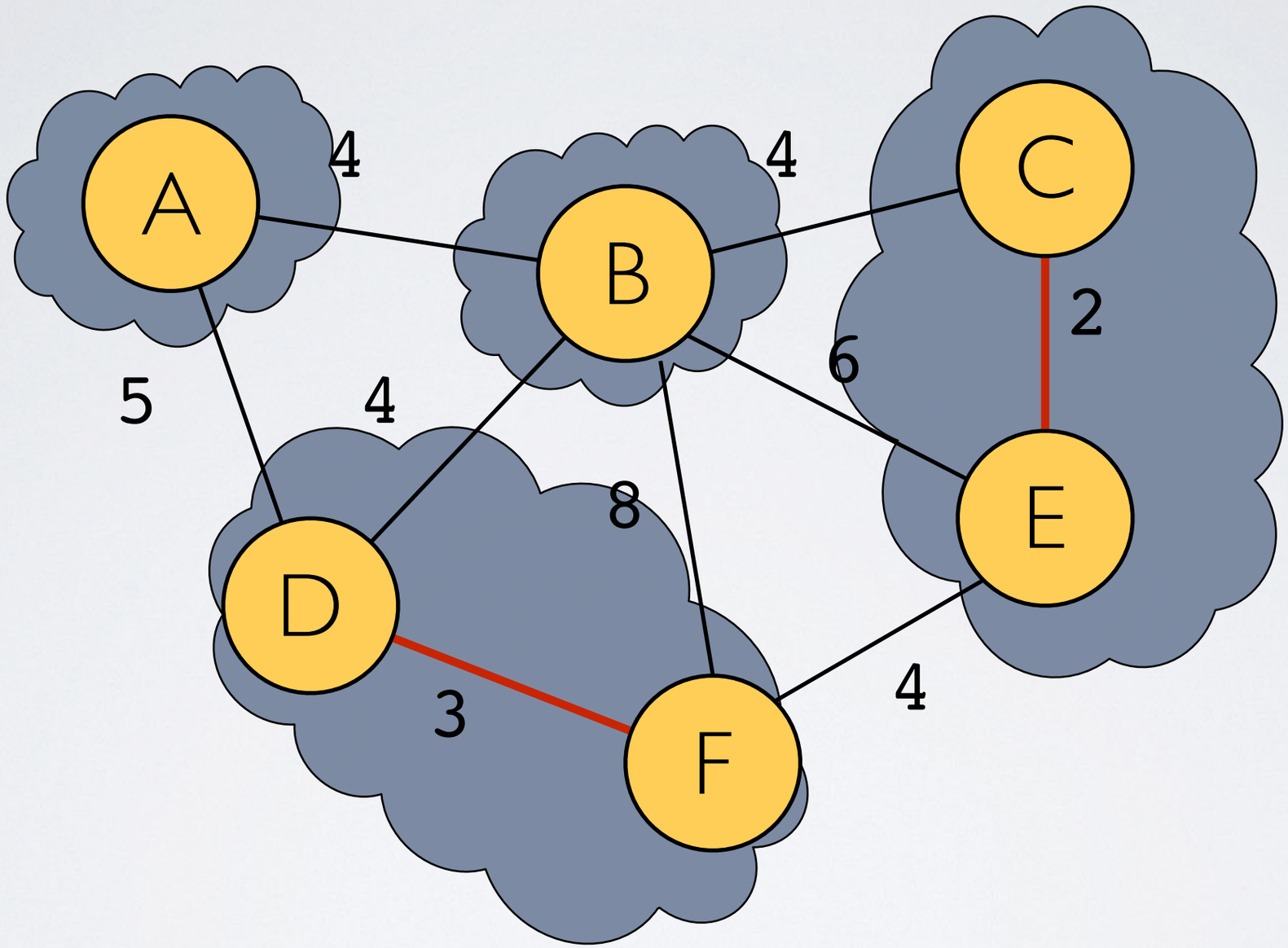
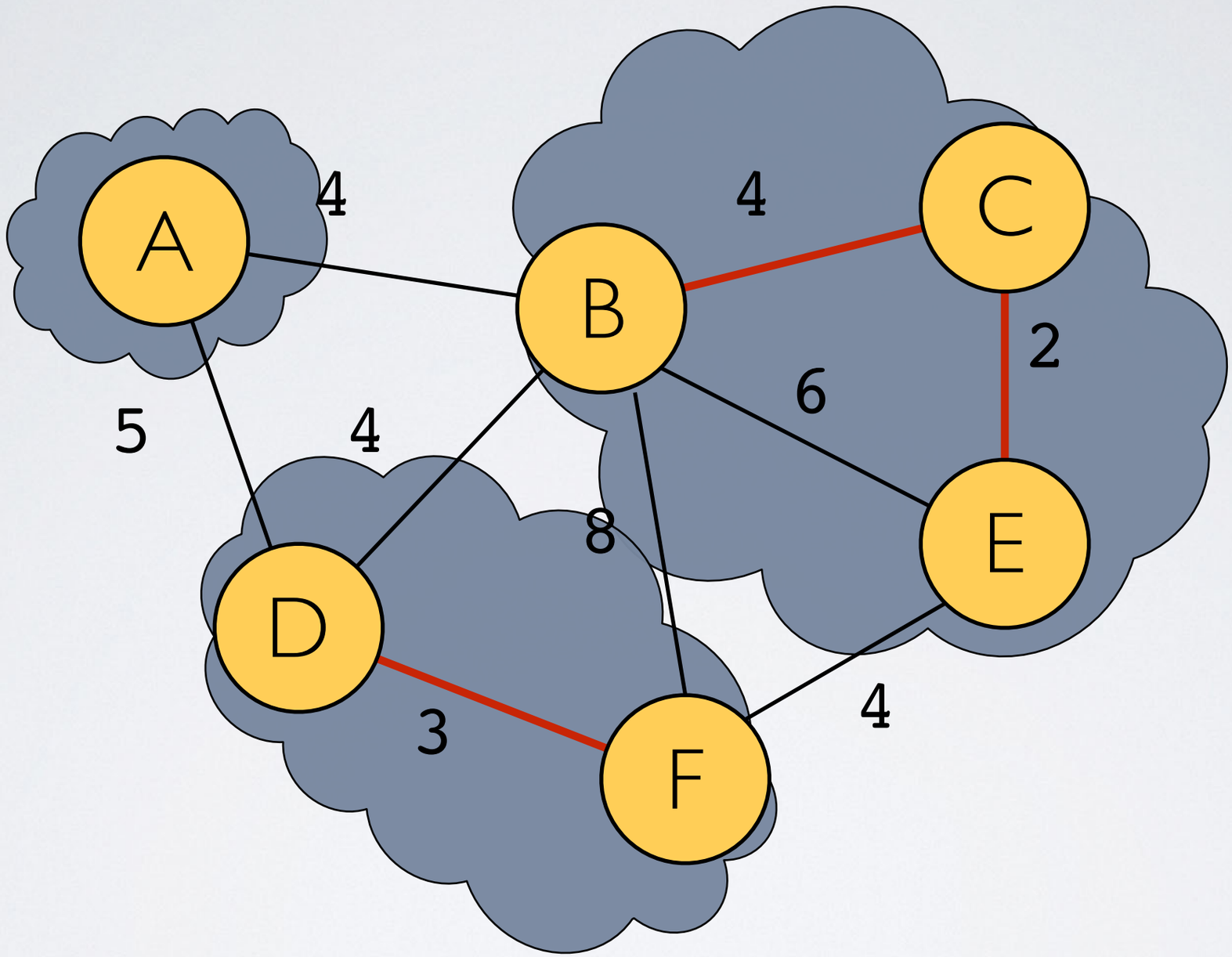
edges = [(D,F),(B,C),(E,F),(B,D),(A,B),(A,D),(B,E),(B,F)]
```

Example



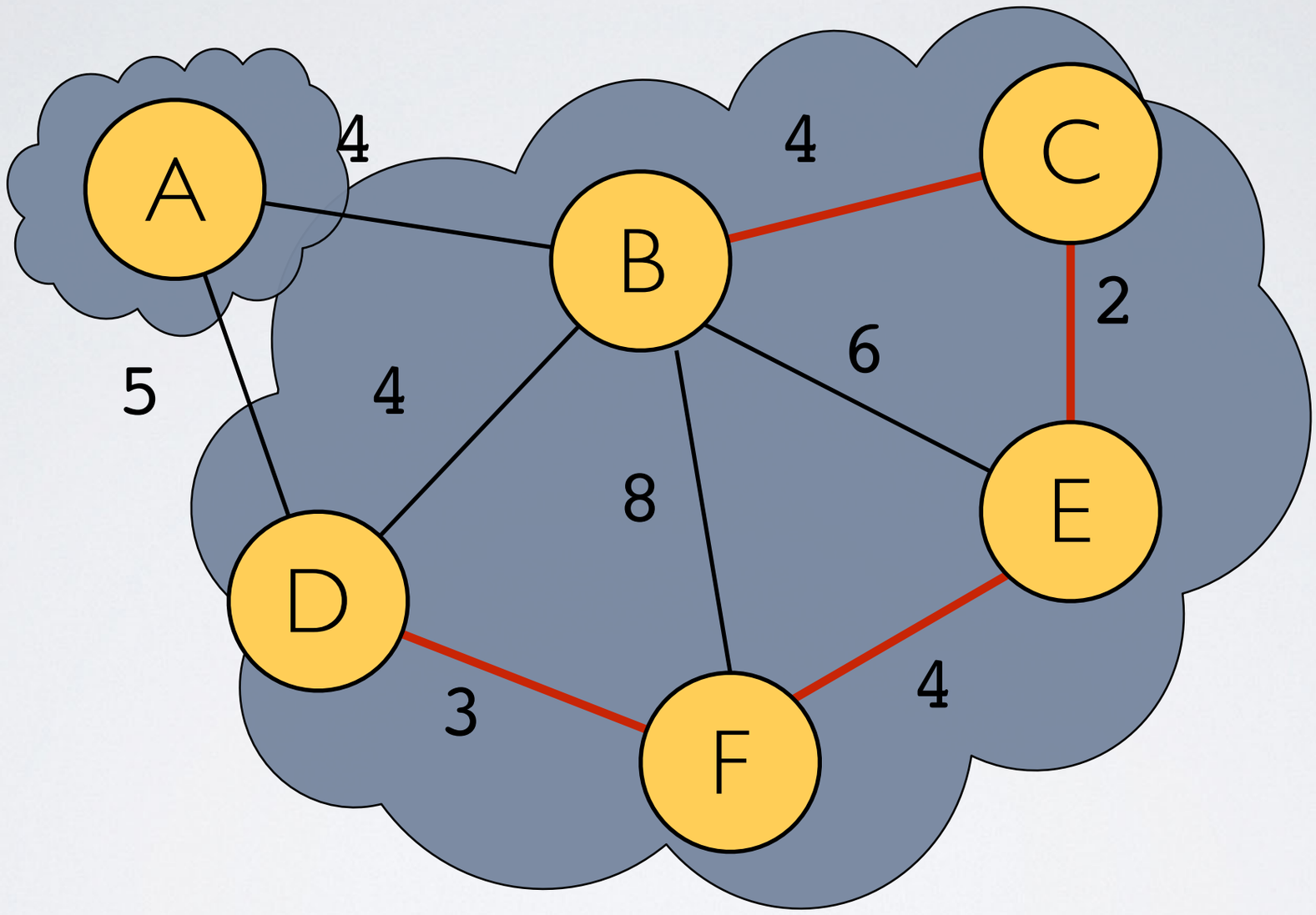

```
edges = [(B,C),(E,F),(B,D),(A,B),(A,D),(B,E),(B,F)]
```

Example



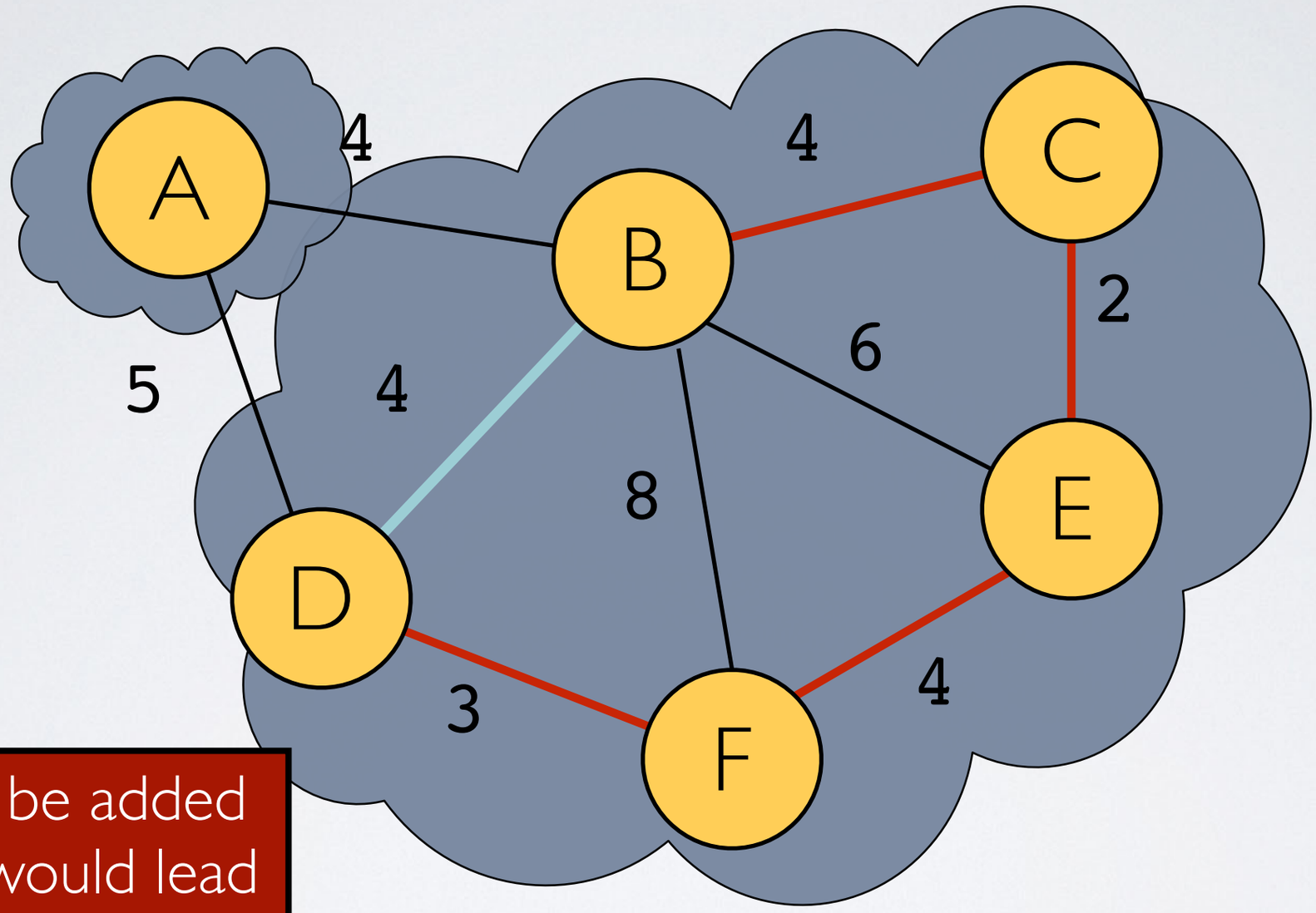

```
edges = [(E,F),(B,D),(A,B),(A,D),(B,E),(B,F)]
```

Example




```
edges = [(B,D),(A,B),(A,D),(B,E),(B,F)]
```

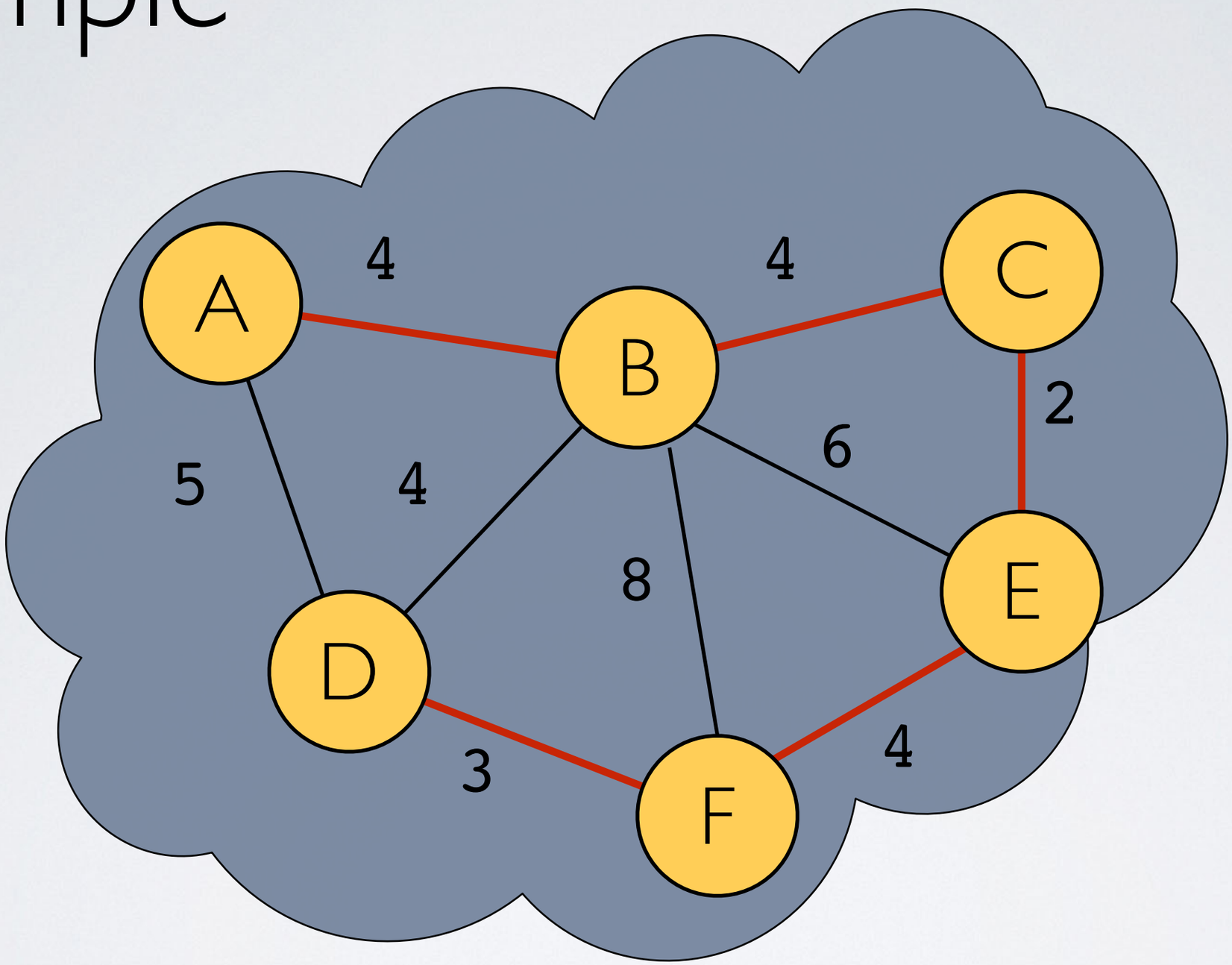
Example



BD cannot be added because it would lead to a cycle

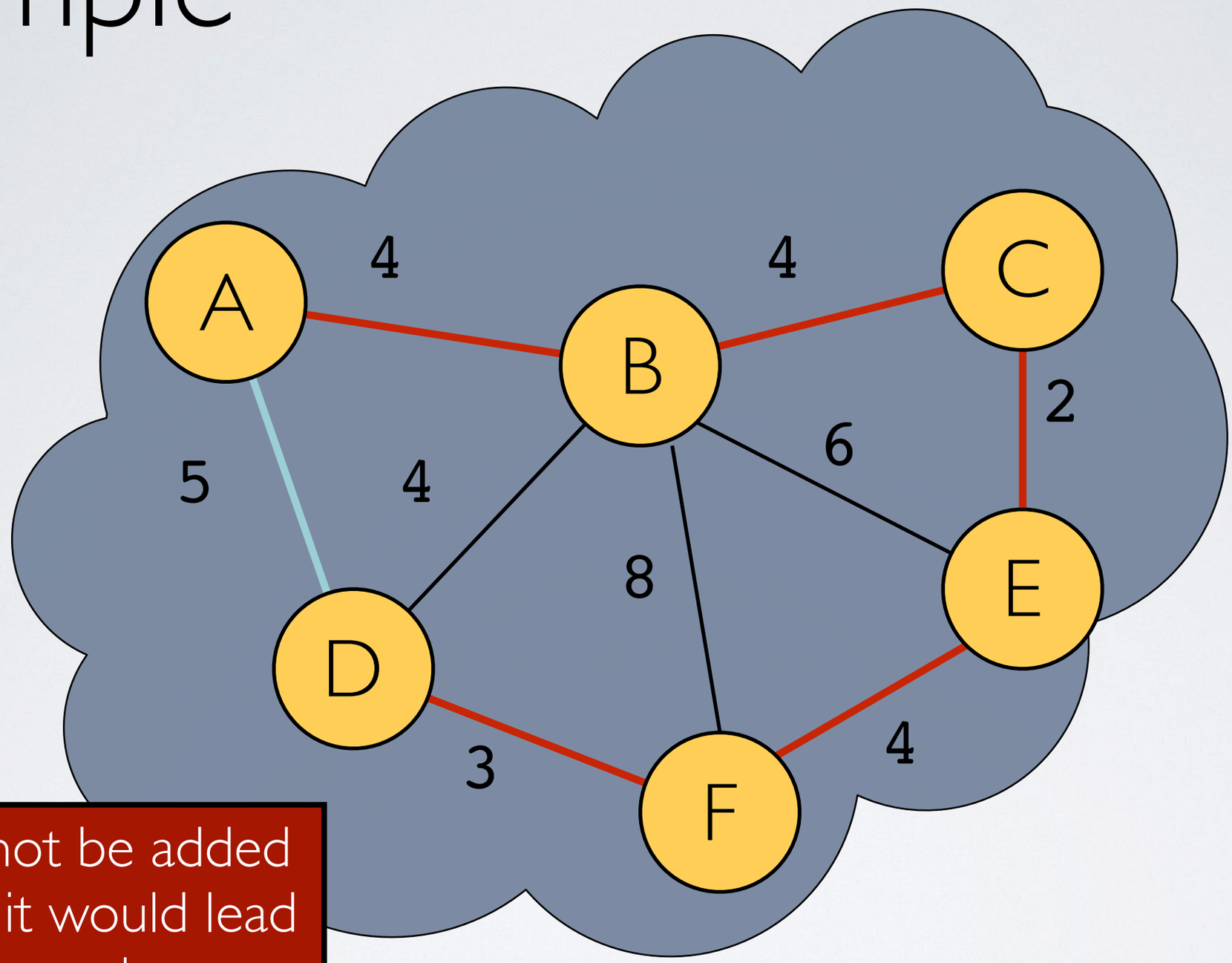
```
edges = [(A,B),(A,D),(B,E),(B,F)]
```

Example




```
edges = [(A,D),(B,E),(B,F)]
```

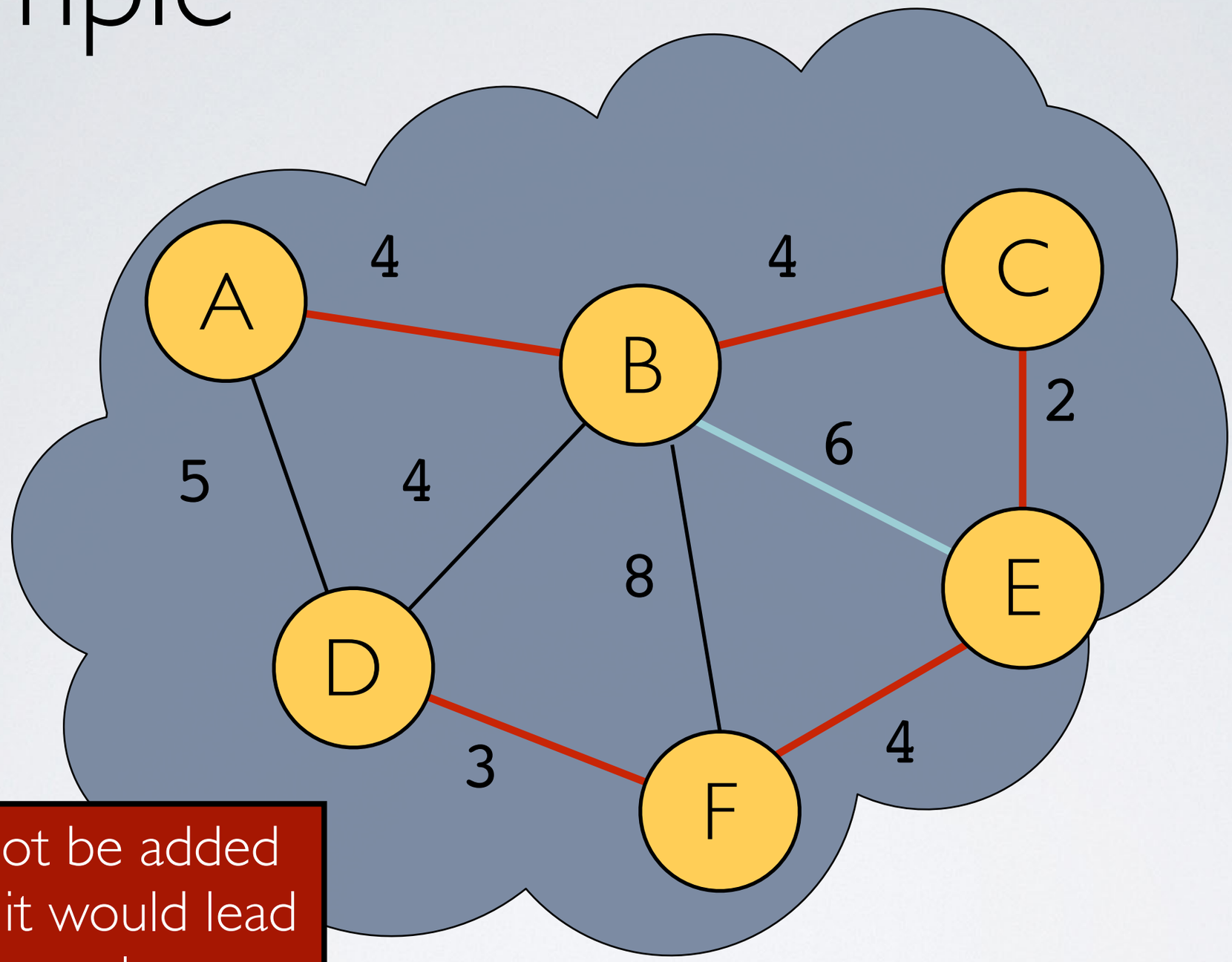
Example



AD cannot be added because it would lead to a cycle

```
edges = [(B,E),(B,F)]
```

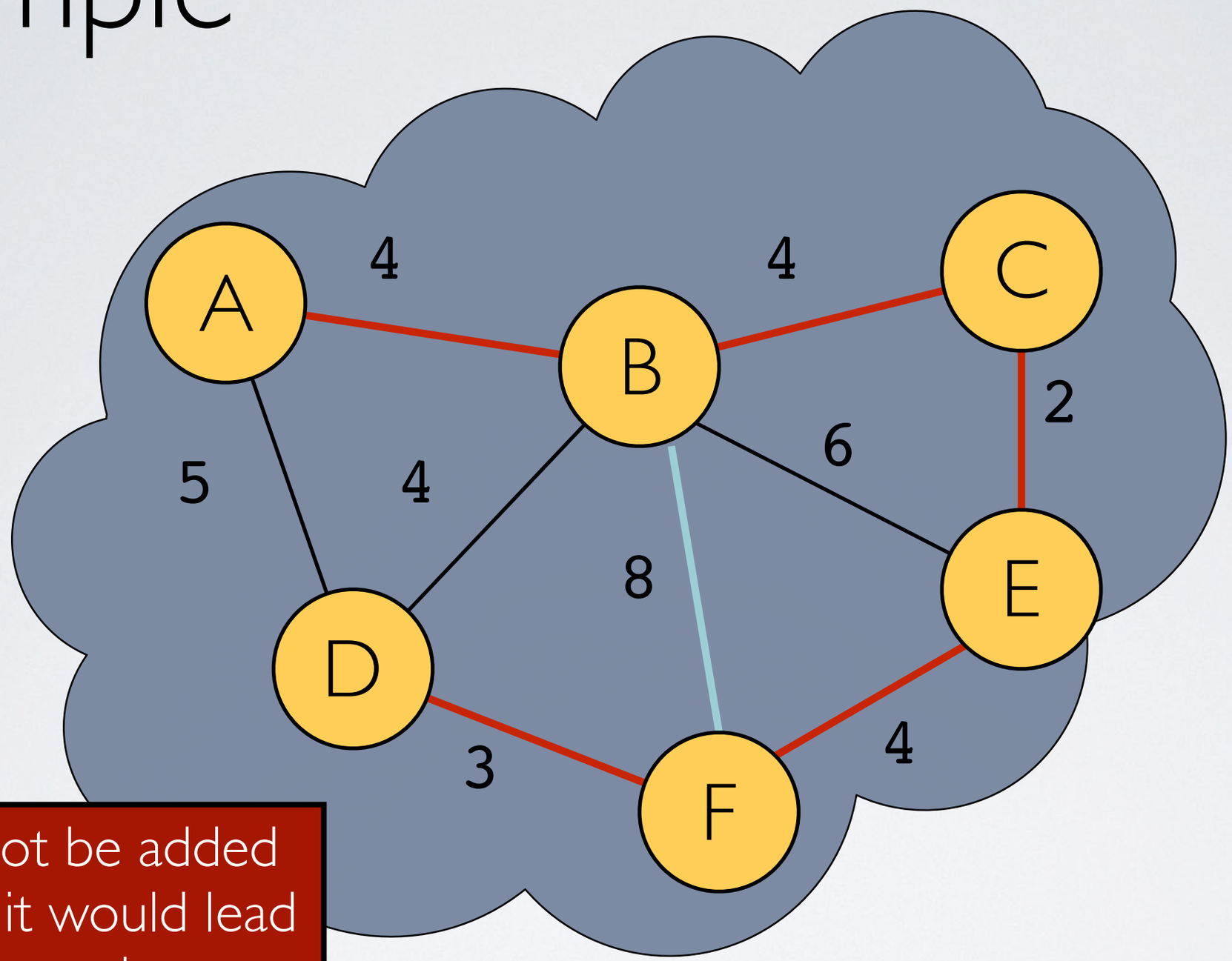
Example



BE cannot be added because it would lead to a cycle

```
edges = [(B,F)]
```

Example



BF cannot be added because it would lead to a cycle

```
edges = [ ]
```

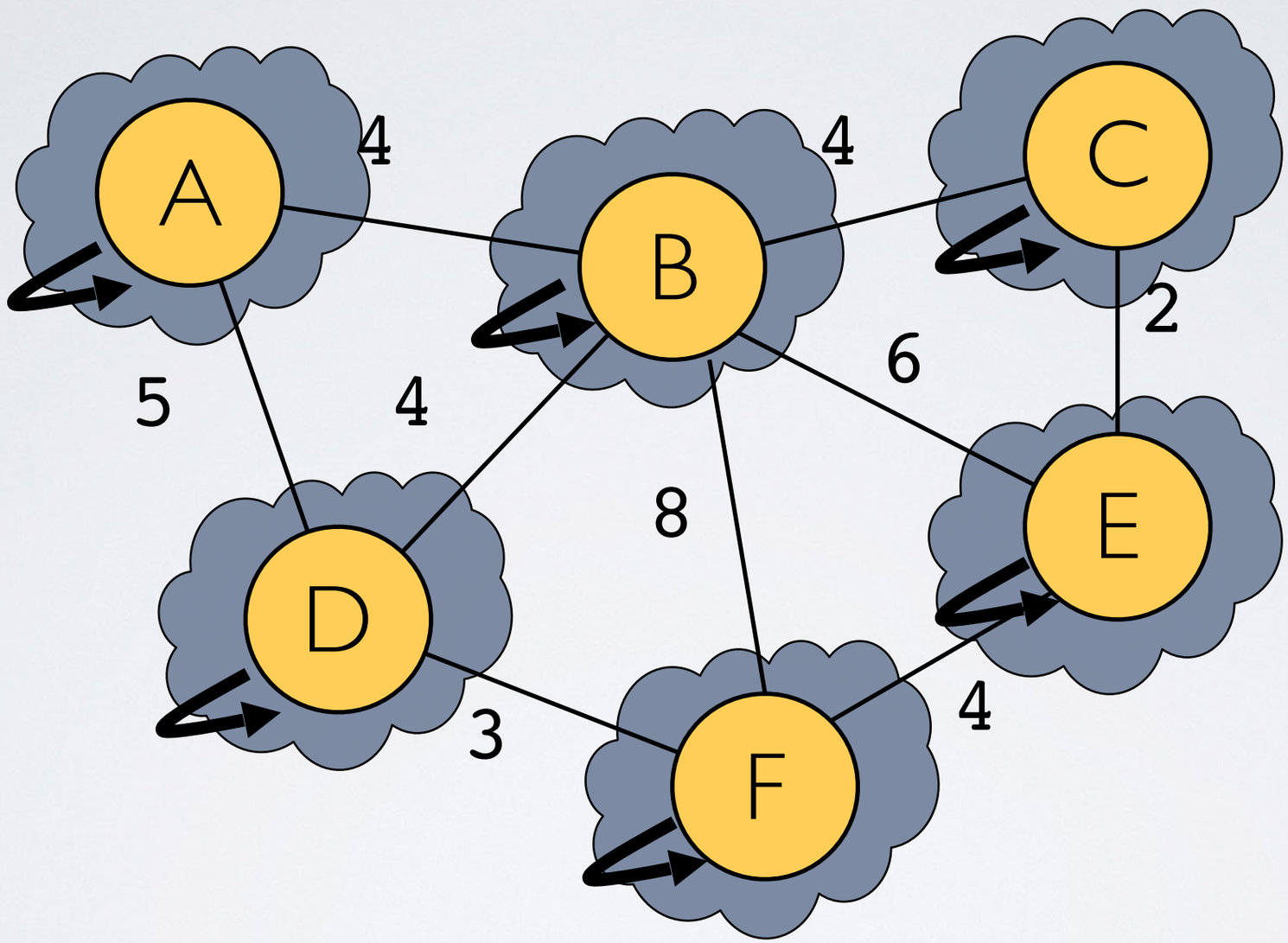
Kruskal Pseudo-Code

```
function kruskal(G):
    // Input: undirected, weighted graph G
    // Output: list of edges in MST
    for vertices v in G:
        makeCloud(v) // put every vertex into it own set
    MST = []
  Sort all edges
    for all edges (u,v) in G sorted by weight:
        if u and v are not in same cloud:
            add (u,v) to MST
            merge clouds containing u and v
    return MST
```

Implementing Clouds: Union-Find

- Let's rethink notion of clouds
  - think of clouds as small trees
  - cloud is represented by the root of its tree
- Every vertex in these trees has
  - a parent pointer that leads up to root of the tree
  - a rank that measures how deep the tree is

Example

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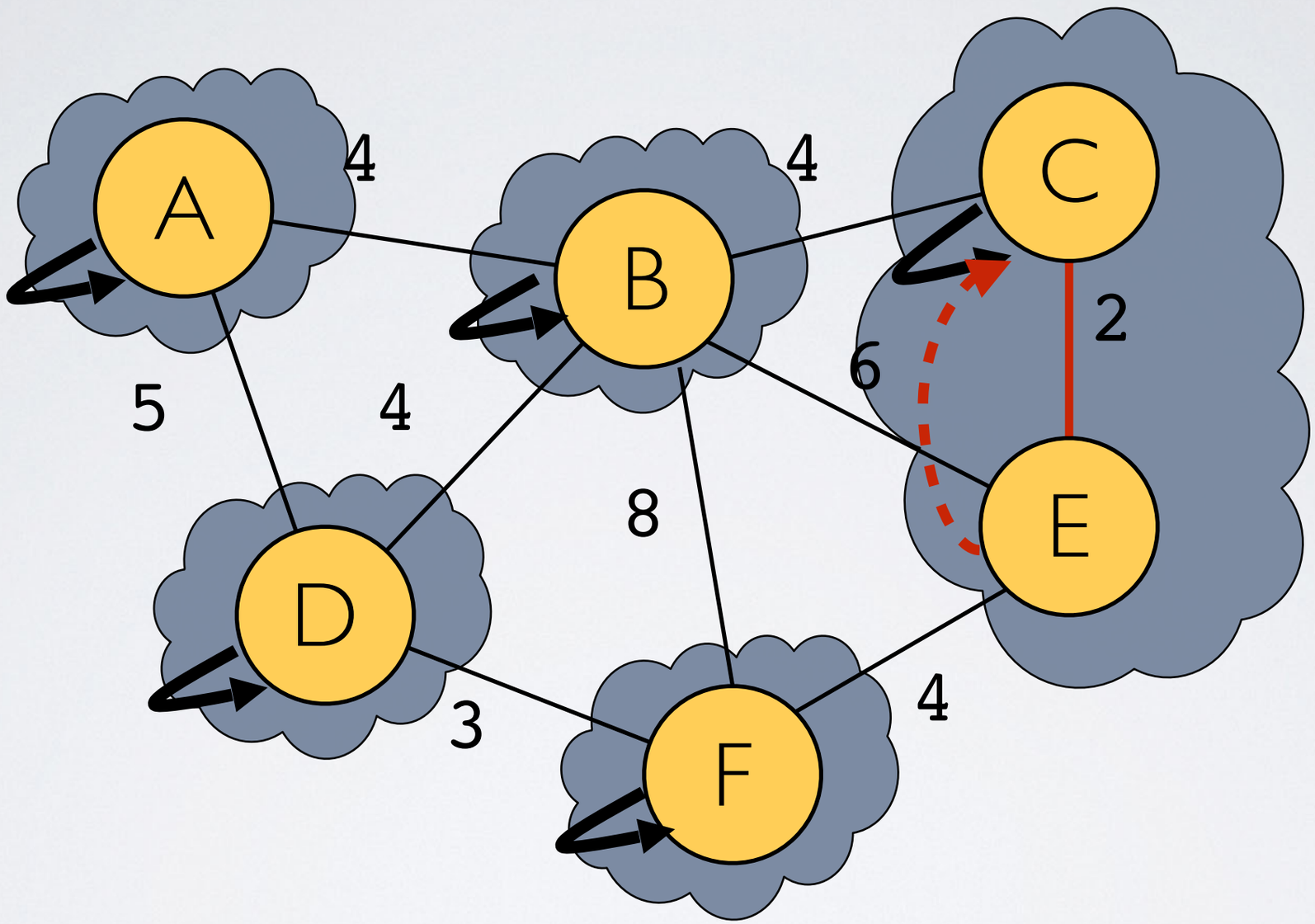


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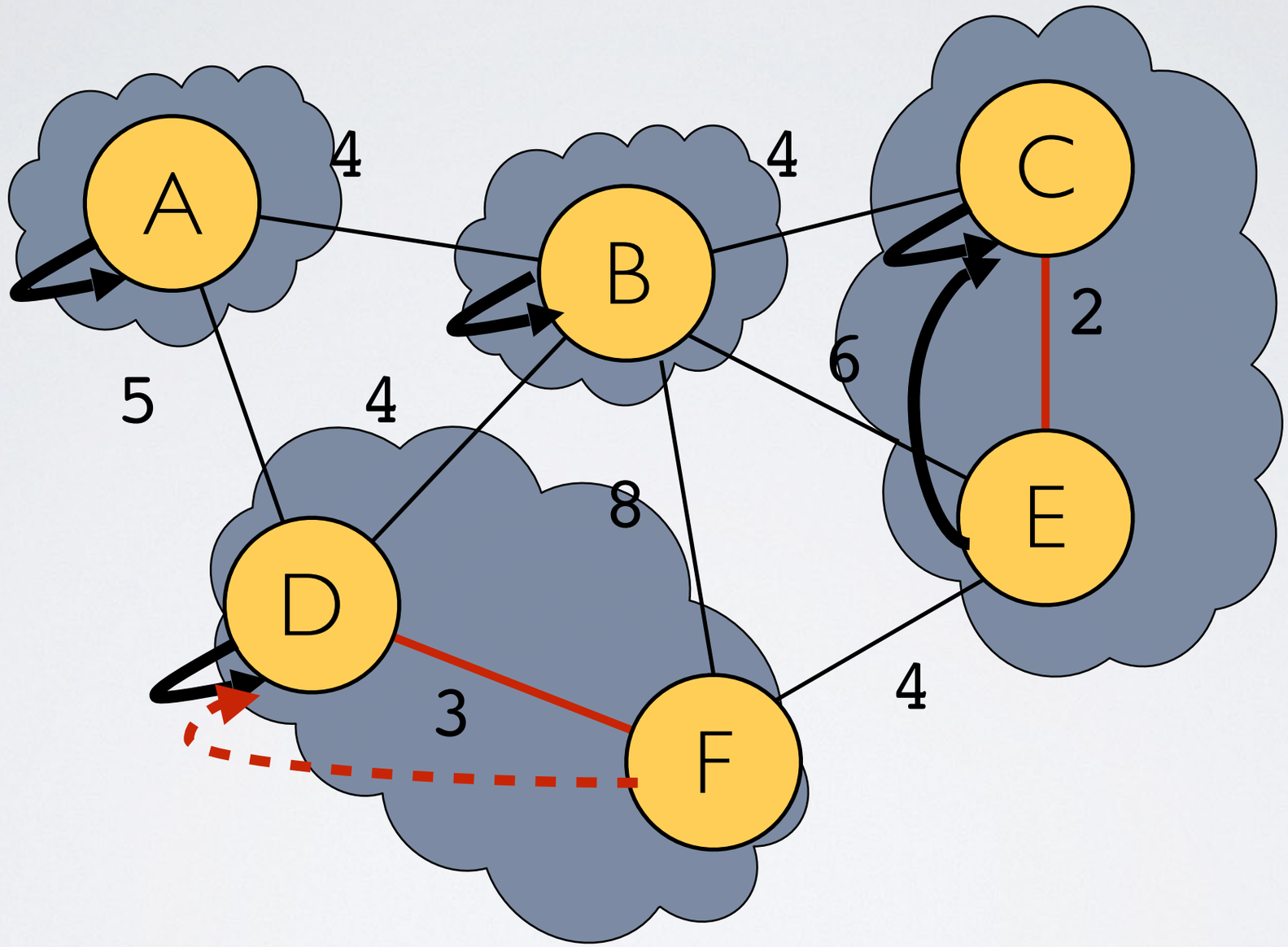
Example




```
edges = [(B,C),(E,F),(B,D),(A,B),(A,D),(B,E),(B,F)]
```

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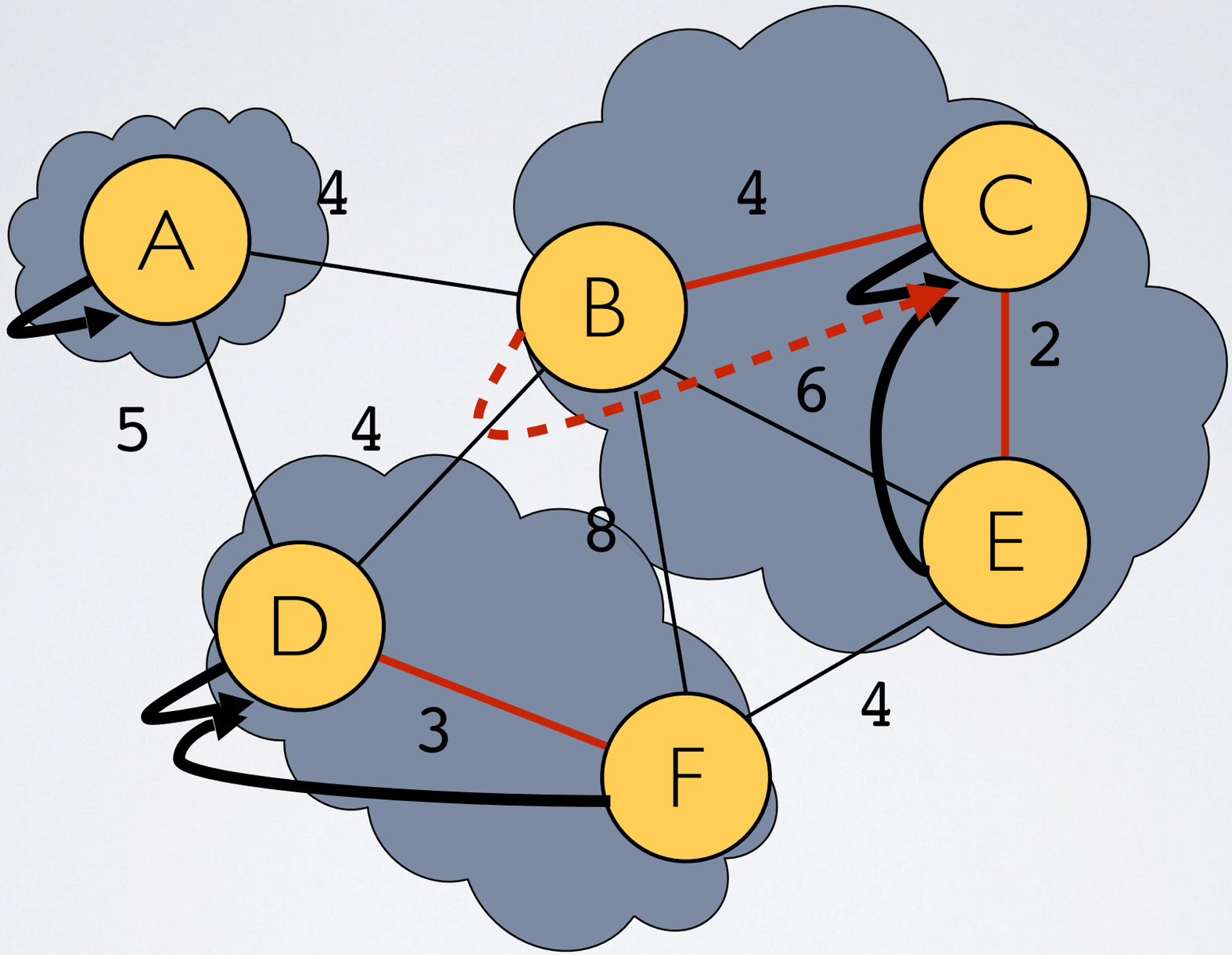
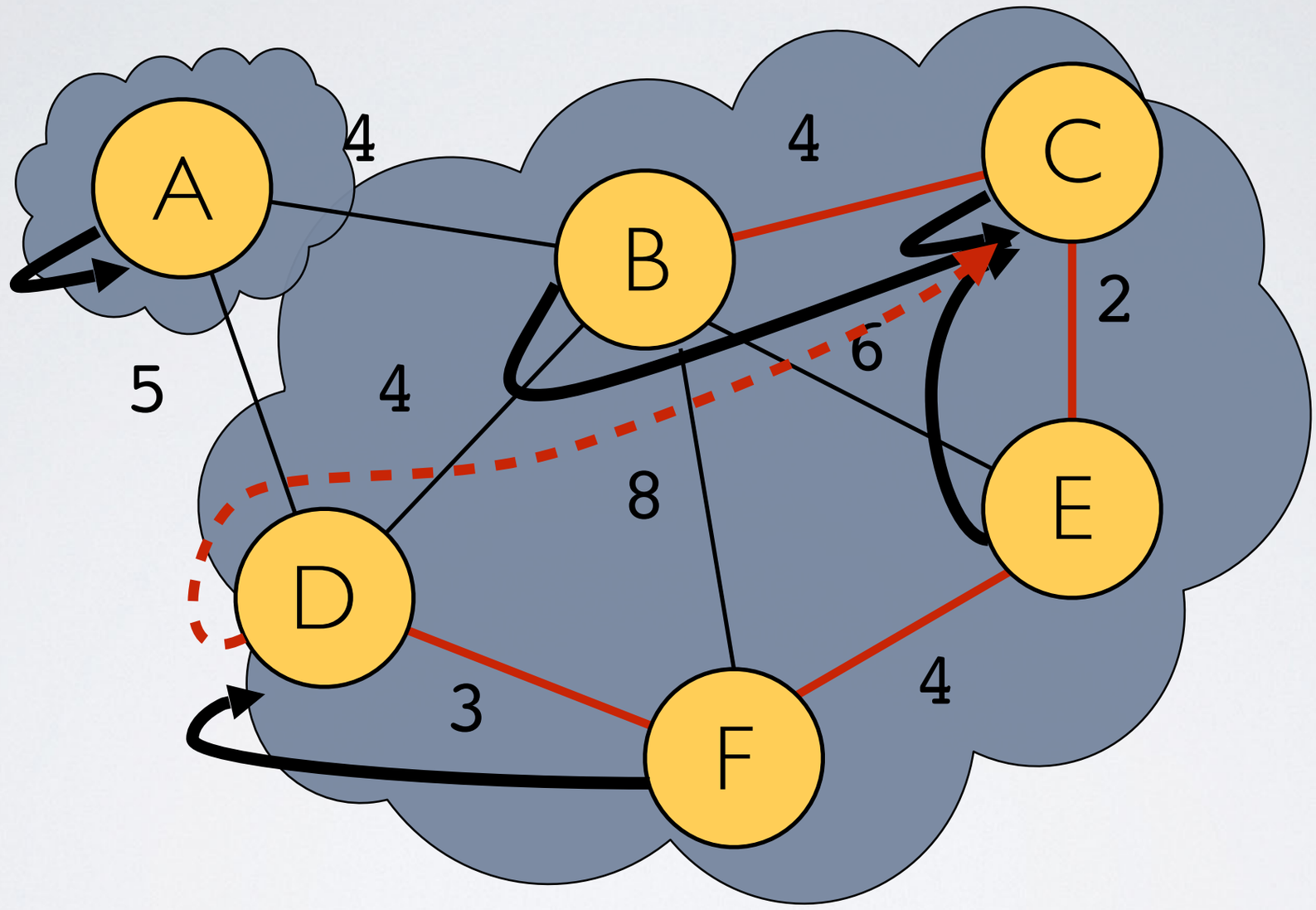


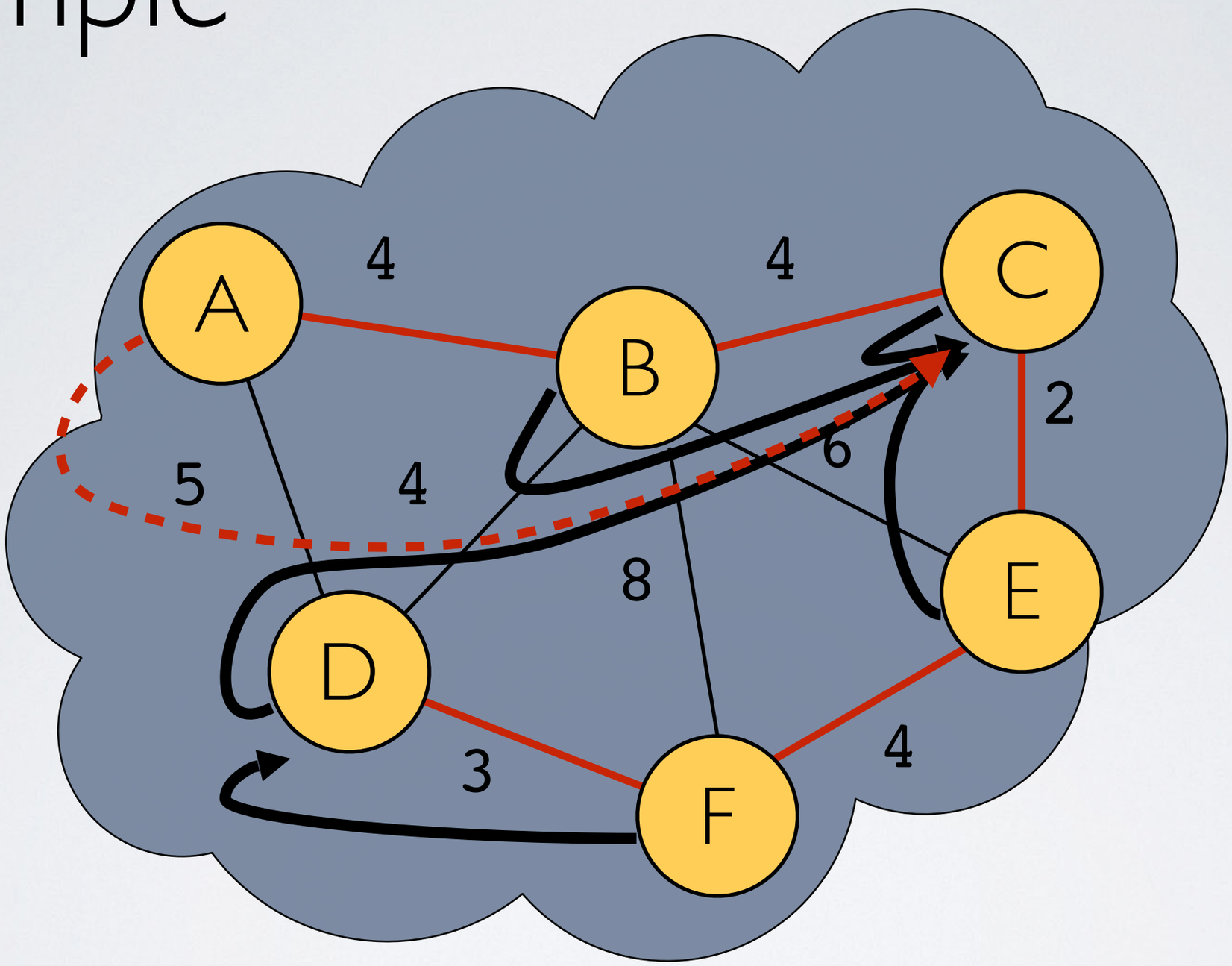

```
edges = [(E,F),(B,D),(A,B),(A,D),(B,E),(B,F)]
```

Example




```
edges = [(B,D),(A,B),(A,D),(B,E),(B,F)]
```

Example

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edges = [(A,D),(B,E),(B,F)]
```
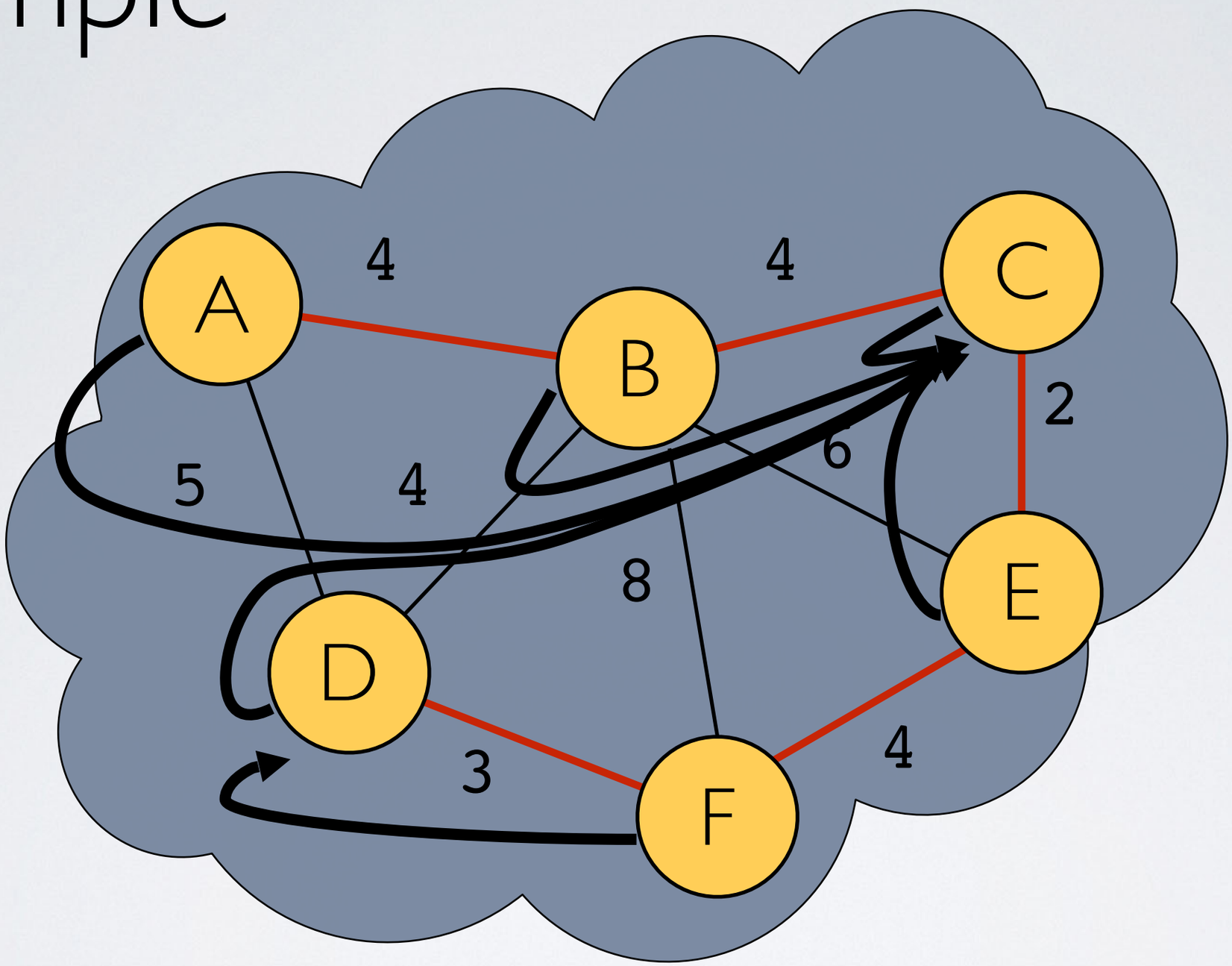

Example




```
edges = [(A,D),(B,E),(B,F)]
```

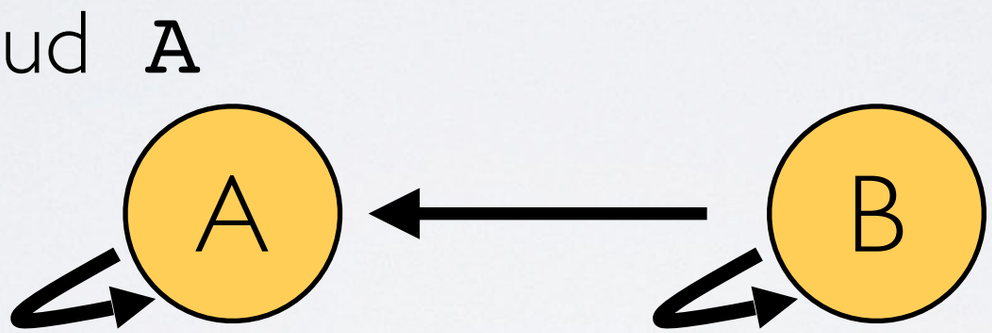
Implementing Union-Find

- At start of Kruskal
  - every node is put into own cloud

```
// Decorates every vertex with its parent ptr & rank
function makeCloud(x):
    x.parent = x
    x.rank = 0
```

Merging two Clouds/Trees

- Suppose **`A`** is in cloud/tree **`A`** and **`B`** is in cloud/tree **`B`**
- To merge the two make **`B`** point to **`A`**



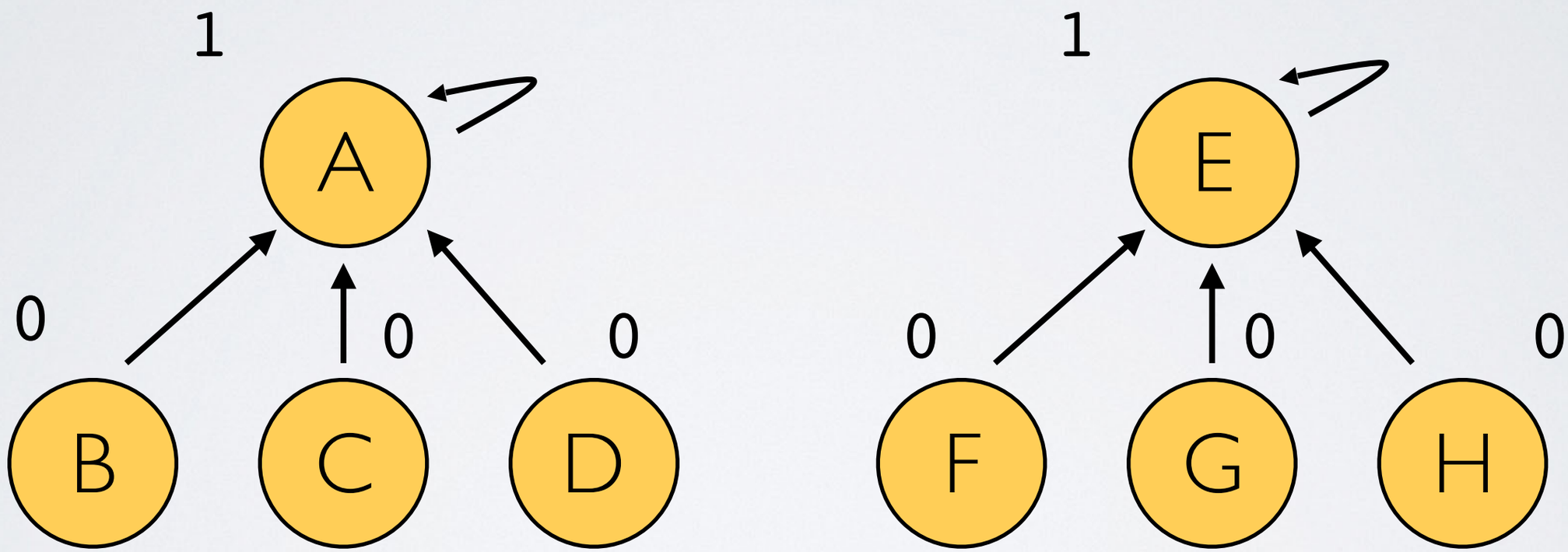
- Given two clouds which one should point to the other?
  - if roots have same rank then it doesn't matter
  - if roots have different rank then it does matter

Merging two Clouds/Trees

- If roots have different rank
  - make lower-ranked root point to higher-ranked root
  - then update rank
- How do we update ranks?
  - For clouds of size **1** root always has rank **0**
  - For clouds of size larger than **1** we increment rank only when merging clouds of same rank

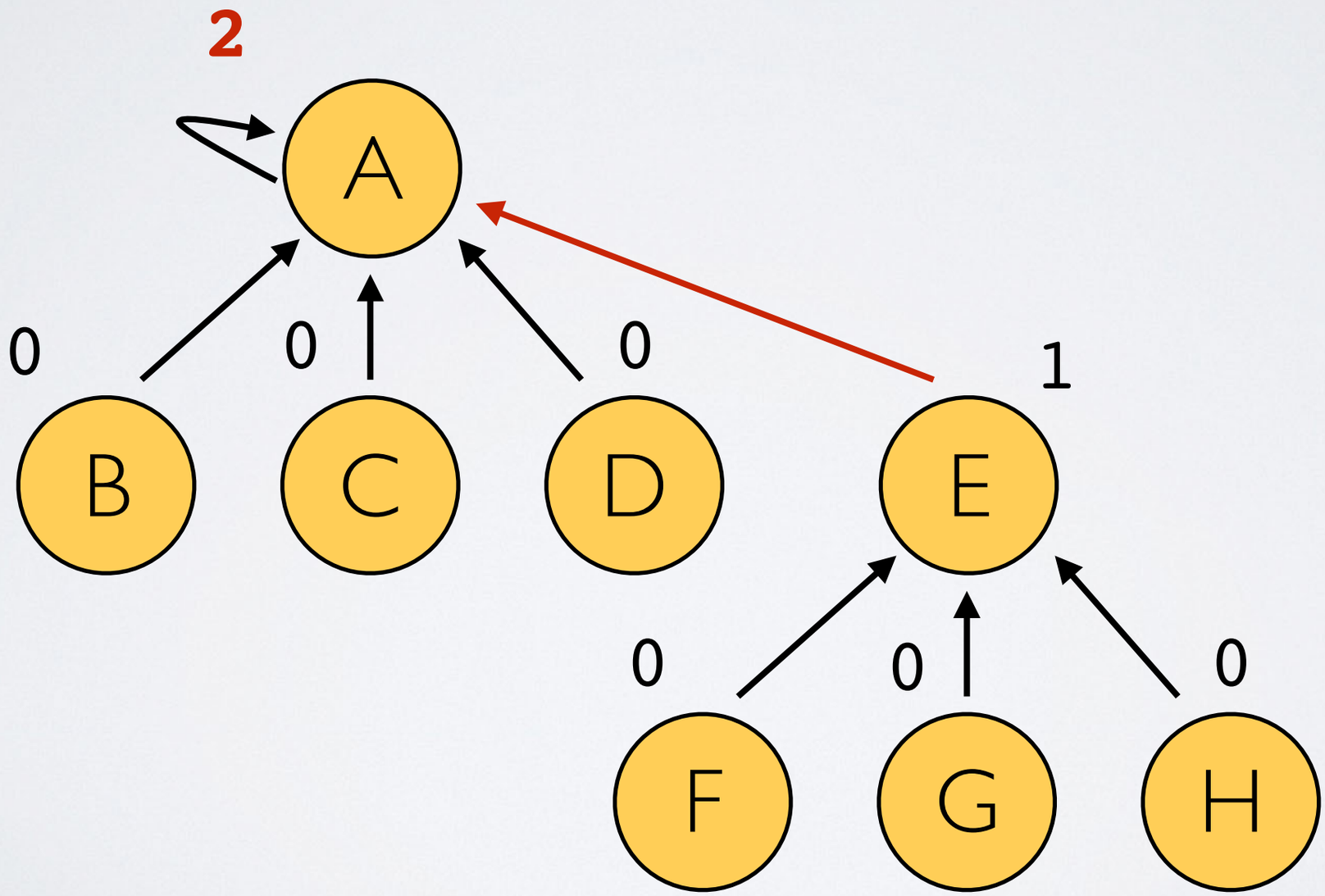
Merging two Clouds/Trees

- Merging trees with same rank



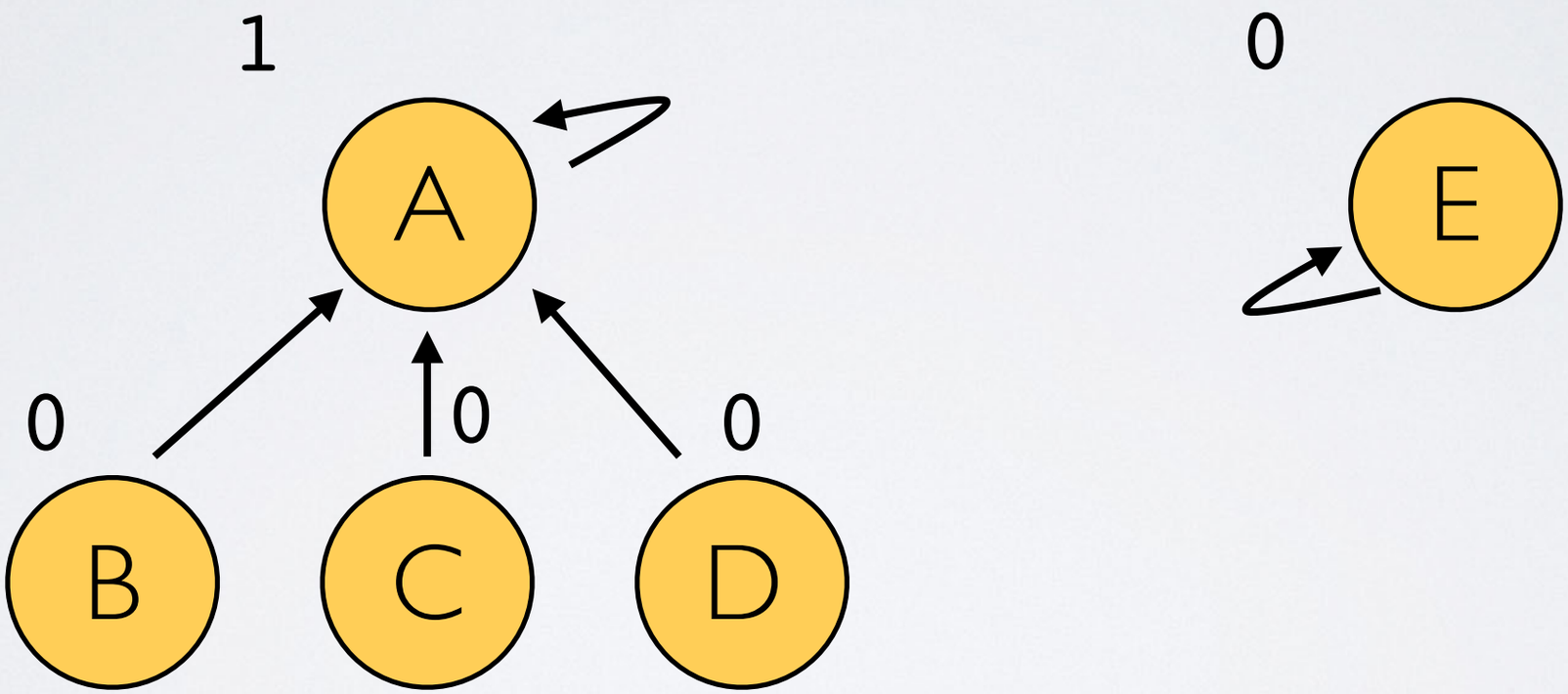
Merging two Clouds/Trees

- Merging trees with same rank



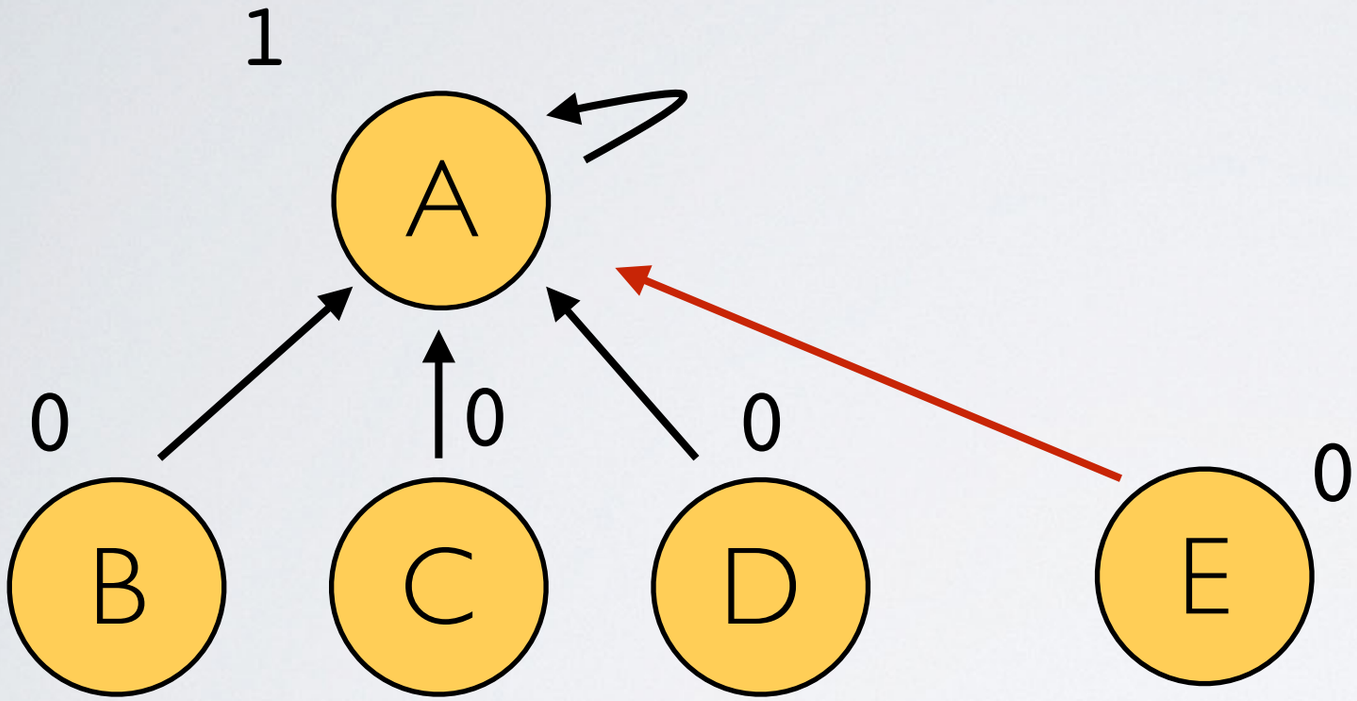
Merging two Clouds/Trees

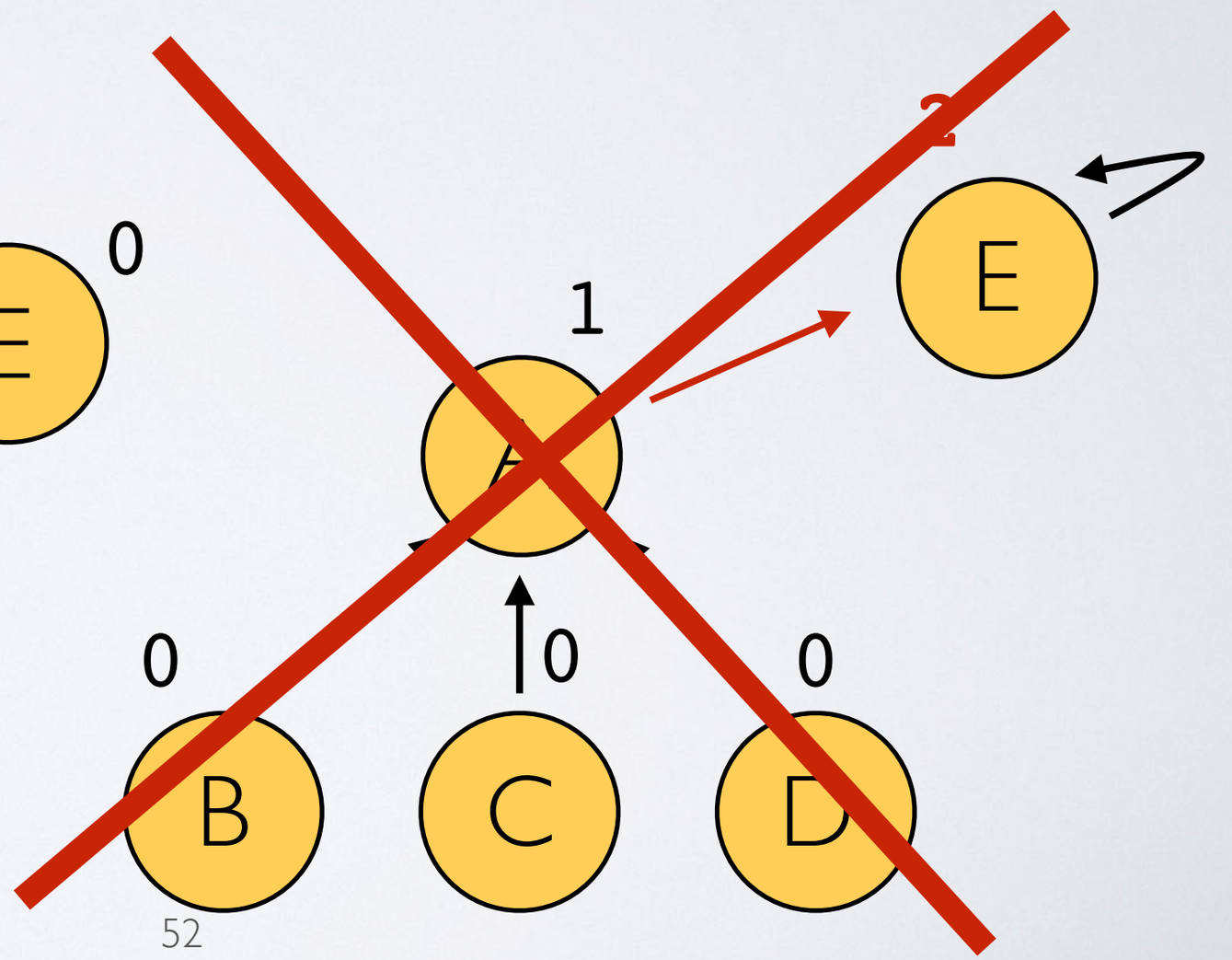
- Merging trees with different ranks



Merging two Clouds/Trees

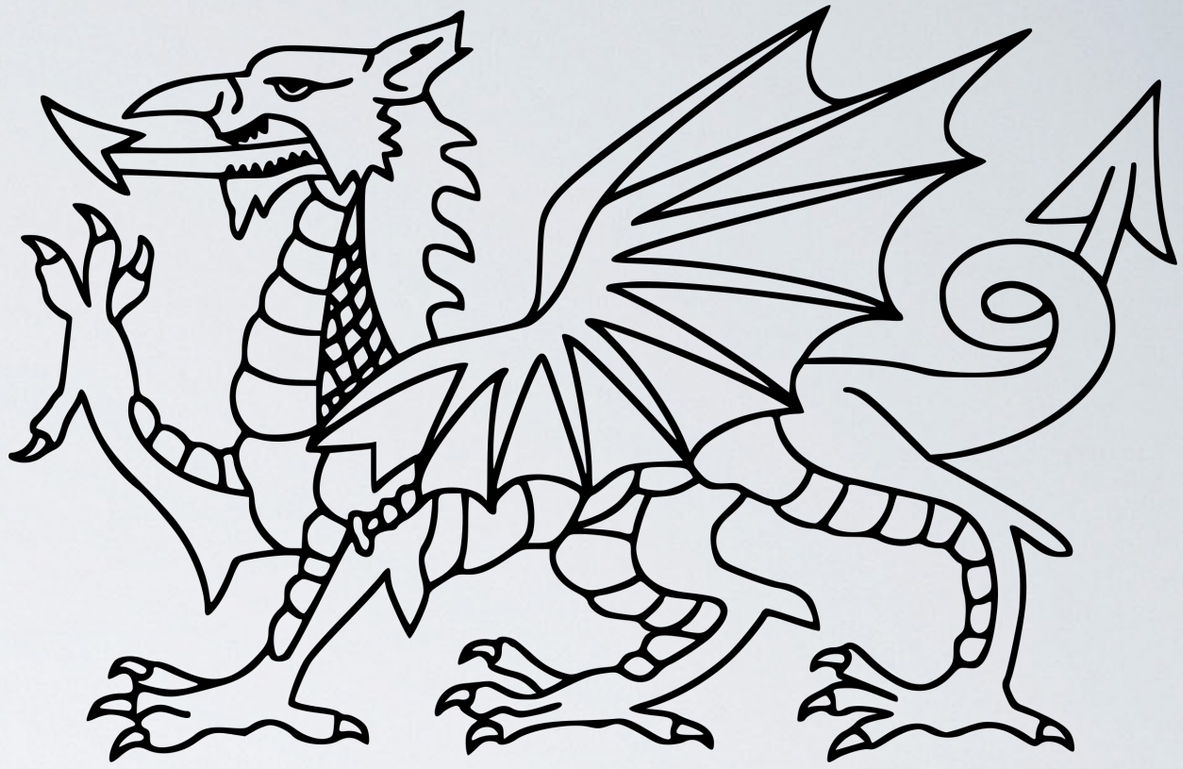
- Merging trees with different ranks





Outline

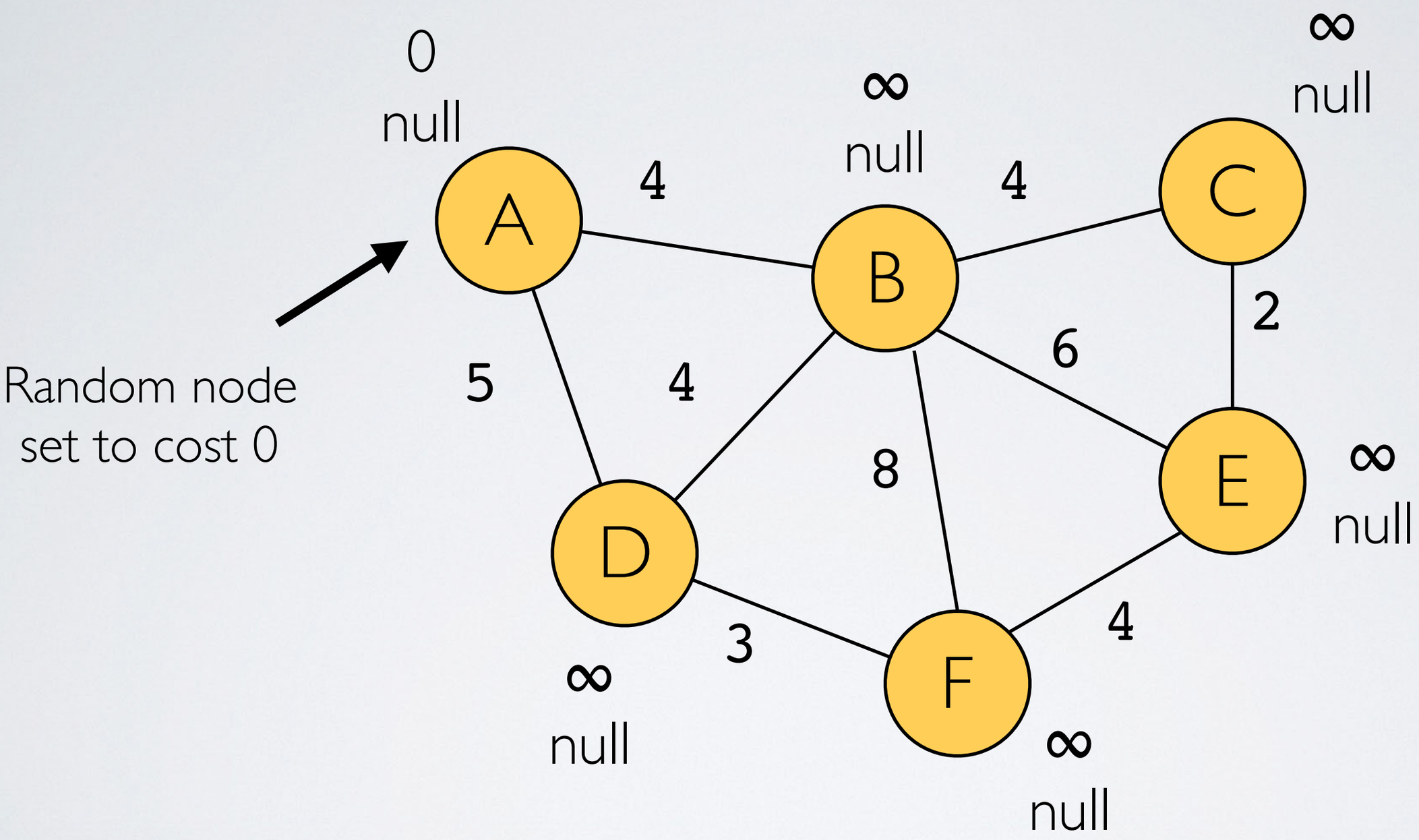
- Dynamic Programming
- PageRank
- Kruskal's Algorithm
- Prim-Jarnik Algorithm
- Functional Programming
- Hardness
- Neural Networks



Prim-Jarnik Algorithm

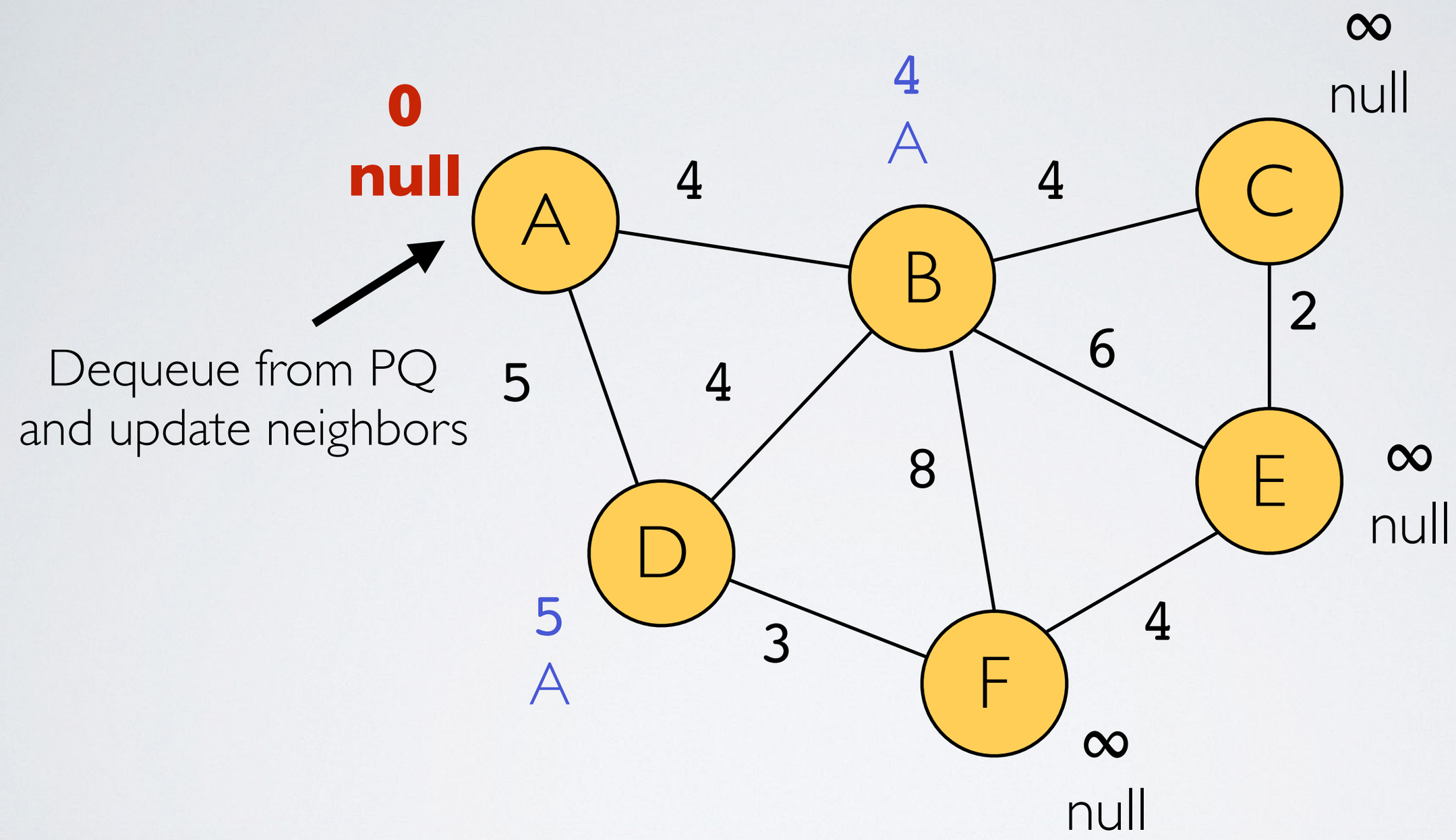
- Traverse **G** starting at any node
  - Maintain priority queue of nodes
  - set priority to weight of the edge that connects them to MST
- Un-added nodes start with priority ∞
- At each step
  - Connect the node with lowest cost
  - Update ("relax") neighbors as necessary
- Stop when all nodes added to MST

Example



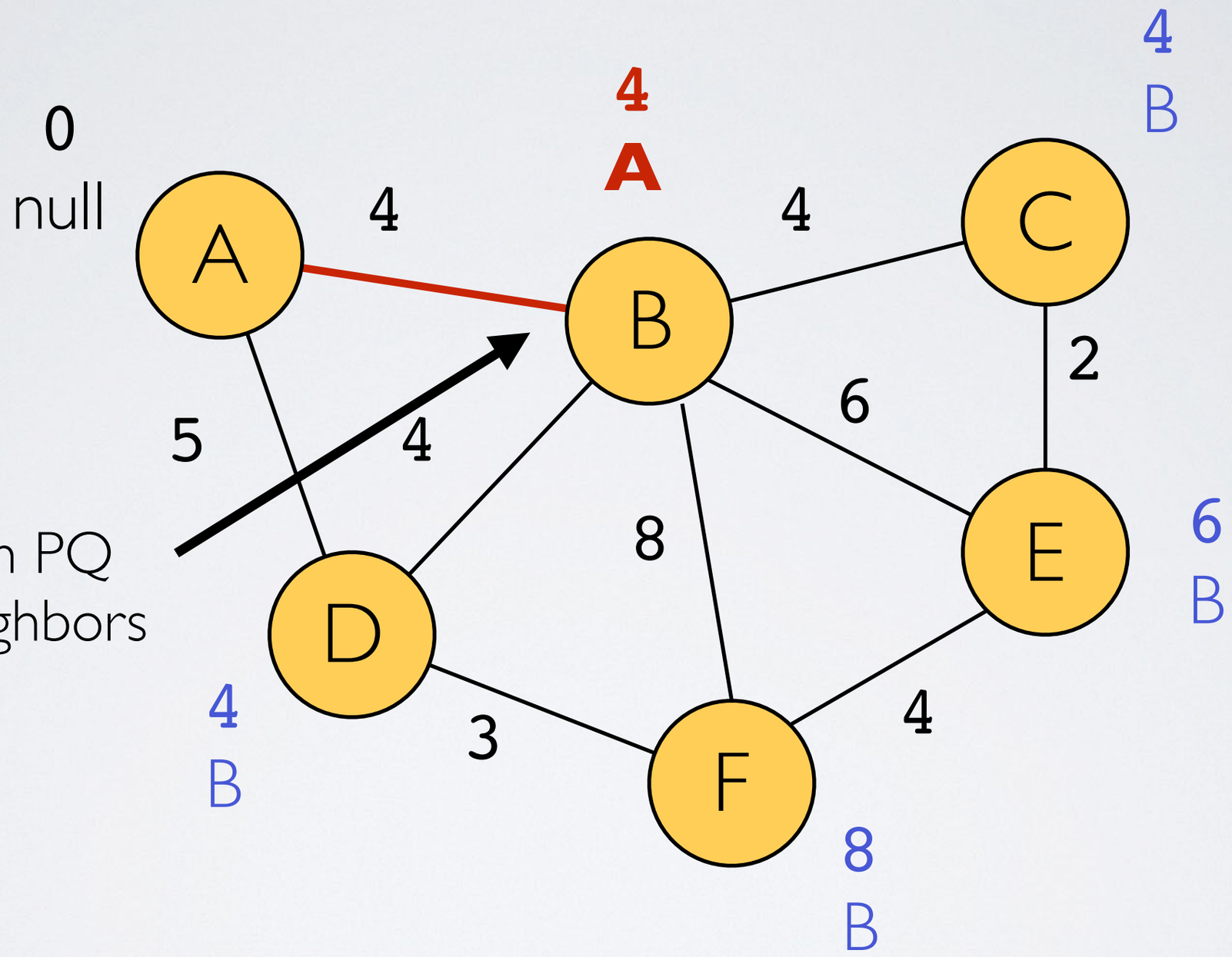

```
PQ = [(0,A),(∞,B),(∞,C),(∞,D),(∞,E),(∞,F)]
```

Example




```
PQ = [(4,B),(5,D),(∞,C),(∞,E),(∞,F)]
```

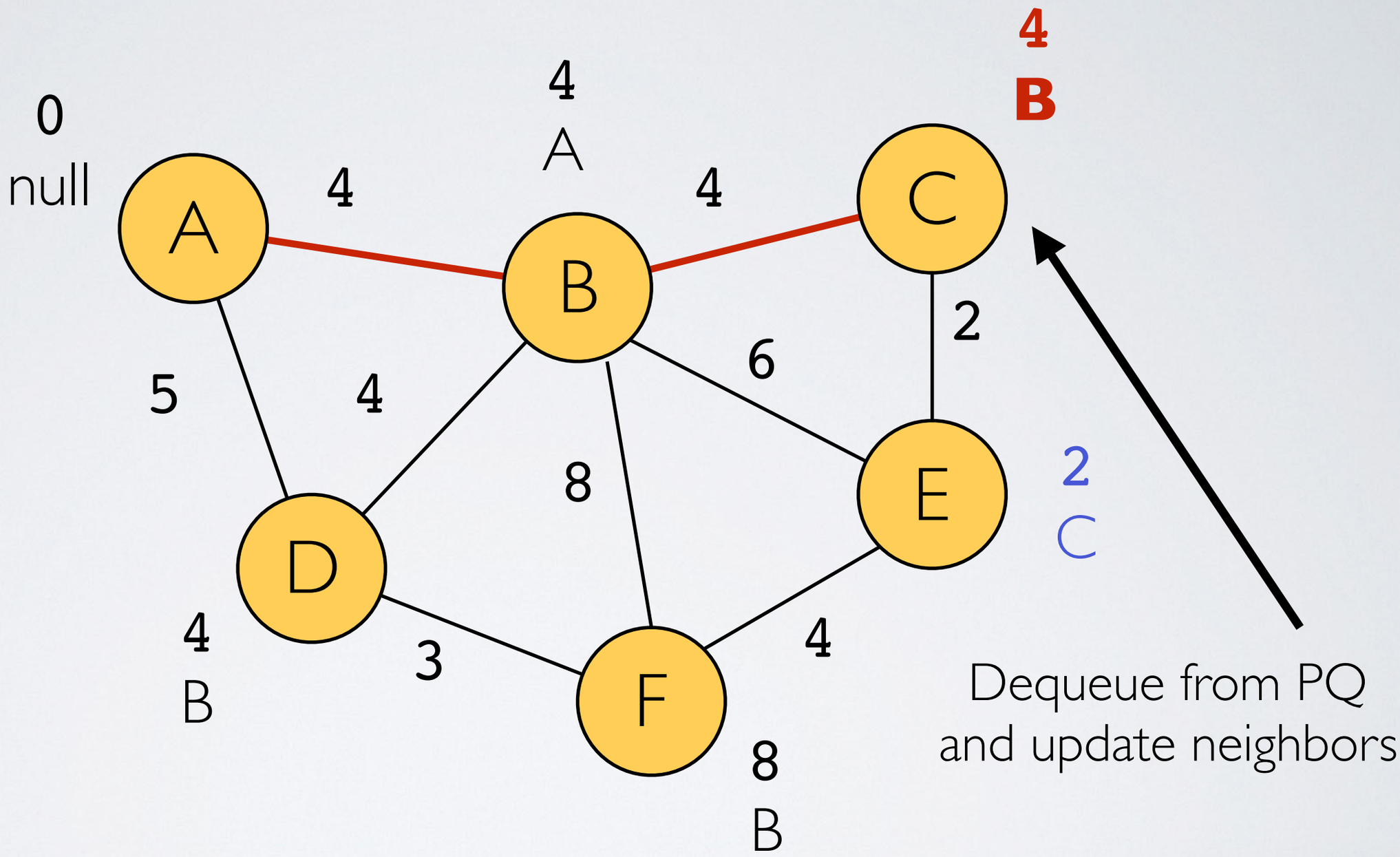
Example



Dequeue from PQ
and update neighbors

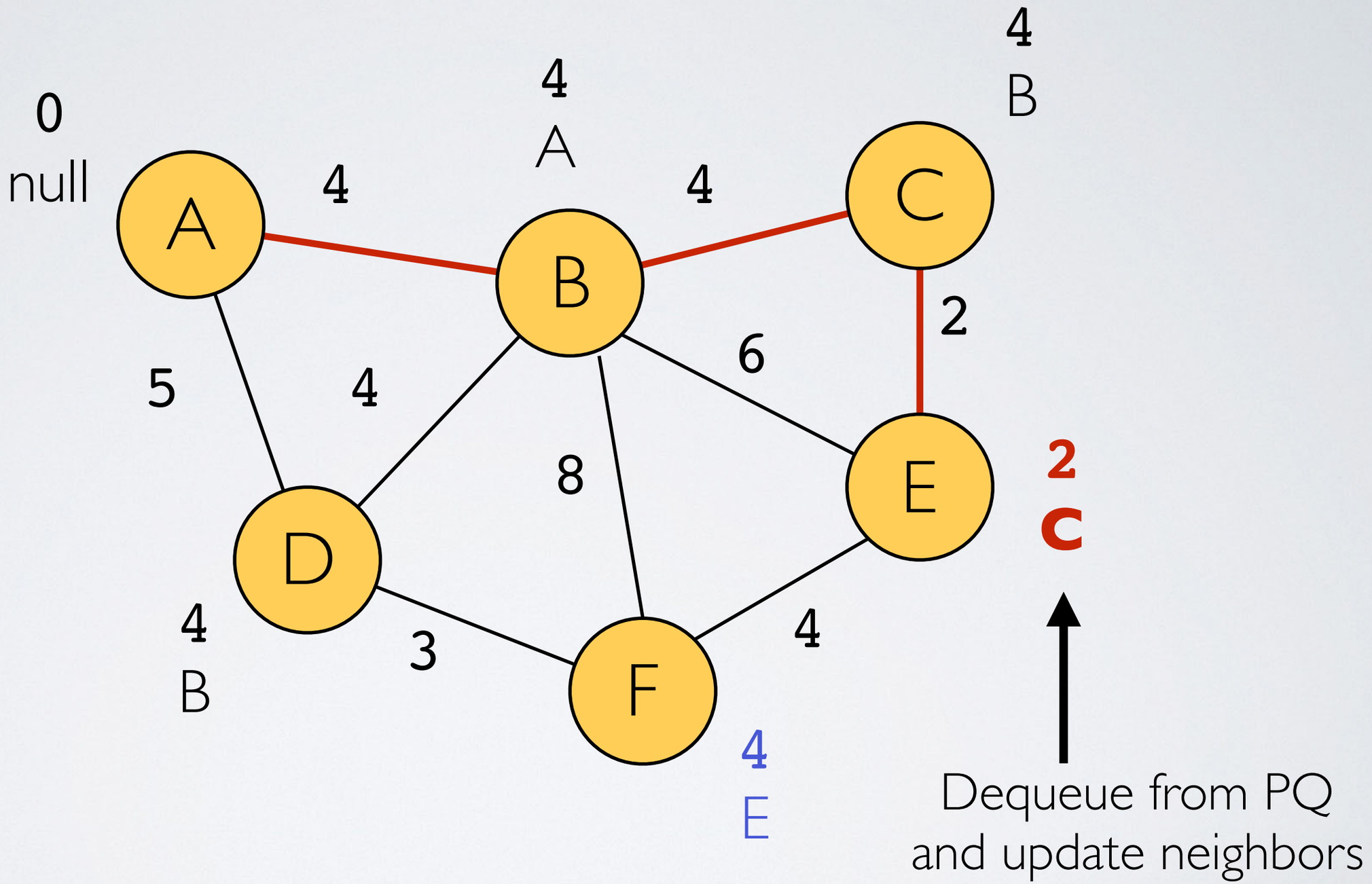
```
PQ = [(4,C),(4,D),(6,E),(8,F)]
```

Example



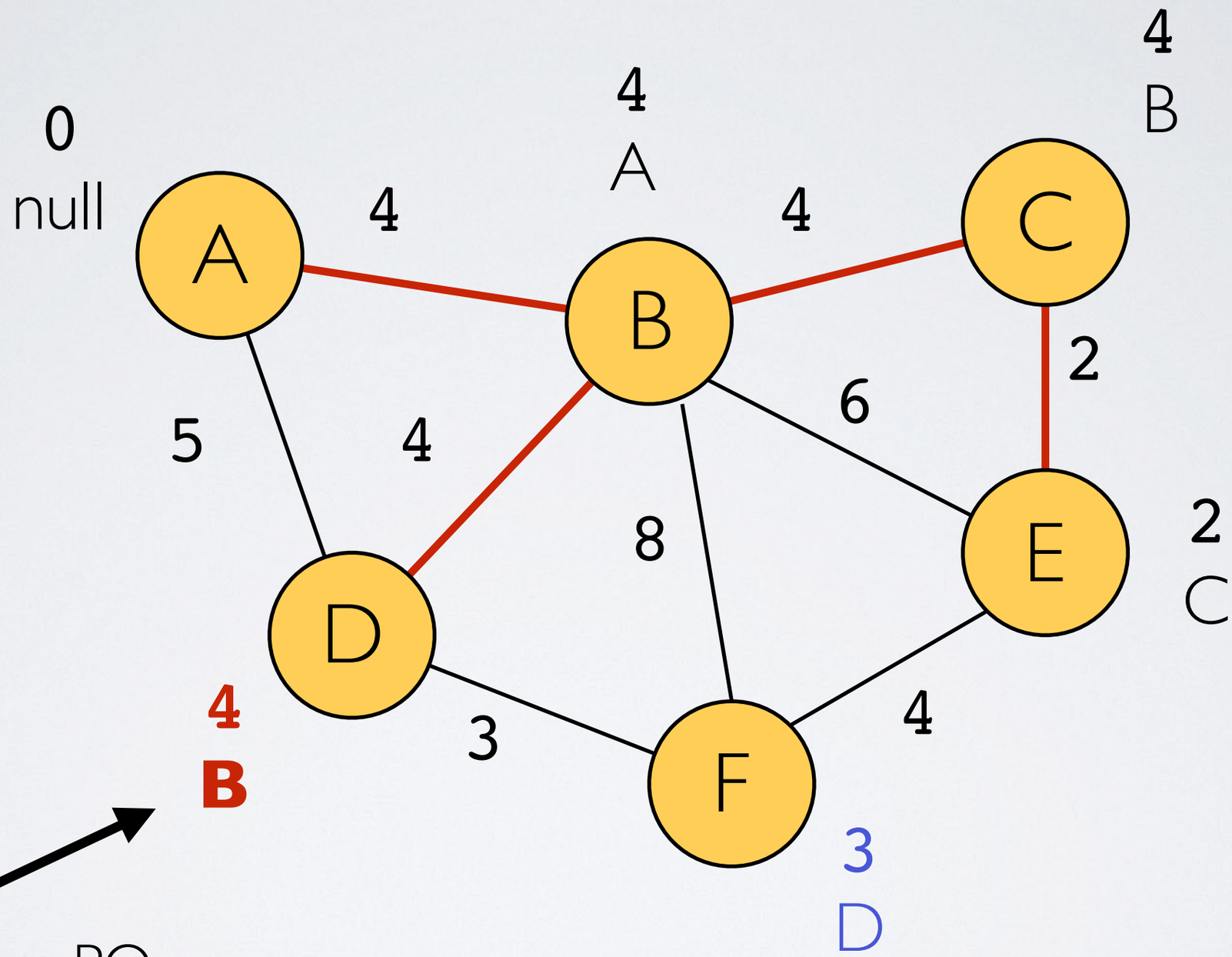

```
PQ = [(2,E),(4,D),(8,F)]
```

Example




```
PQ = [(4,D),(4,F)]
```

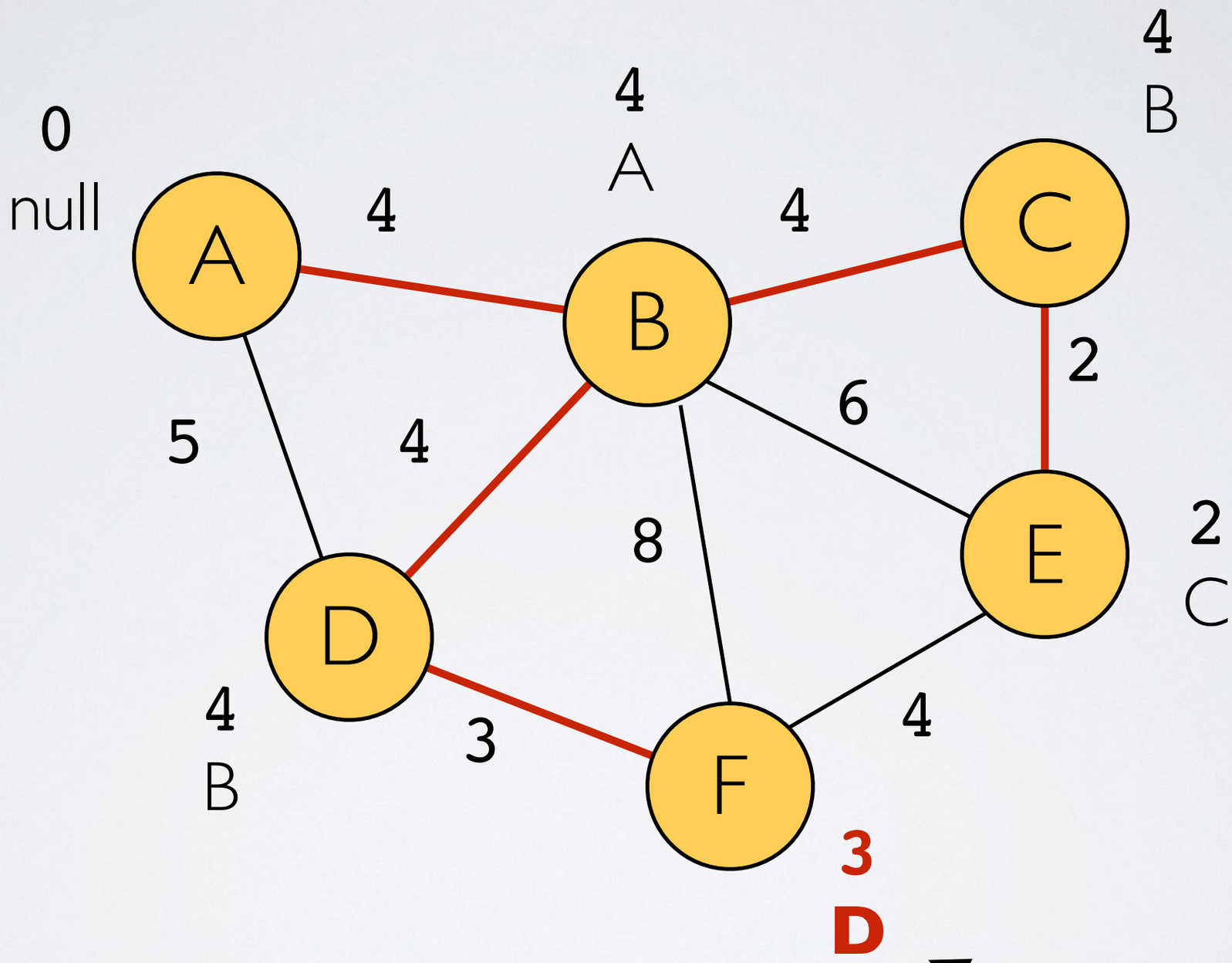
Example



Dequeue from PQ
and update neighbors

```
PQ = [(3,F)]
```

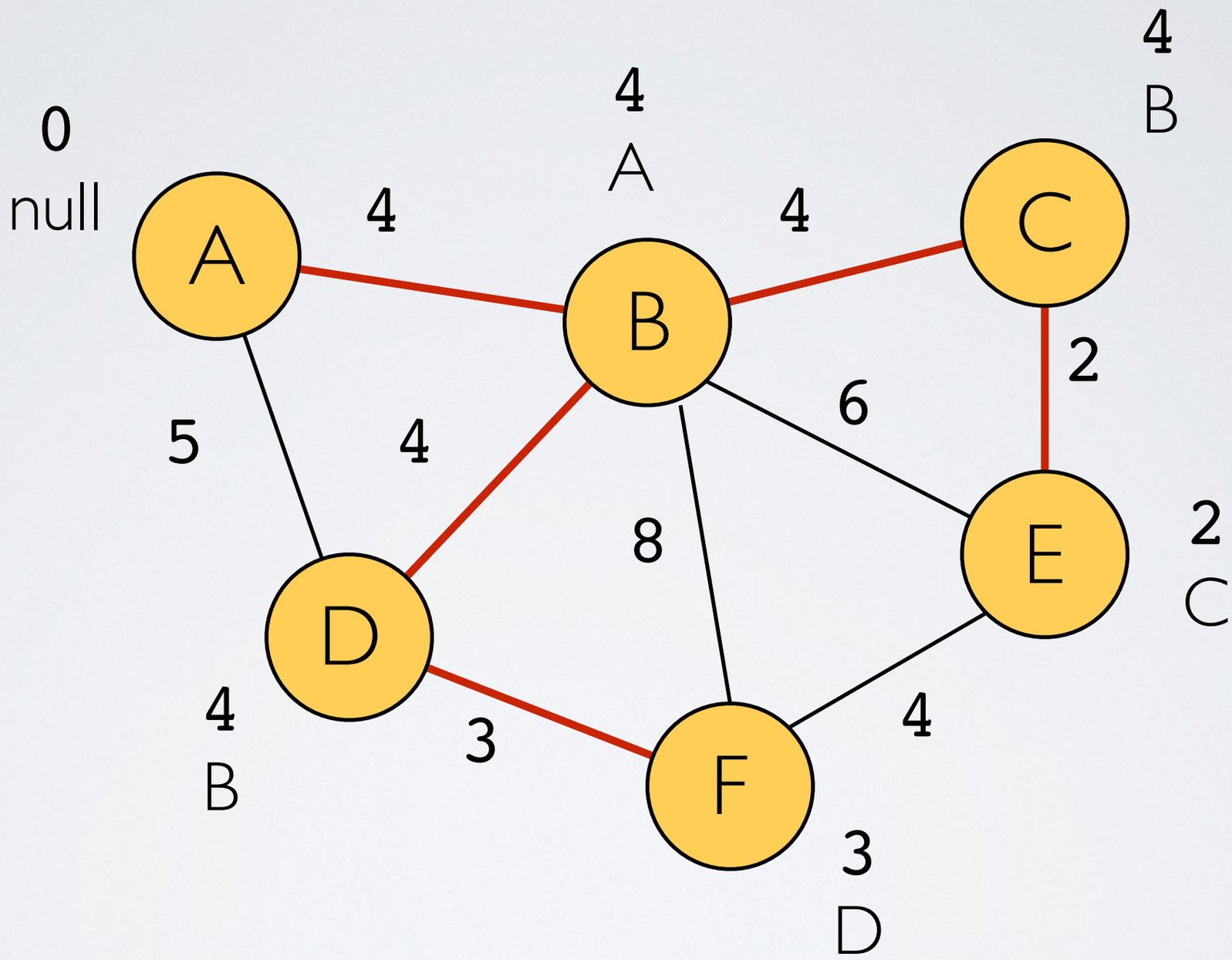
Example



PQ = [ ]

Dequeue from PQ and update neighbors

Example



Pseudo-code

```
function prim(G):
   // Input: weighted, undirected graph G with vertices V
   // Output: list of edges in MST
   for all v in V:
      v.cost = ∞
      v.prev = null
   s = a random v in V // pick a random source s
   s = 0
   MST = []
   PQ = PriorityQueue(V) // priorities will be v.cost values
   while PQ is not empty:
      v = PQ.removeMin()
      if v.prev != null:
         MST.append((v, v.prev))
      for all incident edges (v,u) of v such that u is in PQ:
         if u.cost > (v,u).weight:
            u.cost = (v,u).weight
            u.prev = v
            PQ.decreaseKey(u, u.cost)
return MST
```

Runtime Analysis

- Decorating nodes with distance and previous pointers is $O(|V|)$
- Putting nodes in PQ is $O(|V|\log|V|)$ (really $O(|V|)$ since ∞ priorities)
- While loop runs $|V|$ times
  - removing vertex from PQ is $O(\log|V|)$
  - So $O(|V|\log|V|)$
- For loop (in while loop) runs $|E|$ times ***in total***
  - Replacing vertex's key in the PQ is $\log|V|$
  - So $O(|E|\log|V|)$
- Overall runtime
  - $O(|V| + |V|\log|V| + |V|\log|V| + |E|\log|V|)$
    - $= O((|E| + |V|)\log|V|)$

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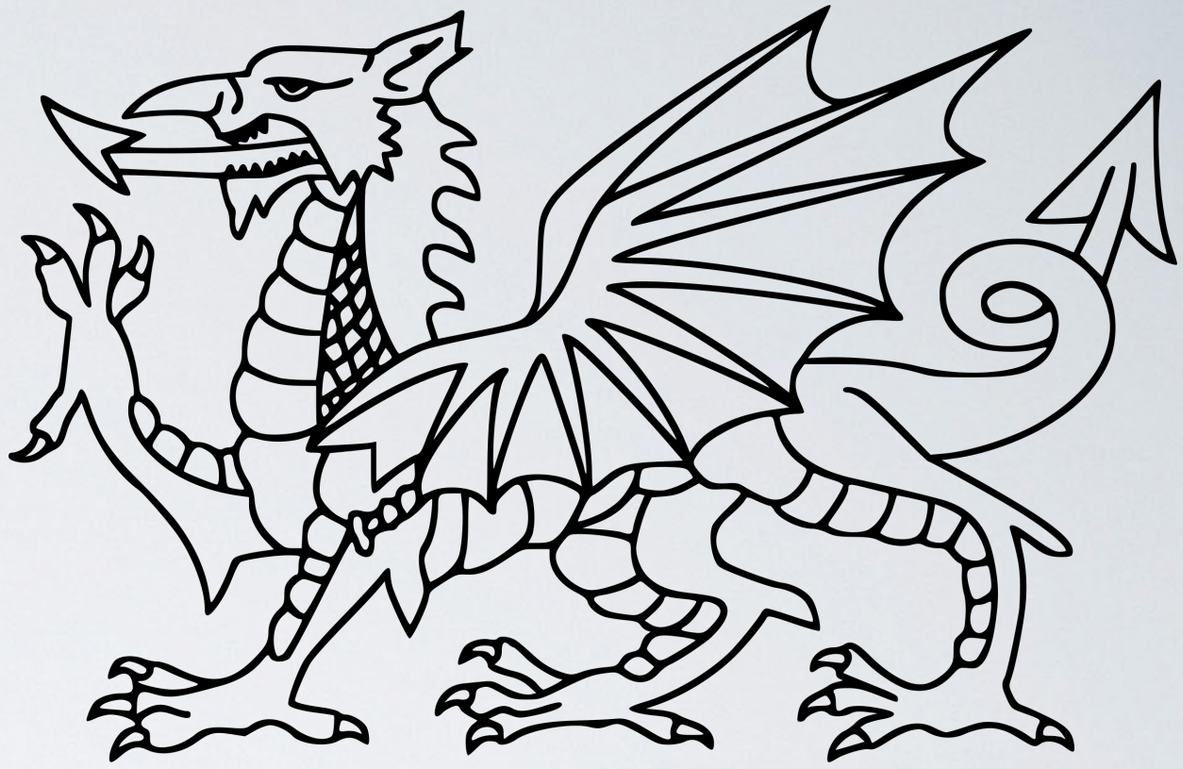


Functional Programming

- Anonymous functions
  - `lambda x: x + 1`
  - `lambda x: 100*x`
- Higher order functions
  - map: applies a function to a list of elements
  - reduce: applies a binary function to pairs of elements from a list, and accumulates the results in an accumulator

Map

```
map(lambda x: x-2, [11,9,24,-5,34,4])
```

Reduce

- `reduce(lambda x, y: x+y, [1,2,3], 0)`
  - `acc = 0 + 1 = 1`
    - `acc = 1 + 2 = 3`
      - `acc = 3 + 3 = 6`

Practice Problems

- function that turns list of nouns into adverbs
  - `loud` ⟶ `loudly`
- function that sums total length of a list of strings
  - `["hi", "cs16"]` ⟶ `6`
- function that counts number of times "dog" appears in a list
- function that removes numbers less than `10` from a list of ints

Practice Problem Answers

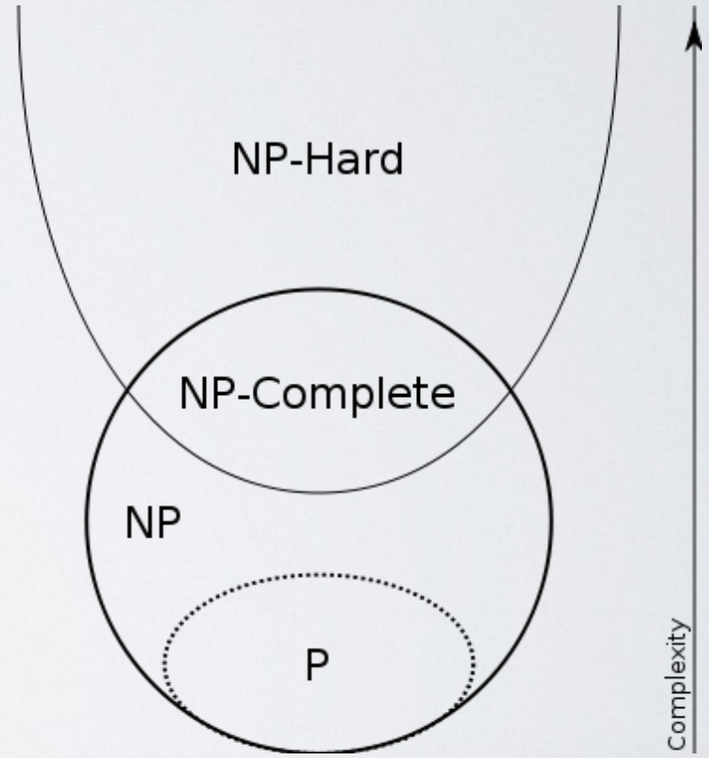
- `map(lambda s: s+”ly”, list)`
- `reduce(lambda acc, s: acc+len(s), list, 0)`
- `reduce(lambda acc, s: acc+1 if s == "dog" else acc, list, 0)`
- `reduce(lambda acc, x: acc+[x] if x > 10 else acc, list, [])`

Hardness

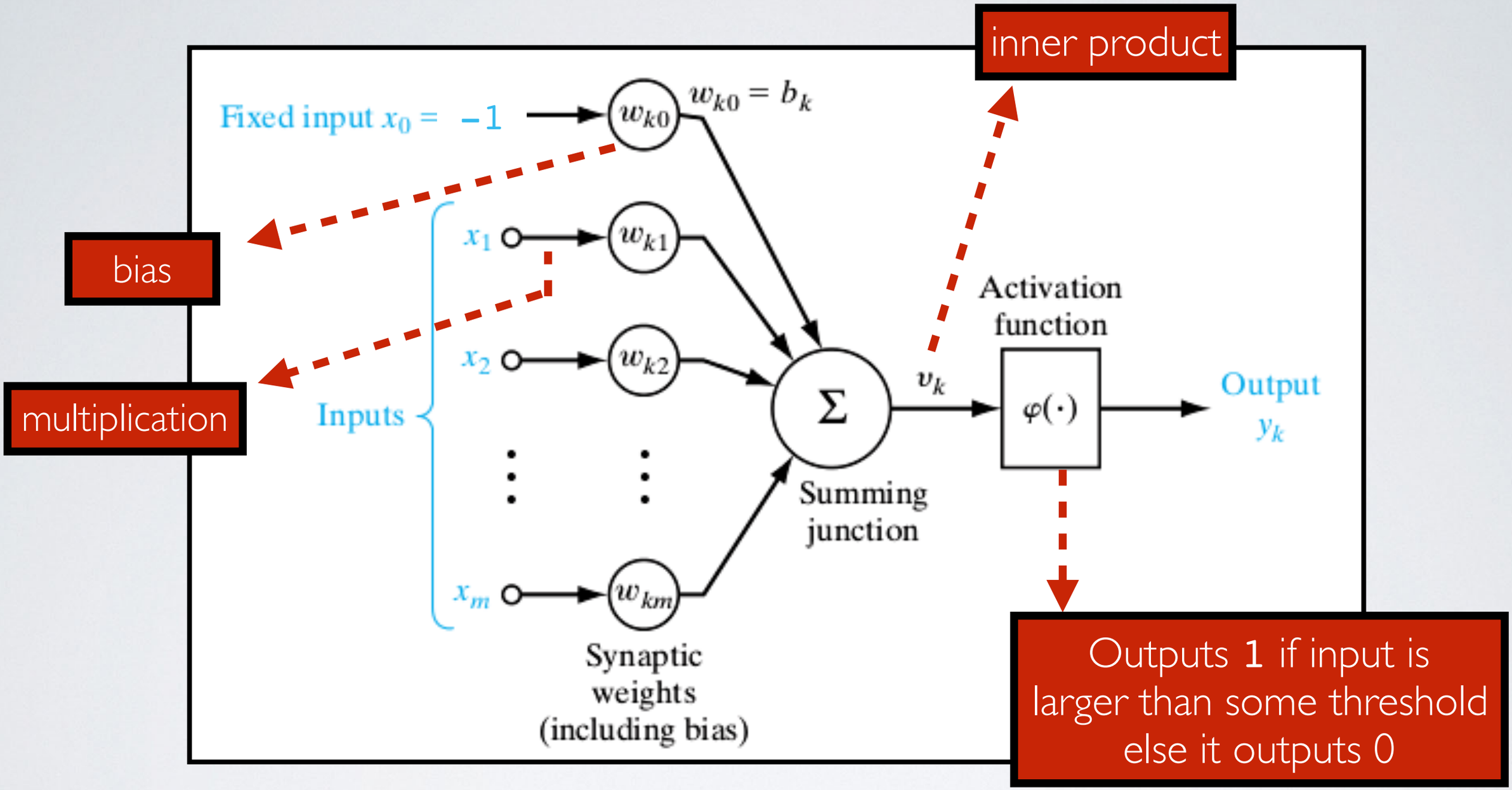
- Hardness of a problem is defined by the runtime of the best known solution
- Hardness of comparative sorting?
  - `O(nlog n)`
    - There are $O(2^n)$ sorting algorithms, but the *sorting problem* is not hard
- Problems that have polynomial time solutions are *tractable*
  - $O(n), O(n^2), O(n^{500}),\ldots$
- Problems with super-polynomial time solutions are *intractable*
  - $O(n!), O(2^n), O(n^n),\ldots$

Categories of Hardness

- **NP**: problems whose solutions can be *verified* in poly-time
- **P**: problems whose solutions can be *found* in poly-time
- **NP-Complete:** the hardest problems in NP
  - Solutions can be verified in poly time
  - if poly-time solution exists for problem, then a poly-time solution exists for all problems in NP
  - not known whether there exist any polynomial time algorithms to solve them
- **NP-Hard:** at least as hard as NP-complete
  - Don't know if solutions can be verified in poly time
  - if poly-time solution exists for problem, then a poly-time solution exists for all problems in NP

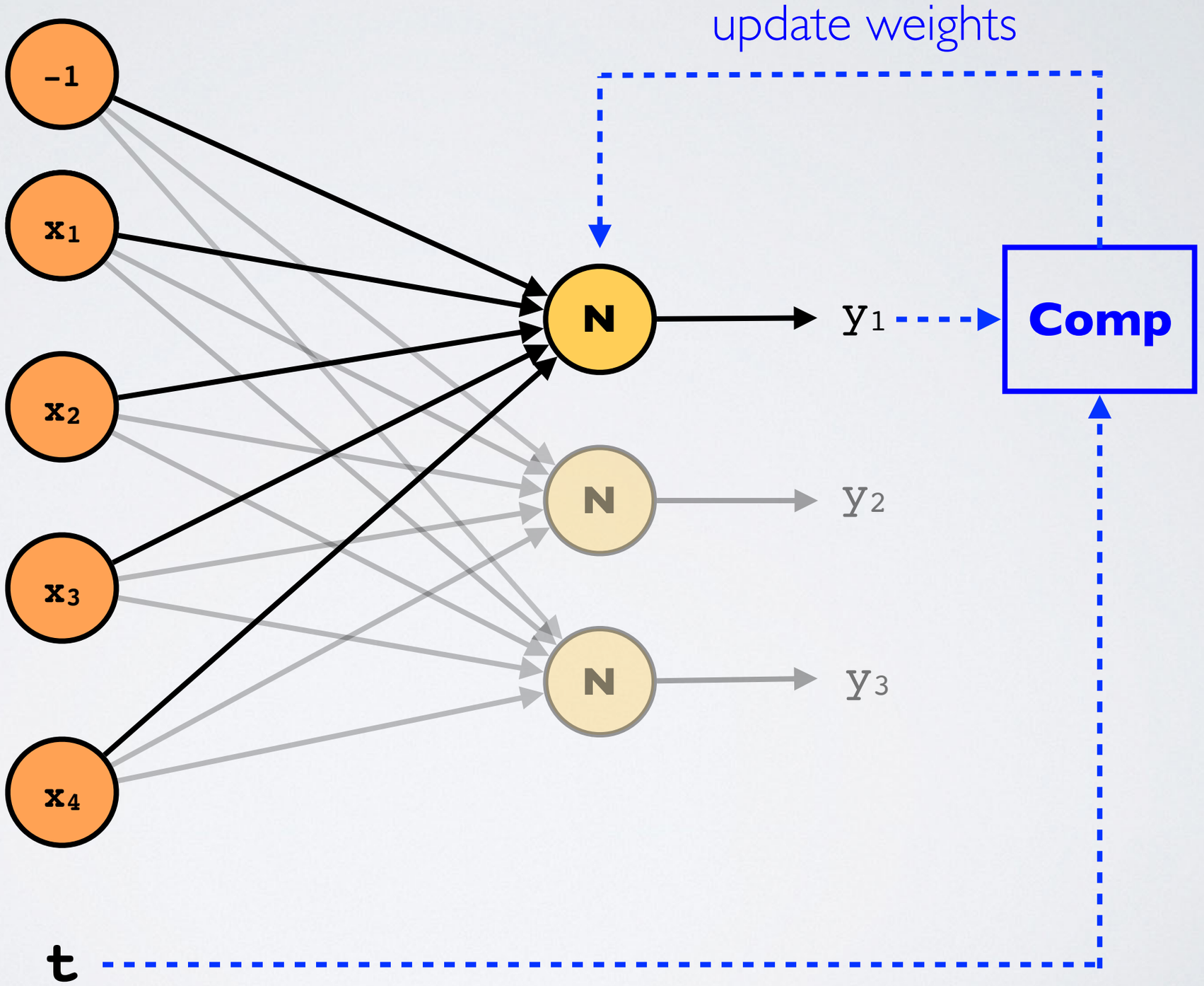


Artificial Neuron

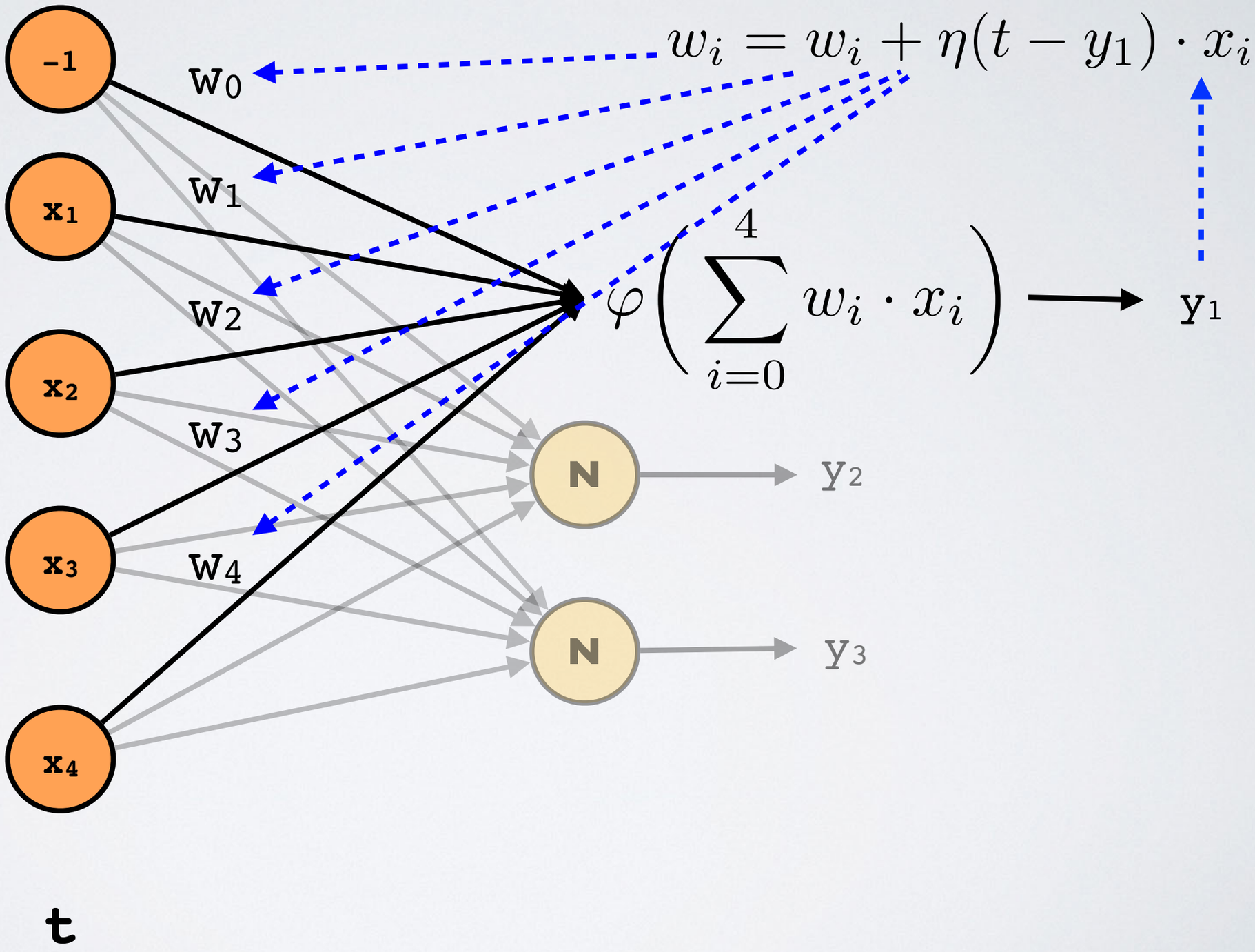


$$y = \varphi\left(\sum_{i=1}^{m} x_i \cdot w_i - b\right) = \varphi(\mathbf{x}^\intercal \cdot \mathbf{w} - b) = \begin{cases} 0, & \text{if } \mathbf{x}^\intercal \cdot \mathbf{w} - b \leq 0 \\ 1, & \text{if } \mathbf{x}^\intercal \cdot \mathbf{w} - b > 0 \end{cases}$$

Perceptron Network



Perceptron Network



Multi-Layer Perceptron