Problem 1

Prove symmetric rounding upper bound on relative error is

$$\left|\frac{x - rd(x)}{x}\right| \leq 2^{-p}$$

* here I set $2^q = 1$ because it is not used in the analysis

For mapping real number x to a nearby machine number in $\mathbb{R}(p,q)$ where p labels mantissa digits & q labels exponent digits.

Case 1: first discarded bit is zero

"problem" ✓ the first bit is bit $p+2$ (first bit that could disagree).

Worst case is all bits starting w/ bit $p+2$ and going to ∞ are 1.

Example for $p=3$

$x = 0.110011\ldots \Rightarrow 0.110$ rounding

0.110 truncation

$p \; p{+}1 \; p{+}2$

General

If $x = \sum_{l=1}^{\infty} b_l 2^{-l}$, then $rnd(x) = \sum_{l=1}^{p} \tilde{b}_l 2^{-l}$

$$\max|x - rnd(x)| = \sum_{l=p+2}^{\infty} b_l 2^{-l}$$

because for case 1 first bit in disagreement is in place $p+2$.

Take worst case that $b_l, l = p+2, \ldots, \infty$ $= 1$

$$= \sum_{l=p+2}^{\infty} 2^{-l} = 2^{-p-2} + 2^{-p-3} + 2^{-p-4} + \ldots$$

$$= 2^{-p-1}(2^{-1} + 2^{-2} + \ldots)$$

which converges to

$$= 2^{-p-1}$$

Now find relative error.

If $x \neq 0$, smallest x can be is 2^{-1} since leading bit is 1.

$$\max\left|\frac{x - rnd(x)}{x}\right| = \frac{2^{-p-1}}{2^{-1}} = \boxed{2^{-p}}$$

Case 2 : first discarded bit is 1

Example for $p=3$

$x = 0.11011\ldots \Rightarrow 0.111$ rounding
0.110 truncation

$p \; p+1 \; p+2$

General

$$x = \sum_{l=1}^{\infty} b_l 2^{-l}, \quad rnd(x) = \sum_{l=1}^{p} \tilde{b}_l 2^{-l}$$

Agree on bits $1, 2, \ldots, p-1$

Disagree for bit p, that is, for bits $(p, p+1)$ we have:
$(0,1) \rightarrow (1,0)$ say $0.01 \rightarrow 0.10$

$p \; p+1$

$$\text{error} = |0.01 - 0.10| = 0.01 = 2^{-p-1} \quad (\star)$$

$p \; p+1$

The worst case in this scenario is that bits $p+2, \ldots, \infty$ are all 0's. In that case, the error is simply coming from $(\star)$ above and it is 2^{-p-1}.

And again, if $x \neq 0$, smallest x can be is 2^{-1},

$$\max\left|\frac{x - rnd(x)}{x}\right| = \frac{2^{-p-1}}{2^{-1}} = \boxed{2^{-p}}$$

Problem 3

(a) To compute x^n, could

(i) repeatedly multiply x

(ii) First find $\ln(x)$, then multiply that by n, then raise e to power $n \ln(x)$

Derive upper-bound for cases (i) and (ii) on relative error resulting from machine arithmetic

Recall $fl(x \circ y) = (x \circ y)(1+\varepsilon)$, $|\varepsilon| \leq eps$

Assume $\begin{cases} fl(\ln(x)) = (\ln x)(1+\varepsilon), & |\varepsilon| \leq eps \\ fl(\exp(x)) = (\exp(x))(1+\varepsilon), & |\varepsilon| \leq eps \end{cases}$

Neglect $O(eps^2)$ & higher terms.

Determine guideline in terms of x and n for when (i) or (ii) is better.

(b) Find ε in x^a when:

(i) x is an exact machine # but a is subject to RE ε_a

(ii) a is exact machine # but x is subject to RE ε_x

where $\varepsilon_x, \varepsilon_a$ could be $>$ eps. Express ε in terms of a, x, ε_a, and ε_x, & neglect terms $O(\varepsilon_a^2)$ or $O(\varepsilon_x^2)$. When could the error become substantial?

(a) (i) Assume $fl(x) = x$

Goal is to find error from multiplying x n times.

$$fl(x \cdot x) = (x \cdot x)(1+\varepsilon_1)$$

$$fl(x \cdot x \cdot x) = fl(x \cdot fl(x \cdot x)) = (x \cdot (xx(1+\varepsilon_1)))(1+\varepsilon_2)$$

$$= (xxx)(1+\varepsilon_1+\varepsilon_2) \quad \text{to 1st order}$$

$$fl(xxxx) = (xxxx)(1+\varepsilon_1+\varepsilon_2+\varepsilon_3)$$

$$\vdots$$

$$fl(\underbrace{xx\ldots x}_{n \text{ times}}) = (\underbrace{xx\ldots x}_{n \text{ times}})\left(1+\sum_{i=1}^{n-1}\varepsilon_i\right)$$

(ii) Goal is to find error from computing $e^{n \ln x}$

$$fl(\ln x) = (\ln x)(1+\varepsilon_{ln})$$
$$fl(n\, fl(\ln x)) = (n\, fl(\ln x))(1+\varepsilon_n)$$
$$= (n \ln x)(1+\varepsilon_{ln})(1+\varepsilon_n)$$
$$= (n \ln x)(1+\varepsilon_{ln}+\varepsilon_n) \quad \text{to 1st order}$$
$$fl(\exp(fl(n\, fl(\ln x)))) = \exp[fl(n \ln x)](1+\varepsilon_{exp})$$
$$= \exp(n \ln x)(1+\varepsilon_{ln}+\varepsilon_n)(1+\varepsilon_{exp})$$
$$= \exp(n \ln x)(1+\varepsilon_{ln}+\varepsilon_n+\varepsilon_{exp})$$

$\Rightarrow$ exponentiation via repeated multiplication is more accurate than the log exponential method when

$$\left(\sum_{i=1}^{n-1} \varepsilon_i\right) < \left(\varepsilon_{ln}+\varepsilon_n+\varepsilon_{exp}\right)$$

which probably occurs for low "n"!

(b) x^a, x positive a nonzero.

(i) x is an exact machine # & a is subject to relative error ε_a.

<u>Case 1</u>: repeated multiplication

$$fl(\underbrace{xx\cdots x}_{a \text{ times}}) = (\underbrace{xx\cdots x}_{a \text{ times}})\left(1+\sum_{i=1}^{a-1}\varepsilon_i\right) \quad [\text{no change}]$$

if "a" is not an integer, but some fractional or irrational quantity ~~best evaluate numerator using above and~~ ~~then denominator using above & divide~~ ~~if a cannot be written as fraction~~, then I don't see how to use repeated multiplication approach.

* in terms of only $a, x, \varepsilon_a, \varepsilon_x$: $fl(xx\cdots x) = (xx\cdots x)$

Case 2: log exponential method

$fl(x) = x$

$fl(a) = (a)(1+\varepsilon_a)$

$fl(\ln x) = (\ln x)(1+\varepsilon_{ln})$

$fl(fl(a)\, fl(\ln x)) = fl(a(1+\varepsilon_a)\ln x(1+\varepsilon_{ln}))$

$= fl(a \ln x (1+\varepsilon_a)(1+\varepsilon_{ln}))$

$= (a \ln x)(1+\varepsilon_a)(1+\varepsilon_{ln})(1+\varepsilon_{a\ln x})$

$\approx (a \ln x)(1+\varepsilon_a+\varepsilon_{ln}+\varepsilon_{a\ln x})$ to first order

$fl(\exp(a \ln x)) = \exp(a\ln x)(1+\varepsilon_a+\varepsilon_{ln}+\varepsilon_{a\ln x}+\varepsilon_{exp})$ to first order

★ in terms of only $x, a, \varepsilon_x, \varepsilon_a$:
$fl(\exp(a\ln x)) = \exp(a\ln x)(1+\varepsilon_a)$

(ii) a is an exact machine # and x has relative error ε_x:

Case 1: repeated multiplication

$fl(x) = (x)(1+\varepsilon_x)$

$fl(a) = a$

$fl(xx) = fl(fl(x)fl(x)) = fl(x(1+\varepsilon_x)\,x(1+\varepsilon_x))$

$\approx fl(xx(1+2\varepsilon_x))$ to 1st order $= xx(1+2\varepsilon_x)(1+\varepsilon_1)$

$\approx xx(1+2\varepsilon_x+\varepsilon_1)$

$fl(fl(x)\,fl[fl(x)fl(x)]) = fl(x(1+\varepsilon_x)\,xx(1+2\varepsilon_x+\varepsilon_1))$

$\approx fl(xxx(1+3\varepsilon_x+\varepsilon_1))$ to 1st order

$= xxx(1+3\varepsilon_x+\varepsilon_1)(1+\varepsilon_2)$

$\approx xxx(1+3\varepsilon_x+\varepsilon_1+\varepsilon_2)$

$fl(\underbrace{xx\ldots x}_{a\text{ times}}) \approx \underbrace{xx\ldots x}_{a\text{ times}}\left(1+a\varepsilon_x+\sum_{i=1}^{a-1}\varepsilon_i\right)$

$\approx xx\ldots x(1+a\varepsilon_x)$ in terms of only $x, a, \varepsilon_x, \varepsilon_a$

Once again, if "a" is not an integer than I don't see how to use the repeated multiplication approach.

Case 2: log exponential method

$fl(x) = x(1+\varepsilon_x)$

$fl(a) = a$

$fl(\ln fl(x)) = fl(\ln x(1+\varepsilon_x))$

$= \ln x(1+\varepsilon_x)(1+\varepsilon_{\ln}) \approx \ln x(1+\varepsilon_x+\varepsilon_{\ln})$

to 1st order

$fl(a\, fl(\ln fl(x))) = fl(a \ln x(1+\varepsilon_x+\varepsilon_{\ln}))$

$= a \ln x(1+\varepsilon_x+\varepsilon_{\ln})(1+\varepsilon_{a\ln x})$

$\approx a \ln x(1+\varepsilon_x+\varepsilon_{\ln}+\varepsilon_{a\ln x})$

$fl(\exp(a\ln x)) \approx \exp(a\ln x)(1+\varepsilon_x+\varepsilon_{\ln}+\varepsilon_{a\ln x}+\varepsilon_{\exp})$

$\approx \exp(a\ln x)(1+\varepsilon_x)$

to first order

Error could become substantial for (ii) case 1, as the error is $a\varepsilon_x$, when "a" is large. Otherwise, these seem like good approaches.

Problem 4

$f(x) = 1 - e^{-x} \quad \in [0, 1]$

(a) Find $(\text{cond } f)(x)$ in terms of x, and show it is less than 1 on interval $[0, 1]$.

$$(\text{Cond } f)(x) = \left| \frac{x f'(x)}{f(x)} \right| = \left| \frac{x(e^{-x})}{1-e^{-x}} \right| = \left| \frac{xe^{-x}}{1-e^{-x}} \right|$$

L'Hopital, for limit $x \to 0$, take derivative of num & denom:

$$\frac{e^{-x} - xe^{-x}}{e^{-x}} = 1 - x$$

limit of 0, cond $f(x)$ goes to 1.

Limit of $x \to 1$,

$$\frac{e^{-1}}{1-e^{-1}} \approx 0.58$$

Limit of 1, cond $f(x)$ goes to

↳ cond $f(x)$ is a decreasing function, so given that in the limit of $x \to 0$, cond$f(x) \to 1$, we know that for $x > 0$, cond$f(x) < 1$.

for $x > 0$

(b) Define A to be algo that:

(i) negates x

(ii) computes e^{-x}

(iii) subtract result from 1

where x is machine # on machine with epsilon eps.

Assume (ii) and (iii) return correct rounding of real ans.

Find upper bound on $(\text{cond } A)(x)$, neglecting quantities of order $O(eps^2)$ or above.

$$\text{Cond}\,A(x) = \frac{1}{eps} \frac{\|x_A - x\|}{\|x\|}$$

First get x_A,

$$fl(x) = x$$
$$fl(-x) = -x$$
$$fl(e^{-x}) = e^{-x}(1 + \varepsilon_{exp})$$

For step (iii), the subtraction, recall from lecture we found

$$fl(x \pm y) = (x \pm y)(1 + \varepsilon_{xy})$$

where $\varepsilon_{xy} = \frac{x}{x \pm y}|\varepsilon_x| \pm \frac{y}{x \pm y}|\varepsilon_y|$

We need to find $fl(1 - fl(e^{-x}))$, so

$x = 1$, $|\varepsilon_x| = 0$ (assumption)

$y = e^{-x}$, $|\varepsilon_y| = |\varepsilon_{exp}|$ (from above)

$$\hookrightarrow fl(1 - e^{-x}) \overset{fl}{=} \left[(1 - e^{-x})\left(1 + \left|\left[\frac{1}{1-e^{-x}}(0) - \frac{e^{-x}}{1-e^{-x}}|\varepsilon_{exp}|\right]\right|\right)\right]$$

$$= \left[(1 - e^{-x})\left(1 + \frac{e^{-x}}{1-e^{-x}}|\varepsilon_{exp}|\right)\right](1 + \varepsilon_{rnd})$$

Assume $|\varepsilon_{exp}| = eps$ and $\varepsilon_{rnd} = eps$,

$$= (1 - e^{-x})\left(1 + \frac{e^{-x}}{1-e^{-x}} eps\right)(1 + \varepsilon_{rnd})$$

$$= (1 - e^{-x})\left(1 + \frac{e^{-x}}{1-e^{-x}} eps + eps\right)$$

to first order

from last page,

$$fl(1-e^{-x}) = (1-e^{-x})\left(1 + \frac{e^{-x}}{1-e^{-x}} eps + eps\right)$$

$$= (1-e^{-x})\underbrace{\left(1 + \frac{1}{1-e^{-x}} eps\right)}_{\varepsilon_A}$$

Substitute $x_A = x(1+\varepsilon_A)$ into CondA(x) expression,

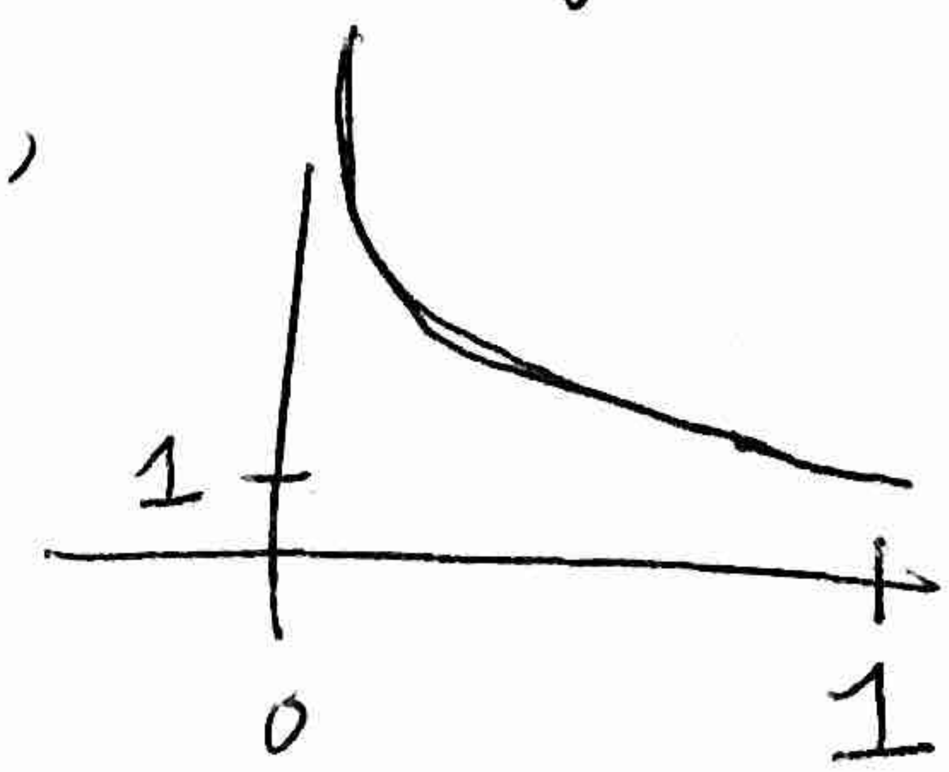
$$\text{Cond}A(x) = \frac{1}{eps}\frac{\|x_A - x\|}{\|x\|} = \frac{1}{eps}\frac{\left\|x\left(1+\frac{1}{1-e^{-x}}eps\right) - x\right\|}{\|x\|}$$

$$= \frac{1}{eps}\frac{\left\|x\left(1+\frac{1}{1-e^{-x}}eps\right) - x\right\|}{\|x\|}$$

$$= \frac{1}{eps}\frac{\left\|x + \frac{x}{1-e^{-x}}eps - x\right\|}{\|x\|}$$

$$= \frac{1}{eps}\frac{\left\|\frac{x}{1-e^{-x}}eps\right\|}{\|x\|} = \boxed{\frac{\|x/1-e^{-x}\|}{\|x\|}}$$

↳ we see that cond A(x) diverges as $x \to 0$, and it approaches $\frac{1}{1-\frac{1}{e}}$ in the limit of $x \to 1$, which is approx 1.6. So, Cond A(x) ≥ 1 everywhere on $[0,1]$, and it looks like this,



So it is particularly poorly conditioned for small x.

(e) Estimate upper bound on relative error in output starting from $x = 0.69$, 0.28, 0.13, 0.06

↳ from part (b) we have

$$fl(1-e^{-x}) = (1-e^{-x})\left(1 + \frac{1}{1-e^{-x}}\, eps\right)$$

So the error is bounded by

$$\varepsilon \leq \frac{1}{1-e^{-x}}\, eps$$

$x = 0.69$: $\varepsilon = \frac{1}{1-e^{-0.69}}\, eps \approx 2eps \approx 1$ bit

$x = 0.28$: $\varepsilon = 4\,eps = 2^2 = 2$ bits

$x = 0.13$: $\varepsilon = 8\,eps = 2^3 = 3$ bits

$x = 0.06$: $\varepsilon = 16\,eps = 2^4 = 4$ bits

Good, as expected

↳ Now to find relative error

$$\varepsilon_{rel} = \left|\frac{f(x) - f(x_A)}{f(x)}\right| = \left|\frac{(1-e^{-x}) - (1-e^{-x})(1+\varepsilon)}{(1-e^{-x})}\right|$$

$= |\varepsilon| =$ same answers as above (circled).

(f) Ideas for better conditioned algorithm?

$$\hookrightarrow 1 - e^{-x} = 1 - \frac{1}{e^x} = \frac{e^x - 1}{e^x}$$

Expand e^x as Taylor series,

$$= \frac{\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots\right) - 1}{e^x} = \frac{\left(x + \frac{x^2}{2} + \frac{x^3}{6} + \dots\right)}{e^x}$$

So, a better way to calculate $(1-e^{-x})$ that avoids subtraction is to calculate it as

$$(1-e^{-x}) = \frac{\sum_{i=1}^{K} \frac{1}{i!} x^{i}}{e^{x}}$$ for some suitable truncation K

Which only involves multiplication (to get Taylor Series terms), addition, and division @ the end.

Problem 7

(e) $p(x) = \sum_{k=0}^{n} a_k x^k$, n free coeffs since $a_n = 1$

Let Ω denote the map from the space of n free coeffs of $p(x)$ to the space of its n roots

$$\Omega = \Omega(a_0, a_1, \ldots, a_{n-1})$$

Such that $\Omega_k = k^{th}$ root of $p(x)$

Define condition matrix Γ_{kl} as condition # for k-th root given changes in l-th coefficient.

Then for some root Ω_k we get condition vector for $l = 0, 1, \ldots, n-1$.

Taking sum of abs val of vector elements leads to

$$(\text{Cond}\,\Omega_k)(a) = \sum_{l=0}^{n-1} (\Gamma_{kl})(a)$$

(i) Find expression for $(\text{cond}\,\Omega_k)(a)$ in terms of Ω_k, $p'(\Omega_k)$ and $a_0, \ldots, a_{n-1}$.

from class,

$$\text{Cond}_{\nu\mu} f \equiv \left| \frac{x_\mu \frac{\partial f_\nu}{\partial x_\mu}(\vec{x})}{f_\nu(\vec{x})} \right|$$

For this problem, we get

As per lecture, define cond. mat..

$$\Gamma_{K\ell} = \frac{a_\ell \dfrac{\partial \Omega_K(\vec{a})}{\partial a_\ell}}{\Omega_K(\vec{a})}$$

where $\Omega_K(\vec{a})$ is map from $\vec{a}$ to the k-th root Ω_K

Sum over ℓ, take norm,

$$\Gamma_K = \text{Cond}\,\Omega_K(\vec{a}) = \sum_{\ell=0}^{n-1} \frac{a_\ell \dfrac{\partial \Omega_K(\vec{a})}{\partial a_\ell}}{\Omega_K(\vec{a})} \longrightarrow$$

goal is to re-express $\Gamma_{K\ell}$ in terms of desired quantities.

Consider $\Gamma_{K\ell}$ with some perturbation Δa_ℓ in a_ℓ.

$$\Gamma_{K\ell} = \frac{a_\ell \dfrac{\partial \Omega_K(\vec{a})}{\partial a_\ell}}{\Omega_K(\vec{a})} \approx \frac{a_\ell \dfrac{\Omega_K(a_\ell + \Delta a_\ell) - \Omega_K(a_\ell)}{\Delta a_\ell}}{\Omega_K(a_\ell)}$$

$$= \frac{a_\ell}{\Delta a_\ell} \frac{\Omega_K(a_\ell + \Delta a_\ell) - \Omega_K(a_\ell)}{\Omega_K(a_\ell)}$$

$$= \frac{a_\ell \left(1/\Omega_K(a_\ell)\right)}{\Delta a_\ell \left(1/(\Omega_K(a_\ell + \Delta a_\ell) - \Omega_K(a_\ell))\right)}$$

Substitute $\frac{1}{\Omega_K} = \frac{\Omega_K^{\ell-1}}{\Omega_K^{\ell}}$

$$= \frac{a_\ell \Omega_K^{\ell-1}(a_\ell)}{\Delta a_\ell \Omega_K^{\ell}(a_\ell) / (\Omega_K(a_\ell + \Delta a_\ell) - \Omega_K(a_\ell))}$$

From last page,

$$\Gamma_{k\ell} = \frac{a_\ell \Omega_k^{\ell-1}(a_\ell)}{\dfrac{\Delta a_\ell \Omega_k^{\ell}(a_\ell)}{\Omega_k(a_\ell+\Delta a_\ell)-\Omega_k(a_\ell)}}$$

Substitute $p(\Omega_k(a_\ell+\Delta a_\ell)) - p(\Omega_k(a_\ell)) = \Delta a_\ell \Omega_k^{\ell}(a_\ell)$

$$= \frac{a_\ell \Omega_k^{\ell-1}(a_\ell)}{\dfrac{p(\Omega_k(a_\ell+\Delta a_\ell))-p(\Omega_k(a_\ell))}{\Omega_k(a_\ell+\Delta a_\ell)-\Omega_k(a_\ell)}} = \boxed{\frac{a_\ell \Omega_k^{\ell-1}(a_\ell)}{p'(\Omega_k)}} = \Gamma_{\ell k}$$

so * $\text{cond}\,\Omega_k(a) = \sum_{\ell=0}^{n-1} \frac{a_\ell \Omega_k^{\ell-1}(\vec{a})}{p'(\Omega_k)}$

(ii) Evaluate cond #'s for roots $r = 14, 16, 17, 20$

↳ see MATLAB

(iii) Could a sufficiently clever algorithm help here?

→ In this case probably <u>no</u> since we treated the problem abstractly and algorithm-independently. A nice algo probably can't fix a fundamentally ill-conditioned problem.

Problem 8

$$y_n \equiv \int_0^1 dx\, x^n e^x \quad n \geq 0$$

recurrence reln $\quad y_{n+1} = e - (n+1)y_n$

(a) Reverse the recurrence reln, imagine it as map g_K from y_N to y_K, $K < N$. Establish upper limit on cond $g_K(y_N)$ in terms of k and N.

$$y_{n+1} = e - (n+1)y_n$$

$$(n+1)y_n = e - y_{n+1}$$

$$(\star) \quad y_n = \frac{e - y_{n+1}}{n+1} \quad \text{or} \quad y_K = g_K(y_N)$$

$$\text{cond } g_K(y_N) = \left| \frac{y_N\, g_K'(y_N)}{g_K(y_N)} \right|$$

$$= \frac{y_N\left(-\frac{1}{N+1}\right)}{\frac{e - y_N}{N+1}} = \underbrace{\frac{-y_N}{e - y_N}}_{\text{just for doing it once}}$$

Starting with $(\star)$, we may seek the general form for finding y_K from $y_N \to y_K$

$$\hookrightarrow y_n = \frac{e - y_{n+1}}{n+1} \rightarrow y_{n-1} = \frac{e - \left(\frac{e - y_{n+1}}{n+1}\right)}{n}$$

$$y_{n-2} = \frac{e - \left(\frac{e - \left(\frac{e - y_{n+1}}{n+1}\right)}{n}\right)}{n-1} = \frac{e}{n-1} - \frac{e}{n(n-1)} - \frac{e}{(n+1)(n)(n-1)} + \frac{y_{n+1}}{(n+1)(n)(n-1)}$$

Setting $K = n-2$,

$$y_K = \frac{e}{K+1} - \frac{e}{(K+2)(K+1)} - \frac{e}{(K+3)(K+2)(K+1)} + \frac{y_{K+3}}{(K+3)(K+2)(K+1)}$$

From this, the general expression can be inferred to be for $N > K$

$$y_K = \sum_{j=K+1}^{N} \frac{e}{(j)!/(K)!} + \frac{y_N}{N!/K!}$$

For this situation, the condition # becomes

(*)

$$\text{cond}\, y_K(y_N) = \frac{y_N\,(K!/N!)}{\sum_{j=K+1}^{N} \frac{e}{j!/K!} + \frac{y_N}{N!/K!}} = \frac{y_N\,K!}{N!\left(\sum_{j=K+1}^{N} \frac{eK!}{j!} + \frac{y_N K!}{N!}\right)}$$

$$= \frac{y_N}{\sum_{j=K+1}^{N} \frac{N!\,e}{j!} + y_N} = \left[\frac{\sum_{j=K+1}^{N} \frac{N!\,e}{j!} + y_N}{y_N}\right]^{-1}$$

$$= \left(\frac{\sum_{j=K+1}^{N} \frac{N!\,e}{j!}}{y_N} + 1\right)^{-1}$$

$$= \left(\frac{1}{y_N} + \frac{1}{\sum_{j=K+1}^{N} \frac{N!\,e}{j!}}\right)^{-1} \Rightarrow \text{can't simplify}$$

↳ Go back to (*)

$$\text{cond}\, y_K(y_N) = \frac{y_N\,(K!/N!)}{y_K} \leq \boxed{\frac{K!}{N!}}$$ because $y_N / y_K < 1$

(b) $\varepsilon_{y_K} \leq \left| \frac{K!}{N!} \right| \varepsilon_{y_N}$

$$\varepsilon \leq \left| \frac{K!}{N!} \right| (1)$$

$$N! = \frac{K!}{\varepsilon}$$

$$N = \frac{K!}{\varepsilon (N-1)!}$$

(c) $N = \frac{20!}{eps(N-1)!} \rightarrow N! = \frac{20!}{eps}$

$N! = 1.0957 \times 10^{34}$

from MATLAB trial & error,

$N \approx 32$