TOPOLOGICAL HOCHSCHILD HOMOLOGY OF THE SECOND TRUNCATED BROWN-PETERSON SPECTRUM III

GABE ANGELINI-KNOLL AND DOMINIC LEON CULVER

Abstract. We complete our computations of topological Hochschild homology of the second truncated Brown-Peterson spectrum $BP\langle 2\rangle$ at the primes $2, 3$ with coefficients in $BP\langle 1\rangle$. At the prime $p = 2$ we use the model for $BP\langle 2\rangle$ constructed by Lawson-Naumann [2] using topological modular forms equipped with a $\Gamma_1(3)$-structure and at $p = 3$ we use the model for $BP\langle 2\rangle$ constructed using a Shimura curve of discriminant 14 due to Hill-Lawson [1]. A key part of the proof involves constructing a structured Postnikov tower and a structured Whitehead tower for $BP\langle 2\rangle$ with filtration quotients in ℓ-modules, which we believe should have independent interest.

Contents

1. An E_∞-Postnikov tower for $BP\langle 2\rangle$ constructed from ℓ-modules

In [?Basterra], the author constructs an E_∞-Postnikov tower out of Eilenberg-Maclane spectra after unpublished work of Kriz. Inspired by this work, we construct an explicit E_∞-Postnikov tower for $BP\langle 2\rangle$ out of ℓ-modules. We first describe the less structured version, which is significantly easier to construct and probably well known.

Theorem 1.1. *There exists a relative Postnkov tower $P_\bullet$ for $BP\langle 2\rangle$ of the form*

$$\begin{tikzcd}
BP\langle 2\rangle \arrow[dr, "f_2"'] \arrow[ddr, "f_1"'] \arrow[dddr, "f_0"'] & {} \arrow[d] & \\
& P_2 \arrow[r, "k_2"] \arrow[d] & \Sigma^{6p^2-6}\ell \\
& P_1 \arrow[r, "k_1"] \arrow[d] & \Sigma^{4p^2-4}\ell \\
& P_0 = \ell \arrow[r, "k_0"] & \Sigma^{2p^2-2}\ell
\end{tikzcd}$$

such that there are fiber sequences

$$P_i \to P_{i-1} \to \Sigma^{(2p^2-2)i}\ell$$

and there is a weak equivalence $BP\langle 2\rangle \simeq \lim P_k$.

Proof. We begin by considering the homotopy cofiber sequence

$$BP\langle 2\rangle \xrightarrow{f_0} \ell \xrightarrow{j_0} C_0$$

which has the property that $C_1 \simeq \Sigma^{2p^2-2}BP\langle 2\rangle$. We can then compose with this weak equivalence and the map $\Sigma^{2p^2-2}f_0\colon \Sigma^{2p^2-2}BP\langle 2\rangle \longrightarrow \Sigma^{2p^2-2}\ell$ to get a map $\ell \xrightarrow{k_0} \Sigma^{2p^2-2}\ell$ and since the map

$$BP\langle 2\rangle \xrightarrow{f_0} \ell \xrightarrow{k_0} \Sigma^{2p^2-2}\ell$$

factors through $j_0 \circ f_0$, it factors through 0 up to homotopy. We therefore get a map $f_1\colon BP\langle 2\rangle \longrightarrow P_1$ where P_1 is defined to be the homotopy fiber of the map $\ell \xrightarrow{k_0} \Sigma^{2p^2-2}\ell$. We then repeat the same process. Observe that the cofiber C_2 in the homotopy cofiber sequence

$$BP\langle 2\rangle \xrightarrow{f_1} P_1 \xrightarrow{j_1} C_1$$

is homotopy equivalent to $\Sigma^{4p^2-4}BP\langle 2\rangle$. We then define k_1 as $\Sigma^{4p^2-4}f \circ j_2$ and again $f_1 \circ k_1$ factors through 0 up to homotopy so there exists a map $f_2\colon BP\langle 2\rangle \to P_2$ where P_2 is the homotopy fiber of the map $k_1\colon P_1 \to \Sigma^{4p^2-4}\ell$. An easy induction completes the proof of the theorem. $\square$

For the more structured version of the theorem, we will build on work of [?Basterra].

Lemma 1.2 (Lemma 8.2. [?Basterra])**.** *Let B be a connective commutative S-algebra and let $f\colon A \longrightarrow B$ be a map of commutative S-algebras which is an n-equivalence; where $n \geq 1$. Then; $TAQ_B(A)$ is n-connected and $\pi_{n+1}(TAQ_B(A)) \cong \pi_n(A)$.*

We will use the notation $\mathrm{CAlg}_{BP\langle 2\rangle/\ell}$ for the category of commutative $BP\langle 2\rangle$-algebras over ℓ.

Theorem 1.3. *There exists a relative E_∞-Postnkov tower $\mathbb{P}_\bullet$ for $BP\langle 2\rangle$ of the form*

$$\begin{tikzcd}
BP\langle 2\rangle \arrow[dr, "f_2"'] \arrow[ddr, "f_1"'] \arrow[dddr, "f_0"'] & {} \arrow[d] & \\
& \mathbb{P}_2 \arrow[r, "\tilde{k}_2"] \arrow[d] & \Sigma^{6p^2-6}\ell \\
& \mathbb{P}_1 \arrow[r, "\tilde{k}_1"] \arrow[d] & \Sigma^{4p^2-4}\ell \\
& \mathbb{P}_0 = \ell \arrow[r, "\tilde{k}_0"] & \Sigma^{2p^2-2}\ell
\end{tikzcd}$$

such that

(1) *each object $\mathbb{P}_k$ and each map $\mathbb{P}_k \to \mathbb{P}_{k-1}$ is in $\mathrm{CAlg}_{BP\langle 2\rangle/\ell}$,*
(2) *after forgetting to $BP\langle 2\rangle$-modules, there are fiber sequences*

$$\mathbb{P}_i \to \mathbb{P}_{i-1} \to \Sigma^{(2p^2-2)i}\ell,$$

(3) *the map* $BP\langle 2\rangle \simeq \lim \mathbb{P}_k$ *is a weak equivalence in* $\mathrm{CAlg}_{BP\langle 2\rangle/\ell}$.

Proof. We begin by composing the unit map with the map which kills higher homotopy on $TAQ_{BP\langle 2\rangle}(\ell)$ to form

$$\ell \to TAQ_{BP\langle 2\rangle}(\ell) \to \Sigma^{2p^2-2}H\mathbb{Z}_{(p)}$$

however, this is not exactly the map we desire. It is easy to see that this composite is nontrivial and therefore must represent an integral lift of (some unit multiple of) the element $Q_2 \in H\mathbb{F}_p^{2p^2-2}(\ell)$. We then compose with the map

$$\Sigma^{2p^2-2}H\mathbb{Z}_{(p)} \longrightarrow \Sigma^{2p^2-2}TAQ_\ell(H\mathbb{Z}_{(p)}) \longrightarrow \Sigma^{2p^2-2}\Sigma^{2p-2}H\mathbb{Z}_{(p)}$$

which is a suspension of the first E_∞ k-invariant of ℓ. Since this is known to be Q_1, we see that, after post composing with the map $\Sigma^{2p^2-2}\Sigma^{2p-2}H\mathbb{Z}_{(p)} \to \Sigma^{2p^2-2}\Sigma^{2p-2}H\mathbb{F}_p$, the composite is Q_2Q_1. By examination of this relation, we see that

$$Q_2Q_1 = 0 \in H\mathbb{F}_p^{2p^2-2+2p-2}(\ell)$$

and therefore this composite factors through the 0 up to homotopy. We therefore get a factorization

$$\ell \to TAQ_{BP\langle 2\rangle}(\ell) \to \Sigma^{2p^2-2}\tau_{\leq 2p-2}\ell.$$

[Gabe: Warning: This next part is hand wavey and should be made more precise]

We repeat this same process and in each case the relevant relation in $H\mathbb{F}_p^{2p^2-2+(2p-2)k}\ell$ implies that we get maps

$$\ell \to TAQ_{BP\langle 2\rangle}(\ell) \to \Sigma^{2p^2-2}\tau_{\leq (2p-2)k}\ell$$

for all k and consequently a map

$$\tilde{k}_0\colon\ \ell \to TAQ_{BP\langle 2\rangle}(\ell) \to \Sigma^{2p^2-2}\ell$$

which we define to be the first E_∞ k-invariant of our relative E_∞-Postnikov tower. We define $\mathbb{P}_0 = \ell$ and define $\mathbb{P}_1$ to be the homotopy fiber of the map $\mathbb{P}_0 \to \mathbb{P}_0 \vee \Sigma^{2p^2-2}\ell$ in the category $\mathrm{CAlg}_{BP\langle 2\rangle/\ell}$. Since the map $BP\langle 2\rangle \xrightarrow{f_0} \ell \xrightarrow{\tilde{k}_0} \Sigma^{2p^2-2}\ell$ factors through the terminal object in the category of $\mathrm{CAlg}_{BP\langle 2\rangle/\ell}$, there is also a map $f_1\colon\ BP\langle 2\rangle \to \mathbb{P}_1$ in $\mathrm{CAlg}_{BP\langle 2\rangle/\ell}$

[Gabe: Warning: This next part is even more hand wavey and should be made more precise]

We then can do the same process and induct up the tower to define objects $\mathbb{P}_m$ as the homotopy fiber of the map $\mathbb{P}_{m-1} \longrightarrow \mathbb{P}_{m-1} \vee \Sigma^{(2p^2-2)m}\ell$ in the category $\mathrm{CAlg}_{BP\langle 2\rangle/\mathbb{P}_{m-1}}$ along with compatible maps $f_m\colon\ BP\langle 2\rangle \to \mathbb{P}_m$. This completes the theorem. □

Remark 1.4. *We should be able to give similar Postnikov filtrations for* $\mathrm{tmf}_0(3)$ *at the prime* 2 *and* tmf *at large primes.*

References

[1] Michael Hill and Tyler Lawson, *Automorphic forms and cohomology theories on Shimura curves of small discriminant* (200902), available at 0902.2603.

[2] Tyler Lawson and Niko Naumann, *Commutativity conditions for truncated Brown-Peterson spectra of height 2* (201101), available at 1101.3897.

Michigan State University, East Lansing
Email address: angelini@math.msu.edu

University of Illinois, Urbana-Champaign
Email address: dculver@illinois.edu