03/02/23 - Symbolic Dynamics, Prof. Wolf

recall: if $X \subseteq \Sigma_d^{\pm}$ a shift space, the entropy of X is defined
$H(X) = H(\sigma|_X) = \lim_{n\to\infty} \frac{1}{n} \log |\mathcal{L}_n(X)|$

we want to make sure this limit is valid

lemma: let $(a_n)_n$ be a sequence of real numbers s.t.
$a_{n+m} \le a_n + a_m \ \forall \ n, m$.
$\Rightarrow \lim \frac{a_n}{n}$ exists and is equal to $\inf_n \frac{a_n}{n}$

claim: $n \longmapsto \log |\mathcal{L}_n(X)|$ is subadditive

let $w = w_1 \dots w_{n+m} \in \mathcal{L}_{n+m}(X)$
$\Rightarrow w = \underbrace{w_1 \dots w_n}_{\in \mathcal{L}_n(X)} \underbrace{w_{n+1} \dots w_{n+m}}_{\in \mathcal{L}_m(X)}$ gives an upper bound on valid words

$\Rightarrow \mathcal{L}_{n+m}(X) \le \mathcal{L}_n(X) \cdot \mathcal{L}_m(X)$, now apply log. $\square$

prop: Let $X \subseteq \Sigma_d^{\pm}$ be a shift space. Then
$h_{top}(\sigma|_X) = H(X)$

recall: $h_{top}(f) = \lim_{\varepsilon\to 0} \lim_{n\to 0} \frac{1}{n} \log |F_n(\varepsilon)|$
where $F_n(\varepsilon)$ is a maximal (n, ε) separated subset of X.

proof: Let $\varepsilon_k = \theta^{k+1}$.
let $x, y \in X$. then $x_{-k} = y_{-k} \dots x_k = y_k$ is equivalent to saying $d(x,y) < \varepsilon_k$.
$\Rightarrow x_{-k} = y_{-k} \dots x_{k+n-1} = y_{k+n-1}$ is equivalent to $d_n(x,y) < \varepsilon_k$.

define ℓ_n = # cylinders in X of length $2k+n$
$\Rightarrow \ell_n = |\mathcal{L}_{n+2k}(X)| = |F_n(\varepsilon_k)|$

So $\lim_{n\to\infty} \frac{1}{n} \log |F_n(\varepsilon_k)| = \lim_{n\to\infty} \frac{1}{n} \log |\mathcal{L}_{2k+n}(X)|$

subadditivity of before gives this exists if the second does
$\le \lim_{n\to\infty} \frac{1}{n} (\log |\mathcal{L}_{2k}(X)| + \log |\mathcal{L}_n(X)|)$
$= \lim_{n\to\infty} \frac{1}{n} \log |\mathcal{L}_n(X)|$.

Also $\lim_{n\to\infty} \frac{1}{n} \log |F_n(\varepsilon_k)| = \lim_{n\to\infty} \frac{1}{n} \log |\mathcal{L}_{n+2k}(X)|$
$\ge \lim_{n\to\infty} \frac{1}{n} \log |\mathcal{L}_n(X)| = H(X)$.

$\Rightarrow \lim_{n\to\infty} \frac{1}{n} \log |F_n(\varepsilon_k)| = \lim_{n\to\infty} \frac{1}{n} \log \mathcal{L}_n(X)|$

we don't need a lim over k here because this term isn't dependent on k. This isn't true in general.

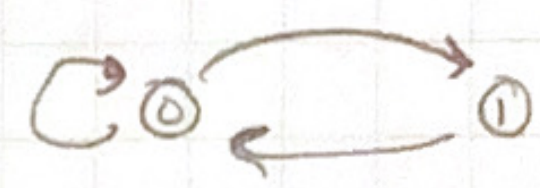
def: let $f: X \to X$ be cont. on a compact metric space X. We say f is expansive if $\exists \ \varepsilon > 0$ s.t. if $d(f^n(x), f^n(y)) < \varepsilon \ \forall n \in \mathbb{N}_0$ then $x = y$. (i.e. if $d(f^n(x), f^n(y)) < \varepsilon \ \forall n \in \mathbb{Z}$ for f homeo).

prop: If f is expansive w/ expansivity constant ε then
$\lim_{n\to\infty} \frac{1}{n} \log |F_n(\frac{\varepsilon}{2})| = h_{top}(f)$.

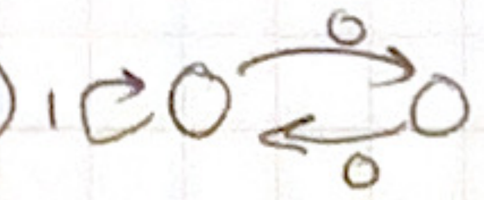
intuitively: entropy is the exponential growth rate of the language

eg. a) $X = \Sigma_d^{\mathbb{Z}}$ full shift

$\Rightarrow h_{top}(X) = \log d$

b) $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ $X = X_A$ golden mean shift  vertex shift

$h_{top}(X) = \log\left(\frac{1+\sqrt{5}}{2}\right)$

c)  even shift

claim: $h_{top}(X) = \log\left(\frac{1+\sqrt{5}}{2}\right)$

neat! comes from the fact that they have the same graph. even shift has just one extra presentation of $\overline{0}$, which goes away in the limit.

this is an example of two systems w/ the same entropy that are not conjugate

Let A be a transition matrix, giving X_A.

lemma: Define $A^n = (a_{ij}^n)_{ij}$. Then $a_{ij}^n = \{|w| : w \in \mathcal{L}_{n+1}(X_A), w_1 = i, w_{n+1} = j\}$,

(the number of paths starting at i, ending at j, of length $n+1$.)

def: A is irreducible if $\forall\, i,j$, $a_{ij}^n \neq 0$ for some n.
(there's some path between every i, j).

lemma: A irreducible iff $\sigma|_{X_A}$ is topologically transitive

↳ this is a "minimality condition" in light of decomposition into simple pieces.

thm: Perron-Frobenius Theorem.
Let A be an irreducible, positive square matrix. Then $\exists$ a positive real eigenvalue λ called the perron eigenvalue s.t.

i) λ is a simple root of the characteristic polynomial
ii) λ has strict, positive left and right eigenvectors
iii) the eigenvectors are unique up to constant multiples
iv) $\lambda \geq |\mu|$ for any other eigenvalue μ of A.
v) If $0 \leq B \leq A$ and β is an eigenvalue of B, then $|\beta| \leq \lambda$ with equality iff $B = A$.

thm: If A is an irreducible transition matrix ~~with~~ then $h_{top}(X_A) = \log \lambda$ where λ is the perron eigenvalue of A.

proof: By the Perron-Frobenius theorem $\exists\, c_1, c_2 > 0$ s.t.
$c_1 \lambda^\ell \leq |\mathcal{L}_\ell(X_A)| \leq c_2 \lambda^\ell$ if ℓ is large enough ∎

def: let $i, j \in 0 \ldots d-1$. we say i, j are related if $\exists\, k, \ell \in \mathbb{N}$ s.t. $a_{ij}^k > 0$ and $a_{ji}^\ell > 0$ (there's some path in each direction)

def: an index i is transient if it's not related to any (itself) index.

this defines an equivalence relation:

$R = \{(i,j) : i$ related to j, i, j not transient$\}$ has equivalence classes that give the irreducible components

The submatrices composed of indices (i,j) which belong to the same equivalence class are the irreducible components of A.

$$\Rightarrow P^{-1}AP = \begin{pmatrix} A_1 & & \cdots \\ & A_2 & \\ 0 & & \ddots & A_r \end{pmatrix}$$ where P is a permutation matrix

cor: If A transition matrix w/ irr. components $A_1 \ldots A_r$ then $h_{top}(X_A) = \log \lambda$ where $\lambda = \max(\lambda_1, \ldots, \lambda_r)$ perron values