4) $f(x) = 1 - e^{-x}$ on $[0,1]$

a) $f'(x) = e^{-x}$. Thus,

$$(\text{cond} f)(x) = \left| \frac{x e^{-x}}{1 - e^{-x}} \right| = \left| \frac{x}{e^x - 1} \right|$$

on $[0,1]$, $e^x \geq 1$ and $e^x \leq e$

As $x \to 0$ $\frac{x}{e^x - 1} \to 1 \leq 1$ and as $x \to 1$,

$$\frac{x}{e^x - 1} \to \frac{1}{e-1} \leq 1$$

It seems that, as $(\text{cond} f)(x)$ has no local minima or maxima, it goes monotonically from 1 to $1/e-1$ on this interval.

b) The condition of A is:

$$(\text{cond } A)(\bar{x}) = \frac{|\bar{x}_A - \bar{x}|}{|\bar{x}|}$$

find first what $f_A(x)$ yields. Then find

$$x_A = f^{-1}(f_A(x))$$

$f_A(x)$: first, $x \to x(1 + \varepsilon_x)$

then, $e^{-x(1+\varepsilon_x)}(1+\varepsilon) = e^{-x}(1 - x\varepsilon_x + \varepsilon)$

and, $1.0 - e^{-x}(1 - x\varepsilon_x + \varepsilon) = f_A(x)$

Assume the 1.0 has no error. So any error only comes from the e^{-x} Let's try to move the error to the exponent for ease of finding x_A.

$$e^{-x}(1 + \varepsilon - x\varepsilon_x) \approx e^{-x} e^{\varepsilon - x\varepsilon_x} \approx e^{-x + \varepsilon - x\varepsilon_x}$$

$$\Rightarrow \quad x_A = x + \varepsilon + x\varepsilon_x$$

$$\Rightarrow (\text{Cond } A)(x) = \frac{|x + x\varepsilon_x + \varepsilon - x|}{|x|} \varepsilon^{-1}$$

$$= \left| \varepsilon_x + \varepsilon/x \right| \varepsilon^{-1}$$

Assume that $\varepsilon = \varepsilon_x$. Then

$$(\text{Cond } A)(x) = \left| 1 + 1/x \right| > 1 \text{ on } 0 \text{ to } 1.$$

In fact as $x \to 0$, $(\text{Cond } A)(x)$ diverges.

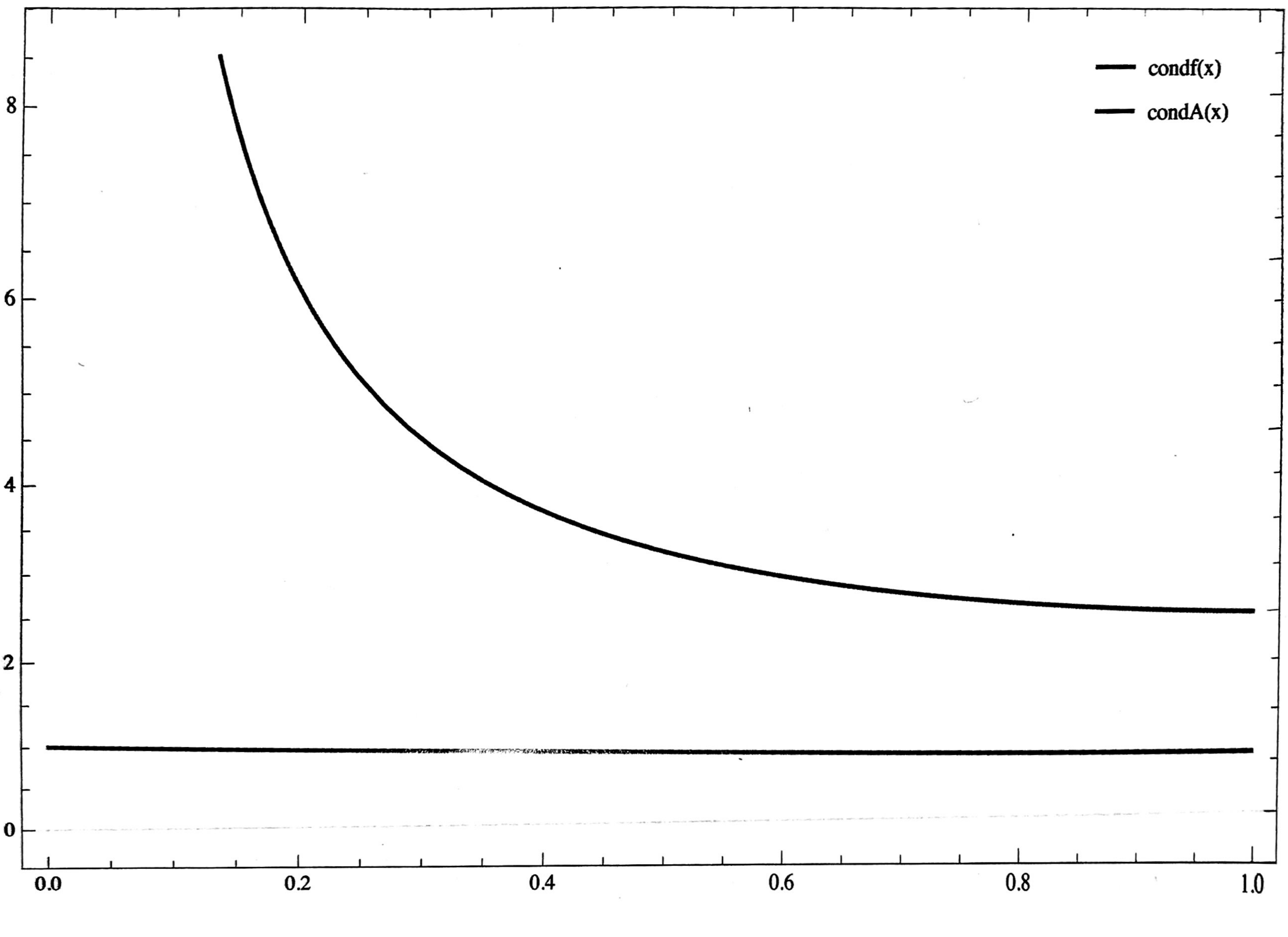
(c) See plot attached.
The root of the problem is that the error from exponentiation, ε, gets comparable to the magnitude of x as x gets small.

d) S'pose we have a p bit mantissa.
Any number on $[0,1]$ can be written as

$$x = \sum_{n=1}^{p} b_n 2^{-n} \quad \text{(rounded)}$$

The last bit of significance is $b_p 2^{-p}$.
This is true even when x is small. Let's at least write $x = 2^e \left(\sum_{n=1}^{p} b_n 2^{-n} \right)$

c.



It's not clear to me if it's meant n bits in xA or in f.
f isn't to sensitive to errors, so I'll assume it's in x.
If $\varepsilon = 2^{-p-1}$, then 2ε is losing one digit of significance, and $2^n\varepsilon$ is losing n bits of significance.

When does $1 + 1/x = 2^n$?

$$(2^n - 1)^{-1} = x_{min}$$

We require $x > x_{min}$ to lose at most n bits of significance. When $n = 1$,

$x > x_{min} = 1$. We always lose one bit of precision.

What if $n = 2, 3, 4$? $1/3$

n	x_{min}
2	$1/3$
3	$1/7$
4	$1/15$

e) We know the relative error is bounded by

$$\operatorname{cond} f(x)\left(\varepsilon + \varepsilon(\operatorname{cond} A)x\right) = err(x)$$

So:

x	$err(x)/\varepsilon$
1	1.746
$1/3$	1.966
$1/7$	1.993
$1/15$	1.999

f) When x is small, we could use an alternative method. Namely,

$$f(x) = 1 - e^{-x} \cdot \frac{1+e^{-x}}{1+e^{-x}} = \frac{1-e^{-2x}}{1+e^{-x}}$$

When x is small, this is well behaved. we see that

$$f(x) \rightarrow \frac{1+2x}{1-x} \text{ for small } x.$$

This is fine.