Binary Search

CS16: Introduction to Data Structures & Algorithms
Spring 2019

Outline

- Binary search
- Pseudo-code
- Analysis
- In-place binary search
- Iterative binary search

Phonebook Search

Activity #1

2 *min*

Phonebook Search

Activity #1

2 min

Phonebook Search

Activity #1

1 min

Phonebook Search

Activity #1

0 min

The Problem

1	1	3	4	7	8	10	10	12	18	19	21	23	23	24

- Is an item **`x`** in a sorted array?
  - ex: is **`5`** in the array above?
- Idea #0
  - scan array to find **`x`**
  - **`O(n)`** running time
- Can we do better?

Let's use the fact that array is sorted...

The Problem

1	1	3	4	7	8	10	10	12	18	19	21	23	23	24

- Observation `#1`
  - we can stop searching for `11` if we reach `12`
  - we can stop searching for **`x`** if we reach **`y > x`**
- Why?
  - since array is sorted, `11` can't be after `12`
  - since array is sorted, **`x`** can't be after **`y`**
- But what if we're looking for `25`?

The Problem

1	1	3	4	7	8	10	10	12	18	19	21	23	23	24

mid

- Observation #`1`
  - we can stop searching for `x` if we reach `y > x`
- Observation #`2`
  - what happens if we compare `x` to middle element?
  - if `x = mid`, then we found `x`
  - if `x < mid`, then `x` cannot be in right half of array
  - if `x > mid`, then `x` cannot be in left half of array

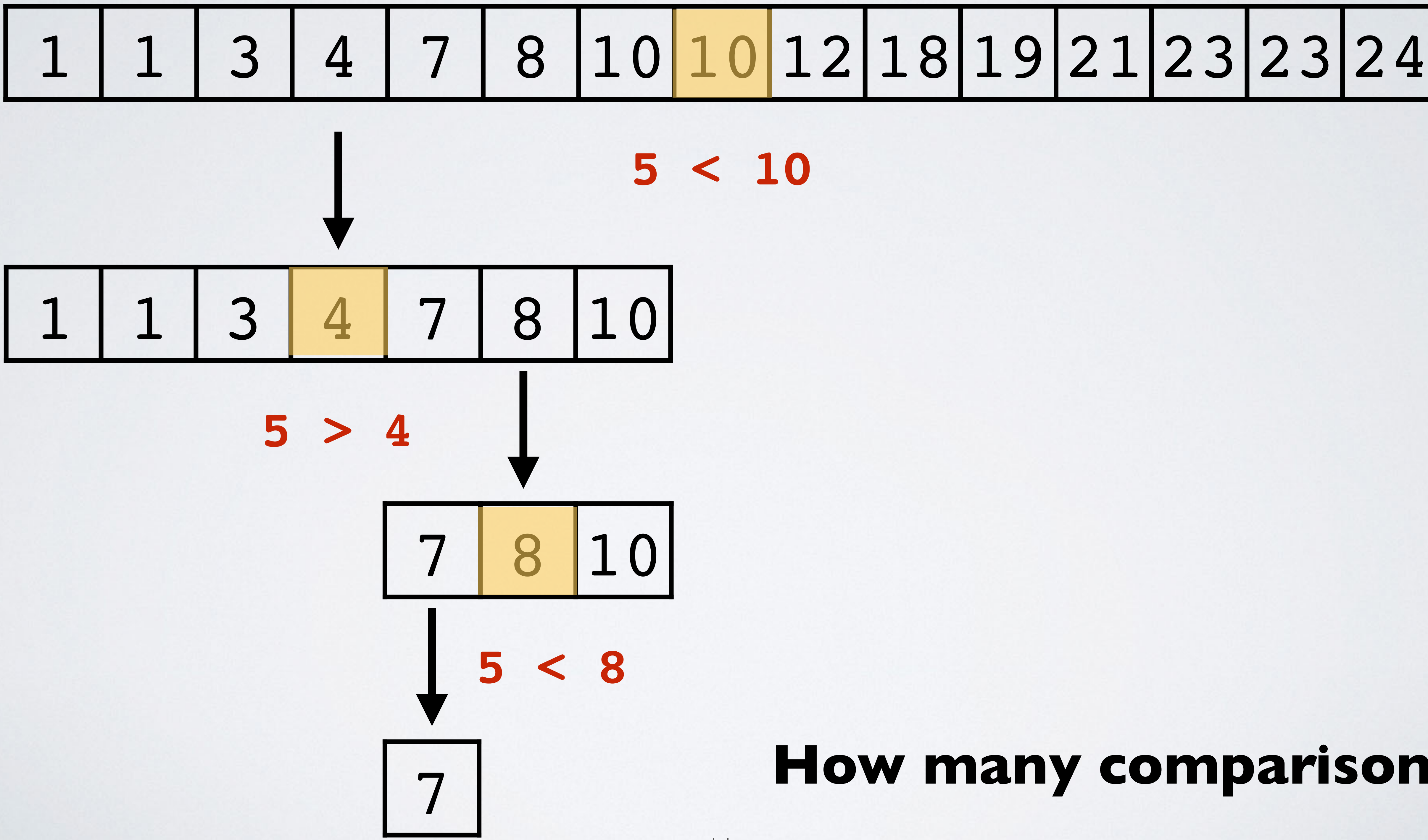
The Problem

1	1	3	4	7	8	10	10	12	18	19	21	23	23	24

- Using observation #2
  - We got rid of half the array!
- What if do it again?
  - same problem…but half the size!
- Does this remind you of something?

The Problem

Find 5



How many comparisons?

Analysis

- How many comparisons on array of size `n`?
  - after each comparison we cut array in half
  - how many times can we split array in `2` before we get array of size `1`?
    - if $n=2^k$ for some `k`, then $\log_2(n)=k$
- So what is runtime of binary search?
  - **O(log n)**?
- Let's look at pseudo-code!

Binary Search Pseudo-Code

```
function binarysearch(A,x):
  if A.size == 0:
    return false
  if A.size == 1:
    return A[0] == x

  mid = A.size / 2

  if x == A[mid]:
    return true
  if x > A[mid]:
    return binarysearch(A[mid+1…end], x)
  if x < A[mid]:
    return binarysearch(A[0…mid-1], x)
```

Assume `A.size` is power of `2`

Binary Search Analysis

- Binary search implementation is recursive…
- So how do we analyze it?
  - write down the recurrence relation
  - solve it with plug & chug + induction
- The recurrence relation of Binary Search is
  - $T(n) = T(n/2) + f(n), \text{with } T(1) = c$
  - where `f(n)` is the work done at each level of recursion
- Where does `T(n/2)` come from?
  - because we cut the problem in half at each level of recursion
- What is `f(n)`?

Binary Search Pseudo-Code

```
function binarysearch(A,x):
  if A.size == 0:                              O(1)
    return false                               O(1)
  if A.size == 1:                              O(1)
    return A[0] == x                           O(1)

  mid = A.size / 2                             O(1)

  if x == A[mid]:                              O(1)
    return true                                O(1)
  if x > A[mid]:                               O(1)
    return binarysearch(A[mid+1…end], x)
  if x < A[mid]:                               O(1)
    return binarysearch(A[0…mid-1], x)
```

copying half the array… is O(n)!!

Binary Search Analysis

- Recurrence relation:

linear

$$T(n) = T(n/2) + \boxed{c_1 n + c_2,} \quad T(1) = c_0$$

- Plug and chug:

$$T(1) = c_0$$

$$T(2) = T(1) + 2c_1 + c_2 = c_0 + 2c_1 + c_2$$

$$T(4) = T(2) + 4c_1 + c_2 = c_0 + (4+2)c_1 + 2c_2$$

$$T(8) = T(4) + 8c_1 + c_2 = c_0 + (8+4+2)c_1 + 3c_2$$

$$T(n) = c_0 + \left(n + \frac{n}{2} + \frac{n}{4} + \cdots + 4 + 2\right) c_1 + (\log n)c_2$$

converges to 2n as n gets large

What is T(n)?

Binary Search Analysis

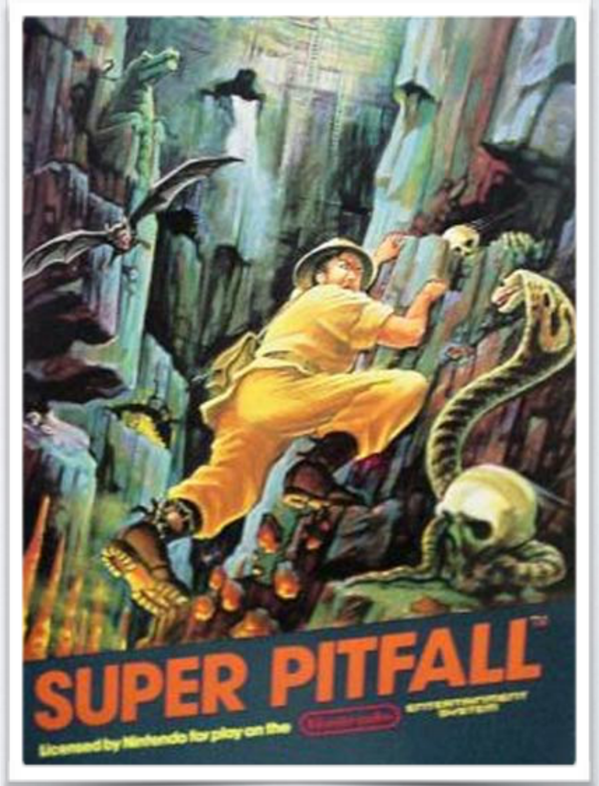
- `T(n)` is `O(n + log n)`
  - is this a proof?
- As bad as scanning array…
  - But in our example it was **O(log n)**!

What happened?

Subtlety in Binary Search!

- In our implementation we copied half the array
  - at each step, this cost us `O(n)`
  - so runtime went back up to `O(n)`

Common pitfall when implementing efficient algorithms



Q: What should we do?

In-Place Binary Search

- We should keep reusing the original array
  - no copying of elements!
- We should implement it “in-place”

In-Place Binary Search Pseudo-Code

```
function binarysearch(A, lo, hi, x):
  if lo >= hi:
    return A[lo] == x

  mid = (lo + hi) /2

  if x == A[mid]:
    return true
  if x > A[mid]:
    return binarysearch(A, mid+1, hi, x)
  if x < A[mid]:
    return binarysearch(A, lo, mid-1, x)
```

In-Place Binary Search

```
A = [0, 3, 8, 10, 10, 15, 18]
x = 7
```

Activity #2

4 min

In-Place Binary Search

```
A = [0, 3, 8, 10, 10, 15, 18]
x = 7
```

Activity #2

4 min

In-Place Binary Search

```
A = [0, 3, 8, 10, 10, 15, 18]
x = 7
```

Activity #2

3 min

In-Place Binary Search

```
A = [0, 3, 8, 10, 10, 15, 18]
x = 7
```

Activity #2

2 min

In-Place Binary Search

```
A = [0, 3, 8, 10, 10, 15, 18]
x = 7
```

Activity #2

1 min

In-Place Binary Search

```
A = [0, 3, 8, 10, 10, 15, 18]
x = 7
```

Activity #2

0 min

In-Place Binary Search

- Does `O(1)` ops at each level of recursion
- Recurrence is now

$$T(n) = T(n/2) + c_1, \text{ with } T(1) = c_0$$

- Plug & Chug:

$$T(1) = c_0$$
$$T(2) = T(1) + c_1 = c_0 + c_1$$
$$T(4) = T(2) + c_1 = c_0 + 2c_1$$
$$T(8) = T(4) + c_1 = c_0 + 3c_1$$

$$T(n) = c_0 + (\log n) \cdot c_1$$

In-Place Binary Search

- So in-place binary search is
  - **O(log n)** !
- Is this a proof?

Iterative Binary Search

```
function binarysearch(A,x):
  lo = 0
  hi = A.size - 1

  while lo < hi
    mid = (lo + hi) / 2
    if A[mid] == x:
      return true
    if A[mid] < x:
      lo = mid + 1
    if A[mid] > x:
      hi = mid - 1

  return [lo] == x
```

- Recursive algorithms can be implemented iteratively