02/01/23 - Symbolic Dynamics - Prof. Wolf

Coding maps from Hooper's class as "change of coordinates"

Initial issues with the idea of partitions to construct codes
→ points on boundaries might have multiple possible codes
→ points that stay close together under iteration might not be distinguishable
↳ need "expanding" so points move away. So called "uniformly hyperbolic sets" (see: stable/unstable manifold)

We want to study more kinds of map though, "non-uniformly hyperbolic systems"
↳ look at diffeomorphisms of manifolds, and look at (countable) partitions that behave properly under all hyperbolic measures
↳ called countable markov shifts, not area of research.

We can relate the dynamics of the first return map to the dynamics of the map itself (e.g. Poincaré Recurrence Thm).

⇒ point: we can use symbolic dynamics to understand smooth dynamical systems too.

I) Basic Symbolic Dynamics

I.1) Shift Spaces

def: alphabet $\mathcal{A} = \{0, \dots, d-1\}$

def: $\Sigma_d^{\pm} = \mathcal{A}^{\mathbb{Z}} = \{x = (x_k)_{k \in \mathbb{Z}} : x_k \in \mathcal{A} \ \forall k\}$ two-sided full shift.

def: $\Sigma_d^{+} = \mathcal{A}^{\mathbb{N}}$ .. one-sided shif.

convention: $x = \dots x_{-2} x_{-1} \bullet x_0 x_1 x_2 \dots$ use a $\bullet$ to mark x_0

def: $x_{[i,j]} = x_i x_{i+1} \dots x_j$ for $\overset{i \leq j}{i,j} \in \mathbb{Z}$, $x \in \Sigma_d^{\pm}$ word

def: if $i > j$, $x_{[i,j]}$ is the empty word.

def: $w = w_1 \dots w_m$, $v = v_1 \dots v_n$, $wv = w_1 \dots w_m v_1 \dots v_n$ the concatenation of words
and $w^k = \underbrace{w \dots w}_{k \text{ times}}$, $w^\infty = www\dots$, ${}^\infty w = \dots www$, ${}^\infty w^\infty = \dots www \dots$

def: shift map $\sigma : \Sigma_d^{\pm} \to \Sigma_d^{\pm}$ by $\sigma(x_k)_k = x_{k+1}$

def: Cylinder $[x]_i^j = \{y = (y_k)_k : y_i = x_i, \dots y_j = x_j\}$ for $x \in \mathcal{A}^{\mathbb{Z}}$, $\overset{i,j \in \mathbb{Z}}{i \leq j}$
for full shift, or for arbitrary shift space
$[x]_i^j = \{y \in \Sigma_d^{+} : y_i = x_i \dots y_j = x_j\}$ for $x \in \Sigma_d^{+}$, $0 \leq i \leq j$.
↳ form basis for the topology.

if w is a word of len n and $i \in \mathbb{Z}$, $[w]_i := \{y \in \Sigma_d^{\pm} : y_i = w_1 \dots y_{i+n} = w_n\}$

def: Product topology on $\Sigma_d^{\pm}$ (or Σ_d^{+}) wrt. the discrete topology on A. (Discrete top $\equiv$ every subset open)
(Product top: coursest top that makes all projections open)
$\Rightarrow \Sigma_d^{\pm}$ is a compact top. space (Tychanov)
$\Rightarrow$ metrizable (many options)
let $0 < \alpha < 1$, $d(x,y) = \begin{cases} \alpha \wedge \min\{|k| : x_k \neq y_k\} & \text{for } x \neq y \\ 0 \end{cases}$

exercise: this metric induces the product topology

Lemma: Cylinders are both open and closed (clopen)

proof: Let $i, j \in \mathbb{Z}$, $i \leq j$, $x \in \Sigma_d^{\pm}$. This gives a cylinder $[x]_i^j = \{y = (y_k)_k : y_i = x_i \dots y_j = x_j\}$.

closed: wts complement open. let $x' \in C[x]_i^j \Rightarrow$
$\exists l \in \{i, \dots, j\}$
$x'_l = x_l$, define $\varepsilon = (\frac{1}{2})^{l+1}$
let $y \in B(x', \varepsilon)$
$\Rightarrow y'_l = x'_l \neq x_l$
$\Rightarrow y \notin [x]_i^j$
$\Rightarrow$ ~~Σ~~ $B(x, \varepsilon) \subseteq C[x]_i^j$ b/c complement is open in a metric space, it must be open.

open: let $\{w^1, \dots, w^r\}$ be the set of all words of length $j-i+1$ (the length of w generating the cylinder) except $x_{[i,j]}$. $\Rightarrow r = d^{j-i+1} - 1$.
clearly $[w^n]_i \cap [x]_i^j \neq \emptyset$
$\Rightarrow C[x]_i^j$ is the finite sum of closed sets
$\Rightarrow$ open ∎

★ Cor: $\Sigma_d^{\pm}$ is totally disconnected

proof: Suppose $x, y \in \Sigma_d^{\pm}$ $x \neq y$. $\exists C \subseteq \Sigma_d^{\pm}$ connected w/ $x, y \in C$
$\Rightarrow \exists l \in C$ s.t. $x_l \neq y_l$. Let $u = [x]_l^l$, $V = Cu$.
$\Rightarrow (u, v)$ is a disconnection of C. ∎

I.2) Subshifts and Languages

def: $X \subseteq \Sigma_d^{\pm}$ is a ~~subshift~~ shift space if X is σ-invariant and closed.
$\sigma(X) = X$

def: $\sigma|_X$ is a subshift.
(the space of subshifts is uncountable!)

fix $m \in \mathbb{N}$. $\mathcal{L}_m(x) = \{w : w$ is a word of length m which occurs in $x \in X\}$
$m = 0$, $\mathcal{L}_0(x) = \{\varepsilon\}$, ε the empty word.

def: $\mathcal{L}(X) = \bigcup_{n=0}^{\infty} \mathcal{L}_n(X)$ is the language of X.

On the other hand, if $\mathcal{L} \subseteq \mathcal{L}(\Sigma_d^{\pm})$ s.t. (*) $\forall w \in \mathcal{L}$, and all subwords v of w we have $v \in \mathcal{L}$, and (*) $\forall w \in \mathcal{L}$, $\exists l \in A$ s.t. $wl \in \mathcal{L}$. then we call $\mathcal{L}$ a one-sided language. For a two-sided language, you need also $\exists l, m \in A$ s.t. $mwl \in \mathcal{L}$.

Lemma: Let $\mathcal{L} \subseteq \mathcal{L}(\Sigma_d^{\pm})$ be a two-sided language. Then $X(\mathcal{L}) = \{x \in \Sigma_d^{\pm} : x_{[i,j]} \in \mathcal{L} \ \forall i,j \in \mathbb{Z}, i \leq j\}$ is a shift space.

proof: The shift invariance is trivial since both $\sigma(X)$ and $\sigma^{-1}(X)$ have the same words as X.
To show $X(\mathcal{L})$ is closed, consider $x \in C_{X(\mathcal{L})} \Rightarrow \exists$ word $w \in x$ s.t. $w \notin \mathcal{L}$. $\Rightarrow \exists\, i,j \in \mathbb{Z}\ i \leq j$ s.t. $x_{[i,j]} = w$.
$\Rightarrow [x]_i^j \subseteq C_{X(\mathcal{L})}$, so $C_{X(\mathcal{L})}$ is open, so $X(\mathcal{L})$ is closed.

idea: shift spaces and languages are the same object.

Shift spaces can also be described by the set of forbidden words.

def: Let $X \subseteq \Sigma_d^{\pm}$ be a shift space. $\mathcal{F}_X = \mathcal{L}(\Sigma_d^{\pm}) \setminus \mathcal{L}(X)$ is the set of forbidden words.
↪ you have to be careful when giving $\mathcal{F}_X$ that its complement is indeed a language.

e.g. $d=2$, $\mathcal{F} = \emptyset \Rightarrow X = \Sigma_2^{\pm}$
e.g. $d=2$, $\mathcal{F} = \{11\}$ the Golden Mean Shift.
e.g. $d=2$, $\mathcal{F}_X = \{10^{2k+1}1 : k \in \mathbb{N}\}$. the Even Shift. (this is not a SFT, note the forbidden set is not finite ← it's an S-Gap)

Let $\mathcal{F} \subseteq \mathcal{L}(\Sigma_d^{\pm})$.
Say $\bar{\mathcal{F}} = \{w \in \mathcal{L}(\Sigma_d^{\pm}) : w \text{ has a subword in } \mathcal{F}\}$.
Then $\bar{\mathcal{F}}$ is a forbidden set, and
$X_{\mathcal{F}} = X(\mathcal{L}(\Sigma_d^{\pm} \setminus \bar{\mathcal{F}})) \Rightarrow X_{\mathcal{F}}$ is a shift space.

if you have an arbitrary subset of words, you can fill it up so that you get a forbidden ~~set~~ that gives a language for a shift space

def: A shift space X is irreducible if $\forall\, u,v \in \mathcal{L}(X)\ \exists\, w \in \mathcal{L}(X)$ s.t. $uwv \in \mathcal{L}(X)$.
(you can connect arbitrary words in the language.

def: We say a shift space $X \subseteq \Sigma_d^{\pm}$ is a subshift of finite type if there exist $\mathcal{F} \subseteq \mathcal{L}(\Sigma_d^{\pm})$ finite s.t. $X_{\mathcal{F}} = X$.
(i.e. defined by a finite set of forbidden words)

fun fact: you can always re-write an SFT to another SFT whose $\mathcal{F}$ has words of max length 2. (Increasing the alphabet) which is nice b/c it makes it so much easier to check if something is in our shift space.

Let X be a shift space, Fix $N \in \mathbb{N}$. Say $\mathcal{B} = \mathcal{L}_N(X)$ (words of length N in the alphabet).
We identify $\mathcal{B}$ by $\{0, \dots, a-1\}$ for $a \in \mathbb{N}$. (each word gets a letter in the new alphabet).

def: $X^{[N]} = \beta_N(X)$ where $\beta_N : X \to \mathcal{B}^{\mathbb{Z}}$ by $(\beta_N(x))_k = x_k \cdots x_{k+N-1}$

the higher-block representation of X and order N.

e.g. give $X = \ldots x_{-3} x_{-2} x_{-1} . x_0 x_1 x_2 x_3 \ldots$, $N=3$. gives

$y = \ldots y_{-3}\, y_{-2}\, y_{-1}\, y_0\, y_1 \ldots$

$$\begin{bmatrix} x_{-1} \\ x_{-2} \\ x_{-3} \end{bmatrix} \begin{bmatrix} x_0 \\ x_{-1} \\ x_{-2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_0 \\ x_{-1} \end{bmatrix} \begin{bmatrix} x_2 \\ x_1 \\ x_0 \end{bmatrix} \begin{bmatrix} x_3 \\ x_2 \\ x_1 \end{bmatrix}$$

prop: $(X^{[N]}, \sigma)$ is a shift space. Moreover, (X, σ) and $(X^{[N]}, \sigma)$ are top. conj. $\forall N \in \mathbb{N}$.

let $u = u_1 \ldots u_N$, $v = v_1 \ldots v_N$. we say u, v progressively overlap if $u_2 \ldots u_N = v_1 \ldots v_{N-1}$

$\Rightarrow$ if uv occurs in the language of the higher block rep. of X, (i.e. in $\beta_N(X)$ for some N) then u and v progressively overlap.

$\Rightarrow$ proof: Suppose X is a shift space and $N \in \mathbb{N}$.

shift invariance:

Let B be the alphabet of $X^{[N]}$

Observe that $\beta_N : A^{\mathbb{Z}} \longrightarrow B^{\mathbb{Z}}$ is a bijection

Furthermore $X^{[N]} \subseteq B^{\mathbb{Z}}$.

Give $y = \ldots y_{-2} y_{-1} . y_0 y_1 y_2 \ldots \in X^{[N]}$

$\Rightarrow \exists x \in X$ s.t. $\beta_N(x) = y$.

We have $\sigma(y) = \ldots y_{-2} y_{-1} y_0 . y_1 y_2 \ldots$

$\sigma(x) = x_{-2} x_{-1} x_0 . x_1 x_2$

☆ $\Rightarrow \beta_N(\sigma(x)) = \sigma(y)$

Since $\sigma(x) = x \Rightarrow \sigma(y) \in X^{[N]}$

closed: let $y \in C_{X^{[N]}} = B^{\mathbb{Z}} \setminus X^{[N]}$

$\exists x \in \Sigma_d^+$ s.t. $\beta_N(x) = y$.

$\Rightarrow x \in C_X \Rightarrow \exists \varepsilon > 0$ s.t. $B(x, \varepsilon) \subseteq C_X$

and since β_N bijective $\Rightarrow \beta_N(B(x, \varepsilon)) \subseteq B^{\mathbb{Z}} \setminus X^{[N]}$

$\Rightarrow \beta_N(B(x, \varepsilon)) = B(y, \varepsilon)$

$\Rightarrow X^{[N]}$ is closed.

We like block representations because they let us go to $\mathcal{F}$ with words of length 2 by moving to a higher alphabet. And then we can represent shifts as matrices because we only ever need to be worried about what is allowed in the immediate next step.