Dealing with Hard Problems

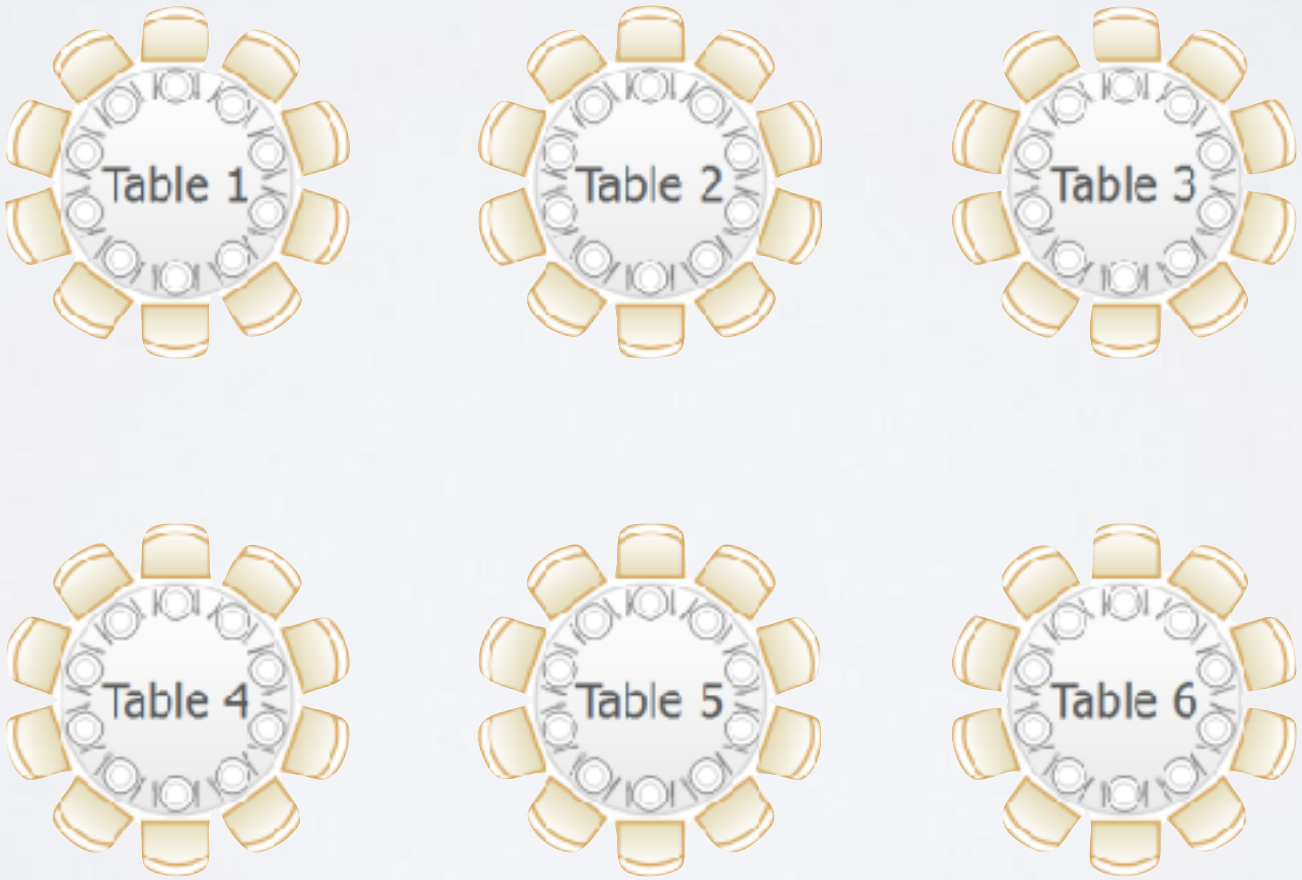
CS16: Introduction to Data Structures & Algorithms
Spring 2019

Outline

- Seating Arrangements
- Problem hardness
- P, NP, NP-Complete, NP-Hard
- Dealing with hard problems
  - Problem translation
  - Genetic Algorithms
  - Approximations
- Travling Salesman Problem

Seating Arrangement Problem

- Your dating algorithms worked!
- You need to plan the seating arrangements for a wedding



Seating Arrangement Problem

- Constraints / goals
  - **k** tables
  - **n** people
  - Avoid antagonistic pairs (exes, rivals, etc) sitting at the same table
  - Maximise overall happiness

Quantifications of Pair-wise Happiness

- Assume each pair of people (A, B) has an associated 'compatibility score'
  - for **friends** comp(A, B) = **10**
  - for **couples** comp(A, B) = **50**
  - for **antagonistic pairs** comp(A, B) = **-500**
- These values are known ahead of time

Quantifications of Table-wise Happiness

- Sum all the compatibility scores for each pair at the table

$$H\left(table\right) = \sum_{pair \in table} comp(pair)$$

Quantification of Total Happiness

- Utilitarian Approach:

$$Total\_H_{utilitarian} = \sum_{t \in tables} H(table)$$

- Egalitarian Approach:

$$Total\_H_{egalitarian} = \min_{t \in tables} H(t)$$

- Many more options!

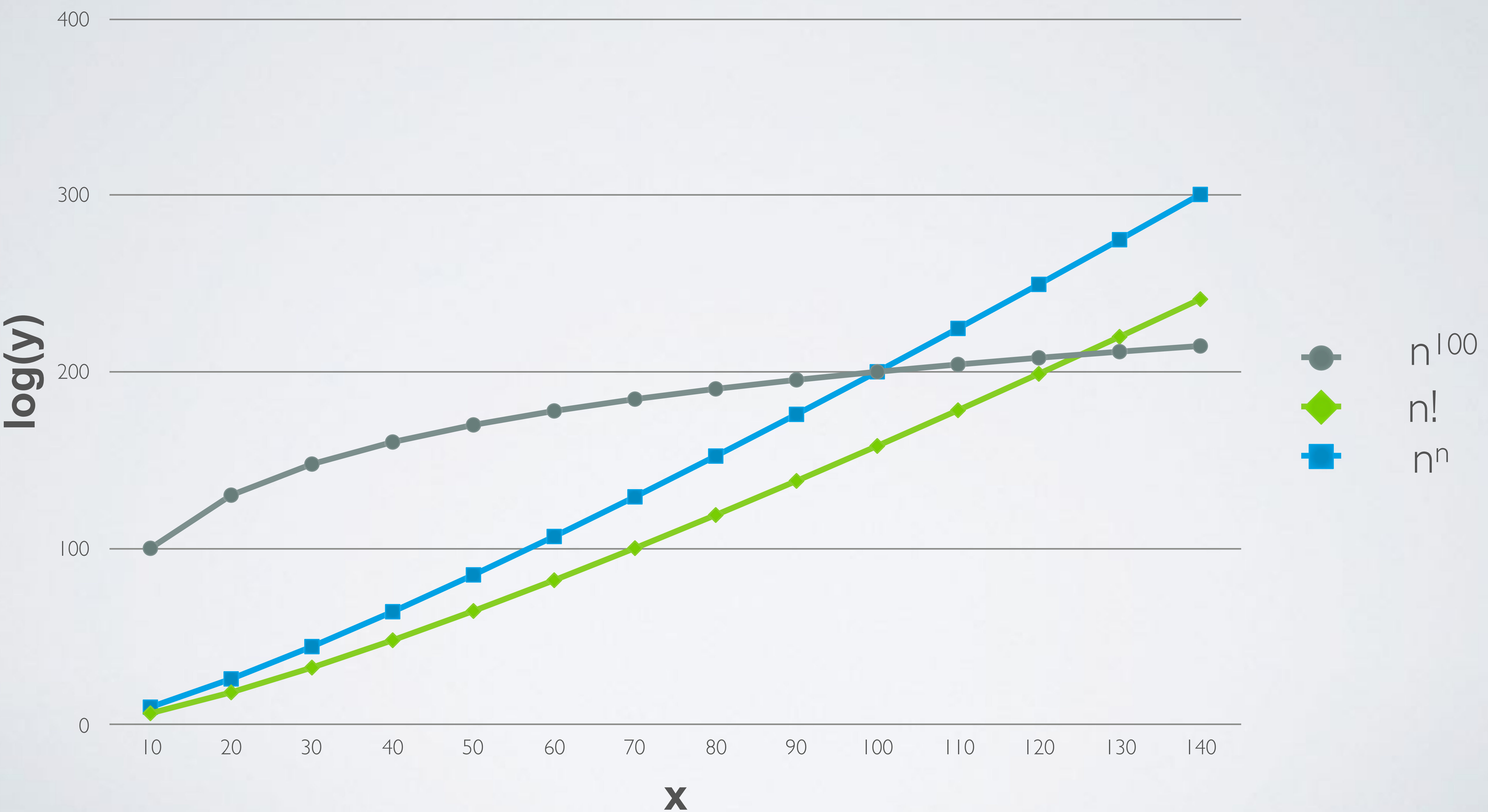
This seems hard

- Could we just try permutations and comparing scores?
- With 60 people, 60! permutations to test
  - 8.32×10^{81}
  - ouch
- This doesn't necessarily mean that the problem *is* hard, however

Defining Problem Hardness

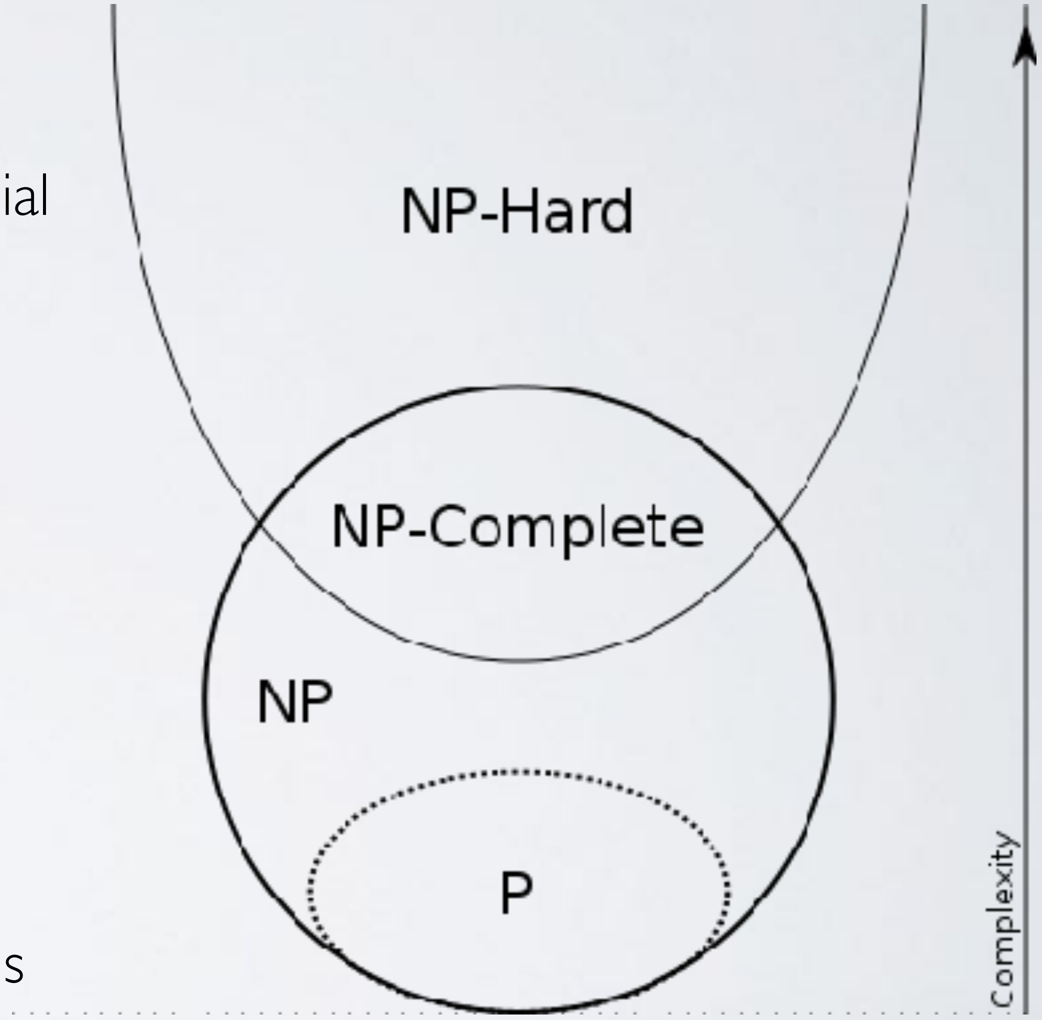
- Hardness of problem is defined by the runtime of the best solution
  - A bad sorting algorithm *could* be O(n!), but sorting in general isn't considered hard, because we have fast algorithms to solve it
- Polynomial Runtimes
  - $O(n)$, $O(n^2)$, $O(n^{500})$
  - Problems with these solutions are **tractable**
- Super-Polynomial Runtimes
  - $O(n!)$, $O(2^n)$, $O(n^n)$
  - Problems with these solutions are **intractable**

Exponential vs. Polynomial Growth Rates



Categories of Hardness

- NP
  - The set of problems for which we can verify the correctness of a solution in polynomial time
- P
  - A subset of NP, where the problem is solvable in polynomial time
- NP-Complete
  - "The hardest problems in NP"
  - Solution is checkable in polynomial time
  - not known whether there exist any polynomial time algorithms to solve them
- NP-Hard
  - Problems that are "at least as hard as the hardest problems in NP"
  - Don't necessarily have solutions that are checkable in polynomial time



Back to our seating arrangement

- To get an intuition as to how hard our problem is, let's see if we can convert it into a problem that has already been proven to be in NP, P, NP-Complete, or NP-Hard
- But… where to start?

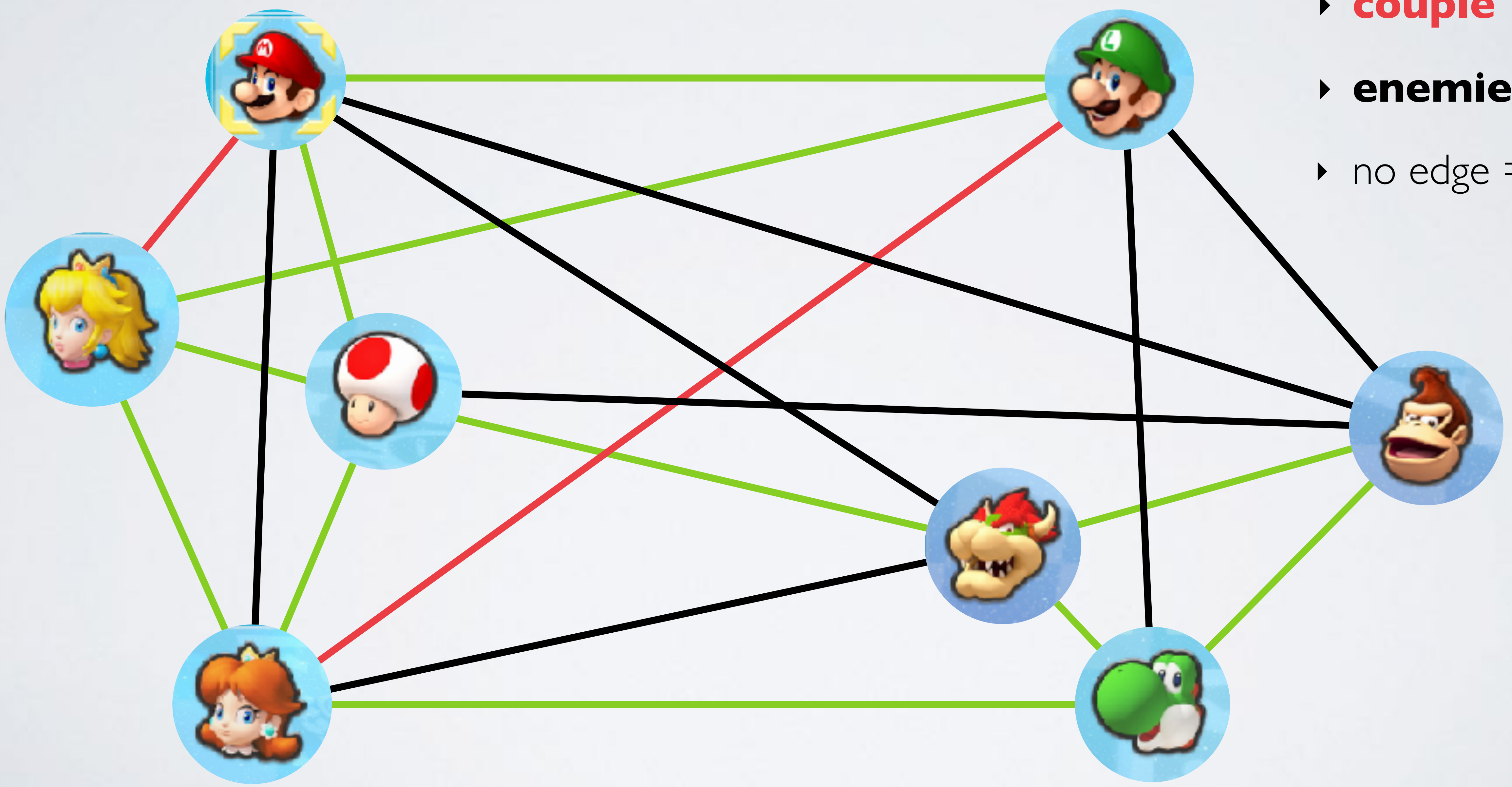
Constraint Relaxation

- See if you can solve an ‘easier’ version of the problem, by removing some of the properties that make the problem hard
- In real life
  - “what would you do if you could not fail?”
  - “which job would you take if they all paid equally?”

Let’s avoid disaster

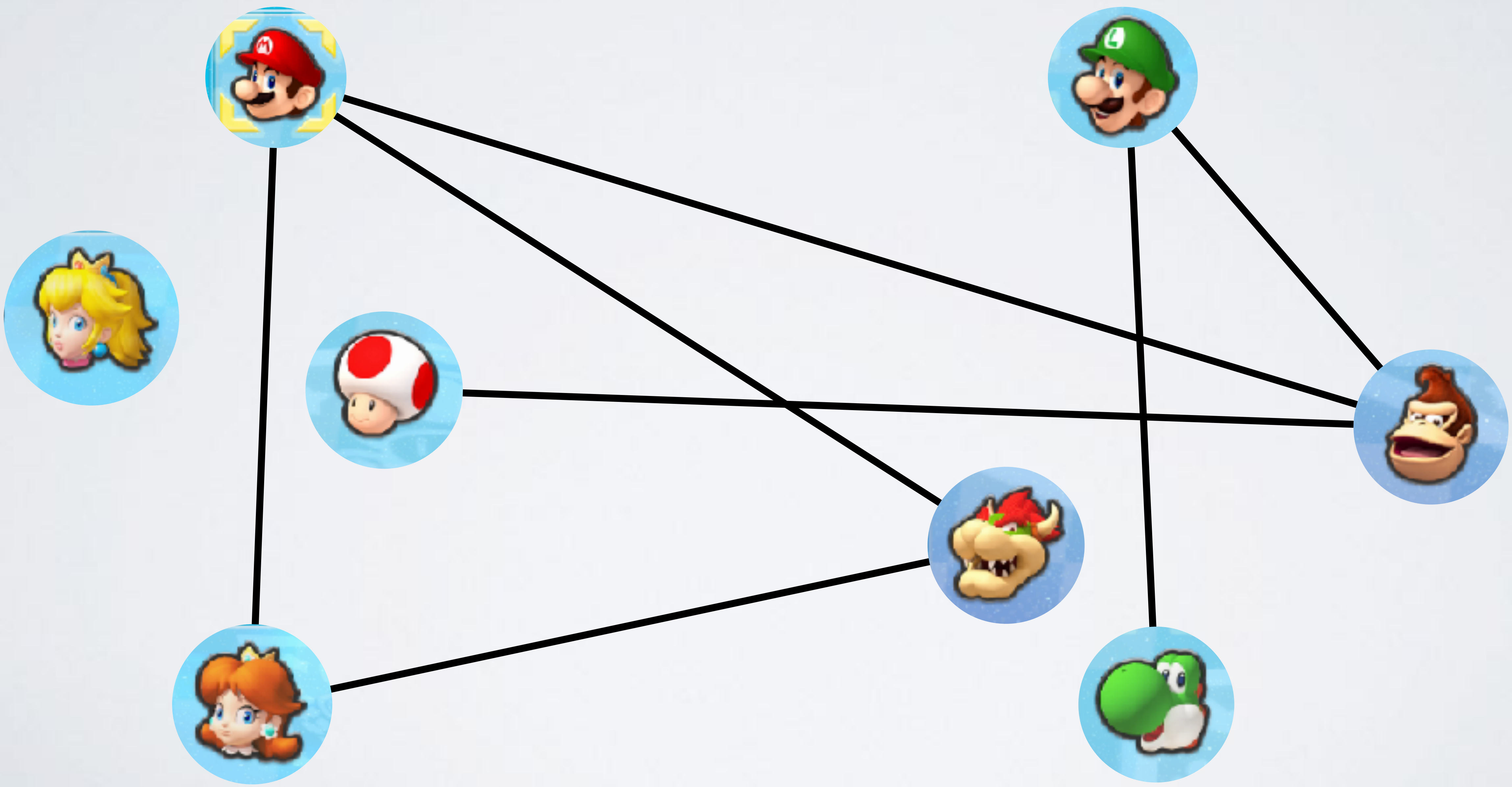
- Constraints / goals
    - # of tables
    - # of people
    - Avoid antagonistic pairs (exes, rivals, etc)
    - ~~Maximise overall happiness~~
- Hopefully, having no tables with antagonistic pairs will put in the right direction for maximising overall happiness

Relationships as a graph



- edge key:
  - **friends**
  - **couple**
  - **enemies**
  - no edge = no prefs

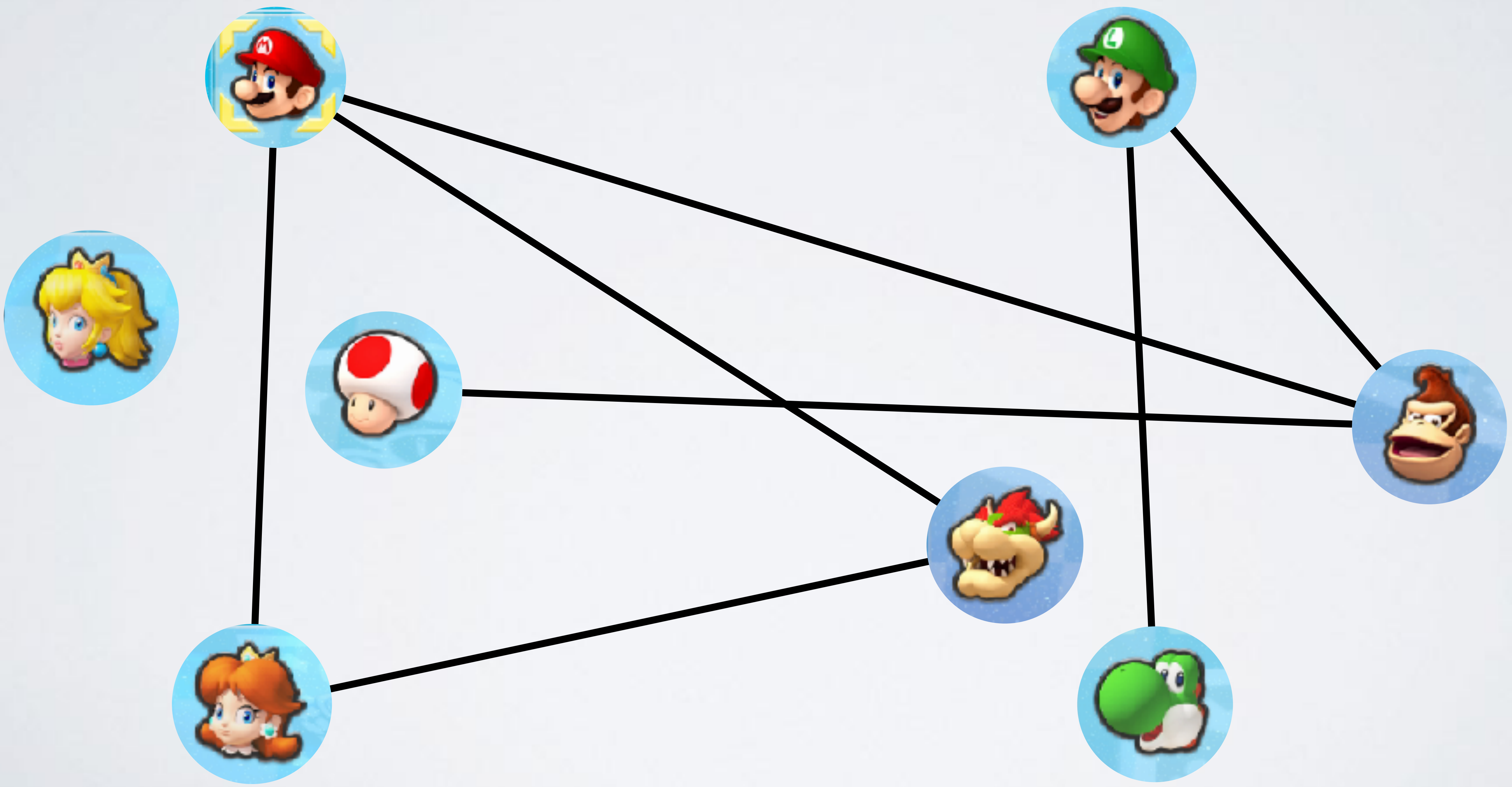
An Antagonism graph



Translating the problem

- Now, we have these antagonistic relationships represented as a graph!
- Question is no longer:
  - Can we avoid antagonistic pairs (exes, rivals, etc) sitting at the same table, given **n** people and **k** tables?
- Instead:
  - Use colours to represent different tables, so:
  - Could we assign 1 of **k** colours to each node in the antagonism graph, such that no two nodes that share an edge have the same colour?

An Antagonism graph



Lecture Activity 3

Try out the Graph k-colouring problem!

2 Mins….

Lecture Activity 3

Try out the Graph k-colouring problem!

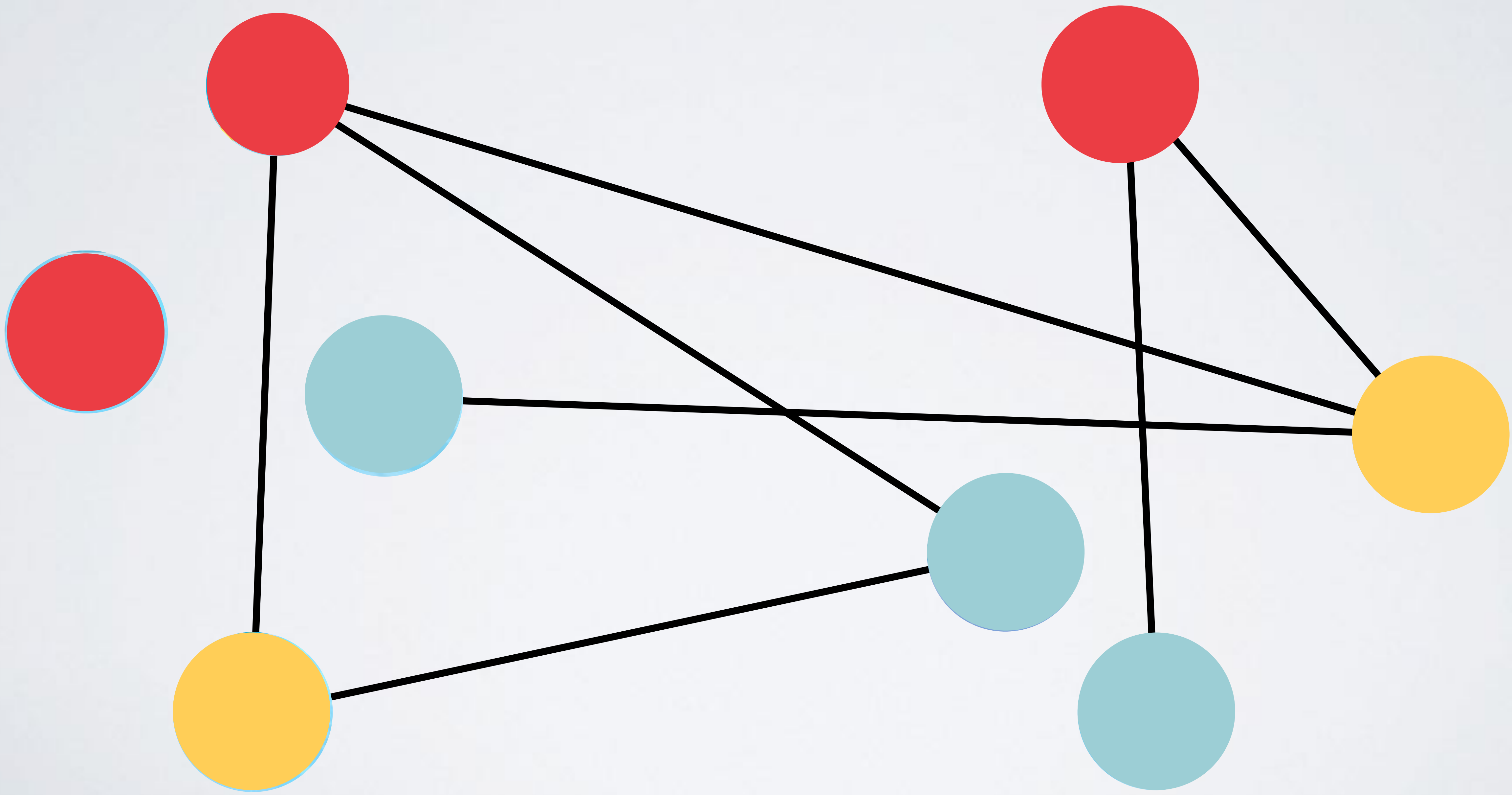
1 Min….

Lecture Activity 3

Try out the Graph k-colouring problem!

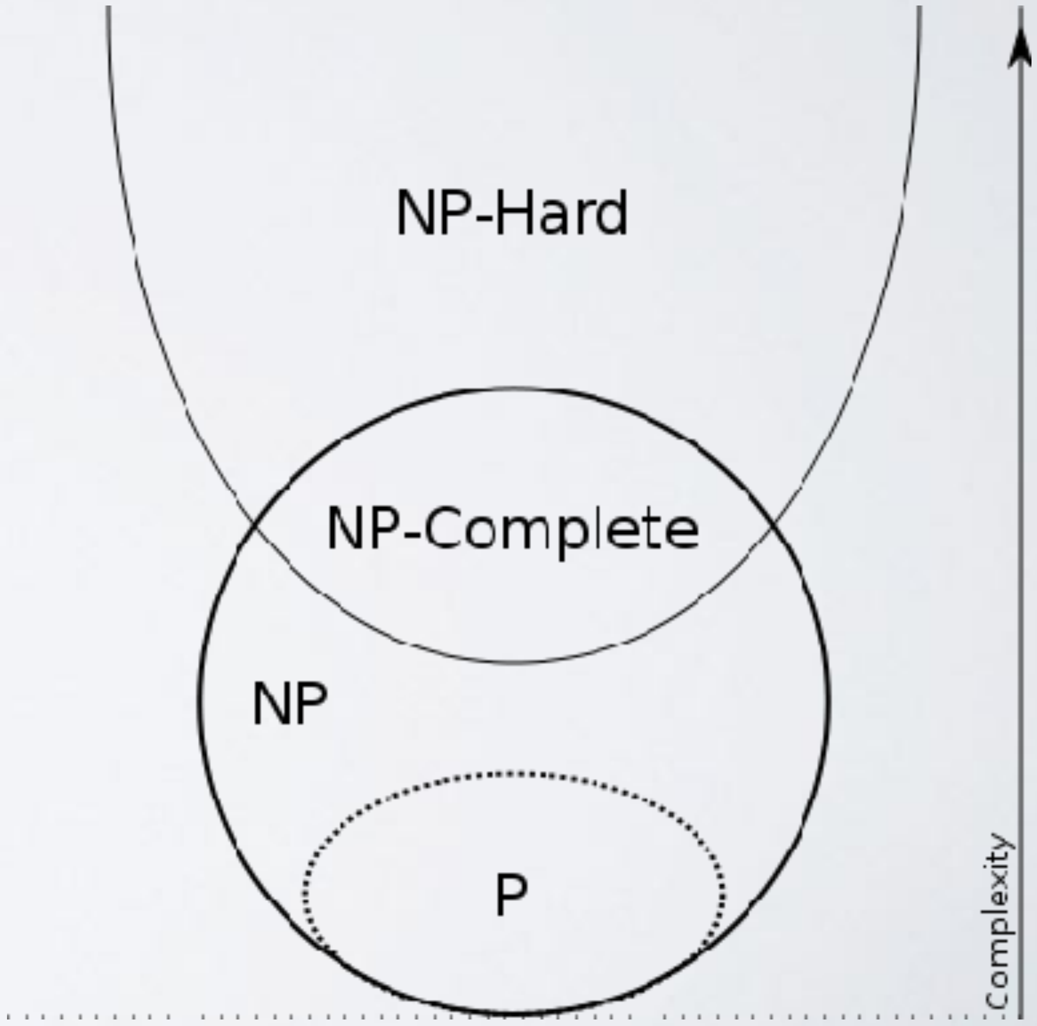
0 Mins….

Graph colouring example



Graph k-colouring

- Generally, the problem of determining whether nodes in a graph can be coloured using up to **k** separate colours, such that no two adjacent vertices share a colour
- This is NP-Complete!
- And thus, even this much easier version of the problem is **very hard**



Are we screwed?

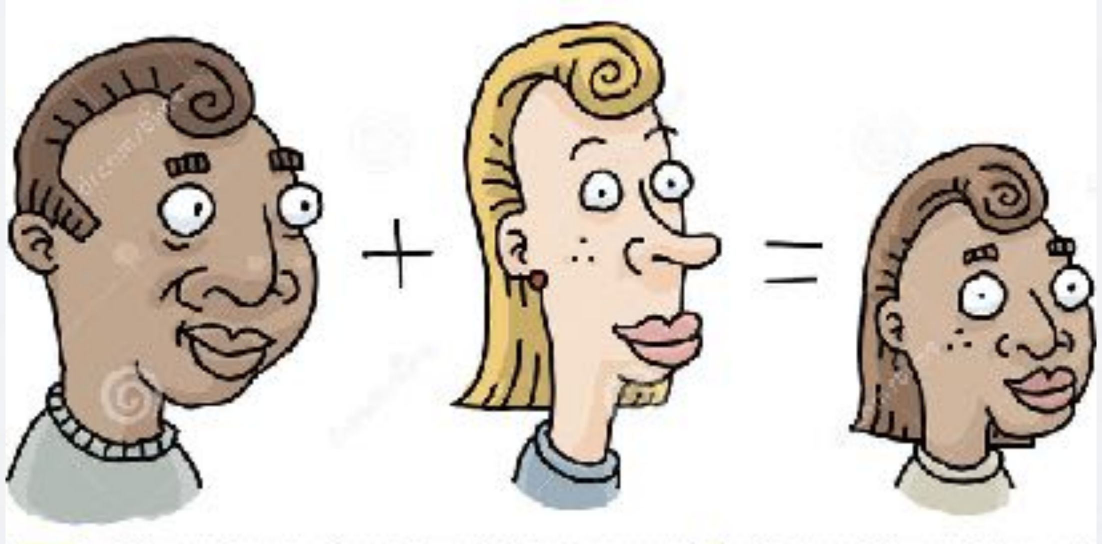
- The best algorithms to solve the graph k-colorability problem take $O(2.445^n)$ time and space
- With 60 guests, 2.445^{60} = ~450 billion
  - which isn't *that* bad
  - Modern computers can handle ~3 billion 'operations' / sec, so this would take more than a couple minutes, probably less than 15
- But we've still only avoided the worst case!

Genetic Algorithms

- A form of 'guess and check', using a number of possible solutions to a problem
- Inspired the process of evolution

Biology Review

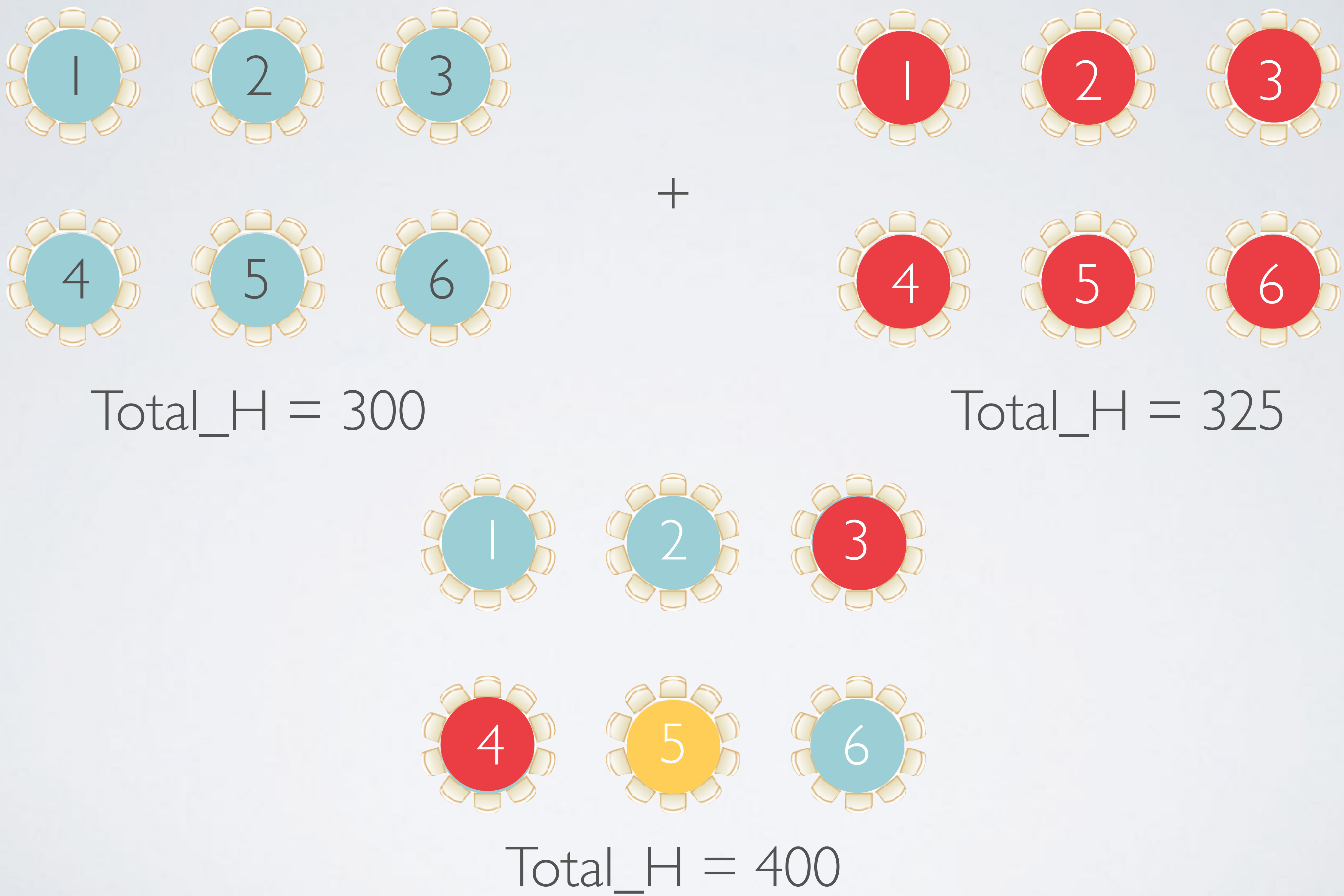
- All organisms are made up of **genes**, where genes (or a combination many genes) interact to produce our **phenotype**, the expression of those genes
- We are all a combination of a mix of our parents genes, and some random mutations



Evolution via Sexual Reproduction, broadly

- There exist an initial population of organisms within a species
- The ‘sexually fit’ organisms reproduce
  - Take some genes of parent A, some of parent B
  - add some random noise
  - this new collection of genes is a new specimen, AB’
- Older + less fit parts of populations die off, leaving the survivors to repeat the reproduction process

Solution Mating



High-Level Genetic Algorithm Pseudocode

```
function geneticAlgo(opt_seed_sols):
   solution_set = opt_seed_sols or || randomly generated initial population of solutions
   init_size = size(solution_set, threshold, time_limit)

   while True:
        new_gen = []
        for some number of iterations:
           A, B = 2 solutions from solution_set, drawn at random
           AB' = a new solution that combines properties of A and B
           randomlyMutate(AB')
           new_gen.append(AB')
        solution_set.addAll(new_gen)
        rank solutions in solution_set based on 'fitness'
        remove all but init_size many best solutions from solution_set
        if best(solution_set) > threshold or time_limit has passed:
           break
  return highest ranking solution from solution_set
```

Genetic Algorithms

- If seeded with 'good' solutions for the initial population of solutions, output is guaranteed to be at least as good as the best of the initial solutions

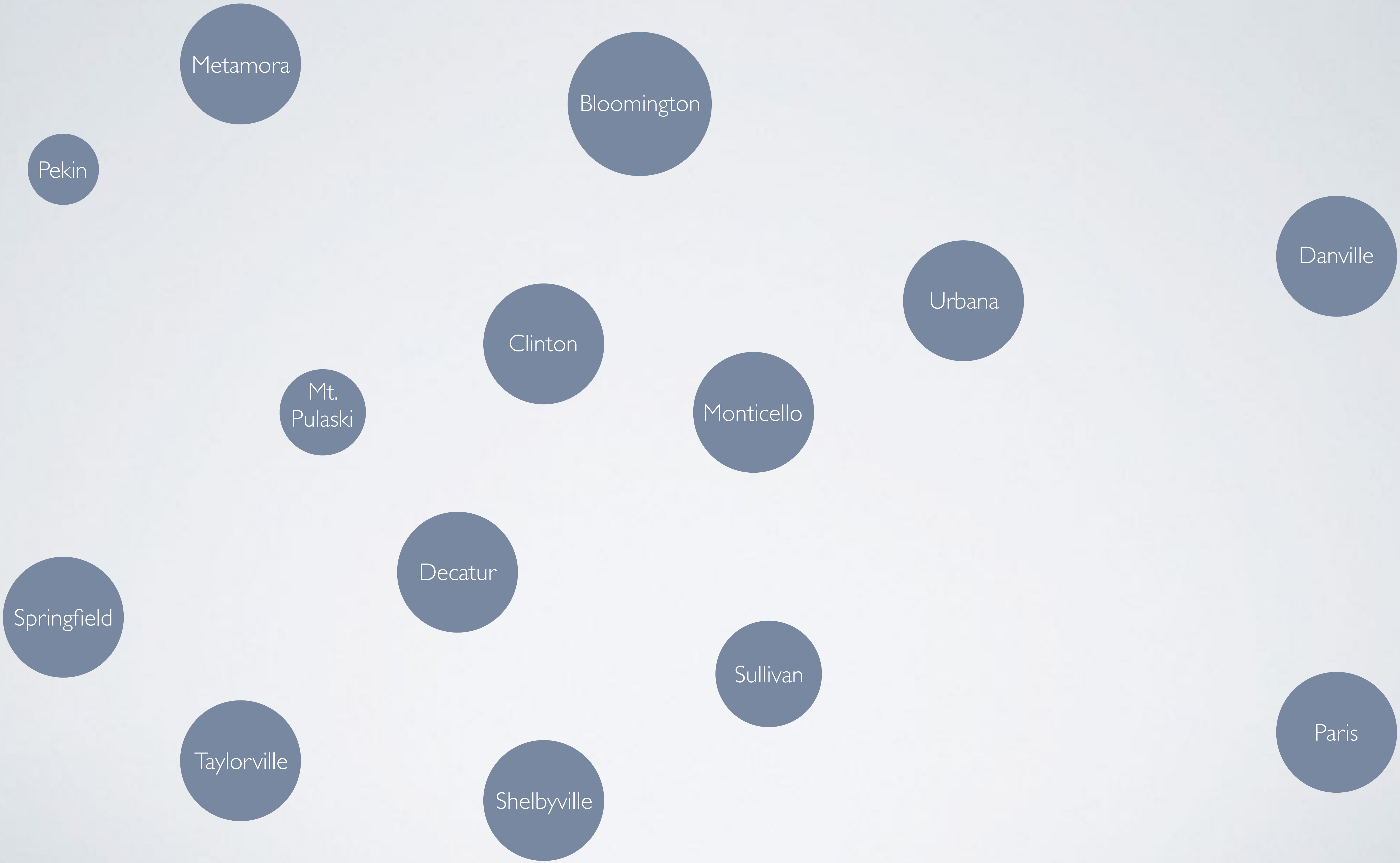
- Can come up with unexpected solutions

- Tend to do really well!

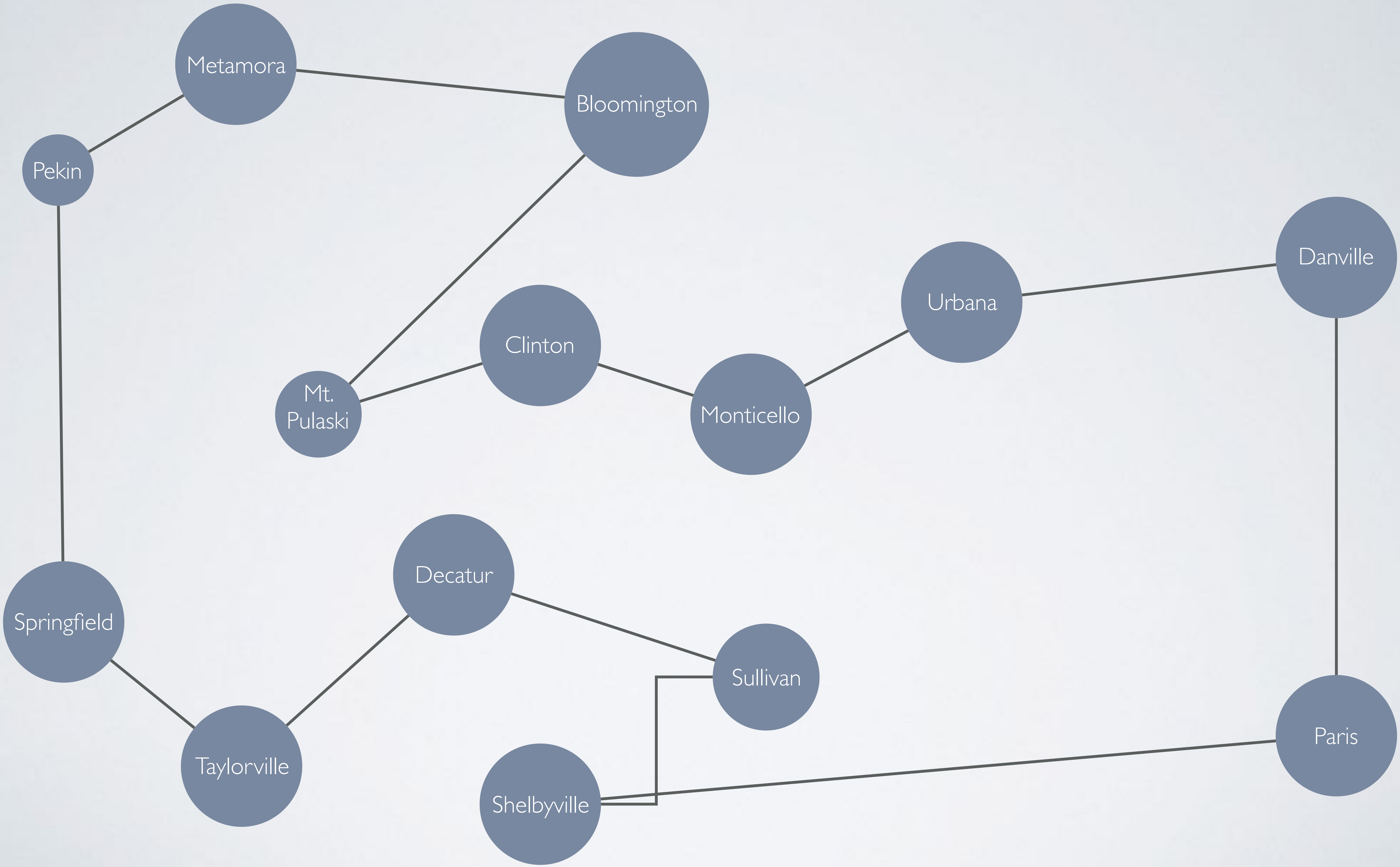
Honeymooning

- Also known as the Traveling Salesman Problem
- TSP, defined: "Given a list of cities and the distances between each pair of cities, what is the shortest possible route that visits each city exactly once and returns to the origin city?"

Cities (not to scale):



Best route:


Metamora
Bloomington
Pekin
Danville
Urbana
Clinton
Mt.
Pulaski
Monticello
Decatur
Springfield
Sullivan
Paris
Taylorville
Shelbyville

TSP Hardness

- Given a graph with **n** nodes
  - we *could* exhaustively try O(n!) possible city-orderings
  - But let's see if we can do any better
- Finding the most optimal route is NP-Hard :(
- Held-Karl algorithm solves it in $O(n^2 \times 2^n)$

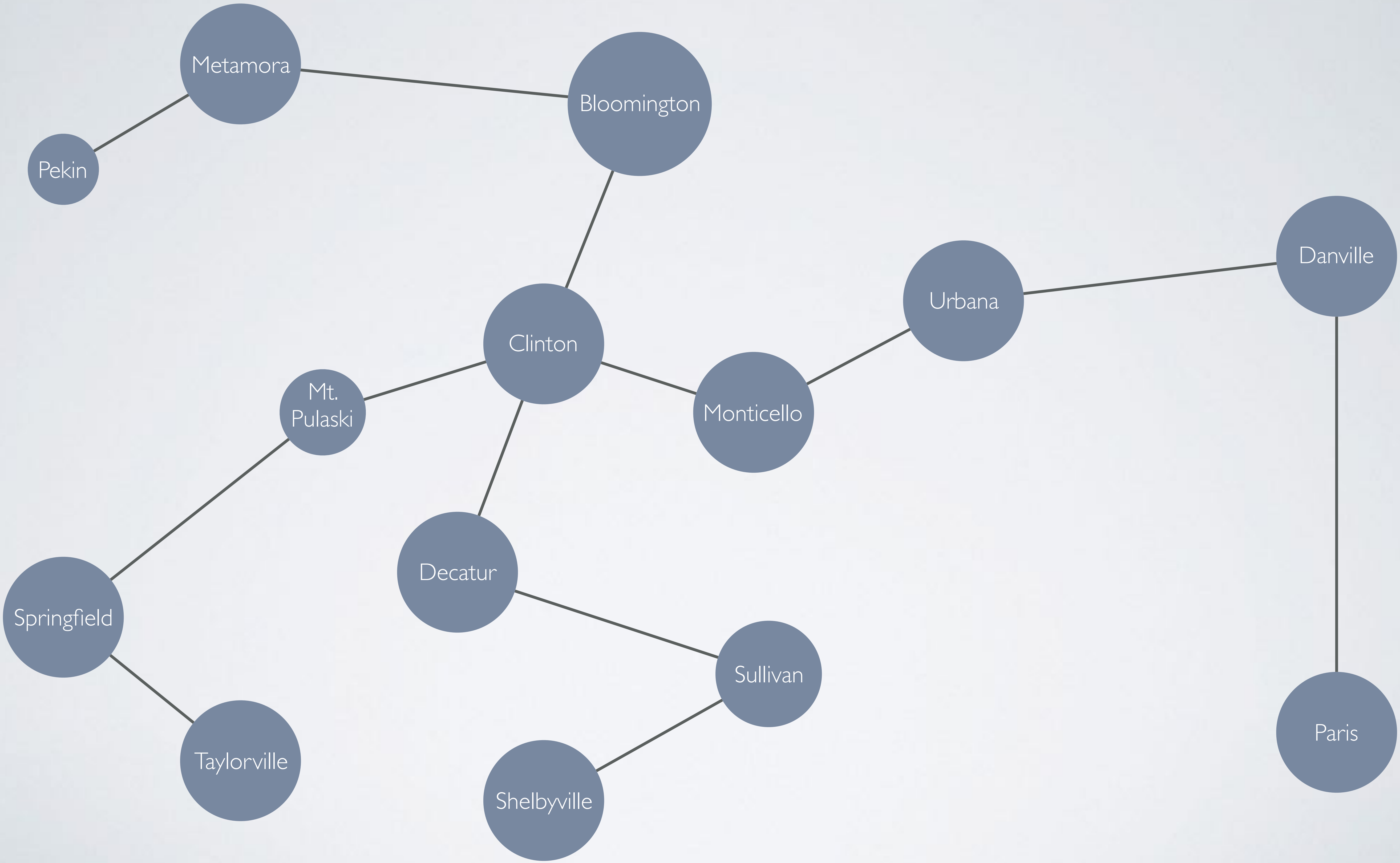
But we’re not totally screwed!

- Again, relaxing constraints…
- What if we were
  - allowed to visit a city more than once, and
  - allowed to retrace your steps for free?
- Sounds like the problem reduces to connecting alls the cities as cheaply as possible - do we know how to solve this problem?

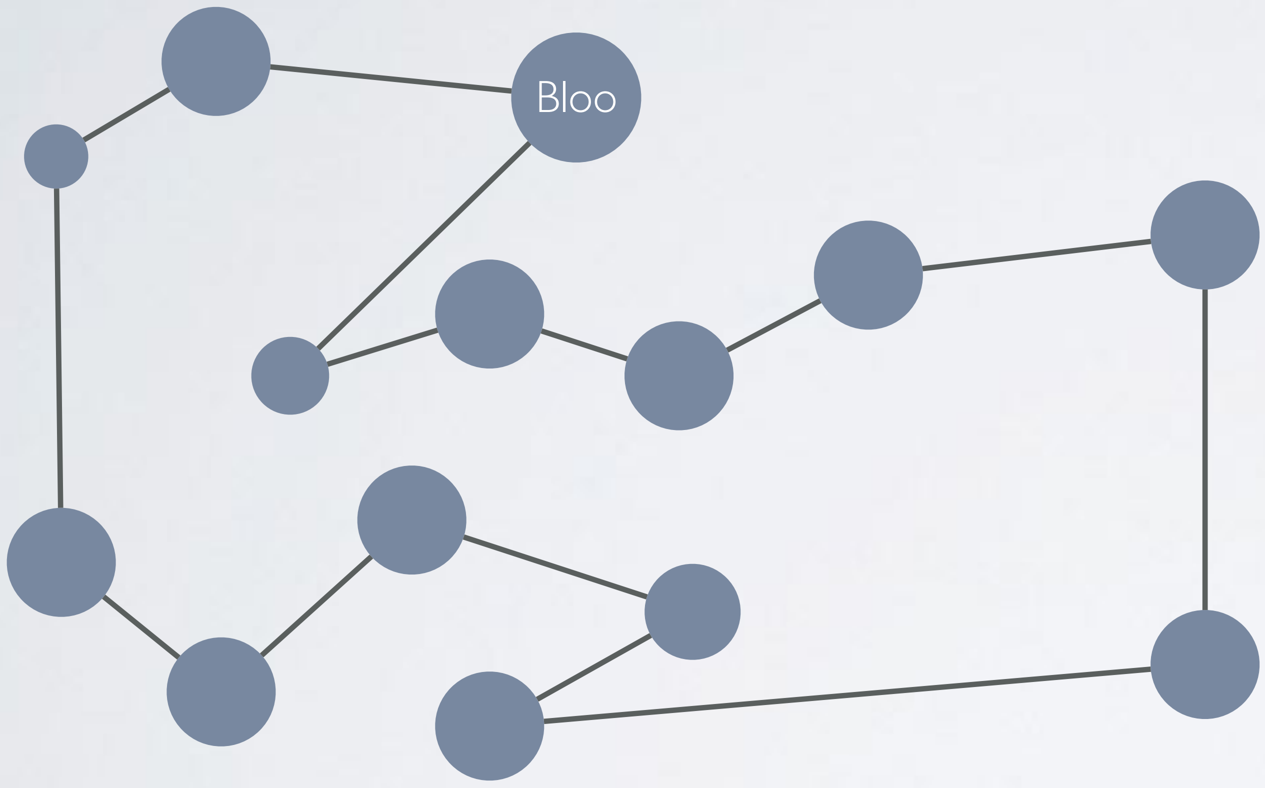
MSTs as a starting point to approximate TSP

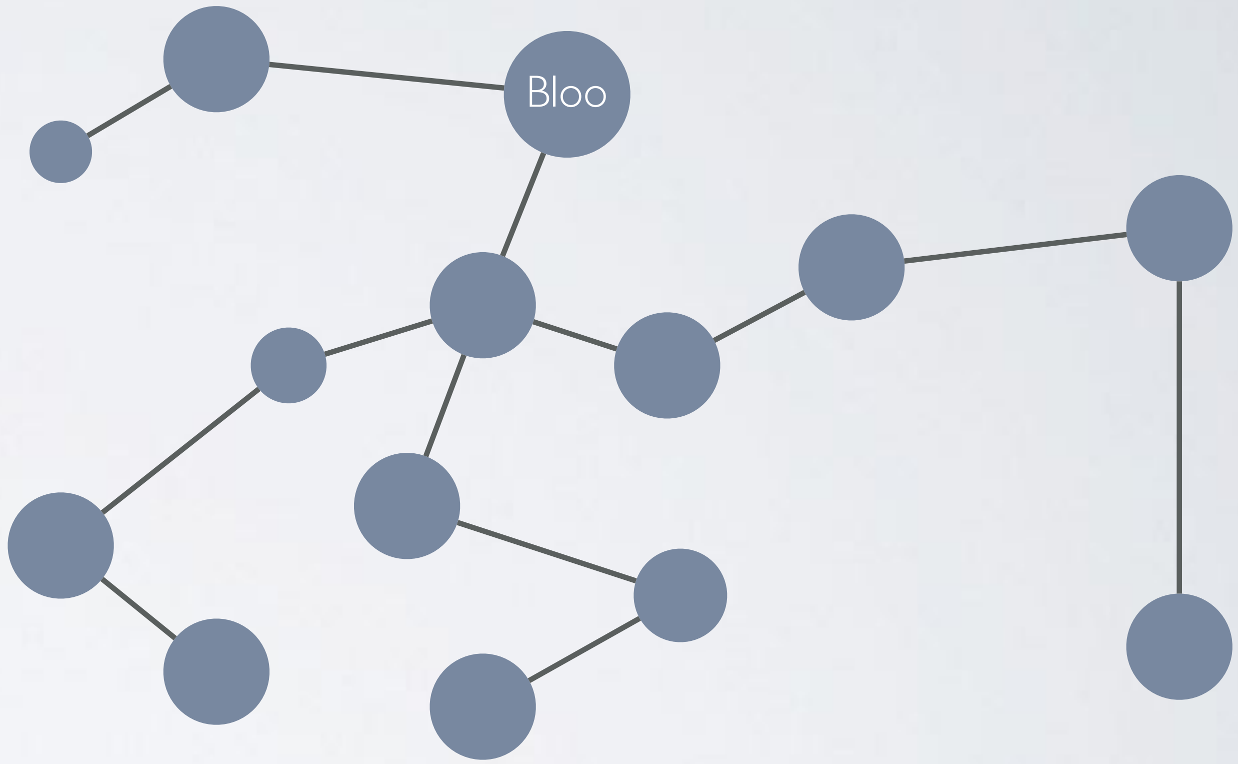
- This is very easy!
- Provides a lower bound for the real solution
  - a solution with free backtracking can’t possibly be worse than a solution that has to follow all the original rules
  - If we find a solution to the original problem, can use the MST as a comparison for how close we might be
    - If an MST for some graph has total 100 mile distance, but a given solution has total distance of 110, we are at most 10% longer than the best solution

MST of cities:



Best route vs. MST





The big takeaway

- Some problem are just plain hard
- But we can get pretty good solutions in a reasonable amount of time anyway
- Sometimes the best approach is to accept that getting the absolute best solution is impossible
  - but we can get reasonably close by solving simpler versions of the problem that we *do* know how to solve