04/26/23 - Symbolic Dynamics - Prof. Wolf

know this

ergodic measure: extreme point in the measure space
or the preimage of a ~~pre~~measurable f-invariant set must be 0 or 1.
e.g. Lebesgue measure not ergodic for a rational rotation of a circle.

last class: for ergodic measures $h_\mu(f) = h_{top}(\mathcal{B}(\mu))$

what is the basin of μ? know this

e.g. let $x \in \Sigma_{inv.}$
$x = f^\alpha(x)$ $\quad x \in P_{\mu_\alpha}(f)$

def: $\mu_x = \frac{1}{\alpha} \sum_{k=0}^{\alpha-1} \delta_{f^k(x)}$ periodic orbit gives you an invariant measure that is ergodic.

Compute entropy using: $h_\mu(x) = \lim_{\varepsilon \to 0} \lim_{n \to \infty} -\frac{1}{n} \log \mu(B_n(x, \varepsilon))$
- x is the only point carrying the measure
- so $B_n(x, \varepsilon) = \frac{1}{\alpha}$
- so entropy $= 0$.

e.g. Let $X = \Sigma_d^+$ $\quad 0 \le p_0, \dots p_{d-1} \le 1$ $\quad \sum p_i = 1$.
define $\mu(C(x_0 \cdots x_k)) = \prod_{i=0}^{k} p_{x_i}$ def: this is the Bernoulli measure on the cylinders → form a subalgebra, so we know for the full space
↳ the Bernoulli measure is shift invariant.
↳ $h_\mu(f) = \sum_{i=0}^{d-1} p_i \log p_i$

$p_i = \frac{1}{d}$

$\Rightarrow h_\mu(f) = \log d = h_{top}(f)$
↳ μ is a measure of maximal entropy, (mme)
one can show that this measure is unique

note: an mme always exists for (an SFT ?)

e.g. Markov measures.
Let $X = X_A$ is a transitive SFT w/ transition matrix A
finite path between all letters
$B \ge 0$ is a stochastic matrix (rows sum to 1)
def: B is faithfully compatible w/ A if $A_{ij} = 0 \Rightarrow B_{ij} = 0$.

Let $M_{stoch}(A) = \{B$ stochastic, faithfully compatible w/ $A\}$
$p = (p_0, \dots, p_{d-1})$ w/ $p_0 + \dots + p_{d-1} = 1$, $\quad P \in M_{stoch}(A)$

define $\mu_{(p,P)}(C(x_0 \cdots x_k)) = p_{x_0} \cdot P_{x_0 x_1} P_{x_1 x_2} \cdots P_{x_{k-1} x_k}$
little *matrix*
↳ invariant measure
↳ called a 1-step Markov measure
→ $h_{\mu_{(p,P)}}(\sigma) = \sum_{i,j} p_i P_{ij} \log P_{ij}$

e.g. $X \subset \Sigma_{inv}$ $\mu \in \mathcal{M}_X$.
$\Rightarrow h_\mu(f) = \lim_{n \to \infty} -\frac{1}{n} \sum_{\tau \in \mathcal{L}_n(X)} \mu([\tau]^{n-1}) \log \mu([\tau]^{n-1})$ for general invariant measures you need to take the limit (use that the cylinders are generating partitions)

check: Pesin Dynamical Systems book or Springer lecture notes that has lots of fundamental dynamics. (on local entropy by Katok and Brin)

(f is a continuous map on a compact metric space)

philisophically: all a scientist can do is observe measurements. There may be an underlying dynamical systems.

↳ ~~inverse~~ dynamical inverse problem: in a fixed phase space, can you recover the dynamical system from observables?
- ⇒ is there an injective map from observables to a dynamical system?
- ⇒ is this map constructive

⇒ usually no. Maybe sometimes you can? using topological pressure.

def: Topological pressure $P_{top}: C(X, \mathbb{R}) \to \mathbb{R}$ (continuous)
in general $f: X \to X$ is a continuous dynamical system.
or work w/ $X \in \Sigma_{inv}$, $\sigma: X \to X$.

$$Z_n(\phi) = \sum_{\tau \in \mathcal{L}_n(X)} \exp\left(\sup_{x \in [\tau]_0^{n-1}} S_n\phi(x)\right)$$

where $S_n\phi(x)$ is the weighted orbit of x along n, i.e. $S_n\phi(x) = \sum_{k=0}^{n-1} \phi(\sigma^k(x))$

this is the n^{th} partition fn

know this know this

we can show $\log Z_n(\phi)$ is subadditive.
⇒ convergent, and the limit is
$P_{top}(\phi) = \lim_{n\to\infty} \frac{1}{n} \log Z_n(\phi)$

↳ we can use this def too on a countable alphabet.

$= \inf_n \{ \cancel{Z_n(\phi)} \log \frac{1}{n} Z_n(\phi) \}$

$\phi = 0$ gives $Z_n(\phi) = |\mathcal{L}_n(X)|$ so ⇒ $P_{top}(\phi) = h_{top}(\sigma)$

note: this is a purely topological construction!

Thm: (Variational principle).

$$P_{top}(\phi) = \sup_{\mu \in M_X} \underbrace{(h_\mu(\sigma) + \int \phi \, d\mu)}_{\text{"free energy" } F_\mu(\phi)}$$

if μ_ϕ satisfies that $P_{top}(\phi) = F_{\mu_\phi}(\phi)$ then we say μ is an equilibrium state of ϕ.

understand

Claim: when X is a shift map there is always one equilibrium state.
how to construct?

$(\mu_n) \subseteq M_X$ s.t. $h_{\mu_n}(\sigma) + \int \phi \, d\mu_n \longrightarrow P_{top}(\phi)$

(upper semicont. / cont. / by def. of weak top...)

↓ has a convergent subsequence that must converge to $P_{top}(\phi)$

upper semi continuous gives convex + closed
↳ so it has extreme points (the set of equilibrium states)

if ϕ is a transitive sophic SFT, is there a unique equilibrium state?
⇒ yes!

04/26/23 - Symbolic Dynamics

it's very difficult to study invariant measures
but if you have a space of potentials w/ equillibrium states then
we can work with observables

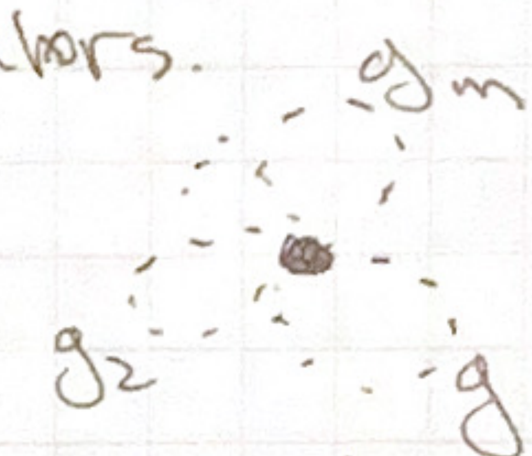
We want to talk about coded shifts, and generalize results from SFTS.
and Sofic shifts

Recall $\mathcal{G} = \{g_1, \dots\}$ generating words.
$X = X(\mathcal{G}) = \overline{X_{seq}}$ closure of all concatenations of $\mathcal{G}$

this is not unique for some X, but theres a "minimal" notion
and look at $\mathcal{G}$ which uniquely represents X.

If $\mathcal{G}$ not finite the closure includes things not in X_{seq},
which we call $X_{residual} = X \setminus X_{sequential}$

Let $X_m = X(\{g_1, \dots g_m\})$ w/ finite generators.
↳ this is a sofic shift: produce the graph
which is finite, directed, labled.



We can compute matrizes of Sofic shifts by finding the
adjacency matrix of the graph + computing like an SFT.

The same is true for pressure. For which coded shifts $X(\mathcal{G})$ and
which potentials $\phi : X \to \mathbb{R}$ do we have
$P_{top}(X_m, \phi|_{X_m}) \longrightarrow P_{top}(X, \phi)$

ϕ lives on all of X but we can restrict it to the first m generators

if $\phi \equiv 0$ does $h_{top}(X_n) \longrightarrow h_{top}(X)$?
↳ this does not hold for all coded shifts

but it does for some!
Recall S-gap shifts: $S \subseteq \mathbb{N}_0$ $\mathcal{G} = \{0^n 1 : n \in S\}$
* know this * Recall β-shift: take β shift on $[0,1]$, do the β expansion of 1,
look at all #s that are smaller than the β-expansion of 1.

↳ even w/ full oracle access and full generators you cannot compute
the entropy (which is a bit surprising)

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* know Borell set * inducing (a common technique in dynamical systems research)
Let $X = \Sigma_{inv}$ $\sigma : X \to X$, $\mu \in \mathcal{M}_X$ invariant, fix $E \in \mathcal{B}(X)$ measurable (Borell set) (smallest Σ-alg containing all open sets)
look at first return map on X:
$\tau_E(x) = \min\{n \geq 1 : \sigma^n(x) \in E\}$ w/ $\min \emptyset = +\infty$.
Look at $x \in X$ that return infinitely often:
$E^\infty = \{x \in E : f^n(x) \in E$ infinitely many $n \in \mathbb{N}\}$.
$\mu(E^\infty) = \mu(E)$ by Poincaré recurrence
$\mu_E(F) = \dfrac{\mu(F \cap E)}{\mu(E)}$ is a prob. measure on E

$\sigma_E : E^\infty \to E^\infty$ by $\sigma_E(x) = \sigma^{\tau_E(x)}(x)$
μ_E is a a σ_E invariant measure and if μ is ergodic $\Rightarrow \mu_E$ ergodic.

this is how we push measures nonzero on E onto the first return map.
↳ Sometimes it's easier to study this induced system.

$$\mathcal{M}_X \qquad \mathcal{M}_{E^\infty}$$
$$\mu \longmapsto \mu_E$$
$$\text{|||}$$
$$i(\mu)$$
(induced)

we have a formula relating entropies for these induced system.

$$h_{i(\mu)}(\sigma_E) = \frac{h_\mu(\sigma)}{\mu(E)}$$

now say you have potential ϕ and take the integral?

$$\int \phi \, d\mu \longmapsto \int_E \Big(\underbrace{\sum_{k=0}^{\tau_E(x)-1} \phi(\sigma^k(x))}_{} \Big) \, d\mu$$

point outside E get 0 measure

this potential is called ϕ_E

so this is fine doesn't need to be μ_E (b/c we want to be able to divide in the next step)

$$\frac{\int_E \phi \, d\mu}{\mu(E)} = \int \phi_E \, d\mu_E$$

we obtain free energy $\dfrac{F_{\mu,\sigma}(\phi)}{\mu(E)} = F_{\mu_E, \sigma_E}(\phi_E)$

gives us a relation to study the pressure of the induced system which is often easier

↳ in the induced system the set is often not more compact.

...

Assume measure $\nu \in \mathcal{M}_{\sigma_E}$ w/ integral over first return is finite:
then we can lift the measure: $\int \tau_E(x) \, d\nu(x) < \infty$. (i.e. τ_E is L^1)

$\tilde{\nu} \in \mathcal{M}_E$ by $\tilde{\nu}(B) = \sum_{k=0}^{\infty} \nu(\{x : \tau_E(x) > k\} \cap f^{-k}(B))$

↳ finite (b/c integral finite)

↳ σ-invariant

↳ not nec. a prob. measure

normalize this measure:

$\ell(\nu) = \dfrac{\tilde{\nu}}{\tilde{\nu}(X)}$ is the lift of ν, and it is a σ-inv. prob. measure.

prop: $i(\ell(\nu)) = \nu$ (lifting and inducing are inverses)

$$\mathcal{M}_{\sigma, X} \underset{\ell}{\overset{i}{\rightleftarrows}} \mathcal{M}_{\sigma_E, E^\infty} \cap \left\{ \int_E \tau_E(x) \, d\nu(x) < \infty \right\}$$

is a natural bijection.

↳ for other systems, a different inducing scheme might be better (choose a different return map).

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why are we doing all of this?

$X_{seq} = \{ x = \dots g_{-2} g_{-1} g_0 g_1 g_2 \dots \}$

induce on set where origin lands between generators

- gives you a full shift on generators (countably infinite)
- then it's good enough to estimate on the first m generators.