THH OF tmf

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ABSTRACT. We compute $Z_*THH(tmf)$ where Z is one of the spectra in the class of spectra $\mathcal{Z}$ of [1] such that $H^*(Z) \cong A(2)//E(Q_2)$.

CONTENTS

1. INTRODUCTION

2. COMPUTING THE BOCKSTEIN SPECTRAL SEQUENCE

Here we give the first step towards the calculation of $\pi_*(THH(tmf))$; i.e., we compute the Bockstein spectral sequence

$$THH_*(tmf; H\mathbb{F}_2)[v_2] \Rightarrow THH_*(tmf; k(2)) \tag{2.1}$$

We will use the fact that there are a class of spectra $\mathcal{Z}$, constructed by Bhattacharya-Egger [1], with the property that for $Z \in \mathcal{Z}$, there is a weak equivalence $Z \wedge tmf \simeq k(2)$.

Lemma 2.2. *There is an isomorphism of spectral sequences between the Bockstein spectral sequence*

$$E_2^{*,*} = THH_*(tmf; H\mathbb{F}_2)[v_2] \Rightarrow THH_*(tmf; k(2)) \tag{2.3}$$

and the Adams spectral sequence

$$\tilde{E}_2^{*,*} = Ext^{*,*}_{\mathcal{A}_*}(\mathbb{F}_p; H_*THH(tmf; H\mathbb{F}_2)) \tag{2.4}$$

Proof. First, note that the there is no room for d_1-differentials so the $E_1^{*,*}$-page is isomorphic to the $E_2^{*,*}$-page of the Bockstein spectral sequence. Due Angeltveit-Rognes [?qx], there is an isomorphism of $\mathcal{A}_*$-comodules and $H_*(tmf)$-Hopf algebras

$$H_*(THH(tmf)) \cong H_*(tmf) \otimes P(\sigma\zeta_1^8, \sigma\zeta_2^4, \sigma\zeta_3^2) \otimes P(\sigma\zeta_4).$$

Recall that $H_*(tmf) \cong \mathcal{A}//A(2)$ and $H_*(Z) \cong A(2)//E(Q_2)$ by definition of Z [1]. Therefore, we deduce that $H_*(Z \wedge tmf) \cong \mathcal{A}//E(Q_2)$ and we see that there is an isomorphism of A_*-comodules

$$\begin{aligned} H_*(Z \wedge THH(tmf)) &\cong H_*(Z \wedge tmf \wedge_{tmf} THH(tmf)) \\ &\cong \mathcal{A}//E(Q_2) \otimes P(\sigma\zeta_1^8, \sigma\zeta_2^4, \sigma\zeta_3^2) \otimes P(\sigma\zeta_4) \end{aligned}$$

by the collapse of the Künneth spectral sequence. We can therefore apply a change of rings isomorphism to produce the isomorphism

$$Ext^{*,*}_{\mathcal{A}_*}(\mathbb{F}_p; H_*THH(tmf; H\mathbb{F}_2)) \cong Ext^{*,*}_{E(Q_2)}(\mathbb{F}; P(\sigma\zeta_1^8, \sigma\zeta_2^4, \sigma\zeta_3^2) \otimes P(\sigma\zeta_4)).$$

However, since Q_2 acts trivially on $P(\sigma\zeta_1^8, \sigma\zeta_2^4, \sigma\zeta_3^2) \otimes P(\sigma\zeta_4)$, which can be seen by computing Margolis homology

$$H_*(P(\sigma\zeta_1^8, \sigma\zeta_2^4, \sigma\zeta_3^2) \otimes P(\sigma\zeta_4); Q_2) \cong P(\sigma\zeta_1^8, \sigma\zeta_2^4, \sigma\zeta_3^2) \otimes P(\sigma\zeta_4).$$

we get an isomorphism

$$\begin{aligned} Ext^{*,*}_{E(Q_2)}(\mathbb{F}; P(\sigma\zeta_1^8, \sigma\zeta_2^4, \sigma\zeta_3^2) \otimes P(\sigma\zeta_4)) &\cong \\ Ext^{*,*}_{E(Q_2)}(\mathbb{F}; \mathbb{F}_2) \otimes P(\sigma\zeta_1^8, \sigma\zeta_2^4, \sigma\zeta_3^2) \otimes P(\sigma\zeta_4) &\cong \\ THH_*(tmf; H\mathbb{F}_2)[v_2]. \end{aligned}$$

□

Remark 2.5. *Note that Z is constructed as a type 2 spectrum, so we may choose a v_2-self map and take the telescope to form $v_2^{-1}Z$. We can also compute $\pi_*(v_2^{-1}Z \wedge THH(tmf))$ using the localized Adams spectral sequence,*

$$v_2^{-1}\tilde{E}_2^{*,*} = v_2^{-1}Ext^{*,*}_{\mathcal{A}_*}(\mathbb{F}_2, H_*Z \wedge THH(tmf)) \Rightarrow \pi_*(v_2^{-1}Z \wedge THH(tmf))$$

and by the same argument as above, this input is isomorphic $THH_*(tmf; H\mathbb{F}_2)[v_2^{\pm 1}]$. This will allow use to see what elements in $THH_*(tmf; Z)$ are v_2-torsion.

We will first show that $K(2)_*THH(tmf)$ has a nice description. To do this, we first compute $K(2)_*tmf$ using the same technique as McClure-Staffeldt [?qx] and Angeltveit-Rognes [?qx].

Lemma 2.6. *There is an isomorphism of $K(2)_*$-modules*

$$K(2)_*tmf \cong K(2)_* \otimes K(2)_0 tmf$$

and K_0tmf is isomorphic as a $\mathbb{F}_p$-algebra to a colimit of finitely generated semisimple $\mathbb{F}_p$-algebras colim B_n

Proof. First, we note that since we can construct $BP\langle 2\rangle$ by killing off the regular sequence $(v_3, v_4, \dots)$ in BP_*, so

$$K(2)_*BP\langle 2\rangle \cong K(2)_*[t_1, t_2, t_3, \dots]/(v_2 t_k^2 - v_2^k t_k)$$

using the right unit formula in Ravenel [?qx]. We note that $K(2)_*BP\langle 2\rangle \cong K(2)_*tmf_1(3)$ up to a change of choice of generators (Is this right Dominic? Does this computation depend on our model for $BP\langle 2\rangle$?). Recall that $tmf \wedge DA(1) \simeq tmf_1(3)$ where $DA(1)$ is the double of $A(1)$ (see Mathew [?qx] for this result as well as the definition of $DA(1)$). We therefore see that

$$\begin{aligned} K(2)_*(tmf \wedge DA(1)) &\cong K(2)_*(tmf) \otimes_{K(2)_*} K(2)_*(DA(1)) \\ &\cong K(2)_*[t_1, t_2, t_3, \dots]/(v_2 t_k^2 - v_2^k t_k | k \geq 1) \end{aligned}$$

We claim that

$$K(2)_*(tmf) \cong K(2)_*[t_2, t_3, \dots]/(v_2 t_k^2 - v_2^k t_k | k \geq 2)$$

(This is my guess so far. Still need to prove that it (or some variation of it) is true).

We now let $u_i = v_2^{\frac{1-p^i}{p^2-1}} t_i$ for $i \geq 2$, then there is an isomorphism

$$K(2)_*(tmf) \cong K(2)_* \otimes \mathbb{F}[u_2, u_3, \dots]/(u_k^2 - u_k | k \geq 2)$$

where $\mathbb{F}_p[u_2, u_3, \dots]/(u_k^2 - u_k | k \geq 2) \cong K_0(tmf)$. This proves the first part of the lemma.

We then define

$$B_n = \mathbb{F}[u_2, u_3, \dots, u_n]/(u_k^2 - u_k | n \geq k \geq 2)$$

and clearly $K_0 tmf \cong \operatorname{colim} B_n$. Note that there is an isomorphism

$$B_n \cong \prod_{i=1}^{2^{n-1}} \mathbb{F}_2$$

of $\mathbb{F}_2$-algebras. This proves the second part of the lemma. □

Corollary 2.7. *The $K(2)$-Bökstedt spectral sequence*

$$HH_*^{K(2)_*}(K(2)_*(tmf)) \Rightarrow K(2)_* THH(tmf). \tag{2.8}$$

collapses and the edge homomorphism

$$K(2)_* tmf \to K(2)_* THH(tmf)$$

is an isomorphism, where this edge homomorphism is induced by the unit map $tmf \to THH(tmf)$. In other words, the map

$$tmf \to THH(tmf)$$

is a $K(2)$-local equivalence.

Proof. This follows easily from Lemma 2.6, by the following argument: since

$$K(2)_*(tmf) \cong K(2)_* \otimes_{\mathbb{F}_p} K(2)_0(tmf) \tag{2.9}$$

where $K_0(tmf) \cong \operatorname{colim} B_i$, where each B_i is isomorphic to $\prod_{i=1}^{2^{i-1}} \mathbb{F}_2$ as $\mathbb{F}_2$-algebras, there are isomorphisms

$$\begin{aligned}
HH_*^{K(2)_*}(K(2)_* tmf) &\cong Tor^{K(2)_* tmf \otimes_{K(2)_*} K(2)_* tmf}(K(2)_* tmf; K(2)_* tmf) \\
&\cong Tor^{K(2)_* \otimes (K_0(tmf) \otimes K_0(tmf))}(K(2)_* tmf; K(2)_* tmf) \\
&\cong K(2)_* \otimes Tor^{K_0(tmf) \otimes K_0(tmf)}(K_0(tmf); K_0(tmf)) \\
&\cong K(2)_* \otimes HH_*^{\mathbb{F}_p}(K(2)_0(tmf)) \\
&\cong K(2)_* \otimes \operatorname{colim} HH_*^{\mathbb{F}_p}(B_i) \\
&\cong K(2)_* \otimes \operatorname{colim} B_i \\
&\cong K(2)_*(tmf)
\end{aligned}$$

This shows that the Bökstedt spectral sequence collapses and therefore the unit map

$$K(2)_* tmf \to K(2)_* THH(tmf)$$

is an isomorphism. Hence, the map $tmf \to THH(tmf)$ is a $K(2)$-local equivalence as desired. □

Lemma 2.10. *If a map $X \to Y$ is a $K(2)$-local equivalence, then*

$$v_2^{-1}Z \wedge X \to v_2^{-1}Z \wedge Y$$

is an equivalence.

Proof. I think this is true, roughly, because of the commutative diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & v_2^{-1}Z \wedge X \arrow[r] \arrow[d] & L_{K(2)}(Z \wedge X) \arrow[d] \\
Y \arrow[r] & v_2^{-1}Z \wedge Y \arrow[r] & L_{K(2)}(Z \wedge Y)
\end{tikzcd}$$

and because Z is type 2, so $L_{K(2)}Z \cong L_{E(2)}(Z)$ and $E(2)$-localization is smashing. We also know that $L_2^f(Z) \cong v_2^{-1}Z$ and L_2^f is smashing. I need to work through the argument of McClure-Staffeldt and Ausoni-Rognes □

As a consequence, we have the following corollary.

Corollary 2.11. *There is an isomorphism*

$$K(2)_* \cong v_2^{-1}Z_*tmf \cong v_2^{-1}Z_*THH(tmf)$$

This tells us that every element in $Z_*THH(tmf)$ is v_2-torsion except for the elements in the subaglebra $P(v_2)$. This also forces certain differentials in the spectral sequence computing $Z_*THH(tmf)$.

Proposition 2.12. *The differentials in the Bockstein spectral sequence*

$$THH_*(tmf; H\mathbb{F}_2)[v_2] \Rightarrow THH_*(tmf; k(2))$$

are the following... This needs to be finished.

Therefore, we get the following answer!

Theorem 2.13. *There is an isomorphism*

$$Z_*THH(tmf) \cong ...$$

This needs to be finished.

References

[1] P. Bhattacharya and P. Egger, *A class of* 2*-local finite spectra which admit a v_2*^{1}*-self-map*, ArXiv e-prints (August 2016), available at `1608.06250`.

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