APC 524 - Homework #1

8) $y_n \equiv \int_0^1 x^n e^x \, dx \quad (n > 0) \qquad y_{n+1} = e - (n+1) y_n$

a) start from y_N to get to $y_k \quad (k < N)$

$$y_{n+1} = e - (n+1) y_n \quad \rightarrow \quad y_n = \frac{e - y_{n+1}}{n+1}$$

then the reverse recurrence is $y_{n-1} = \dfrac{e - y_n}{n}$

try a couple iterations to find the explicit formula

$$y_{n-2} = \frac{e - \left(\frac{e - y_n}{n}\right)}{n-1} = \frac{ne - e + y_n}{n(n-1)}$$

$$y_{n-3} = \frac{e - \left(\frac{ne - e + y_n}{n(n-1)}\right)}{n-2} = \frac{n(n-1)e - ne + e - y_n}{n(n-1)(n-2)}$$

$$y_{n-4} = \frac{e - \left(\frac{n(n-1)e - ne + e - y_n}{n(n-1)(n-2)}\right)}{n-3}$$

$$= \frac{n(n-1)(n-2)e - n(n-1)e + ne - e + y_n}{(n)(n-1)(n-2)(n-3)}$$

so the pattern is

$$y_k = \frac{1}{(n)(n-1)\cdots(n-k+1)} \left(e \sum_{i=k+2}^{N} (-1)^i \prod_{j=i}^{N} j + (-1)^{N-k} y_n\right)$$

$$= \frac{k!}{N!} \left(e \sum_{i=k+2}^{N} (-1)^i \prod_{j=i}^{N} j + (-1)^{N-k} y_n\right)$$

then $\quad \text{cond } g(k)(y_N) = \left|\dfrac{y_N \, g'(k)}{y_k}\right| = \left|\dfrac{y_N}{y_k} \dfrac{k!}{N!}\right|$

we know that $\dfrac{y_N}{y_k}$ must always be less than 1 since the integral

$\int_0^1 x^n e^x dx$ decreases as n increases. Then $\left|\dfrac{y_N}{y_k}\right| \leq 1$

$$\text{cond } g(k)(y_N) \leq \left|\frac{k!}{N!}\right|$$

b)

rel error $y_k \leq$ cond $g(k) \cdot$ rel error y_N

assuming rel error $y_N = 1$ and we want rel error $y_k = \varepsilon$

$$\varepsilon \leq \left|\frac{k!}{N!}\right| (1) \quad \rightarrow \quad N! \leq \frac{k!}{\varepsilon}$$

c) assuming machine epsilon $= 2 \times 10^{-16}$ and $k = 20$

$$2 \times 10^{-16} \leq \frac{20!}{N!}$$

want to find smallest N such that this relationship is true.

by trial and error, $N \simeq 31$