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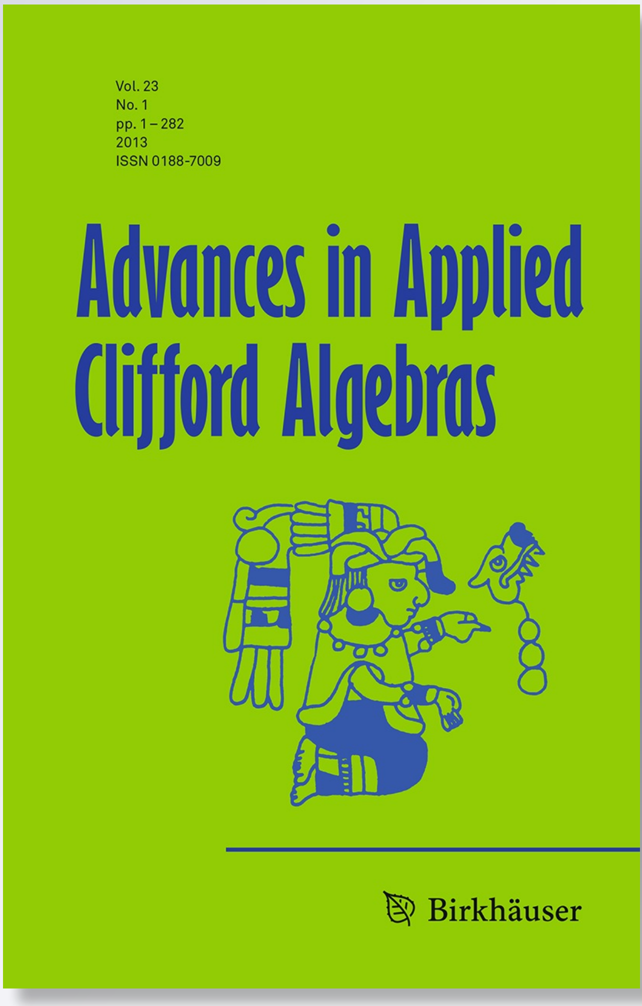

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**Advances in Applied Clifford Algebras**

# The Einstein Relativistic Velocity Model of Hyperbolic Geometry and Its Plane Separation Axiom

Nilgün Sönmez* and A.A. Ungar

**Abstract.** The relativistically admissible velocities of Einstein's special theory of relativity are regulated by the Beltrami-Klein ball model of the hyperbolic geometry of Bolyai and Lobachevsky. It is shown in this expository article that the Einstein velocity addition law of relativistically admissible velocities enables Cartesian coordinates to be introduced into hyperbolic geometry, resulting in the Cartesian-Beltrami-Klein ball model of hyperbolic geometry. Suggestively, the latter is increasingly becoming known as the *Einstein Relativistic Velocity Model* of hyperbolic geometry. Möbius addition is a transformation of the ball linked to Clifford algebra. Einstein addition and Möbius addition in the ball of the Euclidean $n$-space are isomorphic to each other, and they share remarkable analogies with vector addition. Thus, in particular, Einstein (Möbius) addition admits scalar multiplication, giving rise to gyrovector spaces, just as vector addition admits scalar multiplication, giving rise to vector spaces. Moreover, the resulting Einstein (Möbius) gyrovector spaces form the algebraic setting for the Beltrami-Klein (Poincaré) ball model of $n$-dimensional hyperbolic geometry, just as vector spaces form the algebraic setting for the standard Cartesian model of $n$-dimensional Euclidean geometry. As an illustrative novel example special attention is paid to the study of the plane separation axiom (PSA) in Euclidean and hyperbolic geometry.

**Keywords.** Hyperbolic geometry; Plane separation axiom; Einstein velocity addition; Möbius addition; Gyrogroup; Gyrovector Space.

## 1. Introduction

Newtonian velocity addition is both commutative and associative and, as such, it is a commutative group operation. This group operation admits scalar

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multiplication, giving rise to vector spaces that, in turn, form the algebraic setting for the standard Cartesian model of $n$-dimensional Euclidean geometry. Einsteinian velocity addition is given by the Einstein relativistic velocity addition law of the special theory of relativity that Einstein introduced in 1905 [9].

Contrasting Newtonian velocity addition, Einsteinian velocity addition is neither commutative nor associative. Rather, it is both *gyrocommutative* and *gyroassociative* in the sense explained in Sec. 4 in the study of Einstein addition vs. vector addition. As such, in full analogy with Newtonian velocity addition, Einsteinian velocity addition is a gyrocommutative gyrogroup operation. This gyrogroup operation admits scalar multiplication, giving rise to Einstein gyrovector spaces that, in turn, form the algebraic setting for the Cartesian-Beltrami-Klein ball model of $n$-dimensional hyperbolic geometry. Here, the Cartesian-Beltrami-Klein model is the well-known Beltrami-Klein ball model of $n$-dimensional hyperbolic geometry into which Cartesian coordinates are introduced, as illustrated in Fig. 1.

Indeed, Fig. 1 in Sec. 9 presents a *gyrosegment* in an Einstein gyrovector plane, drawn graphically by the use of Cartesian coordinates in the same way we commonly use these coordinates in drawing graphically a segment in a Euclidean plane. In the same way that Cartesian coordinates enable Euclidean geometry to be studied analytically [5], these coordinates enable hyperbolic geometry to be studied analytically as well, as demonstrated in [30, 32, 38].

The Cartesian-Beltrami-Klein ball model is becoming increasingly popular, suggestively being known as the *Einstein relativistic velocity model* of the hyperbolic geometry of Bolyai and Lobachevsky; see, for instance, [2, 3, 4, 26].

Furthermore, Einstein addition is isomorphic to the so called Möbius addition that, in turn, gives rise to Möbius gyrovector spaces. Remarkably, the latter form the algebraic setting for the Poincaré ball model of $n$-dimensional hyperbolic geometry. Classical connections between Möbius addition and Clifford numbers are described by Ahlfors in an expository paper, [1]. Recently, modern connections have been discovered and employed by Ferreira [11, 12, 13].

The aim of this expository article is to present the role that Einstein addition plays in the foundations of the hyperbolic geometry of Bolyai and Lobachevsky. Special attention is paid to the so called *Plane Separation Axiom* (PSA) of metric geometry [23, Chap. 4], and to the translation of its validity from Euclidean into hyperbolic geometry. This translation is interesting and instructive since it involves the translation of the concept of barycentric coordinates from Euclidean geometry into hyperbolic geometry, as explained in Secs. 13 – 15.

In order to emphasize analogies with classical results, we prefix a *gyro* to any term that describes a concept in Euclidean geometry and in associative algebra to mean the analogous concept in hyperbolic geometry and nonassociative algebra. The prefix "gyro" stems from "gyration", which is the mathematical abstraction of the special relativistic effect known as "Thomas precession" [32, Sec. 10.3]. Remarkably, the mere introduction of gyrations translates Euclidean into hyperbolic geometry, as demonstrated in [29, 30, 32, 34, 37, 38], and as indicated in Secs. 4 – 9.

In particular, a sequence of definitions and results that lead in [23] to metric geometry and to the Plane Separation Axiom (PSA) is translated in this paper into analogous concepts that lead to gyrometric geometry, into hyperbolic geometry and into the Hyperbolic Plane Separation Axiom. The latter, presented in Sec. 17, is naturally called in gyrolanguage the Gyroplane Separation Axiom (GPSA).

In Sec. 7 we show that, in the gyrovector sense, Einstein addition is isomorphic to Möbius addition. The latter gives rise to Möbius gyrovector spaces, just as Einstein addition gives rise to Einstein gyrovector spaces. Furthermore, Möbius gyrovector spaces form the algebraic setting for the Cartesian-Poincaré ball model of $n$-dimensional hyperbolic geometry, just as Einstein gyrovector spaces form the algebraic setting for the Cartesian-Beltrami-Klein ball model of $n$-dimensional hyperbolic geometry [31]. The use of gyrogroups and gyrovector spaces described in this paper prove useful in several areas, including (i) hyperbolic geometry, [2], [3], [4], [7], [8], [25], [26], [29], [30], [32], [34], [35], [36], [37], [38]; (ii) relativity physics, [29], [32], [38], [39]; and (ii) Clifford algebra, [11], [13].

## 2. Metric Geometry

We present in this section a sequence of definitions that lead to metric geometry and to the Plane Separation Axiom (PSA) in order to set the stage for analogies that guide the introduction of these concepts into hyperbolic geometry.

**Definition 1. (Abstract Geometry** [23, p. 17]**).** An abstract geometry $\mathcal{A} = \{\mathcal{S}, \mathcal{L}\}$ consists of a set $\mathcal{S}$, whose elements are called points, together with a collection $\mathcal{L}$ of non-empty subsets of $\mathcal{S}$ called lines, such that

1. For every two points $A, B \in \mathcal{S}$ there is a line $L \in \mathcal{L}$ with $A \in \mathcal{L}$ and $B \in \mathcal{L}$.
2. Every line has at least two points.

**Definition 2. (Incidence Geometry** [23, p. 22]**).** An abstract geometry $\{\mathcal{S}, \mathcal{L}\}$ is an incidence geometry if

1. Every two distinct points in $\mathcal{L}$ lie on a unique line.
2. There exist three points $A, B, C \in \mathcal{S}$ which do not lie all on one line.

**Definition 3. (Distance Function** [23, p. 28]**).** A distance function on a set $\mathcal{S}$ is a function $d : \mathcal{S} \times \mathcal{S} \to \mathbb{R}$ such that for all $P, Q \in \mathcal{S}$

1. $d(P, Q) \geq 0$;
2. $d(P, Q) = 0$ if and only if $P = Q$; and
3. $d(P, Q) = d(Q, P)$.

**Definition 4. (Ruler** [23, p. 30]**).** Let $L$ be a line in an incidence geometry $\{\mathcal{S}, \mathcal{L}\}$. Assume that there is a distance function $d$ on $\mathcal{S}$. A function $f : L \to \mathbb{R}$ is a ruler for $L$ if

1. $f$ is a bijection; and
2. for each pair of points $P$ and $Q$ on $L$
$$|f(P) - f(Q)| = d(P, Q) \tag{1}$$
Equation (1) is called the ruler equation and $f(P)$ is called the coordinate of $P$ with respect to $f$.

**Definition 5. (Metric Geometry** [23, p. 30]**).** An incidence geometry $\{\mathcal{S}, \mathcal{L}\}$ together with a distance function $d$ satisfies the Ruler Postulate if every line $L \in \mathcal{L}$ has a ruler. In this case we say $\mathcal{M} = \{\mathcal{S}, \mathcal{L}, d\}$ is a metric geometry.

**Definition 6. (Betweenness in a Metric Geometry** [23, p. 47]**).** $B$ is between $A$ and $C$ if $A, B, C$ are distinct collinear points in a metric geometry $\{\mathcal{S}, \mathcal{L}, d\}$ and if $d(A, B) + d(B, C) = d(A, C)$. We use the notation $A - B - C$.

**Definition 7. (Line Segments** [23, p. 52]**).** If $A$ and $B$ are distinct points in a metric geometry $\{\mathcal{S}, \mathcal{L}, d\}$ then the line segment from $A$ to $B$ is the set
$$AB = \{C \in \mathcal{S} : A - C - B \quad \text{or} \quad C = A \quad \text{or} \quad C = B\} \tag{2}$$

**Definition 8. (Convex Set** [23, p. 63]**).** Let $\{\mathcal{S}, \mathcal{L}, d\}$ be a metric geometry and let $\mathcal{S}_1 \subset \mathcal{S}$. The set $\mathcal{S}_1$ is said to be convex if for every two points $P, Q \in \mathcal{S}_1$, the segment $PQ$ is a subset of $\mathcal{S}_1$.

**Definition 9. (Plane Separation Axiom** (PSA) [23, p. 64]**).** A metric geometry $\{\mathcal{S}, \mathcal{L}, d\}$ satisfies the Plane Separation Axiom (PSA) if for every $L \in \mathcal{L}$ there are two subsets $H_1$ and $H_2$ of $\mathcal{S}$ (called half planes determined by $L$) such that

1. $\mathcal{S} - L = H_1 \cup H_2$;
2. $H_1$ and $H_2$ are disjoint and each is convex;
3. If $A \in H_1$ and $B \in H_2$ then $AB \cap L \neq \phi$.

Millman and Parker remark [23, p. 63] that the Plane Separation Axiom is a careful statement of the very intuitive idea that every line in a plane has "two sides".

## 3. Einstein Addition

Let $c$ be any positive constant and let $\mathbb{R}^n = (\mathbb{R}^n, +, \cdot)$ be the Euclidean $n$-space, $n = 1, 2, 3, \ldots$, equipped with the common vector addition, $+$, and inner product, $\cdot$. The home of all $n$-dimensional Einsteinian velocities is the $c$-ball
$$\mathbb{R}^n_c = \{\mathbf{v} \in \mathbb{R}^n : \|\mathbf{v}\| < c\} \tag{3}$$
It is the open ball of radius $c$, centered at the origin of $\mathbb{R}^n$, consisting of all vectors $\mathbf{v}$ in $\mathbb{R}^n$ with magnitude (norm) $\|\mathbf{v}\| = \sqrt{\mathbf{v}\cdot\mathbf{v}}$ smaller than $c$.

Einstein addition and scalar multiplication play in the ball $\mathbb{R}_c^n$ the role that vector addition and scalar multiplication play in the Euclidean $n$-space $\mathbb{R}^n$.

**Definition 10.** Einstein addition is a binary operation, $\oplus$, in the $c$-ball $\mathbb{R}_c^n$ given by the equation, [29], [27, Eq. 2.9.2], [24, p. 55], [14],

$$\mathbf{u}\oplus\mathbf{v} = \frac{1}{1+\frac{\mathbf{u}\cdot\mathbf{v}}{c^2}} \left\{ \mathbf{u} + \frac{1}{\gamma_{\mathbf{u}}}\mathbf{v} + \frac{1}{c^2}\frac{\gamma_{\mathbf{u}}}{1+\gamma_{\mathbf{u}}}(\mathbf{u}\cdot\mathbf{v})\mathbf{u} \right\} \tag{4}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}_c^n$, where $\gamma_{\mathbf{u}}$ is the Lorentz gamma factor given by the equation

$$\gamma_{\mathbf{v}} = \frac{1}{\sqrt{1-\frac{\|\mathbf{v}\|^2}{c^2}}} \tag{5}$$

where $\mathbf{u}\cdot\mathbf{v}$ and $\|\mathbf{v}\|$ are the inner product and the norm in the ball, which the ball $\mathbb{R}_c^n$ inherits from its space $\mathbb{R}^n$.

A nonempty set with a binary operation is called a *groupoid* so that, accordingly, the pair $(\mathbb{R}_c^n, \oplus)$ is an *Einstein groupoid*.

In the Newtonian limit of large $c$, $c \to \infty$, the ball $\mathbb{R}_c^n$ expands to the whole of its space $\mathbb{R}^n$, as we see from (3), and Einstein addition $\oplus$ in $\mathbb{R}_c^n$ reduces to the ordinary vector addition $+$ in $\mathbb{R}^n$, as we see from (4) and (5).

In physical applications, $\mathbb{R}^n = \mathbb{R}^3$ is the Euclidean 3-space, which is the space of all classical, Newtonian velocities, and $\mathbb{R}_c^n = \mathbb{R}_c^3 \subset \mathbb{R}^3$ is the $c$-ball of $\mathbb{R}^3$ of all relativistically admissible, Einsteinian velocities. Furthermore, the constant $c$ represents in physical applications the vacuum speed of light. Since we are interested in geometry, we allow $n$ to be any positive integer and, sometimes, replace $c$ by $s$.

We naturally use the abbreviation $\mathbf{u}\ominus\mathbf{v} = \mathbf{u}\oplus(-\mathbf{v})$ for Einstein subtraction, so that, for instance, $\mathbf{v}\ominus\mathbf{v} = \mathbf{0}$, $\ominus\mathbf{v} = \mathbf{0}\ominus\mathbf{v} = -\mathbf{v}$. Einstein addition and subtraction satisfy the equations

$$\ominus(\mathbf{u}\oplus\mathbf{v}) = \ominus\mathbf{u}\ominus\mathbf{v} \tag{6}$$

and

$$\ominus\mathbf{u}\oplus(\mathbf{u}\oplus\mathbf{v}) = \mathbf{v} \tag{7}$$

for all $\mathbf{u}, \mathbf{v}$ in the ball $\mathbb{R}_c^n$, in full analogy with vector addition and subtraction in $\mathbb{R}^n$. Identity (6) is called the *gyroautomorphic inverse property* of Einstein addition, and Identity (7) is called the *left cancellation law* of Einstein addition. We may note that Einstein addition does not obey the naive right counterpart of the left cancellation law (7) since, in general,

$$(\mathbf{u}\oplus\mathbf{v})\ominus\mathbf{v} \neq \mathbf{u} \tag{8}$$

However, this seemingly lack of a *right cancellation law* of Einstein addition is repaired, for instance, in [38, Sec. 1.9].

## 4. Einstein Addition Vs. Vector Addition

Vector addition, $+$, in $\mathbb{R}^n$ is both commutative and associative, satisfying

$$\begin{aligned} \mathbf{u}+\mathbf{v} &= \mathbf{v}+\mathbf{u} && \text{Commutative Law} \\ \mathbf{u}+(\mathbf{v}+\mathbf{w}) &= (\mathbf{u}+\mathbf{v})+\mathbf{w} && \text{Associative Law} \end{aligned} \tag{9}$$

for all $\mathbf{u},\mathbf{v},\mathbf{w}\in\mathbb{R}^n$. In contrast, Einstein addition, $\oplus$, in $\mathbb{R}^n_c$ is neither commutative nor associative.

In order to measure the extent to which Einstein addition deviates from associativity we introduce *gyrations*, which are self maps of $\mathbb{R}^n$ that are *trivial* in the special cases when the application of $\oplus$ is associative. For any $\mathbf{u},\mathbf{v}\in\mathbb{R}^n_c$ the gyration $\mathrm{gyr}[\mathbf{u},\mathbf{v}]$ is a map of the Einstein groupoid $(\mathbb{R}^n_c,\oplus)$ onto itself. Gyrations $\mathrm{gyr}[\mathbf{u},\mathbf{v}]\in Aut(\mathbb{R}^n_c,\oplus)$, $\mathbf{u},\mathbf{v}\in\mathbb{R}^n_c$, are defined in terms of Einstein addition by the equation

$$\mathrm{gyr}[\mathbf{u},\mathbf{v}]\mathbf{w} = \ominus(\mathbf{u}\oplus\mathbf{v})\oplus\{\mathbf{u}\oplus(\mathbf{v}\oplus\mathbf{w})\} \tag{10}$$

for all $\mathbf{u},\mathbf{v},\mathbf{w}\in\mathbb{R}^n_c$, and they turn out to be automorphisms of the Einstein groupoid $(\mathbb{R}^n_c,\oplus)$.

We recall that an automorphism of a groupoid $(S,\oplus)$ is a one-to-one map $f$ of $S$ onto itself that respects the binary operation, that is, $f(a\oplus b) = f(a)\oplus f(b)$ for all $a,b\in S$. The set of all automorphisms of a groupoid $(S,\oplus)$ forms a group, denoted $Aut(S,\oplus)$. To emphasize that the gyrations of an Einstein gyrogroup $(\mathbb{R}^n_c,\oplus)$ are automorphisms of the gyrogroup, gyrations are also called *gyroautomorphisms*.

A gyration $\mathrm{gyr}[\mathbf{u},\mathbf{v}]$, $\mathbf{u},\mathbf{v}\in\mathbb{R}^n_c$, is *trivial* if $\mathrm{gyr}[\mathbf{u},\mathbf{v}]\mathbf{w}=\mathbf{w}$ for all $\mathbf{w}\in\mathbb{R}^n_c$. Thus, for instance, the gyrations $\mathrm{gyr}[\mathbf{0},\mathbf{v}]$, $\mathrm{gyr}[\mathbf{v},\mathbf{v}]$ and $\mathrm{gyr}[\mathbf{v},\ominus\mathbf{v}]$ are trivial for all $\mathbf{v}\in\mathbb{R}^n_c$, as we see from (10).

Einstein gyrations, which possess their own rich structure, measure the extent to which Einstein addition deviates from commutativity and associativity as we see from the gyrocommutative and the gyroassociative laws of Einstein addition in the following identities [29, 30, 32]:

$$\begin{aligned} \mathbf{u}\oplus\mathbf{v} &= \mathrm{gyr}[\mathbf{u},\mathbf{v}](\mathbf{v}\oplus\mathbf{u}) && \text{Gyrocommutative Law} \\ \mathbf{u}\oplus(\mathbf{v}\oplus\mathbf{w}) &= (\mathbf{u}\oplus\mathbf{v})\oplus\mathrm{gyr}[\mathbf{u},\mathbf{v}]\mathbf{w} && \text{Left Gyroassociative Law} \\ (\mathbf{u}\oplus\mathbf{v})\oplus\mathbf{w} &= \mathbf{u}\oplus(\mathbf{v}\oplus\mathrm{gyr}[\mathbf{v},\mathbf{u}]\mathbf{w}) && \text{Right Gyroassociative Law} \\ \mathrm{gyr}[\mathbf{u}\oplus\mathbf{v},\mathbf{v}] &= \mathrm{gyr}[\mathbf{u},\mathbf{v}] && \text{Gyration Left Loop Property} \\ \mathrm{gyr}[\mathbf{u},\mathbf{v}\oplus\mathbf{u}] &= \mathrm{gyr}[\mathbf{u},\mathbf{v}] && \text{Gyration Right Loop Property} \\ \mathrm{gyr}[\ominus\mathbf{u},\ominus\mathbf{v}] &= \mathrm{gyr}[\mathbf{u},\mathbf{v}] && \text{Gyration Even Property} \\ (\mathrm{gyr}[\mathbf{u},\mathbf{v}])^{-1} &= \mathrm{gyr}[\mathbf{v},\mathbf{u}] && \text{Gyration Inversion Law} \end{aligned} \tag{11}$$

for all $\mathbf{u},\mathbf{v},\mathbf{w}\in\mathbb{R}^n_c$.

Einstein addition is thus regulated by gyrations to which it gives rise owing to its nonassociativity, so that Einstein addition and its gyrations are inextricably linked. The resulting gyrocommutative gyrogroup structure of Einstein addition was discovered in 1988 [28]. Interestingly, gyrations are the mathematical abstraction of the relativistic effect known as *Thomas precession* [32, Sec. 10.3].

The loop properties in (11) present important gyration identities. These two gyration identities are, however, just the tip of a giant iceberg. The identities in (11) and many other useful gyration identities are studied in [29, 30, 32, 34, 37, 38].

## 5. From Einstein Addition to Gyrogroups

Taking the key features of the Einstein groupoid $(\mathbb{R}^n_c, \oplus)$ as axioms, and guided by analogies with groups, we are led to the formal gyrogroup definition in which gyrogroups turn out to form a most natural generalization of groups.

**Definition 11. (Gyrogroups** [32, p. 17]**).** *A groupoid* $(G, \oplus)$ *is a gyrogroup if its binary operation satisfies the following axioms. In* $G$ *there is at least one element,* $0$*, called a left identity, satisfying*

$(G1)$ $\qquad 0\oplus a = a$

*for all* $a \in G$*. There is an element* $0 \in G$ *satisfying axiom* $(G1)$ *such that for each* $a \in G$ *there is an element* $\ominus a \in G$*, called a left inverse of* $a$*, satisfying*

$(G2)$ $\qquad \ominus a\oplus a = 0\,.$

*Moreover, for any* $a, b, c \in G$ *there exists a unique element* $\mathrm{gyr}[a, b]c \in G$ *such that the binary operation obeys the left gyroassociative law*

$(G3)$ $\qquad a\oplus(b\oplus c) = (a\oplus b)\oplus\mathrm{gyr}[a, b]c\,.$

*The map* $\mathrm{gyr}[a, b] : G \to G$ *given by* $c \mapsto \mathrm{gyr}[a, b]c$ *is an automorphism of the groupoid* $(G, \oplus)$*, that is,*

$(G4)$ $\qquad \mathrm{gyr}[a, b] \in Aut(G, \oplus)\,,$

*and the automorphism* $\mathrm{gyr}[a, b]$ *of* $G$ *is called the gyroautomorphism, or the gyration, of* $G$ *generated by* $a, b \in G$*. The operator* $\mathrm{gyr} : G \times G \to Aut(G, \oplus)$ *is called the gyrator of* $G$*. Finally, the gyroautomorphism* $\mathrm{gyr}[a, b]$ *generated by any* $a, b \in G$ *possesses the left loop property*

$(G5)$ $\qquad \mathrm{gyr}[a, b] = \mathrm{gyr}[a\oplus b, b]\,.$

The gyrogroup axioms $(G1)-(G5)$ in Definition 11 are classified into three classes:

1. The first pair of axioms, $(G1)$ and $(G2)$, is a reminiscent of the group axioms.
2. The last pair of axioms, $(G4)$ and $(G5)$, presents the gyrator axioms.
3. The middle axiom, $(G3)$, is a hybrid axiom linking the two pairs of axioms in (1) and (2).

As in group theory, we use the notation $a\ominus b = a\oplus(\ominus b)$ in gyrogroup theory as well. In full analogy with groups, gyrogroups are classified into gyrocommutative and non-gyrocommutative gyrogroups.

**Definition 12. (Gyrocommutative Gyrogroups).** *A gyrogroup* $(G, \oplus)$ *is gyrocommutative if its binary operation obeys the gyrocommutative law*

$$(G6) \qquad a \oplus b = \mathrm{gyr}[a, b](b \oplus a)$$

*for all* $a, b \in G$.

It was the study of Einstein velocity addition law and its associated Lorentz transformation group of special relativity theory that led to the discovery of the gyrogroup structure in 1988 [28]. However, gyrogroups are not peculiar to Einstein addition [33]. Rather, they are abound in the theory of groups [15, 16, 10], loops [17], quasigroup [18, 22], and Lie groups [19, 20, 21].

## 6. Einstein Scalar Multiplication

The rich structure of Einstein addition is not limited to its gyrocommutative gyrogroup structure. Indeed, Einstein addition admits scalar multiplication, giving rise to the Einstein gyrovector space. Remarkably, the resulting Einstein gyrovector spaces form the setting for the Cartesian-Beltrami-Klein ball model of hyperbolic geometry just as vector spaces form the setting for the standard Cartesian model of Euclidean geometry, as shown in [29, 30, 32, 34, 37, 38] and as indicated in the sequel.

Let $k \otimes \mathbf{v}$ be the Einstein addition of $k$ copies of $\mathbf{v} \in \mathbb{R}^n_c$, that is $k \otimes \mathbf{v} = \mathbf{v} \oplus \mathbf{v} \ldots \oplus \mathbf{v}$ ($k$ terms). Then,

$$k \otimes \mathbf{v} = c \frac{\left(1 + \frac{\|\mathbf{v}\|}{c}\right)^k - \left(1 - \frac{\|\mathbf{v}\|}{c}\right)^k}{\left(1 + \frac{\|\mathbf{v}\|}{c}\right)^k + \left(1 - \frac{\|\mathbf{v}\|}{c}\right)^k} \frac{\mathbf{v}}{\|\mathbf{v}\|} \tag{12}$$

The definition of scalar multiplication in an Einstein gyrovector space requires analytically continuing $k$ off the positive integers, thus obtaining the following definition:

**Definition 13. (Einstein Scalar Multiplication).** *An Einstein gyrovector space* $(\mathbb{R}^n_s, \oplus, \otimes)$ *is an Einstein gyrogroup* $(\mathbb{R}^n_s, \oplus)$ *with scalar multiplication* $\otimes$ *given by*

$$r \otimes \mathbf{v} = s \frac{\left(1 + \frac{\|\mathbf{v}\|}{s}\right)^r - \left(1 - \frac{\|\mathbf{v}\|}{s}\right)^r}{\left(1 + \frac{\|\mathbf{v}\|}{s}\right)^r + \left(1 - \frac{\|\mathbf{v}\|}{s}\right)^r} \frac{\mathbf{v}}{\|\mathbf{v}\|} = s \tanh(r \tanh^{-1} \frac{\|\mathbf{v}\|}{s}) \frac{\mathbf{v}}{\|\mathbf{v}\|} \tag{13}$$

*where* $r$ *is any real number,* $r \in \mathbb{R}$, $\mathbf{v} \in \mathbb{R}^n_s$, $\mathbf{v} \neq \mathbf{0}$, *and* $r \otimes \mathbf{0} = \mathbf{0}$, *and with which we use the notation* $\mathbf{v} \otimes r = r \otimes \mathbf{v}$.

Einstein gyrovector spaces are studied in [29, 30, 32, 34, 37, 38]. Einstein scalar multiplication does not distribute over Einstein addition, but it possesses other properties of vector spaces. For any positive integer $k$, and for all real numbers $r, r_1, r_2 \in \mathbb{R}$ and $\mathbf{v} \in \mathbb{R}^n_s$, we have

$$\begin{aligned} k\otimes\mathbf{v} &= \mathbf{v}\oplus\ldots\oplus\mathbf{v} && k \text{ terms} \\ (r_1+r_2)\otimes\mathbf{v} &= r_1\otimes\mathbf{v}\oplus r_2\otimes\mathbf{v} && \text{Scalar Distributive Law} \\ (r_1 r_2)\otimes\mathbf{v} &= r_1\otimes(r_2\otimes\mathbf{v}) && \text{Scalar Associative Law} \end{aligned} \tag{14}$$

in any Einstein gyrovector space $(\mathbb{R}^n_s, \oplus, \otimes)$.

Additionally, Einstein gyrovector spaces possess the *scaling property*

$$\frac{|r|\otimes\mathbf{a}}{\|r\otimes\mathbf{a}\|} = \frac{\mathbf{a}}{\|\mathbf{a}\|} \tag{15}$$

$\mathbf{a} \in \mathbb{R}^n_s,\ \mathbf{a} \neq \mathbf{0},\ r \in \mathbb{R},\ r \neq 0$, the *gyroautomorphism property*

$$\mathrm{gyr}[\mathbf{u},\mathbf{v}](r\otimes\mathbf{a}) = r\otimes\mathrm{gyr}[\mathbf{u},\mathbf{v}]\mathbf{a} \tag{16}$$

$\mathbf{a}, \mathbf{u}, \mathbf{v} \in \mathbb{R}^n_s$, $r \in \mathbb{R}$, and the identity gyroautomorphism

$$\mathrm{gyr}[r_1\otimes\mathbf{v}, r_2\otimes\mathbf{v}] = I \tag{17}$$

$r_1, r_2 \in \mathbb{R}$, $\mathbf{v} \in \mathbb{R}^n_s$.

Any Einstein gyrovector space $(\mathbb{R}^n_s, \oplus, \otimes)$ inherits an inner product and a norm from its vector space $\mathbb{R}^n$. These turn out to be invariant under gyrations, that is,

$$\begin{aligned} \mathrm{gyr}[\mathbf{a},\mathbf{b}]\mathbf{u}\cdot\mathrm{gyr}[\mathbf{a},\mathbf{b}]\mathbf{v} &= \mathbf{u}\cdot\mathbf{v} \\ \|\mathrm{gyr}[\mathbf{a},\mathbf{b}]\mathbf{v}\| &= \|\mathbf{v}\| \end{aligned} \tag{18}$$

for all $\mathbf{a}, \mathbf{b}, \mathbf{u}, \mathbf{v} \in \mathbb{R}^n_s$.

## 7. From Einstein to Möbius

In this section Einstein addition $\oplus$ is denoted by $\oplus_{\mathrm{E}}$ in order to contrast it with Möbius addition, which is denoted by $\oplus_{\mathrm{M}}$.

**Definition 14. (Möbius Addition** [32, p. 154]**).** Möbius addition $\oplus_{\mathrm{M}}$ in the ball $\mathbb{R}^n_s$ is given in terms of Einstein addition $\oplus_{\mathrm{E}}$ in the ball $\mathbb{R}^n_s$ by the equation

$$\mathbf{u}\oplus_{\mathrm{M}}\mathbf{v} = \tfrac{1}{2}\otimes(2\otimes\mathbf{u}\oplus_{\mathrm{E}}2\otimes\mathbf{v}) \tag{19}$$

We may note that Möbius addition $\oplus_{\mathrm{M}}$ and Einstein addition $\oplus_{\mathrm{E}}$ in (19) are different owing to the result that scalar multiplication, $\otimes$, does not distribute over Einstein addition, $\oplus_{\mathrm{E}}$. It is, indeed, interesting to realize that in terms of gyrovector space scalar multiplication, $\otimes$, Einstein addition and Möbius addition are closely related to each other by (19). More about gyrovector space isomorphism is found in [32, Sec. 6.21].

Möbius addition (19) in the ball $\mathbb{R}^n_s$ is given explicitly by the equation [32, Eq. (3.129)]

$$\mathbf{u}\oplus_{\mathrm{M}}\mathbf{v} = \frac{(1+\frac{2}{s^2}\mathbf{u}\cdot\mathbf{v}+\frac{1}{s^2}\|\mathbf{v}\|^2)\mathbf{u}+(1-\frac{1}{s^2}\|\mathbf{u}\|^2)\mathbf{v}}{1+\frac{2}{s^2}\mathbf{u}\cdot\mathbf{v}+\frac{1}{s^4}\|\mathbf{u}\|^2\|\mathbf{v}\|^2} \tag{20}$$

where $\cdot$ and $\|\cdot\|$ are the inner product and norm that the ball $\mathbb{R}^n_s$ inherits from its space $\mathbb{R}^n$.

Furthermore, Möbius scalar multiplication is identical with Einstein scalar multiplication (13), satisfying the two identities [32, Eq. (6.325)]

$$\begin{aligned} \mathbf{u}\oplus_{\mathrm{M}}\mathbf{v} &= \tfrac{1}{2}\otimes(2\otimes\mathbf{u}\oplus_{\mathrm{E}}2\otimes\mathbf{v}) \\ \mathbf{u}\oplus_{\mathrm{E}}\mathbf{v} &= 2\otimes(\tfrac{1}{2}\otimes\mathbf{u}\oplus_{\mathrm{M}}\tfrac{1}{2}\otimes\mathbf{v}) \end{aligned} \tag{21}$$

for all $\mathbf{u},\mathbf{v}\in\mathbb{R}^n_s$. Identities (21) exhibit the gyrovector space isomorphism between Einstein gyrovector spaces and Möbius gyrovector spaces.

In an expository paper [1, Eq. (3.3)], Ahlfors presents the so called *standard conformal mapping* of the ball $\mathbb{R}^n_s$ on itself, $T_a(x)$, which by Def. 14, can be written as

$$T_a(x) = \ominus_{\mathrm{M}}a\oplus_{\mathrm{M}}x \tag{22}$$

Clearly, $T_{-a}(x) = a\oplus_{\mathrm{M}}x$ is nothing else but Möbius addition. Ahlfors, however, does not call it "addition" since the results that (i) Möbius addition is both gyrocommutative and gyroassociative; and (ii) is closely related to Einstein addition became known fairly recently [29, Chap. 6].

Furthermore, in his expository paper [1] Ahlfors also presents an important relationship between Möbius addition and Clifford numbers. A useful Clifford algebra approach to Möbius gyrogroups was recently presented by Ferreira and Ren in [13].

## 8. From Einstein Scalar Multiplication to Gyrovector Spaces

Taking the key features of Einstein scalar multiplication as axioms, and guided by analogies with vector spaces, we are led to the formal gyrovector space definition in which gyrovector spaces turn out to form a most natural generalization of vector spaces.

**Definition 15. (Real Inner Product Gyrovector Spaces** [32, p. 154]**).** A real inner product gyrovector space $(G,\oplus,\otimes)$ (gyrovector space, in short) is a gyrocommutative gyrogroup $(G,\oplus)$ that obeys the following axioms:

(1) $G$ is a subset of a real inner product vector space $\mathbb{V}$ called the carrier of $G$, $G\subset\mathbb{V}$, from which it inherits its inner product, $\cdot$, and norm, $\|\cdot\|$, which are invariant under gyroautomorphisms, that is,

(V1) $\mathrm{gyr}[\mathbf{u},\mathbf{v}]\mathbf{a}\cdot\mathrm{gyr}[\mathbf{u},\mathbf{v}]\mathbf{b} = \mathbf{a}\cdot\mathbf{b}$ Inner Product Gyroinvariance

for all points $\mathbf{a},\mathbf{b},\mathbf{u},\mathbf{v}\in G$.

(2) $G$ admits a scalar multiplication, $\otimes$, possessing the following properties. For all real numbers $r,r_1,r_2\in\mathbb{R}$ and all points $\mathbf{a}\in G$:

(V2) $1\otimes\mathbf{a} = \mathbf{a}$ Identity Scalar Multiplication

(V3) $(r_1 + r_2)\otimes\mathbf{a} = r_1\otimes\mathbf{a}\oplus r_2\otimes\mathbf{a}$ Scalar Distributive Law

(V4) $(r_1 r_2)\otimes\mathbf{a} = r_1\otimes(r_2\otimes\mathbf{a})$ Scalar Associative Law

(V5) $\dfrac{|r|\otimes\mathbf{a}}{\|r\otimes\mathbf{a}\|} = \dfrac{\mathbf{a}}{\|\mathbf{a}\|}, \quad \mathbf{a} \neq \mathbf{0},\ r \neq 0$ Scaling Property

(V6) $\mathrm{gyr}[\mathbf{u},\mathbf{v}](r\otimes\mathbf{a}) = r\otimes\mathrm{gyr}[\mathbf{u},\mathbf{v}]\mathbf{a}$ Gyroautomorphism Property

(V7) $\mathrm{gyr}[r_1\otimes\mathbf{v}, r_2\otimes\mathbf{v}] = I$ Identity Gyroautomorphism.

(3) Real, one-dimensional vector space structure $(\|G\|, \oplus, \otimes)$ for the set $\|G\|$ of one-dimensional "vectors" (see, for instance, [6])

(V8) $\|G\| = \{\pm\|\mathbf{a}\| : \mathbf{a} \in G\} \subset \mathbb{R}$ Vector Space

with vector addition $\oplus$ and scalar multiplication $\otimes$, such that for all $r \in \mathbb{R}$ and $\mathbf{a}, \mathbf{b} \in G$,

(V9) $\|r\otimes\mathbf{a}\| = |r|\otimes\|\mathbf{a}\|$ Homogeneity Property

(V10) $\|\mathbf{a}\oplus\mathbf{b}\| \leq \|\mathbf{a}\|\oplus\|\mathbf{b}\|$ Gyrotriangle Inequality.

## 9. Gyrolines – The Hyperbolic Lines

In applications to geometry it is convenient to replace the notation $\mathbb{R}^n_c$ for the $c$-ball of an Einstein gyrovector space by the $s$-ball, $\mathbb{R}^n_s$. Moreover, it is understood that $n \geq 2$, unless specified otherwise.

Let $A, B \in \mathbb{R}^n_s$ be two distinct points of the Einstein gyrovector space $(\mathbb{R}^n_s, \oplus, \otimes)$, and let $t \in \mathbb{R}$ be a real parameter. Then, the graph of the set of all points

$$A\oplus(\ominus A\oplus B)\otimes t \tag{23}$$

$t \in \mathbb{R}$, in the Einstein gyrovector space $(\mathbb{R}^n_s, \oplus, \otimes)$ is a chord of the ball $\mathbb{R}^n_s$. As such, it is a geodesic line of the Beltrami-Klein ball model of hyperbolic geometry, shown in Fig. 1 for $n = 2$. The geodesic line (23) is the unique gyroline that passes through the points $A$ and $B$. It passes through the point $A$ when $t = 0$ and, owing to the left cancellation law, (7), it passes through the point $B$ when $t = 1$. Furthermore, it passes through the midpoint $m_{A,B}$ of $A$ and $B$ when $t = 1/2$. Accordingly, the *gyrosegment* $AB$ that joins the points $A$ and $B$ in Fig. 1 is obtained from gyroline (23) with $0 \leq t \leq 1$.

Gyrolines (23) are the geodesics of the Beltrami-Klein ball model of hyperbolic geometry. Similarly, gyrolines (23) with Einstein addition $\oplus$ replaced by Möbius addition $\oplus_{\mathrm{M}}$ are the geodesics of the Poincaré ball model of hyperbolic geometry. This interesting result is established by methods of differential geometry in [31].

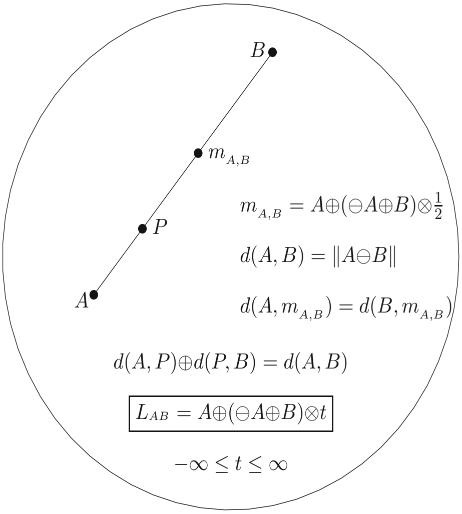


*Figure 1.*

**Figure 1.** Gyroline, the hyperbolic line. The gyroline $L_{AB} = A\oplus(\ominus A\oplus B)\otimes t$, $t \in \mathbb{R}$, in an Einstein gyrovector space $(\mathbb{R}^n_s, \oplus, \otimes)$ is a geodesic line in the Cartesian-Beltrami-Klein ball model of hyperbolic geometry, fully analogous to the straight line $A + (-A + B)t,\ t \in \mathbb{R}$, in the Cartesian model of the Euclidean geometry of $\mathbb{R}^n$. The points $A$ and $B$ correspond to $t = 0$ and $t = 1$, respectively. The point $P$ is a generic point on the gyroline through the points $A$ and $B$ lying between these points. The Einstein sum, $\oplus$, of the gyrodistance from $A$ to $P$ and from $P$ to $B$ equals the gyrodistance from $A$ to $B$. The point $m_{A,B}$ is the gyromidpoint of the points $A$ and $B$, corresponding to $t = 1/2$. The gyrosegment $AB$ that joins the point $A$ to $B$ is obtained from the gyroline $L_{AB}$ with $0 \le t \le 1$.

Each point of (23) with $0 < t < 1$ is said to lie *between* $A$ and $B$. Thus, for instance, the point $P$ in Fig. 1 lies between the points $A$ and $B$. As such, the points $A$, $P$ and $B$ obey the *gyrotriangle equality* according to which

$$d(A,P)\oplus d(P,B) = d(A,B) \tag{24}$$

in full analogy with Euclidean geometry, where

$$d(A,B) = \|\ominus A\oplus B\| \tag{25}$$

$A, B \in \mathbb{R}^n_s$, is the Einstein *gyrodistance function*, also called the Einstein *gyrometric*. This gyrodistance function in Einstein gyrovector spaces corresponds bijectively to a standard hyperbolic distance function, as demonstrated in Sec. 10, and it gives rise to the well-known Riemannian line element of the Beltrami-Klein ball model of hyperbolic geometry, as demonstrated in [31].

The gyrotriangle equality (24) suggests the following gyro-counterpart of Defs. 6 and 7:

**Definition 16. (Betweenness in a Gyrometric Geometry).** $B$ is between $A$ and $C$ if $A, B, C$ are distinct gyrocollinear points in a gyrometric geometry and if $d(A, B) \oplus d(B, C) = d(A, C)$. We use the notation $A - B - C$.

**Definition 17. (Gyroline Gyrosegments).** If $A$ and $B$ are distinct points in a gyrometric geometry then the gyroline gyrosegment from $A$ to $B$ is the set

$$AB = \{C \in \mathcal{S} : A - C - B \quad \text{or} \quad C = A \quad \text{or} \quad C = B\} \tag{26}$$

## 10. Turning the Einstein Gyrometric into a Metric

An Einstein gyrovector space $(\mathbb{R}^n_s, \oplus, \otimes)$ will turn out to be a gyrometric geometry with gyrometric given by the Einstein gyrodistance function (25), satisfying the gyrotriangle inequality, [32, p. 222],

$$\begin{aligned} d(A, C) &\leq d(A, B) \oplus d(B, C) \\ &= \frac{d(A, B) + d(B, C)}{1 + \frac{1}{s^2} d(A, B) d(B, C)} \\ &= s \frac{\tanh \phi_{B \ominus A} + \tanh \phi_{C \ominus B}}{1 + \tanh \phi_{B \ominus A} \tanh \phi_{C \ominus B}} \\ &= s \tanh(\phi_{B \ominus A} + \phi_{C \ominus B}) \end{aligned} \tag{27}$$

where

$$d(A, C) = \|C \ominus A\| = s \tanh \phi_{C \ominus A} \tag{28}$$

etc. Hence, by (27),

$$\tanh \phi_{C \ominus A} \leq \tanh(\phi_{B \ominus A} + \phi_{C \ominus B}) \tag{29}$$

or, equivalently,

$$\phi_{C \ominus A} \leq \phi_{B \ominus A} + \phi_{C \ominus B} \tag{30}$$

where

$$\phi_{B \ominus A} = \tanh^{-1} \frac{\|B \ominus A\|}{s} \tag{31}$$

is the rapidity of $B \ominus A$, etc.

Inequality (30) suggests the introduction of the Einstein distance function

$$h(A, B) = \tanh^{-1} \frac{d(A, B)}{s} = \frac{1}{2} \ln \frac{s + d(A, B)}{s - d(A, B)} \tag{32}$$

known as the Bergman metric on the ball $\mathbb{R}^n_s$ [40, p. 25], since it allows the replacement of the gyrotriangle inequality by a corresponding triangle inequality. Indeed, Einstein distance function (32) turns in (27) the gyrotriangle inequality into a corresponding triangle inequality (30) that, by (31)–(32), takes the form

$$h(A, C) \leq h(A, B) + h(B, C) \tag{33}$$

Hence, we see from (27) and (33) that the gyrometric (25) of an Einstein gyrovector space $(\mathbb{R}^n_s, \oplus, \otimes)$ and its gyrotriangle inequality (27),

$$d(A, C) \leq d(A, B) \oplus d(B, C) \tag{34}$$

are equivalent to the metric (32) of the Einstein gyrovector space and its triangle inequality (33).

In order to capture analogies with classical results, Einstein gyrovector spaces are studied along with their gyrometric rather than metric. Accordingly, an Einstein gyrovector space with its Einstein's gyrodistance forms a *gyrometric geometry*. In order to obtain a definition of a gyrometric geometry analogous to Def. 5 of metric geometry, we must define the *gyroruler* as the gyro-counterpart of the ruler in Def. 4, and demonstrate that any Einstein gyrovector space satisfies the gyroruler postulate. This is, accordingly, the task we face in Sec. 11.

## 11. The Gyroruler

**Definition 18. (Gyroruler).** Let $L$ be a gyroline in an Einstein gyrovector space $(\mathbb{R}^n_s, \oplus, \otimes)$ with the gyrodistance function (27). A function $f : L \to \mathbb{R}$ is a ruler for $L$ if:

1. f is bijective; and
2. for each pair of points $P_1, P_2 \in L$

$$|f(P_1) \ominus f(P_2)| = d(P_1, P_2) \tag{35}$$

Equation (35) is called the gyroruler equation, and $f(P)$ is called the coordinate of $P$ on the gyroline $L$ with respect to the gyroruler $f$.

We will now construct the Einstein gyroruler. Let $P_1$ and $P_2$ be any two points on a gyroline $L$,

$$L = A \oplus (\ominus A \oplus B) \otimes t \tag{36}$$

$t \in \mathbb{R}$, in the Einstein gyrovector space $(\mathbb{R}^n_s, \oplus, \otimes)$ so that

$$\begin{aligned} P_1 &= A \oplus (\ominus A \oplus B) \otimes t_1 \\ P_2 &= A \oplus (\ominus A \oplus B) \otimes t_2 \end{aligned} \tag{37}$$

for some $t_1, t_2 \in \mathbb{R}$.

Furthermore, we define the function $f : L \to \mathbb{R}$ by the equation

$$f(P) = f(A \oplus (\ominus A \oplus B) \otimes t) = t \otimes d(A, B) \tag{38}$$

for any $P \in L$.

In the following chain of equations, which are numbered for subsequent explanation, we show that $f$ is a gyroruler:

$$
\begin{aligned}
d(P_1,P_2) &\overbrace{=\!=}^{(1)} \|P_2\ominus P_1\| \\
&\overbrace{=\!=}^{(2)} \|\{A\oplus(\ominus A\oplus B)\otimes t_2\}\ominus\{A\oplus(\ominus A\oplus B)\otimes t_1\}\| \\
&\overbrace{=\!=}^{(3)} \|\mathrm{gyr}[A,\ominus A\oplus B)\otimes t_2]\{(\ominus A\oplus B)\otimes t_2\}\ominus\{(\ominus A\oplus B)\otimes t_1\}\| \\
&\overbrace{=\!=}^{(4)} \|(\ominus A\oplus B)\otimes t_2\ominus(\ominus A\oplus B)\otimes t_1\| \\
&\overbrace{=\!=}^{(5)} \|(\ominus A\oplus B)\otimes(t_2-t_1)\| \\
&\overbrace{=\!=}^{(6)} |t_2-t_1|\otimes\|\ominus A\oplus B\| \\
&\overbrace{=\!=}^{(7)} |t_2-t_1|\otimes d(A,B) \\
&\overbrace{=\!=}^{(8)} |(t_2-t_1)\otimes d(A,B)| \\
&\overbrace{=\!=}^{(9)} |t_2\otimes d(A,B)\ominus t_1\otimes d(A,B)| \\
&\overbrace{=\!=}^{(10)} |f(P_2)\ominus f(P_1)|
\end{aligned}
\tag{39}
$$

Derivation of the numbered equalities in (39) follows:

1. Follows from the definition of the gyrodistance function $d$ in (25).
2. Follows from the definition of the points $P_1$ and $P_2$ in (37).
3. Follows from the Gyrotranslation Theorem [32, Theorem 3.13, p. 57] for gyrocommutative gyrogroups.
4. Follows from the result that Einstein gyrations preserve the norm [32, p. 92].
5. Follows from the scalar distributive law of gyrovector spaces, Def. 15 (V3).
6. Follows from the homogeneity property of gyrovector spaces, Def. 15 (V9).
7. Follows from the definition of the gyrodistance function $d$ in (25).
8. Follows from the homogeneity property.
9. Follows from the scalar distributive law.
10. Follows from the definition of $f : L \to \mathbb{R}$ in (38).

We are now in the position to suggest the gyro-counterpart of Def. 5.

**Definition 19. (Gyrometric Geometry).** An Einstein gyrovector space $(\mathbb{R}^n_s,\oplus,\otimes)$ together with its gyrodistance function $d$ satisfies the Gyroruler

Postulate if every gyroline in the space has a ruler. In this case we say that the space is a gyrometric geometry.

Following Defs. 18 and 19, and the result in (39), it is now clear that any Einstein gyrovector space $\mathbb{R}^n_s$ is a gyrometric geometry, just as any Euclidean space $\mathbb{R}^n$ is a metric geometry.

## 12. Gyroangles – The Hyperbolic Angles

The analogies between lines and gyrolines suggest corresponding analogies between angles and gyroangles. In full analogy with the notions of distance and angle, the notion of the gyroangle is deduced from the notion of the gyrodistance. Let $O$, $A$ and $B$ be any three distinct points in an Einstein gyrovector space $(\mathbb{R}^n_s, \oplus, \otimes)$.

Following the analogies between gyrolines and lines, the radian measure of gyroangle $\alpha$ is, suggestively, given by the equation

$$\cos\alpha = \frac{\ominus O \oplus A}{\|\ominus O \oplus A\|} \cdot \frac{\ominus O \oplus B}{\|\ominus O \oplus B\|} \tag{40}$$

Here, $(\ominus O \oplus A)/\|\ominus O \oplus A\|$ and $(\ominus O \oplus B)/\|\ominus O \oplus B\|$ are unit *gyrovectors*, and cos is the common cosine function of trigonometry, which we apply to the inner product between unit gyrovectors rather than unit vectors. Accordingly, in the context of gyrovector spaces rather than vector spaces, we refer the function “cosine” of trigonometry to as the function “gyrocosine” of gyrotrigonometry. Similarly, all the other elementary trigonometric functions and their interrelationships survive unimpaired in their transition from the common trigonometry in Euclidean spaces $\mathbb{R}^n$ to a corresponding gyrotrigonometry in Einstein gyrovector space $\mathbb{R}^n_s$.

The center $\mathbf{0} = (0, \ldots, 0) \in \mathbb{R}^n_s$ of the ball $\mathbb{R}^n_s = (\mathbb{R}^n_s, \oplus, \otimes)$ is conformal (to Euclidean geometry) in the sense that the measure of any gyroangle with vertex $\mathbf{0}$ is equal to the measure of its Euclidean counterpart. Indeed, if $O = \mathbf{0}$ then (40) reduces to

$$\cos\alpha = \frac{A}{\|A\|} \cdot \frac{B}{\|B\|} \tag{41}$$

which is indistinguishable from its Euclidean counterpart.

Unlike Euclidean geometry, where the differences $-O + A$ and $A - O$ are equivalent, in general, the gyrodifferences $\ominus O \oplus A$ and $A \ominus O$ are distinct. The presence of the gyrodifferences $\ominus O \oplus A$ and $\ominus O \oplus B$ in the gyroangle definition (40), rather then $A \ominus O$ and $B \ominus O$, is dictated by the demand that gyroangles be *gyroinvariant*, that is, invariant under gyromotions, as stated in the gyroangle gyrocovariance theorem in [38, Theorem 6.1, p. 129]. Being invariant under the gyromotions of $\mathbb{R}^n_s$, which are left gyrotranslations and rotations about the origin, gyroangles are geometric objects of the hyperbolic geometry of the Einstein gyrovector space $(\mathbb{R}^n_s, \oplus, \otimes)$.

## 13. Euclidean and Hyperbolic Barycentric Coordinates

**Definition 20. (Euclidean Barycentric Coordinates).** Let $S = \{A_1, \ldots, A_N\}$ be a set of $N$ points in $\mathbb{R}^n$. Then, the real numbers $m_1, \ldots, m_N$, satisfying

$$\sum_{k=1}^{N} m_k \neq 0 \tag{42}$$

are barycentric coordinates of a point $P \in \mathbb{R}^n$ with respect to the set $S$ if

$$P = \frac{\sum_{k=1}^{N} m_k A_k}{\sum_{k=1}^{N} m_k} \tag{43}$$

Equation (43) is said to be a barycentric coordinate representation of $P$ with respect to the set $S = \{A_1, \ldots, A_N\}$.

Barycentric coordinates are homogeneous in the sense that the barycentric coordinates $(m_1, \ldots, m_N)$ of the point $P$ in (43) are equivalent to the barycentric coordinates $(\lambda m_1, \ldots, \lambda m_N)$ for any real nonzero number $\lambda \in \mathbb{R}$, $\lambda \neq 0$. Since in barycentric coordinates only ratios of coordinates are relevant, the barycentric coordinates $(m_1, \ldots, m_N)$ are also written as $(m_1 : \ldots : m_N)$ so that

$$(m_1 : m_2 : \ \ldots \ : m_N) = (\lambda m_1 : \lambda m_2 : \ \ldots \ : \lambda m_N) \tag{44}$$

for any real $\lambda \neq 0$.

It is easy to see from (43) that barycentric coordinates are independent of the choice of the origin of their vector space, that is,

$$X + P = \frac{\sum_{k=1}^{N} m_k (X + A_k)}{\sum_{k=1}^{N} m_k} \tag{45}$$

for all $X \in \mathbb{R}^n$. The proof that (45) follows from (43) is trivial, owing to the result that scalar multiplication in vector spaces distributes over vector addition. Interestingly, however, the hyperbolic counterpart, (49) below, of (45) is far away from being trivial because it involves both ordinary vector addition, $+$ (implicit in the $\Sigma$ notation for vector summation), and Einstein vector addition, $\oplus$.

It follows from (45) that the barycentric coordinate representation (43) of a point $P$ is *covariant* with respect to translations of $\mathbb{R}^n$ in the sense that the point $P$ and the points $A_k$, $k = 1, \ldots, N$, of its generating set $S = \{A_1, \ldots, A_N\}$ vary in (45) together under translations.

Hyperbolic barycentric coordinates in Einstein gyrovector spaces, fully analogous to Euclidean barycentric coordinates are called *gyrobarycentric coordinates.*

**Definition 21. (Gyrobarycentric Coordinates in Einstein Gyrovector Spaces [37, p. 179][38, p. 89]).** Let $S = \{A_1, \ldots, A_N\}$ be a pointwise independent set of $N \geq 2$ points in an Einstein gyrovector space $(\mathbb{R}^n_s, \oplus, \otimes)$. The $N$ real numbers $m_1, \ldots, m_N$ are gyrobarycentric coordinates of a point $P \in \mathbb{R}^n_s$ with respect to $S$ if

$$\sum_{k=1}^{N} m_k \gamma_{A_k} \neq 0 \tag{46}$$

and

$$P = \frac{\sum_{k=1}^{N} m_k \gamma_{A_k} A_k}{\sum_{k=1}^{N} m_k \gamma_{A_k}} \tag{47}$$

Gyrobarycentric coordinates are homogeneous in the sense that the gyrobarycentric coordinates $(m_1, \ldots, m_N)$ of the point $P$ in (47) are equivalent to the gyrobarycentric coordinates $(\lambda m_1, \ldots, \lambda m_N)$ for any $\lambda \neq 0$. Since in gyrobarycentric coordinates only ratios of coordinates are relevant, the gyrobarycentric coordinates $(m_1, \ldots, m_N)$ are also written as $(m_1 :, \ldots, : m_N)$ so that

$$(m_1 : m_2 : \ \ldots \ : m_N) = (\lambda m_1 : \lambda m_2 : \ \ldots \ : \lambda m_N) \tag{48}$$

for any real $\lambda \neq 0$.

The point $P$ given by (47) is said to be a gyrobarycentric combination of the points of the set $S$, possessing the gyrobarycentric coordinate representation (47).

Surprisingly, the analogies that barycentric coordinates and gyrobarycentric coordinates share include covariance. In full analogy with the covariance under translations of barycentric coordinates, gyrobarycentric coordinates are covariant under left gyrotranslations; a property called *gyrocovariance*.

Indeed, if a point $P \in \mathbb{R}^n_s$ possesses the gyrobarycentric coordinate representation (47), then it obeys the identity

$$X \oplus P = \frac{\sum_{k=1}^{N} m_k \gamma_{X \oplus A_k} (X \oplus A_k)}{\sum_{k=1}^{N} m_k \gamma_{X \oplus A_k}} \tag{49}$$

for any $X \in \mathbb{R}^n_s$, where the point $P$ and the points $A_k$, $k = 1, \ldots, N$, of its generating set $S = \{A_1, \ldots, A_N\}$ vary together under left gyrotranslations.

The proof of the gyrocovariance of gyrobarycentric coordinates in a more general context is found in [38, Theorem 4.6, p. 90] and in [37, Sec. 4.3]. The study of gyrobarycentric coordinates along with their use in hyperbolic geometry is presented in [38] for Einstein gyrovector spaces, and in [37] for both Einstein and Möbius gyrovector spaces.

## 14. Example I – The Euclidean Segment

The example in this section illustrates Definition 20 of barycentric coordinates in the form used by Millman and Parker in their study of the validity of the Plane Separation Axiom (PSA) in the Euclidean plane [23, Chap. 4]. The aim of this Example I is to set the stage for the presentation of its hyperbolic counterpart, Example II in Sec. 15.

Let $A, B \in \mathbb{R}^2$ be two distinct points of the Euclidean plane $\mathbb{R}^2$, and let $P \in \mathbb{R}^2$ be a point on the segment $AB$ that joins $A$ to $B$. Then, the barycentric coordinate representation of $P$ with respect to the points $A$ and $B$ is

$$P = \frac{m_1 A + m_2 B}{m_1 + m_2} \tag{50}$$

with barycentric coordinates $m_1$ and $m_2$ satisfying $m_1 \geq 0$, $m_2 \geq 0$ and $m_1 + m_2 \neq 0$.

1. If $m_1 = 0$, then $P = B$.
2. If $m_2 = 0$, then $P = A$.
3. If $m_1, m_2 > 0$, then $P$ lies on the interior of segment $AB$, that is, between $A$ and $B$.

Owing to the homogeneity of barycentric coordinates, these can be normalized by the condition

$$m_1 + m_2 = 1 \tag{51}$$

so that, for instance, we can use the notation $m_1 = t$ and $m_2 = 1 - t$, $0 < t < 1$. In that case, the point $P$ possesses the barycentric coordinate representation

$$P = tA + (1 - t)B \tag{52}$$

Finally, owing to the covariance of barycentric coordinate representations with respect to translations, the barycentric coordinate representation (52) of $P$ obeys the Identity

$$X + P = t(X + A) + (1 - t)(X + B) \tag{53}$$

for all $X \in \mathbb{R}^2$. The derivation of Identity (53) from (52) is trivial. However, Identity (53) serves as an illustration of its hyperbolic counterpart in (57) below, which is far away from being trivial.

## 15. Example II – The Hyperbolic Segment

The example in this section illustrates Definition 21 of gyrobarycentric coordinates in the form used in Sec. 17 of this article in our present study of the validity of the Gyroplane Separation Axiom (GPSA) in the Einsteinian gyrovector plane, that is, in the Cartesian-Beltrami-Klein plane model of hyperbolic geometry.

Let $A, B \in \mathbb{R}_s^2$ be two distinct points of the Einstein gyrovector plane $\mathbb{R}_s^2 = (\mathbb{R}_s^2, \oplus, \otimes)$, and let $P \in \mathbb{R}_s^2$ be a point on the gyrosegment $AB$ that joins $A$ to $B$. Then, the gyrobarycentric coordinate representation of $P$ with respect to the points $A$ and $B$ is

$$P = \frac{m_1 \gamma_A A + m_2 \gamma_B B}{m_1 \gamma_A + m_2 \gamma_B} \tag{54}$$

with gyrobarycentric coordinates $m_1$ and $m_2$ satisfying $m_1 \geq 0$, $m_2 \geq 0$ and $m_1 \gamma_A + m_2 \gamma_B \neq 0$.

1. If $m_1 = 0$, then $P = B$.

2. If $m_2 = 0$, then $P = A$.
3. If $m_1, m_2 > 0$, then $P$ lies on the interior of gyrosegment $AB$, that is, between $A$ and $B$.

Owing to the homogeneity of barycentric coordinates, these can be normalized by the condition

$$m_1 + m_2 = 1 \tag{55}$$

so that, for instance, we can use the notation $m_1 = t$ and $m_2 = 1 - t$, $0 \le t \le 1$. In that case, the point $P$ possesses the gyrobarycentric coordinate representation

$$P = \frac{t\gamma_A A + (1-t)\gamma_B B}{t\gamma_A + (1-t)\gamma_B} \tag{56}$$

Finally, owing to the gyrocovariance of gyrobarycentric coordinate representations with respect to left gyrotranslations, the barycentric coordinate representation (56) of $P$ obeys the Identity

$$X \oplus P = \frac{t\gamma_{X\oplus A}(X\oplus A) + (1-t)\gamma_{X\oplus B}(X\oplus B)}{t\gamma_{X\oplus A} + (1-t)\gamma_{X\oplus B}} \tag{57}$$

for all $X \in \mathbb{R}^2_s$. Unlike its Euclidean counterpart (53), Identity (57) is, indeed, far away from being trivial.

## 16. Hyperbolic Plane Separation

The Euclidean Plane Separation Axiom (PSA) is studied by Millman and Parker [23, Chap. 4] in terms of inner products of vectors. Guided by analogies, we study in this section and in Sec. 17 the hyperbolic Gyroplane Separation Axiom (GPSA) in terms of inner products of gyrovectors in Einstein gyrovector spaces.

The mere introduction of gyrations turns Euclidean into hyperbolic geometry, emphasizing analogies and giving rise to our gyrolanguage, in which we prefix a gyro to terms that describe concepts in Euclidean geometry to mean the analogous concepts in hyperbolic geometry. Accordingly, in the following definitions we present the well-known plane separation axiom in gyrolanguage, in which hyperbolic planes are called gyroplanes and hyperbolic lines are called gyrolines, *etc.* (However, points in gyrolanguage usually remain points rather than gyropoints. In contrast, the hyperbolic counterparts of some special points in Euclidean geometry do have a gyro-term in gyrolanguage as, for instance, Euclidean centroids become hyperbolic gyrocentroids, Euclidean incenters and excenters of triangles become hyperbolic ingyrocenters and exgyrocenters of gyrotriangles, and Euclidean triangle orthocenters become hyperbolic gyrotriangle orthogyrocenters, *etc.*; see [29, 30, 32, 34, 37, 38]). The gyro-counterparts of Defs. 8 and 9 follow:

**Definition 22. (Gyroconvex Sets).** Let $(G, \oplus, \otimes)$ be a gyrovector space, and let $S \subset G$. $S$ is said to be gyroconvex if for every two points $P, Q \in S$, the gyrosegment $PQ$ is a subset of $S$.

**Definition 23. (Gyroplane Separation Axiom (GPSA)).** A gyrovector space $(G, \oplus, \otimes)$ satisfies the gyroplane separation axiom (GPSA) if for every gyroline $L$ in $G$ there are two subsets $H_1$ and $H_2$ of $G$ (called half gyroplanes determined by $L$) such that

1. $G - L = H_1 \cup H_2$;
2. $H_1$ and $H_2$ are disjoint and each is gyroconvex;
3. If $A \in H_1$ and $B \in H_2$ then $AB \cap L \neq \phi$, $AB$ being the gyrosegment that joins $A$ to $B$.

Definition 23 demands that the gyroline $L$ have two sides, $H_1$ and $H_2$, each of which is gyroconvex. Condition (3) of the definition says that any gyrosegment from one side to the other must cut across $L$. Let $L_{AB}$ be the unique gyroline that passes through the points $A$ and $B$. Obviously, $L_{AB} \cap L$ can only have one point in it, otherwise $L_{AB} = L$ since two gyrolines that share two points coincide.

The following theorem is analogous to a theorem that Millman and Parker present in [23, Theorem 4.1.1, p. 65] and, accordingly, our proof is motivated by analogies that our proof shares with the proof that Millman and Parker present.

**Theorem 24.** *Let $L$ be a gyroline in a gyrovector space $(G, \oplus, \otimes)$. If both $H_1, H_2$ and $H_1', H_2'$ satisfy the conditions of GPSA for a gyroline $L$, then either $H_1 = H_1'$ (and $H_2 = H_2'$) or $H_1 = H_2'$ (and $H_2 = H_1'$).*

*Proof.* Let $A \in H_1$. Since $A \notin L$, either $A \in H_1'$ or $A \in H_2'$. Suppose that $A \in H_1'$. We will show that in this case $H_1 = H_1'$. The case where $A \in H_2'$ yields $H_1 = H_2'$ in a manner similar to what follows.

To show that $H_1 \subset H_1'$ let $B \in H_1$. If $B = A$ then clearly $B \in H_1'$. Suppose that $B \neq A$. If $B \notin H_1'$ then $B \in H_2'$ since $B \notin L$. This means that $AB \cap L \neq \phi$ since $A \in H_1'$ and $B \in H_2'$.

On the other hand, $A$ and $B$ belong to the gyroconvex set $H_1$ so that $AB \subset H_1$ and $AB \cap L = \phi$.

The resulting contradiction shows that $B \in H_1'$. Hence, $B \in H_1 \Rightarrow B \in H_1'$, so that $H_1 \subset H_1'$.

The proof that $H_1' \subset H_1$ is similar. Hence, $H_1 = H_1'$. Finally, $H_2 = G - L - H_1 = G - L - H_1' = H_2'$. □

## 17. GPSA for the Einstein Gyroplane

In the Cartesian model of the Euclidean plane $\mathbb{R}^2$, points of the plane $\mathbb{R}^2$ are given by pairs $(x, y)$ of real numbers $x$ and $y$ such that $x^2 + y^2 < \infty$. In full analogy, in the Cartesian model of the hyperbolic geometry that underlies the Einstein gyrovector plane $(\mathbb{R}^2_s, \oplus, \otimes)$, points of the gyroplane $\mathbb{R}^2_s$ are given by pairs $(x, y)$ of real numbers $x$ and $y$ such that $x^2 + y^2 < s^2$.

Following Millman and Parker [23, Sec. 4.2], for

$$X = (x, y) \in \mathbb{R}^2_s \tag{58a}$$

we use the notation

$$X^{\perp} = (-y, x) \in \mathbb{R}^2_s \tag{58b}$$

Using $X^{\perp}$ we will give an alternative description of a gyroline in the Einstein gyrovector plane $(\mathbb{R}^2_s, \oplus, \otimes)$ in Theorem 26 below. As Millman and Parker remark [23, Sec. 4.2], our motivation here is the idea from linear algebra that a line can be described by giving one point on the line and a vector normal to it. Here we extend the idea from planes in the standard Cartesian model of Euclidean geometry to gyroplanes in the Cartesian-Beltrami-Klein model of hyperbolic geometry, studied in [32, 37, 38].

**Lemma 25.** *Let $X \in \mathbb{R}^2_s$.*

1. *Then $X{\cdot}X^{\perp} = 0$.*
2. *If, additionally, $X \neq (0,0)$ then $Z{\cdot}X^{\perp} = 0$ implies that $Z = r{\otimes}X$ for some $r \in \mathbb{R}$.*

*Proof.* Part (1) is obvious, $X{\cdot}X^{\perp} = (x,y){\cdot}(-y,x) = 0$. For part (2) we proceed as follows. Let $X = (x,y)$ and $Z = (z,w)$, where $x^2 + y^2 < s^2$ and $z^2 + w^2 < s^2$, so that $X, Z \in \mathbb{R}^2_s$ and $X^{\perp} = (-y,x) \in \mathbb{R}^2_s$. Then the equation $Z{\cdot}X^{\perp} = 0$ is equivalent to the equation

$$-zy + wx = 0 \tag{59}$$

Since $X \neq (0,0)$, one of $x$ and $y$ is not zero.

1. If $x \neq 0$ then, by (59), $w = zy/x$ so that $Z = tX$ with $t = z/x$.
2. If $y \neq 0$ then, by (59), $z = xw/y$ so that $Z = tX$ with $t = w/y$.

Either way, $Z = tX$ for some $t \in \mathbb{R}$.

Since $Z \in \mathbb{R}^2_s$, we have $s^2 > \|Z\|^2 = t^2\|X\|^2$ so that $|t|\|X\|/s < 1$; and since $X \in \mathbb{R}^2_s$, we have $s^2 > \|X\|^2$. The resulting two inequalities

$$-1 < \frac{t\|X\|}{s} < 1 \qquad \text{and} \qquad 0 < \frac{\|X\|}{s} < 1 \tag{60}$$

enable $r \in \mathbb{R}$ to be defined by the equation

$$r = \frac{\tanh^{-1} \frac{t\|X\|}{s}}{\tanh^{-1} \frac{\|X\|}{s}} \tag{61}$$

so that, by (13),

$$Z = tX = r{\otimes}X \tag{62}$$

as desired. □

**Theorem 26.** *Let $P, Q \in \mathbb{R}^2_s$ be two distinct points in an Einstein gyrovector plane $(\mathbb{R}^2_s, \oplus, \otimes)$, and let $L_{PQ}$ be the gyroline that passes through these points, that is*

$$L_{PQ} = P{\oplus}({\ominus}P{\oplus}Q){\otimes}t \tag{63}$$

*$-\infty < t < \infty$. Furthermore, let*

$$M_{PQ} = \{A \in \mathbb{R}^2_s \;:\; ({\ominus}P{\oplus}A){\cdot}({\ominus}P{\oplus}Q)^{\perp} = 0\} \tag{64}$$

*Then,*

$$M_{PQ} = L_{PQ} \tag{65}$$

*Proof.* Let $A \in L_{PQ}$. Then, as explained in Example II, Sec. 15, $A$ possesses the gyrobarycentric coordinate representation

$$A = \frac{m_1\gamma_{_P} P + m_2\gamma_{_Q} Q}{m_1\gamma_{_P} + m_2\gamma_{_Q}} \tag{66}$$

with respect to the points $P$ and $Q$ for some gyrobarycentric coordinates $m_1, m_2 \in \mathbb{R}$. Owing to the gyrocovariance of gyrobarycentric coordinate representations with respect to left gyrotranslations, (66) and (49) yield the equation

$$\begin{aligned} \ominus P \oplus A &= \frac{m_1\gamma_{_{\ominus P\oplus P}}(\ominus P\oplus P) + m_2\gamma_{_{\ominus P\oplus Q}}(\ominus P\oplus Q)}{m_1\gamma_{_{\ominus P\oplus P}} + m_2\gamma_{_{\ominus P\oplus Q}}} \\ &= \frac{m_2\gamma_{_{\ominus P\oplus Q}}(\ominus P\oplus Q)}{m_1 + m_2\gamma_{_{\ominus P\oplus Q}}} \end{aligned} \tag{67}$$

noting that $\ominus P\oplus P = \mathbf{0}$ and that $\gamma_{_{\ominus P\oplus P}} = \gamma_{_\mathbf{0}} = 1$.

Hence, by (67) and part (1) of Lemma 25,

$$(\ominus P\oplus A)\cdot(\ominus P\oplus Q)^\perp = \frac{m_2\gamma_{_{\ominus P\oplus Q}}}{m_1 + m_2\gamma_{_{\ominus P\oplus Q}}}(\ominus P\oplus Q)\cdot(\ominus P\oplus Q)^\perp = 0 \tag{68}$$

so that $A \in M_{PQ}$. Hence, $L_{PQ} \subset M_{PQ}$.

To prove the reverse containment, assume that $A \in M_{PQ}$. Clearly, $\ominus P\oplus Q \neq 0$ since $P \neq Q$. Then, by Lemma 25 there is a real number $r \in \mathbb{R}$ such that

$$\ominus P\oplus A = (\ominus P\oplus Q)\otimes r \tag{69}$$

implying by the left cancellation law (7) of Einstein addition,

$$A = P\oplus(\ominus P\oplus Q)\otimes r \tag{70}$$

so that $A \in L_{PQ}$. Hence, $L_{PQ} \supset M_{PQ}$.

Having containment in both directions, the sets are equal, $L_{PQ} = M_{PQ}$, as desired. □

**Definition 27.** Let $L_{PQ}$ be a gyroline in the Einstein gyrovector plane $(\mathbb{R}_s^2, \oplus, \otimes)$. The Einsteinian half gyroplanes determined by $L_{PQ}$ are

$$\begin{aligned} H^+ &= \{A \in \mathbb{R}_s^2 \;:\; (\ominus P\oplus A)\cdot(\ominus P\oplus Q)^\perp > 0\} \\ H^- &= \{A \in \mathbb{R}_s^2 \;:\; (\ominus P\oplus A)\cdot(\ominus P\oplus Q)^\perp < 0\} \end{aligned} \tag{71}$$

**Theorem 28.** *The Einsteinian half gyroplanes determined by a gyroline $L_{PQ}$ are gyroconvex.*

*Proof.* Let $A, B \in H^+$ be two distinct points of $H^+$, so that

$$\begin{aligned} (\ominus P\oplus A)\cdot(\ominus P\oplus Q)^\perp > 0 \\ (\ominus P\oplus B)\cdot(\ominus P\oplus Q)^\perp > 0 \end{aligned} \tag{72}$$

and let $C$ be any point between $A$ and $B$. Then, as explained in Item (3) of Example II, Sec. 15, $C$ possesses the gyrobarycentric coordinate representation

$$C = \frac{m_1\gamma_{_A} A + m_2\gamma_{_B} B}{m_1\gamma_{_A} + m_2\gamma_{_B}} \tag{73}$$

with respect to $A$ and $B$, with positive gyrobarycentric coordinates $m_1, m_2 > 0$.

By the gyrocovariance of gyrobarycentric coordinate representations, (49), we have

$$\ominus P \oplus C = \frac{m_1\gamma_{_{\ominus P\oplus A}}(\ominus P\oplus A) + m_2\gamma_{_{\ominus P\oplus B}}(\ominus P\oplus B)}{m_1\gamma_{_{\ominus P\oplus A}} + m_2\gamma_{_{\ominus P\oplus B}}} \tag{74}$$

Hence, by (72) and (74), noting that $m_1, m_2 > 0$, we have

$$\begin{aligned}
&(\ominus P\oplus C)\cdot(\ominus P\oplus Q)^{\perp} \\
&= \frac{m_1\gamma_{_{\ominus P\oplus A}}(\ominus P\oplus A) + m_2\gamma_{_{\ominus P\oplus B}}(\ominus P\oplus B)}{m_1\gamma_{_{\ominus P\oplus A}} + m_2\gamma_{_{\ominus P\oplus B}}}\cdot(\ominus P\oplus Q)^{\perp} \\
&= \frac{m_1\gamma_{_{\ominus P\oplus A}}(\ominus P\oplus A)\cdot(\ominus P\oplus Q)^{\perp}}{m_1\gamma_{_{\ominus P\oplus A}} + m_2\gamma_{_{\ominus P\oplus B}}} + \frac{m_2\gamma_{_{\ominus P\oplus B}}(\ominus P\oplus B)\cdot(\ominus P\oplus Q)^{\perp}}{m_1\gamma_{_{\ominus P\oplus A}} + m_2\gamma_{_{\ominus P\oplus B}}} \\
&> 0
\end{aligned} \tag{75}$$

so that $C \in H^+$, as desired. The proof that $A, B \in H^- \Rightarrow C \in H^-$ is similar. □

**Theorem 29.** *The Einstein gyrovector plane satisfies GPSA.*

*Proof.* Let $L_{PQ}$ be a gyroline in an Einstein gyrovector plane $(\mathbb{R}^2_s, \oplus, \otimes)$ and let $A \in \mathbb{R}^2_s$. Then

1. either
$$(\ominus P\oplus A)\cdot(\ominus P\oplus Q)^{\perp} > 0 \tag{76}$$
so that, by (71), $A \in H^+$,
2. or
$$(\ominus P\oplus A)\cdot(\ominus P\oplus Q)^{\perp} = 0 \tag{77}$$
so that, by (65), $A \in L_{PQ}$,
3. or
$$(\ominus P\oplus A)\cdot(\ominus P\oplus Q)^{\perp} < 0 \tag{78}$$
so that, by (71), $A \in H^-$.

Hence,

$$\mathbb{R}^2_s - L_{PQ} = H^+ \cup H^- \tag{79}$$

Clearly, $H^+$ and $H^-$ are disjoint and, by Theorem 28 they are gyroconvex. Hence, we need only show that condition (3) of GPSA in Def. 23 holds.

Let $A \in H^+$ and $B \in H^-$. To show that $AB \cap L_{PQ} \neq \phi$ we must find a point $C$ on gyrosegment $AB$ that lies on gyroline $L_{PQ}$.

Accordingly, let $C \in AB$ be a point on gyrosegment $AB$. As in (73), let

$$C = \frac{m_1\gamma_{_A} A + m_2\gamma_{_B} B}{m_1\gamma_{_A} + m_2\gamma_{_B}} \tag{80}$$

be the gyrobarycentric coordinate representation of $C$ with respect to $A$ and $B$, where the gyrobarycentric coordinates $m_1, m_2 > 0$ are to be determined in (85) below.

Since gyrobarycentric coordinates are homogeneous, they can be normalized by the condition

$$m_1 + m_2 = 1 \tag{81}$$

According to Theorem 26, $C \in L_{PQ}$ if and only if

$$(\ominus P \oplus C)\cdot(\ominus P \oplus Q)^{\perp} = 0 \tag{82}$$

Substituting $\ominus P \oplus C$ from (74) into (82) we obtain, similar to (75), the equation

$$\frac{m_1\gamma_{_{\ominus P\oplus A}}(\ominus P\oplus A)\cdot(\ominus P\oplus Q)^{\perp} + m_2\gamma_{_{\ominus P\oplus B}}(\ominus P\oplus B)\cdot(\ominus P\oplus Q)^{\perp}}{m_1\gamma_{_{\ominus P\oplus A}} + m_2\gamma_{_{\ominus P\oplus B}}} = 0 \tag{83}$$

or, equivalently,

$$m_1\gamma_{_{\ominus P\oplus A}}(\ominus P\oplus A)\cdot(\ominus P\oplus Q)^{\perp} + m_2\gamma_{_{\ominus P\oplus B}}(\ominus P\oplus B)\cdot(\ominus P\oplus Q)^{\perp} = 0 \tag{84}$$

Solving (81) and (84) for the unknowns $m_1$ and $m_2$, we have

$$\begin{aligned} m_1 &= \frac{-\gamma_{_{\ominus P\oplus B}}(\ominus P\oplus B)\cdot(\ominus P\oplus Q)^{\perp}}{\gamma_{_{\ominus P\oplus A}}(\ominus P\oplus A)\cdot(\ominus P\oplus Q)^{\perp} - \gamma_{_{\ominus P\oplus B}}(\ominus P\oplus B)\cdot(\ominus P\oplus Q)^{\perp}} \\ m_2 &= \frac{\gamma_{_{\ominus P\oplus A}}(\ominus P\oplus A)\cdot(\ominus P\oplus Q)^{\perp}}{\gamma_{_{\ominus P\oplus A}}(\ominus P\oplus A)\cdot(\ominus P\oplus Q)^{\perp} - \gamma_{_{\ominus P\oplus B}}(\ominus P\oplus B)\cdot(\ominus P\oplus Q)^{\perp}} \end{aligned} \tag{85}$$

Both $m_1$ and $m_2$ in (85) are positive, noting that gamma factors are positive and that

$$(\ominus P\oplus A)\cdot(\ominus P\oplus Q)^{\perp} > 0 \tag{86a}$$

since $A \in H^+$, and

$$(\ominus P\oplus B)\cdot(\ominus P\oplus Q)^{\perp} < 0 \tag{86b}$$

since $B \in H^-$.

With the values of the gyrobarycentric coordinates $m_1$ and $m_2$ given by (85), we have the point $C$ given by (80) and (85), such that $C \in AB \cap L_{PQ}$, so that $AB \cap L_{PQ} \neq \phi$, and the proof is complete. □

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