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Recall: Coded shifts: $\mathcal{A} = \{0, \dots, d-1\}$

$\mathcal{G} \subseteq \mathcal{L}(\Sigma_d^{\pm})$ generating set

$X = X(\mathcal{G})$ is the smallest shift space which contains all bi-infinite concatenations of generators.

e.g. $X_{seq} = \{\dots g_{i-2} g_{i-1} g_{i_0} g_{i_1} g_{i_2} \dots\}$ sequential set

then $X = \overline{X}_{seq}$ ← topological closure, which makes it a shift space

$X_{res} = X \setminus X_{seq}$ residual set

↳ empty iff $\mathcal{G}$ infinite

↙ maybe X_{res}?

So if $\mathcal{G}$ finite $X(\mathcal{G})$ is a sofic shift, which we understand, so we focus on $|\mathcal{G}| = \infty$.

$x \in X_{seq} \iff \exists$ strictly increasing sequence $(k_i)_{i \in \mathbb{Z}}$ s.t. $x_{(k_i, k_{i+1})} \in \mathcal{G}$. (*)

e.g. $x = \dots x_{-3} \underbrace{x_{-2} x_{-1} . x_0 x_1 x_2}_{\text{maybe this is } g_e} x_3 x_4 \dots$

$k_i = -2$ $\quad k_{i+1} = 3$

$X(\mathcal{G})$ is uniquely representable if $\forall x \in X_{seq}$. $\exists! (k_i)_i$ satisfying (*)

"$\mathcal{G}$ is a unique representation"

↳ i.e. there's only one way to write each bi-infinite sequence.

A coded shift can be described by a countable, labled graph.

$G = (V, E, L)$ a countable labled graph means

V is countable, $L : E \to \mathcal{A}$

we can assign each bi-infinite path a point in the space.

↳ finite $V \Rightarrow$ shift space

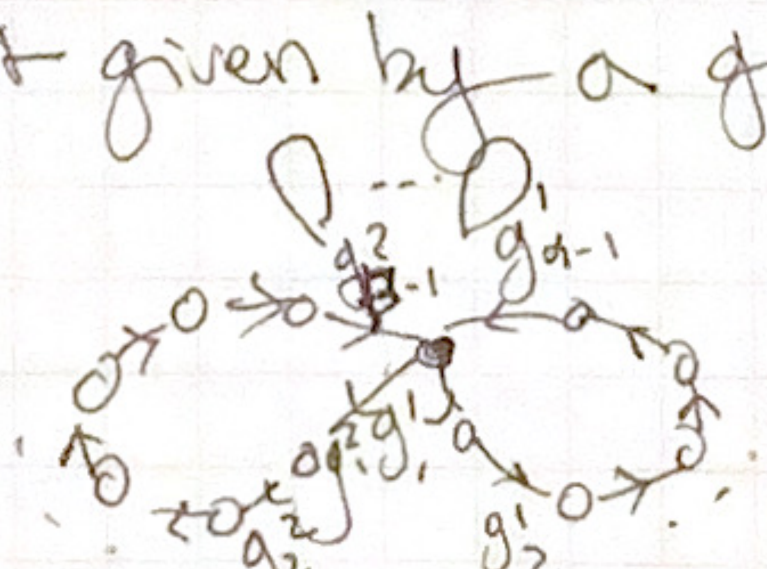
infinite $V \Rightarrow$ take closure to get shift space

coded shifts are exactly the ones that are strongly connected

... TODO: fill in

to build the graph of a shift given by a generating set: (specifically X_{seq})

"infinite flower graph"



e.g. a) S-gap shift. $S \subseteq \mathbb{N}_0$ the shift space X_S is the coded shift w/ generating set $\mathcal{G} = \{0^s 1 : s \in S\}$

here $X_{res} = X_+ \cup X_- \cup X_0$ ← $\{0^\infty_\infty\}$ ← limit point of longer and longer 0s.

same on the left

$X_+ = \{\dots 10^{s-2} 10^{s-1} 1 \dots 10^\infty\}$

We'll see today that all the complexity happens on X_{seq}.

b) Beta shifts. $\beta > 1$ $\beta \notin \mathbb{N}$. Consider $T_\beta : [0,1) \circlearrowleft$ by $x \mapsto \beta x \pmod 1$

The coding space of T_β is given by:
for $x \geq 0$ let $\{x\}$ be the fractional part of x, and $\lfloor x \rfloor$ be the integer part of x.
for $x \in (0,1)$ define $a_1(x) = \{\beta x\}$ and $a_i(x) = \{\beta(a_{i-1}(x))\}$
and define $x_1(x) = \lfloor \beta x \rfloor$ and $x_i(x) = \lfloor \beta a_{i-1}(x) \rfloor$

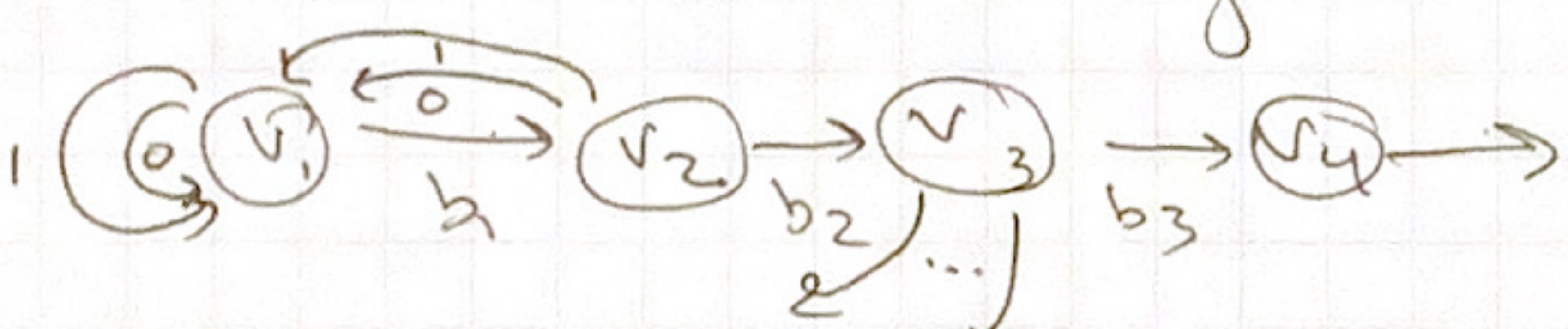
We call $(x_i)_{i \in \mathbb{N}}$ the Beta expansion of X, with $x_i \in \{0, \dots, \lfloor \beta \rfloor\}$

The beta shift X_β is the closure of all beta expansions for $x \in [0,1)$.

it's perfectly alright that this defines a one-sided shift space, because we end up studying the same way.
Here is an alternate definition:

We denote by $\preceq$ the lexicographic order on the one-sided full shift. Then $\exists$ sequence $b = b(\beta) = (b_1 b_2 b_3 \dots)$ which is the lexicographic supremum of the equation $\sum_{j=1}^{\infty} b_j \beta^{-j} = 1$ (the beta expansion of 1) in case there are duplicates.

$x \in X_\beta \Leftrightarrow \sigma^m(x) \preceq b(\beta) \; \forall m$
(under iteration always $\preceq$ the β expansion of 1)



→ strongly connected graph
→ coded shift associated

back to V_1, all things less than V_3

Generating set for the corresponding coded shift is
$\mathcal{G}_\beta = \{b_1 \dots b_\ell j : \ell \in \mathbb{N}, j < b_{\ell+1}\} \cup \{i : i < b_1\}$ } → all loops starting and ending at V_1

and $X_\beta = X(\mathcal{G}_\beta)$

thm: a β shift is coded iff β eventually periodic
↳ can pick finitely many generators from graph

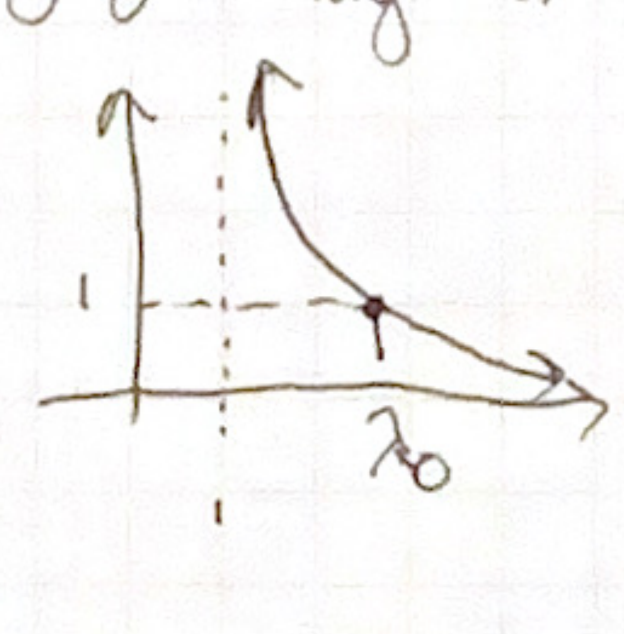
Entropy

Characteristic Equation of a coded shift:

$$\sum_{g \in \mathcal{G}} \lambda^{-|g|} = 1$$

$|g|$: length of word g

→ only representative of the shift space if $\mathcal{G}$ is a unique representation of $X(\mathcal{G})$



λ_0 is the solution of the characteristic equation.

hope: $\log(\lambda_0)$ = entropy of shift space.
(not true! but hopeful b/c lots of e.g.s

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e.g. $X = \Sigma_d^+$, then $\mathcal{G} = \mathcal{L} = \{0, \dots, d-1\}$ unique representation

then CE is $\sum_{0=1}^{d-1} \lambda^{-1} = 1$

$\frac{d}{\lambda} = 1$

$\lambda_0 = d \Rightarrow \log \lambda_0 = h_{top}(X)$ (b/c we know $= \log d$)

e.g. Even shift. $\mathcal{G} = \{1, 00\}$, $X_{res} = \emptyset$
then CE ~~is~~ $\sum_{g \in G} \lambda^{-|g|} = 1$ gives

$\frac{1}{\lambda} + \frac{1}{\lambda^2} = 1$ w/ solutions $\lambda_0 = \frac{1 \pm \sqrt{5}}{2}$

follows $h_{top}(X) = \log \lambda_0 = \log \frac{1+\sqrt{5}}{2}$

e.g. (less trivial) S-gap shift (main ingredients of proof, actual proof too long). (this will be an advanced combinatorial proof)

$S \subseteq \mathbb{N}_0$, $X_S = X(\mathcal{G})$ w/ $\mathcal{G} = \{0^s 1, : s \in S\}$.

let $a_n = ~~\#~~ |\mathcal{L}_n(X_S)|$

We define a generating function $H(z) = \sum_{n=1}^{\infty} a_n z^n$

Note: the radius of convergence of H, ~~is~~ $r = e^{-h_{top}(X_S)}$

Suppose $|z| < r$, then $\beta = \frac{|z|}{r} < 1$ and $|a_n z^n| = \underbrace{\frac{|\mathcal{L}_n(X_S)|}{e^{n h_{top}(X)}}}\cdot \beta^n$

$h_{top}(X) = \lim_{n \to \infty} \frac{1}{n} \log |\mathcal{L}_n(X)|$ so b/c $e^{n h_{top}(X)} \to \mathcal{L}_n(X_S)$,
this term $\to 1$, and since $\beta < 1$, β^n causes $H(z)$ to converge.

Similarly if $|z| > r$ then it diverges

idea: if you can get a handle on radius of convergence, you can figure out entropy.

Define $F_1(z) = \sum_{n \in S} z^{n+1}$ (basically the C.E.)

goal: ~~show that [illegible] of $H(z)$~~ radius of convergence

i.e. a relationship between H and F s.t. the radius of convergence of H is the solution of the equation $F_1(z) = 1$

Want to find a combinatorial way to count the words of length n.
Let $k, n \in \mathbb{N}$, let A_n^k be the # of words in $\{0,1\}^n$ which are concatenations of precisely k words in $\mathcal{G}$.

e.g. $0^{n_1} 1 0^{n_2} 1 0^{n_3} 1$ would have $k = 3$

$n = n_1 + n_2 + n_3 + 3$

Observe $A_n^1 = 1_S(n-1)$

$\Rightarrow F_1(z) = \sum_{n=1}^{\infty} A_n^1 z^n$ and similarly for $F_k(z) = \sum_{n=1}^{\infty} A_n^k z^n$

Fact: $F_k(z) \cdot F_\ell(z) = F_{k+\ell}(z)$.
$\Rightarrow F_k(z) = (F_1(z))^k$

note: this is a finite sum b/c can't have $k > n$.

idea: hope $|\mathcal{L}_n(X)| = \sum_{k \geq 1} A_n^k$ (★★)

↳ problem: a word can start and/or end w/ a part of a generator.

but maybe this computation is good enough?
turns out these extra uncounted words are not enough to change the exponential growth rate, so they're the same in the limit.

(★★) being true implies $H(z) = \sum_{k \geq 1} F_k(z)$
$\Rightarrow H(z) = \sum_{k \geq 1} F_1(z)^k$
which converges if $F_1(z) < 1$ and diverges if $F_1(z) > 1$.

thm: Let $S \subseteq \mathbb{N}_0$. Then $h_{top}(X_S) = \log \lambda_0$ where λ_0 is the solution of $\sum_{n \in S} \lambda^{-(n+1)}$.

It would be nice if $h_{top}(X_g) = \log \lambda_0$ where λ_0 is the solution of CE, holds for all uniquely representable coded shifts $X(g)$
↳ turns out this is wrong!
if X_{seq} has more entropy than X_{res} this works
if they are $=$, may or may not
if $X_{residual}$ entropy is larger, then entropy of X_{seq} is ...

The briefest review of measure theory:

X a set, $\mathcal{B} \subseteq P(X)$ is a sigma algebra if:
i) $X \in \mathcal{B}$
ii) if $A \in \mathcal{B} \Rightarrow C_A = X \setminus A \in \mathcal{B}$
iii) if $A_1, A_2, A_3, \ldots \in \mathcal{B}$ then $\bigcup_{i \in \mathbb{N}} A_i \in \mathcal{B}$ ← sigma comes from "countable"

If X a topological space we call the smallest sima algebra which contains all open sets the sigma algebra of Borel sets.

A function $\mu: \mathcal{B} \to [0, \infty]$ is a measure if $(A_i) \subseteq \mathcal{B}$ (pairwise disjoint)
Then $\mu(\bigcup A_i) = \sum_i \mu(A_i)$ "sigma additivity".

If $\mu(X) = 1$, μ is a probability measure.

~~Suppose X is compact, we denote by M the~~

Suppose $f: X \circlearrowleft$ is cont. w/ (X, d) a compact metric space.
We denote by $\mathcal{M}_X$ the space of all f-invariant Borel probability measures on X.
f invariant $\Leftrightarrow \mu(A) = \mu(f^{-1}(A)) \; \forall A \in \mathcal{B}(X)$
^set of Borel measures

We endow $\mathcal{M}_X$ w/ the weak★ topology, i.e. the smallest topology on $\mathcal{M}_X$ which makes the functions $\mu \mapsto \int \phi \, d\mu$ $\phi \in C(X, \mathbb{R})$ continuous

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i.e. $\mu_n \xrightarrow{\text{in weak *}} \mu$ iff. $\int \phi \, d\mu_n \longrightarrow \int \phi \, d\mu \quad \forall \phi \in C(X, \mathbb{R})$

continuous fns from $X \to \mathbb{R}$

Some nice properties of weak * topology:
- metrizable (by $w_1(\mu_1, \mu_2) = \sup\limits_{\phi \in 1\text{-Lip}(X)} \left| \int \phi \, d\mu_1 - \int \phi \, d\mu_2 \right|$

where $1\text{-Lip}(X) = \{ \phi : X \to \mathbb{R}$ Lipschitz w/ constant $1 \}$

w_1 is the Wasserstein-Kantorovich metric

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next time:
- e.g.s of invariant measures
- measures of maximal entropy
- entropies of X_{seq} and X_{res}.

TODO: review standard measure theory book.