04/19/23 – Symbolic Dynamics – Prof Wolf

Final:
- informal oral exam
- favorite proof, idea how to prove it, where you might go from there
- how you liked the class

An entire class of Ergodic theory in one lecture

$X \longmapsto h_{top}$

general shift space

only computable at X where entropy is 0.

What about more specific shift spaces? e.g. coded shifts
$\mathcal{G}$ a generating set $\longrightarrow X(\mathcal{G})$ a coded shift
notion of uniquely representable coded shifts.
Consider $\mathcal{U} = \{\mathcal{G}$ generating sets : $X(\mathcal{G})$ uniquely representable$\}$
map $\mathcal{G} \mapsto X(\mathcal{G}) \longrightarrow h_{top}(X(\mathcal{G}))$ for $\mathcal{G} \in \mathcal{U}$.
in general this fn is not computable, but there is a classification of some that are computable.

↳ to do this classification, we need a lot of ergodic theory.

Let (X, d) a compact metric space, $f: X \to X$ continuous. $\mu(X) = 1$.
invariant probability measures: $\mathcal{M} = \{\mu : \mu$ is f-invariant Borel prob. measure$\}$
$\mu(f^{-1}(A)) = \mu(A) \ \forall A \in \mathcal{B}(X)$
endowed w/ weak* topology $\phi \in C(X, \mathbb{R})$ continuous fns $\mu \mapsto \int \phi \, d\mu$ functional
↳ look for smallest topology that makes all these functionals continuous.
in weak*, things behave a little confusingly
$\mu_n \to \mu$ iff. $\int \phi \, d\mu_n \to \int \phi \, d\mu \ \forall \phi \in C(X, \mathbb{R})$

Fact: $\mathcal{M}$ is compact in this topology

does such a measure exist? i.e. is $\mathcal{M} = \emptyset$?

thm: (Kryglov-Bololulov)
$\mathcal{M} \neq \emptyset$.

proof (idea): Pick $x \in X$. let (φ_m) be dense in $C(X, \mathbb{R})$ (using the sup norm)
fix $m \in \mathbb{N}$. $\Rightarrow \frac{1}{n} \sum_{k=0}^{n-1} \varphi_m(f^k(x))$
So $\exists n_k$ s.t. $\lim_{k \to \infty} \frac{1}{n_k} \sum_{k=0}^{n_k - 1} \varphi_m(f^k(x))$ converges (standard existence of a convergent subsequence)
now we need convergence for every m
using a diagonal argument, $\exists n_k$ s.t. $\lim_{k \to \infty} \frac{1}{n_k} \sum_{k=0}^{n_k - 1} \varphi_m(f^k(x))$ for $\forall m \in \mathbb{N}$ $= J(\varphi_m)$.
since φ_m is dense, and this limit exist $\forall \varphi_m$, we can prove it converges for all φ. (by "standard arguments") $J(\varphi) \ \exists \ \forall \varphi \in C(X, \mathbb{R})$. ∎

note $J: C(X, \mathbb{R}) \to \mathbb{R}$ is linear, bounded, positive.
$\Rightarrow \exists \mu$ Borel probability measure s.t. $J(\varphi) = \int \varphi \, d\mu \ \forall \varphi$ } Riesz Representation Theorem

One can now use the construction to show that μ is invariant.

Simplest example of a <u>uniquely ergodic system</u>:
- irrational rotation ↳ unique measure
  (measure is the Lebesgue measure)

def: $\mathcal{M}_X$ <u>convex</u> if $\mu_1, \mu_2 \in \mathcal{M}_X$, $\alpha \in [0,1] \Rightarrow \alpha\mu_1 + (1-\alpha)\mu_2 \in \mathcal{M}_X$

if you have a compact, convex top. space, you can define an extreme point.

def: $\underline{\mathcal{M}_E} \subseteq \mathcal{M}_X$ is the subset of extreme points in $\mathcal{M}_X$.

def: μ is <u>ergodic</u> if $\underbrace{A = f^{-1}(A)}_{\text{invariant set}}$ means $\mu(A) = 0$ or $= 1$.

↳ turns out $\mathcal{M}_E$ is also the set of all Ergodic measures.
(otherwise you could decompose μ into nontrivial μ_1 and μ_2)

Ergodic decomposition
$\mu \in \mathcal{M}_X \Rightarrow \exists\, \lambda_\mu$ borel prob. measure in $\mathcal{M}_X$ s.t.
$\lambda(\mathcal{M}_E) = 1$ and $\int \phi \, d\mu = \iint \phi \, d\nu \, d\lambda_\nu \quad \forall \phi \in C(X, \mathbb{B})$
arbitrary measure
(the non-ergodic measures form a set of measure 0 for λ)

Poincaré Recurrence Theorem:
Let $\mu \in \mathcal{M}_X$, $A \in \mathcal{B}(X)$ with $\mu(A) > 0$.
$\Rightarrow \mu(\underbrace{\{x \in A : \exists N \text{ s.t. } f^n(x) \notin A \ \forall n \geq N\}}) = 0$
the set of points that only return a finite number of times
equivalently:
$\mu(\{x \in A : f^n(x) \in A \text{ infinitely many times}\}) = \mu(A)$

↳ what this lets us do is look at what

Berkhoff Ergodic Theorem
$\mu \in \mathcal{M}_X$, let $\phi \in C^1(X, \mu)$. Then the time average, given by
$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \phi(f^k(x)) = \phi_f(x)$ exists for x "μ almost everywhere"
also $\phi_f \in C^1(X, \mu)$. furthermore if $\mu \in \mathcal{M}_E$ then ϕ_f is constant a.e.
and $\underbrace{\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \phi(f^k(x))}_{\text{time average}} = \underbrace{\int \phi \, d\mu}_{\text{space average}}$ for a.e. $x \in X$

cor: let $B(\mu)$ the basin of μ $= \{x \in X : \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \delta_{f^k(x)} = \mu\}$.
If $\mu \in \mathcal{M}_E$ then $\mu(B(\mu)) = 1$

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def: let $\mu, \nu \in \mathcal{M}_X$. define $W_1(\mu, \nu) = \sup \left| \int \phi \, d\mu - \int \phi \, d\nu \right|$ $\phi \in 1\text{-Lip}(X, \mathbb{B})$
the Wasserstein–Kantorovitch distance. (one-lipschitz cont.)
↳ this metric induces weak ★

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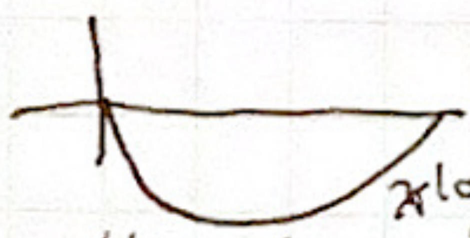
let X be a shift space, $\mu \in \mathcal{M}_X$.



$x \log x$
(has 6 great properties)

def: $\lim_{n\to\infty} -\frac{1}{n} \sum_{\tau^n \in \mathcal{L}_n(X)} \mu(\tau^n) \log \mu(\tau^n)$

take all words of length n, take the measure of the cylinder, then $\mu(\,)\log\mu(\,)$

Thm: (Katok – Brin) Let $\mu \in \mathcal{M}_X$. Then the limit
$$\lim_{r\to 0} \lim_{n\to\infty} -\frac{1}{n} \log \mu(B_n(x,r)) = h_\mu(x) \text{ exists } \mu\text{-a.e. } x.$$

recall $= \max_{k\in\{0,\dots,n-1\}} d(f^k(x), f^k(y))$

idea: for this to be positive the measure must live on points that are pushed apart.

also $x \mapsto h_\mu(x)$ is μ-measurable and if $\mu \in \mathcal{M}_E$, $h_\mu()$ is const. μ-a.e., and $h_\mu(f) = \int h_\mu(x)\, d\mu(x)$

Connection between metric and topological entropy:

Variational Principle:
$$h_{top}(f) = \sup_{\mu\in\mathcal{M}_X} h_\mu(f)$$
and if $h_\mu(f) = h_{top}(f)$ then μ is a measure of maximal entropy (mme).

in some systems there is no mme! (b/c sup).
but in symbolic systems there is.

shift maps are always expansive.

nice property: f expansive means $\mu \mapsto h_\mu(f)$ is upper semi continuous

$\mu_n \to \mu \Rightarrow \limsup_{n\to\infty} h_{\mu_n}(f) \le h_\mu(f)$
(in the limit you can only go up).

Let $X = X(\mathcal{G})$ be a coded shift.

$h_{seq}(X(\mathcal{G})) = \sup\{h_\mu(\sigma) : \mu \in \mathcal{M}_X, \mu(X_{seq}) = 1\}$

$h_{res}(X(\mathcal{G})) = \sup\{h_\mu(\sigma) : \mu \in \mathcal{M}_X, \mu(X_{res}) = 1\}$

idea: entropy computable if $h_{seq} > h_{res}$, and sometimes if $h_{seq} = h_{res}$.
where does the entropy live? is it on the concatenations? then a computer can find it, if its in the limits, a computer cannot find it.

works if there's a mme that lives on h_{seq}.