Problem 7

Part (a)

In[64]:=
```
p = Product[(x - k), {k, 1, 20}] // Expand (* This gives us the polynomial *)
```

Out[64]=
$$2\,432\,902\,008\,176\,640\,000 - 8\,752\,948\,036\,761\,600\,000\,x + 13\,803\,759\,753\,640\,704\,000\,x^2 - 12\,870\,931\,245\,150\,988\,800\,x^3 + 8\,037\,811\,822\,645\,051\,776\,x^4 - 3\,599\,979\,517\,947\,607\,200\,x^5 + 1\,206\,647\,803\,780\,373\,360\,x^6 - 311\,333\,643\,161\,390\,640\,x^7 + 63\,030\,812\,099\,294\,896\,x^8 - 10\,142\,299\,865\,511\,450\,x^9 + 1\,307\,535\,010\,540\,395\,x^{10} - 135\,585\,182\,899\,530\,x^{11} + 11\,310\,276\,995\,381\,x^{12} - 756\,111\,184\,500\,x^{13} + 40\,171\,771\,630\,x^{14} - 1\,672\,280\,820\,x^{15} + 53\,327\,946\,x^{16} - 1\,256\,850\,x^{17} + 20\,615\,x^{18} - 210\,x^{19} + x^{20}$$

Part (b)

In[65]:=
```
(* To get the coefficients, we do this: *)
coeffs = N[CoefficientList[p, x]] (* Evaluates coeffs to machine precision *)
```

Out[65]=
$$\{2.4329\times10^{18}, -8.75295\times10^{18}, 1.38038\times10^{19}, -1.28709\times10^{19}, 8.03781\times10^{18}, -3.59998\times10^{18}, 1.20665\times10^{18}, -3.11334\times10^{17}, 6.30308\times10^{16}, -1.01423\times10^{16}, 1.30754\times10^{15}, -1.35585\times10^{14}, 1.13103\times10^{13}, -7.56111\times10^{11}, 4.01718\times10^{10}, -1.67228\times10^{9}, 5.33279\times10^{7}, -1.25685\times10^{6}, 20615., -210., 1.\}$$

In[66]:=
```
flp[x_] := Sum[coeffs[[n]] x^n, {n, 1, Length[coeffs]}];
(* Define the new float polynomial *)
```

In[ ]:=
```
(x /. N[FindRoot[flp[x], {x, 21.}], 16.]) - 20. // Quiet (* We see that the
  root is somewhat different from 20. It's a bit smaller. Not by much,
 but by some. Mathematica will display it as 20 unless you coax it
  into telling you the difference. It will even tell you that it
  requires more than machine precision to be sure of this result. *)
```

Out[ ]= -0.0000485129

Part (c)

```
In[ ]:= (* Now let's add a little delta. *)
delta = {10.^(-8), 10.^(-6), 10.^(-4), 10.^(-2)};
perturbedRoots = {};
For[i = 1, i ≤ Length[delta], i++,
 newcoeffs = coeffs;
 newcoeffs[[-1]] += delta[[i]];
 flp[x_] := Sum[newcoeffs[[n]] x^n, {n, 1, Length[coeffs]}];
 AppendTo[perturbedRoots,
  {delta[[i]], (x /. N[Quiet[FindRoot[flp[x], {x, 21.}]], 16.]) - 20.}]
]
Print[perturbedRoots]
(* We see that even a small perturbation in the coefficient
 makes a large difference in the position of the largest root. *)
```

$\{\{1. \times 10^{-8}, -10.4129\}, \{1. \times 10^{-6}, -12.2473\}, \{0.0001, -14.0307\}, \{0.01, -14.5304\}\}$

Part (d)

First, we should see what roots 16 and 17 were in the first place.

```
In[ ]:= flp[x_] := Sum[coeffs[[n]] x^n, {n, 1, Length[coeffs]}]
```

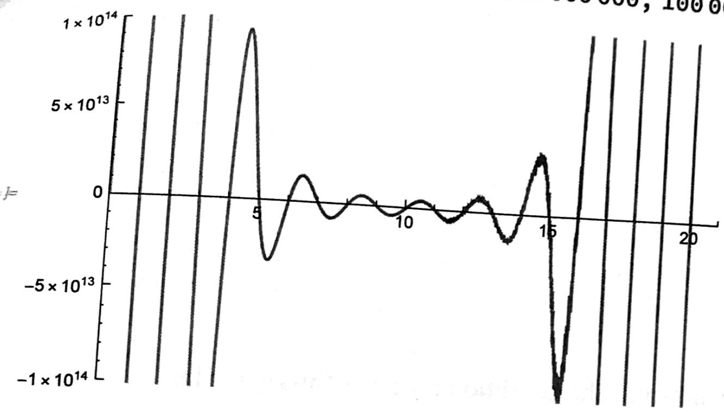
```
In[ ]:= {(*16 *)
  (x /. N[FindRoot[flp[x], {x, 16.1}]]),
  (*17*)
  (x /. N[FindRoot[flp[x], {x, 17.1}]])
 } // Quiet

Out[ ]= {15.9945, 17.0007}
```

Furthermore, a plot of the function (somewhat enlightening, as we can visually see the roots) reveals the following structure:

```
Plot[flp[x], {x, 0, 21},
 PlotRange → {{0, 21}, {-100 000 000 000 000, 100 000 000 000 000}}]
```

Out[•]=



Clearly, the roots are wrong to some error in the third or fourth decimal place. However, When we instead perturb the second coefficient, we have the following result:

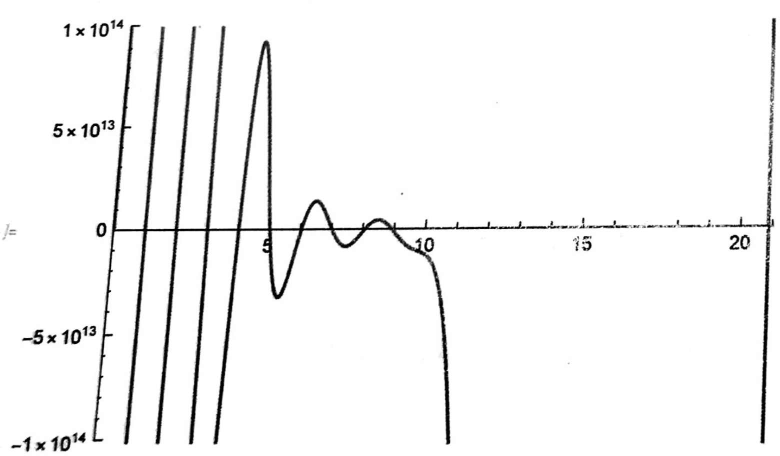
```
In[•]:= newcoeffs = coeffs;
newcoeffs[[-2]] += -2.^(-23); (* Add the perturbation *)
flp[x_] := Sum[ newcoeffs[[n]] x^n, {n, 1, Length[coeffs]}]
{(*16 *)
  (x /. N[FindRoot[flp[x], {x, 16.1}]]),
  (*17*)
  (x /. N[FindRoot[flp[x], {x, 17.00001}]])
 } // Quiet
```

```
Out[•]= {8.00724, 8.0072}
```

The roots become nearly degenerate, differing only in the fourth decimal place, both being near 8 instead. Actually, it seems even quite sensitive to the guess that I provide, having tried several different guesses. A plot now reveals someth8ing odd has happened to the later half of the plot. Roots 16 and 17 appear to have actually disappeared entirely. Only the first half of the roots seem roughly intact.

```
In[•]:= Plot[flp[x], {x, 0, 21},
 PlotRange → {{0, 21}, {-100 000 000 000 000, 100 000 000 000 000}}]
```

Out[•]=



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Part (e)

Part (i)

In[35]:=
```
Needs["MaTeX`"]
```

We have that the polynomial is given by:

$p(x) = \prod_{k=1}^{n} (x - \Omega_k) = \sum_{l=0}^{n-1} a_l \, x^l$

So given this, we can compute the coefficients. The condition matrix is thus given by:

In[48]:=
```
MaTeX["\\Gamma_{k \\ell} = \\bigg(\\frac{\\partial \\log a_{\\ell}}{\\partial
  \\log \\Omega_{k}}\\bigg)^{-1} = \\lim_{\\Delta\\rightarrow 0}
  \\frac{a_{\\ell}}{\\Omega_k}\\frac{\\Omega_k (a_{\\ell}+\\Delta
  a_{\\ell})-\\Omega_k (a_{\\ell})}{\\Delta a_{\\ell}}", FontSize → 20]
```

Out[48]=
$$\Gamma_{k\ell} = \left(\frac{\partial \log a_\ell}{\partial \log \Omega_k}\right)^{-1} = \lim_{\Delta\to 0} \frac{a_\ell}{\Omega_k} \frac{\Omega_k(a_\ell + \Delta a_\ell) - \Omega_k(a_\ell)}{\Delta a_\ell}$$

Now lets expand terms. First of all, the perturbed polynomial (p with a hat on top) can be expanded as:

In[49]:=
```
MaTeX["\\hat{p}(\\Omega_k ) = -\\Delta \\Omega_k
  \\hat{p}'(\\Omega_k + \\Delta \\Omega_k)", FontSize → 24]
```

Out[49]=
$$\hat{p}(\Omega_k) = -\Delta\Omega_k \hat{p}'(\Omega_k + \Delta\Omega_k)$$

We thus have that the terms in the matrix are given by:

```
MaTeX["\\Gamma_{k\\ell} = \\lim_{\\Delta\\rightarrow
  0} \\bigg|\\frac{a_{\\ell} \\hat{p}(\\Omega_k )}{\\Omega_k
  \\Delta a_{\\ell} \\hat{p}'(\\Omega_k + \\Delta \\Omega_k
  )}\\bigg| = \\lim_{\\Delta\\rightarrow 0} \\bigg|\\frac{a_{\\ell}
  \\big(p(\\Omega_k )+\\Delta a_{\\ell}\\Omega_k^{\\ell}
  \\big)}{\\Omega_k \\Delta a_{\\ell} \\hat{p}'(\\Omega_k
  + \\Delta \\Omega_k )}\\bigg|", FontSize → 24]
MaTeX["= \\lim_{\\Delta\\rightarrow 0} \\bigg|\\frac{a_{\\ell} \\Omega_k^{\\ell
  - 1}}{\\hat{p}'(\\Omega_k + \\Delta \\Omega_k )}\\bigg| = \\bigg|
  \\frac{a_{\\ell}\\Omega_k^{\\ell - 1}}{p'(\\Omega_k)} \\bigg|", FontSize → 24]
```

Out[59]= $$\Gamma_{k\ell} = \lim_{\Delta\rightarrow 0} \bigg|\frac{a_{\ell} \hat{p}(\Omega_k)}{\Omega_k \Delta a_{\ell} \hat{p}'(\Omega_k + \Delta \Omega_k)}\bigg| = \lim_{\Delta\rightarrow 0} \bigg|\frac{a_{\ell} \big(p(\Omega_k)+\Delta a_{\ell}\Omega_k^{\ell} \big)}{\Omega_k \Delta a_{\ell} \hat{p}'(\Omega_k + \Delta \Omega_k)}\bigg|$$

Out[60]= $$= \lim_{\Delta\rightarrow 0} \bigg|\frac{a_{\ell} \Omega_k^{\ell - 1}}{\hat{p}'(\Omega_k + \Delta \Omega_k)}\bigg| = \bigg| \frac{a_{\ell}\Omega_k^{\ell - 1}}{p'(\Omega_k)} \bigg|$$

Therefore, the cond Ω is given by,

In[63]:=
```
MaTeX["(\\text{cond }\\Omega_k) = \\sum_{\\ell} \\bigg|
  \\frac{a_{\\ell}\\Omega_k^{\\ell - 1}}{p'(\\Omega_k)} \\bigg|", FontSize → 24]
```

Out[63]= $$(\text{cond }\Omega_k) = \sum_{\ell} \bigg| \frac{a_{\ell}\Omega_k^{\ell - 1}}{p'(\Omega_k)} \bigg|$$

Part (ii)

Now we'll use our function for the polynomial, flp[x], as well as the list coeffs to calculate the condition numbers.

In[69]:=
```
cond[x_] :=
  Sum[Abs[coeffs[[l]] x^(l - 1) / (D[flp[c], c] /. (c → x))], {l, 1, Length[coeffs]}];
```

In[70]:=
```
{cond[14.],
 cond[16.],
 cond[17.],
 cond[20.]}
```

Out[70]= $\{5.54962 \times 10^{13}, 3.5461 \times 10^{13}, 1.81787 \times 10^{13}, 1.37838 \times 10^{11}\}$

We see that the condition numbers here are enormous, and horrible. Even small errors can be multiplied by many orders of magnitude.

Part (iii)