03/22/23 - Symbolic Dynamics - Prof. Wolf

$\Sigma_{inf} \ni X \longmapsto h_{top}(X)$ is computable at X_0 iff. $h_{top}(X_0) = 0$.

e.g. we can compute entropy for Golden mean shift, but this is different than this fn being computable at that point b/c we can find an approximating SFT w/ periodic orbit w/ entropy 0.

How do we know periodic points dense in transitive SFTs?

X is transitive if $\forall\, v, u \in \mathcal{L}(X)\ \exists\, w \in \mathcal{L}(X)$ s.t. $vwu \in \mathcal{L}(X)$

↳ we can show there's a dense orbit (a point that contains every word)

def: Let X be a top. space, $f: X \to X$ cont. (don't have compact metric) We say f is topologically transitive if $\forall\, U, V \subseteq X$ open, $\neq \emptyset$ $\exists\, n \in \mathbb{N}$ s.t. $f^{(n)}(U) \cap V \neq \emptyset$.

Facts: a) f transitive is a topological invariant
b) if f has a dense orbit then f is transitive
c) if (X, d) is a compact metric space, then f transitive $\Leftrightarrow$ f has a dense orbit.

Lemma: Let X be a transitive SFT, then $\overline{\mathrm{Per}(\sigma|_X)} = X$.
(the set of periodic orbits is dense in X).

proof: After conjugation we may assume X is 1-step (if m-step, take cylinders of length m as new alphabet, prove this is a conjugacy). X 1-step $\Rightarrow$ transition matrix $A = (a_{ij})$ s.t. $X = X_A$.
Let $x \in X$ and $m \in \mathbb{N}$. Write $x_{[-m,m]} = x_{-m} \dots x_m = w$.
Let $v \in \mathcal{L}(X)$ s.t. $x_m v x_{-m} \in \mathcal{L}(X)$ (which we know exists b/c transitive), so $p = \overline{wv}$ is periodic $\in X$.
$\Rightarrow d(\sigma^m(p), x) < 2^{-m}$.

Shift Spaces beyond SFT & Sofic

def: We say a shift space X is coded if $\exists$ a set of words $\mathcal{G}$ called the generating set s.t. X is the smallest shift space ~~concatenat~~ containing all concatenation of generators.

$X_{seq} = \{ \cdots g_{-i_2} g_{-i_1} g_{i_0} g_{i_1} g_{i_2} \cdots \}$

$X_{seq}(\mathcal{G})$

$X = X(\mathcal{G}) \implies X = \overline{X_{seq}}$

(smallest shift space that contains all biinfinite concatenations)

X_{seq} is definitely shift invariant, but not nec. closed. So we take the closure, which must be the smallest set containing X_{seq} that is a shift space. (Closed iff. G finite $\Rightarrow$ sofic shift).

We call $X_{res} = X \setminus X_{seq}$ the residual set (the things added by closure)

def: We say $X(\mathcal{G})$ is uniquely representable if $\forall\, x \in X_{seq}\ \exists$ unique concatenation of operations that represent x

↳ notes very related to prefix free codes def: $\mathcal{G}$ is uniquely decipherable if $\forall g_{i_1},\dots,g_{i_k}$ the word $w=g_{i_1}\cdots g_{i_k}$ can only be represented in one way (finitely many concatenations of generators).

Let V be a countable set of vertices, E set of edges, $L: E \to \mathcal{A}$
Then $G=(V,E,L)$ is a countable graph.
↳ try taking all paths to form a shift space, but have issues again with closure.

Thm: let X be a shift space. Then X is coded iff one of the following holds:
a) $X = X(\mathcal{G})$ for some uniquely decipherable generating set $\mathcal{G}$
*recent! b) $X = X(\mathcal{G})$ for some uniquely representable generating set $\mathcal{G}$.
c) $\exists$ countable irreducible graph (V,E,L) which represents X.
d) $\exists$ increasing sequence of sofic shifts X_m s.t. $\overline{\bigcup X_m} = X$.

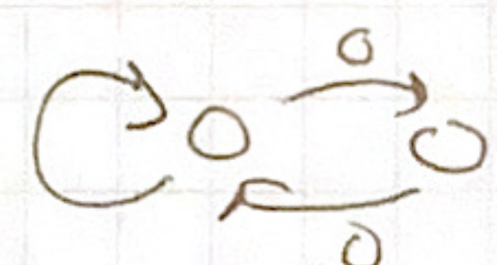
~~recent result~~

References:
- Blanchard & Hansel, System Codes, Theoretical Computer Science 1986
- Fiebig & Fiebig, Covers for Coded Systems, Contemporary Mathematics 1990s.
- Beal & Restivo, Unambiguous coded systems, European Journal of Combinatorics, 2022.

e.g. $\mathcal{G}$ finite $X = X_{seq}(\mathcal{G})$ sofic

e.g. $\mathcal{G}$ co-finite SFT

e.g. even shift, $\mathcal{G} = \{10^{2n}1 : n \in \mathbb{N}\}$,



e.g. S-gap shifts, $S \subseteq \mathbb{N}_0$ $X_S = X_S(\mathcal{G}_S)$ where
$\mathcal{G}_S = \{0^n 1 : n \in S\}$
$\dots 1\underbrace{0\dots0 1}\underbrace{0\dots0 1}1\dots$

$X_{res} = \{\bar{0}, \bar{0}\dots, \dots\bar{0}, \bar{0}\dots\bar{0}\}$

e.g. Generalized gap shifts, Let $d \geq 2$, $S_0,\dots,S_{d-1} \subseteq \mathbb{N}_0$ nonempty.
$\Pi \subseteq P_d$ (permutations on the alphabet, some subset not nec. a subgroup)
Define $X_{S_0,\dots,S_{d-1},\Pi} = X(\mathcal{G}_{S_0,\dots,S_{d-1},\Pi})$
where $\mathcal{G}_{S_0\dots S_{d-1},\Pi} = \{p(0)^{s_{p(0)}}\cdots p(d-1)^{s_{p(d-1)}}d : p \in \Pi, s_{p(j)} \in S_{p(j)}\}$
this tells us where a generator ends

e.g. Number theory β maps.
let $\beta > 0$, then $T_\beta : [0,1) \to [0,1)$ $x \mapsto \beta x \bmod$ is the β shift.
Consider $\beta \notin \mathbb{Z}$, let ~~$x \in (0,1)$~~ $x > 1$, call $\lfloor x \rfloor$ the integer part of x and $\{x\}$ the fractional part.
now let $x \in (0,1)$. call $a_1(x) = \{\beta x\}$ $\quad x_1(x) = \lfloor \beta x \rfloor$
$a_i(x) = \{\beta(a_{i-1}(x))\}$ $\quad x_i(x) = \lfloor \beta(x_{i-1}(x)) \rfloor$
we call $x_1 x_2 x_3 \dots$ the β-expansion x.
X_β is the closure of all β-expansions of all points in $[0,1)$.
↳ the alphabet is $A_\beta = \{0,\dots,\lfloor\beta\rfloor\}$

Another way: let $b = b_1 b_2 b_3 \dots$. The lexicographic supremum of equation $\sum_{j=1}^{\infty}$ by $\beta^{-j} = 1$.

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Fact: $x \in X_\beta \iff \sigma^{n}(x) \leq b(\beta) \ \forall\, n \in \mathbb{N}_0$.

graph Γ_β

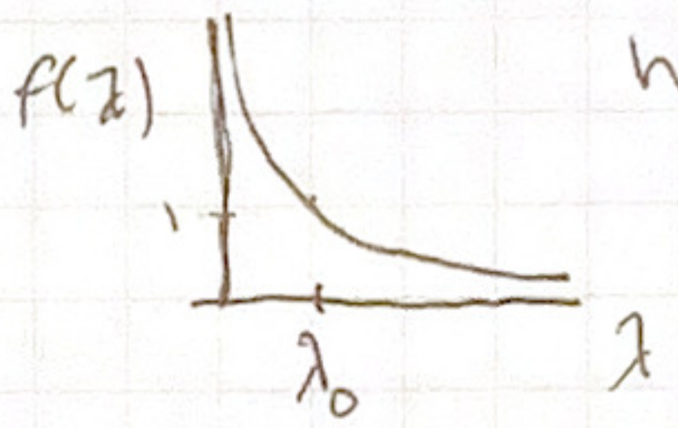
the β-shift is 1 sided (starting at v_1) but we can get a 2-sided shift from this graph:

$$\mathcal{G}_\beta = \{b_1 b_2 \dots b_\ell i : \ell \in \mathbb{N},\ i < b_{\ell+1}\} \cup \{k : k < b_1\}$$

turns out $h_{top}(X_\beta) = \log \beta$.

Every $\mathcal{G}$ (uniquely representable) defines a characteristic equation:

$$\underbrace{\sum_{g \in \mathcal{G}} \lambda^{-|g|}}_{f(\lambda)} = 1$$



hope: $h_{top}(X(\mathcal{G})) = \log(\lambda_0)$

we don't know when this is true.