APC 524 - Numerical Algorithms

Homework #1

3)

a)

i) repeatedly multiplying by x (assume x is an error-free machine #)

consider first $x \cdot x$

$fl(x \cdot x) = x^2(1+\varepsilon_{x^2})$, where ε_{x^2} is the rounding error from representing the product of two machine numbers

now consider $x \cdot x \cdot x$ -

$fl(x^2(1+\varepsilon_{x^2})x) = x^2(1+\varepsilon_{x^2})x(1+\varepsilon_{x^3}) = x^3(1+\varepsilon_{x^2}+\varepsilon_{x^3}+\cancel{\varepsilon_{x^2}\varepsilon_{x^3}})$

$= x^3(1+\varepsilon_{x^2}+\varepsilon_{x^3})$

→ the rounding error from representing $x^2(1+\varepsilon_{x^2}) \cdot x$

and for $x \cdot x \cdot x \cdot x$ (x^4)

$fl(x^3(1+\varepsilon_{x^2}+\varepsilon_{x^3})x) = x^4(1+\varepsilon_{x^2}+\varepsilon_{x^3})(1+\varepsilon_{x^4})$

$= x^4(1+\varepsilon_{x^2}+\varepsilon_{x^3}+\varepsilon_{x^4}+\cancel{O(\varepsilon^2)})$

$= x^4(1+\varepsilon_{x^2}+\varepsilon_{x^3}+\varepsilon_{x^4})$

then repeatedly multiplying x to calculate x^n results in the following relative error

rel error $= \varepsilon_1+\varepsilon_2+\dots+\varepsilon_{n-1}$, where all ε_i represent the errors from each multiplication

the upper bound on this error is found by assuming all ε_i are maximized. Based on the ansatz that $fl(x \circ y) = (x \circ y)(1+\varepsilon)$ $|\varepsilon| \le eps$ then the above expression can be written

max rel error $\le (n-1)\,eps$

ii) calculate x^n by $\exp(n\ln x)$

first consider $\ln x$ (assuming x is error free machine #)

$fl(\ln x) = \ln x(1+\varepsilon_{\ln x})$

now consider $n\ln x$

$fl(n\ln x(1+\varepsilon_{\ln x})) = n\ln x(1+\varepsilon_{\ln x})(1+\varepsilon_n)$

$= n\ln x(1+\varepsilon_{\ln x}+\varepsilon_n+\cancel{\varepsilon_{\ln x}\varepsilon_n})$

$= n\ln x(1+\varepsilon_{\ln x}+\varepsilon_n)$

finally consider $\exp(n\ln(x))$

$fl(\exp(n\ln x(1+\varepsilon_{\ln x}+\varepsilon_n))) = \exp(n\ln x(1+\varepsilon_{\ln x}+\varepsilon_n))(1+\varepsilon_{\exp})$

$= \exp(n\ln x)\exp(n\ln x(\varepsilon_{\ln x}+\varepsilon_n))(1+\varepsilon_{\exp})$

examine $\exp(n\ln x(\varepsilon_{\ln x}+\varepsilon_n))$

this should be very small since ε is very small. Then we can approximate this with a first order Taylor expansion

$\exp(n\ln x(\varepsilon_{\ln x}+\varepsilon_n)) \approx 1 + n\ln x(\varepsilon_{\ln x}+\varepsilon_n) + \dots$ (higher order terms)

plugging back into the error equation

$fl(\exp(n\ln x(\varepsilon_{\ln x}+\varepsilon_n))) = \exp(n\ln x)(1+n\ln x(\varepsilon_x+\varepsilon_n))(1+\varepsilon_{exp})$

$= \exp(n\ln x)(1+n\ln x(\varepsilon_x+\varepsilon_n)+\varepsilon_{exp} + n\ln x(\cancel{\varepsilon_x+\varepsilon_n})\varepsilon_{exp})$

$= x^n(1+n\ln x(\varepsilon_{\ln x}+\varepsilon_n)+\varepsilon_{exp})$

so the max relative error (bounded by eps) is

max rel error $= (1+2|n\ln x|)\,eps$

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3)

a) continued

If we want to create a guide for using repeated multiplication or the method of $e^{n\ln x}$ we can set the errors equal and get relationship between x and n

$(n-1)\,\text{eps} = (1 + 2|n\ln x|)\,\text{eps} \rightarrow n-1 = 1 + 2|n\ln x|$

solving for x -

$x = \exp\left(\frac{n-2}{2n}\right)$ this is the relationship between n and x for which both methods produce the same error.

then if x is between 1 and $\exp\left(\frac{n-2}{2n}\right)$, the error will be smaller by using the $e^{n\ln x}$ method.

$$\text{guide} = \begin{cases} e^{n\ln x}, & 1 \le x \le \exp\left(\frac{n-2}{2n}\right);\ n \ge 2 \\ x \cdot x \cdot x \cdots, & \text{otherwise} \end{cases}$$

b) $x > 0$ $a \neq 0$

$fl(x^a) = x^{a(1+\varepsilon_a)} = x^a x^{a\varepsilon_a}$

ii) $fl(x^a) = (x(1+\varepsilon_x))^a = x^a(1+\varepsilon_x)^a$

$(1+\varepsilon_x)^a \approx 1 + a\varepsilon_x$

$fl(x^a) \rightarrow x^a(1 + a\varepsilon_x)$

this case has relative error of $a\varepsilon_x$, which should be small because ε_x will always be small.