01/25/23 – Symbolic Dynamics Prof. Wolf

recently Prof. Wolf has been applying symbolic dynamics to biology.
recent research in catagorification of entropy.
Course format:
- Introduction
- Sofic Shifts (Shifts of finite types, entropy, factors)
- Coded Shifts (thermodynamic formalism (entropy, pressure, invariant measures))
- Papers (entropy beyond h_t, equillibrium + Gibbs measures, computability, complexity)

To pass this course with an A – come to every lecture, and a 15 minute conversation at the end of the semester (what was your favorite theorm?)
This is a topics class, you get as much out of it as you want.
No mandatory homework, problems suggested if you want to do them.

Office hours 12–2 Wed. 4208.

References:
1. Introduction to Symbolic Dynamics (Lind & Marcus)
2. Symbolic Dynamics (Kitchens)
3. Graph directed Markov systems (Mauldin & Ubanski)
4. Introduction to Ergodic Theory (Waltus)
5. Modern theory of Dynamical Systems (Katok & Hasselblatt)
6. An introduction to Chaotic Dynamical Systems (Devaney)

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0. Introduction & Comments

The mathematics of objects that change over time.
Ingredients:
1. Phase space X – all possible states the system can be in
2. time T – discrete or continuous (for deterministic systems)
   $T = \mathbb{N}$ $\quad$ $T = \mathbb{R}$.
3. time evolution law – family of maps (ϕ_t) $t \in T$, $\phi_t : X \to X$.
   under some assumptions:
   a) $\phi_0 = id_X$
   b) $\phi_{s+t} = \phi_s \circ \phi_t$ (flow property)
   in the discrete case, this implies by induction $\phi_m(x_0) = \underbrace{\phi_1 \circ \cdots \circ \phi_1}_{m \text{ times}}(x_0)$
   So we say $f = \phi_1$, $\phi_m = f^m(x_0)$ $\forall\, m \in \mathbb{N}$.
   i.e. in the discrete case all we need is a map on the phase space, which we iterate.

   forward orbit $O^+(x) = \{x, f(x), f^2(x), \dots\}$
   backward orbit $O^-(x) = \{x, f^{-1}(x), f^{-2}(x), \dots\}$

Subfields:
a) (X, Ω, μ) a measure space, $f : X \to X$ measurable
   $\Rightarrow$ Measurable Dynamical Systems or Ergodic Theory.
b) (X, τ) topological space, f continuous
   $\Rightarrow$ topological Dynamics
c) X a differentiable manifold, f smooth
   $\Rightarrow$ differentiable dynamics.
d) X complex structure, f holomorphic
   $\Rightarrow$ Complex dynamics

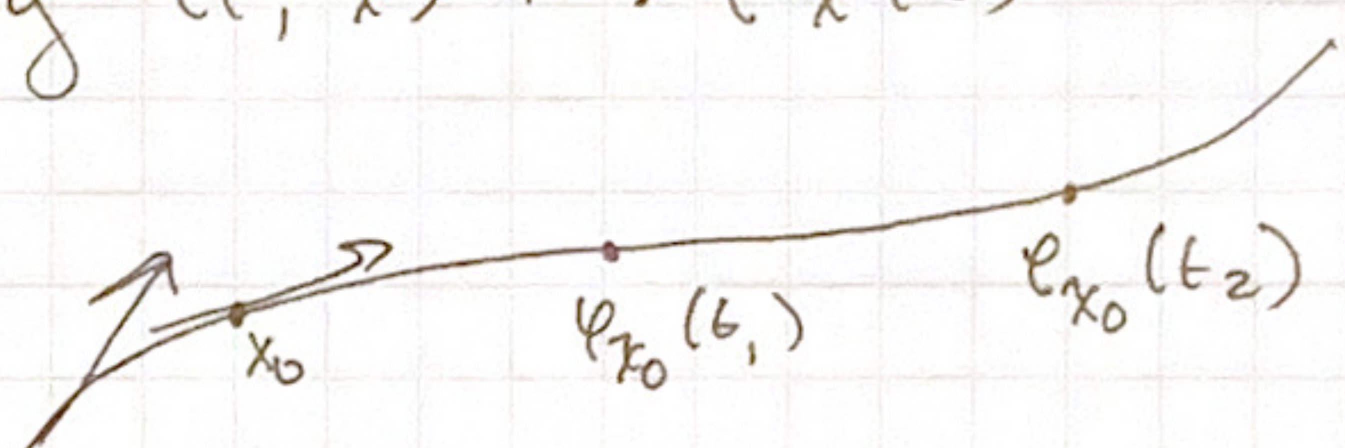
"I believe PDEs are a subfield of dynamical systems"

Some subfields of Dynamics
- ODEs
- Ergodic Theory
- Algebraic Dynamics
- Topological Dynamics
- Computability in D.S.
- Complex Dynamics
- PDEs
- Dimension Theory in D.S.
- Smooth Ergodic Theory

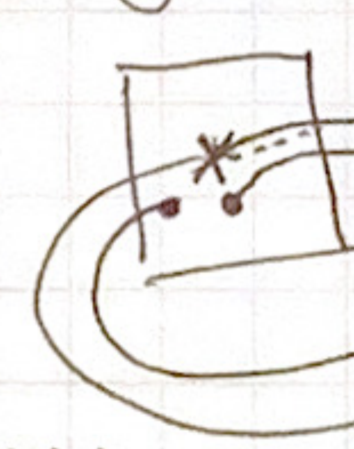
Significant open problem
- what does a generic dynamical system look like on a Riemannian manifold.

0.2 Examples

a) flow $V: \mathbb{R}^n \to \mathbb{R}^n$ vector field, Lipschitz continuous.
for $x_0 \in \mathbb{R}^n$ consider the initial value problem
i) $x(0) = x_0$
ii) $\dot{x}(t) = v(x(t))$ $\qquad \dot{x}$ means derivative
Lip. cont. gives solution (standard odes)
$\varphi_{x_0}\ I_{x_0} \hookrightarrow \mathbb{R}$ $\qquad$ (Picard-Lindelöf)
$0 \in I_{x_0}$ any open interval in $\mathbb{R}_+$
Suppose $I_{x_0} = \mathbb{R}$ $\forall x_0 \in \mathbb{R}^n$
$\phi: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ by $(t, x) \mapsto \varphi_x(t)$
this fn is called a flow.



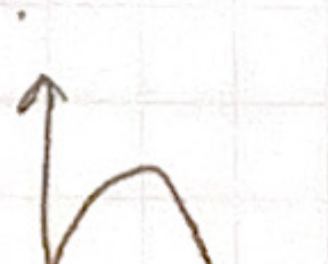
The time 1 map $f = \phi(1, \cdot)$
is a diffeomorphism, which you can iterate. And once you understand the dynamics of f you know the dynamics of the entire flow.



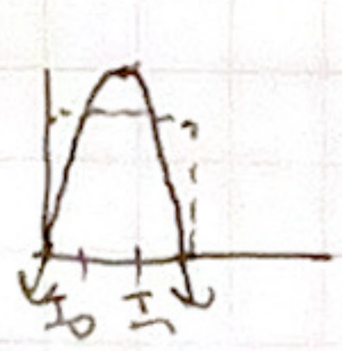
Poincare flow, look at the intersection of a flow map w/ the hyperplane and look at the first return map, gives a diffeomorphism. (return time is continuous on the distance)

$\Rightarrow$ application to the n-body problem.
n point masses, need a point in $\mathbb{R}^6$ to describe position/motion.

b) Symbolic systems.
alphabet $A = \{0, 1\}$ $\quad X = A^{\mathbb{N}_0}$ i.e. $X = \{(x_k)_{k \in \mathbb{N}_0} : x_k \in A\}$
canonical metric $d(x, y) = (\frac{1}{2})^{\inf\{k : x_k \neq y_k\}}$ and $d(x, x) = 0$.
makes (X, d) a compact metric space, (induces the topology)
dynamical system $\sigma: \cancel{X} \to X$ shift map to the left.
$(\sigma(x_k)_k)_\ell = x_{\ell+1}$
this system is chaotic, w/ entropy $\log 2$, and many periodic points.
def: $x \in X$ is a periodic point $\exists\, p \in \mathbb{N}$ s.t. $\sigma^p(x) = x$, p is a period and the smallest period is called the prime period.



c) logistic map.
$\mu > 0$, $f_\mu : \mathbb{R} \to \mathbb{R}$ by $x \mapsto \mu x(1-x)$
as μ increases 1 p.p., 2 p.p., period doubling bifurcations, chaos.



Case $\mu = 4$ is when vertex hits 1.
Case $\mu > 4$:

01/25/22 - Symbolic dynamics - cont.

Properties of the $\mu > 4$ case:
$\forall x \in \mathbb{R}$ s.t. $\exists\, n \in \mathbb{N}$ w/ $f^{\ell}(x) \notin [0,1] \Rightarrow f^{n}(x) \to -\infty$.

def: $\Lambda = \{ x \in [0,1] : f^{n}(x) \in [0,1]\ \forall n \}$ e.g. 0.

What does Λ look like? want to say it's the same as a shift map
give $h : \Lambda \to \{0,1\}^{\mathbb{N}_0}$ by $h(y) = (x_m)_n$ $\quad f_{\mu}^{m}(x) \in I_{x_m}$

exercise: prove h is a homeomorphism.

So we can say
$$\sigma \circ h = h \circ f_{\mu}$$
$$\sigma = h \circ f_{\mu} \circ h^{-1}$$
$$\sigma^{n} = h \circ f_{\mu}^{n} \circ h^{-1}$$

h is just a change of coordinates. These systems act the same, we can use symbolic dynamics to understand smooth dynamical systems.

* polynomials have an infinite number of periodic points? $p(x) - id = 0$

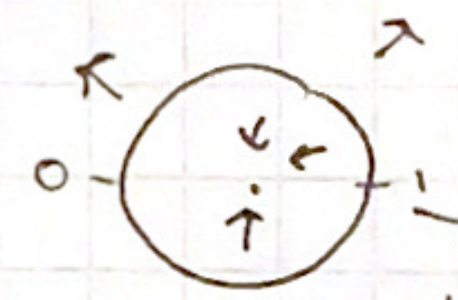
d) Complex dynamics
$p : \mathbb{C} \to \mathbb{C}$ polynomial degree ≥ 2.
$p(z) = c_d z^{d} + c_{d-1} z^{d-1} + \dots + c_0$ (general form of a complex poly of degree d).
w/ $c_0, \dots, c_d \in \mathbb{C}$.
(this is a bit overkill, you only actually need $d-1$ complex numbers to characterize).

e.g. $d = 2$, $p(z) = z^2 + c \quad c \in \mathbb{C}$.
even quadratic polynomials have complicated behavior.

def: Filled-in Julia Set $K_p = \{ z \in \mathbb{C} : O^{+}(z) \text{ bounded} \}$
$J_p = \partial K$ is the Julia Set.

also get one here as long as you watch the points on the boundary (that hit 0 or ∞)

e.g. $P(z) = z^2$, (doubling map) gives $K_p = \overline{B(0,1)}$
$J_p = S^1$ (unit circle)



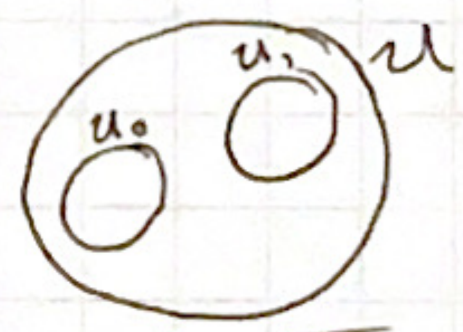
inside attracted to 0
outside attracted to ∞

e.g. $P(z) = z^2 - 2$ gives $K_p = J_p = [-2, 2]$ & empty interior, but J_p is connected.

(the number of c where J_p is connected is the Mandelbrot set) has to do w/ convergence of critical point at 0.

* this is a Markov partition of the Julia set b/c U_0, U_1 disjoint

Case k large (≥ 4) (solenoid)



U_0 conformally mapped onto U by the P_0 inverse, and same for U_1 w/ P_1^{-1}.

So $J \subseteq U_0 \cup U_1$, so we can relate the dynamics of J onto the dynamics of a symbolic system (conformal) iterated function system

$h : \{0,1\}^{\mathbb{N}_0} \to J \quad h((x_k)_k) = z$ provided $p^{k}(z) \in U_{x_k}$
this is a conjugacy

$$\begin{tikzcd} J_p \arrow[r, "p"] & J_p \\ \{0,1\}^{\mathbb{N}_0} \arrow[u, "h"] \arrow[r, "\sigma"'] & \{0,1\}^{\mathbb{N}_0} \arrow[u, "h"'] \end{tikzcd}$$

$\Rightarrow$ dynamics on the Julia set are the same up to change of coordinates has the same dynamics.

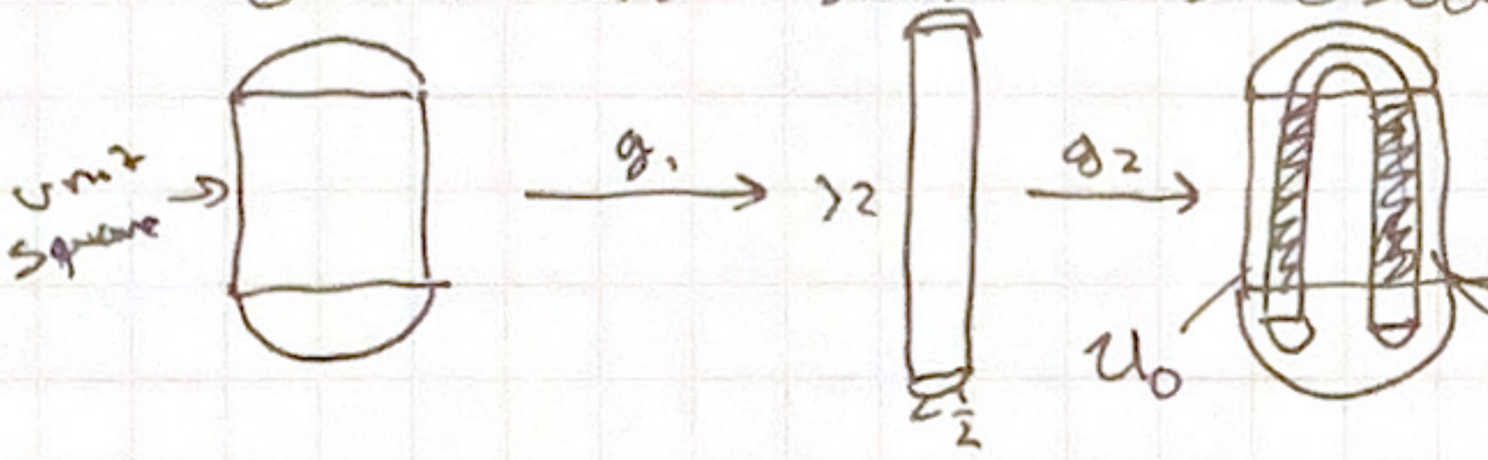
e) higher dimensions – Smale – Horseshoe

$X = \{0,1\}^{\mathbb{Z}}$ bi-infinite sequences.

$d(x,y) = \left(\frac{1}{2}\right)^{\inf\{|k| : x_k \neq y_k\}}$ $d(x,x) = 0$

induces product topology and compact metric space.

$\sigma : X \to X$ shift as usual.

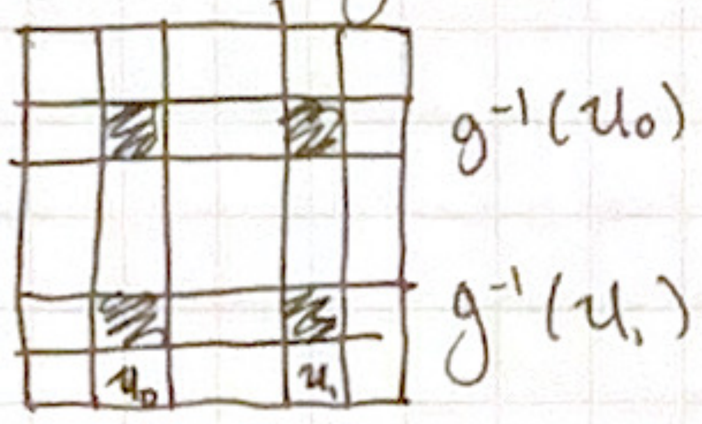


$g = g_2 \circ g_1$

on these two strips we need in the vertical direction derivative expanding by factor >2, horizontal direction derivative contracting by factor $< \frac{1}{2}$. (doesn't have to be linear!)

$\Lambda = \{x \in S : g^{m}(x) \in S \ \forall m \in \mathbb{Z}\}$

U_0, U_1 give a markov partition



first step in constructing the cantor set.

Conjugacy is $h : \Lambda \to \{0,1\}^{\mathbb{Z}}$ by $h(x) = (x_k)_k$ where $g^k(x) \in U_{x_k}$

$$\begin{array}{ccc} \Lambda & \xrightarrow{g} & \Lambda \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\ \{0,1\}^{\mathbb{Z}} & \xrightarrow{\sigma} & \{0,1\}^{\mathbb{Z}} \end{array}$$

key: we can extract a lot of the significant dynamics from the symbolic system.

next week: axioms of symbolic dynamics, and SFT.

note: no class feb 8.