TOPOLOGICAL HOCHSCHILD HOMOLOGY OF THE SECOND TRUNCATED BROWN-PETERSON SPECTRUM I

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Abstract. We compute topological Hochschild homology of the second truncated Brown-Peterson spectrum $BP\langle 2\rangle$ at the prime 3 with coefficients in $BP\langle 1\rangle$. We use the model for $BP\langle 2\rangle$ constructed using a Shimura curve of discriminant 14 due to Hill-Lawson, which an instance of topological automorphic forms.

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1. Introduction

Topological Hochschild homology is a rich invariant of rings, or more generally ring spectra, with applications to such fields as deformation theory, string topology, and integral p-adic Hodge theory. It is also a first order approximation to algebraic K-theory in a sense made precise using Goodwillie's calculus of functors, which is our primary motivation. Algebraic K-theory of ring spectra that arise in chromatic stable homotopy theory are of particular interest because of the program of Ausoni-Rognes [4] which, in a broad sense, suggests that the arithmetic of structured ring spectra encoded in algebraic K-theory is intimately connected to chromatic complexity.

One of the most fundamental objects in chromatic stable homotopy theory is the Brown-Peterson spectrum BP, which is a complex oriented cohomology theory associated to the universal p-typical formal group. The coefficients of BP are a polynomial algebra over $\mathbb{Z}_{(p)}$ on generators v_i for $i \geq 1$, and we may form truncated versions of BP, denoted $BP\langle n\rangle$ by coning off a regular sequence $(v_{n+1}, v_{n+2}, \ldots)$. By convention $BP\langle -1\rangle = H\mathbb{F}_p$ and when $n = 0, 1$, we can produce models for $BP\langle n\rangle$ by letting $BP\langle 0\rangle = H\mathbb{Z}_{(p)}$, and $BP\langle 1\rangle = \ell$ where ℓ is the Adams summand of complex topological K-theory ku. Until recently, the previous list exhausted the examples of $BP\langle n\rangle$ that were known to have models as E_∞-ring spectra. However, in the last decade, models for $BP\langle 2\rangle$ as an E_∞-ring spectrum were constructed at the prime $p = 2$ by Lawson-Naumann [12] and at the prime $p = 3$ by Hill-Lawson [9]. Lawson-Naumann [12] use the theory of topological Modular forms with a $\Gamma_1(3)$-structure to construct an E_∞ model for $BP\langle 2\rangle$ at the prime 2 and Hill-Lawson [9] use the theory of topological automorphic forms associated to a Shimura curve of discriminant 14 to construct an E_∞ model for $BP\langle 2\rangle$ at the prime $p = 3$. This is especially interesting in view of recent work of Lawson [11] at $p = 2$ and Senger [18] for odd primes, where they prove that no model for $BP\langle n\rangle$ as an E_∞-ring spectrum exists for $n \geq 4$.

The main theorem of this paper is a computation of topological Hochschild homology of $BP\langle 2\rangle$ with coefficients in $BP\langle 1\rangle$ at the prime 3. For small values of n, the calculations of $THH_*(BP\langle n\rangle)$ are known and of fundamental importance. The first known computations of topological Hochschild homology are Bökstedt's calculations of $THH_*(BP\langle -1\rangle)$ and $THH_*(BP\langle 0\rangle)$ in [6]. To illustrate how fundamental these computations are, we point out the computation

$$THH_*(BP\langle -1\rangle) \cong P(\mu_0)$$

where $|\mu_0| = 2$, now referred to as Bökstedt periodicity, is the linchpin for new proof of Bott periodicity [10].

In McClure-Staffledt [14], they compute the Bockstein spectral sequence

$$THH_*(BP\langle 1\rangle; H\mathbb{F}_p)[v_1] \Rightarrow THH_*(BP\langle 1\rangle; k(1)).$$

This result is extended by Angeltveit-Hill-Lawson [2] where they compute the square of spectral sequences

$$\begin{tikzcd}
THH_*(BP\langle 1\rangle; H\mathbb{F}_p)[v_0, v_1] \arrow[r, Rightarrow] \arrow[d, Rightarrow] & THH_*(BP\langle 1\rangle; H\mathbb{Z}_{(p)})_p[v_1] \arrow[d, Rightarrow] \\
THH_*(BP\langle 1\rangle; k(1))[v_0] \arrow[r, Rightarrow] & THH_*(BP\langle 1\rangle; BP\langle 1\rangle)_p.
\end{tikzcd}$$

This gives a complete answer for the "integral" calculation $THH_*(BP\langle 1\rangle)$.

When $n = 2$, the calculation $THH_*(BP\langle 2\rangle; H\mathbb{F}_p)$ follows naturally from [3] as we discuss in Section 2.1, but no further results towards $THH_*(BP\langle 2\rangle)$ are known.

In the present paper, we compute the square of spectral sequences

$$\begin{tikzcd} THH_*(BP\langle 2\rangle; H\mathbb{F}_p)[v_0,v_1] \arrow[r, Rightarrow] \arrow[d, Rightarrow] & THH_*(BP\langle 2\rangle; H\mathbb{Z}_p)[v_1] \arrow[d, Rightarrow] \\ THH_*(BP\langle 2\rangle; k(1))[v_0] \arrow[r, Rightarrow] & THH_*(BP\langle 2\rangle; BP\langle 1\rangle)_p, \end{tikzcd}$$

which is slightly more complex computationally (in a precise sense) than the result of Angeltveit-Hill-Lawson [2], though many of the techniques developed in [2] and [14] carry over.

We apply a new tool, however, introduced by the first author and Salch, called the topological Hochschild-May spectral sequence [1]. This allows one to compute

$$THH_*(BP\langle 2\rangle, BP\langle 1\rangle)$$

directly. This will not replace the Bockstein spectral sequence, however, because the computational difficulty is elevated. Instead we think of it as computing the diagonal of the square and we compare the diagonal to both paths around the square in order to complete the calculation. Combining all three ways of computing the output therefore allows us to compute all of the differentials and hidden extensions.

[Gabe: Include statements of main results.]

1.1. **Outline of the strategy.**

[Gabe: Rewrite this section to reflect current strategy.]

Conventions. Fix $p \in \{2,3\}$ throughout. We will write $H_*(-)$ for homology with $\mathbb{F}_p$ coefficients, or in other words, the functor $\pi_*(H\mathbb{F}_p \wedge -)$. We write $\doteq$ to mean that an equality holds up to multiplication by a unit. Specifically, $BP\langle 2\rangle$ will denote the E_∞-model for the second truncated Brown-Peterson spectrum constructed by [12] at $p=2$ and [9] at $p=3$. We also note that by coning off v_2 on $BP\langle 2\rangle$ we may construct $BP\langle 1\rangle$ as an E_∞-$BP\langle 2\rangle$-algebra by [5] and since the E_∞-ring spectrum structure on $BP\langle 1\rangle$ is unique after p-completion, this is equivalent to the E_∞ ring spectrum model constructed in [14] after p-completion. Similarly, we may construct $H\mathbb{F}_p$ as an E_∞-$BP\langle 2\rangle$-algebra by [5]. Let $k(n)$ denote an A_∞-ring spectrum model for the connective cover of the Morava K-theory spectrum $K(n)$.

When not otherwise specified, tensor products will be taken over $\mathbb{F}_p$ and $HH_*(A)$ denotes the Hochschild homology of a graded $\mathbb{F}_p$-algebra relative to $\mathbb{F}_p$. We will let $P(x)$, $E(x)$ and $\Gamma(x)$ denote a polynomial algebra, exterior algebra, and divided power algebra over $\mathbb{F}_p$ on a generator x.

The dual Steenrod algebra will be denoted $\mathscr{A}_*$ with coproduct $\Delta\colon \mathscr{A}_* \to \mathscr{A}_* \otimes \mathscr{A}_*$. Given a right $\mathscr{A}_*$-comodule M, its right coaction will be denoted $\nu\colon \mathscr{A} \to \mathscr{A} \otimes M$ where the comodule M is understood from the context. The antipode $\chi\colon \mathscr{A}_* \to \mathscr{A}_*$, will not play a role except that we will write $\bar{\xi}_i := \chi(\xi_i)$ and $\bar{\tau}_i := \chi(\tau_i)$.

2. First two Bockstein spectral sequences

2.1. **Preliminary results.** The homology of topological Hochschild homology of $BP\langle 2\rangle$ is a straightforward application of results of [3, 6, 7] and it appears in [3, Thm. 5.12]. Recall that there is an isomorphism

$$H_*(BP\langle 2\rangle) \cong \begin{cases} P(\bar{\xi}_1, \bar{\xi}_2, \ldots) \otimes E(\bar{\tau}_3, \bar{\tau}_4, \ldots) \text{ if } p \geq 3 \\ P(\bar{\xi}_1^2, \bar{\xi}_2^2, \bar{\xi}_3^2, \bar{\xi}_4, \ldots) \text{ if } p = 2 \end{cases}$$

of $\mathscr{A}_*$-comodules. Then by [3, Thm. 5.12, Cor. 5.12] there is an isomorphism

$$H_*(\mathrm{THH}(BP\langle 2\rangle)) \cong \begin{cases} H_*BP\langle 2\rangle \otimes E(\sigma\bar{\xi}_1, \sigma\bar{\xi}_2, \sigma\bar{\xi}_3) \otimes P(\sigma\bar{\tau}_3) \text{ if } p = 3 \\ H_*BP\langle 2\rangle \otimes E(\sigma\bar{\xi}_1^2, \sigma\bar{\xi}_2^2, \sigma\bar{\xi}_3^2) \otimes P(\sigma\bar{\xi}_4) \text{ if } p = 2 \end{cases} \tag{2.1}$$

of primitively generated $H_*BP\langle 2\rangle$-Hopf algebras and $\mathscr{A}_*$-comodules. We also note the coaction on $H_*(\mathrm{THH}(BP\langle 2\rangle))$ as a comodule over $\mathscr{A}_*$ computed in [3, Thm. 5.12]

$$\nu(\sigma\bar{\tau}_3) = 1 \otimes \sigma\bar{\tau}_3 + \bar{\tau}_0 \otimes \sigma\bar{\xi}_3 \tag{2.2}$$

at $p = 3$ and

$$\nu(\sigma\bar{\xi}_4) = 1 \otimes \sigma\bar{\xi}_4 + \bar{\xi}_1 \otimes \sigma\bar{\xi}_3^2. \tag{2.3}$$

at $p = 2$. These both follow from the formula

$$\nu \circ \sigma = (1 \otimes \sigma) \circ \nu \tag{2.4}$$

in [3, Eq. 5.11] and the well known $\mathscr{A}_*$-coaction on $H_*(BP\langle 2\rangle)$. By the same argument, $\sigma\xi_i$ is primitive at $p = 3$ and $\sigma\xi_i^2$ is primitive at $p = 2$ for $i = 1, 2, 3$.

2.1.1. *THH of $BP\langle 2\rangle$ modulo (p, v_1, v_2).* We now compute

$$\mathrm{THH}_*(BP\langle 2\rangle; H\mathbb{F}_p).$$

By [3, Lem. 4.1], it suffices to compute the sub-algebra of co-mododule primitives in

$$H_*(\mathrm{THH}(BP\langle 2\rangle; H\mathbb{F}_p))$$

since $\mathrm{THH}(BP\langle 2\rangle; H\mathbb{F}_p)$ is an $H\mathbb{F}_p$-algebra. Since $BP\langle 2\rangle$ and $H\mathbb{F}_p$ are commutative ring spectra there is a weak equivalence of commutative ring spectra

$$\mathrm{THH}(BP\langle 2\rangle; H\mathbb{F}_p) \simeq \mathrm{THH}(BP\langle 2\rangle) \wedge_{BP\langle 2\rangle} H\mathbb{F}_p.$$

Since $H_*(\mathrm{THH}(BP\langle 2\rangle))$ is free over $H_*BP\langle 2\rangle$ by (2.1), the Eilenberg-Moore spectral sequence

$$\mathrm{Tor}_{*,*}^{H_*(BP\langle 2\rangle)}(H_*(THH(BP\langle 2\rangle)), H_*(H\mathbb{F}_p)) \Rightarrow H_*(THH(BP\langle 2\rangle; H\mathbb{F}_p))$$

collapses immediately implying

$$H_*(\mathrm{THH}(BP\langle 2\rangle; H\mathbb{F}_p)) \cong \begin{cases} \mathscr{A}_* \otimes E(\sigma\bar{\xi}_1, \sigma\bar{\xi}_2, \sigma\bar{\xi}_3) \otimes P(\sigma\bar{\tau}_3) \text{ if } p = 3 \\ \mathscr{A}_* \otimes E(\sigma\bar{\xi}_1^2, \sigma\bar{\xi}_2^2, \sigma\bar{\xi}_3^2) \otimes P(\sigma\bar{\xi}_3) \text{ if } p = 2. \end{cases} \tag{2.5}$$

The $\mathscr{A}_*$ coaction on elements in $\mathscr{A}_*$ is given by the coproduct and the remaining coactions are determined by the formula (2.4) and the fact that the coaction is a ring map. Therefore, these coactions are the same as in (2.2) and (2.3). We write $\lambda_i = \sigma\bar{\xi}_i$ at $p = 3$ and $\lambda_i = \sigma\bar{\xi}_i^2$ are $p = 2$. We also define

$$\mu_3 = \begin{cases} \sigma\bar{\tau}_3 - \bar{\tau}_0 \otimes \sigma\bar{\xi}_3 \text{ if } p = 3 \\ \sigma\bar{\xi}_4 - \bar{\xi}_1 \otimes \sigma\bar{\xi}_3^2 \text{ if } p = 2 \end{cases}.$$

Then it is clear that the algebra of comodule primitives in $H_*(\mathrm{THH}(BP\langle 2\rangle; H\mathbb{F}_p))$ is generated by μ_3 and λ_i for $1 \leq i \leq 3$. We therefore produce the following isomorphism of graded $\mathbb{F}_p$-algebras

$$\mathrm{THH}_*(BP\langle 2\rangle; H\mathbb{F}_p) \cong E(\lambda_1, \lambda_2, \lambda_3) \otimes P(\mu_3). \tag{2.6}$$

The degrees of the algebra generators are $|\lambda_i| = 2p^i - 1$ for $1 \leq i \leq 3$ and $|\mu_3| = 2p^3$.

2.1.2. *Rational homology.*

[Gabe: Change $E_2^{*,*}$ to $E^2_{*,*}$ for Bökstedt spectral sequence throughout.]

Next, we compute the rational homology of $\mathrm{THH}(BP\langle 2\rangle)$ to locate the torsion free component of $\mathrm{THH}_*(BP\langle 2\rangle)$. Towards this end, we will use the $H\mathbb{Q}$-based Bökstedt spectral sequence. This is a spectral sequence of the form

$$E^2_{**} = \mathrm{HH}^{\mathbb{Q}}_*(H\mathbb{Q}_*(BP\langle 2\rangle)) \implies H\mathbb{Q}_* \, \mathrm{THH}(BP\langle 2\rangle)).$$

Recall that the rational homology of $BP\langle 2\rangle$ is

$$H\mathbb{Q}_*(BP\langle 2\rangle) \cong P_{\mathbb{Q}}(v_1, v_2).$$

Thus the E^2-term of the Bökstedt spectral sequence is

$$P_{\mathbb{Q}}(v_1, v_2) \otimes E_{\mathbb{Q}}(\sigma v_1, \sigma v_2)$$

and the bidegree of σv_i is $(1, 2(p^i - 1))$. Note that $BP\langle 2\rangle$ is a commutative ring spectrum, so by [3, Prop. 4.3] the Bökstedt spectral sequence is multiplicative. All the algebra generators are in Bökstedt filtration 0 and 1 and the d^2 differential shifts Bökstedt filtration by two, so there is no room for differentials. Thus, the E^2-term is isomorphic to the E^∞-term as graded $\mathbb{Q}$-algebras. There are clearly no additive extensions since the abutment is a $\mathbb{Q}$-algebra. There are no multiplicative extensions for bidegree reasons. Thus, there is an isomorphism of graded $\mathbb{Q}$-algebras

$$\mathrm{THH}_*(BP\langle 2\rangle) \otimes \mathbb{Q} \cong P_{\mathbb{Q}}(v_1, v_2) \otimes_{\mathbb{Q}} E_{\mathbb{Q}}(\sigma v_1, \sigma v_2)$$

where $|\sigma v_i| = 2p^i - 1$. At $p = 2, 3$, there is an E_2-ring map

$$BP \to BP\langle 2\rangle.$$

To see this, we note that our E_∞ ring spectrum models for $BP\langle 2\rangle$ are clearly complex oriented and therefore come equipped with formal groups. It is also clear that these formal groups are p-typical. There is therefore an associated E_1 ring map

$$BP \to BP\langle 2\rangle$$

and then by [8, Thm. 1.2] this E_1-ring map can be lifted to an E_2-ring map. Rationally, this map

$$H\mathbb{Q}_*(BP) \to H\mathbb{Q}_*(BP\langle 2\rangle)$$

sends v_1 and v_2 to the generators of the same name. We therefore produce a multiplicative map of rational Bökstedt spectral sequences

$$\begin{tikzcd} HH_*^{\mathbb{Q}}(H\mathbb{Q}_*(BP)) \arrow[r, Rightarrow] \arrow[d] & H\mathbb{Q}_*(THH(BP)) \arrow[d] \\ HH_*^{\mathbb{Q}}(H\mathbb{Q}_*(BP\langle 2\rangle)) \arrow[r, Rightarrow] & H\mathbb{Q}_*(THH(BP\langle 2\rangle)) \end{tikzcd}$$

where on E_2 pages the map

$$P_{\mathbb{Q}}(v_i \mid i \geq 1) \otimes E_{\mathbb{Q}}(\sigma v_i \mid \geq 1) \to P_{\mathbb{Q}}(v_1, v_2) \otimes E_{\mathbb{Q}}(\sigma v_1 \sigma v_2)$$

sends v_i to v_i and σv_i to σv_i for $i = 1, 2$. By [17, Thm. 1.1],

$$\sigma v_1 = p\lambda_1$$

$$\sigma v_2 = p\lambda_2 - v_1^p \lambda_1 - v_1^p \sigma v_1$$

in $THH_*(BP)$. Since the map

$$THH_*(BP) \to THH_*(BP\langle 2\rangle)$$

sends λ_1 and λ_2 to classes of the same name, we have the same relations in $THH_*(BP\langle 2\rangle)$.

[Gabe: This part isn't proven yet. I think we can prove it, but maybe this part belongs later. Do we even use this?]

Consequently, up to a change of basis,

$$THH_*(BP\langle 2\rangle; H\mathbb{Q}) \cong E_{\mathbb{Q}}(\lambda_1, \lambda_2). \tag{2.7}$$

Also, we conclude that

$$L_0 \operatorname{THH}(BP\langle 2\rangle) \simeq L_0 BP\langle 2\rangle \vee \Sigma^{2p-1} L_0 BP\langle 2\rangle \vee \Sigma^{2p^2-1} L_0 BP\langle 2\rangle \vee \Sigma^{2p^2+2p-2} L_0 BP\langle 2\rangle$$

where $L_0 = L_{H\mathbb{Q}}$, since L_0 is a smashing localization and $L_0 S = H\mathbb{Q}$.

2.2. The $H\mathbb{Z}$-Bockstein spectral sequence.

Recall that there is an isomorphism of $\mathcal{A}_*$-comodules

$$H_*(S/p \wedge THH(BP\langle 2\rangle)) \cong \begin{cases} E(\bar{\tau}_0) \otimes H_*(BP\langle 2\rangle) \otimes E(\sigma\bar{\xi}_1, \sigma\bar{\xi}_2, \sigma\bar{\xi}_3) \otimes P(\sigma\bar{\tau}_3) & \text{if } p = 3 \\ E(\bar{\xi}_1) \otimes H_*(BP\langle 2\rangle) \otimes E(\sigma\bar{\xi}_1^2, \sigma\bar{\xi}_2^2, \sigma\bar{\xi}_3^2) \otimes P(\sigma\bar{\xi}_4) & \text{if } p = 2 \end{cases}$$

where the coaction on $x \in \mathcal{A}_*$, denoted $\nu(x)$,is given by the restriction of the coproduct Δ of the dual Steenrod algebra to $H_*(BP\langle 2\rangle/p) \subset \mathcal{A}_*$ and the remaining coactions follow from (2.4) along with multiplicativity of the coaction. In this section, we compute the Bockstein spectral sequence

$$E^1_{*,*} = THH_*(BP\langle 2\rangle; \mathbb{F}_p)[v_0] \Rightarrow THH(BP\langle 2\rangle; \mathbb{Z}_{(p)})_p. \tag{2.8}$$

As a direct consequence the coaction on $\sigma\bar{\tau}_3$, there is a differential

$$d_1(\mu_3) = v_0 \lambda_3. \tag{2.9}$$

in the $H\mathbb{Z}$-Bockstein spectral sequence (2.8).

The following lemma follows from [13, Prop. 6.8] by translating to the E_∞-context (cf. the proof of [2, Lem. 3.2]).

Lemma 2.10. *If $d_j(x) \neq 0$ in the $H\mathbb{Z}$-Bockstein spectral sequence (2.8) then*

$$d_{j+1}(x^p) = v_0 x^{p-1} d_j(x)$$

if $p > 2$ or if $p = 2$ and $j \geq 2$. If $p = 2$ and $j = 1$ then

$$d_{j+1}(x^p) = v_0 x^{p-1} d_j(x) + Q^{|x|}(d_1(x))$$

When $p = 2$, we have the differential

$$d_1(\mu) = v_0 \lambda_3.$$

Therefore, the error term for $d_2(\mu_3^2)$ is

$$Q^{16}\lambda_3 = Q^{16}(\sigma\bar{\xi}_3^2) = \sigma(Q^{16}(\bar{\xi}_3^2)) = \sigma((Q^8\bar{\xi}_3)^2) = \sigma(\bar{\xi}_4^2) = 0.$$

The first equality holds by definition of λ_3, the second equality holds because σ commutes with Dyer-Lashoff operations by [6], the third equality holds by [7], and the last equality holds because σ is a derivation [3].

Corollary 2.11. *When $p = 2, 3$, there are differentials*

$$d_{i+1}(\mu^{p^i}) = v_0^{i+1}\mu_3^{p^i-1}\lambda_3.$$

Consequently, there are differentials

$$d_{\nu_p(k)+1}(\mu^k) \doteq v_0^{\nu_p(k)+1}\mu^{k-1}\lambda_3$$

where $\nu_p(k)$ denotes the p-adic valuation of k.

Proof. Let $\alpha = \nu_p(k)$. We have that $k = p^\alpha j$ where p does not divide j. So by the Leibniz rule

$$d_{\alpha+1}(\mu_3^k) = d_{\alpha+1}((\mu_3^{p^\alpha})^j) = j\mu_3^{p^\alpha(j-1)} d_{\alpha+1}(\mu_3^{p^\alpha}) = jv_0^{\alpha+1}\mu^{p^\alpha(j-1)}\mu^{p^\alpha-1}\lambda_3 = jv_0^{\alpha+1}\mu^{j-1}\lambda_3.$$

Since j is not divisible by p, it is a unit mod p. ∎

Now recall from (2.7) that $THH_*(BP\langle 2\rangle; H\mathbb{Q}) \cong E_{\mathbb{Q}}(\lambda_1, \lambda_2)$. In fact the map, $THH_*(B; H\mathbb{Z}_{(p)}) \to THH_*(B; H\mathbb{Q})$ sends λ_i to λ_i for $i = 1, 2$. Therefore, the elements λ_1, λ_2 are p-torsion free and there are no further differentials in the $H\mathbb{Z}$-Bockstein spectral sequence. We rename the following classes as follows

$$\begin{aligned} c_i^{(1)} &:= \lambda_3\mu_3^{i-1}, & d_i^{(1)} &:= \lambda_1 c_i^{(1)}, \\ c_i^{(2)} &:= \lambda_2 c_i^{(1)}, & d_i^{(2)} &:= \lambda_2 d_i^{(1)}. \end{aligned} \tag{2.12}$$

Thus we have the following

Corollary 2.13. *There is an isomorphism of $\mathbb{Z}_{(p)}$-algebras*

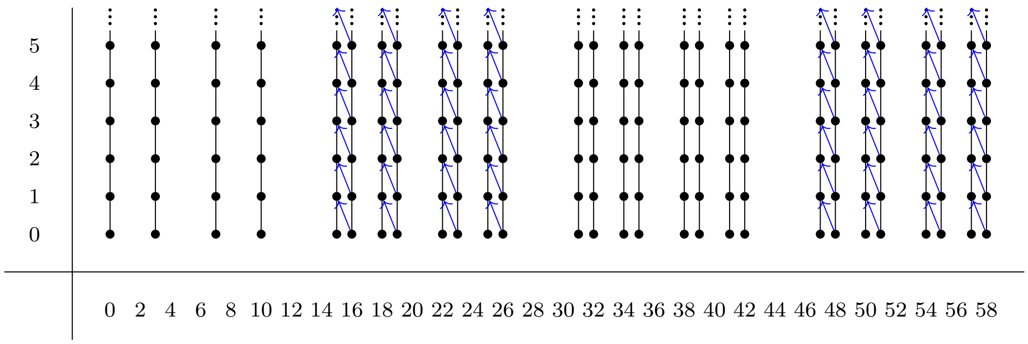
$$THH_*(BP\langle 2\rangle; H\mathbb{Z}_{(p)}) \cong E_{\mathbb{Z}_{(p)}}(\lambda_1, \lambda_2) \oplus T_0$$

where T_0 is a torsion $\mathbb{Z}_{(p)}$-module defined by

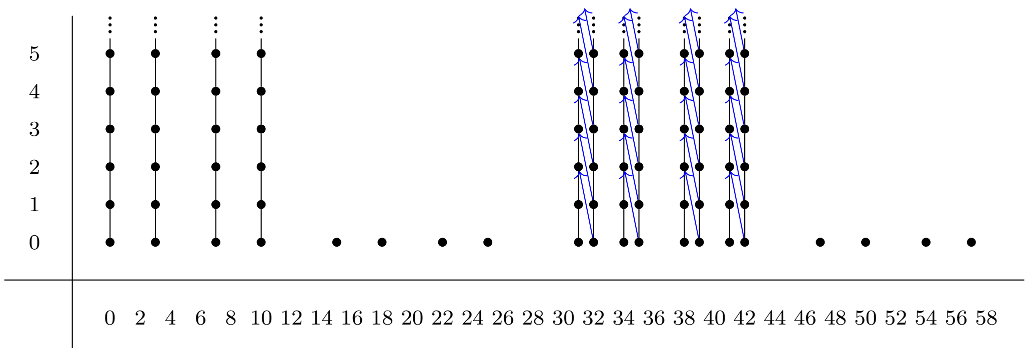
$$T_0 = \left(\mathbb{Z}_{(p)}\{c_i^{(k)}, d_i^{(k)} \mid i \geq 1, 1 \leq k \leq 2\}\right) / (p^j c_i^{(k)}, p^j d_i^{(k)} \mid j = \nu_p(i)+1, i \geq 1, 1 \leq k \leq 2)$$

where the products on the elements $c_i^{(k)}$, $d_i^{(k)}$ are specified by Formula (2.12) and by letting all other products be zero.

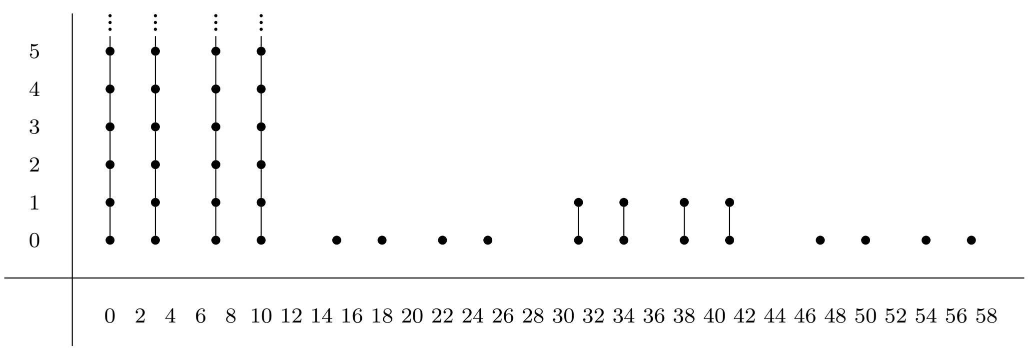
E_1-page of v_0-Bockstein Spectral Sequence at $p = 2$



E_2-page of v_0-Bockstein Spectral Sequence at $p = 2$



E_3-page of v_0-Bockstein Spectral Sequence at $p = 2$



2.3. **Topological Hochschild homology of $BP\langle 2\rangle$ with $K(1)$ coefficients.** We begin by computing $K(1)_*(BP\langle 2\rangle)$. This requires determining $\eta_R(v_{2+n})$ in $K(1)_*(BP)$ modulo the ideal generated by $(\eta_R(v_3), \ldots, \eta_R(v_{1+n}))$. We will need the following.

[Gabe: Double check reference]

Lemma 2.14. [16, Lemma A.2.2.5] *Let v_n denote the Araki generators. Then there is the following equality in $BP_*(BP)$*

$$\sum_{i,j\geq 0}^{F} t_i \eta_R(v_j)^{p^i} = \sum_{i,j\geq 0}^{F} v_i t_j^{p^i}$$

In this section, the distinction between Hazewinkel generators and Araki generators is unimportant, as the two sets of generators coincide modulo p. In $K(1)_*BP$, we have killed all v_i's except v_1, which gives us the following equation

$$\sum_{i,j\geq 0}^{F} t_i\eta_R(v_j)^{p^i} = \sum_{k\geq 0}^{F} v_1 t_k^p$$

Note that the following degrees of the terms:

$$|v_1 t_j^p| = 2(p^{j+1}-1)$$
$$|t_i\eta_R(v_j)^{p^i}| = 2(p^{i+j}-1)$$

Since we are interested in the term $\eta_R(v_{2+n})$, we collect all the terms on the left of degree $2(p^{2+n}-1)$. Thus we are summing over the ordered pairs (i,j) such that $i+j=2+n$. Since we only care about $\eta_R(v_{2+n})$ modulo $\eta_R(v_3),\ldots,\eta_R(v_{1+n})$ we only need to collect the terms where $j=1,2$, or $2+n$. This shows that

$$t_{1+n}\eta_R(v_1)^{p^{n+1}} + t_n\eta_R(v_2)^{p^n} + \eta_R(v_{n+2}) = v_1 t_{n+1}^p$$

The value of η_R on v_1 and v_2 can also be computed by Lemma 2.14. One obtains, in $K(1)_*(BP)$, the following

[Gabe: Double check these formulas.]

$$\eta_R(v_1) = v_1$$
$$\eta_R(v_2) = v_1 t_1^p - t_1 v_1^p.$$

Combining these observations, we obtain

Lemma 2.15. *In $K(1)_*(BP)$, the following congruence is satisfied*

$$\eta_R(v_{2+n}) \equiv v_1 t_{n+1}^p - v_1^{p^n} t_1^{p^{n+1}} t_n + v_1^{p^{n+1}}(t_1^{p^n} t_n - t_{n+1}) \mod (\eta_R(v_3),\ldots,\eta_R(v_{1+n}))$$

for $n\geq 1$.

Consequently, we have the following corollary.

Corollary 2.16. *There is an isomorphism of $K(1)_*$-algebras*

$$K(1)_*(BP\langle 2\rangle) \cong K(1)_*(BP)/(v_1 t_{n+1}^p - v_1^{p^n} t_1^{p^{n+1}} t_n + v_1^{p^{n+1}}(t_1^{p^n} t_n - t_{n+1}) \mid n\geq 1)$$

Define elements

$$u_n := v_1^{\frac{1-p^n}{p-1}} t_n.$$

These elements are in degree 0, and therefore there is an isomorphism of $K(1)_*$-algebras

$$K(1)_*(BP\langle 2\rangle) \cong K(1)_* \otimes_{\mathbb{F}_p} K(1)_0(BP\langle 2\rangle).$$

The calculations above imply the following corollary.

Corollary 2.17. *There is an isomorphism of $\mathbb{F}_p$-algebras*

$$K(1)_0(BP\langle 2\rangle) \cong P(u_i \mid i\geq 1)/(u_{n+1}^p - u_1^{p^{n+1}} u_n + u_1^{p^n} u_n - u_{n+1} \mid n\geq 1).$$

Our goal is to use this and the $K(1)$-based Bökstedt spectral sequence to compute the $K(1)$-homology of $\mathrm{THH}(BP\langle 2\rangle)$. This is a spectral sequence of the form

$$E^2_{s,t} = \mathrm{HH}_s^{K(1)_*}(K(1)_*(BP\langle 2\rangle)) \implies K(1)_{s+t}(\mathrm{THH}(BP\langle 2\rangle)).$$

The above considerations imply

$$E^2_{*,*} \cong K(1)_* \otimes \mathrm{HH}_*^{\mathbb{F}_p}(K(1)_0 BP\langle 2\rangle).$$

The following results will be useful for our calculation.

Lemma 2.18 ([15]). *Let $V = \mathrm{Spec}(A)$ be a nonsingular affine variety over a field k. Let W be the subvariety of $V \times \mathbb{A}^n$ defined by equations*

$$g_i(Y_1, \dots, Y_n) = 0,\ g_i \in A[Y_1, \dots, Y_n],\ i = 1, \dots, n.$$

Then the projection map $W \to V$ is étale at a point $(P; b_1, \dots, b_n)$ of W if and only if the Jacobian matrix $\left(\frac{\partial g_i}{\partial Y_j}\right)$ is a nonsingular matrix at $(P; b_1, \dots, b_n)$.

Theorem 2.19 (Étale Descent, [19]). *Let $A \hookrightarrow B$ be an étale extension of commutative k-algebras. Then there is an isomorphism*

$$\mathrm{HH}_*(B) \cong \mathrm{HH}_*(A) \otimes_A B$$

Example 2.20. Consider the subalgebra

$$\mathbb{F}_p[u_1, u_2]/(u_2^p - u_1^{p^2+1} + u_1^{p+1} - u_2 = f_1).$$

We will regard this as a $\mathbb{F}_p[u_1]$-algebra. The partial derivative $\partial_{u_2} f_1$ is $-1 \pmod p$, and therefore a unit at every point. Then Lemma 2.18 tells us that this algebra is then étale over $\mathbb{F}_p[u_1]$.

By the same argument given above, we claim that there are a sequence of subalgebras A_n of

$$A := K(1)_0(BP\langle 2\rangle) \cong \mathbb{F}_p[u_i \mid i \geq 1]/(u_{n+1}^p - u_1^{p^{n+1}} u_n + u_1^{p^n} u_n - u_{n+1} \mid n \geq 1)$$

such that each map $A_i \hookrightarrow A_{i+1}$ is an étale extension. Here

$$A_0 := \mathbb{F}_p[u_1]$$

$$A_n := \mathbb{F}_p[u_1, u_2, \dots u_n, u_{n+1}]/(u_{k+1}^p - u_1^{p^{k+1}} u_k + u_1^{p^k} u_k - u_{k+1} = f_k \mid 1 \leq k \leq n)$$

and the partial derivative

$$\partial_{u_k} f_k = -1 \pmod p$$

for all $1 < k \leq n$ and therefore a unit at each point. The claim then follows by Lemma 2.18.

By the étale base change formula for Hochschild homology in Theorem 2.19, there is an isomorphism

$$\mathrm{HH}_*^{\mathbb{F}_p}(A_{i+1}) \cong \mathrm{HH}_*^{\mathbb{F}_p}(A_i) \otimes_{A_i} A_{i+1}$$

and since the functors $\mathrm{HH}_*(-)$ and $\mathrm{HH}_*^{\mathbb{F}_p}(A_1)\otimes_{A_1}(-)$ commute with filtered colimits of $\mathbb{F}_p$-algebras, there are isomorphisms

$$\begin{aligned}\mathrm{HH}_*^{\mathbb{F}_p}(A) &\cong \mathrm{HH}_*^{\mathbb{F}_p}(\operatorname{colim} A_n)\\ &\cong \operatorname{colim}\mathrm{HH}_*^{\mathbb{F}_p}(A_n)\\ &\cong \operatorname{colim}\mathrm{HH}_*^{\mathbb{F}_p}(A_0)\otimes_{A_0} A_n\\ &\cong \mathrm{HH}_*^{\mathbb{F}_p}(A_0)\otimes_{A_0} A.\end{aligned}$$

Consequently,

$$\mathrm{HH}_*^{K(1)_*}(K(1)_*(BP\langle 2\rangle))\cong K(1)_*\otimes E(\sigma t_1)\otimes K_0(BP\langle 2\rangle)$$

and therefore, since $\sigma t_1 \doteq \lambda_1 \bmod p$,

$$K(1)_*(\mathrm{THH}(BP\langle 2\rangle))\cong K(1)_*(BP\langle 2\rangle)\otimes E(\lambda_1)$$

and

$$\mathrm{THH}_*(BP\langle 2\rangle;K(1))\cong K(1)_*\otimes E(\lambda_1).$$

In other words,

$$THH_*(BP\langle 2\rangle;k(1))\cong F\oplus T$$

where F is a free $P(v_1)$-module generated by 1 and λ_1 and T is a torsion $P(v_1)$-module.

In summary, we have proven the following theorem.

Theorem 2.21. *The following hold:*

1. *The $K(1)$-homology of $\mathrm{THH}(BP\langle 2\rangle;K(1))$ is $K(1)_*K(1)\otimes E(\lambda_1)$*
2. *There is a weak equivalence*
$$K(1)\vee\Sigma^{2p-1}K(1)\simeq \mathrm{THH}(BP\langle 2\rangle;K(1)).$$
3. *The v_1-torsion free part of $\mathrm{THH}(BP\langle 2\rangle;k(1))$ is generated by 1 and λ_1.*

2.4. **The v_1-Bockstein spectral sequence.** We now analyze the v_1-BSS. Recall that this spectral sequence is of the form

$$\mathrm{THH}(BP\langle 2\rangle;\mathbb{F}_p)[v_1]\implies \mathrm{THH}(BP\langle 2\rangle;k(1)).$$

Thus the E_1-page is

$$E(\lambda_1,\lambda_2,\lambda_3)\otimes P(\mu_3)\otimes P(v_1). \tag{2.22}$$

Since the λ_i are all in odd total degree and since v_1^k are known to be survive to the E_∞-term, the λ_i are all permanent cycles. If μ_3 were a permanent cycle, then by multiplicativity, the spectral sequence would collapse. This would contradict Theorem 2.21. Therefore, the element μ_3 must support a differential. The only possibility for bidegree reasons is

$$d_{p^2}(\mu_3)\doteq v_1^{p^2}\lambda_2.$$

Thus, we obtain

$$v_1^{-1}E_{p^2+1}\cong K(1)_*\otimes E(\lambda_1,\lambda_3,\lambda_4')\otimes P(\mu_3^p)$$

where

$$\lambda_4' := \lambda_2\mu_3^{p-1}.$$

So the bidegree of λ_4' is given by

$$|\lambda_4'| = (2p^4 - 2p^3 + 2p^2 - 1, 0).$$

For analogous reasons, the class λ_4' is a permanent cycle, and μ_3^p cannot be a permanent cycle. Based on degree considerations, there are two possible differentials,

$$d_{p^2}(\mu_3^p) \doteq v_1^{p^2}\lambda_4' \quad \text{or} \quad d_{p^3}(\mu_3^p) \doteq v_1^{p^3}\lambda_3.$$

The first would contradict the Leibniz rule for d_{p^2} and the fact that $d_{p^2}(\mu_3) \doteq v_1^{p^2}\lambda_2$. Thus,

$$v_1^{-1}E_{p^3+1} \cong K(1)_* \otimes E(\lambda_1, \lambda_4', \lambda_5') \otimes P(\mu_3^{p^2})$$

where

$$\lambda_5' := \lambda_3\mu_3^{p(p-1)}.$$

The bidegree of λ_5' is

$$|\lambda_5'| = (2p^5 - 2p^4 + 2p^3 - 1, 0).$$

For degree reasons, the class λ_5' is a permanent cycle. As before, the class $\mu_3^{p^2}$ must support a differential. Degree considerations, again, give two possibilities

$$d_{p^3}(\mu_3^{p^2}) \doteq v_1^{p^3}\lambda_5'$$

or

$$d_{p^4+p^2}(\mu_3^{p^2}) \doteq v_1^{p^4+p^2}\lambda_4'.$$

The former would contradict the Leibniz rule, leaving the latter as the only possibility. This gives us

$$v_1^{-1}E_{p^4+p^2+1} \cong K(1)_* \otimes E(\lambda_1, \lambda_5', \lambda_6') \otimes P(\mu_3^{p^3})$$

where $\lambda_6' := \lambda_4'\mu_3^{p^2(p-1)}$. We will continue via induction. First we need some notation. We will recursively define classes λ_n' by

$$\lambda_n' := \begin{cases} \lambda_n & n = 1,2,3 \\ \lambda_{n-2}'\mu_3^{p^{n-4}(p-1)} & n \geq 4 \end{cases}$$

We let $d'(n)$ denote the topological degree of λ_n'. Then this function is given recursively by

$$d'(n) = \begin{cases} 2p^n - 1 & n = 1,2,3 \\ 2p^n - 2p^{n-1} + d'(n-2) & n > 3 \end{cases}$$

Thus, by a simple induction, one has

$$d'(n) = \begin{cases} 2p^n - 1 & n = 1,2,3 \\ 2p^n - 2p^{n-1} + 2p^{n-2} - 2p^{n-3} + \cdots + 2p^2 - 1 & n \equiv 0 \mod 2,\ n > 3 \\ 2p^n - 2p^{n-1} + 2p^{n-2} - 2p^{n-3} + \cdots + 2p^3 - 1 & n \equiv 1 \mod 2,\ n > 3 \end{cases}.$$

Observe that the integers $2p^{n+2} - d(n+1) - 1$ and $2p^{n+2} - d(n+2) - 1$ are divisible by $|v_1|$. Let $r'(n)$ denote the integer

$$r'(n) := |v_1|^{-1}(|\mu_3^{p^{n-1}}| - |\lambda_{n+1}'| - 1) = |v_1|^{-1}(2p^{n+2} - d'(n+1) - 1).$$

Then a simple induction shows that

$$r'(n) = \begin{cases} p^{n+1} + p^{n-1} + p^{n-3} + \cdots + p^2 & n \equiv 1 \mod 2 \\ p^{n+1} + p^{n-1} + p^{n-3} + \cdots + p^3 & n \equiv 0 \mod 2 \end{cases}.$$

We can now describe the differentials in the v_1-BSS.

Theorem 2.23. *In the v_1-BSS, the following hold:*

(1) *The only nonzero differentials are in $v_1^{-1}E_{r'(n)}$.*
(2) *The $r'(n)$-th page is given by*

$$v_1^{-1}E_{r'(n)} \cong K(1)_* \otimes E(\lambda_1, \lambda'_{n+1}, \lambda'_{n+2}) \otimes P(\mu_3^{p^{n-1}})$$

and the classes $\lambda'_{n+1}, \lambda'_{n+2}$ are permanent cycles.
(3) *The differential $d_{r'(n)}$ is uniquely determined by multiplicativity of the BSS and the differential*

$$d_{r'(n)}(\mu_3^{p^{n-1}}) \doteq v_1^{r'(n)} \lambda'_{n+1}.$$

for $n \geq 1$.

Proof. We proceed by induction. We have already shown the theorem for $n \leq 4$. Assume that

$$v_1^{-1}E_{r'(n-1)} \cong K(1)_* \otimes E(\lambda_1, \lambda'_n, \lambda'_{n+1}) \otimes P(\mu_3^{p^{n-2}}).$$

and λ'_n is an infinite cycle.

Since $\lambda'_n, \lambda'_{n+1}$ are both in odd topological degree, λ'_{n+1} cannot support a differential hitting the v_1-towers on λ'_i for $i < n+1$. Thus, the only possibility is that λ'_{n+1} supports a differential into the v_1-tower on 1 or λ_1. But this would contradict Theorem 2.21. Therefore, the class λ'_{n+1} is a permanent cycle.

The class $\mu_3^{p^{n-1}}$ must support a differential, for if it did not, then the spectral sequence would collapse. This would lead to a contradiction of Theorem 2.21. Degree considerations show that the following differentials are possible

$$d_{\ell(n)}(\mu_3^{p^{n-1}}) \doteq v_1^{\ell(n)} \lambda'_{n+2}$$

and

$$d_{r(n)}(\mu_3^{p^{n-1}}) \doteq v_1^{r'(n)} \lambda'_{n+1}$$

where

$$\ell(n) = |v_1^{-1}|(2p^{n+2} - |\lambda'_{n+2}|).$$

An elementary inductive computation shows that

$$\ell(n) = r'(n-1).$$

We claim that the former differential cannot occur. This follows because, by the inductive hypothesis,

$$d_{r'(n-1)}(\mu_3^{p^{n-2}}) \doteq \lambda'_n,$$

and the former differential would contradict the Leibniz rule. So we must conclude the latter differential occurs. □

We now state the main result of this section.

Theorem 2.24. *For each $n \geq 2$ and each nonnegative integer m with $m \not\equiv p-1 \mod p$ there are elements $z_{n,m}$ and $z'_{n,m}$ in $THH_*(BP\langle 2\rangle; k(1))$ such that*

1. *$z_{n,m}$ projects to $\lambda'_n \mu_3^{mp^{n-2}}$ in $E_\infty^{*,0}$*
2. *$z'_{n,m}$ projects to $\lambda'_n \lambda'_{n+1} \mu_3^{mp^{n-2}}$ in $E_\infty^{*,0}$*

As a $P(v_1)$-module, $THH_*(BP\langle 2\rangle; k(1))$ is generated by the unit element 1, λ_1, and the elements $\lambda_1^\epsilon z_{n,m}, \lambda_1^\epsilon z'_{n,m}$ where $\epsilon \in \{0,1\}$. The only relations are

$$v_1^{r'(n-1)} \lambda_1^\epsilon z_{n,m} = v_1^{r'(n-1)} \lambda_1^\epsilon z'_{n,m} = 0.$$

To prove this, we first need to prove a couple lemmas. We first introduce notation. Let $P(m)$ denote a free rank one $P(v_m)$-module and let $P(m)_i$ denote the $P(v_m)$-module $P(m)/v_m^i$. Let X be a $BP\langle 2\rangle$-module such that

$$H_*(X) \cong H_*(BP\langle 2\rangle) \otimes H_*(\overline{X})$$

as a $H_*(BP\langle 2\rangle)$-module and consider the Adams spectral sequence

$$E_2^{*,*}(X) = Ext^{*,*}_{E(Q_m)_*}(\mathbb{F}_p, H_*(\overline{X})) \Rightarrow \pi_*(X \wedge_{BP\langle 2\rangle} k(m))_p \tag{2.25}$$

and the v_m-inverted Adams spectral sequence

$$v_m^{-1} E_2^{*,*}(X) = v_m^{-1} Ext^{*,*}_{E(Q_m)_*}(\mathbb{F}_p, H_*(\overline{X})) \Rightarrow \pi_*(X \wedge_{BP\langle 2\rangle} K(m))_p. \tag{2.26}$$

Consider the map of spectral sequences

$$E_2^{*,*}(X) \longrightarrow v_m^{-1} E_2^{*,*}(X)$$

induced by the localization map $k(m) \to v_m^{-1} k(m) = K(m)$.

Lemma 2.27. *For $r \geq 2$, the E_r-page of the Adams spectral sequence (2.25) for $X = THH(BP\langle 2\rangle; k(1))$ and $m = 1$ is generated by elements in filtration 0 as a $P(1)$-module and $E_r^{*,*}$ is a direct sum of copies of $P(1)$ and $P(1)_i$ for $i \leq r$.*

Proof. We will begin by proving the first statement by induction. Note that (2.22) implies the base case in the induction when $r = 2$, since the E_2-page of the Adams spectral sequence with signature

$$E_2(THH(BP\langle 2\rangle; k(1))) \Rightarrow THH_*(BP\langle 2\rangle; k(1))$$

is isomorphic to the E_1-page of the Bockstein spectral sequence

$$P(v_1) \otimes E(\lambda_1, \lambda_2, \lambda_3) \otimes P(\mu_1)$$

which is finitely generated in each degree and therefore can be written as direct sum of (suspsensions) of copies of $P(v_1) = P(1)$. Suppose the statement holds for some r. Choose a basis y_i for the $\mathbb{F}_p$-vector space V_r such that

$$V_r = \{x \in E_r^{*,0} \mid v_1^{r-1} x = 0\}.$$

Then $d_r(y_i)$ is in filtration r and since the differentials are v_1-linear, $v_1^{r-1} d_r(y_i) = 0$. However, this contradicts the induction hypothesis because the induction hypothesis implies that all elements in filtration r are v_1-torsion-free. Thus, each basis element y_i is a d_r-cycle. Next choose a set of elements $\{y'_j\} \subset E_r^{*,0}$ such that $\{d_r(y'_j)\}$ is a basis for $\mathrm{im}(d_r\colon E_r^{*,0} \to E_r^{*,r})$. Choose $y''_j \in E_r^{*,0}$ such that $v_1^r y''_j = d_r(y'_j)$. Then y''_j are d_r-cycles and y''_j and y_j are linearly independent. We can therefore choose d_r-cycles y'''_j such that $\{y_j\} \cup \{y''_j\} \cup \{y'''_j\}$ are a basis for the d_r-cycles in $E_r^{*,0}$. Then $\{y_j\} \cup \{y'_j\} \cup \{y''_j\} \cup \{y'''_j\}$ are a basis for $E_r^{*,0}$ and the differential is completely determined by the formulas

$$d_r(y_i) = 0, \quad d_r(y'_j) = v_1^r y''_j, \quad d_r(y''_j) = 0, \text{ and } \quad d_r(y'''_j) = 0.$$

Thus, $E_r^{*,*}$ is generated as a $P(1)$-module by y_i, y''_i, and y'''_i where $v_1^{r-1}y_i = 0$ and $v_1^r y''_i = 0$ and y'''_i is v_1-torsion free. □

Corollary 2.28. *For each $r \geq 2$ the localization map*

$$E_r(THH(BP\langle 2\rangle; k(1)) \to v_1^{-1}E_r^{s,t}(THH(BP\langle 2\rangle; k(1))$$

is a monomorphism in filtration t for $t \geq r-1$. Consequently, for each $r \geq 2$, the differentials in the source spectral sequence are determined by those in the target.

Proof. This follows by applying Lemma 2.27 and [14, Thm. 7.1] as in the remark after the proof of loc. cit. □

Proof of Theorem 2.24. For brevity, we will let $\delta_{n,m}$ denote $\lambda'_n\mu_3^{mp^{n-2}}$ and we will let $\delta'_{n,m}$ denote $\lambda'_n\lambda'_{n+1}\mu_3^{mp^{n-2}}$. By Corollary 2.28, to prove that the elements $\delta_{n,m}$, $\delta'_{n,m}$, $\lambda_1\delta_{n,m}$, and $\lambda_1 delta'_{n,m}$ are infinite cycles, it suffices to show that

(1) the elements $\lambda_1^{\epsilon}\delta_{n,m}$, and $\lambda_1^{\epsilon}\delta'_{n,m}$ for $\epsilon = 0, 1$ together with 1, form a basis for $E_\infty^{*,0}$ as an $\mathbb{F}_p$-vector space, and
(2) that each of $\delta_{n,m}$, $\delta'_{n,m}$, $\lambda_1\delta_{n,m}$, and $\lambda_1\delta'_{n,m}$ are killed by $v_1^{r'(n-1)}$.

By induction on n, we will prove

$$E_{r(n-1)}(THH(BP\langle 2\rangle; k(1))) \cong M_n \oplus E(\lambda_1, \lambda'_n, \lambda'_{n+1}) \otimes P(\mu_3^{p^{n-2}})$$

where M_n is generated by $\{\lambda_1^{\epsilon}\delta_{k,m}, \lambda_1^{\epsilon}\delta'_{k,m} \mid k < n, \epsilon = 0, 1\}$ modulo the relations

$$v_1^{r'(k-1)}\lambda_1^{\epsilon}\delta_{k,m} = v_1^{r'(k-1)}\lambda_1^{\epsilon}\delta'_{k,m} = 0.$$

This statement holds for $n = 2$ by (2.22). Assume the statement holds for all integers less than or equal to some $N \geq 2$. Lemma 2.28, Lemma 2.27, and Theorem 2.23 imply that the only nontrivial differentials with source in $E_{r(N)}^{*,0}$ are the differentials

$$d_{r(N)}(\lambda_1^{\epsilon}\mu_3^{(m+1)p^{N-1}}) = (m+1)v_1^{r'(N)}\lambda_1^{\epsilon}\lambda'_N\mu_3^{mp^{N-2}} \doteq v_1^{r'(N)}\lambda_1^{\epsilon}\delta_{N,m},$$

and the differentials

$$d_{r'(N)}(\lambda'_{N+1}\lambda_1^{\epsilon}\mu_3^{(m+1)p^{N-1}}) = (m+1)v_1^{r'(N)}\lambda_1^{\epsilon}\lambda'_N\lambda'_{N+1}\mu_3^{mp^{N-2}} \doteq v_1^{r'(N)}\lambda_1^{\epsilon}\delta'_{N,m}$$

where $m \not\equiv p-1 \mod p$.

[Gabe: Need to give an argument for why λ_1 is a permanent cycle using previous results.]

Combining this with Lemma 2.27 and Lemma 2.28, this implies that

$$E_{r'(N)+1}(\mathrm{THH}(BP\langle 2\rangle)) \cong M_n \oplus V_{N+1} \oplus \Big(P(1) \otimes E(\lambda_1, \lambda'_N, \lambda'_N\mu_3^{(p-1)p^{N-2}}) \otimes P(\mu_3^{p^N})\Big)$$

where V_{N+1} has generators $\delta_{N,m}$ and $\delta'_{N,m}$ and relations

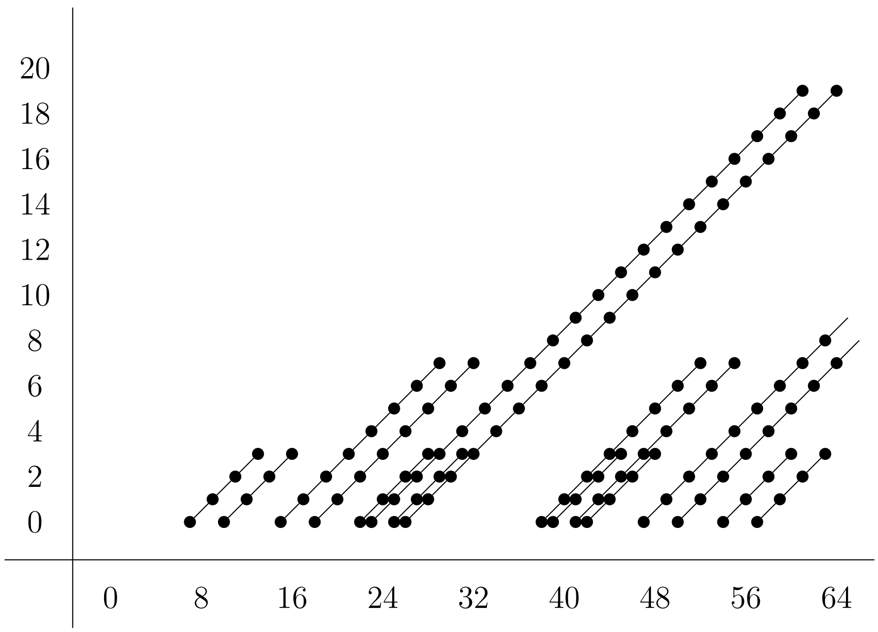
$$v_2^{r'(N)}\delta_{N,m} = v_2^{r'(N)}\delta'_{N,m} = 0.$$

By Lemma 2.28, Lemma 2.27, and Theorem 2.23 there is an isomorphism

$$E_{r(N)+1}(\mathrm{THH}(BP\langle 2\rangle)) \cong E_{r(N+1)}(\mathrm{THH}(BP\langle 2\rangle)).$$

Also, note that $M_N \oplus V_{N+1} = M_{N+1}$ and $\lambda'_N\mu_3^{(p-1)p^{N-2}} = \lambda'_{N+2}$ by definition. This completes the inductive step and consequently the proof. $\square$

v_1-torsion in the E_∞-page of v_1-Bockstein Spectral Sequence for $0 \leq x \leq 64$



Remark 2.29. One may attempt to run the same arguments as in [2], but then one runs into the issue that the E_2-pages of the remaining Bockstein spectral sequences are more dense and therefore the “vanishing column” arguments that are essential to completing the results by their methods do not apply here. We therefore use the topological Hochschild-May spectral sequence in the next section instead to determine the first family of differentials.

3. Topological Hochschild-May spectral sequences

3.1. **Preliminaries.** Here we briefly summarize joint work of the first author with Salch [1] focusing on the aspects of the paper that are relevant for the present computation. Let $\mathbb{N}$ be the category of natural numbers regarded as a partially ordered set. A *decreasingly filtered commutative monoid* in Sp is a cofibrant object in the category $\mathrm{CAlg}(\mathsf{Sp}^{\mathbb{N}^{\mathrm{op}}})$ of commutative monoids in the functor category $\mathsf{Sp}^{\mathbb{N}^{\mathrm{op}}}$ or equivalently, by work of Day [?Day], the category of lax symmetric monoidal functors from $\mathbb{N}^{\mathrm{op}}$ to Sp.

Example 3.1. The Whitehead filtration

$$\cdots \longrightarrow \tau_{\geq 3}R \longrightarrow \tau_{\geq 2}R \longrightarrow \tau_{\geq 1}R \longrightarrow \tau_{\geq 0}R$$

of a connective commutative ring spectrum R can be constructed as a cofibrant object in $\mathrm{CAlg}(\mathsf{Sp}^{\mathbb{N}^{\mathrm{op}}})$ by [1, Thm. 4.2.1].

To an object I in $\mathrm{CAlg}(\mathsf{Sp}^{\mathbb{N}^{\mathrm{op}}})$ corresponds an associatiated graded commutative monoid spectrum E_0I [1, Def. 3.1.6]. In the case of Example 3.1, $E_0I = H\pi_*R$ where $H\pi_*R$ is the generalized Eilenberg-Maclane spectrum associated to the commutative differential graded algebra π_*R. When M is an R-module the Whitehead filtration also provides a *decreasingly filtered* $\tau_{\geq\bullet}R$*-module*, by a straightforward generalization of [1, Thm. 4.2.1]. Associated to the pair (R, M) there is a spectral sequence

$$E^1_{*,*}(R, M) = THH_*(H\pi_*R; H\pi_*M) \Rightarrow THH_*(R; M) \tag{3.2}$$

where the second grading on the input comes from the May filtration, which in this case is the Whitehead filtration. This spectral sequence is multiplicative when M is a commutative R-algebra. We will refer to this as the HMSS associated to the pair (R, M); note we have also chosen a particular filtration and this will not be explicitly stated since we use the Whitehead filtration throughout. Computing $THH(BP\langle 2\rangle)$ using other filtrations of $BP\langle 2\rangle$ is an interesting question that we plan to return to in future work.

3.2. The E^1-page.

[Gabe: Rewrite this whole section to reflect current approach using Eva's argument.]

The first goal will always be to compute the E^1-page and therefore we will do this in all the cases of interest. First, we will write $B = BP\langle 2\rangle$ and consider the case where $R = BP\langle 2\rangle$ and $M = H\mathbb{F}_p$. Observe that the E^1-term can be expressed as the homotopy groups of

$$H\mathbb{F}_p \wedge_{H\pi_*B} \mathrm{THH}(H\pi_*BP\langle 2\rangle).$$

since $H\pi_*BP\langle 2\rangle$ is a commutative ring spectrum.

Proposition 3.3. *Let $p = 3$. There are isomorphisms of graded $\mathbb{Z}_{(p)}$-algebras*

$$\mathrm{THH}_*(H\pi_*BP\langle 2\rangle) \cong \mathrm{THH}_*(H\mathbb{Z}_{(p)}) \otimes_{\mathbb{Z}_{(p)}} P_{\mathbb{Z}_{(p)}}(v_1, v_2) \otimes E_{\mathbb{Z}_{(p)}}(\sigma v_1, \sigma v_2)$$

Proof. **[Gabe: Add Eva's argument for why**

$$THH_*(H\pi_*BP\langle 2\rangle) = \pi_*\left(THH(H\mathbb{Z}_{(p)}) \wedge THH(S[v_1, v_2])\right)$$

is an equivalence of $HZ_{(p)}$-algebras here.] ∎

Remark 3.4. *Note that the May filtration of an element corresponds to where it appears in the Whitehead filtration. So the May filtration of $v_1, \sigma v_1$ is $2(p-1)$ and of $v_2, \sigma v_2$ is $2(p^2-1)$. Throughout, we follow the convention that the May filtration is always reindexed by dividing by $2(p-1)$ without further mention.*

Lemma 3.5. *The E^1-term of the HMSS associated to the pair $(B; H\mathbb{F}_p)$ is given by*

$$E^1_{*,*}(B; H\mathbb{F}_p) = \mathrm{THH}_*(H\mathbb{Z}_p; H\mathbb{F}_p) \otimes E(\sigma v_1, \sigma v_2),$$

where the classes in $\mathrm{THH}_*(H\mathbb{Z}_p; H\mathbb{F}_p)$ are in May filtration 0 and where the bidegree of σv_i is $(2p^i - 1, (i-1)p + 1)$.

Proof. Adapt the proof of the previous result accordingly. □

Lemma 3.6. *The E^1-term of the HMSS for the pair $(B, H\mathbb{Z}_p)$ is given by*

$$E^1_{*,*}(BP\langle 2\rangle; H\mathbb{Z}_p) = \mathrm{THH}_*(H\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} E_{\mathbb{Z}_p}(\sigma v_1, \sigma v_2),$$

where the classes in $\mathrm{THH}_*(H\mathbb{Z}_p)$ are in May filtration 0 and where the bidegree of σv_i is $(2p^i - 1, (i-1)p + 1)$.

Proof. Adapt the proof before accordingly. Perhaps we can rewrite this in a way that we don't need a separate lemma for each result still? □

We also need the following result of Bökstedt.

Theorem 3.7. *(Bökstedt) There is an isomorphism of graded $\mathbb{F}_p$-algebras*

$$\mathrm{THH}_*(H\mathbb{Z}_p; H\mathbb{F}_p) \cong E(\lambda_1) \otimes P(\mu_1),$$

there are isomorphisms of groups

$$\pi_t \,\mathrm{THH}(\mathbb{Z}) \cong \begin{cases} \mathbb{Z}\{1\} & t = 0 \\ \mathbb{Z}/n\{\gamma_n\} & t = 2n-1 > 0 \\ 0 & else \end{cases}$$

and the map

$$THH_*(H\mathbb{Z}) \to THH_*(H\mathbb{Z}; H\mathbb{F}_p)$$

sends γ_n to $\lambda_1\mu_1^{k-1}$ when $n = pk$ for some integer $k \geq 1$ and to 0 otherwise. This is also a map of graded rings where the former has a graded ring structure by letting $\gamma_i \cdot \gamma_j = 0$ for all i, j.

Corollary 3.8. *Taking the p-localization yields*

$$\pi_t \,\mathrm{THH}(\mathbb{Z}_{(p)}) \cong \begin{cases} \mathbb{Z}_{(p)} & t = 0 \\ \mathbb{Z}/p^{\nu_p(n)}\{\gamma_n\} & t = 2n-1 > 0 \\ 0 & else \end{cases}$$

where ν_p denotes the p-adic valuation and the map $THH_*(\mathbb{Z}) \to THH_*(\mathbb{Z}_{(p)})$ sends γ_n to γ_n if $p \mid n$ and zero otherwise, so the map of graded $\mathbb{Z}_{(p)}$-algebras

$$THH_*(H\mathbb{Z}_{(p)}) \to THH_*(H\mathbb{Z}_{(p)}; H\mathbb{F}_p)$$

is sends γ_{pk} to $\lambda_1\mu_1^{k-1}$ as before with $\gamma_i \cdot \gamma_j = 0$ for all i, j as before.

By functoriality of the topological Hochschild-May spectral sequence there is a map of spectral sequences

$$\begin{tikzcd} E^1(B; H\mathbb{Z}_{(p)}) = THH_*(H\pi_*B; H\mathbb{Z}_{(p)}) \arrow[r, Rightarrow] \arrow[d] & THH_*(H\pi_*B; H\mathbb{Z}_{(p)}) \arrow[d] \\ E^1(B; H\mathbb{F}_p) = THH_*(H\pi_*B; H\mathbb{F}_p) \arrow[r, Rightarrow] & THH_*(H\pi_*B; H\mathbb{F}_p) \end{tikzcd}$$

which is induced by the canonical quotient map $E_{\mathbb{Z}_{(p)}}(\sigma v_1, \sigma v_2) \to E(\sigma v_1, \sigma v_2)$ sending σv_i to σv_i tensored with the map

$$THH_*(H\mathbb{Z}_{(p)}) \to THH_*(H\mathbb{Z}_{(p)}; H\mathbb{F}_p).$$

computed by Bökstedt and described in Corollary 3.8.

[Gabe: Do we need a more precise proof of this with the current setup?]

Lemma 3.9. *The E^1-term of the* THH*-May spectral sequence for* $\mathrm{THH}(B;\ell)$ *is given by*

$$E^1_{*,*}(B,\ell) = \mathrm{THH}_*(H\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} P_{\mathbb{Z}_p}(v_1) \otimes_{\mathbb{Z}_p} E_{\mathbb{Z}_p}(\sigma v_1, \sigma v_2),$$

where the classes in $\mathrm{THH}_*(H\mathbb{Z}_p)$ *are in May filtration 0 and where the bidegree of σv_i is $(2p^i - 1, (i-1)p+1)$ and the bidegree of v_1 is $(2p-2, 1)$.*

Proof. Adapt previous proof again. ☐

Essentially the same proofs also give the E^1-term with $k(1)$-coefficients, which we leave to the reader in the interest of brevity.

Lemma 3.10. *The E^1-term of the HMSS associated to $(B, k(1))$ is given by*

$$E^1_{*,*}(B, k(1)) = \mathrm{THH}_*(H\mathbb{Z}_p; H\mathbb{F}_p) \otimes P(v_1) \otimes E(\sigma v_1, \sigma v_2),$$

where the classes in $\mathrm{THH}_*(H\mathbb{Z}_p; H\mathbb{F}_p)$ *are in May filtration 0 and where the bidegree of σv_i is $(2p^i - 1, (i-1)p+1)$ and the bidegree of v_1 is $(2p-2, 1)$.*

Note that some care must be taken with multiplicativity of this E_1-page. We will not use multiplicativity, so we make no claims about the description above being multiplicative at the moment.

Again, essentially the same proofs provide us with the E^1 terms for ℓ with coefficients in $H\mathbb{F}_p$, $H\mathbb{Z}_p$, $k(1)$, and ℓ.

Lemma 3.11. *The E^1-term of the HMSS associated to $(\ell, H\mathbb{F}_p)$ is given by*

$$E^1_{*,*}(\ell, H\mathbb{F}_p) = \mathrm{THH}_*(H\mathbb{Z}_p; H\mathbb{F}_p) \otimes E(\sigma v_1),$$

where the classes in $\mathrm{THH}_*(H\mathbb{Z}_p; H\mathbb{F}_p)$ *are in May filtration 0 and where the bidegree of σv_1 is $(2p-1, 1)$. The E^1-term of the HMSS associated to $(\ell, H\mathbb{Z}_p)$ is given by*

$$E^1_{*,*}(\ell, H\mathbb{Z}_p) = \mathrm{THH}_*(H\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} E_{\mathbb{Z}_p}(\sigma v_1),$$

where the classes in $\mathrm{THH}_*(H\mathbb{Z}_p)$ *are in May filtration 0 and where the bidegree of σv_1 is $(2p-1, 1)$. The E^1-term of the HMSS associated to $(\ell, k(1))$ is given by*

$$E^1_{*,*}(\ell, k(1)) = \mathrm{THH}_*(H\mathbb{Z}_p; H\mathbb{F}_p) \otimes P(v_1) \otimes E(\sigma v_1),$$

where the classes in $\mathrm{THH}_*(H\mathbb{Z}_p)$ *are in May filtration 0 and where the bidegree of σv_1 is $(2p-1, 1)$ and the bidegree of v_1 is $(2p-2, 1)$. The E^1-term of the HMSS associated to (ℓ, ℓ) is given by*

$$E^1_{*,*}(\ell) = \mathrm{THH}_*(H\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} P_{\mathbb{Z}_p}(v_1) \otimes_{\mathbb{Z}_p} E_{\mathbb{Z}_p}(\sigma v_1),$$

where the classes in $\mathrm{THH}_*(H\mathbb{Z}_p)$ *are in May filtration 0 and where the bidegree of σv_1 is $(2p-1, 1)$ and the bidegree of v_1 is $(2p-2, 1)$.*

3.3. **The topological Hochschild-May spectral sequence with $\mathbb{F}_p$-coefficients.** We computed in (insert internal reference qx)

$$\mathrm{THH}_*(BP\langle 2\rangle;\mathbb{F}_p)\cong P(\mu_3)\otimes E(\lambda_1,\lambda_2,\lambda_3)$$

where $|\lambda_i| = 2p^i - 1$ and $|\mu_3| = 2p^3$. This forces differentials in the topological Hochschild-May spectral sequence, which we can then import into other spectral sequences. The following lemma follows easily from these considerations.

Lemma 3.12. *In the HMSS for $(BP\langle 2\rangle; H\mathbb{F}_p)$,*

$$E^1_{*,*}(BP\langle 2\rangle; H\mathbb{F}_p) = P(\mu_1)\otimes E(\lambda_1,\sigma v_1,\sigma v_2) \implies \mathrm{THH}_*(B;H\mathbb{F}_p)$$

the differentials are uniquely determined by multiplicativity and the differentials

$$d^1(\mu_1) = \sigma v_1, d^{p+1}(\mu_1^p) = \sigma v_2.$$

The classes λ_2 and λ_3 are detected by $\mu_1^{p-1}\cdot\sigma v_1$ and $\mu_1^{p(p-1)}\sigma v_2$, respectively and μ_3 is detected by $\mu_1^{p^2}$. There are no hidden extensions.

We will use this computation to build up to more complicated coefficients.

3.4. **The topological Hochschild-May spectral sequence with $H\mathbb{Z}_{(p)}$-coefficients.** Recall that $E^1_{*,*}(BP\langle 2\rangle; H\mathbb{Z}_{(p)})$ is isomorphic to

$$\mathrm{THH}_*(H\mathbb{Z}_{(p)})\otimes_{\mathbb{Z}_{(p)}} E_{\mathbb{Z}_{(p)}}(\sigma v_1,\sigma v_2)$$

and the map $E^1_{*,*}(BP\langle 2\rangle;\mathrm{H}\mathbb{Z}_{(p)})\to E^1_{*,*}(BP\langle 2\rangle; H\mathbb{F}_p)$ is determined by the map

$$\mathrm{THH}_*(H\mathbb{Z}_{(p)})\to \mathrm{THH}_*(H\mathbb{Z}_{(p)}, H\mathbb{F}_p)$$

tensored with the reduction mod p map $E_{\mathbb{Z}_{(p)}}(\sigma v_1,\sigma v_2)\to E(\sigma v_1,\sigma v_2)$. We therefore determine the following d^1-differentials and d^{p+1}-differentials

Lemma 3.13. *In the HMSS for the pair $(BP\langle 2\rangle; H\mathbb{Z}_{(p)})$*

$$E^1_{*,*}(BP\langle 2\rangle; H\mathbb{Z}_{(p)}) = THH_*(\mathbb{Z}_{(p)})\otimes_{\mathbb{Z}_{(p)}} E_{\mathbb{Z}_{(p)}}(\sigma v_1,\sigma v_2)$$

there is a d^1-differential

$$d^1(\gamma_{pk})\doteq(k-1)\sigma v_1\gamma_{p(k-1)},$$

and, consequently, an isomorphism

$$E^2_{*,*}(BP\langle 2\rangle; H\mathbb{Z}_{(p)}) = E^2_{*,*}(\ell; H\mathbb{Z}_{(p)})\otimes_{\mathbb{Z}_{(p)}} E(\sigma v_2)$$

where the classes λ_1, a_i and b_i are detected by γ_p, $p\gamma_{p^2 i}$ and $\gamma_{p^2 i}\sigma v_1$, respectively. There are then differentials

$$d^{p+1}(a_k)\doteq(k-1)\sigma v_2 a_{k-1}, \qquad d^{p+1}(b_k)\doteq(k-1)\sigma v_2 b_{k-1}$$

and hidden additive extensions $p\lambda_1 = \sigma v_1$ and $p\lambda_2 := pa_1 = \sigma v_2$. Using the naming convention of (cite previous theorem), we see that pa_{pn} detects $c_n^{(1)}$, pb_{pn} detects $d_n^{(1)}$, $\sigma v_2 a_{p(n-1)}$ detects $c_n^{(2)}$ and $\sigma v_2 b_{p(n-1)}$ detects $d_n^{(2)}$ for $n\geq 1$.

Proof. Since γ_{pk} maps to $\lambda_1\mu_1^{k-1}$ the differential $d^1(\lambda_1\mu_1^{k-1}) = (k-1)\lambda_1\mu_1^{k-1}$ pulls back when $p \nmid k-1$.

[Gabe: Fix proof for when $p|k$.]

Then $d^1(p\gamma_{p^2k}) = p(pk-1)\sigma v_1\gamma_{p(k-1)} = 0$ since the order of $\gamma_{p(pk-1)}$ is $p^{1+\nu_p(pk-1)} = p$. Since the order of γ_{p^2k} is $p^{1+\nu_p(pk)} \geq p^2$, the element $p\gamma_{p^2k}$ is a d^1-cycle and detects a_k. We also observe that when $p|k-1$ so that $k-1 = pj$ for some integer j there is a differential $d^1(\gamma_{p(pj+1)}) = pj\sigma v_1\gamma_{p^2j}$ and therefore for $j \geq 1$ the element $\sigma v_1\gamma_{p^2j}$ is not the target of a differential. Also, $d^1(\sigma v_1\gamma_{p^2j}) = 0$, so $\sigma v_1\gamma_{p^2j}$ must survive to the next page and it is pj-torsion.

Now the element $p\gamma_{p^2k}$ maps to zero, so we cannot determine a differential on $p\gamma_{p^2k}$ in this same way. However, $\sigma v_1\gamma_{p^2j}$ maps to $\sigma v_1\lambda_1\mu_1^{pj-1}$ so the differential

$$d^{p+1}(\sigma v_1(\lambda_1\mu_1^{p-1})\mu_1^{p(j-1)}) = (j-1)\sigma v_2\sigma v_1(\lambda_1\mu_1^{p-1})\mu_1^{p(j-2)}$$

pulls back to the differential

$$d^{p+1}(b_j) = (j-1)\sigma v_2 b_{j-1}.$$

[Gabe: Again, this only works when $p \not| j-1$ fix proof when $p|j$.]

To determine the differential on a_i we cheat a bit and use our work on the Bockstein spectral sequence. In that spectral sequence, we computed $THH_*(B;\mathbb{Z}_p)$ is

$$E(\lambda_1,\lambda_2) \oplus \mathbb{Z}_p\{c_i^{(k)}, d_i^{(k)} | k=1,2, i\geq 1\}/(p^{\nu_p(i)+1}c_i^{(k)} = p^{\nu_p(i)+1}d_i^{(k)} = 0 | k=1,2,i\geq 1)$$

where $c_i^{(1)} = \lambda_3\mu_3^{i-1}$, $c_i^{(2)} = \lambda_1 c_i^{(1)}$, $d_i^{(1)} = \lambda_2 c_i^{(1)}$ and $d_i^{(2)} = \lambda_1 d_i^{(1)}$.

This implies that there must be differentials on a_i the only possibility is that is consistent with the known answer is that

$$d^{p+1}(a_k) \doteq (k-1)\sigma v_2 a_{k-1}$$

So a_1 is a permanent cycle and when $k = pj$ for some positive integer j we observe that pa_k is a permanent cycle since $\sigma v_2 a_{k-1}$ has order p in this case. Therefore, a_{pj} must detect $c_j^{(1)}$. We also see that when $k-1 = pj$, then $\sigma v_2 a_{pj}$ is a permanent cycle because $d^{p+1}(a_{pj+1}) = pj\sigma v_2 a_{k+1}$ for $j \geq 1$ and therefore $\sigma v_2 a_{pj}$ is not a boundary. The element $\sigma v_2 a_{pj}$ must detect $d_j^{(1)}$ for degree reasons. Finally, the same argument can be made for pb_{pj} and $\sigma v_2 b_{pj}$ so they are permanent cycles and they must detect $c_j^{(2)}$ and $d_j^{(2)}$, respectively, for degree reasons. □

3.5. The topological Hochschild-May spectral sequence with $k(1)$-coefficients.

We will also use the HMSS with $k(1)$-coefficients. Recall that the E^1-term is given by

$$E^1_{*,*}(B,k(1)) \cong \mathrm{THH}(H\mathbb{Z}_p; H\mathbb{F}_p) \otimes P(v_1) \otimes E(\sigma v_1, \sigma v_2).$$

We will now use the map of spectral sequences

$$E^1_{*,*}(B;k(1)) \to E^1_{*,*}(B;H\mathbb{F}_p)$$

to lift differentials.

[Gabe: Even though

$$E^1_{*,*}(B,k(1)) \Rightarrow THH(BP\langle 2\rangle; k(1))$$

may not be multiplicaitve, I think it is not hard to show that it should be a module over the spectral sequence

$$E^1_{*,*}(S, k(1)) \Rightarrow THH(S; k(1))$$

which collapses so that differentials are v_1-linear. A discussion of this is in order if it is actually used.]

Proposition 3.14. *We can lift the d^1 and d^{p+1}-differentials from the $\mathbb{F}_p$-coefficient May spectral sequence. We have that*

$$E^{p+2}_{*,*}(B, k(1)) \cong P(\mu_3) \otimes P(v_1) \otimes E(\lambda_1, \lambda_2, \lambda_3)$$

where $\lambda_2 = \mu_1 \sigma v_1$, $\lambda_3 = (\mu_1^p)^{p-1}\sigma v_2$, and $\mu_3 = \mu_1^{p^2}$.

Proof. We clearly can lift the d^1-differentials, which shows that

$$E^2_{*,*}(B, k(1)) \cong P(\mu_2, v_1) \otimes E(\lambda_1, \lambda_2, \sigma v_2)$$

where $\lambda_2 = \mu_1^{p-1}\sigma v_1$ and $\mu_2 = \mu_1^p$. We would like to lift the d^{p+1}-differentials, so we must exclude the possibility of an earlier differential.

Observe that for bidegree reasons that $v_1, \lambda_1, \lambda_2$ and σv_2 are all infinite cycles. For bidegree reasons, the first class that could be a target of a differential supported by μ_2 is σv_2 and there are no other elements in additive generators in this bidegree. Thus we can lift the d^{p+1}-differential

$$d^{p+1}(\mu_2) = \sigma v_2$$

from HMSS for the pair $(B, H\mathbb{F}_p)$. We then let $\mu_3 = \mu_1^{p^2}$ and $\lambda_3 = (\mu_1^p)^{p-1}\sigma v_2$. □

Remark 3.15. *Note that we did not need the HMSS for the pair $(B, k(1))$ to be multiplicative because so far we have pulled back all of our differentials from a spectral sequence that is multiplicative.*

Corollary 3.16. *The HMSS for the pair $(B, k(1))$ is a reindexed version of the v_1-Bockstein spectral sequence from the E^{p+2}-page onward.*

We recall the differentials computed in (cite previous result), but here we write them in their reindexed form. There are differentials

$$d_{r'(n)+\epsilon}(\mu_3^{p^{n-1}}) \doteq v_1^{r'(n)}\lambda'_{n+1}$$

where

$$r'(n) = \begin{cases} p^{n+1} + p^{n-1} + p^{n-3} + \cdots + p^2 & n \equiv 1 \mod 2 \\ p^{n+1} + p^{n-1} + p^{n-3} + \cdots + p^3 & n \equiv 0 \mod 2, \end{cases}$$

the integer $\epsilon = n + 1 \mod 2$, and

$$\lambda'_n := \begin{cases} \lambda_n & n = 1, 2, 3 \\ \lambda'_{n-2}\mu_3^{p^{n-4}(p-1)} & n \geq 4 \end{cases}$$

3.6. **The topological Hochschild-May spectral sequence with ℓ-coefficients.** Recall that the E^1-page of the HMSS associated to the pair $(BP\langle 2\rangle, \ell)$ is

$$\mathrm{THH}_*(\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} P_{\mathbb{Z}_p}(v_1) \otimes_{\mathbb{Z}_p} E_{\mathbb{Z}_p}(\sigma v_1, \sigma v_2).$$

We will start by computing the maps

$$E^1_{*,*}(BP\langle 2\rangle, \ell) \to E^1_{*,*}(BP\langle 2\rangle, H\mathbb{Z}_p), \tag{3.17}$$

$$E^1_{*,*}(BP\langle 2\rangle, \ell) \to E^1_{*,*}(BP\langle 2\rangle, k(1)) \tag{3.18}$$

with the aim of lifting differentials.

Proposition 3.19. *The map*

$$E^1_{*,*}(BP\langle 2\rangle; \ell) \to E^1_{*,*}(BP\langle 2\rangle; H\mathbb{Z}_p)$$

is the projection map induced by sending v_1 to 0.

Proof. The way we computed the E^1-page was entirely functorial since the map

$$H\pi_*\ell \wedge_{H\pi_*BP\langle 2\rangle} THH(H\pi_*BP\langle 2\rangle) \to H\mathbb{Z}_p \wedge_{H\pi_*BP\langle 2\rangle} THH(H\pi_*BP\langle 2\rangle)$$

is given by $f \wedge_{H\pi_*BP\langle 2\rangle} THH(H\pi_*BP\langle 2\rangle)$ where f is the projection $f\colon H\pi_*\ell \to H\mathbb{Z}_p$, which is equivalent to the map $H\mathbb{Z}_p \wedge \mathbb{S}[v_1] \to H\mathbb{Z}_p \wedge \mathbb{S}$, we conclude that after rearranging colimits functorially that the map is the one stated. □

Proposition 3.20. *The map*

$$E^1_{*,*}(B, \ell) \to E^1_{*,*}(BP\langle 2\rangle, k(1))$$

is induced by modding out by p and the map $\mathrm{THH}(\mathbb{Z}_p) \to \mathrm{THH}(\mathbb{Z}_p; \mathbb{F}_p)$.

Proof. We prove this in the same way as above. The map is induced by the map

$$H\pi_*\ell \to H\pi_*k(1)$$

which is equivalent to the map $H\mathbb{Z}_p \wedge \mathbb{S}[v_1] \to H\mathbb{F}_p \wedge \mathbb{S}[v_1]$. The conclusion then follows in the same way as before. □

We will now find an infinite family of d^{p+1}-differentials in the May spectral sequence for $\mathrm{THH}(BP\langle 2\rangle; \ell)$. We will now be careful about p-adic units, which will always be written using Greek letters to differentiate them. First, we prove hat there are no possible nontrivial differentials d_r for $1 < r < p+1$. This will give a template for our approach on later pages.

Lemma 3.21. *In the HMSS for the pair $(BP\langle 2\rangle, \ell)$ there is an isomorphism*

$$E^2_{*,*}(BP\langle 2\rangle; \ell) \cong E^{p+1}_{*,*}(BP\langle 2\rangle; \ell)$$

Proof. We rename the following classes: $\gamma_p = \lambda_1$, $p\gamma_{p^2 i} =: a_i$ and $\sigma v_1 \gamma_{p^2 i} =: b_i$. The only possible differentials are those with source a_i or b_i since we know λ_1 is permanent cycle and the remaining generators are infinite cycles for bidegree reasons. We therefore need to eliminate the possibility that there are elements x such that

$$|x| = (2p^2 i - 2, m)$$

for $1 < m < p+1$ and

$$|x| = (2p^2 + 2p - 3, m)$$

for $1 < m < p+1$.

We first consider the first case. We may immediately rule out all σv_2 divisible elements since the May filtration of σv_2 is $p+1$. The only possible targets are then $v_1^k a_j$ or $v_1^\ell b_{j'}$ for some pairs of positive integers (k, j) or (ℓ, j'). Since $2p^2 i - 2$ is even and $v_1^k a_j$ is in an odd stem, we may eliminate $v_1^k a_j$. Therefore, the only possibility is $v_1^\ell b_j$ in stem $(2p-2)\ell + 2p^2 j + 2p - 2$. We therefore need the equalities

$$\begin{aligned}(2p-2)\ell + 2p^2 j + 2p - 2 &= 2p^2 i - 2\\ (2p-2)\ell + 2p &= 2p^2(i-j)\end{aligned}$$

to hold, which means $p^2 z = (p-1)\ell + 1$ for some nonnegative integer z. If $z = 0$ this clearly cannot hold and if $z = 1$ then $p^2 = (p-1)\ell + p$ implies $\ell = p$, which then fails the condition on May filtration since $b_{j'}$ is in positive May filtration and v_1^p is in May filtration p. Similarly, if $z > 1$ then $\ell > p$ and v_1^ℓ has May filtration greater than p. Therefore, there is no possible target of a differential on a_i of length less than $p+1$.

We now consider the second case. As before, our element x cannot be divisible by σv_2 because of the restriction on the May filtration and x cannot be $v_1^\ell b_j$ because the stem of x is odd and the stem of $v_1^\ell b_j$ is odd. It suffices to consider $v_1^k a_j$, which has stem $(2p-2)k + 2p^2 j - 1$. By restriction on May filtration $1 < k < p+1$. We would need the equalities

$$\begin{aligned}(2p-2)k + 2p^2 j - 1 &= 2p^2 i + 2p - 3\\ (2p-2)k + 2 &= 2p^2(i-j)\end{aligned}$$

to hold so we would have $(p-1)k + 1 = p^2 y$ for some nonnegative integer y. If $y = 0$ then this clearly cannot hold. If $y = 1$ then $k = p+1$, but this cannot be the case by the restriction on k. If $y > 1$ then again $k > p+1$ and this cannot hold. □

Proposition 3.22. *We have the following differentials in the HMSS for the pair (B, ℓ):*

$$\begin{aligned}d_{p+1}(a_i) =& \alpha_{i-1}(i-1)v_1^p b_{i-1} + \beta_{i-1}(i-1)\sigma v_2 \cdot a_{i-1},\\ d_{p+1}(b_i) =& \beta_{i-1}(i-1)\sigma v_2 b_{i-1}.\end{aligned}$$

for some p-adic units α_i, β_i for $i \geq 0$.

Proof. Note that $\sigma v_2 a_{i-1}$ and $v_1^p b_{i-1}$ are the only two classes in the appropriate bidegree, so the d_{p+1}-differential on a_i is necessarily a linear combination of these two classes

$$A\sigma v_2 \cdot a_{i-1} + Bv_1^p b_{i-1}$$

so it suffices to determine A and B. When we project to the HMSS for the pair $(B; H\mathbb{Z}_p)$ we know that $Bv_1^p b_{i-1}$ maps to zero, and we know the differential must be

$$d_{p+1}(a_i) \doteq (i-1)\sigma v_2 \cdot a_{i-1}$$

so we determine that $A = \alpha_{i-1}(i-1)$ for some unit α_{i-1}. When we project to the HMSS for the pair (ℓ, ℓ), we know that $A\sigma v_2 \cdot a_{i-1}$ maps to zero and we have the differential

$$d_{p+1}(a_i) \doteq (i-1)v_1^p b_{i-1}$$

so we determine that $B = \gamma_{i-1}(i-1)$ for some unit γ_{i-1}. We also determine that

$$d_{p+1}(b_i) = \beta_{i-1}(i-1)\sigma v_2 b_{i-1}$$

for some unit β_{i-1} in the HMSS for $(BP\langle 2\rangle, \ell)$ by this method. Since

$$d_{p+1}(a_{i+1}) \doteq A v_1^p b_i + B\sigma v_2 \cdot a_i$$

we know that $d_{p+1}(Av_1^p b_i + B\sigma v_2 \cdot a_i) = 0$. We can therefore determine that

$$Av_1^p \beta_{i-1}(i-1)\sigma v_2 b_{i-1} + B(Av_1^p b_{i-1}\sigma v_2) = 0$$

so $A\beta_{i-1} - AB = A(\beta_{i-1}(i-1) - B) = 0$. Since we know A is a unit and there are no zero divisors other than zero in $\mathbb{Z}/p^k$ for any k, we know that $B = \beta_{i-1}(i-1)$. □

This allows us to deduce the following differential by the Leibniz rule and the fact that σv_2 is a d_{p+1}-cycle.

Corollary 3.23. *We have the differentials*

$$d_{p+1}(a_i \sigma v_2) = \beta_{i-1}(i-1) v_1^p b_{i-1}\sigma v_2.$$

The following lemma will be useful for describing relations imposed by the previous differential.

Lemma 3.24. *For all i, we have that*

$$\nu_p(i-1) = \max\{\nu_p(i-1) - \nu_p(i), 0\}$$

Proof. If the max is 0, then $\nu_p(i) > \nu_p(i-1) \geq 0$. This implies that $\nu_p(i-1) = 0$ because either $\nu_p(i) = \nu_p(i-1) = 0$ or $\nu_p(i) > 0$, in which case $\nu_p(i-1) = 0$. If the max is not 0, then $\nu_p(i-1) > \nu_p(i) \geq 0$, which implies that $\nu_p(i) = 0$ by essentially the same argument. □

This allows us to deduce the following.

Corollary 3.25. *In $E^{p+2}_{*,*}(BP\langle 2\rangle, \ell)$ we have the relations*

$$p^{\nu_p(i-1)} v_1^p b_{i-1} \doteq p^{\nu_p(i-1)}\sigma v_2 \cdot a_{i-1}.$$

and

$$p^{\nu_p(i-1)}\sigma v_2 b_{i-1} = 0.$$

Note that $d_{p+1}(b_{i-1}) = \beta_{i-2}(i-2)\sigma v_2 b_{i-2}$ so b_{i-2} does not survive. However, when $i-1 = p\ell$ for some integer ℓ, then the order of the target is p and therefore $pb_{p\ell}$ survives, and we denote this class $d_\ell^{(1)}$. We also have

$$d_{p+1}(a_{i-1}\sigma v_2) = \beta_{i-1}(i-1)v_1^p b_{i-1}\sigma v_2$$

so $pa_{p\ell}\sigma v_2$ survives as well, and we denote this class $c_\ell^{(1)}$. Additionally, we observe that $b_{pk}\sigma v_2$ survives and we denote this class $d_k^{(2)}$ and the class

$$\alpha_{pk}\sigma v_2 \cdot a_{pk} + \beta_{pk} v_1^p b_{pk}$$

survives and we denote this $c_k^{(2)}$

This seems to cause an issue, but there are possible longer differentials

$$d_{p^2+p+1}(p\beta_{p\ell}b_{p\ell}) = \beta_{\ell,2}\beta_{p\ell}(\ell-1)v_1^{p^2}\sigma v_2 b_{p(\ell-1)}$$

and

$$d_{p^2+p+1}(p\alpha_{p\ell}a_{p\ell}) = \beta_{\ell,2}\alpha_{p\ell}(k-1)v_1^{p^2+p}b_{(\ell-1)p} + \alpha_{2,k}\alpha_{p\ell}v_1^{p^2}\sigma v_2 a_{(\ell-1)p}.$$

so by the Leibniz rule

$$d_{p^2+p+1}(v_1^p(pb_{p\ell})\beta_{p\ell}) = \beta_{p\ell}\beta_{\ell,2}(\ell-1)v_1^{p^2+p}\sigma v_2 b_{p(\ell-1)}$$

and

$$d_{p^2+p+1}(pa_{p\ell}\sigma v_2\alpha_{p\ell}) = \alpha_{p\ell}\beta_{\ell,2}(\ell-1)v_1^{p^2+p}b_{(\ell-1)p}\sigma v_2$$

so the difference $p(\beta_{p\ell}b_{p\ell}v_1^p - \alpha_{p\ell}a_{p\ell}\sigma v_2)$ would be a d_{p^2+p+1}-cycle as long as $\beta_{p\ell} = \alpha_{p\ell}$. Consequently, we can assume $\beta_{p\ell} = \alpha_{p\ell}$ and simply let

$$c_k^{(2)} = \sigma v_2 \cdot a_{pk} + v_1^p b_{pk}.$$

We will now make this more precise. We first argue that there is no possible shorter differential on $pb_{p\ell}$ for bidegree reasons.

Lemma 3.26. *There is an isomorphism*

$$E_{p+2}(BP\langle 2\rangle, \ell) \cong E_{p^2+p}(BP\langle 2\rangle, \ell).$$

Proof. We use the same brute force method as before. The elements that are possibly the source of a differential are the elements pa_{kp}, $pb_{p\ell}$, $v_1^p b_{pk} + \sigma v_2 a_{kp}$, and $\sigma v_2 b_{p\ell}$ with bidegrees $(2p^3\ell-1, 0)$, $(2p^3+2p-2, 1)$, $(2p^3\ell+2p^2-2, p+1)$, and $(2p^3+2p^2+2p-3, p+2)$ respectively. We therefore need to check that there no elements x in any of the bidegrees

$$(2p^3\ell-2, m), (2p^3+2p-3, m+1), (2p^3\ell+2p^2-2, m+p+1), (2p^3+2p^2+2p-3, m+p+2)$$

for $p+1 < m < p^2+p+1$. The possible targets are elements of the form

$$v_1^{i_1}(\sigma v_2)^{\epsilon_1}pa_{kp}, v_1^{i_2}(\sigma v_2)^{\epsilon_2}pb_{p\ell}, v_1^{i_3}(\sigma v_2)^{\epsilon_3}(v_1^p b_{pk} + \sigma v_2 a_{kp}), v_1^{i_4}(\sigma v_2)^{\epsilon_4}\sigma v_2 b_{p\ell}$$

in bidegrees

$$(2p^3\ell - 1 + (2p-2)i_1 + (2p^2-1)\epsilon_1, i_1 + (p+1)\epsilon_1), \tag{3.27}$$

$$(2p^3 + 2p - 2 + (2p-2)i_2 + (2p^2-1)\epsilon_2, 1 + i_2 + (p+1)\epsilon_2), \tag{3.28}$$

$$(2p^3\ell + 2p^2 - 2 + (2p-2)i_3 + (2p^2-1)\epsilon_3, p + 1 + i_3 + (p+1)\epsilon_3), \tag{3.29}$$

$$(2p^3 + 2p^2 + 2p - 3 + (2p-2)i_4 + (2p^2-1)\epsilon_4, p + 2 + i_4 + (p+1)\epsilon_4) \tag{3.30}$$

respectively. We split into four cases and in each of these, four subcases.

Case 1: We show that there are no elements x in bidegree $(2p^3\ell-2, m)$ for $p+1 < m < p^2+p+1$. If such an x existed it would have to be of the form (3.27) or (3.30) for $\epsilon_i = 1$ or of the form (3.28) or (3.29) for $\epsilon_i = 0$.

In the case (3.27), we have equalities

$$\begin{aligned} 2p^3j - 1 + (2p-2)i_1 + 2p^2 - 1 &= 2p^3\ell - 2 \\ (2p-2)i_1 + 2p^2 &= 2p^3(\ell - j) \\ (p-1)i_1 + p^2 &= p^3(\ell - j) \end{aligned}$$

and $\ell - j \geq 1$ or else the equality could not hold. If $\ell - j \geq 1$, however then $i_1 \geq p^2$ and then the May filtration of this element is at least $p^2 + p + 1$, which is already too large.

In the case (3.28), we have the equalities

$$\begin{aligned} 2p^3j + 2p - 2 + (2p-2)i_2 &= 2p^3\ell - 2 \\ 2p + (2p-2)i_2 &= 2p^3(\ell - j) \\ p + (p-1)i_2 &= p^3(\ell - j) \end{aligned}$$

where $\ell - j \geq 1$ or else this could not hold. If $\ell - j \geq 1$ then $i_2 \geq p^2 + p$ and the May filtration of this element is at least $1 + p^2 + p$, which is already too large.

In case (3.29), we see that the equalities

$$\begin{aligned} 2p^3j + 2p^2 - 2 + (2p-2)i_3 &= 2p^3\ell - 2 \\ 2p^2 + (2p-2)i_3 &= 2p^3(\ell - j)p^2 + (p-1)i_3 = p^3(\ell - j) \end{aligned}$$

would have to hold. Therefore, $p^2 + (p-1)i_3 = p^3 z$ for some positive integer z, since obviously this does not hold when $z = 0$. If $z = 1$, then $i_3 = p^2$ would make this hold, however then the May filtration would be $p+1+p^2$, which does not meet the restriction on May filtration. Again, if $z > 1$, then $i_3 > p^2$ and again $p + 1 + i_3 \geq p^2 + p + 1$, which cannot be the case.

In the case (3.30), we have equalities

$$\begin{aligned} 2p^3j + 2p^2 + 2p - 3 + (2p-2)i_4 + 2p^2 - 1 &= 2p^3\ell - 2 \\ (2p-2)i_4 + 2p^2 + 2p - 2 &= 2p^3(\ell - j) \\ (p-1)(i_4 + 1) + p^2 &= p^3(\ell - j) \end{aligned}$$

and again we must have $\ell - j \geq 1$ for this to possibly hold. However, when $\ell - j \geq 1$, then we must have $i_4 + 1 \geq p^2$ and so $i_4 \geq p^2 - 1$ and then the May filtration is already greater or equal to $p + 2 + p^2 - 1 + p + 1 \geq p^2 + p + 1$.

Case 2: We show that there are no elements x in bidegree $(2p^3\ell + 2p^2 - 2, m + p + 1)$ for $p + 1 < m < p^2 + p + 1$. If such an x existed it would have to be of the form (3.27) or (3.30) for $\epsilon_i = 1$ or of the form (3.28) or (3.29) for $\epsilon_i = 0$.

In the case (3.27), we have equalities

$$\begin{aligned} 2p^3j - 1 + (2p-2)i_1 + 2p^2 - 1 &= 2p^3\ell + 2p^2 - 2 \\ (2p-2)i_1 &= 2p^3(\ell - j) \\ (p-1)i_1 &= p^3(\ell - j) \end{aligned}$$

and $\ell - j \geq 1$ or else the equality could not hold. We observe that p^3 must therefore divide i_1, but then $p^3 + p + 1 \geq p^2 + p + 1$ and therefore this cannot be the case by the restriction on May filtration.

In the case (3.28), we have the equalities

$$\begin{aligned} 2p^3j + 2p - 2 + (2p-2)i_2 &= 2p^3\ell + 2p^2 - 2 \\ 2p + (2p-2)i_2 - 2p^2 &= 2p^3(\ell - j) \\ p + (p-1)i_2 - p^2 &= p^3(\ell - j). \end{aligned}$$

If $i_2 = p$ and $\ell = j$, then this holds but then the May filtration is $p+1$, which is too small. If $i - j \geq 1$ then $i_2 \geq p^3 + p$ and the May filtration of this element is at least $1 + p^3 + p$, which is too large.

In case (3.29), we see that the equalities

$$\begin{aligned} 2p^3j + 2p^2 - 2 + (2p-2)i_3 &= 2p^3\ell + 2p^2 - 2 \\ (2p-2)i_3 &= 2p^3(\ell - j)(p-1)i_3 = p^3(\ell - j) \end{aligned}$$

would have to hold. We see that p^3 must divide i_3 and therefore the May filtration of this element is greater or equal to $p^3 + p + 1 > p^2 + p + 1$.

In the case (3.30), we have equalities

$$\begin{aligned} 2p^3j + 2p^2 + 2p - 3 + (2p-2)i_4 + 2p^2 - 1 &= 2p^3\ell + 2p^2 - 2 \\ (2p-2)i_4 + 2p - 2 &= 2p^3(\ell - j) \\ (p-1)(i_4 + 1) &= p^3(\ell - j) \end{aligned}$$

and again we must have $\ell - j \geq 1$ for this to possibly hold. However, when $\ell - j \geq 1$, then we must have that p^3 divides $i_4 + 1$ and so $i_4 \geq p^3 - 1$ and then the May filtration is already greater or equal to $p + 2 + p^3 - 1 + p + 1 > p^2 + p + 1$.

[Gabe: Gosh. This is so elementary and tedious, but it seems to work. Halfway done.]

□

Lemma 3.31. *There are infinite families of differentials of length $p^2 + p + 1$*

$$d_{p^2+p+1}(pb_{p\ell}) = \beta_{\ell,2}(\ell - 1)v_1^{p^2+p}\sigma v_2 b_{p(\ell-1)}$$

and

$$d_{p^2+p+1}(pa_{kp}) = \beta_{k-1,2}(k-1)v_1^{p^2}b_{(k-1)p} + \alpha_{k-1,2}v_1^{p^2}\sigma v_2 a_{(k-1)p}.$$

in the HMSS for the pair (B, ℓ) and in fact $\beta_{pk,2} = \alpha_{pk,2}$.

Proof. We know that pa_{kp} maps to pa_{kp} in the HMSS for (ℓ, ℓ) and in that spectral sequence it is the source of a differential hitting some unit times $(k-1)v_1^{p^2}b_{(k-1)p}$. We therefore choose a lift of the element $(k-1)v_1^{p^2+p^2}b_{(k-1)p}$ to the HMSS for the pair (B, ℓ), which we know must be a linear combination of $v_1^{p^2}b_{(k-1)p}$ and $v_1^{p^2}\sigma v_2 a_{(k-1)p}$. If the coefficient of the first term were zero, then it would not map to $(k-1)v_1^{p^2+p^2}b_{(k-1)p}$ as desired and if the coefficient of the second term were zero, it would lead to a contradiction because then this element is known to die at an earlier page. Therefore, we may choose our lift to be of the form $\omega_{k,2}(k-1)\left(v_1^{p^2+p^2}b_{(k-1)p}+\alpha_{2,k}v_1^{p^2}\sigma v_2 a_{(k-1)p}\right)$ and then the differential is forced. We also note that this differential implies a differential

$$d_{p^2+p+1}(\omega_{k-1,2}(k-1)\alpha_{k-1,2}v_1^{p^2}\sigma v_2 a_{(k-1)p}) = \omega_{k-1,2}(k-1)\alpha_{k-1,2}v_1^{p^2}\sigma v_2(\omega_{k-2,2}(k-2)\left(v_1^{p^2}b_{(k-2)p}+\alpha_{k-2,2}v_1^{p^2}\sigma v_2 a_{(k-2)p}\right)$$

which reduces to

$$d_{p^2+p+1}(\omega_{k-1,2}(k-1)\alpha_{k-1,2}v_1^{p^2}\sigma v_2 a_{(k-1)p}) = \omega_{k-1,2}(k-1)\alpha_{k-1,2}v_1^{p^2}\sigma v_2(\omega_{k-2,2}(k-2)v_1^{p^2}b_{(k-2)p})$$

so in order for $\omega_{k-1,2}(k-1)\left(v_1^{p^2}b_{(k-1)p}+\alpha_{k-1,2}v_1^{p^2}\sigma v_2 a_{(k-1)p}\right)$ to be a d_{p^2+p+1}-cycle and not have a contradiction, we need

$$d_{p^2+p+1}(\omega_{k-1,2}(k-1)v_1^{p^2}b_{(k-1)p}) = \omega_{k-1,2}(k-1)\alpha_{k-1,2}v_1^{p^2}\sigma v_2(\omega_{k-2,2}(k-2)v_1^{p^2}b_{(k-2)p})$$

which implies the other family of differentials

$$d_{p^2+p+1}(pb_{kp}) = p\alpha_{k,2}\omega_{k-1,2}(k-1)v_1^{p^2}\sigma v_2 b_{(k-1)p}$$

where $\alpha_{2,k+1}\omega_{k,2} =: \delta_{\ell,2}$ □

4. Topological Hochschild homology of $BP\langle 2\rangle$ with L coefficients

In this section we calculate the homotopy groups of $\mathrm{THH}(B;L)$. For notational simplicity, we will write $(-)_E$ for the Bousfield localization functor L_E when E is a homology theory. We will calculate $\mathrm{THH}(B;L)$ using the arithmetic fracture square

$$\begin{tikzcd} \mathrm{THH}(B;L) \arrow[r] \arrow[d] & \prod_q \mathrm{THH}(B;L)_{S/q} \arrow[d] \\ \mathrm{THH}(B;L)_{\mathbb{Q}} \arrow[r] & \left(\prod_q \mathrm{THH}(B;L)_{S/q}\right)_{\mathbb{Q}} \end{tikzcd}.$$

where the product ranges over all primes q. We now identify the homotopy type of $\mathrm{THH}(B;L)_{S/q}$ and $\mathrm{THH}(B;L)_{\mathbb{Q}}$. First, note that the class λ_1 survives to $\mathrm{THH}(B;\ell)$ since it must be a permanent cycle in the HMSS for the pair $(B;\ell)$ for bidegree reasons. Since $\mathrm{THH}(B;L)$ is an L-module, we have a morphism of L-modules

$$L\vee\Sigma^{2p-1}L\to\mathrm{THH}(B;L).$$

Proposition 4.1. *The map above induces an isomorphism in $K(1)$-homology.*

Proof. Recall the equivalence

$$\mathrm{THH}(B;L)\simeq L\wedge_B\mathrm{THH}(B).$$

The EMSS thus collapses at E_2 and gives an isomorphism

$$K(1)_*(\mathrm{THH}(B;L))\cong K(1)_*L\otimes_{K(1)_*B}K(1)_*\mathrm{THH}(B).$$

We have previously seen that

[Gabe: Insert internal citation]
$K(1)_* \operatorname{THH}(B) \cong K(1)_* B \otimes_{K(1)_*} E(\lambda_1)$, and so we have

$$K(1)_* \operatorname{THH}(B; L) \cong K(1)_* L \otimes_{K(1)_*} E(\lambda_1).$$

This implies the map is a $K(1)$-isomorphism. □

Corollary 4.2. *The map above induces an equivalence*

$$\left(L \vee \Sigma^{2p-1} L\right)_{K(1)} \to (\operatorname{THH}(B; L))_{K(1)}.$$

Remark 4.3. *Recall that (cf. Ravenel "localization...") that the Bousfield class of $v_1^{-1} B$ is the same as the Bousfield class of L, and that the Bousfield class of L is the Bousfield class of $H\mathbb{Q} \vee K(1)$. So we also need to check this map induces an isomorphism on $H\mathbb{Q}$-homology.*

Since $\operatorname{THH}(B; L)$ is an L-module it is L-local. We know from Prop 2.11 of Bousfield that there is an isomorphism of functors

$$(-)_{K(1)} \cong ((-)_L)_{S/p}.$$

Thus, we can write the above equivalence as

$$((L \vee \Sigma^{2p-1} L)_L)_{S/p} \xrightarrow{\simeq} ((\operatorname{THH}(B; L)_L)_{S/p}.$$

But both $L \vee \Sigma^{2p-1} L$ and $\operatorname{THH}(B; L)$ are L-local. Since $\operatorname{THH}(B; L_p)$ is a L_p-module, it is in particular p-complete there is an equivalence

$$\operatorname{THH}(B; L_p) \simeq \operatorname{THH}(B; L)_p$$

so we may conclude the following corollary.

Corollary 4.4. *There is an equivalence*

$$L_p \vee \Sigma^{2p-1} L_p \to \operatorname{THH}(B; L)_p.$$

and consequently an equivalence

$$(L_p)_{\mathbb{Q}} \vee (\Sigma^{2p-1} L_p)_{\mathbb{Q}} \to (\operatorname{THH}(B; L)_p)_{\mathbb{Q}} \simeq \operatorname{THH}(B; (L_p)_{\mathbb{Q}})$$

where the last equivalence holds because $(-)_{\mathbb{Q}}$ is a smashing localization.

Consequently, we know that λ_1 is v_1-torsion free.

We now compute $THH(B; L)$ rationally. There is a Bökstedt spectral sequence

$$E_2^{*,*} = HH_*^{\mathbb{Q}}(H\mathbb{Q}_* B; H\mathbb{Q}_* L) \Rightarrow H\mathbb{Q}_*(THH(B; L)) \cong \pi_*(THH(B; L)_{H\mathbb{Q}})$$

with input

$$HH_*^{\mathbb{Q}}(P_{\mathbb{Q}}(v_1, v_2); P_{\mathbb{Q}}(v_1^{\pm 1})) \cong P_{\mathbb{Q}}(v_1^{\pm 1}) \otimes_{\mathbb{Q}} E_{\mathbb{Q}}(\sigma v_1, \sigma v_2).$$

the spectral sequence collapses at the E_2-page since the generators are all in Bökstedt filtration zero or one. Thus,

$$\pi_* THH(B; L)_{\mathbb{Q}} \cong E_{L_* \otimes \mathbb{Q}}(\sigma v_1, \sigma v_2).$$

we therefore observe that

$$THH(B; L)_{\mathbb{Q}} \simeq L_{\mathbb{Q}} \vee \Sigma^{2p-1} L_{\mathbb{Q}} \vee \Sigma^{2p^2-1} L_{\mathbb{Q}} \vee \Sigma^{2p^2+2p-2} L_{\mathbb{Q}}.$$

Since there is a pullback

$$\begin{tikzcd} L \arrow[r] \arrow[d] & L_p \arrow[d] \\ L_{\mathbb{Q}} \arrow[r] & (L_p)_{\mathbb{Q}} \end{tikzcd}$$

when we apply these results to the arithmetic fracture square we get the following corollary.

Corollary 4.5. *There is an equivalence*

$$THH(B;L) \simeq L \vee \Sigma^{2p-1}L \vee \Sigma^{2p^2-1}L_{\mathbb{Q}} \vee \Sigma^{2p^2+2p-2}L_{\mathbb{Q}}.$$

Consequently, we know that σv_1, σv_2, and $\sigma v_1 \sigma v_2$ are v_1-torsion free.

5. Topological Hochschild cohomology of $BP\langle 2\rangle$

We will write $THH^*_S(BP\langle 2\rangle; M)$ for topological Hochschild cohomology of $BP\langle 2\rangle$ with coefficients in a $BP\langle 2\rangle \wedge BP\langle 2\rangle^{\mathrm{op}}$-module M, which is defined to be

$$THH^*_S(BP\langle 2\rangle; M) := \pi_*\left(F_{BP\langle 2\rangle^e}(BP\langle 2\rangle, M)\right)$$

where $BP\langle 2\rangle^e := BP\langle 2\rangle \wedge BP\langle 2\rangle^{\mathrm{op}}$. We recall that there is a universal coefficient spectral sequence (UCSS) computing the homotopy groups of $F_{BP\langle 2\rangle^e}(BP\langle 2\rangle, BP\langle 2\rangle)$

$$Ext^{*,*}_{\pi_*(BP\langle 2\rangle^e)}(BP\langle 2\rangle_*, BP\langle 2\rangle_*) \Rightarrow THC^*(BP\langle 2\rangle),$$

but this is usually not computable. With coefficients in $H\mathbb{F}_p$, however, we can compute $THC^*(BP\langle 2\rangle; H\mathbb{F}_p)$ by a different means. First, note that

$$THH_*(BP\langle 2\rangle; H\mathbb{F}_p) \cong E(\lambda_1, \lambda_2, \lambda_3) \otimes P(\mu_3)$$

is a finite type graded $\mathbb{F}_p$-algebra and $THH(BP\langle 2\rangle; H\mathbb{F}_p)$ is an $H\mathbb{F}_p$-algebra. Given a map of commutative ring spectra $f\colon R \to S$ there is an associated adjunction $f_! \dashv f^*$ where $f_!(M) = M \wedge_R S$ is extension of scalars and f^* is restriction. By the adjunction $f_! \dashv f^*$ associated to the map of commutative ring spectra $f\colon BP\langle 2\rangle^e \to H\mathbb{F}_p$, there is an equivalence

$$F_{H\mathbb{F}_p}(THH(BP\langle 2\rangle; H\mathbb{F}_p), H\mathbb{F}_p) \simeq F_{BP\langle 2\rangle^e}(BP\langle 2\rangle, H\mathbb{F}_p).$$

The UCSS

$$\mathrm{Ext}^{*,*}_{\mathbb{F}_p}(\pi_*(\mathrm{THH}(BP\langle 2\rangle; H\mathbb{F}_p)), \mathbb{F}_p) \Rightarrow \mathrm{THH}^*_S(BP\langle 2\rangle; H\mathbb{F}_p)$$

collapses and

$$\mathrm{THH}^*_S(BP\langle 2\rangle, H\mathbb{F}_p) \cong \mathrm{Hom}_{\mathbb{F}_p}(\mathrm{THH}_*(BP\langle 2\rangle; H\mathbb{F}_p), \mathbb{F}_p) \tag{5.1}$$

where

$$\mathrm{Hom}_{\mathbb{F}_p}(THH_*(BP\langle 2\rangle; H\mathbb{F}_p), \mathbb{F}_p) \cong E(x_1, x_2, x_3) \otimes \Gamma(c_1)$$

and $|x_i| = 2p^i - 1$ and $|c_3| = 2p^3$. The classes x_i are dual to λ_i and the class $c_i = \gamma_i(c_1)$ is dual to μ_3^i, adopting the notation conventions of [2]. In fact, many of the results of this section are straighforward generalizations of those in [2], but we include all the details we need for completeness.

5.1. **Relative topological Hochschild cohomology of $BP\langle 2\rangle$.** Recall that there is an isomorphism

$$H_*(MU) \cong P(b_k \mid k \geq 1)$$

and the map

$$H_*(MU) \to H_*(BP) \cong P(\bar{\xi}_k | k \geq 1)$$

sends b_j to $\bar{\xi}_k$ for $k \geq 1$ if $j = p^k - 1$ and zero otherwise at $p = 3$.

Lemma 5.2. *There is an isomorphism of rings*

$$\pi_* \left(H\mathbb{F}_p \wedge_{MU} BP\langle 2\rangle\right) \cong E(\tau_3, \tau_4, \ldots) \otimes E(\delta b_i \mid i \not\equiv 0 \pmod{p^k - 1}, k \geq 1)$$

and the map from $H_*(BP\langle 2\rangle)$ is given by the canonical quotient

$$H_*(BP\langle 2\rangle) \cong P(\xi_1, \xi_2, \ldots) \otimes E(\tau_3, \tau_4, \ldots) \to E(\tau_3, \tau_4, \ldots)$$

tensored with the unit map

$$\mathbb{F}_p \to E(\delta b_i \mid i \not\equiv 0 \pmod{p^k - 1}, k \geq 1).$$

Here $|\delta b_i| = 1 + |b_i|$.

Proof. First note that there is an equivalence of commutative $H\mathbb{F}_p$-algebras

$$H\mathbb{F}_p \wedge_{MU} BP\langle 2\rangle \simeq H\mathbb{F}_p \wedge_{H\mathbb{F}_p \wedge MU} H\mathbb{F}_p \wedge BP\langle 2\rangle.$$

The Künneth spectral sequence has input

$$\begin{aligned} \mathrm{Tor}_*^{H_*MU}(\mathbb{F}_p, H_*BP\langle 2\rangle) &\cong \mathrm{Tor}_*^{P(\bar{\xi}_1, \bar{\xi}_2, \ldots)}(\mathbb{F}_p, H_*BP\langle 2\rangle) \otimes \mathrm{Tor}^{P(b_i : i \not\equiv 0 \bmod p^k - 1, k \geq 1)}(\mathbb{F}_p, \mathbb{F}_p) \\ &= E(\tau_3, \tau_4, \ldots) \otimes E(\delta b_i : i \not\equiv 0 \bmod p^k - 1, k \geq 1) \end{aligned}$$

and since MU is a commutative ring spectrum, it is a multiplicative spectral sequence. The Künneth spectral sequence collapses because all the algebra generators are in filtration $0, 1$ and the differentials shift filtration by at least 2. By factoring the relevant map as

$$H_*(BP\langle 2\rangle) \to \pi_*(H\mathbb{F}_p \wedge_{BP} BP\langle 2\rangle) \to \pi_*(H\mathbb{F}_p \wedge_{MU} BP\langle 2\rangle)$$

and computing $\pi_*(H\mathbb{F}_p \wedge_{BP} BP\langle 2\rangle) \cong E(\tau_3, \tau_4, \ldots)$ by the same argument, we see that the map is the composite of the canonical quotient with the identity tensored with the unit map as desired. $\square$

Recall from Lemma 2.4 [2] that when $R \to Q$ is a map of E-algebras and M is a $Q \wedge_E R^{\mathrm{op}}$-module, with an $R \wedge_E R^{\mathrm{op}}$-module structure by pullback, then

$$THH_E(R; M) \simeq F_{Q \wedge_E R^{\mathrm{op}}}(Q, M).$$

Lemma 5.3. *The following hold:*

(1) *There is an isomorphism of rings*

$$THH^*_{MU}(BP\langle 2\rangle; H\mathbb{F}_p) \cong P(e_i \mid i \geq 3) \otimes P(g_i \mid i \not\equiv 0 \pmod{p^k - 1}, k \geq 1)$$

where $|e_i| = |\tau_i| + 1$ and $|g_i| = 2i + 2$.

(2) *Consequently, $THC^*_{MU}(BP\langle 2\rangle)$ is isomorphic to*

$$BP\langle 2\rangle_*(e_i \mid i \geq 3) \otimes P(g_i \mid i \not\equiv 0 \pmod{p^k - 1}, k \geq 1).$$

(3) *The map*

$$THH^*_{MU}(BP\langle 2\rangle) \to THH^*_{MU}(BP\langle 2\rangle; H\mathbb{F}_p)$$

is induced by the quotient by (p, v_1, v_2).

(4) *The map*

$$\mathrm{THH}^*_{MU}(BP\langle 2\rangle; H\mathbb{F}_p) \to \mathrm{THH}^*_{S}(BP\langle 2\rangle; H\mathbb{F}_p)$$

sends e_i *to* $c_{p^{i-3}}$ *for* $i \geq 3$.

(5) *Consequently, the elements* $c_{p^{i-3}}$ *pull back to elements in* $THC^*(BP\langle 2\rangle)$.

Proof. From the setup before this lemma, we may consider the UCSS computing

$$\mathrm{THH}_E(R; M) \simeq F_{Q\wedge_E R^{\mathrm{op}}}(Q, M)$$

with input

$$\mathrm{Ext}^{*,*}_{\pi_*(Q\wedge_E R^{\mathrm{op}})}(Q_*, M_*.)$$

When $E = MU$, $R = BP\langle 2\rangle$ and $M = Q = H\mathbb{F}_p$. The UCSS computing

$$\mathrm{THH}^*_{MU}(BP\langle 2\rangle; H\mathbb{F}_p)$$

has input

$$\mathrm{Ext}^*_{\pi_* H\mathbb{F}_p \wedge_{MU} BP\langle 2\rangle}(\mathbb{F}_p, \mathbb{F}_p) \cong P(e_i \mid i \geq 3) \otimes P(g_i \mid i \not\equiv 0 \pmod{p^k - 1}, k \geq 1)$$

where $|e_i| = 2p^i$ and $|g_i| = 2i + 2$. Note that by Tor duality and Koszul duality

$$\mathrm{Tor}^{P(b)}_*(\mathbb{F}_p, \mathbb{F}_p) \cong E(\delta b_i)$$

where $|\delta b_i| = |b_i| + 1$ and

$$\mathrm{Ext}^*_{E(\delta b_i)}(\mathbb{F}_p, \mathbb{F}_p) \cong P(g_i)$$

where $|g_i| = |b_i| + 2$ for all i. Since all elements are in even total degree there is no room for differentials and the spectral sequence collapses. This proves the first statement.

There are three Bockstein spectral sequences to go from $\mathrm{THH}^*_{MU}(BP\langle 2\rangle; H\mathbb{F}_p)$ to $\mathrm{THH}^*_{MU}(BP\langle 2\rangle)$, but in each case all elements are in even columns and the spectral sequences collapse since there is an Adams style differential convention. This proves the second statement and the third statement.

Now, by the commutative diagram

$$\begin{tikzcd}
\mathrm{THH}^*_{MU}(BP\langle 2\rangle) \arrow[r] \arrow[d] & \mathrm{THH}^*_{MU}(BP\langle 2\rangle; H\mathbb{F}_p) \arrow[d] \\
\mathrm{THH}^*_{S}(BP\langle 2\rangle) \arrow[r] & \mathrm{THH}^*_{S}(BP\langle 2\rangle; H\mathbb{F}_p)
\end{tikzcd}$$

the fifth statement follows from the fourth statement. It therefore remains to show that the map

$$\mathrm{THH}^*_{MU}(BP\langle 2\rangle; H\mathbb{F}_p) \to \mathrm{THH}^*_{S}(BP\langle 2\rangle; H\mathbb{F}_p)$$

sends e_i to $c_{p^{i-3}}$ for $i \geq 3$. Recall the map $H_*(BP\langle 2\rangle) \to \pi_*\left(H\mathbb{F}_p \wedge_{MU} BP\langle 2\rangle\right)$ sends τ_i to τ_i for $i \geq 3$. Tracing this through the induced map of universal coefficient spectral sequences produces the desired result. $\square$

5.2. **Computation of the cap product.** Next we determine the p-torsion of c_k in

$$\mathrm{THH}_S^*(BP\langle 2\rangle; H\mathbb{Z}_{(p)}).$$

Our first approach will be to compute the UCSS

$$\mathrm{Ext}^{*,*}_{\mathbb{Z}_{(p)}}(\mathrm{THH}_*(BP\langle 2\rangle;\mathbb{Z}_{(p)}),\mathbb{Z}_{(p)}) \Rightarrow \mathrm{THH}_S^*(BP\langle 2\rangle; H\mathbb{Z}_{(p)})$$

which collapses to the zero and one line since $\mathbb{Z}_{(p)}$ is a PID. Additionally for bidegree reasons there are no possible additive extensions.

Recall that there is an isomorphism

$$\mathrm{THH}_*(BP\langle 2\rangle;\mathbb{Z}_{(p)}) \cong F_0 \oplus T_0$$

where F_0 is the free graded $\mathbb{Z}_{(p)}$-module

$$F_0 = E_{\mathbb{Z}_{(p)}}(\lambda_1,\lambda_2)$$

and T_0 is the graded torsion $\mathbb{Z}_{(p)}$-module

$$T_0 = \mathbb{Z}_{(p)}\{c_i^{(k)}, d_i^{(k)} : k=1,2; i\geq 1\}/(p^{\nu_p(k)+1}c_i^{(k)} = p^{\nu_p(k)+1}d_i^{(k)} = 0)$$

where $|c_i^{(1)}| = 2p^3(i+1)-1$, $|c_i^{(2)}| = 2p^3(i+1)+2p-2$, $|d_i^{(1)}| = 2p^3(i+1)+2p^2-2$ and $|d_i^{(2)}| = 2p^3(i+1)+2p^2+2p-3$. We therefore observe the following lemma.

Lemma 5.4. *There is an isomorphism of graded $\mathbb{Z}_{(p)}$-modules*

$$\mathrm{THH}_S^*(BP\langle 2\rangle; H\mathbb{Z}_{(p)}) \cong E(x_1,x_2) \oplus \left(Ext^{1,*}_{\mathbb{Z}_{(p)}}(T_0,\mathbb{Z}_{(p)})\right)_{*+1}$$

where

$$Ext^{1,*}_{\mathbb{Z}_{(p)}}(T_0,\mathbb{Z}_{(p)}) \cong \mathbb{Z}_{(p)}\{\delta c_i^{(k)}, \delta d_i^{(k)} : k=1,2; i\geq 1\}/(p^{\nu_p(k)+1}c_i^{(k)} = p^{\nu_p(k)+1}d_i^{(k)} = 0)$$

so $|\delta c_i^{(k)}| = |c_i^{(k)}|+1$ and $|\delta d_i^{(k)}| = |d_i^{(k)}|+1$ and x_i is the $\mathbb{Z}_{(p)}$-linear dual of λ_i.

Lemma 5.5. *There is an isomorphism of $\mathbb{Z}_{(p)}$ Hopf algebras*

$$\mathrm{THH}_S^*(BP\langle 2\rangle; H\mathbb{Z}_{(p)}) \cong E_{\mathbb{Z}_{(p)}}(x_1,x_2)\otimes \Gamma_{\mathbb{Z}_{(p)}}(c_1)/(pc_1).$$

Proof. In general, if R is a commutative ring spectrum and $H\mathbb{Z}_{(p)}$ is a commutative R-algebra, then $\mathrm{THH}^S_*(R, H\mathbb{Z}_{(p)})$ is a $\mathbb{Z}_{(p)}$ Hopf-algebra spectrum whenever the $\mathbb{Z}_{(p)}$-modules $THH_k(R;H\mathbb{Z}_{(p)})$ are a finitely generated for all k. By Corollary 2.13 and Lemma 5.4, $\mathrm{THH}^S_*(BP\langle 2\rangle; H\mathbb{Z}_{(p)})$ is a finitely generated $\mathbb{Z}_{(p)}$-algebra in each degree. Also, since $THH_S^*(BP\langle 2\rangle; H\mathbb{Z}_{(p)})$ is finitely generated in each degree and the Bockstein spectral sequence

$$THH_S^*(BP\langle 2\rangle; H\mathbb{F}_p)[v_0] \Rightarrow THH_S^*(BP\langle 2\rangle; H\mathbb{Z}_{(p)})_p$$

converges. In order for the isomorphism

$$\mathrm{Ext}^*_{\mathbb{Z}_{(p)}}(\mathrm{THH}^S_*(BP\langle 2\rangle; H\mathbb{Z}_{(p)});\mathbb{Z}_{(p)}) \cong \mathrm{THH}_S^*(BP\langle 2\rangle; H\mathbb{Z}_{(p)})$$

to hold the differentials

$$d_{i+1}(c_{p^i-1}x_3) \doteq v_0^{i+1}c_{p^i}$$

are forced for $i \geq 0$ where $c_0 = 1$ by convention. It is also clear from the isomorphism

$$\mathrm{Ext}^*_{\mathbb{Z}_{(p)}}(\mathrm{THH}^S_*(BP\langle 2\rangle; H\mathbb{Z}_{(p)}); \mathbb{Z}_{(p)}) \cong \mathrm{THH}^*_S(BP\langle 2\rangle; H\mathbb{Z}_{(p)})$$

that the additive extensions are as stated.

To resolve multiplicative extensions, we also need to know, for example, that $p!c_p = c_1^p$. In this case, we know that $c_1^p = 0$ in the E_∞-page so, if it is nonzero in the abutment, it must be in higher filtration. The only element in higher filtration that it could be in this degree is then the element detected by $v_0 c_p$, which is $p!c_p$ (up to multiplication by a unit) by the additive extension that we already determined. We therefore, just need to show that c_1^p is nonzero.

[Gabe: Hmm... thought I could fix this, but still not sure how to resolve multiplicative extensions.]

□

Recall that there is a cap product

$$THH^k_S(BP\langle 2\rangle) \otimes THH^S_m(BP\langle 2\rangle; H\mathbb{Z}_{(p)}) \to THH^S_{m-k}(BP\langle 2\rangle; H\mathbb{Z}_{(p)})$$

and we showed that the elements c_k lift to elements in $THH^*_S(BP\langle 2\rangle)$.

[Gabe: In [2], they claim that they lift to torsion free elements. Do we need this? Can we prove this here if we do?]

We now remark on two facts that will be useful for computing the cap product.

Lemma 5.6. *The cap product commutes with scalars so that*

$$c_k \cap (\alpha \cdot x) = \alpha \cdot (c_k \cap x)$$

and the the cap product $c_k \cap -$ induces a map of Bockstein spectral sequences

$$\begin{tikzcd} THH^S_*(BP\langle 2\rangle; H\mathbb{F}_p)[v_0] \arrow[r, "c_k \cap -"] \arrow[d, Rightarrow] & THH^S_*(BP\langle 2\rangle; H\mathbb{F}_p)[v_0] \arrow[d, Rightarrow] \\ THH^S_*(BP\langle 2\rangle; H\mathbb{Z}_{(p)}) \arrow[r, "c_k \cap -"] & THH^S_*(BP\langle 2\rangle; H\mathbb{Z}_{(p)}) \end{tikzcd}$$

which is compatible with the edge homomorphism

$$\begin{tikzcd} THH_*(BP\langle 2\rangle; H\mathbb{Z}_{(p)}) \arrow[r, "c_k \cap -"] \arrow[d] & THH_*(BP\langle 2\rangle; H\mathbb{Z}_{(p)}) \arrow[d] \\ THH_*(BP\langle 2\rangle; H\mathbb{F}_p) \arrow[r] & THH_*(BP\langle 2\rangle; H\mathbb{F}_p) \end{tikzcd}$$

where the edge homomorphism in the Bockstein spectral sequence is simply the canonical quotient map given by reduction modulo p.

Proof. The fact that the cap product commutes with scalars is clear by construction. Since the cap product is defined by sending a $R \wedge R^{\mathrm{op}}$-linear map $f\colon R \to R$ to the map $M \wedge_{R \wedge R^{\mathrm{op}}} f$, it is clear that capping with class induces a map of Bockstein spectral sequences and the last fact is a direct consequence. □

Corollary 5.7. *For $k < n$, the cap product satisfies the following formulae*

$$c_k \cap c_n^{(m)} \doteq p^{\gamma(n,k)} c_{n-k}^{(m)}$$

$$c_k \cap d_n^{(m)} \doteq p^{\gamma(n,k)} d_{n-k}^{(m)}$$

for $1 \leq m \leq 2$ where $\gamma(n,k) = \min\{\nu_p(n) - \nu_p(n-k), 0\}$.

Proof. We showed that the classes $c_n^{(k)}$ and $d_n^{(k)}$ in $THH_*(BP\langle 2\rangle; H\mathbb{Z}_{(p)})$ are each $p^{\nu_p(n)+1}$-torsion for $k = 1, 2$. We will just prove the case for $c_n^{(1)}$ since the other cases are exactly the same. Since $c_n^{(1)}$ maps to $\lambda_3\mu_3^{n-1}$ in $THH_*(BP\langle 2\rangle; H\mathbb{F}_p)$ via the edge homomorphism and $c_k \cap \lambda_3\mu_3^{n-1} = \lambda_3\mu_3^{n-k-1}$ we see that observe that

$$c_k \cap c_n^{(1)} = \beta \cdot c_{n-k}^{(1)}$$

for some $\beta \in \mathbb{Z}_{(p)}$.

By Lemma 5.6, it is clear that

$$c_k \cap (p^{\nu_p(n)+1} \cdot c_n^{(1)}) = p^{\nu_p(n)+1} \cdot (c_k \cap c_n^1) = 0.$$

Therefore, $p^{\nu_p(n)+1}\beta \cdot c_{n-k}^{(1)} = 0$,

[Gabe: There is a problem here. How do we know β itself isn't zero then, for example?]

so since $p^{\nu_p(n-k)+1} c_{n-1}^{(1)} = 0$, we know that $p^{\nu_p(n-k)+1} \cdot \beta \equiv 0 \mod p^{\nu_p(n)+1}$, so in other words $\beta \doteq p^{\gamma(n,k)}$ as desired. □

6. Remaining two Bockstein spectral sequences

We now compute the remaining two Bockstein spectral sequences

$$THH_*(BP\langle 2\rangle; H\mathbb{Z}_{(p)})[v_1] \Rightarrow THH_*(BP\langle 2\rangle; BP\langle 1\rangle) \tag{6.1}$$

and

$$THH_*(BP\langle 2\rangle; k(1))[v_0] \Rightarrow THH_*(BP\langle 2\rangle; BP\langle 1\rangle)_p. \tag{6.2}$$

First, we claim that the spectral sequence (6.1) is multiplicative because it is equivalent to a multiplicative relative Adams spectral sequence. To see this, recall that

$$\pi_*(H\mathbb{Z}_p \wedge_{BP\langle 1\rangle} H\mathbb{Z}_p) \cong E_{\mathbb{Z}_p}(\tau_1)$$

and

$$\pi_*(H\mathbb{Z}_p \wedge_{BP\langle 1\rangle} BP\langle 1\rangle \wedge_{BP\langle 2\rangle} THH(BP\langle 2\rangle) \cong \pi_* THH(BP\langle 2\rangle; \mathbb{Z}_p)$$

so the relative Adams spectral sequence with signature

$$\mathrm{Ext}^{*,*}_{\pi_*(\mathbb{Z}_p \wedge_{BP\langle 1\rangle} \mathbb{Z}_p)}(\mathbb{Z}_p, \pi_*(THH(BP\langle 2\rangle; H\mathbb{Z}_p))) \Rightarrow THH_*(BP\langle 2\rangle, BP\langle 1\rangle)$$

is isomorphic to the spectral sequence (6.1). The spectral sequence (6.2), however is not known to be multiplicative so we will just use it as a comparison tool.

We recall that the input (6.1) is

$$\left(E_{\mathbb{Z}_{(p)}}(\lambda_1, \lambda_2) \oplus \mathbb{Z}_{(p)}\{c_i^{(k)}, d_i^{(k)} : i \geq 1, k = 1, 2\}/\sim\right) \otimes P_{\mathbb{Z}_{(p)}}(v_1)$$

where $p^{\nu_p(i)}c_i^{(k)} \sim p^{\nu_p(i)}d_i^{(k)} \sim 0$ for all $i \geq 1, k = 0, 1$. See the appendix for a table of bidegrees of these elements. (Add this table to an appendix.) We immediately conclude that the v_1-towers on 1,λ_1, and λ_2 are permanent cycles.

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