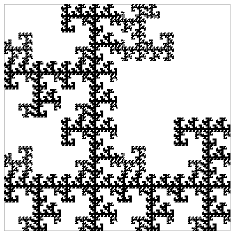
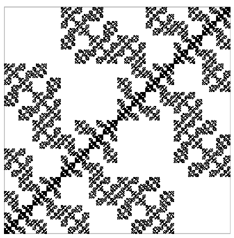
Math 190 Midterm Solutions

1. Here ar the IFS rules to generate these fractals.



r	s	θ	φ	e	f
0.5	0.5	90	90	0.5	0
0.5	0.5	90	90	1.0	0
0.5	0.5	90	90	0.5	0.5
0.25	0.25	0	0	0.5	0.75



r	s	θ	φ	e	f
0.5	0.5	0	0	0	0
0.5	0.5	0	0	0.5	0.5
0.25	−0.25	0	0	0.5	0.25
0.25	−0.25	0	0	0.75	0.5
0.25	−0.25	0	0	0	0.75
0.25	−0.25	0	0	0.25	1.0

2. (a) This fractal consists of 3 copies scaled by 0.5 and 1 copy scaled by 0.25, so the Moran equation becomes

$$3 \cdot 0.5^d + 0.25^d = 1$$

Substituting $x = 0.5^d$, the Moran equation becomes $3x + x^2 = 1$. The positive root is $x = (-3 + \sqrt{13})/2$ and so $d = \log((-3 + \sqrt{13})/2)/\log(1/2)$.

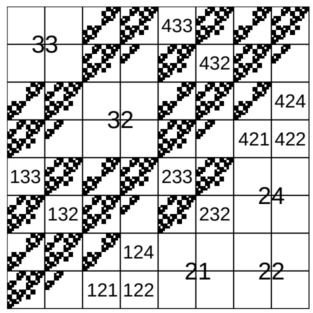
(b) This fractal consists of 2 copies scaled by 0.5 and 4 copies scaled by 0.25, so the Moran equation becomes

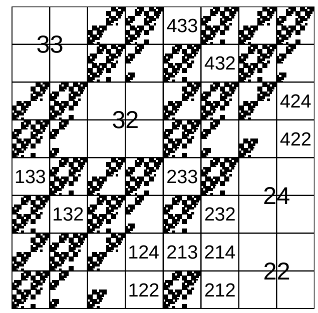
$$2 \cdot 0.5^d + 4 \cdot 0.25^d = 1$$

Substituting $x = 0.5^d$, the Moran equation becomes $2x + 4x^2 = 1$. The positive root is $x = (-1 + \sqrt{5})/4$ and so $d = \log((-1 + \sqrt{5})/4)/\log(1/2)$.

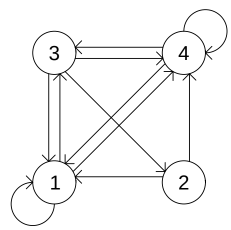
3. (a) For (i) the forbidden pairs are 21, 22, 24, 32, and 33. The forbidden triples that are not contained in these pairs are 121, 122, 124, 132, 133, 232, 233, 421, 422, 424, 432, and 433. Each of these contains a forbidden pair, so to the level of forbidden triples, (i) is generated by forbidden pairs.

For (ii) the forbidden pairs are 22, 24, 32, and 33. The forbidden triples that are not contained in these pairs are 122, 124, 132, 133, 212, 213, 214, 232, 233, 422, 424, 432, and 433. Note that 212, 213, and 214 do not contain forbidden pairs, so (ii) is not generated by forbidden pairs.

(i) 

(ii) 

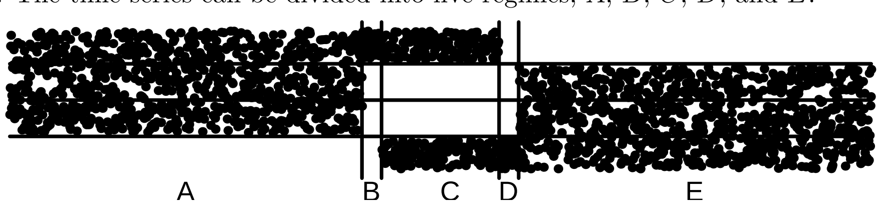
(b) The transition graph for (i) is obtained by filling in each arrow corresponding to an allowed pair, taking care that the pair ij corresponds to the transition $j \to i$.



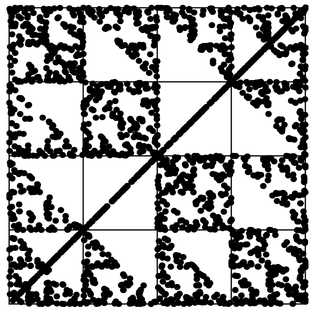
(c) For (i), 1 and 4 are romes; there are paths from a rome to each non-rome: $1 \to 3$, $1 \to 3 \to 2$, and also $4 \to 3$, $4 \to 3 \to 2$; and there are no loops among non-romes. Consequently, the attractor of this IFS with memory also can be generated by an IFS without memory. The allowed compositions, T_1, T_2, $T_3 \circ T_1$, $T_3 \circ T_4$, $T_2 \circ T_3 \circ T_1$, and $T_2 \circ T_3 \circ T_4$, determine the IFS rules

r	s	θ	φ	e	f
0.5	0.5	0	0	0	0
0.5	0.5	0	0	0.5	0.5
0.25	0.25	0	0	0	0.5
0.25	0.25	0	0	0.25	0.75
0.125	0.125	0	0	0.5	0.25
0.125	0.125	0	0	0.625	0.375

4. The time series can be divided into five regimes, A, B, C, D, and E.



Regime A generates points on the gasket with corners 2, 3, and 4. Regime B produces a sequence of points converging to corner 4. Regime C produces points on the line between corners 1 and 4. Regime D produces points converging to corner 1. Regime E produces points on the gasket with corners 1, 2, and 3. So the driven IFS shows two gaskets, sharing the segment between $(0,1)$ and $(1,0)$, together with the segment between $(0,0)$ and $(1,1)$.



5. For typical placements of A and B in the plane, the intersection formula gives

$$\dim(A \cap B) = \dim(A) + \dim(B) - 2$$

With $\dim(A \cap B) = 1$ and the specification of B as consisting of $N = 2$ pieces each scaled by r, the intersection formula simplifies to

$$3 - \frac{\log(3)}{\log(2)} = \dim(B) = \frac{\log(2)}{\log(1/r)}$$

Simplifying, we obtain

$$\frac{3\log(2) - \log(3)}{\log(2)} = \frac{\log(2)}{\log(1/r)}$$

$$\frac{\log(8) - \log(3)}{\log(2)} = \frac{\log(2)}{\log(1/r)}$$

$$\frac{\log(8/3)}{\log(2)} = \frac{\log(2)}{\log(1/r)}$$

$$\log(1/r) = \frac{(\log(2))^2}{\log(8/3)}$$

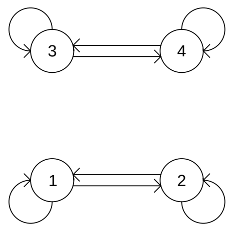
$$\log(r) = -\frac{(\log(2))^2}{\log(8/3)}$$

and finally

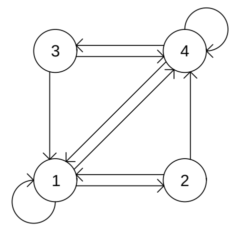
$$r = 10^{-((\log(2))^2)/\log(8/3)}$$

6. (a) From the IFS table

	r	s	θ	φ	e	f
T_1	0.5	0.5	0	0	0	0
T_2	0.5	0.5	0	0	0.5	0
T_3	0.5	0.5	0	0	0	0.5
T_4	0.5	0.5	0	0	0.5	0.5



(i)



(ii)

(a) Transition graph (i) produces two line segments. The segment between $(0,0)$ and $(1,0)$ is produced by T_1 and T_2; the segment between $(0,1)$ and $(1,1)$ is produced by T_3 and T_4. Both consist of $N = 2$ pieces, each scaled by $r = 1/2$, and so have dimension $\log(2)/\log(1/(1/2)) = 1$. Then by the dimension of unions rule, graph (i) produces a shape of dimension 1.

(b) First note that 1 and 4 are romes, there is a transition from 1 to 2 and a transition from 4 to 3, and there are no loops among 2 and 3. Consequently, this IFS with memory can be realized by an IFS without memory, consisting of two pieces, 1 and 4, scaled by $r = 1/2$, and two pieces, 21 and 34, scaled by 1/4. Applying the Moran equation, we find the dimension d is given by

$$2(1/2)^d + 2(1/4)^d = 1$$

Letting $x = (1/2)^d$, this becomes the quadratic equation

$$2x + 2x^2 = 1$$

The positive solution is $x = (-1 + \sqrt{3})/2$, so $d = \log((-1 + \sqrt{3})/2)/\log(1/2)$.