WEEKLY LOG

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1. Week 1

- overview of class
- some examples

Week 2

Summary: This week we introduced the basic terminology of shift spaces and subshifts of finite type. We saw how to characterize a subshift by its language or by its set of forbidden words. Finally we introduced the notion of a higher block representation of an SFT, which for any SFT lets us find an alphabet that gives a forbidden word set with word length at most two. This property will be necessary in order to view SFTs as matrices.

I.1 Shift Spaces.

Definition 1.0.1 (Basic terminology)**.** // TODO Alphabet $\mathcal{A}$, two sided full shift $\Sigma_d^{\pm}$, word, empty word, concatenation, shift map, Cylinder $[x]_i^j$.

// TODO: an interlude into topology Note: the Product topology is the coarsest topology that makes all projections open.

Definition 1.0.2. The product topology on $\Sigma_d^{\pm}$ with regards to the discrete topology on $\mathcal{A}$. This implies that $\Sigma_d^{\pm}$ is a compact topological space. It is also metrizable, for example let $0 < \alpha < 1$, then the following induces the product topology:

$$d(x,y) = \begin{cases} \alpha^{\min\{|k|:x_k \neq y_k\}} & \text{for } x \neq y \\ 0 \end{cases}$$

Lemma 1.0.3. *Cylinders are both open and closed.*

Proof. // tODO □

Corollary 1.0.4. *The $\Sigma_d^{\pm}$ is totally disconnected.*

Proof. // TODO □

I.2 Subshifts and Languages.

Definition 1.0.5. $X \subseteq \Sigma_d^{\pm}$ is a shift space if X is σ-invariant and closed.

Definition 1.0.6. $\sigma|_X$ is a subshift.

Definition 1.0.7 (Languages)**.** Let $n \in \mathbb{N}$, $\mathcal{L}_n(x) = \{w : |w| = n, w \in x\}$. The empty word is denoted ε and the *language* of X is $\mathcal{L}(X) = \bigcup_{n=0}^{\infty} \mathcal{L}_n(X)$.

We call a subset $\mathcal{L} \subseteq \mathcal{L}(\Sigma_d^{\pm})$ a *one-sided language* if

- $\forall w \in \mathcal{L}$, all subwords v of w are also in $\mathcal{L}$, and
- $\forall w \in \mathcal{L}$ there exists some letter $l \in \mathcal{A}$ such that $wl \in \mathcal{L}$.

For a *two-sided language* the second condition needs two letters $m, l \in \mathcal{A}$ such that $mwl \in \mathcal{L}$.

A fundamental idea: shift spaces and languages are the same object:

Lemma 1.0.8. *Let $\mathcal{L} \subseteq \mathcal{L}(\Sigma_d^{\pm})$ be a two-sided language. Then $X(\mathcal{L}) = \{x \in \Sigma_d^{\pm} : x_{[i,j]} \in \mathcal{L} \forall i \leqslant j \in \mathbb{Z}\}$ is a shift space.*

Proof. // TODO □

Shift spaces can also be described by the set of forbidden words.

Definition 1.0.9. Let $X \subseteq \Sigma_d^{\pm}$ be a shift space. $\mathcal{F}_x = \mathcal{L}(\Sigma_d^{\pm})\ \mathcal{L}(X)$ is the set of forbidden words.

You have to be careful when giving $\mathcal{F}_x$ that it's complement is indeed a language.

e.g. $d = 2, \mathcal{F} = \varnothing \Rightarrow x = \Sigma_d^{\pm}$

e.g. $d = 2, \mathcal{F} = \{11\}$ The Golden Mean Shift

e.g. $d = 2, \mathcal{F}_x = \{10^{2k+1}1 : k \in \mathbb{N}\}$ The even shift. (Note: this is not a SFT, because the forbidden set is not finite. It is what's called an S-Gap shift)

Given an arbitrary subset of words, you can fill in the set to obtain a forbidden word set that gives the language of a shift space. Let $\mathcal{F} \subseteq \mathcal{L}(\Sigma_d^{\pm})$. Call $\overline{\mathcal{F}} = \{w \in \mathcal{L}(\Sigma_d^{\pm}) : w \text{ has a subword in } \mathcal{F}\}$.Then $\overline{\mathcal{F}}$ is a forbidden set and $X_{\overline{\mathcal{F}}} = X(\mathcal{L}(\Sigma_d^{\pm} \backslash \overline{\mathcal{F}}))$ is a shift space.

Definition 1.0.10. A shift space X is irreducible if $\forall u, , v \in \mathcal{L}(X)$ there exists a word $w \in \mathcal{L}(X)$ such that $uwv \in \mathcal{L}(X)$.

The idea of irreducibility is you can connect arbitrary words in the language.

// TODO: make typesetting math L&M

Definition 1.0.11. We say a shift space $\mathsf{X} \subseteq \Sigma_d^{\pm}$ is a *subshift of finite type* if there exist a finite set of forbidden words $\mathcal{F} \subseteq \mathcal{L}(\Sigma_d^{\pm})$ such that $X_{\mathcal{F}} = \mathsf{X}$.

You can always rewrite an SFT to another SFT whose forbidden words have a max length of 2 by increasing the alphabet. This is nice because it makes it much easier to check if something is in the shift space: you only need to look at adjacent elements (window of length 2).

Definition 1.0.12. Let X be a shift space, $n \in \mathbb{N}$. We call $B = \mathcal{L}_n(X)$ the set of words of length n in the alphabet. We assign every element of B a letter in a new alphabet $\{a_0, \dots, a_{n-1}\}$.

// tODO: what is this defn, why do we care

Definition 1.0.13. $X^{[N]} = \beta_N(X)$ where $\beta_N : X \to B^{\mathbb{Z}}$ by $(\beta_N(X))_k = x_k \dots x_{k+N-1}$.

We can create a *higher block representation* of X and order N by e.g. give $x = \dots x_{-3}x_{-2}x_{-1}.x_0x_1x_2\dots$ and $N = 3$,

$$y = \dots \begin{bmatrix} x_{-1} \\ x_{-2} \\ x_{-3} \end{bmatrix} \begin{bmatrix} x_0 \\ x_{-1} \\ x_{-2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_0 \\ x_{-1} \end{bmatrix} . \begin{bmatrix} x_2 \\ x_1 \\ x_0 \end{bmatrix} \begin{bmatrix} x_3 \\ x_2 \\ x_1 \end{bmatrix} \dots$$

Definition 1.0.14. Let $u = u_1 \dots u_N, v = v_1 \dots v_N$. We say u and v *progressively overlap* if $u_2 \dots u_N = v_1 \dots v_{N-1}$.

// tODO: what does this mean? i.e. If uv occurs in the language of the higher blog representation of X (i.e. in $\beta_N(X)$ for some N) then u and v progressively overlap.

Proposition 1.0.15. $\left(X^{[N]}, \sigma\right)$ *is a shift space. Moreover it is topologically conjugate to* (X, σ) *for all* $N \in \mathbb{N}$

Proof. // TODO □

We like block representations because they let us go to $\mathcal{F}$ with words of length 2 by moving to a higher alphabet. And then we can represent shifts as matrices because we only ever need to be worried about what is allowed in the immediate next step.

TODO: - go back and fill in proofs - resolve misunderstandings noted - identify relevant sections of L&M for exercises TODO: - do problems - integrate with L&M notes Exercises: - the metric given here induces the product topology