02/22/23 - Symbolic Dynamics - Prof. Wolf

Review: 1-step SFTs are described by a transition matrix.
A graph can be described by an adjacency matrix where A_{ij} is the number of edges from $i \to j$, ~~or a transition matrix~~
This is different from a transition matrix which cannot describe parallel edges, only if there is an edge.

An edge shift is described by an adjacency matrix where the alphabet is given by the edges, and $e_1 e_2$ can occur if $t(e_1) = i(e_2)$.
This is a 1-step SFT.

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def: A graph $G = (V, \mathcal{E})$ is essential if no vertex is stranded, where stranded means either there are no incomming or outgoing edges from the vertex. (would mean a bi-infinite sequence would terminate at that node)

def: A graph $H \subseteq G$ is a subgraph of G if $V(H) \subseteq V(G)$, $\mathcal{E}(H) \subseteq \mathcal{E}(G)$, and all edges in H start and end at the same vertices in G.

prop: let G be a graph. Then $\exists$ a unique subgraph H of G s.t. H is essential and $X_H = X_G$.

def: A labled graph $\mathcal{G} = (G, L)$ where G is a graph and $L: \mathcal{E} \to \mathcal{A}$, for alphabet $\mathcal{A} = \{0, \dots d-1\}$.

A labled graph $\mathcal{G}$ is irreducible if G is irreducible. (there's a path in each direction between any two nodes).

note: every edge shift is also a labled graph using $L = id$.

typically lables are not 1-1.

move between biinfinite paths on G and sequences in a shift space by applying the labling function.

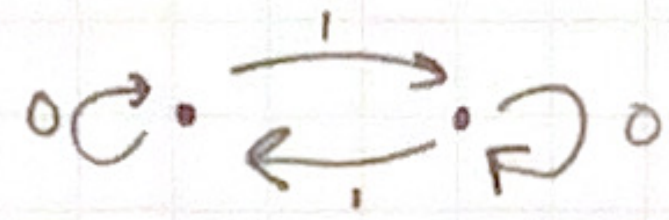
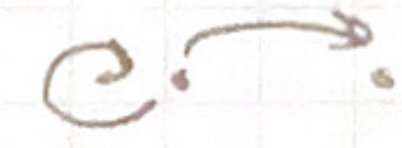
claim: the set generated by applying a labeling function to a graph is a shift space.

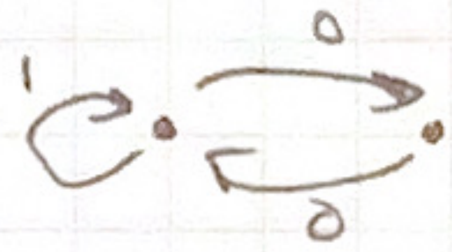
def: we say a subset X of the full shift is a sofic shift if $X = X_{\mathcal{G}}$ for some labeled graph $\mathcal{G}$.

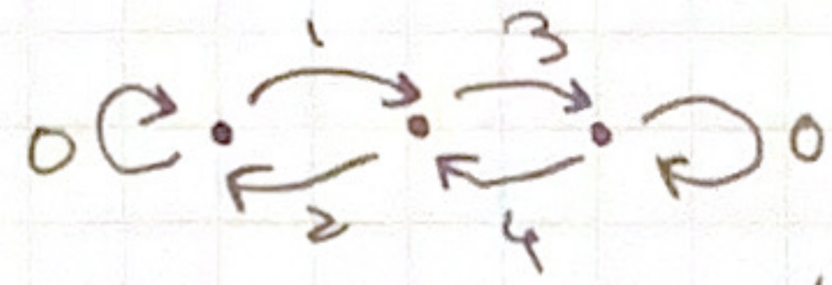
observation: every edge shift is sofic. (usind the id labeling)

$\Rightarrow$ Every SFT is sofic.
(SFT $\to$ higher block code (1-step) $\to$ edgeshift $\to$ id labeling)

def: Let $w \in \mathcal{L}(X)$. We say a block π on $\mathcal{G}$ is a presentation of w if $L(\pi_1) \dots L(\pi_\ell) = w$. $\in L(\pi)$
(i.e. words correspond to finite paths on $\mathcal{G}$ under L).

fact: blocks w might have finitely many presentations e.g.  if $w = 01$, then  or  work.

e.g. The even shift  is not an SFT, but is a sofic shift.

e.g.  claim: this is not an SFT.

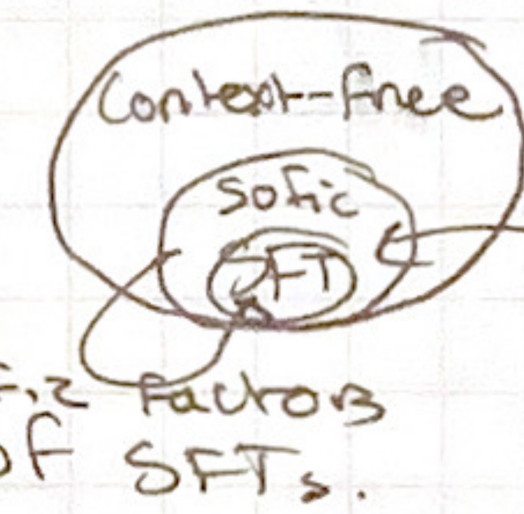
use lemma: $30^M \in \mathcal{L}(X)$, but $30^M 1 \notin \mathcal{L}(X)$
$0^M 1 \in \mathcal{L}(X)$, $\forall M$.
X an SFT means X M-step for some M

note: countable SFTs (countable $\mathcal{A}$, countable words)
countable sofic (countable directed finite graphs, countable lablings)
all shifts are uncountable, even over $\mathcal{A} = \{0,1\}$.
$\to$ most shift spaces are not sofic.

e.g. of a non-sofic shift: context free shift. $\mathcal{A} = \{a, b, c\}$.
$X \subseteq \mathcal{A}^{\mathbb{Z}}$ by $ab^m c^n a \in \mathcal{L}(X)$ iff. $m = n$.
this is a language, $\Leftrightarrow$ so it's a shift space
✓ subwords valid ✓ can extend in either direction.

proof: if X sofic, $\exists$ labeled graph $\mathcal{G}(G, L)$ w/ $X = X_G$.
Set $r = |V|$. Then consider $w = ab^{r+1} c^{r+1} a \in \mathcal{L}(X)$. So there's a path $\pi \in G$ presenting w.
Let τ be the subpath presenting b^{r+1}. Since V has r vertices, τ must contain a repeated vertex. So we can write $\tau = \tau_1 \tau_2 \tau_3$ where τ_2 is a cycle. So $\tilde{\tau} = \tau_1 \tau_2 \tau_2 \tau_3$ is also a path on G. Replacing τ with $\tilde{\tau}$ in π gives a path $\tilde{\pi}$ in G. But $L(\tilde{\pi}) = ab^{r+1+n} c^{r+1} a$ for $|\tau| = n$, which is not allowed in $\mathcal{L}(X)$.

note: every shift space can be represented by an infinite graph, but you have to be careful with closure.



def: Let $f: X \to X$, $g: Y \to Y$ be dynamical systems (X, Y compact metric, f, g cont.).
We say g is a factor of f if $\exists$ $h: X \to Y$ cont. surj. s.t.
$g \circ h = h \circ f$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "h"] & X \arrow[d, "h"] \\ Y \arrow[r, "g"] & Y \end{tikzcd}$$

(h is a factor map, also called a semi-conjugacy).
"(just like top conj. but can be many: one)
f is an extension of g"

e.g. X a sofic shift, $\mathcal{L}_\infty : X_G \to X_{\mathcal{G}}$ is a factor map.
for any labling of G.

cor: X is sofic iff. X is a factor of an SFT.
this is easier to define.

def: (Weiss) let X be a shift space, $w \in \mathcal{L}(X)$. We call $F_{O_X}(w) = \{v \in \mathcal{L}(X) : wv \in \mathcal{L}(X)\}$ the follow words of w wrt. X.
The follow set of a set X is $F_0(X) = \{F_{O_X}(w) : w \in \mathcal{L}(X)\}$
(look at all prefix words and see what can follow).

thm (Weiss) X is sofic $\Leftrightarrow$ $F_0(X)$ is finite.

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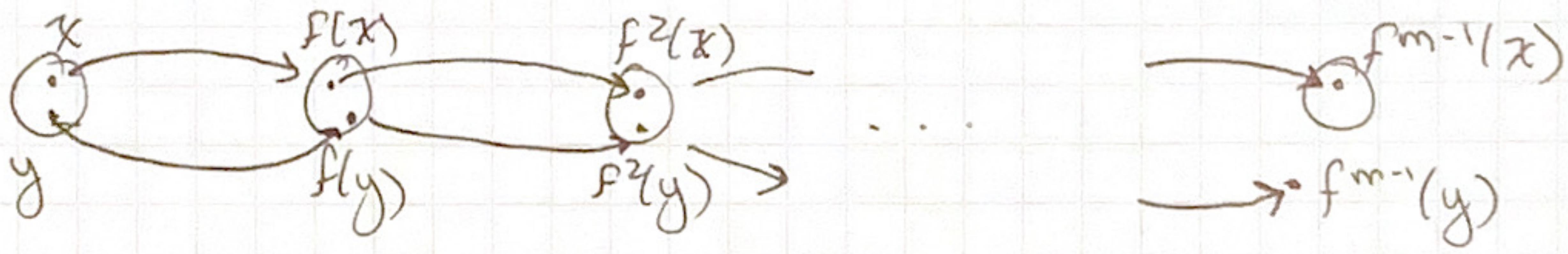
proof: if X is an m-step SFT, $\mathcal{F}_0(X) = \{F_{\sigma_X}(w) : w\, \mathcal{L}_m(X)\}$
for sofic... see L & M.

note: this gives us our third equivalent def. of sofic shift.

*? note: factors can have different entropy of their extensions, but for the special case of sofic/SFTs, these dynamical properties are preserved.

Entropy
Let $f: X \to X$ dynamical system
We're interested in counting the distinguishable orbits under some margin of measurement (e.g. pixels on a screen)
Let $\varepsilon > 0$ be the margin of measurement. Do two points stay in an epsilon ball after m iterations?



Fix $n \in \mathbb{N}$ number of iterations. define
$d_n(x,y) = \max\{d(f^k(x), f^k(y)) : k \in [0 \dots m-1]\}$
then d_n is an equivalent metric to d (same topology)
(note geometrically balls in d and d_m could look very different)

$F \subseteq X$ is (n, ε)-separated if $\forall\, x, y \in F$, $x \neq y$
we have $d_n(x,y) \geq \varepsilon$.

Let $F_n(\varepsilon)$ be a maximal (n,ε)-separated set (w.r.t. the inclusion)

idea of entropy: measure the exponential growth of $F_n(\varepsilon)$ as $n \to \infty$.

def: $\lim_{\varepsilon \to 0} \left(\limsup_{n \to \infty} \frac{1}{n} \log |F_n(\varepsilon)| \right) = h_{top}(f)$ the topological entropy of f.

def: $R \subseteq X$ is an (n,ε)-spanning set if $\forall\, x \in X\ \exists\, y \in R : d_n(x,y) \leq \varepsilon$.
let $R_m(\varepsilon)$ be a minimal (m, ε)-spanning set.

def: $\lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log |R_n(\varepsilon)| = h_{top}(f)$ (i.e. you can span or you can separate w/ the same result)

def: let X be a shift space. Define ~~$H(X)$~~
$H(X) = \lim_{n \to \infty} \frac{1}{n} \log |\mathcal{L}_n(X)|$

Positive entropy means chaotic behavior, but lots of research on trying to classify maps w/ 0 entropy.