02/15/23 - Symbolic Dynamics - Prof Wolf

Recall: a shift space $X \subseteq \Sigma_d^{\pm} = \{ (x_k)_k : k \in \mathbb{Z},\ x_k \in \mathcal{A} \}$
- shift invariant and closed, so we can do dynamics on it utilizing invariant measures

higher block shifts: let $\mathcal{B} = \mathcal{L}_N(X)$ be the words of length N in X. The function $\beta_N : X \to \mathcal{L}_N(X)^{\mathbb{Z}}$ gives a recoding of X to a new alphabet by $(\beta_m(x))_k = x_k \cdots x_{k+N-1} = [x]_k^{k+N-1}$ where the new words must progressively overlap when in sequence with each other. We call $X^{[N]} = \beta_m(X)$ the N^{th} higher block shift, and we have a conjugacy

$$\begin{array}{ccc} X & \xrightarrow{\sigma} & X \\ \beta_N \downarrow & & \downarrow \beta_N \\ X^{[N]} & \xrightarrow{\sigma} & X^{[N]} \end{array}$$

and β_m is a homeo. b/c $\beta_N^{-1}([y_1 \dots y_m]_e) = [x]_e^{e+m-1+N}$

So $\forall x \in X$ $\beta_N(x) \in [y_1 \dots y_m]_e$, a cylinder set.

note: invariant measures and sets are preserved by conjugacies (e.g. top. entropy is the same for conjugate systems).

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def: Let $X \subseteq \Sigma_d^{\pm}$ be a shift space. We say X is a Shift of Finite Type if $\exists\ \mathcal{F} \subseteq \mathcal{L}(\Sigma_d^{\pm})$ finite s.t. $X = X_{\mathcal{F}}$ (it can be defined by a finite set of forbidden words).

e.g. Full shift, $\mathcal{F} = \emptyset$

e.g. Golden mean shift $\mathcal{F} = \{11\}$

e.g. Let $A : \{0, \dots d-1\} \times \{0, \dots d-1\} \to \{0,1\}$ be a matrix.
Define $X_A = \{ (x_k)_{k \in \mathbb{Z}} : a_{x_k x_{k+1}} = 1 \}$

goal today: show every SFT up to conj. looks like this

def: an SFT is m-step if $\exists$ forbidden word set $\mathcal{F}$ of words length $m+1$ s.t. $X = X_{\mathcal{F}}$.

e.g. if A is a transition matrix, X_A is 1-step

lemma: Every SFT is M step for some $M \geq 0$.

thm: A sift space X is a M-step SFT iff. whenever $uv \in \mathcal{L}(X)$ and $vw \in \mathcal{L}(X)$ and $|v| \geq M$, then $uvw \in \mathcal{L}(X)$.

claim: the even shift is not an SFT

proof: $X_{even} = \{ (x_k)_{k \in \mathbb{Z}} : x_k \in \{0,1\}$ if 10^k1 a word, k even $\}$

If X_{even} was an SFT, then it would be M-step, for some m

Then since $10^{2m+1} \in \mathcal{L}$, $0^{2m+1}1 \in \mathcal{L}$, $\Rightarrow 10^{2m+1}1 \in \mathcal{L}$, but it cannot be since $2m+1$ is odd.

S-gap shift. Give $S \subseteq \mathbb{N}_0$, generators $\{10^n : n \in S\}$, and then
$X_{seq}(S) = \{ \dots g_{n-2}\, g_{n-1}\, g_{n_0}\, g_{n_1}\, g_{n_2} \dots \}$
$X = \overline{X_{seq}}$ is an S-gap shift.

what is ~~add~~ in the complement?
$\bar{0}$, $\bar{0} g_i \dots$, $\dots g_i \bar{0}$, $\bar{0} g_i \dots g_n \bar{0}$

you can do this with an arbitrary generating set $\mathcal{G}$, what you get is called a coded shift. (have irreducible graphs)

↳ we don't know a lot about these, good to think more about in research ~~area~~ paper section of class.

turns out you can find a generating set that gives unique sequences for points (recent result)

can you prove something made up of irreducible graphs (coding shifts) constitutes nonwandering sets and you can say something about complexity?

proof of thm: Suppose X is an M-step SFT

$\Rightarrow X = X_{\mathcal{F}}$ for $\mathcal{F}$ finite, length $M+1$ words.

suppose $uv, vw \in \mathcal{L}(X)$, $|v| \geq M = m$

So $\exists\, x, y \in X$ s.t. $x_{[-k,m]} = uv$, $y_{[1,\ell)} = vw$.

$\Rightarrow x_{[1,m]} = y_{[1,m]} = v$.

we claim $z = x_{(-\infty,0]}\, v\, y_{[m+1,\infty)} \in X$.

If ~~...~~, z contains a forbidden word, that word must occur either in $x_{(-\infty,0]}v$ or $vy_{[m+1,\infty)}$, which is not possible because $x, y \in X$.

So $uvw \in \mathcal{L}(X)$.

now suppose X is a shift space and $M \in \mathbb{N}$ s.t. $uv, vw \in \mathcal{L}(X)$

$|v| \geq M \Rightarrow uvw \in \mathcal{L}(X)$.

Let $\mathcal{F}$ be the set of words of length $M+1$ that do not occur in X.

We claim $X = X_{\mathcal{F}}$.

Let $x \in X$, then no subword of x can occur in $\mathcal{F} \rightarrow x \in X_{\mathcal{F}}$.

Let $x \in X_{\mathcal{F}}$, then $x_{[0,M]}, x_{[1,M+1]} \in \mathcal{L}(X) \Rightarrow x_{[0,M+1]} \in X$.

Also $x_{[2,M+2]} \in \mathcal{L}(X)$, $\Rightarrow x_{[0,M+2]} \in X$.

This works for any finite word in either direction,

so every word in x belongs to X, $\Rightarrow x \in X$. ∎

thm: Let X be an SFT, let Y be a shift space conjugate to X.
Then Y is an SFT.

proof: see L&M.

cor: every SFT is conjugate to a 1-step SFT

cor: every SFT is conjugate to an SFT X_A for some transition matrix A.

Shift spaces via (directed) graph.

def: A graph is a set $G = (V, \mathcal{E})$ where V is a finite set of vertices and $\mathcal{E}$ is a finite set of edges (v_1, v_2) between vertices in V.
(written $e = (i(e), t(e))$ initial state / terminal state)

if two edges have the same vertices, they are called multiple edges.

if $I \in V$, we call $\mathcal{E}_I$ the set of outgoing edges of I, and $\mathcal{E}^I$ the set of incoming edges.

$|\mathcal{E}_I|$ is the out degree, and $|\mathcal{E}^I|$ is the in degree.

def: Let $G = (V, \mathcal{E})$ be a graph. For vertices $I, J \in V$ let A_{IJ} be the # of edges from $I \to J$. Then we call $A_G = [A_{IJ}]$ the adjacency matrix of G.

Similarly, every $r \times r$ matrix A over $\mathbb{N}_0$ defines a ~~matrix~~ graph G_A by $V(G_A) = \{1 \dots r\}$, $\forall I, J \in V$ $\exists$ A_{IJ} edges $I \to J$.

So $A_G = A(G_A)$

def: Let $G = (V, \mathcal{E})$ be a graph, w/ adjacency matrix A. Then the edge shift X_G or X_A is defined by bi-infinite walks on the graph. (the edges are the alphabet).

$X_G = \{ (\xi_i)_{i \in \mathbb{Z}} \in \mathcal{E}^{\mathbb{Z}} : t(\xi_i) = i(\xi_{i+1}) \ \forall i \}$ (these are finite markov shifts)

note: if you lable the edges, it's possible for multiple edges to have the same labels, then you get a sofic shift.

w/o lables you get an SFT.

moral: SFT $\equiv$ transition matrix $\equiv$ graph

note: transition matrix vs adjacency matrix.