APC 524 - Homework # 1

4)

$f(x) = 1 - e^{-x}$

a) $\text{cond } f = \left| \frac{x f'(x)}{f(x)} \right|$ when $x, y \neq 0$

$\text{cond } f = |f'(x)|$ when $x = 0$

for $0 < x < 1$,

$\text{cond } f = \left| \frac{x f'(x)}{f(x)} \right| = \left| \frac{x e^{-x}}{1 - e^{-x}} \right|$ this function is monotonically decreasing in the interval $(0,1)$

and at $x = 0$

$\text{cond } f = |f'(x)| = |e^{-x}| = 1$, when $x = 0$

since we know $\frac{x e^{-x}}{1 - e^{-x}}$ is always decreasing on $(0,1)$, we know cond f is always less than 1 on the interval

b)

$fl(-x) = -x$

$fl(e^{-x}) = e^{-x}(1 + \varepsilon_{exp})$

$fl(1 - e^{-x}(1 + \varepsilon_{exp})) = (1 - e^{-x}(1 + \varepsilon_{exp})(1 + \varepsilon_{sub}))(1 + \varepsilon_{rd})$

$= (1 - e^{-x}(1 + \varepsilon_{exp} + \varepsilon_{sub} + \cancel{\varepsilon_{exp}\varepsilon_{sub}}))(1 + \varepsilon_{rd})$

$= 1 - e^{-x} + \varepsilon_{rd} - e^{-x}(\varepsilon_{exp} + \varepsilon_{sub}) - e^{-x}(\cancel{\varepsilon_{exp}\varepsilon_{sub}})\varepsilon_{rd}$

$= 1 - e^{-x} + \varepsilon_{rd} + |e^{-x}(\varepsilon_{exp} + \varepsilon_{sub})|$

$(1 - e^{-x})(1 + \varepsilon_t) =$

$1 + \varepsilon_t = 1 + \frac{\varepsilon_{rd} + e^{-x}(\varepsilon_{exp} + \varepsilon_{sub})}{1 - e^{-x}}$

assuming $\varepsilon_{rd}, \varepsilon_{exp}, \varepsilon_{sub} \leq eps$

$1 + \varepsilon_t = 1 + eps\left(\frac{1 + 2e^{-x}}{1 - e^{-x}}\right)$

to find cond A, want x_A such that $f(x_A) = f_A(x)$

$(1 - e^{-x})\left(1 + eps\left(\frac{1 + 2e^{-x}}{1 - e^{-x}}\right)\right) = 1 - e^{-x_A}$

$\cancel{1} - e^{-x} + eps\left(\frac{1 + 2e^{-x}}{1 - e^{-x}}\right) + \left|e^{-x} eps\left(\frac{1 + 2e^{-x}}{1 - e^{-x}}\right)\right| = \cancel{1} - e^{-x_A}$

$e^{-x}\left(1 + eps\left(\frac{1 + 2e^{-x}}{1 - e^{-x}}\right)e^{x} + eps\left(\frac{1 + 2e^{-x}}{1 - e^{-x}}\right)\right) = e^{-x_A}$

$e^{-x}\left(1 + (1 + e^{x})\, eps\left(\frac{1 + 2e^{-x}}{1 - e^{-x}}\right)\right) = e^{-x_A}$

$x_A = x - \ln\left(1 + (1 + e^{x})\, eps\left(\frac{1 + 2e^{-x}}{1 - e^{-x}}\right)\right)$

$\ln(1 + x) \approx x$

$x_A = x - \left((1 + e^{x})\, eps\left(\frac{1 + 2e^{-x}}{1 - e^{-x}}\right)\right)$

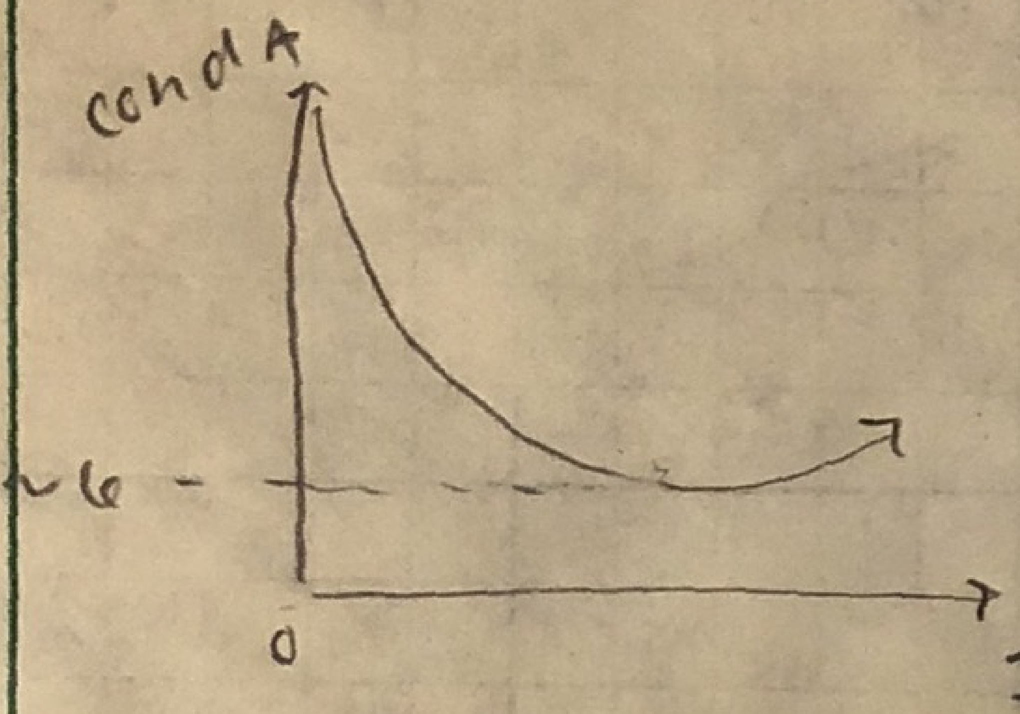
then $\text{cond } A = \frac{\|x - x_A\|}{\|x\|}\frac{1}{eps} = \frac{(1 + e^{x})}{x}\left(\frac{1 + 2e^{-x}}{1 - e^{-x}}\right)$

APC 523 -

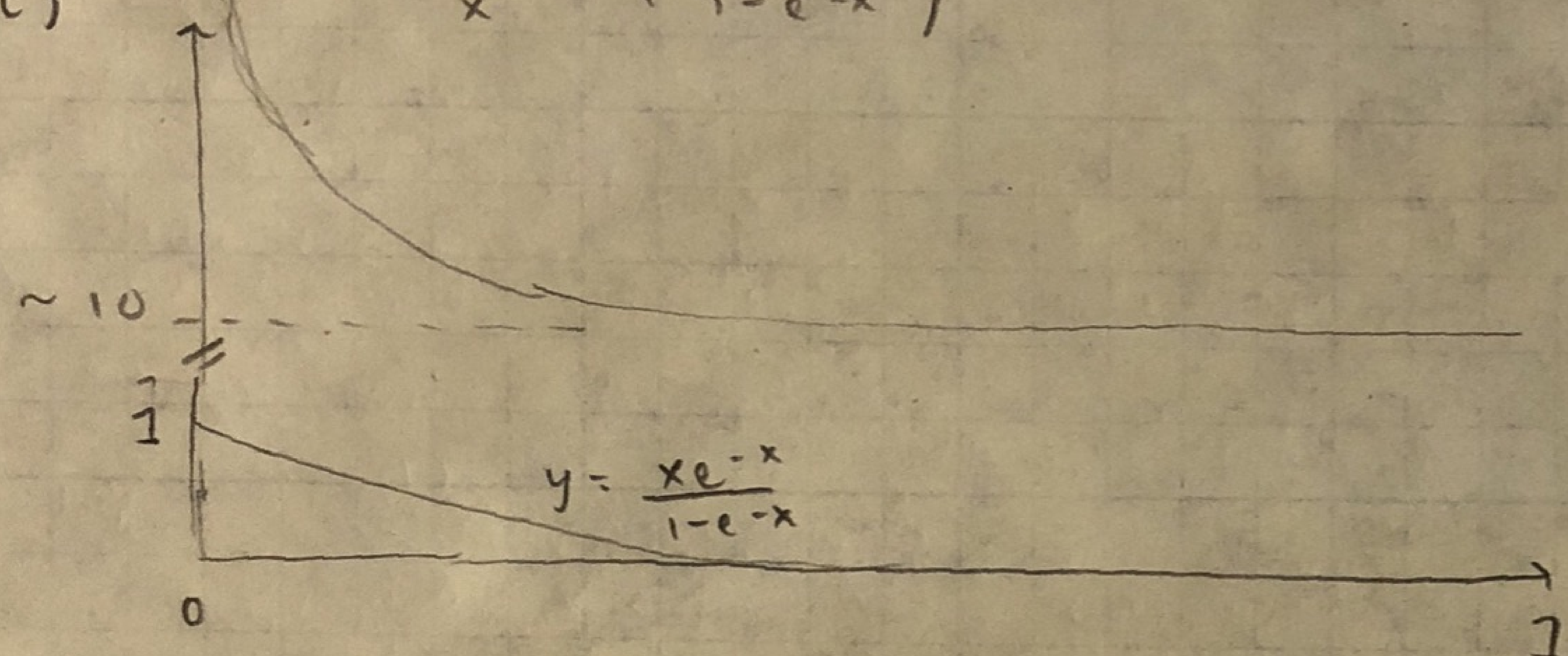
4)

b) continued.

This function $\frac{(1+e^{x})}{x}\left(\frac{1+2e^{-x}}{1-e^{-x}}\right) > 1$ on the interval $(0,1)$



c) $y = \frac{(1+e^{-x})}{x}\left(\frac{1+2e^{-x}}{1-e^{-x}}\right)$



the root cause of this ill conditioning is the subtraction term, which causes more and more error as x gets smaller

d)

if $x-y$, $2^{-a} \leq \frac{x-y}{x} \leq 2^{-b}$

if we only want to lose 1 bit

$\frac{1-e^{-x}}{1} \leq 2^{-1} \longrightarrow x = 0.693$

2 bits, $x = 0.2877$

3 bits, $x = 0.1335$

4 bits, $x = 0.0645$

e) $\frac{\|y_A - y\|}{\|y\|} \leq \text{cond } f \left(\underbrace{\frac{\|x - x_A\|}{\|x\|}}_{\text{rounding}} + \text{cond } A \right)$

$\frac{\|y_A - y\|}{\|y\|} \longrightarrow \left(\frac{(0.693)e^{-0.693}}{1-e^{-0.693}}\right)\left(\varepsilon + \left(\frac{1+e^{0.693}}{0.693}\left(\frac{1+2e^{-0.693}}{1-e^{-0.693}}\right)\right)\right)$

$= 12.0$

for $x = 0.1335$, rel error $= 330$

$x = 0.0645$, rel error $= 1427.2$

I think this is wrong ... :(

f. $(1-e^{-x})\left(\frac{e^{x}}{e^{x}}\right) = \frac{e^{x}-1}{e^{x}}$

if we take the Taylor expansion $e^{x} \approx 1 + x + x^2/2 + x^3/3! + \ldots$

we get $1-e^{-x} = \frac{x + x^2/2! + x^3/3! \ldots}{1 + x + x^2/2! + \ldots} \longrightarrow$ no subtraction