TOPOLOGICAL HOCHSCHILD HOMOLOGY OF tmf WITH COEFFICIENTS IN $k(2)$

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1. Introduction

The purpose of this short document is to carry out a computation of $\mathrm{THH}(\mathrm{tmf};k(2))$ at the prime $p=2$. We take advantage of the fact that Bhattacharya-Egger have recently constructed a finite spectrum Z with the property that it has a v_2^1-self map and gives an equivalence

$$\mathrm{tmf}\wedge Z\simeq k(2),$$

where $k(2)$ denotes the connective second Morava K-theory.

2. Calculations

Let's start the calculation. We begin by computing the $K(2)$-homology of tmf. After that we run the v_2-Bockstein spectral sequence.

2.1. $K(2)_*\mathrm{tmf}$. There are two steps to this calculation. We start by computing the Adams spectral sequence for $k(2)_*\mathrm{tmf}$, and then invert v_2. This gives an associated graded calculation of $K(2)_*\mathrm{tmf}$. Then, to resolve hidden extensions, we determine the map

$$K(2)_*\mathrm{tmf}\to K(2)_*BP\langle 2\rangle,$$

and use the known calculation of $K(2)_*BP\langle 2\rangle$.

The Adams spectral sequence for $k(2)\wedge\mathrm{tmf}$ takes the form

$$\mathrm{Ext}_{A_*}(k(2)\wedge\mathrm{tmf})\implies k(2)_*\mathrm{tmf}.$$

A change-of-rings isomorphism allows us to express the E_2-page as

$$E_2\cong\mathrm{Ext}_{E(Q_2)}(H_*\mathrm{tmf})\cong(P(v_2)\otimes M_*(\mathrm{tmf};Q_2))\oplus(v_2\text{-torsion}).$$

Recall that the mod 2 homology of tmf is

$$H_*(\mathrm{tmf})\cong A/\!\!/A(2)_*\cong P(\zeta_1^8,\zeta_2^4,\zeta_3^2,\zeta_4,\ldots).$$

The Q_2-action on $H_*\mathrm{tmf}$ is

$$Q_2(\zeta_k)=\zeta_{k-3}^8.$$

Thus the Margolis homology is

$$M_*(\mathrm{tmf}; Q_2) \cong P(\zeta_2^4, \zeta_3^2, \zeta_4^2, \zeta_5^2, \ldots)/(\zeta_2^8, \zeta_3^8, \zeta_4^8, \ldots).$$

It follows from this that the Adams spectral sequence collapses immediately.

Inverting v_2 kills the v_2-torsion, and so we get that

$$v_2^{-1}\operatorname{Ext}_{A_*}(k(2) \wedge \mathrm{tmf}) \cong K(2)_* \otimes P(\zeta_2^4, \zeta_3^2, \zeta_4^2, \zeta_5^2, \ldots)/(\zeta_2^8, \zeta_3^8, \zeta_4^8, \ldots).$$

Now onto the hidden extensions. First, note that there is a map of E_∞-rings[1]

$$\mathrm{tmf} \to \mathrm{tmf}_1(3).$$

A similar analysis with the ASS shows that the E_∞-page for $K(2) \wedge \mathrm{tmf}$ is

$$K(2)_* \otimes P(\zeta_1^2, \zeta_2^2, \zeta_3^2, \ldots)/(\zeta_1^8, \zeta_2^8, \zeta_3^8, \ldots).$$

The morphism from tmf to $\mathrm{tmf}_1(3)$ induces the obvious map on v_2-inverted Ext groups. Thus we know the morphism

$$K(2)_*\mathrm{tmf} \to K(2)_*\mathrm{tmf}_1(3)$$

up to associated graded.

We also need to use the fact that $\mathrm{tmf}_1(3)$ is a *form* of $BP\langle 2\rangle$: that is there is (canonical?) equivalence

$$BP\langle 2\rangle \simeq \mathrm{tmf}_1(3)$$

after completing at 2. Since localizing with respect to $K(2)$ also p-completes, we have that

$$L_{K(2)}BP\langle 2\rangle \simeq L_{K(2)}\mathrm{tmf}_1(3),$$

and hence

$$K(2)_*(BP\langle 2\rangle) \cong K(2)_*(\mathrm{tmf}_1(3)).$$

We need to recall the following computation.

Theorem 2.1. *There is an isomorphism of graded rings*

$$K(2)_*BP\langle 2\rangle \cong K(2)_*[t_1, t_2, \ldots]/(v_2 t_k^4 - v_2^{2^k} t_k \mid k \geq 1)$$

[1]This map is induced by taking global sections associated to the map of moduli stacks $\mathscr{M}_1(3) \to \mathscr{M}_{\mathrm{ell}}$ from the moduli of elliptic curves with level $\Gamma_1(3)$ to the moduli of all elliptic curves.

We relate this calculation to the one we obtained through the Adams spectral sequence. First, note there is a canonical map of commutative ring spectra

$$BP \to BP\langle 2\rangle,$$

which leads to a homomorphism

$$BP_*BP \to H_*\mathrm{tmf}_1(3)$$

sending t_i to ζ_i^2. Since ζ_i^2 is sent to ζ_i^2 under

$$v_2^{-1}\operatorname{Ext}_A(k(2)\wedge \mathrm{tmf}) \to v_2^{-1}\operatorname{Ext}_A(k(2)\wedge \mathrm{tmf}_1(3)),$$

we obtain the following.

Proposition 2.2. *The $K(2)$-homology of* tmf *is given by*

$$K(2)_*\mathrm{tmf} \cong K(2)_*[t_2^2, t_3, t_4, \dots]/()$$

I am not sure what the relation on t_2^2 is supposed to be...

References

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