THH OF $BP\langle 2\rangle$ WITH COEFFICIENTS IN $k(2)$

G. ANGELINI-KNOLL ÜND D. CULVER

Let $BP\langle 2\rangle$ be the truncated Brown-Peterson spectrum with coefficients $BP\langle 2\rangle_* \cong \mathbb{Z}_{(p)}[v_1, v_2]$ and let $k(2)$ be connective Morava K-theory with coefficients $k(2)_* \cong \mathbb{F}_p[v_2^{\pm 1}]$. At the moment, we will let p be any prime number. Whenever we assume that there is a model for $BP\langle 2\rangle$ that is E_∞ we will assume that $p = 2$ or 3.

The goal is to compute $THH(BP\langle 2\rangle; k(2))$ via the Bockstein spectral sequence

$$THH_*(BP\langle 2\rangle; H\mathbb{F}_p)[v_2] \Rightarrow THH(BP\langle 2\rangle; k(2)).$$

The first goal is to show

$$K(2)_* \cong THH_*(BP\langle 2\rangle; K(2)) \tag{0.1}$$

which will imply that all the classes except the classes in the subalgebra $P(v_2) \subset THH_*(BP\langle 2\rangle; H\mathbb{F}_p)[v_2]$ are v_2-torsion and this will force differentials in the spectral sequence. Our approach is entirely analogous to the calculation of McClure-Staffeldt except for one minor difference, which we will point out.

To compute $THH_*(BP\langle 2\rangle; K(2))$, we can first compute

$$K(2)_* THH(BP\langle 2\rangle; K(2))$$

and then use the fact that $THH(BP\langle 2\rangle; K(2))$ is a free $K(2)$-module (since $K(2)$ is a field spectrum) and the collapse of the $K(2)$-based Adams spectral sequence to finish the computation.

We use the $K(2)$-based Bökstedt spectral sequence to compute $K(2)_* THH(BP\langle 2\rangle; K(2))$; i.e. the spectral sequence

$$HH_*^{K(2)_*}(K(2)_* BP\langle 2\rangle; K(2)) \Rightarrow K(2)_* THH(BP\langle 2\rangle; K(2)).$$

the first goal will be to compute the input.

Lemma 0.2. *There is an isomorphism of graded rings*

$$K(n)_* BP\langle n\rangle \cong K(n)_*[t_1, t_2, \dots]/(v_n t_k^{p^n} - v_n^{p^k} t_k | k \geq 1).$$

Proof. We adapt the proof in McClure-Staffeldt. First $K(n)_* BP \cong K(n)_* \otimes_{BP_*} BP_* BP$ because BP is Landweber exact. Furthermore, $K(n)_* \otimes_{BP_*} BP_* BP \cong K(n)_*[t_1, t_2, \dots]$ and we can restrict $\eta_R : BP_* \to BP_* BP$ to $K(n)_* \otimes_{BP_*} BP_* BP$ to produce the map $\bar{\eta}_R$ and by Ravenel

$$\bar{\eta}_R(v_{n+k}) = v_n t_k^{p^n} - v_n^{p^k} t_k \text{ mod } (\bar{\eta}_R(v_{n+1}), \bar{\eta}_R(v_{n+2}), \dots \bar{\eta}_R(\dots, v_{n+k-1})) \tag{0.3}$$

We can then construct $BP\langle n\rangle$ using Baas-Sullivan theory and the effect is that

$$K(n)_*BP\langle n\rangle \cong K(n)_* \otimes_{BP_*} BP_*BP/(\bar{\eta}_R(v_{n+1}), \bar{\eta}_R(v_{n+2}), \dots)$$

or in other words, by (0.3)

$$K(n)_*BP\langle n\rangle \cong K(n)_*[t_1, t_2, \dots]/(v_n t_k^{p^n} - v_n^{p^k} t_k | k \geq 1)$$

as desired. $\square$

Remark 0.4. *Note that the model of $BP\langle 2\rangle$ that we used in this lemma is not the same model that gives you an E_∞-structure, but since there is a weak equivalence $BP\langle 2\rangle \simeq tmf_1(3)$ the map $K(n)_*BP\langle 2\rangle \cong K(n)_*tmf_1(3)$ as $K(n)_*$-modules. (Is the equivalence $BP\langle 2\rangle \simeq tmf_1(3)$ known to be an equivalence of E_2-algebras? Does this reasoning make sense to you?)*

We now describe the structure of $K(2)_*[t_1, t_2, \dots]/(v_2 t_k^{p^2} - v_2^{p^k} t_k | k \geq 1)$. Note that it can be written as

$$K(2)_*[t_1, t_2, \dots]/(v_2 t_k^{p^2} - v_2^{p^k} t_k | k \geq 1) \cong \bigotimes_{k\geq 1} K(2)_*[t_k]/(v_2 t_k^{p^2} - v_2^{p^k} t_k) \tag{0.5}$$

where the tensor is taken over $K(2)_*$. Note that $|t_k| = 2p^k - 2$ and $2p^2 - 2 | 2p^k - 2$ when $2|k$ and $2p - 2 | 2p^k - 2$ for all k. Therefore,

$$K(2)_*[t_k]/(v_2 t_k^{p^2} - v_2^{p^k} t_k) \cong K(2)_* \otimes \mathbb{F}_p[u_k]/(u_k^{p^2} - v_2^{p^k - 1} u_k]$$

where $u_k = t_k v_2^{m(k)}$ where $m(k) = -p^{k-2} - p^{k-4} - \dots p^2 - 1$ when $2|k$ and

$$K(2)_*[t_k]/(v_2 t_k^{p^2} - v_2^{p^k} t_k) \cong K(2)_* \otimes \mathbb{F}_p[w_k]/(w_k^{p^2} - v_2^{p^k - 1} w_k]$$

where $w_k = t_k v_2^{\ell(k)}$ where $\ell(k) = -p^{k-2} - p^{k-4} - \dots - p$ and k is odd so that $|w_k| = 2p - 2$.

Lemma 0.6. *There is an isomorphism*

$$K(2)_*K(2) \cong HH_*^{K(2)_*}(K(2)_*BP\langle 2\rangle; K(2)_*K(2))$$

and hence the $K(2)_*$-based Adams spectral sequence collapses with no room for hidden extensions and the natural map

$$K(2)_*K(2) \to K(2)_*THH(BP\langle 2\rangle; K(2))$$

is an isomorphism

Proof. Since $K(2)_*BP\langle 2\rangle$ is flat over $K(2)_*$

$$HH_*^{K(2)_*}(K(2)_*BP\langle 2\rangle) \cong Tor_*^{K(2)_*BP\langle 2\rangle \otimes_{K(2)_*} K(2)_*BP\langle 2\rangle}(K(2)_*BP\langle 2\rangle; K(2)_*BP\langle 2\rangle).$$

Also, by (0.5),

$$Tor_*^{(K(2)_*BP\langle 2\rangle)^e}(K(2)_*BP\langle 2\rangle; K(2)_*BP\langle 2\rangle) \cong$$
$$\bigotimes_{k\geq 1} Tor_*^{K(2)_*[t_k]/(v_2t_k^{p^2}-v_2^{p^k}t_k))^e}(K(2)_*[t_k]/(v_2t_k^{p^2}-v_2^{p^k}t_k); K(2)_*[t_k]/(v_2t_k^{p^2}-v_2^{p^k}t_k)) \cong$$
$$\bigotimes_{k\geq 1; k|2} K(2)_*\otimes HH_*^{K(2)_*}(K(2)_*[u_k]/(v_2u_k^{p^2}-v_2^{p^k}u_k))\otimes$$
$$\bigotimes_{k\geq 1; (k+1)|2} K(2)_*\otimes HH_*^{K(2)_*}(K(2)_*[w_k]/(v_2w_k^{p^2}-v_2^{p^k}w_k))$$

By Cartan-Eilenberg, for $k\geq 0$ an odd integer

$$HH_*^{K(2)_*}(K(2)_*[w_k]/(v_2w_k^{p^2}-v_2^{p^k}w_k) \cong K(2)_*[w_k]/(v_2w_k^{p^2}-v_2^{p^k}w_k)\otimes_{K(2)_*} Tor^{K(2)_*K(2)_*[w_k]/(v_2w_k^{p^2}-v_2^{p^k}w_k}(K$$

and by an elementary calculation,

$$Tor^{K(2)_*K(2)_*[w_k]/(v_2w_k^{p^2}-v_2^{p^k}w_k}(K(2)_*, K(2)_*) \cong K(2)_*$$

and therefore

$$HH_*^{K(2)_*}(K(2)_*[w_k]/(v_2w_k^{p^2}-v_2^{p^k}w_k) \cong K(2)_*[w_k]/(v_2w_k^{p^2}-v_2^{p^k}w_k).$$

Also, there is an isomorphism

$$HH_*^{K(2)_*}(K(2)_*[u_k]/(v_2u_k^{p^2}-v_2^{p^k}u_k)) \cong K(2)_*\otimes HH_*(\mathbb{F}_p[u_k]/(u_k^{p^2}-u_k))$$

and since

$$\mathbb{F}_p[u_k]/(u_k^{p^2}-u_k$$

is isomorphic as a $\mathbb{F}_p$-algebra to a product of finite field extensions of $\mathbb{F}_p$ (We should be more precise here.) and since Hochschild homology commutes with limits and $HH_*(\mathbb{F}_{p^n})\cong\mathbb{F}_{p^n})$,

$$HH_*(\mathbb{F}_p[u_k]/(u_k^{p^2}-u_k)) \cong \mathbb{F}_p[u_k]/(u_k^{p^2}-u_k).$$

Putting this all together, we produce an isomorphism

$$K(2)_*BP\langle 2\rangle \cong HH_*^{K(2)_*}(K(2)_*BP\langle 2\rangle)$$

and since

$$HH_*^{K(2)_*}(K(2)_*BP\langle 2\rangle; K(2)_*K(2)) \cong K(2)_*K(2)\otimes_{K(2)_*BP\langle 2\rangle} HH_*^{K(2)_*}(K(2)_*BP\langle 2\rangle)$$

we produce the desired isomorphism

$$K(2)_*K(2) \cong HH_*^{K(2)_*}(K(2)_*BP\langle 2\rangle; K(2)_*K(2))$$

The Bökstedt spectral sequence

$$HH_*^{K(2)_*}(K(2)_*BP\langle 2\rangle; K(2)_*K(2)) \Rightarrow K(2)_*THH(BP\langle 2\rangle; K(2))$$

therefore collapses with no room for hidden extensions and hence the map

$$K(2)\to THH(BP\langle 2\rangle; K(2))$$

induces a $K(2)_*$-equivalence. □

Corollary 0.7. *The map $K(2) \to THH(BP\langle 2\rangle; K(2))$ is a weak equivalence and therefore*

$$THH_*(BP\langle 2\rangle; k(2)) \cong P(v_2) \otimes T$$

where T is a v_2-torsion $P(v_2)$-module.

Proof. Since the map $K(2) \to THH(BP\langle 2\rangle; K(2))$ induces an isomorphism $K(2)_*K(2) \cong K(2)_*THH(BP\langle 2\rangle; K(2))$, the $K(2)$-based Adams spectral sequence for $THH(BP\langle 2\rangle; K(2))$ converges and collapses to the zero line and the map of $K(2)$-based Adams spectral sequences induces an isomorphism

$$K(2)_* \to THH_*(BP\langle 2\rangle; K(2)).$$

Since we have a map that induces an isomorphism on homotopy groups the Whitehead theorem for spectra implies that the map $K(2) \to THH(BP\langle 2\rangle; K(2))$ is a weak equivalence.

Alternatively, we could compute $THH_*(BP\langle 2\rangle; K(2))$ using the v_2-inverted classical Adams spectral sequence, which is equivalent to the Bockstein spectral sequence

$$THH_*(BP\langle 2\rangle; H\mathbb{F}_p)[v_2^{\pm 1}] \Rightarrow THH_*(BP\langle 2\rangle; K(2))$$

and by the computation we just did, we know that all the classes must die except those in $P(v_2^{\pm 1})$. There is also a map of spectral sequences

$$\begin{array}{ccc} THH_*(BP\langle 2\rangle; H\mathbb{F}_p)[v_2^{\pm 1}] & \Longrightarrow & THH_*(BP\langle 2\rangle; K(2)) \\ \uparrow & & \uparrow \\ THH_*(BP\langle 2\rangle; H\mathbb{F}_p)[v_2] & \Longrightarrow & THH_*(BP\langle 2\rangle; k(2)) \end{array}$$

because $v_2^{-1}(-)$ is a localization. This implies that

$$THH_*(BP\langle 2\rangle; k(2)) \cong P(v_2) \otimes T$$

and forces differentials in the bottom spectral sequence above. □

Corollary 0.8. *There are differentials $d_{r(n)}(\mu^{r(n)}) = \lambda_{[n]} v_2^{r(n)}$ where $r(n)$ is . . . , and $\lambda_{[n]}$ is . . . .*

Finish the corollary above.

References

University of Illinois, Urbana-Champaign

E-mail address: dculver@nd.edu

Michigan State University, East Lansing

E-mail address: angelini@math.msu.edu