3.

a) i. Repeatedly multiplying by x. Each x has an error ε_x. Since n just counts how many operations we do, it doesn't necessarily have any error.

However, if it's a parameter we input, then in principle it could.

Thankfully, error propagation in multiplication is additive. So,

$$[x(1+\varepsilon_x)]^{n(1+\varepsilon_n)} \simeq [x^n(1+n\varepsilon_x)]^{(1+\varepsilon_n)}$$

$$\simeq [x^n(1+n\varepsilon_x)](1+\varepsilon_n)$$

$$\simeq x^n(1+\varepsilon_n+n\varepsilon_x)$$

So the relative error is:

$$\boxed{\varepsilon_{x^n} = \varepsilon_n + n\varepsilon_x}$$

ii. What's the error in $\log x$?

Well, $\log x(1+\varepsilon_x) = \log x + \log(1+\varepsilon_x)$
$\simeq \log x + \varepsilon_x$

multiply by $n(1+\varepsilon_n)$.

$$n(1+\varepsilon_n)\log x(1+\varepsilon_x) = n\log x + n\varepsilon_x + n\varepsilon_n \log x$$

We needed to get an ε in $\log x$.

So $\log x(1+\varepsilon_x) = [\log x](1+\varepsilon) + \varepsilon_x = \log x + \varepsilon\log x + \varepsilon_x$

Then we have:

$$fl(n\log x) = n\log x + n\varepsilon_x + n\varepsilon_n \log x + n\varepsilon\log x$$

Now, $fl(e^{n\log x}) = (1+\varepsilon)\, e^{n\log x + n(\varepsilon_n+\varepsilon)\log x + n\varepsilon_x}$

$= e^{n \log x}(1+\varepsilon)(1+n\varepsilon_x+n(\varepsilon+\varepsilon_n)\log x)$

$= e^{n\log x}(1+\varepsilon+n\varepsilon_x+n(\varepsilon+\varepsilon_n)\log x)$

$\Rightarrow \quad \varepsilon_{tot} = \varepsilon + n\varepsilon_x + n(\varepsilon+\varepsilon_n)\log x$

How does this compare to $\varepsilon_n + n\varepsilon_x$?
If all errors are of the same magnitude, it's surely worse.

$$\varepsilon + n\varepsilon < \varepsilon + n\varepsilon + 2n\varepsilon \log x$$

Therefore, (i) is preferable to (ii).

As $\varepsilon_x n$ comes into each side, it all hinges on ε_n and $\log x$. When $\log x$ is small, then

$\varepsilon_{tot} - n\varepsilon_x \approx \varepsilon$, while

$\varepsilon_{x^n} - n\varepsilon_x \approx \varepsilon_n$. ε_n may often be zero. If it is indeed zero, then $e^{n\log x}$ is bigger. Otherwise they're similar. Now, if $\log x$ is fairly large, then $e^{n\log x}$ is much worse than x^n. Assume $\varepsilon_{tot} \approx n(\varepsilon+\varepsilon_n)\log x \gg \varepsilon_n + n\varepsilon_x$

We need quite a large x for this to happen.

b) The advantage that $e^{a\log x} = x^a$ has is that it works for non-integers. Let's assume a is not necessarily an integer.

i) $$fl(x^a) = \left[e^{a\log x}\right]^{(1+\varepsilon_a+\varepsilon)} \quad \text{from the log.}$$

$$= e^{a\log x}\left(1 + (\varepsilon_a+\varepsilon)a\log x\right)$$

$$\Rightarrow \quad \boxed{\varepsilon_{tot} = (\varepsilon_a+\varepsilon)a\log x}$$

(ii) $$fl(x^a) = e^{a(1+\varepsilon)\log[x(1+\varepsilon_x)]}$$

$$= e^{a\log x + \varepsilon\log x + a\varepsilon_x}$$

$$= e^{a\log x}e^{\varepsilon\log x + a\varepsilon_x}$$

$$\approx e^{a\log x}(1+\varepsilon\log x + a\varepsilon_x)$$

$$\Rightarrow \quad \boxed{\varepsilon_{tot} = a\varepsilon_x + \varepsilon\log x}$$

We technically must add ε to each of these for the error in exp, but that won't change our comparison.

In case (i), if $a \gtrsim 1/(\varepsilon_a+\varepsilon)$ then the error could be substantial (provided $\log x \gtrsim 1$)

In case (ii), if $a \gtrsim 1/\varepsilon_x$, we could have substantial error. $\log x$ will never be huge, so a is the thing to worry about.