Midterm Review

CS16: Introduction to Data Structures & Algorithms

Spring 2019

What is Running Time?

Worst-case running time
=
T(n): *Number of elementary operations*
on worst-case input
as a function of input size n

Q: how do we compare running times?

n	$\log n$	n	$n \log n$	n^2	n^3	2^n
8	3	8	24	64	512	256
16	4	16	64	256	4,096	65,536
32	5	32	160	1,024	32,768	4,294,967,296
64	6	64	384	4,096	262,144	1.84×10^{19}
128	7	128	896	16,384	2,097,152	3.40×10^{38}
256	8	256	2,048	65,536	16,777,216	1.15×10^{77}
512	9	512	4,608	262,144	134,217,728	1.34×10^{154}

Comparing Running Times

Comparing running times
=
$T_A(n)$ is better than $T_B(n)$ if
$T_A(n)$ grows slower than $T_B(n)$

Running Times

Constant

independent of input size

Linear

depends on input size

Quadratic

depends on square of input size

Running Times

O(1)
independent of input size

O(n)
depends on input size

O(n²)
depends on square of input size

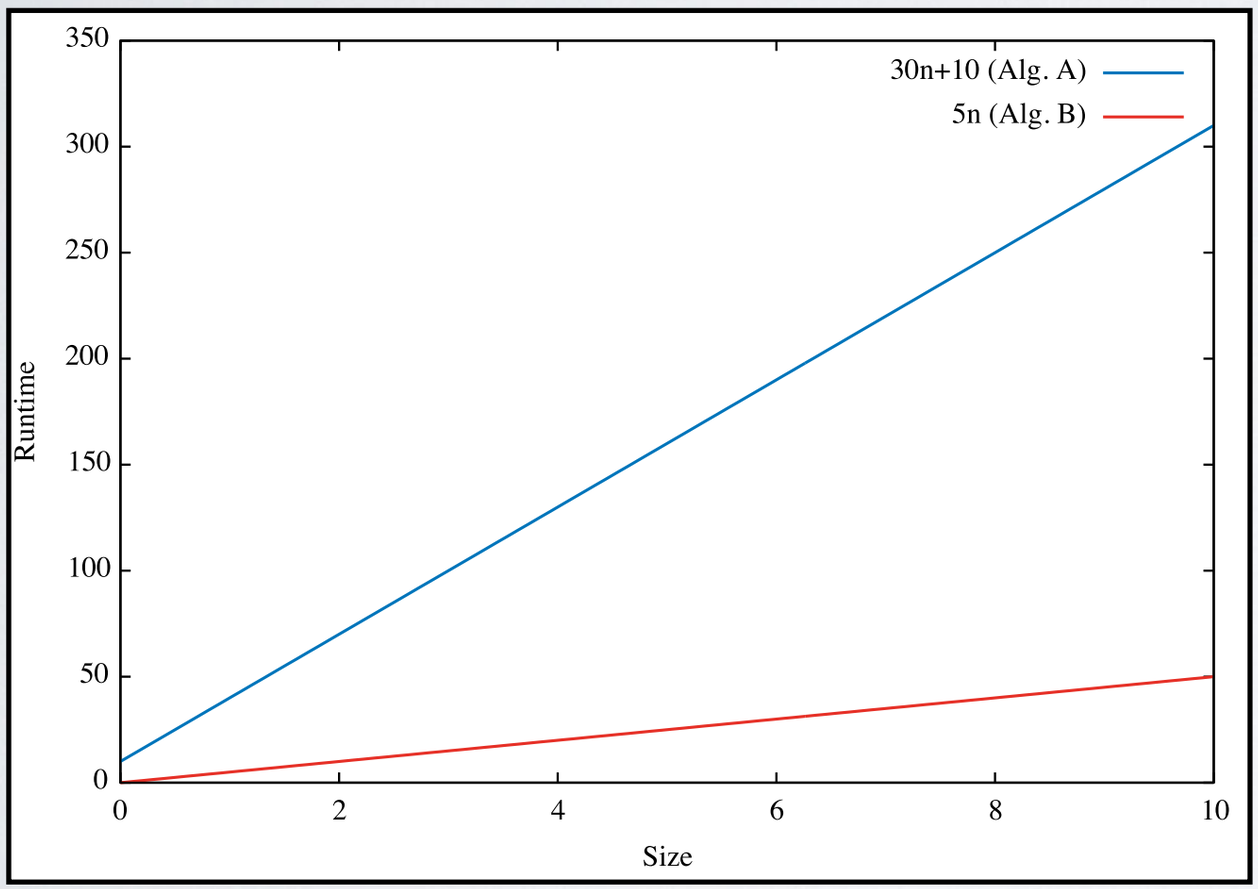
O(n³)
depends on cube of input size

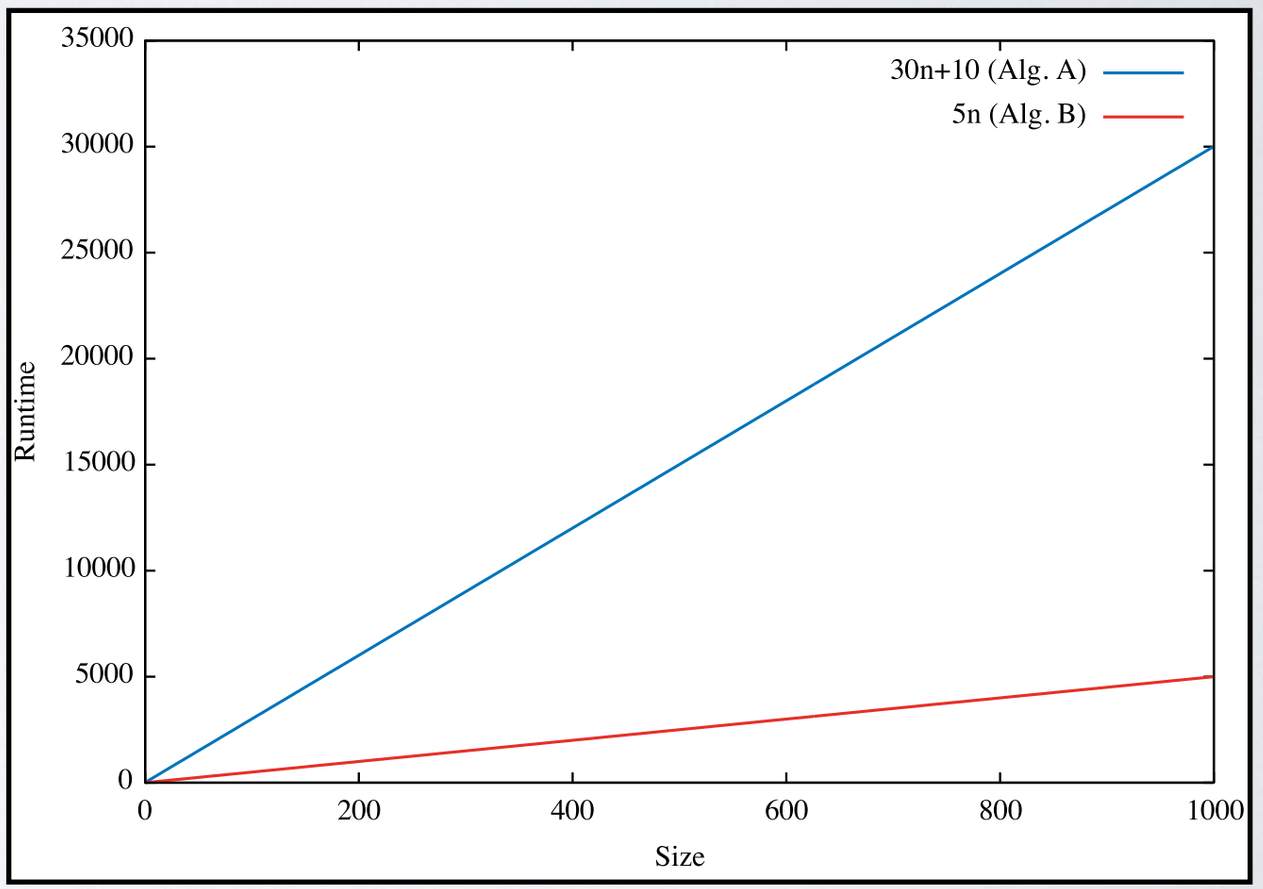
O(n⁷⁰)
depends on 70th power of input size

O(2ⁿ)
exponential in input size

Which Algorithm is Better?

- Algorithm A takes $T_A(n)=30n+10$ ops
- Algorithm B takes $T_B(n)=5n$ ops

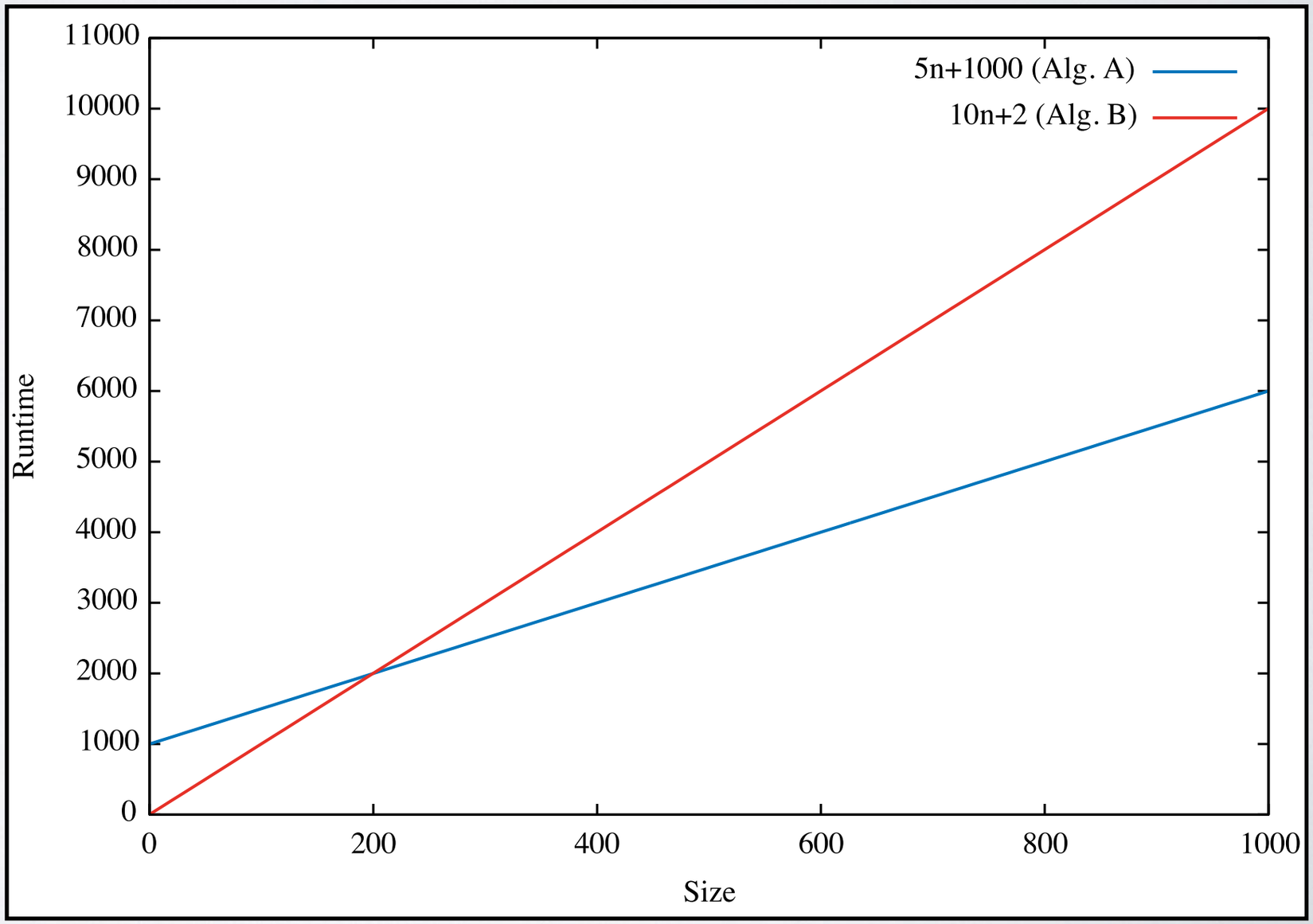




Which Algorithm is Better?

- Alg A takes $T_A(n)=5n+1000$ ops
- Alg B takes $T_B(n)=10n+2$ ops
- It depends on n

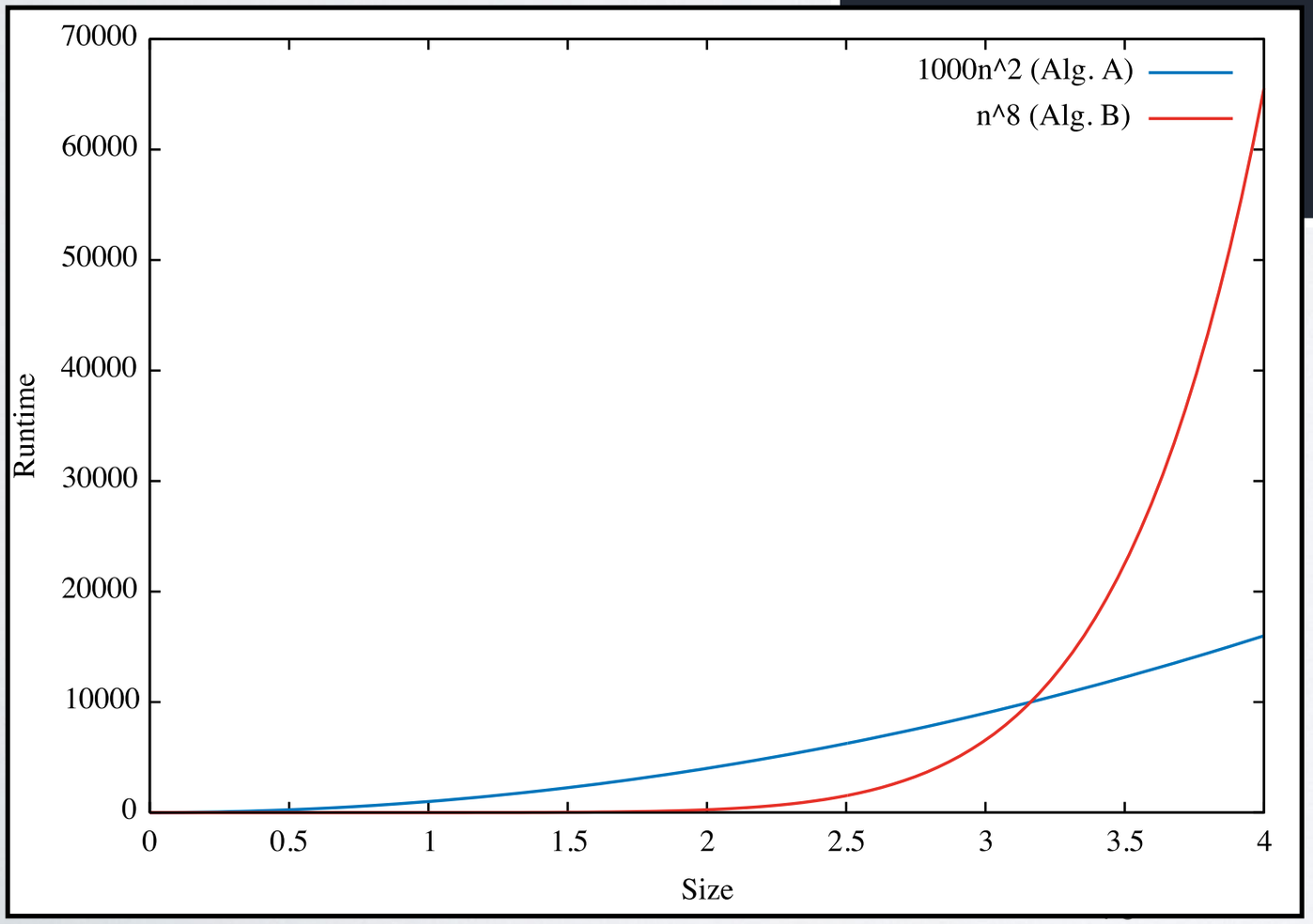
$$\begin{aligned} \text{rtime(A)} < \text{rtime(B)} &\iff 5n+1000 < 10n+2 \\ &\iff 5n > 998 \\ &\iff n > 199.6 \end{aligned}$$

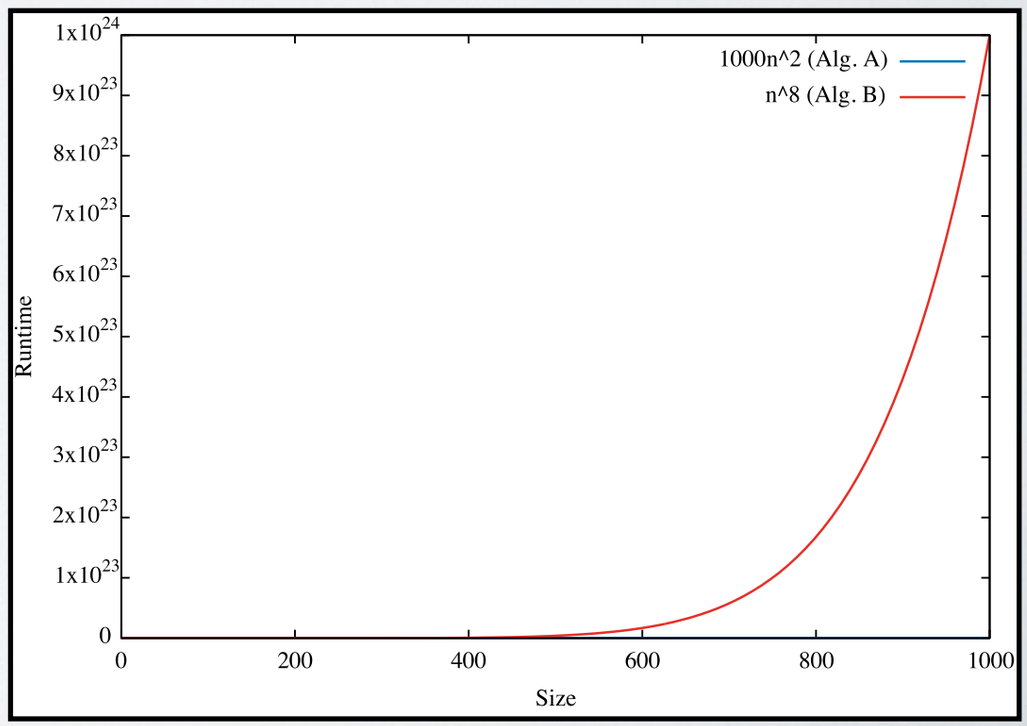


Which Algorithm is Better?

- Alg A takes $T_A(n)=1000n^2$ ops
- Alg B takes $T_B(n)=n^8$ ops
- It depends on n

$$
\begin{aligned}
rtime(A) < rtime(B) &\iff 1000n^2 < n^8 \\
&\iff 1000n^2 - n^8 < 0 \\
&\iff n^2(1000 - n^6) < 0 \\
&\iff 1000 - n^6 < 0 \\
&\iff n > 1000^{1/6} \\
&\iff n > 3.16\ldots
\end{aligned}
$$





Comparing Running Times

Comparing ***asymptotic*** running times

=

$T_A(n)$ *is better than* $T_B(n)$ *if*

for large enough n

$T_A(n)$ *grows slower than* $T_B(n)$

Q: can we formalize all this mathematically?

Big-O

Definition (Big-O): $T_A(n)$ is $O(T_B(n))$ if there exists positive constants c and n_0 such that:

$$T_A(n) \leq c \cdot T_B(n)$$

for all $n \geq n_0$

- $T_A(n)$'s order of growth is at most $T_B(n)$'s order of growth
- Examples
  - $2n+10$ is $O(n)$
  - $n^{10}+2019$ is $O(n^{10})$ and also $O(n^{50})$

Big-O

- How do we find "the Big-O of something"?
  - Usually you "eyeball" it
  - Then you try to prove it
  - (though most of the time in CS16 it will be simple enough that you don't need to prove it)

Eyeballing Big-O

- If **`T(n)`** is a polynomial of degree **`d`** then **`T(n)`** is **$O(n^d)$**
- In other words you can ignore
  - lower-order terms
  - constant factors
- Examples
  - **$1000n^2+400n+739$** is **$O(n^2)$**
  - **$n^{80}+43n^{72}+5n+1$** is **$O(n^{80})$**
- *For Big-O, use the smallest upper bound*
  - **`2n`** is **$O(n^{50})$** but that's not really a useful bound
  - instead it is better to say that **`2n`** is **`O(n)`**

Big-Omega

Definition (Big-Ω): $T_A(n)$ is $\Omega(T_B(n))$ if there exists positive constants c and n_0 such that:

$$T_A(n) \geq c \cdot T_B(n)$$

for all $n \geq n_0$

- $T_A(n)$'s growth rate is lower bounded by $T_B(n)$'s growth rate
- What about an upper **and** a lower bound?
  - We use Big-**Θ** notation

Eyeballing Big-Omega

- If $T(n)$ is a polynomial of degree d then $T(n)$ is $\Omega(n^d)$
- In other words you can ignore
  - lower-order terms
  - constant factors
- Examples
  - $1000n^2+400n+739$ is $\Omega(n^2)$
  - $n^{80}+43n^{72}+5n+1$ is $\Omega(n^{80})$
- *For the Big-*Ω*, use the largest upper bound*
  - $2n$ is $\Omega(\log n)$ but that's not really a useful bound
  - instead it is better to say that $2n$ is $\Omega(n)$

Big-Theta

Definition (Big-Θ): $T_A(n)$ is $\Theta(T_B(n))$ if it is $O(T_B(n))$ and $\Omega(T_B(n))$.

- $T_A(n)$'s growth rate is the same as $T_B(n)$'s

Eyeballing Big-Theta

- If $T(n)$ is a polynomial of degree d then $T(n)$ is $\Theta(n^d)$
- In other words you can ignore
  - lower-order terms
  - constant factors
- Examples
  - $1000n^2+400n+739$ is $\Theta(n^2)$ since it is $O(n^2)$ and $\Omega(n^2)$
  - $n^{80}+43n^{72}+5n+1$ is $\Theta(n^{80})$ since it is $O(n^{80})$ and $\Omega(n^{80})$

Dynamic Programming

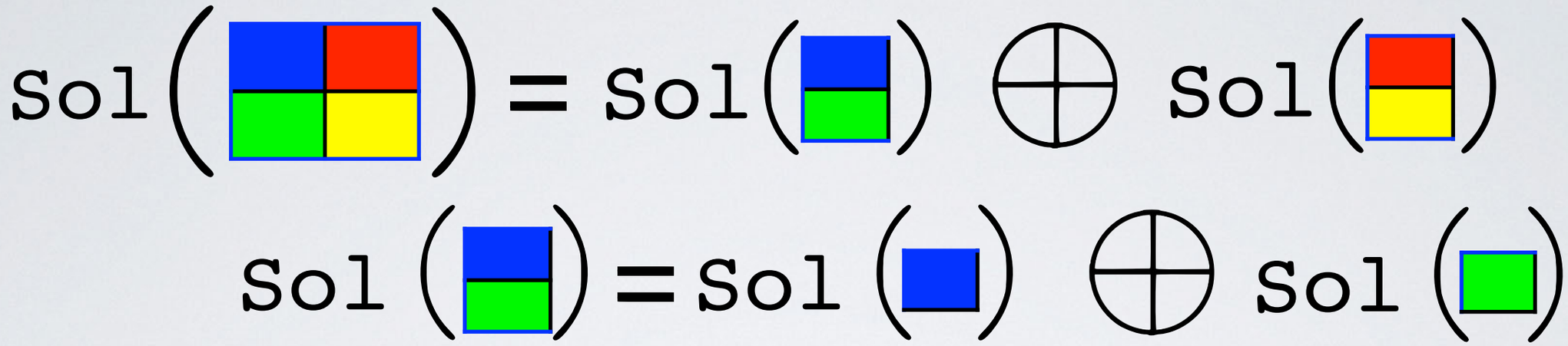
What is Dynamic Programming?

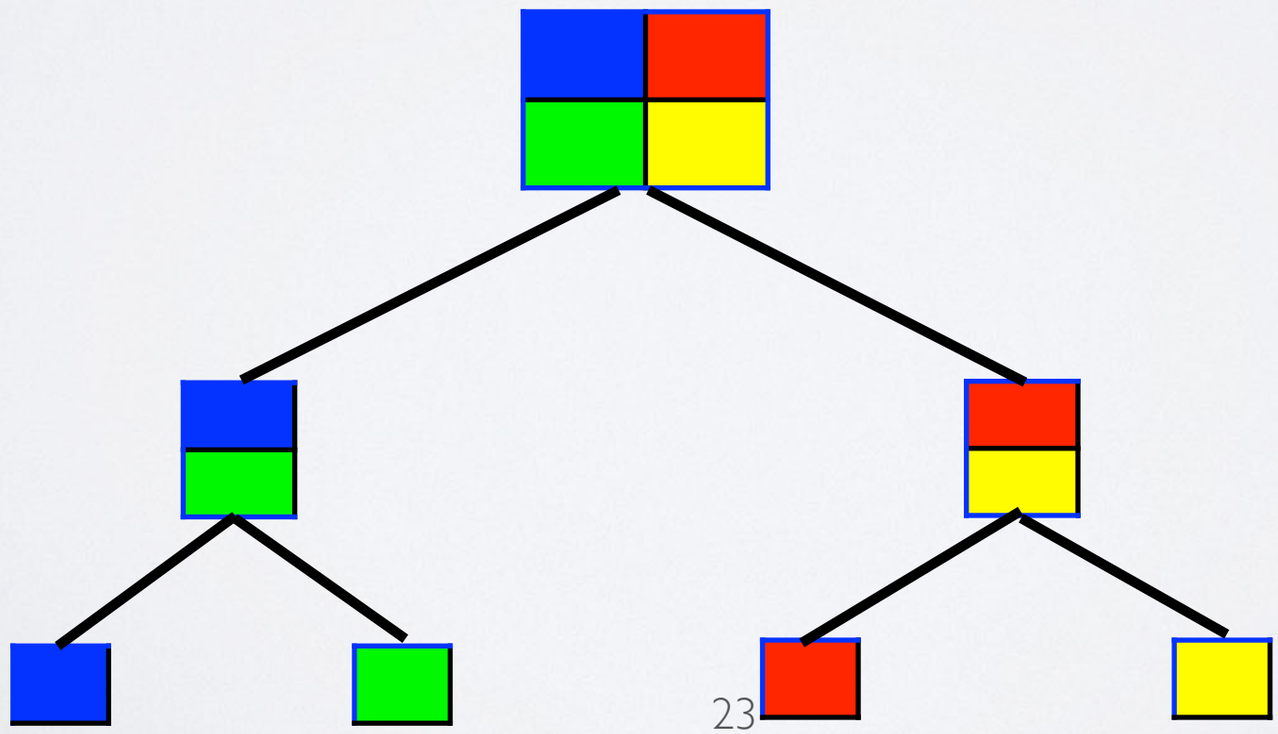
- Algorithm design paradigm/framework
  - Design efficient algorithms for optimization problems
- Optimization problems
  - "find the **best** solution to problem **`X`**"
  - "what is the **shortest** path between **`u`** and **`v`** in **`G`**"
  - "what is the **minimum** spanning tree in **`G`**"
- Can also be used for non-optimization problems

When is Dynamic Programming Applicable?

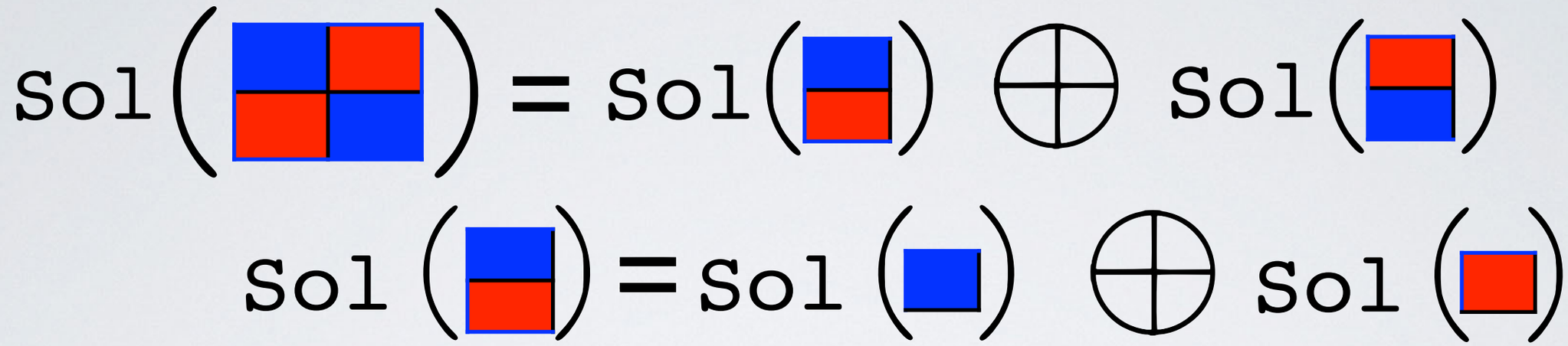
- Condition #1: sub-problems
  - The problem can be solved recursively
  - Can be solved by solving sub-problems
- Condition #2: overlapping sub-problems
  - Same sub-problems need to be solved many times

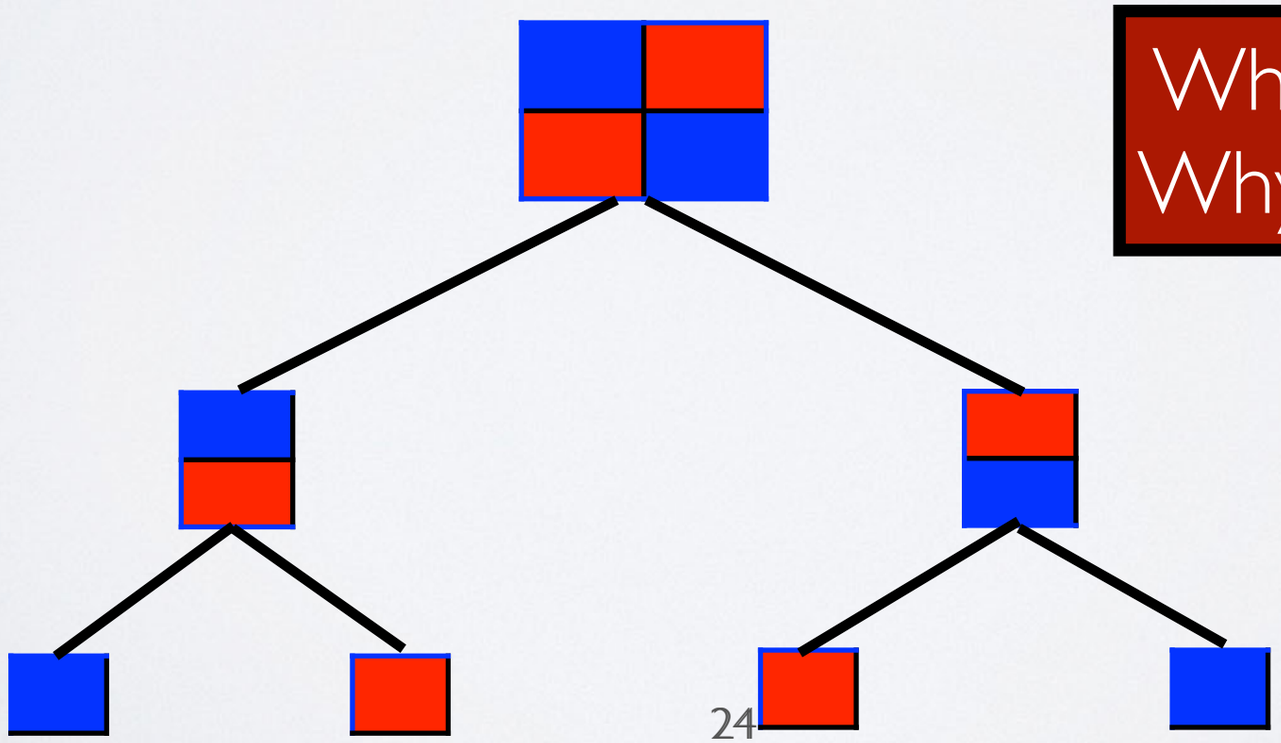
Sub-Problems





Overlapping Sub-Problems





Why solve red twice?
Why solve blue twice?

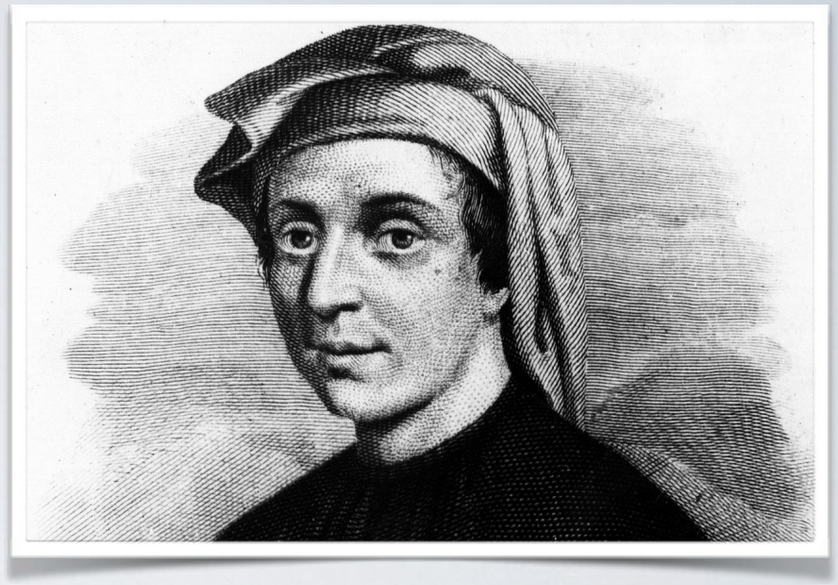
When is Dynamic Programming Applicable?

- Core idea
  - Decompose problem into its sub-problems
  - and if sub-problems are overlapping then
  - solve each sub-problem once and store the solution
  - use stored solution when you need to solve sub-problem again

Steps to Solving a Problem w/ DP

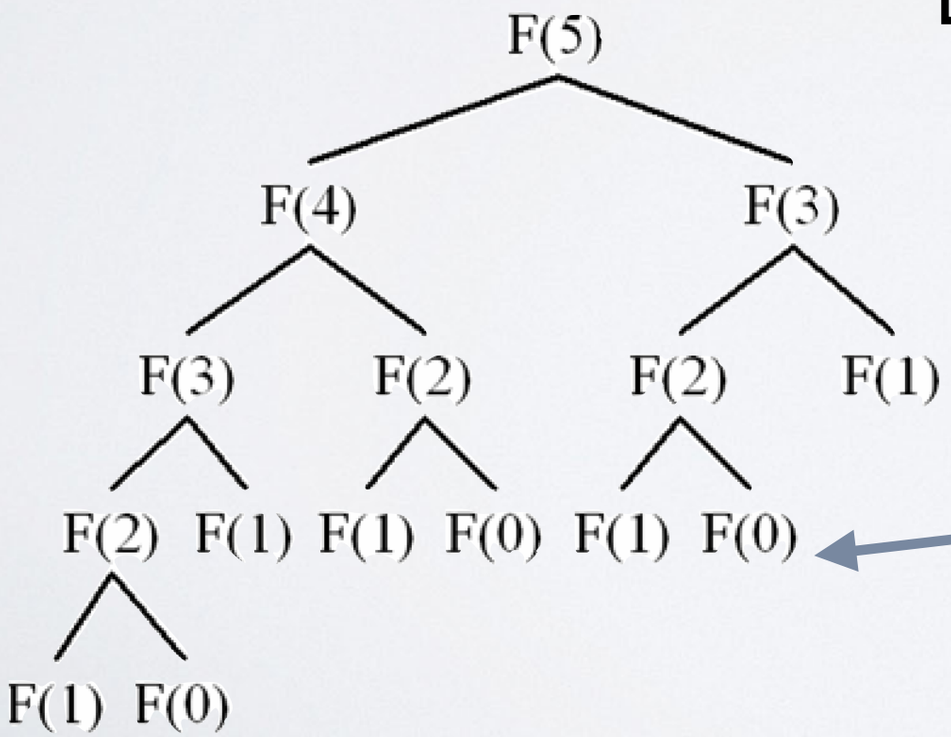
- What are the **sub-problems**?
- What is the "**magic**" step?
  - Given solution to a sub-problem…
  - …how do I *combine* them to get solution to the problem?
- Which **(topological) order** on sub-problems can I use?
  - so that solutions to sub-problems available before I need them
- Design iterative **algorithm**
  - that solves sub-problems in order and stores their solution

Fibonacci



`0,1,1,2,3,5,8,13,21,34,…`

$$F(n) = F(n-1) + F(n-2)$$



base cases:

$$F(0) = 0 \;\&\; F(1) = 1$$

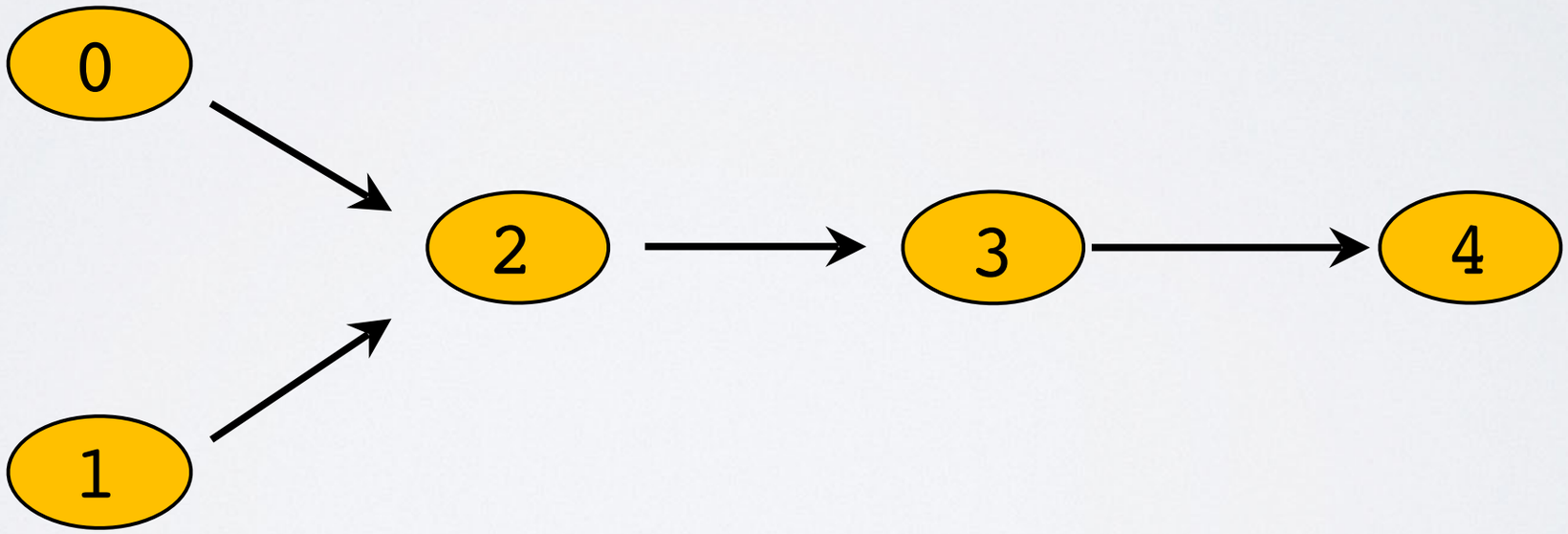
Fibonacci (Dynamic Programming)

- Given **`n`** compute
  - Fib(**`n`**) = Fib(**`n-1`**)+Fib(**`n-2`**)
  - with base cases Fib(**`0`**) = **`0`** and Fib(**`1`**) = **`1`**
- What are the **sub-problems**?
  - Fib(**`n-1`**), Fib(**`n-2`**), …, Fib(**`1`**), Fib(**`0`**)
- What is the **magic** step?
  - Fib(**`n`**) = Fib(**`n-1`**)+Fib(**`n-2`**)

Magic step is usually not provided!!

Fibonacci (Dynamic Programming)

- Which **topological order** should I use?
  - Fib(`0`), Fib(`1`), …,Fib(`n-1`), Fib(`n`)



Fibonacci (Dynamic Programming)

- Design iterative **algorithm**

```
function Fib(n):
    fibs = []
    fibs[0] = 0
    fibs[1] = 1

    for i from 2 to n:
        fibs[i] = fibs[i-1] + fibs[i-2]

    return fibs[n]
```

Fibonacci (Dynamic Programming)

```
function Fib(n):
    fibs = []
    fibs[0] = 0
    fibs[1] = 1

    for i from 2 to n:
        fibs[i] = fibs[i-1] + fibs[i-2]

  return fibs[n]
```

- What's the runtime of **`Fib( )`**?
  - Calculates Fibonacci numbers from **`2`** to **`n`**
    - Performs **`O(1)`** ops for each one
  - Runtime is **`O(n)`**