05/03/23 - Symbolic Dynamics - Prof. Wolf

idea: (for fun) entropy as dimension.
$f : X \to X$, $\mu \in \mathcal{M}_X$.
define $\dim_H \mu = \inf \{ \dim_H A : A \subseteq X \text{ Borel}, \mu(A) = 1 \}$

e.g. $f : J \to J$ J the Julia set of a polynomial,
$\mu \in \mathcal{M}_{J,E}$
define lyapunov exponent: $\lambda_\mu = \int \log |f'| d\mu$
say hyperbolic measure if $\lambda_\mu > 0$.
from manning: $\dim_H \mu = \frac{h_\mu(f)}{\lambda(\mu)}$.
here μ ergodic and hyperbolic.

hausdorff dimension of an invariant measure on a Julia set.

What if we want hausdorff dim. for μ not ergodic?
↪ idea is to use ergodic decomposition
↪ but it turns out this doesn't work.
instead:
$$\dim_H \mu = \text{ess.sup}_{\alpha} \{ \dim_H \tau : \tau \in \mathcal{M}_{f,E} \}$$

ignore sets of measure 0 (essential supremum)

(for each μ you get a diff. α_μ) the ergodic decomposition, a measure on the set of measures
$\alpha_\mu : \Omega(\mathcal{M}_f) \longrightarrow \mathbb{R}^+$ $\alpha_\mu(\mathcal{M}_{f,E}) = 1$.
sigma alg

X compact metric space, $f : X \to X$ cont., $\mu \in \mathcal{M}_X$, $E \in X$ Borel, also $\mu(E) > 0$.

In this set up we can induce a system
$E^\infty = \{ x \in E : f^n(x) \in E \text{ for infinitely many } n \}$
and define induced map
$f_E : E^\infty \to E^\infty$ by $f_E(x) = f^{\tau_E(x)}(x)$ ← first return time.
define $\mu_E(A) = \frac{\mu(A \cap E)}{\mu(E)}$, which
is a Borel prob. measure invariant under f_E

> **Research Question**
> - Studying sets that don't hit an interval "hole"
> - e.g. $f : S^1 \to S^1$ expanding some interval α
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> 
>
> - Study the map $\alpha \mapsto \dim_H \mathcal{M}_\alpha$ ($\mathcal{M}_\alpha$ is the set that never hits α)
> For expanding maps

Then given some potential (observable) $\phi \in C(X, \mathbb{R})$, we can relate topological pressure to measure-theoretic entropy; and then the induced system

$$P_\mu(\phi) = h_\mu(f) + \int \phi \, d\mu$$
$$= \left(h_{\mu_E}(f_E) + \int \phi_E \, d\mu_E \right) \mu(E) \quad \text{Abrimov's Formula}$$

where $\phi_E(x) = \sum_{k=0}^{\tau_E(x)-1} \phi(f^k(x))$

This lets us go $\mu \to \mu_E$, but how would we go the other direction? If we have μ_E and want a measure on X?

We let $\tau \in \mathcal{M}_{f_E}$ w/ $\int \tau_E(x) \, d\tau(x) < \infty$, then define an invariant measure on X:

$$\tilde{\nu}(B) = \sum_{k=0}^{\infty} \nu(\{x : \tau_E(x) > k\} \cap f^{-k}(B))$$

and to make it a probability measure:

$$\ell(\nu) = \frac{\tilde{\nu}}{\tilde{\nu}(X)} \qquad \text{so } \ell(\nu) \in \mathcal{M}_X.$$

Also, $i(\ell(\nu)) = \nu$.

$Y = Y^{\pm}_{\mathbb{N}}$ ← biinfinite ← over an countable alphabet. $g : Y \to Y$ shift map. ← doesn't have to be full shift, can be an SFT.

we can give a metric $d : Y \times Y \to \mathbb{R}^+_0$ by $d(x,y) = \left(\frac{1}{2}\right)^{\min\{|k| : x_k \neq y_k\}}$.

this is a noncompact Polish space (homeomorphic to a separable complete metric space).

Can still define topological pressure:

Let $\phi \in C(Y, \mathbb{R})$, then $P_{top}(\phi) = \lim_{n\to\infty} \frac{1}{n} \log Z_n(\phi)$

where $Z_n(\phi) = \sum_{\tau \in \mathcal{L}_n(X)} \exp(\sup_{x\in[\tau]^{n-1}} S_n\phi(x))$ ← maximal statistical sum

For infinite alphabet many things can go wrong but for an "acceptible potential" it's basically the same as the finite case.

def: ϕ is acceptible if
- ϕ uniformly continuous
- the oscillation of ϕ, $osc(\phi) = \sup\{\sup_{x\in[e]}\phi(x) - \inf_{x\in[e]}\phi(x) : e \in \mathbb{N}\} < \infty$ ← cyl of length 1 over all characters e in alphabet

thm: (Mauldin / Morbanski) ← locally constant
If ϕ acceptable, ~~then~~ $\phi \in LC(Y, \mathbb{R})$ then
- (variational principle):

$$P_{top}(\phi) = \sup\{h_\mu(g) + \int\phi\, d\mu : \mu \in \mathcal{M}_g \text{ w/ } \int\phi\, d\mu > -\infty\}.$$

- $P_{top}(\phi) = \lim_{m\to\infty} P_{top}(Y_m, \phi|_{Y_m})$

where Y_m is the full shift w/ the first m characters of the countable alphabet

We can really compute this! full shift w/ constant potential, easy to compute the pressure.

This is the machinery we need to compute entropy for Coded Shifts.

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known unique rep

Thm: Let $X = X(\mathcal{G})$ be a coded shift which is uniquely represented by $\mathcal{G}$. Let $\phi \in LC(X, \mathbb{R})$ (i.e. $\exists k \in \mathbb{N}$ s.t. $\phi|_{[x]^k_{-k}} \equiv$ const. $\forall x \in X$)

(need an eq. state that lives on X_{seq})

Suppose $\exists \mu_\phi \in ES(\phi)$ ← equilibrium states, w/ $\mu_\phi(X_{seq}) = 1$.

Then $P_{top}(\phi) = \lim_{m\to\infty} P_{top}(X(\mathcal{G}_m), \phi|_{X(\mathcal{G}_m)})$ ← this is a sophic shift

$$\overset{\shortparallel}{P_{seq}(\phi)} \overset{def}{=} \sup\{h_\mu(f) + \int\phi\, d\mu : \mu \text{ put full measure on } X_{seq},\ \mu(X_{seq}) = 1\}$$

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Every shift space has at least one eq. state b/c X is expansive. (might be more)

it's hard to say if there's an ~~measure~~ equillibrium state that gives full measure to X_{seq}. So we compare...

proof: (of thm)
(We want to look at places where pressure is 0.)

1) Replace ϕ with $\phi - P_{top}(\phi)$ ← this is a constant

   So $P_{top}(\phi) = 0$ (this is called normalization). ← new ϕ

2) We can ignore X_{res} for the purpose of pressure. So let $E = \{x = \cdots g_{i_{-2}} g_{i_{-1}} . g_{i_0} g_{i_1} g_{i_2} \cdots, \ x \in X_{seq}\}$.

   ~~either centered~~ only consider x centered like this.

   map $g: E \to E$ by $g(x) = \sigma^{|g_{i_0}(x)|}$ ← unbounded as # of generators get longer.

3) a measure that has full measure on X_{seq} will give E positive measure.
   ↳ so we can induce on every measure
   if $\mu \in \mathcal{M}_X$, $\mu(X_{seq}) = 1 \implies \mu(E) > 0 \Rightarrow i(\mu)$

4) $0 = P_{top}(\phi) = \sup_{\mu \in \mathcal{M}_X} \left(h_\mu(f) + \int \phi \, d\mu \right) = h_{\mu_\phi}(\sigma) + \int \phi \, d\mu_\phi$

   $= \mu_\phi(E) \cdot P_{i(\mu)}(\phi_E)$

   strictly pos. (for $\mu_\phi(E)$) — so this must be 0. (for $P_{i(\mu)}(\phi_E)$) — requires the assumption that there is some equillibrium state.

5) conclude $P_{top}(E, \phi_E) = 0$.

6) X_{seq} is the full shift over the alphabet $\mathcal{G}_m$.

$$\begin{tikzcd} E \arrow[r, "\sigma_E"] \arrow[d, "h"'] & E \arrow[d] \\ Y \arrow[r, "g"'] & Y \end{tikzcd}$$

   $h \in$ natural identification $g_i \to i$ (homeo)

   Apply Morbanski/Mouldin theorem to get:
   $P_{top}(E, \phi) = \sup_{m \in \mathbb{N}} P_{top}(E_m, \phi|_{E_m})$

7) pull back to original result by lifting the measure

   $\Rightarrow P_{top}(\phi) = \lim_{m \to \infty} P_{top}(X(\mathcal{G}_m), \phi|_{X(\mathcal{G}_m)})$

8) this doesn't consider X_{inf}, the points that are made of infinitely many generators, and that's ok!

   ($X(\mathcal{G}_m)$:) this is a sofic shift, for which you can compute the pressure of a locally constant potential.

thm: $X = X(\mathcal{G})$ is a coded shift, $\mathcal{G}$ unique representation.

$h_{seq}(X) = \sup\{h_\mu(\sigma) : \mu(X_{seq}) = 1\}$

$h_{res}(X) = \sup\{h_\mu(\sigma) : \mu(X_{res}) = 1\}$.

Suppose $\mathcal{L}(X)$, $\mathcal{G}$ are given by oracles. Then:

case 1: $h_{seq}(X) > h_{res}(X) \Rightarrow h_{top}(X)$ computable

case 2: $h_{res}(X) > h_{seq}(X) \Rightarrow h_{top}(X)$ not computable

(case 3: $h_{seq}(X) = h_{res}(X)$ computable if eq. state on h_{seq}.)

proof:

(case 1) $\phi = -h_{top}(f) \Longrightarrow P_{top}(\phi) = 0$ (normalize)

always know $n \longmapsto \min\{\frac{1}{k}\log|\mathcal{L}_k(X)| \quad k \le n\} \downarrow$ converges from above to $h_{top}(f)$.

$\Rightarrow h_{top}(X)$ is upper semi-computable

by the prev. thm, $h_{top}(X) = \lim_{m\to\infty} h_{top}(X_m)$

so this is a computable sequence converging from below to $h_{top}(X)$.

recall: entropy for a sofic shift $\sum_{k=1}^{m} \lambda^{-|g_k|} = 1$.

$\Rightarrow$ ~~as~~ $\log\lambda = h_{top}(X_m)$ is computable.

Since ~~there~~ it is computable from above and below, it is computable.

in this case there must be an eq. state that puts full measure on X_{seq}.

(case 2) the sequence in $\lim_{m\to\infty} h_{top}(X_m)$ converges from above to something smaller than entropy.

↳ to prove noncomputable mess w/ the to get any value in the gap

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Some open problems:

1) Let X be a transitive SFT, $\phi \in C^{\alpha}(X, \mathbb{R}) \longrightarrow \exists \mu_\phi \in ES(\phi)$.

Is $\phi \longmapsto \mu_\phi$ computable?

probably not in general, but what would be the largest class for which it is?

2) Zero temperature measures.

$\phi \in C^\alpha(X, \mathbb{R}) \longrightarrow \mu_\phi$.

Consider $\phi_\beta = \beta(\phi)$ where $\beta = \frac{1}{T}$ inverse temperature

↳ μ_β. what happens if $\beta \to \infty$? (i.e. $T \to 0$)

if it exists, $\mu_{\infty,\phi}$.

if it doesn't exist, look at ground state set, set of accumulation points.

this is only solved for locally constant fns.

find families of Holder potentials where you can explicitly decide if a zero temperature measure exists or not?

↳ (almost no explicit results in the literature)