TOPOLOGICAL HOCHSCHILD HOMOLOGY OF THE SECOND TRUNCATED BROWN-PETERSON SPECTRUM

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Abstract. We compute mod p topological Hochschild homology of the second truncated Brown-Peterson spectrum $BP\langle 2\rangle$ at the prime 3. We use the model for $BP\langle 2\rangle$ constructed using Shimura curves of small discriminant due to Hill-Lawson [7]. This is a first step towards a program of studying potential chromatic height three information in mod p algebraic K-theory of $BP\langle 2\rangle$.

Contents

1. Introduction

The Brown-Peterson spectrum BP is a complex oriented cohomology theory associated to universal p-typical formal group law. The cohomology of the associated Hopf algebroid (BP_*, BP_*BP) is the input for an Adams spectral sequence computing π_*S_p. This spectral sequence lead to significant new computations of the homotopy groups of spheres at odd primes. The coefficients of BP are a polynomial algebra over

$\mathbb{Z}_{(p)}$ on generators v_i for $i \geq 1$. By coning off the regular sequence $(v_{n+1}, v_{n+2}, \ldots)$, one can the n-th truncated Brown-Peterson spectrum $BP\langle n\rangle$ where $BP\langle 0\rangle = H\mathbb{F}_p$, $BP\langle 0\rangle = H\mathbb{Z}_{(p)}$, and $BP\langle 1\rangle$ is the Adams summand ℓ of p-local complex K-theory $ku_{(p)}$. Until the last ten years, no analogous interpretation of $BP\langle 2\rangle$ was known, but then in [?qx] Lawson-Nauman showed that there is an E_∞-model for $BP\langle 2\rangle$ at the prime 2 using topological modular forms with level structure. More recently in [?qx], Hill-Lawson also give an E_∞-model for $BP\langle 2\rangle$ at the prime 3 using spectra associated to Shimura curves of small discriminant. This is especially interesting in view of recent groundbreaking work of [?qx], where Lawson proves that at the prime 2 no such E_∞-model for $BP\langle n\rangle$ exists for $n \geq 4$, which was extended to odd primes in [?qx].

[Gabe: At the moment this introduction is here to fill space. We should discuss how we want to pitch our results.]

We compute mod p topological Hochschild homology of $BP\langle 2\rangle$ at the primes 2 and 3 where, by work of [7,8], it is known that an E_∞-ring spectrum model for $BP\langle 2\rangle$ exists.

1.1. **Outline of strategy.** Working from our calculation of $\mathrm{THH}(BP\langle 2\rangle; \mathbb{F}_p)$ we will analyze the cube of Bockstein spectral sequences corresponding to the diagram

$$\begin{tikzcd}
H\mathbb{F}_p \arrow[rr, no head] \arrow[dr, no head] \arrow[dd, no head] & & H\mathbb{Z}_{(p)} \arrow[dr, no head] \arrow[dd, no head] & \\
& k(1) \arrow[rr, no head] \arrow[dd, no head] & & BP\langle 1\rangle \arrow[dd, no head] \\
k(2) \arrow[rr, no head] \arrow[dr, no head] & & BP\langle 2\rangle/v_1 \arrow[dr, no head] & \\
& BP\langle 2\rangle/p \arrow[rr, no head] & & BP\langle 2\rangle
\end{tikzcd} \quad . \tag{1.1}$$

We begin with the three Bockstein spectral sequences:

$$\mathrm{THH}_*(BP\langle 2\rangle; \mathbb{F}_p)[v_0] \implies \mathrm{THH}_*(BP\langle 2\rangle; \mathbb{Z}_{(p)})_p^\wedge \tag{1.2}$$

$$\mathrm{THH}_*(BP\langle 2\rangle; \mathbb{F}_p)[v_1] \implies \mathrm{THH}_*(BP\langle 2\rangle; k(1)) \tag{1.3}$$

$$\mathrm{THH}_*(BP\langle 2\rangle; \mathbb{F}_p)[v_2] \implies \mathrm{THH}_*(BP\langle 2\rangle; k(2)) \tag{1.4}$$

and then compare the three pairs of spectral sequences

$$\mathrm{THH}_*(BP\langle 2\rangle; \mathbb{Z}_{(p)})_p[v_1] \implies \mathrm{THH}_*(BP\langle 2\rangle; BP\langle 1\rangle) \tag{1.5}$$

$$\mathrm{THH}_*(BP\langle 2\rangle; k(1))[v_0] \implies \mathrm{THH}_*(BP\langle 2\rangle; BP\langle 1\rangle) \tag{1.6}$$

$$\mathrm{THH}_*(BP\langle 2\rangle; k(2))[v_1] \implies \mathrm{THH}_*(BP\langle 2\rangle; BP\langle 2\rangle/p) \tag{1.7}$$

$$\mathrm{THH}_*(BP\langle 2\rangle; k(1))[v_2] \implies \mathrm{THH}_*(BP\langle 2\rangle; BP\langle 2\rangle/p) \tag{1.8}$$

$$\mathrm{THH}_*(BP\langle 2\rangle; \mathbb{Z}_{(p)})[v_2] \implies \mathrm{THH}_*(BP\langle 2\rangle; BP\langle 2\rangle/v_1) \tag{1.9}$$

$$\mathrm{THH}_*(BP\langle 2\rangle; k(2))[v_0] \implies \mathrm{THH}_*(BP\langle 2\rangle; BP\langle 2\rangle/v_1) \tag{1.10}$$

and finally we will compute the spectral sequences

$$\mathrm{THH}_*(BP\langle 2\rangle; BP\langle 1\rangle)[v_2] \implies \mathrm{THH}_*(BP\langle 2\rangle) \tag{1.11}$$

$$\mathrm{THH}_*(BP\langle 2\rangle; BP\langle 2\rangle/v_1)[v_1] \implies \mathrm{THH}_*(BP\langle 2\rangle) \tag{1.12}$$

$$\mathrm{THH}_*(BP\langle 2\rangle; BP\langle 2\rangle/p)[v_0] \implies \mathrm{THH}_*(BP\langle 2\rangle)_p. \tag{1.13}$$

In addition, we use the topological Hochschild May spectral sequence

$$S/p_*THH(H\pi_*BP\langle 2\rangle) \Rightarrow S/p_*THH(BP\langle 2\rangle)$$

to go directly across the diagonal of the left-hand side of the cube (1.1).

Our methods for these calculations are inspired by [3, 4, 10] along with the first author's paper [1].

Conventions. Throughout, we will write $H_*(-)$ for the functor $\pi_*(H\mathbb{F}_p \wedge -)$. We write $\doteq$ to mean that an equality holds up to multiplication by a unit. We will write $BP\langle n\rangle$ for the n-th truncated Brown-Peterson spectrum. In particular, $BP\langle 1\rangle$ denotes the E_∞-ring spectrum model for the connective Adams summand [?qx]. Also, $BP\langle 2\rangle$ will denote the E_∞-model for the second truncated Brown-Peterson spectrum constructed by [?qx] at $p = 2$ and [?qx] at $p = 3$. When $p > 3$, $BP\langle 2\rangle$ will denote the A_∞-model for $BP\langle 2\rangle$ constructed in [?qx]. We will let $BP\langle 2\rangle'$ be the model for second truncated Brown-Peterson spectrum constructed by coning off a regular sequence of generators $(v_3, v_4, \ldots)$. We let $k(n)$ denote an A_∞-ring spectrum model for the connective cover of the Morava K-theory spectrum $K(n)$, which exists by [?qx] for $p \geq 3$.

When not otherwise specified, tensor products will be taken over $\mathbb{F}_p$ and $HH_*(A)$ denotes the Hochschild homology of a graded $\mathbb{F}_p$-algebra relative to $\mathbb{F}_p$. We will let $P(x)$, $E(x)$ and $\Gamma(x)$ denote a polynomial algebra, exterior algebra, and divided power algebra over $\mathbb{F}_p$ on a generator x.

The dual Steenrod algebra will be denoted $\mathscr{A}_*$ with coproduct $\Delta\colon \mathscr{A}_* \to \mathscr{A}_* \otimes \mathscr{A}_*$. Given a right $\mathscr{A}_*$-comodule M, its right coaction will be denoted $\nu\colon \mathscr{A} \to \mathscr{A} \otimes M$ where the comodule M is understood from the context.

2. Preliminary results

The homology of topological Hochschild homology of $BP\langle 2\rangle$ is a straightforward application of work Steinberger [5], Bökstedt [6], and Angeltveit-Rognes [4], and it appears in [4, Thm. 5.12]. Recall that

$$H_*(BP\langle 2\rangle) \cong \begin{cases} P(\bar{\xi}_1, \bar{\xi}_2, \ldots) \otimes E(\bar{\tau}_3, \bar{\tau}_4, \ldots) \text{ if } p \geq 3 \\ P(\bar{\xi}_1^2, \bar{\xi}_2^2, \bar{\xi}_3^2, \bar{\xi}_4, \ldots) \text{ if } p = 2 \end{cases}$$

then by [4, Thm. 5.12] there is an isomorphism

$$H_*(\mathrm{THH}(BP\langle 2\rangle)) \cong \begin{cases} H_*BP\langle 2\rangle \otimes E(\sigma\bar{\xi}_1, \sigma\bar{\xi}_2, \sigma\bar{\xi}_3) \otimes P(\sigma\bar{\tau}_3) \text{ if } p = 3 \\ H_*BP\langle 2\rangle \otimes E(\sigma\bar{\xi}_1^2, \sigma\bar{\xi}_2^2, \sigma\bar{\xi}_3^2) \otimes P(\sigma\bar{\xi}_4) \text{ if } p = 2 \end{cases} \tag{2.1}$$

of primitively generated $H_*BP\langle 2\rangle$-Hopf algebras. We also note the coaction on $H_*\,\mathrm{THH}(BP\langle 2\rangle)$ as a comodule over $\mathscr{A}_*$ computed in [4, Thm. 5.12]

$$\nu(\sigma\bar{\tau}_m) = 1 \otimes \sigma\bar{\tau}_m + \bar{\tau}_0 \otimes \sigma\bar{\xi}_m \tag{2.2}$$

at $p=3$ and

$$\nu(\sigma\bar{\xi}_{m+1}) = 1 \otimes \sigma\bar{\xi}_{m+1} + \bar{\xi}_1 \otimes \sigma\bar{\xi}_2^2. \tag{2.3}$$

at $p=2$. These both follow from the formula

$$\nu \circ \sigma = (1 \otimes \sigma) \circ \nu \tag{2.4}$$

in [4, Eq. 5.11] and the well known $\mathscr{A}_*$-coaction on $H_*BP\langle 2\rangle$.

2.1. **Multiplicativity of Bockstein spectral sequences.** As described in our outline, the first goal will be to compute the Bockstein spectral sequences

$$\mathrm{THH}_*(BP\langle 2\rangle;\mathbb{F}_p)[v_0] \implies \mathrm{THH}_*(BP\langle 2\rangle;\mathbb{Z}_{(p)})^\wedge_p \tag{2.5}$$

$$\mathrm{THH}_*(BP\langle 2\rangle;\mathbb{F}_p)[v_1] \implies \mathrm{THH}_*(BP\langle 2\rangle;k(1)) \tag{2.6}$$

$$\mathrm{THH}_*(BP\langle 2\rangle;\mathbb{F}_p)[v_2] \implies \mathrm{THH}_*(BP\langle 2\rangle;k(2)). \tag{2.7}$$

We will first argue that these Bockstein spectral sequences are multiplicative when $p=3$. Recall that $THH(BP\langle 2\rangle;M) \simeq THH(BP\langle 2\rangle) \wedge_B M$. Consider the Adams spectral sequences

$$E_2^{*,*} = \mathrm{Ext}^{**}_{\mathscr{A}}(\mathbb{F}_p, H_*(THH(BP\langle 2\rangle) \wedge_{BP\langle 2\rangle} k(i))) \Rightarrow \pi_*THH(BP\langle 2\rangle;k(i))_p \tag{2.8}$$

where $k(0)=\mathbb{Z}_{(p)}$ and $k(1)=\ell/p$ and $k(2)$ is the connective cover of $K(2)$. By (2.1), we know that $H_*THH(BP\langle 2\rangle)$ is free over $H_*(BP\langle 2\rangle)$. Since $BP\langle 2\rangle$ is a commutative ring spectrum, there is a splitting $THH(BP\langle 2\rangle;M) \simeq BP\langle 2\rangle \vee \overline{THH}(BP\langle 2\rangle;M)$ for any $BP\langle 2\rangle$-module and consequently the input becomes

$$\mathrm{Ext}^{**}_{\mathscr{A}}(\mathbb{F}_p, H_*(\overline{THH}(BP\langle 2\rangle)) \otimes H_*(k(i)))$$

which is isomorphic to

$$\mathrm{Ext}^{**}_{E(\tau_i)_*}(\mathbb{F}_p, H_*(\overline{THH}(BP\langle 2\rangle))).$$

By (2.2), the τ_i coaction on $H_*(\overline{THH}(BP\langle 2\rangle))$ is trivial and therefore the input becomes

$$\mathrm{Ext}^{**}_{E(\tau_i)_*}(\mathbb{F}_p,\mathbb{F}_p) \otimes H_*(\overline{THH}(BP\langle 2\rangle)).$$

and $\mathrm{Ext}^{**}_{E(\tau_i)_*}(\mathbb{F}_p,\mathbb{F}_p) \cong P(v_i)$ for $0 \le i \le 3$. Since the spectra $k(i)$ are A_∞-ring spectra and $BP\langle 2\rangle$ is an E_∞-ring spectrum, $THH(BP\langle 2\rangle;k(i))$ is ring spectrum. Thus, the Adams spectral sequence is multiplicative. Since the Adams spectral sequence (2.8) is equivalent to the corresponding Bockstein spectral sequence, each of these Bockstein spectral sequences is multiplicative.

We will also consider Bockstein spectral sequences

$$\mathrm{THH}_*(BP\langle 2\rangle;\mathbb{Z}_{(p)})_p[v_1] \implies \mathrm{THH}_*(BP\langle 2\rangle;BP\langle 1\rangle) \tag{2.9}$$

$$\mathrm{THH}_*(BP\langle 2\rangle;k(2))[v_1] \implies \mathrm{THH}_*(BP\langle 2\rangle;BP\langle 2\rangle/p) \tag{2.10}$$

$$\mathrm{THH}_*(BP\langle 2\rangle;k(1))[v_2] \implies \mathrm{THH}_*(BP\langle 2\rangle;BP\langle 2\rangle/p) \tag{2.11}$$

$$\mathrm{THH}_*(BP\langle 2\rangle;\mathbb{Z}_{(p)})[v_2] \implies \mathrm{THH}_*(BP\langle 2\rangle;BP\langle 2\rangle/v_1) \tag{2.12}$$

$$\mathrm{THH}_*(BP\langle 2\rangle;k(1))[v_0] \implies \mathrm{THH}_*(BP\langle 2\rangle;BP\langle 1\rangle) \tag{2.13}$$

$$\mathrm{THH}_*(BP\langle 2\rangle;k(2))[v_0] \implies \mathrm{THH}_*(BP\langle 2\rangle;BP\langle 2\rangle/v_1) \tag{2.14}$$

each of which is also multiplicative except the last two by using an Adams spectral sequence in an appropriate module category and choosing a minimal resolution to get an equality of spectral sequences at the E_1-page.

[Gabe: I need to verify this claim still. This is what AHL do.]

Finally, Bockstein spectral sequences

$$\mathrm{THH}_*(BP\langle 2\rangle; BP\langle 1\rangle)[v_2] \implies \mathrm{THH}_*(BP\langle 2\rangle) \tag{2.15}$$

$$\mathrm{THH}_*(BP\langle 2\rangle; BP\langle 2\rangle/v_1)[v_1] \implies \mathrm{THH}_*(BP\langle 2\rangle) \tag{2.16}$$

$$\mathrm{THH}_*(BP\langle 2\rangle; BP\langle 2\rangle/p)[v_0] \implies \mathrm{THH}_*(BP\langle 2\rangle). \tag{2.17}$$

are also multiplicative except the third one by the same kind of argument as above.

[Gabe: At the moment, this is written as though we will compute $THH_*(BP\langle 2\rangle)$ and it will need to be adjusted if we decide to finish the paper at $S/p_*THH(BP\langle 2\rangle)$.]

2.2. **THH of $BP\langle 2\rangle$ modulo (p, v_1, v_2).** We now compute topological Hochschild homology $BP\langle 2\rangle$ modulo (p, v_1, v_2). This amounts to a computation of

$$\mathrm{THH}_*(BP\langle 2\rangle; H\mathbb{F}_p)$$

since $BP\langle 2\rangle/(p, v_1, v_2) = H\mathbb{F}_p$. By [4, Lem. 4.1], it suffices to compute the subalgebra of co-mododule primitives in $H_*(\mathrm{THH}(BP\langle 2\rangle; H\mathbb{F}_p))$. Since $BP\langle 2\rangle$ and $H\mathbb{F}_p$ are commutative there is a weak equivalence of commutative ring spectra

$$\mathrm{THH}(BP\langle 2\rangle; H\mathbb{F}_p) \simeq \mathrm{THH}(BP\langle 2\rangle) \wedge_{BP\langle 2\rangle} H\mathbb{F}_p.$$

Since $H_*(\mathrm{THH}(BP\langle 2\rangle))$ is free over $H_*BP\langle 2\rangle$ by (2.1), the Eilenberg-Moore spectral sequence and [4, Cor. 5.13] immediately implies

$$H_*(\mathrm{THH}(BP\langle 2\rangle; H\mathbb{F}_p)) \cong \begin{cases} \mathscr{A}_* \otimes E(\sigma\bar{\xi}_1, \sigma\bar{\xi}_2, \sigma\bar{\xi}_3) \otimes P(\sigma\bar{\tau}_3) \text{ if } p = 3 \\ \mathscr{A}_* \otimes E(\sigma\bar{\xi}_1^2, \sigma\bar{\xi}_2^2, \sigma\bar{\xi}_3^2) \otimes P(\sigma\bar{\xi}_3) \text{ if } p = 2. \end{cases} \tag{2.18}$$

The $\mathscr{A}_*$ coaction on elements in $\mathscr{A}_*$ is given by the coproduct and the remaining coactions are determined by the formula (2.4) and are therefore the same as in (2.2) and (2.3). We write $\lambda_i = \sigma\bar{\xi}_i$ at $p = 3$ and $\lambda_i = \sigma\bar{\xi}_i^2$ are $p = 2$. We also define

$$\mu_3 = \begin{cases} \sigma\bar{\tau}_3 - \bar{\tau}_0 \otimes \sigma\bar{\xi}_3 \text{ if } p = 3 \\ \sigma\bar{\xi}_4 - \bar{\xi}_1 \otimes \sigma\bar{\xi}_3^2 \text{ if } p = 2 \end{cases}.$$

Then it is clear that the algebra of comodule primitives in $H_*(\mathrm{THH}(BP\langle 2\rangle; H\mathbb{F}_p))$ is generated by μ_3 and λ_i for $1 \le i \le 3$. We therefore produce the following isomorphism of graded $\mathbb{F}_p$-algebras

$$\mathrm{THH}_*(BP\langle 2\rangle; H\mathbb{F}_p) \cong E(\lambda_1, \lambda_2, \lambda_3) \otimes P(\mu_3). \tag{2.19}$$

The degrees are $|\lambda_i| = 2p^i - 1$ for $1 \le i \le 3$ and $|\mu_3| = 2p^3$.

2.3. **Rational homology.** Next, we compute the rational homology of $\mathrm{THH}(BP\langle 2\rangle)$ to locate the torsion free component of $\mathrm{THH}_*(BP\langle 2\rangle)$. Towards this end, we will use the $H\mathbb{Q}$-based Bökstedt spectral sequence. This is a spectral sequence of the form

$$E_2^{**} = \mathrm{HH}_*^{\mathbb{Q}}(H\mathbb{Q}_*BP\langle 2\rangle) \implies H\mathbb{Q}_* \mathrm{THH}(BP\langle 2\rangle).$$

Recall that the rational homology of $BP\langle 2\rangle$ is

$$H\mathbb{Q}_*BP\langle 2\rangle \cong P_{\mathbb{Q}}[v_1, v_2].$$

Thus the E_2-term of the Bökstedt spectral sequence is

$$P_{\mathbb{Q}}(v_1, v_2) \otimes E_{\mathbb{Q}}(\sigma v_1, \sigma v_2)$$

and the bidegree of σv_i is $(1, 2(p^i - 1))$. Note that $BP\langle 2\rangle$ is a commutative ring spectrum, so by [4, Prop. 4.3] the Bökstedt spectral sequence is multiplicative. All the algebra generators are in Bökstedt filtration 0 and 1 and the d^2 differential shifts Bökstedt filtration by two, so there is no room for differentials. Thus, the E_2-term is isomorphic to the E_∞-term as graded $\mathbb{Q}$-algebras. There are also no hidden extensions. Thus, there is an isomorphism of graded $\mathbb{Q}$-algebras

$$\mathrm{THH}_*(BP\langle 2\rangle) \otimes \mathbb{Q} \cong P_{\mathbb{Q}}(v_1, v_2) \otimes_{\mathbb{Q}} E_{\mathbb{Q}}(\sigma v_1, \sigma v_2)$$

where $|\sigma v_i| = 2p^i - 1$. Consequently,

$$L_0 \mathrm{THH}(BP\langle 2\rangle) \simeq L_0BP\langle 2\rangle \wedge \Sigma^{2p-1}L_0BP\langle 2\rangle \vee \Sigma^{2p^2-1}L_0BP\langle 2\rangle \vee \Sigma^{2p^2+2p-2}L_0BP\langle 2\rangle$$

where $L_0 = L_{H\mathbb{Q}}$, since L_0 is a smashing localization and $L_0S = H\mathbb{Q}$.

3. The $H\mathbb{Z}$-Bockstein spectral sequence

Recall that there is an isomorphism of $\mathcal{A}_*$-comodules

$$H_*(S/p \wedge THH(BP\langle 2\rangle)) \cong \begin{cases} E(\bar{\tau}_0) \otimes H_*(BP\langle 2\rangle) \otimes E(\sigma\bar{\xi}_1, \sigma\bar{\xi}_2, \sigma\bar{\xi}_3) \otimes P(\sigma\bar{\tau}_3) & \text{if } p = 3 \\ E(\bar{\xi}_1) \otimes H_*(BP\langle 2\rangle) \otimes E(\sigma\bar{\xi}_1^2, \sigma\bar{\xi}_2^2, \sigma\bar{\xi}_3^2) \otimes P(\sigma\bar{\xi}_4) & \text{if } p = 2 \end{cases}$$

where the coaction on $x \in \mathcal{A}_*$ is $\nu(x) = \Delta(x)$ and the remaining coactions follow from (2.4). In this section, we compute the Bockstein spectral sequence

$$E^1_{*,*} = THH_*(BP\langle 2\rangle; \mathbb{F}_p)[v_0] \Rightarrow THH(BP\langle 2\rangle; \mathbb{Z}_{(p)})_p. \tag{3.1}$$

As a direct consequence the coaction on $\sigma\bar{\tau}_3$, there is a differential

$$d_1(\mu_3) = v_0\lambda_3. \tag{3.2}$$

in the $H\mathbb{Z}$-Bockstein spectral sequence (3.1).

The following lemma follows from [9, Prop. 6.8] by translating to the E_∞-context, see the proof of [3, Lem. 3.2].

Lemma 3.3. *If $d_j(x) \neq 0$ in the $H\mathbb{Z}$-Bockstein spectral sequence (3.1) then*

$$d_{j+1}(x^p) = v_0x^{p-1}d_j(x)$$

if $p > 2$ or if $p = 2$ and $j \geq 2$. If $p = 2$ and $j = 1$ then

$$d_{j+1}(x^p) = v_0x^{p-1}d_j(x) + Q^{|x|}(d_1(x))$$

When $p = 2$, we have the differential

$$d_1(\mu) = v_0\lambda_3.$$

Therefore, the error term for $d_2(\mu^2)$ is

$$Q^{16}\lambda_3 = Q^{16}(\sigma\bar{\xi}_3^2) = \sigma(Q^{16}(\bar{\xi}_3^2)) = \sigma((Q^8\bar{\xi}_3)^2) = \sigma(\bar{\xi}_4^2) = 0.$$

The first equality holds by definition of λ_3, the second equality holds because σ commutes with Dyer-Lashoff operations by [6], the third equality holds by [5], and the last equality holds because σ is a derivation [4].

Corollary 3.4. *When the $p = 2, 3$, there are differentials*

$$d_{i+1}(\mu^{p^i}) = v_0^{i+1}\mu^{p^i-1}\lambda_3.$$

Consequently, there are differentials

$$d_{\nu_p(k)+1}(\mu^k) \doteq v_0^{\nu_p(k)+1}\mu^{k-1}\lambda_3$$

where $\nu_p(k)$ denotes the p-adic valuation of k.

Proof. Let $\alpha = \nu_p(k)$. We have that $k = p^\alpha j$ where p does not divide j. So by the Leibniz rule

$$d_{\alpha+1}(\mu^k) = d_{\alpha+1}((\mu^{p^\alpha})^k) = k\mu^{p^\alpha(k-1)}d_{\alpha+1}(\mu^\alpha) = kv_0^{\alpha+1}\mu^{p^\alpha(k-1)}\mu^{p^\alpha-1}\lambda_3 = kv_0^{\alpha+1}\mu^{k-1}\lambda_3.$$

Since k is not divisible by p, it is a unit mod p. □

Thus we have the following,

Corollary 3.5. *The E_∞ page of of the $H\mathbb{Z}$-Bockstein spectral sequence for R is the algebra*

$$P(v_0) \otimes E(\lambda_1, \lambda_2, \lambda_3) \otimes P(\mu)/(v_0^{\nu_p(k)+1}\mu^k\lambda_3 \mid i \geq 1).$$

[Dom: Insert picture.]

4. Computation of $THH(BP\langle 2\rangle; k(2))$

The goal of this section is to compute the homotopy groups of $\mathrm{THH}(BP\langle 2\rangle; k(2))$. We achieve this through an analysis of the v_2-Bockstein spectral sequence (2.7). We first outline our strategy.

In [10], McClure-Staffeldt compute $\mathrm{THH}_*(\ell, k(1))$ by first arguing that upon inverting v_1, there is an isomorphism

$$v_1^{-1}\,\mathrm{THH}_*(\ell; k(1)) \cong K(1)_* \tag{4.1}$$

when $p \geq 3$. This implies that in the abutment of the v_1-Bockstein spectral sequence

$$\mathrm{THH}_*(\ell; \mathbb{F}_p)[v_1] \implies S/p_*\,\mathrm{THH}_*(\ell)$$

all classes are v_1-torsion besides the powers of v_1. It turns out that there is only one of pattern of differentials that makes this possible, which gives a complete description of this spectral sequence.

In this section we will use a similar method to compute $THH(BP\langle 2\rangle; k(2))$. In particular, we will prove that in the abutment of the v_2-Bockstein spectral sequence

$$\mathrm{THH}_*(BP\langle 2\rangle; \mathbb{F}_p)[v_2] \implies \mathrm{THH}_*(BP\langle 2\rangle; k(2))$$

all elements in the abutment are v_2-torsion except the powers of v_2. Generalizing naively, we would replace S/p with $V(1)$, however, at $p = 2$ the spectrum $V(1)$ does not exist and at $p = 3$ the spectrum $V(1)$ is not a ring spectrum.

This potential issue can be easily avoided using the fact that

$$S/p_* THH(\ell) \cong THH(\ell; k(1))$$

and in our case we therefore compute $THH_*(BP\langle 2\rangle; k(2))$.

Since $k(2)$ has a v_2-self map, there is an induced v_2-self map of $THH(BP\langle 2\rangle; k(2)$ and we define $v_2^{-1}THH(BP\langle 2\rangle; k(2))$ to be the colimit of iterations of this self-map. It is therefore clear that

$$v_2^{-1}\,\mathrm{THH}(BP\langle 2\rangle; k(2)) \simeq \mathrm{THH}(BP\langle 2\rangle; K(2)).$$

and there is a canonical unit morphism

$$K(2) \to \mathrm{THH}(BP\langle 2\rangle; K(2)).$$

We will therefore argue that this unit map is a $K(2)$-equivalence. Since the source and target are both $K(2)$-modules, and hence $K(2)$-local, this will show that the map is in fact an equivalence of spectra.

To establish this, we just need to argue that

$$K(2)_*\,\mathrm{THH}(BP\langle 2\rangle; K(2)) \cong K(2)_*K(2).$$

Note that

$$\mathrm{THH}(BP\langle 2\rangle; K(2)) \simeq K(2) \wedge_{BP\langle 2\rangle \wedge BP\langle 2\rangle} BP\langle 2\rangle$$

so their is an Eilenberg-Moore spectral sequence

$$\mathrm{Tor}_{s,t}^{(K(2)_*BP\langle 2\rangle)^e}(K(2)_*K(2), K(2)_*BP\langle 2\rangle) \implies K(2)_{s+t}(\mathrm{THH}(BP\langle 2\rangle; K(2))). \tag{4.2}$$

where

$$(K(2)_*BP\langle 2\rangle)^e \cong (K(2)_*BP\langle 2\rangle) \wedge_{K(2)_*} (K(2)_*BP\langle 2\rangle).$$

Here we are using the fact that $K(2)_*BP\langle 2\rangle$ is flat over $K(2)_*$, which follows because all $K(2)$-modules are free.

[Dom: Convergence issues in E-M spectral sequence?]

4.1. **The $K(2)$-homology of $\mathrm{THH}(BP\langle 2\rangle; K(2))$.** To begin, we need to compute $K(2)_*BP\langle 2\rangle$. Since the Johnson-Wilson theory $E(2)$ is Landweber exact, one has

$$E(2)_*BP\langle 2\rangle \cong E(2)_* \otimes_{BP_*} BP_*BP \otimes_{BP_*} BP\langle 2\rangle_*.$$

It is known that

$$BP_*BP \otimes_{BP_*} BP\langle 2\rangle_* \cong BP_*[t_1, t_2, \ldots]/(\eta_R(v_i) \mid i \geq 3)$$

where $\eta_R : BP_* \to BP_*BP$ denotes the right unit. Thus,

$$E(2)_*BP\langle 2\rangle \cong E(2)_*[t_i \mid i \geq 1]/(\eta_R(v_i) \mid i \geq 3).$$

Since $K(2)$ is obtained from $E(2)$ by coning off p and v_1, we find that

$$K(2)_*BP\langle 2\rangle \cong K(2)_*[t_i \mid i \geq 1]/(\eta_R(v_i) \mid i \geq 3).$$

We have the following congruences

$$\eta_R(v_{2+k}) \equiv v_2 t_k^{p^2} - v_2^{p^k} t_k \mod (\eta_R(v_3), \ldots, \eta_R(v_{k+1})).$$

in $K(2)_*BP$ for all $k \geq 1$ (cf. formula 6.1.13 of [12]). Thus, the following lemma follows.

Lemma 4.3. *There is an isomorphism of graded rings*

$$K(2)_*BP\langle 2\rangle \cong K(2)_*[t_1, t_2, \ldots]/(v_2 t_k^{p^2} - v_2^{p^k} t_k \mid k \geq 1) \tag{4.4}$$

We proceed by analyzing the Eilenberg-Moore spectral sequence (4.2). First, we note that the E^2-page is

$$E^2_{*,*} \cong \mathrm{HH}_*^{K(2)_*}(K(2)_*BP\langle 2\rangle; K(2)_*K(2)).$$

We compute this E^2-page in the following proposition.

Theorem 4.5. *The unit map induces an isomorphism of $K(2)_*$-modules*

$$K(2)_*K(2). \cong \mathrm{HH}_*^{K(2)_*}(K(2)_*BP\langle 2\rangle; K(2)_*K(2)).$$

Proof. First, we note that the input of (4.2) is isomorphic to

$$K(2)_*K(2) \otimes_{K(2)_*} \mathrm{Tor}^{(K(2)_*BP\langle 2\rangle)}(K(2)_*; K(2)_*)$$

by Cartan-Eilenberg [?qx] and the same argument as appears in the proof of [?qx]. We will now argue that

$$\mathrm{Tor}^{(K(2)_*BP\langle 2\rangle)}(K(2)_*; K(2)_*) \cong K(2)_*.$$

Recall that the topological degree of t_k is $2(p^k - 1)$, and that the degree of v_2 is $2(p^2 - 1)$. Thus $|v_2|$ divides $|t_k|$ if and only if k is even. We observe that there is an isomorphism of $K(2)_*$-algebras

$$K(2)_*BP\langle 2\rangle \cong \bigotimes_k K(2)_*[t_k \mid k \geq 1]/(v_2 t_k^{p^2} - v_2^{p^k} t_k).$$

Let $u_k = v_2^{m(k)} t_k$ where

$$m(k) = -p^{k-2} - p^{k-4} - \cdots - p^2 - 1$$

when $2 \mid k$ and and let $u_k = v_2^{\ell(k)} t_k$ where

$$\ell(k) = -p^{k-2} - p^{k-4} - \cdots - p$$

when k is odd. Thus

$$|u_k| = \begin{cases} 0 & k \equiv 0 \mod 2 \\ 2(p-1) & k \equiv 1 \mod 2 \end{cases}.$$

Define A_n to be the subalgebra of $K(2)_*BP\langle 2\rangle$ generated by $t_1,\ldots,t_n$, and let $A(t_k)$ denote the subalgebra generated by t_k so that there is an isomorphism of $K(2)_*$-algebras

$$A_n \cong \bigotimes_{k=1}^{n} A(t_k).$$

Then there is an isomorphism of $K(2)_*$-algebras

$$\mathrm{Tor}_*^{A_n}(K(2)_*;K(2)_*) \cong \bigotimes_{k=1}^{n} \mathrm{Tor}_*^{A(t_k)}(K(2)_*;K(2)_*)$$

Since the functor $\mathrm{Tor}_*^{(-)}(K(2)_*;K(2)_*)$ commutes with filtered colimits of $K(2)_*$-algebras and $K(2)_*BP\langle 2\rangle = \mathrm{colim}\, A_n$, it follows that there is an isomorphism of $K(2)_*$-algebras

$$\mathrm{Tor}^{K(2)_*BP\langle 2\rangle}(K(2)_*, K(2)_*) \cong \bigotimes_{k=1}^{\infty} \mathrm{Tor}^{A(t_k)}(\mathbb{F}_p).$$

Thus, it suffices to compute $\mathrm{Tor}_*^{A(t_k)}(K(2)_*, K(2)_*)$. When k is even, we have

$$A(t_k) = K(2)_* \otimes \mathbb{F}_p[u_k]/(u_k^{p^2} - u_k),$$

in which case there is an isomorphism of $K(2)_*$-algebras

$$\mathrm{Tor}^{A(t_k)}(K(2)_*, K(2)_*) \cong K(2)_* \otimes \mathrm{Tor}^{\mathbb{F}_p[u_k]/(u_k^{p^2}-u_k)}(\mathbb{F}_p, \mathbb{F}_p)$$

by the base-change formula for Tor. Since the $\mathbb{F}_p$-algebra

$$\mathbb{F}_p[u_k]/(u_k^{p^2} - u_k)$$

is étale over $\mathbb{F}_p$, it follows that

$$\mathrm{HH}^{\mathbb{F}_p}(\mathbb{F}_p[u_k]/(u_k^{p^2} - u_k)) \cong \mathbb{F}_p[u_k]/(u_k^{p^2} - u_k)$$

by [?qx] and since

$$\mathrm{HH}^{\mathbb{F}_p}(\mathbb{F}_p[u_k]/(u_k^{p^2} - u_k)) \cong \mathbb{F}_p[u_k]/(u_k^{p^2} - u_k) \otimes \mathrm{Tor}_*^{\mathbb{F}_p[u_k]/(u_k^{p^2}-u_k)}(\mathbb{F}_p, \mathbb{F}_p)$$

we see that

$$\mathrm{Tor}_*^{\mathbb{F}_p[u_k]/(u_k^{p^2}-u_k)}(\mathbb{F}_p, \mathbb{F}_p) \cong \mathbb{F}_p.$$

When k is odd, there is again an isomorphism

$$\mathrm{Tor}^{\mathbb{F}_p[u_k]/(u_k^{p^2}-u_k)}(\mathbb{F}_p, \mathbb{F}_p) \cong \mathbb{F}_p$$

because $\mathbb{F}_p[u_k]/(u_k^{p^2} - u_k)$ is étale over $\mathbb{F}_p$.

Consequently, there is an isomorphism of $K(2)_*$-algebras

$$\mathrm{Tor}^{K(2)_*BP\langle 2\rangle}(K(2)_*, K(2)_*) \cong K(2)_*$$

completing the proof. □

The following corollary is immediate.

Corollary 4.6. *The $K(2)$-homology of* $\mathrm{THH}(BP\langle 2\rangle; K(2))$ *is isomorphic to* $K(2)_*K(2)$. *Since* $\mathrm{THH}(BP\langle 2\rangle; K(2))$ *is a free $K(2)$-module, it follows that the unit morphism*

$$K(2) \to \mathrm{THH}(BP\langle 2\rangle; K(2))$$

is a weak equivalence.

4.2. **Differentials in the v_2-BSS.** We now turn to analyzing the v_2-BSS (2.7). In particular, we will argue that Corollary 4.6 implies a unique pattern of differentials in the spectral sequence. We adapt the proof of [10] to our setting.

Recall that the E_2-term of the v_2-BSS is

$$\mathrm{THH}_*(BP\langle 2\rangle; \mathbb{F}_p)[v_2] \cong P(v_2) \otimes E(\lambda_1, \lambda_2, \lambda_3) \otimes P(\mu_3),$$

where $|\lambda_i| = (2p^i - 1, 0)$ and $|\mu_3| = (2p^3, 0)$. It will be more convenient to work in the v_2-localized Bockstein spectral sequence. Since the λ_i are in odd total degree and 1 is v_2-torsion free, they cannot support a differential. If μ_3 were a infinite cycle as well, then by multiplicativity of the Bockstein spectral sequence, it would follow that the spectral sequence collapses at the E_1-page. However, this would contradict Corollary 4.6. Therefore, μ_3 supports a differential, the only possible differential for bi-degree reasons is

$$d_p(\mu_3) \doteq v_2^p \lambda_1.$$

Thus,

$$v_2^{-1} E_{p+1}^{*,*} \cong K(2)_* \otimes E(\lambda_2, \lambda_3, \lambda_4) \otimes P(\mu_3^p),$$

where $\lambda_4 := \lambda_1 \mu_3^{p-1}$. Note that the bidegree of λ_4 is

$$|\lambda_4| = (2p^4 - 2p^3 + 2p - 1, 0).$$

In particular, its total degree is odd. So this class cannot support a differential which truncates the the v_2-tower on λ_2 or λ_3. So this class is an infinite cycle. By multiplicativity again, if μ_3^p were an infinite cycle, then the spectral sequence would collapse at E_{p+1}, which would contradict Corollary 4.6. So μ_3^p supports a differential. The only possibility is

$$d_{p^2}(\mu_3^p) \doteq v_2^{p^2} \lambda_2.$$

Thus, there is an isomorphism

$$v_2^{-1} E_{p^2+1}^{*,*} \cong K(2)_* \otimes E(\lambda_3, \lambda_4, \lambda_5) \otimes P(\mu_3^{p^2})$$

where

$$\lambda_5 := \lambda_2 \mu_3^{p^2-p}.$$

The bidegree of this class is

$$|\lambda_5| = (2p^5 - 2p^4 + 2p^2 - 1, 0).$$

Since $\lambda_3, \lambda_4, \lambda_5$ all have odd total degree, they are necessarily infinite cycles. As before, the class $\mu_3^{p^2}$ must support a differential. The only possibility is

$$d_{p^3}(\mu_3^{p^2}) \doteq v_2^{p^3} \lambda_3.$$

This shows that

$$v_2^{-1} E_{p^3+1}^{*,*} \cong K(2)_* \otimes E(\lambda_4, \lambda_5, \lambda_6) \otimes P(\mu_3^{p^4})$$

where

$$\lambda_6 := \lambda_3\mu_3^{p^2(p-1)} = \lambda_3\mu_3^{p^3-p^2},$$

so that the bidegree of λ_6 is

$$|\lambda_6| = (2p^6 - 2p^5 + 2p^3 - 1, 0).$$

Consequently, as we saw before, the class λ_6 cannot support a differential, and hence is an infinite cycle. As before, the class $\mu_3^{p^3}$ must support a differential. An elementary calculation shows the only possibility is

$$d_{p^4+p}\mu^{p^3} \doteq v_2^{p^4+p}\lambda_4$$

Recursively define a function $d(n)$ by

$$d(n) := \begin{cases} 2p^n - 1 & \text{if } 1 \le n \le 3 \\ 2p^3(p^{n-3} - p^{n-4}) + d(n-3) & \text{if } n > 3 \end{cases}$$

and recursively define classes λ_n by

$$\lambda_n := \begin{cases} \lambda_n & 1 \le n \le 3 \\ \lambda_{n-3}\mu^{p^{n-4}(p-1)} & n > 3 \end{cases}.$$

Then a simple inductive argument shows that the bidegree of λ_n is given by

$$|\lambda_n| = (d(n), 0). \tag{4.7}$$

Notice that $d(n)$ is always odd, and so λ_n is always in odd total degree. A simple induction shows that

$$d(n) = \begin{cases} 2p^n - 2p^{n-1} + 2p^{n-2} - 2p^{n-3} + \cdots + 2p - 1 & n \equiv 1 \mod 3 \\ 2p^n - 2p^{n-1} + 2p^{n-2} - 2p^{n-3} + \cdots + 2p^2 - 1 & n \equiv 2 \mod 3 \\ 2p^n - 2p^{n-1} + 2p^{n-2} - 2p^{n-3} + \cdots + 2p^3 - 1 & n \equiv 0 \mod 3 \end{cases}.$$

Lemma 4.8. *The integer $2p^{n+2} - d(n) - 1$ is divisible by $|v_2|$.*

Proof. We proceed by induction. The base case easily holds because $2p^3 - 2p$ is divisible by $2p^2 - 2$. Since

$$(2p^{n+3} - 1) - d(n+1) = (2p^2 - 2)p^n + (2p^n - d(n-2) - 1),$$

the induction hypothesis implies $(2p^{n+3} - 1) - d(n+1)$ is divisible by $2p^2 - 2$. □

Now let $r(n)$ be the function given by

$$r(n) := |v_2|^{-1}(2p^{n+2} - d(n) - 1).$$

Then we obtain as a corollary to the lemma,

Corollary 4.9. *The function $r(n)$ is given by*

$$r(n) = \begin{cases} p^n + p^{n-3} + \cdots + p^4 + p & n \equiv 1 \mod 3 \\ p^n + p^{n-3} + \cdots + p^5 + p^2 & n \equiv 2 \mod 3 \\ p^n + p^{n-3} + \cdots + p^6 + p^3 & n \equiv 0 \mod 3 \end{cases}.$$

We are now in a position to determine the differentials in the spectral sequence.

Theorem 4.10. *In the v_2-BSS, one has*

(1) *The only nonzero differentials are in $v_2^{-1}E_{r(n)}$.*
(2) *The page $v_2^{-1}E_{r(n)}$ is given by*

$$v_2^{-1}E_{r(n)} \cong K(2)_* \otimes E(\lambda_n, \lambda_{n+1}, \lambda_{n+2}) \otimes P(\mu_3^{p^{n-1}}).$$

Moreover, $\lambda_n, \lambda_{n+1}, \lambda_{n+2}$ are infinite cycles.
(3) *The differential $d_{r(n)}$ is determined by the multiplicativity of the BSS and*

$$d_{r(n)}\mu_3^{p^{n-1}} = v_2^{r(n)}\lambda_n.$$

Proof. We proceed by induction, having already shown the theorem for $n \leq 4$. Assume inductively that

$$v_2^{-1}E_{r(n)}^{*,*} \cong K(2)_* \otimes E(\lambda_n, \lambda_{n+1}, \lambda_{n+2}) \otimes P(\mu_3^{p^{n-1}}).$$

By the inductive hypothesis, λ_n, λ_{n+1} are infinite cycles. Since $\lambda_n, \lambda_{n+1}, \lambda_{n+2}$ all have odd total degree, it follows that a differential on λ_{n+2} cannot truncate the v_2-towers on λ_n or λ_{n+1}. Therefore, the only possibility is that λ_n supports a differential hitting v_2^j for some positive integer j. But that would contradict Corollary 4.6. So λ_{n+2} must also be a cycle.

If the class $\mu_3^{p^{n-1}}$ does not support a differential then by multiplicativity the spectral sequence would collapse at $E_{r(n)}$, and this would contradict Corollary 4.6. Thus $\mu_3^{p^{n-1}}$ supports a differential. Lemma 4.8 and a simple modular arithmetic argument shows that the only possibility is

$$d_{r(n)} \doteq v_2^{r(n)}\lambda_n.$$

Since the differential satisfies the Leibniz rule, this gives

$$v_2^{-1}E_{r(n)+1} \cong K(2)_* \otimes E(\lambda_{n+1}, \lambda_{n+2}, \lambda_{n+3}) \otimes P(\mu_3^{p^n}).$$

This completes the inductive step, proving the theorem. □

We now state the main theorem of this section.

Theorem 4.11. *For each $n \geq 2$ and each nonnegative integer m with $m \not\equiv p-1$* mod p *there are elements $y_{n,m}$ and $y'_{n,m}$ and $y''_{n,m}$ in $THH_*(BP\langle 2\rangle; k(2))$ such that*

(1) *$y_{n,m}$ projects to $\lambda_n\mu^{mp^{n-1}}$ in $E_\infty^{*,0}$*
(2) *$y'_{n,m}$ projects to $\lambda_n\lambda_{n+1}\mu^{mp^{n-1}}$ in $E_\infty^{*,0}$*
(3) *$y''_{n,m}$ projects to $\lambda_n\lambda_{n+1}\lambda_{n+2}\mu^{mp^{n-1}}$ in $E_\infty^{*,0}$*

As a $P(v_2)$-module, $THH_*(BP\langle 2\rangle; k(2))$ is generated by the unit element 1 and the elements $y_{n,m}, y'_{n,m}, y''_{n,m}$. The only relations are

$$v_n^{r(n)}y_{n,m} = v_n^{r(n)}y'_{n,m} = v_n^{r(n)}y''_{n,m} = 0.$$

This theorem will follow from the previous results and two additional lemmas. Let $P(m)$ denote a free rank one $P(v_m)$-module and let $P(m)_i$ denote the $P(v_m)$-module $P(m)/v_m^i$. Let X be a $BP\langle n\rangle$-module such that $H_*X \cong H_*BP\langle n\rangle \otimes H_*(\overline{X})$ as a H_*X-module and consider the Adams spectral sequence

$$E_2^{*,*}(X) = Ext_{E(Q_m)}^{*,*}(\mathbb{F}_p, H_*(\overline{X})) \Rightarrow \pi_*(X \wedge_{BP\langle n\rangle} k(m))_p \qquad (4.12)$$

and the v_n-inverted Adams spectral sequence

$$v_m^{-1}E_2^{*,*}(X) = v_m^{-1}Ext^{*,*}_{E(Q_m)}(\mathbb{F}_p, H_*(\overline{X})) \Rightarrow \pi_*(X \wedge_{BP\langle n\rangle} K(m))_p \tag{4.13}$$

There is a map of spectral sequences

$$E_2^{*,*}(X) \longrightarrow v_m^{-1}E_2^{*,*}(X)$$

induced by the localization map $k(m) \to v_m^{-1}k(m) = K(m)$.

Lemma 4.14. *Let $r \geq 2$. Suppose the $E_r(X)$-page of the Adams spectral sequence (4.12) is generated by elements in filtration 0 as a $P(k)$-module and $E_r^{*,*}(X)$ is a direct sum of copies of $P(k)$ and $P(k)_i$ with $i \leq r$ as a $P(k)$-module. Then*

(1) *the map of E_r-pages*

$$E_r^{s,t}(X) \to v_k^{-1}E_r^{s,t}(X)$$

is a monomorphism when $t \geq r+1 \geq 3$.

(2) *Also, the differentials in $E_{r+1}^{*,*}$ are the same as their image in $v_k^{-1}E_{r+1}^{*,*}$.*

Proof. Statement (1) is a consequence of our assumptions since elements in filtration $r+1$ are v_k-torsion free. To prove statement (2) it suffices to prove the following: if $x \in E_r(X)$ maps to a cycle $\bar{x} \in v_k^{-1}E_r(X)$,then x is a cycle. By our assumption, there is an $a \in E_r^{*,0}$ such that $x = v_k^m a$. Statement (1) then implies $d_{r+1}(a) = 0$ so since the differentials are v_k-linear the result follows. □

[Gabe: In MS, they claim that the proof works for $t \geq r-1 \geq 1$ and $E_r^{*,*}$, but I don't see why they get indices instead of the ones I have here.]

Remark 4.15. *The Lemma above is a generalization of part (a) and (b) of Lemma qx [?qx]. This level of generality of the lemma was certainly known to the McClure-Staffeldt and it is mentioned in Remark [?qx].*

Lemma 4.16. *For $r \geq 2$ and $n = 2$, the $E_r(THH(BP\langle 2\rangle))$-page of the Adams spectral sequence is generated by elements in filtration 0 as a $P(2)$-module and $E_r^{*,*}$ is a direct sum of copies of $P(2)$ and $P(2)_i$ for $i \leq r$.*

Proof. We will begin by proving the first statement by induction. Note that (2.18) implies the base case in the induction when $r = 2$. Suppose the statement holds for some r. Choose a basis z_i for the $\mathbb{F}_p$-vector space V_r such that $V_r = \{x \in E_r^{*,0} \mid v_2^{r-1}x = 0\}$. Then $d_r(z_i)$ is in filtration r and since the differentials are v_2-linear, $v_2^{r-1}d_r(z_i) = 0$. However, this contradicts the induction hypothesis because the induction hypothesis implies that all elements in filtration r are v_2-torsion-free. Thus, each basis element z_i is a d_r-cycle. Next choose a set of elements $\{z_j'\} \subset E_r^{*,0}$ such that $\{d_r(z_j')\}$ is a basis for $\mathrm{im}(d_r\colon E_r^{*,0} \to E_r^{*,r})$. Choose $z_j'' \in E_r^{*,0}$ such that $v_2^r z_j'' = d_r(z_j')$. Then z_j'' are d_r-cycles and z_j'' and z_j are linearly independent. We can therefore choose d_r-cycles z_j''' such that $\{z_j\} \cup \{z_j''\} \cup \{z_j'''\}$ are a basis for the d_r-cycles in $E_r^{*,0}$. Then $\{z_j\} \cup \{z_j'\} \cup \{z_j''\} \cup \{z_j'''\}$ are a basis for $E_r^{*,0}$ and the differential is completely determined by the formulas

$$d_r(z_i) = 0, \quad d_r(z_j') = v_2^r z_j'', \quad d_r(z_j'') = 0, \text{ and } \quad d_r(z_j''') = 0.$$

Thus, $E_r^{*,*}$ is generated as a P-module by z_i, z_i'', and z_i''' where $v_2^{r-1}z_i = 0$ and $v_2^r z_i'' = 0$ and z_i''' is v_2-torsion free. $\square$

Proof of Theorem 4.11. For brevity, we will write $\gamma_{n,m} = \lambda_n\mu^{mp^{n-1}}$, $\gamma'_{n,m} = \lambda_n\lambda_{n+1}\mu^{mp^{n-1}}$ and $\gamma''_{n,m} = \lambda_n\lambda_{n+1}\lambda_{n+2}\mu^{mp^{n-1}}$. By Lemma 4.16 and Lemma 4.14 it suffices to prove that the elements $\gamma_{n,m}$, $\gamma'_{n,m}$, and $\gamma''_{n,m}$ are infinite cycles, that together with 1 they form a basis for $E_\infty^{*,0}$ as an $\mathbb{F}_p$-vector space, and that each of $\gamma_{n,m}$, $\gamma'_{n,m}$ and $\gamma''_{n,m}$ are killed by $v_2^{r(n)}$. By induction on n, we will prove

$$E_{r(n)}(THH(BP\langle 2\rangle)) \cong M_n \oplus E(\lambda_n, \lambda_{n+1}, \lambda_{n+2}) \otimes P(\mu^{p^{n-1}})$$

where M_n is generated by $\{\gamma_{k,m}, \gamma'_{k,m}\gamma''_{k,m} \mid k < n\}$ modulo the relations

$$v_2^{r(k)}\gamma_{k,m} = v_2^{r(k)}\gamma'_{k,m} = v_2^{r(k)}\gamma''_{k,m} = 0.$$

This statement holds for $n = 1$ by (2.18). Assume the statement holds for all integers less than or equal to some $N > 1$. Lemma 4.16, Lemma 4.14 and Theorem 4.10 imply that the only nontrivial differentials with source in $E_{r(N)}^{*,0}$ are the differentials

$$d_{r(N)}(\mu^{(m+1)r(N)} = (m+1)v_2^{r(N)}\lambda_N\mu^{mp^{N-1}} \doteq \gamma_{N,m},$$

the differentials

$$d_{r(N)}(\lambda_{N+1}\mu^{(m+1)r(N)} = (m+1)v_2^{r(N)}\lambda_n\lambda_{N+1}\mu^{mp^{N-1}} \doteq \gamma'_{N,m}$$

and the differentials

$$d_{r(N)}(\lambda_{N+2}\lambda_{N+1}\mu^{(m+1)r(N)} = (m+1)v_2^{r(N)}\lambda_N\lambda_{N+1}\lambda_{N+2}\mu^{mp^{N-1}} \doteq \gamma''_{N,m}$$

where $m \not\equiv p-1 \mod p$. Combining this with Lemma 4.16 and Lemma 4.14, this implies that

$$E_{r(N)+1}(\mathrm{THH}(BP\langle 2\rangle)) \cong M_n \oplus V_{N+1} \oplus \Big(P(2) \otimes E(\lambda_{N+1}, \lambda_{N+2}, \lambda_N\mu_3^{(p-1)p^N-1}) \otimes P(\mu^{p^N})\Big)$$

where V_{N+1} has generators $\gamma_{N,m}$, $\gamma'_{N,m}$, and $\gamma''_{N,m}$ and relations

$$v_2^{r(N)}\gamma_{N,m} = v_2^{r(N)}\gamma'_{N,m} = v_2^{r(N)}\gamma''_{N,m} = 0.$$

By Lemma 4.16, Lemma 4.14 and Theorem 4.10 there is an isomorphism

$$E_{r(N)+1}(\mathrm{THH}(BP\langle 2\rangle)) \cong E_{r(N+1)}(\mathrm{THH}(BP\langle 2\rangle)).$$

Also, note that $M_N \oplus V_{N+1} = M_{N+1}$ and $\lambda_N\mu_3^{(p-1)p^N-1} = \lambda_{N+3}$ by definition. This completes the inductive step and consequently the proof. $\square$

5. The v_1-Bockstein spectral sequence

In this section we will begin our analysis of the v_1-Bockstein spectral sequence (2.6) for computing the homotopy of $\mathrm{THH}(BP\langle 2\rangle; k(1))$. We will take a similar approach to the previous section. To start, we need to compute $K(1)_*BP\langle 2\rangle$. This requires determining $\eta_R(v_{2+n})$ in $K(1)_*BP$ modulo the ideal generated by $(\eta_R(v_3), \ldots, \eta_R(v_{1+n}))$. We will need the following.

Lemma 5.1. [12, Lemma A.2.2.5] *Let v_n denote the Araki generators. Then there is the following equality in BP_*BP*

$$\sum_{i,j\geq 0}^{F} t_i\eta_R(v_j)^{p^i} = \sum_{i,j\geq 0}^{F} v_i t_j^{p^i}$$

In our context, the distinction between Hazewinkel generators and Araki generators is unimportant, as the two sets of generators coincide modulo p. In $K(1)_*BP$, we have killed all v_i's except v_1, which gives us the following equation

$$\sum_{i,j\geq 0}^{F} t_i\eta_R(v_j)^{p^i} = \sum_{k\geq 0}^{F} v_1 t_k^{p}$$

Note that the following degrees of the terms:

$$|v_1t_j^p| = 2(p^{j+1}-1)$$
$$|t_i\eta_R(v_j)^{p^i}| = 2(p^{i+j}-1)$$

Since we are interested in the term $\eta_R(v_{2+n})$, we collect all the terms on the left of degree $2(p^{2+n}-1)$. Thus we are summing over the ordered pairs (i,j) such that $i+j=2+n$. Since we only care about $\eta_R(v_{2+n})$ modulo $\eta_R(v_3),\dots,\eta_R(v_{1+n})$ we only need to collect the terms where $j=1,2$, or $2+n$. This shows that

$$t_{1+n}\eta_R(v_1)^{p^{n+1}} + t_n\eta_R(v_2)^{p^n} + \eta_R(v_{n+2}) = v_1t_{n+1}^p$$

The value of η_R on v_1 and v_2 can also be computed by Lemma 5.1. One obtains, in $K(1)_*BP$, the following

$$\eta_R(v_1) = v_1$$
$$\eta_R(v_2) = v_1t_1^p - t_1v_1^p.$$

Combining these observations, we obtain

Lemma 5.2. *In $K(1)_*BP$, the following congruence is satisfied*

$$\eta_R(v_{2+n}) \equiv v_1t_{n+1}^p - v_1^{p^n}t_1^{p^{n+1}}t_n + v_1^{p^{n+1}}(t_1^{p^n}t_n - t_{n+1}) \mod (\eta_R(v_3),\dots,\eta_R(v_{1+n}))$$

for $n\geq 1$.

Consequently, we have the following corollary.

Corollary 5.3. *There is an isomorphism of $K(1)_*$-algebras*

$$K(1)_*BP\langle 2\rangle \cong K(1)_*BP/(v_1t_{n+1}^p - v_1^{p^n}t_1^{p^{n+1}}t_n + v_1^{p^{n+1}}(t_1^{p^n}t_n - t_{n+1}) \mid n\geq 1)$$

Define elements

$$u_n := v_1^{\frac{p^n-1}{p-1}}t_n.$$

These elements are in degree 0, and therefore there is an isomorphism

$$K(1)_*BP\langle 2\rangle \cong_{K(1)_*} K(1)_*\otimes_{\mathbb{F}_p} K(1)_0BP\langle 2\rangle.$$

The calculations above imply the following corollary.

Corollary 5.4. *There is an isomorphism of $\mathbb{F}_p$-algebras*

$$K(1)_0R \cong \mathbb{F}_p[u_i \mid i \geq 1]/(u_{n+1}^p - u_1^{p^{n+1}}u_n + u_1^{p^n}u_n - u_{n+1} \mid n \geq 1).$$

Our goal is to use this and the $K(1)$-based Bökstedt spectral sequence to compute the $K(1)$-homology of $\mathrm{THH}(R)$. This is a spectral sequence of the form

$$E^2_{s,t} = \mathrm{HH}_s^{K(1)_*}(K(1)_*R) \implies K(1)_{s+t}\,\mathrm{THH}(R).$$

The above considerations tell us that the E^2-page is

$$E^2 \cong K(1)_* \otimes \mathrm{HH}^{\mathbb{F}_p}(K(1)_0R).$$

The following will be useful for our calculation.

Lemma 5.5 ([11]). *Let $V = \mathrm{Spec}(A)$ be a nonsingular affine variety over a field k. Let W be the subvariety of $V \times \mathbb{A}^n$ defined by equations*

$$g_i(Y_1,\dots,Y_n) = 0,\ g_i \in A[Y_1,\dots,Y_n],\ i = 1,\dots,n.$$

Then the projection map $W \to V$ is étale at a point $(P; b_1,\dots,b_n)$ of W if and only if the Jacobian matrix $\left(\frac{\partial g_i}{\partial Y_j}\right)$ is a nonsingular matrix at $(P; b_1,\dots,b_n)$.

Theorem 5.6 (Étale Descent, [13]). *Let $A \hookrightarrow B$ be an étale extension of commutative k-algebras. Then there is an isomorphism*

$$\mathrm{HH}_*(B) \cong \mathrm{HH}_*(A) \otimes_A B$$

Example 5.7. Consider the subalgebra

$$\mathbb{F}_p[u_1,u_2]/(u_2^p - u_1^{p^2+1} + u_1^{p+1} - u_2 = f_1).$$

We will regard this as a $\mathbb{F}_p[u_1]$-algebra. The partial derivative $\partial_{u_2}f_1$ is $-1 \pmod p$, and therefore a unit at every point. Then Lemma 5.5 tells us that this algebra is then étale over $\mathbb{F}_p[u_1]$.

By the same argument given above, we claim that there are a sequence of subalgebras A_n of

$$K(1)_0R \cong \mathbb{F}_p[u_i \mid i \geq 1]/(u_{n+1}^p - u_1^{p^{n+1}}u_n + u_1^{p^n}u_n - u_{n+1} \mid n \geq 1) =: A.$$

such that each map $A_i \hookrightarrow A_{i+1}$ is an étale extension. Here

$$A_n := \mathbb{F}_p[u_1,u_2,\dots u_n]/(u_{k+1}^p - u_1^{p^{k+1}}u_k + u_1^{p^k}u_k - u_{k+1} = f_k \mid 1 \leq k \leq n)$$

and the partial derivative $\partial_{u_k}f_k = -1 \pmod p$ for all $1 < k \leq n$ and therefore a unit at each point. The claim then follows by Lemma 5.5.

By the étale base change formula for Hochschild homology in Theorem 5.6, there is an isomorphism

$$\mathrm{HH}_*^{\mathbb{F}_p}(A_{i+1}) \cong \mathrm{HH}_*^{\mathbb{F}_p}(A_i) \otimes_{A_i} A_{i+1}$$

and since the functors $HH_*(-)$ and $\mathrm{HH}_*^{\mathbb{F}_p}(A_1)\otimes_{A_1}(-)$ commute with filtered colimits of $\mathbb{F}_p$-algebras, there are isomorphisms

$$\begin{aligned}\mathrm{HH}_*^{\mathbb{F}_p}(A) &\cong \mathrm{HH}_*^{\mathbb{F}_p}(\operatorname{colim} A_n)\\ &\cong \operatorname{colim}\mathrm{HH}_*^{\mathbb{F}_p}(A_n)\\ &\cong \operatorname{colim}\mathrm{HH}_*^{\mathbb{F}_p}(A_1)\otimes_{A_1} A_n\\ &\cong \mathrm{HH}_*^{\mathbb{F}_p}(A_1)\otimes_{A_1} A.\end{aligned}$$

Consequently,

$$\mathrm{HH}_*(K(1)_*BP\langle 2\rangle)\cong K(1)_*\otimes E(\sigma t_1)\otimes K_0(R)$$

and therefore, since $\sigma t_1 \doteq \lambda_1$,

$$K(1)_*\,\mathrm{THH}(BP\langle 2\rangle)\cong K(1)_*BP\langle 2\rangle\otimes E(\lambda_1)$$

and

$$\mathrm{THH}_*(BP\langle 2\rangle;K(1))\cong K(1)_*\otimes E(\lambda_1).$$

In other words,

$$THH_*(BP\langle 2\rangle;k(1))\cong F\oplus T$$

where F is a free $P(v_1)$-module generated by 1 and λ_1 and T is a torsion $P(v_1)$-module. In summary, we have proven the following theorem.

Theorem 5.8.

(1) *The* $K(1)$*-homology of* $\mathrm{THH}(R;K(1))$ *is* $K(1)_*K(1)\otimes E(\lambda_1)$

(2) *Their is a weak equivalence*

$$K(1)\vee\Sigma^{2p-1}K(1)\simeq \mathrm{THH}(R;K(1)).$$

(3) *The only* v_1*-torsion free classes in* $\mathrm{THH}(R;k(1))$ *are* v_1^k *and* $\lambda_1 v_1^k$ *for* $k\geq 0$.

5.1. **Differentials in the v_1-BSS.** We now analyze the v_1-BSS (2.6). Recall that this spectral sequence is of the form

$$\mathrm{THH}(BP\langle 2\rangle;\mathbb{F}_p)[v_1]\implies \mathrm{THH}(B;k(1)).$$

Thus the E_1-page is

$$E(\lambda_1,\lambda_2,\lambda_3)\otimes P(\mu_3)\otimes P(v_1). \tag{5.9}$$

Since the λ_i are all in odd total degree and since v_1^k are known to be v_1-torsion free for all k, the λ_i are all permanent cycles. If μ_3 were a permanent cycle, then by multiplicativity, the spectral sequence would collapse. This would contradict Theorem 5.8. Therefore, the element μ_3 must support a differential. The only possibility for bidegree reasons is

$$d_{p^2}(\mu_3)\doteq v_1^{p^2}\lambda_2.$$

Thus, we obtain

$$v_1^{-1}E_{p^2+1}\cong K(1)_*\otimes E(\lambda_1,\lambda_3,\lambda_4')\otimes P(\mu_3^p)$$

where

$$\lambda_4' := \lambda_2\mu_3^{p-1}.$$

So the bidegree of λ_4' is given by

$$|\lambda_4'| = (2p^4-2p^3+2p^2-1,0).$$

For analogous reasons, the class λ_4' is a permanent cycle, and μ_3^p cannot be a permanent cycle. Based on degree considerations, there are two possible differentials,

$$d_{p^2}(\mu_3^p)\doteq v_1^{p^2}\lambda_4' \quad \text{or} \quad d_{p^3}(\mu_3^p)\doteq v_1^{p^3}\lambda_3.$$

The first would contradict the Leibniz rule for d_{p^2} and the fact that $d_{p^2}(\mu_3)\doteq v_1^{p^2}\lambda_2$. Thus,

$$v_1^{-1}E_{p^3+1} \cong K(1)_* \otimes E(\lambda_1, \lambda_4', \lambda_5') \otimes P(\mu_3^{p^2})$$

where

$$\lambda_5' := \lambda_3\mu_3^{p(p-1)}.$$

The bidegree of λ_5' is

$$|\lambda_5'| = (2p^5 - 2p^4 + 2p^3 - 1, 0).$$

For degree reasons, the class λ_5' is a permanent cycle. As before, the class $\mu_3^{p^2}$ must support a differential. Degree considerations, again, give two possibilities

$$d_{p^3}(\mu_3^{p^2})\doteq v_1^{p^3}\lambda_5'$$

or

$$d_{p^4+p^2}(\mu_3^{p^2})\doteq v_1^{p^4+p^2}\lambda_4'.$$

The former would contradict the Leibniz rule, leaving the latter as the only possibility. This gives us

$$v_1^{-1}E_{p^4+p^2+1} \cong K(1)_* \otimes E(\lambda_1, \lambda_5', \lambda_6') \otimes P(\mu_3^{p^3})$$

where $\lambda_6' := \lambda_4'\mu_3^{p^2(p-1)}$. We will continue via induction. First we need some notation. We will recursively define classes λ_n' by

$$\lambda_n' := \begin{cases} \lambda_n & n = 1, 2, 3 \\ \lambda_{n-2}'\mu_3^{p^{n-4}(p-1)} & n \geq 4 \end{cases}$$

We let $d'(n)$ denote the topological degree of λ_n'. Then this function is given recursively by

$$d'(n) = \begin{cases} 2p^n - 1 & n = 1, 2, 3 \\ 2p^n - 2p^{n-1} + d(n-2) & n > 3 \end{cases}$$

Thus, by a simple induction, one has

$$d'(n) = \begin{cases} 2p^n - 1 & n = 1, 2, 3 \\ 2p^n - 2p^{n-1} + 2p^{n-2} - 2p^{n-3} + \cdots + 2p^2 - 1 & n \equiv 0 \mod 2,\ n > 3 \\ 2p^n - 2p^{n-1} + 2p^{n-2} - 2p^{n-3} + \cdots + 2p^3 - 1 & n \equiv 1 \mod 2,\ n > 3 \end{cases}.$$

Observe that the integers $2p^{n+1} - d(n) - 1$ and $2p^{n+1} - d(n+1) - 1$ are both divisible by $|v_1|$. Let $r'(n)$ denote the integer

$$r'(n) := |v_1|^{-1}(|\mu_3^{p^{n-1}}| - |\lambda_n'| - 1) = |v_1|^{-1}(2p^{n+2} - d'(n+1) - 1).$$

Then a simple induction shows that

$$r'(n) = \begin{cases} p^{n+1} + p^{n-1} + p^{n-3} + \cdots + p^2 & n \equiv 1 \mod 2 \\ p^{n+1} + p^{n-1} + p^{n-3} + \cdots + p^3 & n \equiv 0 \mod 2 \end{cases}.$$

We can now describe the differentials in the v_1-BSS.

Theorem 5.10. *In the v_1-BSS, the following hold:*

(1) *The only nonzero differentials are in $v_1^{-1}E_{r'(n)}$.*
(2) *The $r'(n)$th page is given by*

$$v_1^{-1}E_{r'(n)} \cong K(1)_* \otimes E(\lambda_1, \lambda_n', \lambda_{n+1}') \otimes P(\mu_3^{p^{n-1}})$$

and the classes $\lambda_n', \lambda_{n+1}'$ are permanent cycles.
(3) *The differential $d_{r'(n)}$ is uniquely determined by multiplicativity of the BSS and the differential*

$$d_{r'(n)}(\mu_3^{p^{n-1}}) \doteq v_1^{r'(n)}\lambda_n'.$$

Proof. We proceed by induction. We have already shown the theorem for $n \leq 4$. Assume inductively that

$$v_1^{-1}E_{r'(n)} \cong K(1)_* \otimes E(\lambda_1, \lambda_n', \lambda_{n+1}') \otimes P(\mu_3^{p^{n-1}}).$$

By the inductive hypothesis, λ_n' is an infinite cycle.

Since $\lambda_n', \lambda_{n+1}'$ are both in odd topological degree, λ_{n+1}' cannot support a differential hitting the v_1-towers on λ_i' for $i < n+1$. Thus, the only possibility is that λ_{n+1}' supports a differential into the v_1-tower on 1 or λ_1. But this would contradict Theorem 5.8. Therefore, the class λ_{n+1}' is a permanent cycle.

The class $\mu_3^{p^{n-1}}$ must support a differential, for if it did not, then the spectral sequence would collapse. This would lead to a contradiction of Theorem 5.8. Degree considerations show that the following differentials are possible

$$d_{k(n)}(\mu_3^{p^{n-1}}) \doteq v_1^{k(n)}\lambda_{n+1}'$$

and

$$d_{r(n)}(\mu_3^{p^{n-1}}) \doteq v_1^{r'(n)}\lambda_n'$$

where

$$\ell(n) = |v_1^{-1}|(2p^{n+2} - |\lambda_{n+1}'|).$$

An elementary inductive computation shows that

$$\ell(n) = r'(n-1).$$

The former differential cannot occur, for by the inductive hypothesis,

$$d_{r'(n-1)}(\mu_3^{p^{n-2}}) \doteq \lambda_{n-1}',$$

and the former differential would contradict the Leibniz rule. So we must conclude the latter differential occurs. This concludes proof. □

We now state the main result of this section.

Theorem 5.11. *For each $n \geq 2$ and each nonnegative integer m with $m \not\equiv p-1$ mod p there are elements $z_{n,m}$ and $z_{n,m}'$ in $THH_*(BP\langle 2\rangle; k(1))$ such that*

(1) *$z_{n,m}$ projects to $\lambda_n'\mu_3^{mp^{n-2}}$ in $E_\infty^{*,0}$*
(2) *$z_{n,m}'$ projects to $\lambda_n'\lambda_{n+1}'\mu_3^{mp^{n-2}}$ in $E_\infty^{*,0}$*

As a $P(v_1)$-module, $THH_*(BP\langle 2\rangle; k(1))$ is generated by the unit element 1, λ_1, and the elements $z_{n,m}, z'_{n,m}$. The only relations are

$$v_1^{r'(n)} z_{n,m} = v_1^{r'(n)} z'_{n,m} = 0.$$

To prove this, we first need to prove a lemma.

Lemma 5.12. *For $r \geq 2$ with $n = 2$ and $k = 1$, the $E_r(THH(BP\langle 2\rangle))$-page of the Adams spectral sequence (**??**) is generated by elements in filtration 0 as a $P(1)$-module and $E_r^{*,*}$ is a direct sum of copies of $P(1)$ and $P(1)_i$ for $i \leq r$.*

Proof. We will begin by proving the first statement by induction. Note that (5.9) implies the base case in the induction when $r = 2$. Suppose the statement holds for some r. Choose a basis y_i for the $\mathbb{F}_p$-vector space V_r such that $V_r = \{x \in E_r^{*,0} \mid v_1^{r-1}x = 0\}$. Then $d_r(y_i)$ is in filtration r and since the differentials are v_1-linear, $v_1^{r-1} d_r(y_i) = 0$. However, this contradicts the induction hypothesis because the induction hypothesis implies that all elements in filtration r are v_1-torsion-free. Thus, each basis element y_i is a d_r-cycle. Next choose a set of elements $\{y'_j\} \subset E_r^{*,0}$ such that $\{d_r(y'_j)\}$ is a basis for $\mathrm{im}(d_r\colon\ E_r^{*,0} \to E_r^{*,r})$. Choose $y''_j \in E_r^{*,0}$ such that $v_1^r y''_j = d_r(y'_j)$. Then y''_j are d_r-cycles and y''_j and y_j are linearly independent. We can therefore choose d_r-cycles y'''_j such that $\{y_j\} \cup \{y''_j\} \cup \{y'''_j\}$ are a basis for the d_r-cycles in $E_r^{*,0}$. Then $\{y_j\} \cup \{y'_j\} \cup \{y''_j\} \cup \{y'''_j\}$ are a basis for $E_r^{*,0}$ and the differential is completely determined by the formulas

$$d_r(y_i) = 0, \quad d_r(y'_j) = v_1^r y''_j, \quad d_r(y''_j) = 0, \text{ and } \quad d_r(y'''_j) = 0.$$

Thus, $E_r^{*,*}$ is generated as a $P(1)$-module by y_i, y''_i, and y'''_i where $v_1^{r-1} y_i = 0$ and $v_1^r y''_i = 0$ and y'''_i is v_1-torsion free. □

Proof of Theorem 5.11. For brevity, we will let $\delta_{n,m}$ denote $\lambda'_n \mu^{mp^{n-2}}$ and we will let $\delta'_{n,m}$ denote $\lambda'_n \lambda'_{n+1} \mu^{mp^{n-2}}$. By Lemma 5.12 and Lemma 4.14 it suffices to prove that the elements $\delta_{n,m}$, and $\delta'_{n,m}$ are infinite cycles that, together with 1 and λ_1, form a basis for $E_\infty^{*,0}$ as an $\mathbb{F}_p$-vector space, and that each of $\delta_{n,m}$, $\delta'_{n,m}$ are killed by $v_1^{r'(n)}$. By induction on n, we will prove

$$E_{r(n)}(THH(BP\langle 2\rangle)) \cong M_n \oplus E(\lambda_1, \lambda'_n, \lambda'_{n+1}) \otimes P(\mu_3^{p^{n-1}})$$

where M_n is generated by $\{\delta_{k,m}, \delta'_{k,m} \mid k < n\}$ modulo the relations

$$v_2^{r(k)} \delta_{k,m} = v_2^{r(k)} \delta'_{k,m} = 0.$$

This statement holds for $n = 1$ by (5.9). Assume the statement holds for all integers less than or equal to some $N \geq 1$. Lemma 5.12, Lemma 4.14 and Theorem 4.10 imply that the only nontrivial differentials with source in $E_{r(N)}^{*,0}$ are the differentials

$$d_{r(N)}(\mu^{(m+1)r'(N)}) = (m+1) v_1^{r'(N)} \lambda'_N \mu^{mp^{N-1}} \doteq \delta_{N,m},$$

and the differentials

$$d_{r'(N)}(\lambda'_{N+1} \mu^{(m+1)r'(N)}) = (m+1) v_1^{r'(N)} \lambda'_n \lambda'_{N+1} \mu^{mp^{N-2}} \doteq \delta'_{N,m}$$

where $m \not\equiv p-1 \mod p$. Combining this with Lemma 5.12 and Lemma 4.14, this implies that

$$E_{r(N)+1}(\mathrm{THH}(BP\langle 2\rangle)) \cong M_n \oplus V_{N+1} \oplus \left(P(2) \otimes E(\lambda_1, \lambda_N', \lambda_N' \mu_3^{(p-1)p^{N-2}}) \otimes P(\mu^{p^N})\right)$$

where V_{N+1} has generators $\delta_{N,m}$ and $\delta'_{N,m}$ and relations

$$v_2^{r(N)}\delta_{N,m} = v_2^{r(N)}\delta'_{N,m} = 0.$$

By Lemma 5.12, Lemma 4.14 and Theorem 5.10 there is an isomorphism

$$E_{r(N)+1}(\mathrm{THH}(BP\langle 2\rangle)) \cong E_{r(N+1)}(\mathrm{THH}(BP\langle 2\rangle)).$$

Also, note that $M_N \oplus V_{N+1} = M_{N+1}$ and $\lambda'_N\mu_3^{(p-1)p^{N-2}} = \lambda'_{N+2}$ by definition. This completes the inductive step and consequently the proof. $\square$

6. The v_2-inverted homotopy of $\mathrm{THH}(BP\langle 2\rangle)$

In this section, we briefly describe the calculation of $\mathrm{THH}(BP\langle 2\rangle; E(2))$ based on recent work of Ausoni-Richter [?qx]. In [?qx], they assume that an E_∞-model for $E(2)$ exists, so for this section we will also make this assumption. There is a map

$$\mathrm{THH}(BP\langle 2\rangle; E(2)) \to \mathrm{THH}(E(2))$$

induced by the localization map

$$BP\langle 2\rangle \to v_2^{-1}BP\langle 2\rangle \simeq E(2).$$

We first recall the main theorem of [?qx]

Theorem 6.1 (Ausoni-Richter [?qx])**.** *There is an equivalence*

$$\mathrm{THH}(E(2)) \simeq E(2) \vee \Sigma^{2p-1}L_1E(2) \vee \Sigma^{2p^2-1}L_0E(2) \vee \Sigma^{2p^2+2p-2}L_0E(2).$$

Since there is an equivalence $THH(BP\langle 2\rangle) \wedge_{BP\langle 2\rangle} E(2) \simeq THH(E(2))$, we get the following corollay.

Corollary 6.2. *There is an equivalence*

$$THH(BP\langle 2\rangle; E(2)) \simeq E(2) \vee \Sigma^{2p-1}L_1E(2) \vee \Sigma^{2p^2-1}L_0E(2) \vee \Sigma^{2p^2+2p-2}L_0(E(2)$$

Note that this is consistent with our computation of $L_0THH(BP\langle 2\rangle)$, our computation of $THH_*(BP\langle 2\rangle; E(2)/(p, v_1))$, and out computation of

$$THH_*(BP\langle 2\rangle; BP\langle 1\rangle)$$

which we computed independently and therefore did not depend on $E(2)$ being a commutative ring spectrum.

Note that $\pi_*(v_1^{-1}S/p \wedge E(2)) \cong P(v_1^{\pm 1}, v_2^{\pm 1})$. In particular, this also implies the following.

Corollary 6.3. *There is an equivalence*

$$THH(BP\langle 2\rangle; E(2)/p) \simeq E(2)/p \vee \Sigma^{2p-1}(L_1E(2))/p$$

and consequently, an equivalence

$$v_1^{-1}S/p \wedge THH(BP\langle 2\rangle; E(2)) \simeq v_1^{-1}E(2)/p \vee \Sigma^{2p-1}v_1^{-1}(L_1E(2))/p$$

and therefore the free part of $THH_*(BP\langle 2\rangle; BP\langle 2\rangle/p)$ *as a* $P(v_1, v_2)$*-module is generated by* 1 *and* λ_1*.*

7. Topological Hochschild comology of $BP\langle 2\rangle$ with coefficients

We will write $THC(BP\langle 2\rangle)$ for topological Hochschild cohomology of $BP\langle 2\rangle$, which may be defined as

$$THC(BP\langle 2\rangle) = F_{BP\langle 2\rangle^e}(BP\langle 2\rangle, BP\langle 2\rangle)$$

where $BP\langle 2\rangle^e := BP\langle 2\rangle \wedge BP\langle 2\rangle^{\mathrm{op}}$. We recall that there is a UCT spectral sequence computing $F_{BP\langle 2\rangle^e}(BP\langle 2\rangle, BP\langle 2\rangle)$ with input

$$Ext_{\pi_*(BP\langle 2\rangle^e)}(\pi_*(BP\langle 2\rangle), \pi_*(BP\langle 2\rangle)) \Rightarrow THC^*(BP\langle 2\rangle),$$

but this is usually not computable. With coefficients in $H\mathbb{F}_p$, however, it is easily computable in this case. First, note that

$$THH_*(BP\langle 2\rangle; H\mathbb{F}_p) \cong E(\lambda_1, \lambda_2, \lambda_3) \otimes P(\mu_3)$$

is a finite type graded $\mathbb{F}_p$-algebra and $THH(BP\langle 2\rangle; H\mathbb{F}_p))$ is an $H\mathbb{F}_p$-algebra. By adjunction,

$$F_{H\mathbb{F}_p}(THH(BP\langle 2\rangle), H\mathbb{F}_p) \simeq THC(BP\langle 2\rangle, H\mathbb{F}_p).$$

Consequently, the UCT spectral sequence collapses and

$$THC^*(BP\langle 2\rangle, H\mathbb{F}_p) \cong Hom_{\mathbb{F}_p}(THH_*(BP\langle 2\rangle; H\mathbb{F}_p), \mathbb{F}_p) \cong E(x_1, x_2, x_3) \otimes \Gamma(c_1)$$

where $|x_i| = 2p^i - 1$ and $|c_1| = 2p^3$. The classes x_i are dual to λ_i and the class $c_i = \gamma_i(c_1)$ is dual to μ_3^i.

8. The mod p homotopy of $\mathrm{THH}(BP\langle 2\rangle)$

[Gabe: I am currently working on this section. There is a lot of work to do here still. No need to read this at the moment since it is still a work in progress.]

In this section, we begin our study of the mod p homotopy of $\mathrm{THH}(BP\langle 2\rangle)$. We will assume that $p = 3$, since in this case the mod 3 Moore spectrum $V(0)$ is a ring spectrum. Our approach to this computation will be to make use of the *THH-May spectral sequence*, which was developed by the first author and Andrew Salch in [2] and applied by the first author in [1].

Let us briefly describe the strategy we will employ. The mod 3 homotopy of $\mathrm{THH}(BP\langle 2\rangle)$ is exactly the homotopy groups

$$\pi_*(\mathrm{THH}(BP\langle 2\rangle); \mathbb{Z}/3) := \pi_*(V(0) \wedge \mathrm{THH}(BP\langle 2\rangle)) = V(0)_* \mathrm{THH}(BP\langle 2\rangle).$$

To compute this, we will use the $V(0)$-based THH-May spectral sequence. Using the Whitehead filtration for $BP\langle 2\rangle$ as developed in [2], the THH-May spectral sequence based on $V(0)$ takes the form

$$\pi_*(\mathrm{THH}(H\pi_*BP\langle 2\rangle);\mathbb{Z}/3) \implies \pi_*(\mathrm{THH}(BP\langle 2\rangle);\mathbb{Z}/3) \tag{8.1}$$

In order to obtain the first several differentials in this spectral sequence, we will consider the $H\mathbb{F}_p \wedge V(0)$-based THH-May spectral sequence. This takes the form

$$H_*(V(0)\wedge \mathrm{THH}(H\pi_*BP\langle 2\rangle)) \implies H_*(V(0)\wedge \mathrm{THH}(BP\langle 2\rangle)). \tag{8.2}$$

The abutment of this spectral sequence is known from which we will determine the differentials in (8.2).

The morphism $V(0) \to H\mathbb{F}_p \wedge V(0)$ of spectra induces a morphism of THH-May spectral sequences

$$\begin{tikzcd} \pi_*(\mathrm{THH}(H\pi_*BP\langle 2\rangle);\mathbb{Z}/3) \arrow[r, Rightarrow] \arrow[d, hook] & \pi_*(\mathrm{THH}(BP\langle 2\rangle);\mathbb{Z}/3) \arrow[d] \\ H_*(V(0)\wedge \mathrm{THH}(H\pi_*BP\langle 2\rangle)) \arrow[r, Rightarrow] & H_*(V(0)\wedge \mathrm{THH}(BP\langle 2\rangle)) \end{tikzcd} \tag{8.3}$$

We will argue that the map on E^1-terms is injective, which will allow us to determine the first several pages of the $V(0)$-based THH-May spectral sequence.

8.1. **Review of the THH-May spectral sequence.** The THH-May spectral sequence takes as input a cofibrant decreasingly filtered commutative monoid I in spectra (specifically symmetric spectra of pointed simplicial sets with the positive stable flat model structure) and produces a spectral sequence

$$E_1^{*,*} = E_*THH(E_0^*I) \Rightarrow E_*THH(I_0)$$

for any connective generalized homology theory E First, recall the definition of a cofibrant decreasingly filtered commutative monoid in spectra.

Definition 8.4. A cofibrant decreasingly filtered commutative monoid in spectra I is a lax symmetric monoidal functor $\mathbb{N}^{\mathrm{op}} \to \mathcal{S}$, which is cofibrant in the projective model structure on the functor category.

Remark 8.5. Note that this differs slightly from the original definition in [2], every cofibrant decreasingly filtered commutative monoid in spectra in the sense of 8.4 is in particular a cofibrant decreasingly filtered commutative monoid in spectra in the sense of [2, Def. 3.1.2]. Also, in the final version of [2] the authors added a couple assumptions to the definition, but these turn out to be redundant in symmetric spectra of pointed simplicial sets with the positive stable flat model structure, so we do not include them here.

To a cofibrant decreasingly filtered commutative monoid I we can associate its associated graded commutative ring spectrum E_0^*I. It is constructed as a commutative ring spectrum in [2, Def. 3.16] and its underlying spectrum is defined to be $E_0^*I = \vee I_i/I_{i+1}$ where I_i is our decreasingly filtered commutative monoid evaluated at a natural number i, I_i/I_{i+1} is the cofiber of the cofibration $I_{i+1} \to I_i$ and the wedge is taken over all natural numbers. Given an object in $\mathsf{Fun}(\mathbb{N}^{\mathrm{op}}, \mathcal{S})$ one may easily produce an object $\mathsf{Fun}(d\mathbb{N}^{\mathrm{op}}, \mathcal{S})$, where $d\mathbb{N}^{\mathrm{op}}$ is the discrete category of natural numbers, and then take the colimit to produce E_0^*I additively. In [2, Def. 3.1.2], we explicitly describe how to make this construction multiplicative as well. The main result needed to construct the THH-May spectral sequence is the identification of the E^1-page as $E^1_{*,*} = E_{*,*}THH(E_0^*I)$.

In order to make use of this spectral sequence, one would like a large supply of cofibrant decreasingly filtered commutative ring spectra, and this is provided by [2, Thm. 4.2.1]. In other words, there is a model for the Whitehead tower of a connective cofibrant commutative ring spectrum A, written

$$\to \tau_{\geq 3}A \to \tau_{\geq 2}A \to \tau_{\geq 1}A \to \tau_{\geq 0}A$$

which is a cofibrant decreasingly filtered commutative monoid in spectra. The associated grade of this filtration can be identified with $H\pi_*A$, the generalized Eilenberg-MacLane spectrum of the differential graded algebra π_*A.

8.2. The THH-May spectral sequence with coefficients in $k(2)$.

[Gabe: We need to be more careful about multiplicativity of this spectral sequence. Just because $V(1)_*THH(BP\langle 2\rangle)$ is a ring spectrum doesn't mean that the spectral sequence is multiplicative. For example, $V(1)_*THH(BP\langle 2\rangle)^{hS^1}$ is a ring spectrum but the homotopy fixed point spectral sequence computing it is not multiplicative. This is an unfortunate consequence of the rigid requirements of the THH-May spectral sequence, but by a change of framework there is an easy fix.]

In this section, we analyze THH-May spectral sequence with coefficients in the $\tau_{\geq\bullet}BP\langle 2\rangle$-algebra $\tau_{\geq\bullet}k(2)$. For a general A_∞ $BP\langle 2\rangle$-algebra M, there is an associated $\tau_{\geq\bullet}BP\langle 2\rangle$-algebra $\tau_{\geq\bullet}M$ and a multiplicative spectral sequence

$$E^1(BP\langle 2\rangle; M) = THH_{*,*}(H\pi_*BP\langle 2\rangle; H\pi_*M) \Rightarrow THH_*(BP\langle 2\rangle; M). \tag{8.6}$$

We will show that $E^5(BP\langle 2\rangle; k(2)$ is a reindexed version of the v_2-Bockstein spectral sequence converging to $\mathrm{THH}_*(BP\langle 2\rangle; k(2))$ which will tell us the remaining differentials in this spectral sequence. There is a map of THH-May spectral sequences

$$\begin{array}{ccc} E^1(BP\langle 2\rangle; BP\langle 2\rangle/p) & \Longrightarrow & \mathrm{THH}(BP\langle 2\rangle; BP\langle 2\rangle/p) \\ \downarrow & & \downarrow \\ E^1(BP\langle 2\rangle; k(2)) & \Longrightarrow & \mathrm{THH}(BP\langle 2\rangle; k(2)) \end{array}$$

induced by the map

$$\tau_{\geq\bullet}B/p \to \tau_{\geq\bullet}k(2)$$

of $\tau_{\geq\bullet}BP\langle 2\rangle$-algebras which we will use to import differentials into the THH-May spectral sequence with coefficients in $BP\langle 2\rangle/p$.

Our goal is to show that

$$E^5(BP\langle 2\rangle; k(2)) \cong E(\lambda_1, \lambda_2, \lambda_3) \otimes P(\mu_3) \otimes P(v_2).$$

We begin by analyzing the $H\mathbb{F}_p$-THH-May spectral sequence:

$$H_*(\mathrm{THH}(H\pi_*BP\langle 2\rangle; H\pi_*k(2))) \implies H_*(THH(BP\langle 2\rangle; k(2))). \tag{8.7}$$

The abutment is known as a consequence of [4]

$$H_*(\mathrm{THH}(BP\langle 2\rangle; k(2))) \cong E(\widetilde{\tau}_0, \widetilde{\tau}_1) \otimes A/\!\!/E(2)_* \otimes E(\lambda_1, \lambda_2, \lambda_3) \otimes P(\mu_3).$$

We first determine the E^1-page using the Bökstedt spectral sequence

$$\mathrm{HH}_*(H_*(H\pi_*BP\langle 2\rangle)) \implies H_*(\mathrm{THH}(H\pi_*BP\langle 2\rangle)).$$

We need to determine the E^1-term of this Bökstedt spectral sequence. In the following lemma, we will use the notation $R[x]$ for the ring spectrum $\bigvee_{i\geq 0} \Sigma^{|x|i}R$ and $R[x,y]$ for $R[x] \wedge_R R[y]$ where $|x| \geq 0$, $|y| \geq 0$ and R is a ring spectrum.

Lemma 8.8. *The mod 3 homology of $H\pi_*BP\langle 2\rangle$ is $A/\!\!/E(0)_* \otimes P(v_1, v_2)$ where v_1 and v_2 are comodule primitives. Consequently, $HH_*(H_*(H\pi_*BP\langle 2\rangle))$ is isomorphic to*

$$H_*(H\pi_*BP\langle 2\rangle) \otimes E(\sigma\bar{\xi}_n \mid n \geq 1) \otimes \Gamma(\sigma\overline{\tau}_k \mid k \geq 1) \otimes E(\sigma v_1, \sigma v_2).$$

as $\mathscr{A}_*$-comodule $H_*BP\langle 2\rangle$-Hopf algebras.

Proof. There is an equivalence of $H\mathbb{Z}_{(p)}$-algebras

$$H\pi_*BP\langle 2\rangle \simeq H\mathbb{Z}_{(p)}[v_1, v_2]$$

where $|v_1| = 2p-2$ and $|v_2| = 2p^2-2$. Thus, there is an isomorphism of rings

$$H_*H\pi_*BP\langle 2\rangle \cong \pi_*(H\mathbb{F}_p \wedge H\mathbb{Z}_{(3)}[v_1, v_2]) \cong \pi_*((H\mathbb{F}_p \wedge H\mathbb{Z}_{(3)} \wedge_{H\mathbb{F}_p} H\mathbb{F}_p[v_1, v_2])$$

and the result follows by the Künneth isomorphism. The element v_1 in homology arises from the inclusion of the summand indexed by v_1. Applying homology to this map takes 1 to v_1. As this is a map of comodules, v_1 is necessarily primitive. A similar argument shows that v_2 is primitive.

The Hochschild homology of an exterior algebra $E(x)$ with a generator in x odd degree is $E(x) \otimes \Gamma(\sigma x)$ where $\Gamma(\sigma x)$ is a divided power algebra on a generator σx with $|\sigma x| = 1 + |x|$, by Koszul duality. The Hochschild homology of a polynomial algebra on a class y in even degree is $P(y) \otimes E(\sigma y)$ where $|\sigma y| = 1 + |y|$. (See [10] for details). Therefore, the result follows by the Künneth isomorphism for Hochschild homology. □

Proposition 8.9. *There is an isomorphism*

$$H_*\,\mathrm{THH}(H\pi_*BP\langle 2\rangle) \cong H_*(H\pi_*BP\langle 2\rangle) \otimes E(\lambda_1) \otimes P(\mu_1) \otimes E(\sigma v_1, \sigma v_2)$$

of $\mathscr{A}_*$-comodule $H_*H\pi_*BP\langle 2\rangle$-Hopf algebras.

Proof. The map of commutative rings $H\mathbb{Z}_p \to H\pi_*BP\langle 2\rangle$ induces a multiplicative map of Bökstedt spectral sequences. This completely determines the differentials and hidden extensions in the Bökstedt spectral sequence

$$HH_*(H_*(H\pi_*BP\langle 2\rangle)) \Rightarrow H_*THH(H\pi_*BP\langle 2\rangle)$$

□

Note that the May filtration of σv_1 and σv_2 are $2(p-1)$ and $2(p^2-1)$ respectively. Since the May filtration is always divisible by $2(p-1)$, we reindex to give σv_1 and σv_2 May filtration 1 and 4 respectively.

Since $H_*H\pi_*k(2) \cong \mathscr{A}_* \otimes P(v_2)$ and $H_*\,\mathrm{THH}(H\pi_*BP\langle 2\rangle)$ is free over $H_*(H\pi_*BP\langle 2\rangle)$ the collapse of the Eilenberg-Moore spectral sequence implies the isomorphism

$$H_*(\mathrm{THH}(H\pi_*BP\langle 2\rangle) \wedge_{H\pi_*BP\langle 2\rangle} H\pi_*k(2))) \cong \mathscr{A}_* \otimes P(v_2) \otimes E(\lambda_1) \otimes P(\mu_1) \otimes E(\sigma v_1, \sigma v_2)$$

Note that the coaction on $\sigma\tau_1$ is

$$\nu(\mu_1) = 1 \otimes \sigma\tau_1 + \overline{\tau}_0 \otimes \sigma\xi_1$$

and $\sigma\bar{\xi}_1$ is primitive. We therefore let λ_1 denote $\sigma\bar{\xi}_1$ and define

$$\mu_1 := \sigma\tau_1 - \overline{\tau}_0\lambda_1,$$

which is a comodule primitive.

We can then establish

Proposition 8.10. *In the spectral sequence (8.7), we have the differentials*

(1) $d_1\mu_1 \doteq \sigma v_1$,
(2) $d^{p+1}\overline{\tau}_2 \doteq v_2$
(3) $d^{p+1}\mu_1^3 \doteq \sigma v_2$

and the classes λ_2, λ_3 are detected by ${\widetilde{\mu_1}}^2\sigma v_1$ and ${\widetilde{\mu_1}}^6\sigma v_2$ respectively[1].

Since $H\pi_*k(2)$ is an $H\mathbb{F}_p$-algbera, $THH(H\pi_*BP\langle 2\rangle; H\pi_*k(2))$ is also an $H\mathbb{F}_p$-algebra.

Corollary 8.11. *There is an isomorphism*

$$E^1(BP\langle 2\rangle; k(2)) \cong E(\lambda_1, \sigma v_1, \sigma v_2) \otimes P(\widetilde{\mu_1}, v_2).$$

There is a differential

$$d^1(\widetilde{\mu_1}) \doteq \sigma v_1$$

and multiplicativity of the spectral sequence implies all remaining d^1 differentials.

From this we can determine the E^2-page,

Corollary 8.12. *One has*

$$E^2(BP\langle 2\rangle; k(2)) \cong E(\lambda_1, \sigma v_2, \widetilde{u}_1^2\sigma v_1) \otimes P(\widetilde{\mu}_1^3, v_2).$$

Furthermore, the map on the E^2-term induced by the Hurewicz map is injective. Since the $H\mathbb{F}_p$-THH-May spectral sequence has no d^3 or d^4 differentials, the map on E^3 and E^4-terms is again injective. The next differential is

$$d^4(\mu_1^3) \doteq \sigma v_2$$

and there are no further d^4 differentials except those implied by the Leibniz rule. This results in

$$E^5(BP\langle 2\rangle; k(2)) \cong E(\lambda_1, {\widetilde{\mu_1}}^2\sigma v_1, {\widetilde{\mu_1}}^6\sigma v_2) \otimes P({\widetilde{\mu_1}}^9).$$

[1]We needed the class $\widetilde{\mu_1}$ as opposed to μ_1 since λ_2, λ_3 are comodule primitives.

Note that

$$\mu_1^9 = \mu_3.$$

Thus we can rename μ_1^9 as μ_3. Renaming classes, the E^5-page is

$$E^5 \cong E(\lambda_1, \lambda_2, \lambda_2) \otimes P(\mu_3, v_2)$$

where v_2 is in May filtration 4. Thus the E^5-page is a reindexed form of the v_2-BSS, and this determines the rest of the $V(1)$-May spectral sequence.

[Dom: Here is an idea: to make things easier for the reader, maybe we should have a table somewhere in the paper with all the names of the various elements, their representatives, their coactions, etc.]

8.3. **The THH-May spectral sequence for $BP\langle 2\rangle$ with coefficients in $BP\langle 2\rangle/p$.** We now study the THH-May spectral sequence (8.6) for $M = BP\langle 2\rangle/p$.

[Gabe: Do we know $BP\langle 2\rangle/p$ is A_∞? We know it is a ring spectrum for $p = 3$ by Oka, which may be enough.]

We begin by determining the E^1-term. We recall the following standard fact.

Lemma 8.13 (cf. [1])**.** *Let M be an $H\mathbb{F}_p$-module. Then M is equivalent to a wedge of suspensions of $H\mathbb{F}_p$, and the Hurewicz map*

$$\pi_*M \to H_*M$$

is an injection onto the A_*-comodule primitives.

Proposition 8.14. *The E^1-term of (8.1) is isomorphic to*

$$E(\lambda_1) \otimes P(\mu_1) \otimes P(v_1, v_2) \otimes E(\sigma v_1, \sigma v_2),$$

and the spectral sequence has only the d^1 differential, $d^1\mu_1 \doteq \sigma\, v_1$. Consequently, the E^2-term of (8.1) is

$$E(\lambda_1, \mu_1^2\sigma v_1) \otimes P(\mu_1^3) \otimes P(v_1, v_2) \otimes E(\sigma v_2).$$

Proof. The description of the E^1-term follows directly from the lemma and Proposition 8.9. Because the map on E^1-terms is injective, we can pull back differentials, which provides the state d_1 differential. [Note that that the Hurewicz map is NOT injective at the E^2-page so the same argument doesn't work for the later differential! This was my mistake in my initial calculation. In particular, there could be differentials hitting v_1^k times some element for some k. I'm starting to think that one of these differentials may actually occur.] ∎

We now use the fact that $\mu_1^2\sigma v_1$ detects λ_2 to rename this class. We also rename the class μ_1^p by μ_2.

Proposition 8.15. *There is a d^3 differential*

$$d^3(\mu_1^p) = v_1^3\lambda_1$$

and no further differentials of this length. The E^4 term of (8.1) is

$$H_*(E(\lambda_1) \otimes P(v_1) \otimes P(\mu_2); d^3(\mu_2) = v_1^3\lambda_1) \otimes E(\lambda_2) \otimes P(v_2) \otimes E(\sigma v_2)$$

The reader may be concerned at this point that $v_1^3\lambda_1$ dies and yet it survived in the first Bockstein spectral sequence computing $THH_*(BP\langle 2\rangle; k(1))$. However, note that in the THH-May spectral sequence the names of classes often change and there is still a class, namely σv_2, which survives in the degree of $v_1^3\lambda_1$.

Proof. Note that there is a map of THH-May spectral sequences with abutment

$$S/3_*THH(BP\langle 2\rangle) \to S/3_*THH(BP\langle 1\rangle)$$

and with input

$$P(\lambda_1) \otimes P(\mu_1) \otimes P(v_1, v_2) \otimes E(\sigma v_1, \sigma v_2) \to P(\lambda_1) \otimes P(\mu_1) \otimes P(v_1) \otimes E(\sigma v_1)$$

and by inspection all classes map to classes of the same name except v_2 and σv_2, which map to zero. In the target spectral sequence, we compute the differential $d_1(\mu_1) = \sigma v_1$ by the same means as we did before. Therefore, the map of E^2-terms is

$$P(\lambda_1, \lambda_2) \otimes P(\mu_2) \otimes P(v_1, v_2) \otimes E(\sigma v_2) \to P(\lambda_1, \lambda_2) \otimes P(\mu_2) \otimes P(v_1)$$

and again the classes all map to classes of the same name except v_2 and σv_2, which map to zero. Note that this verfies that the renaming of λ_2 and μ_2 is reasonable. The target of this map is exactly the same as the input of the Bökstein spectral seqeunce computing $THH_*(BP\langle 1\rangle; k(1))$ and therefore we know what the remaining differentials have to be by McClure-Staffeldt [10]. In particular, there is a differential $d^3(\mu_2) = v_1^3\lambda_1$ and this is the only differential of this length. This implies that the same differential takes place in the source spectral sequence. To see that there are no further differentials of this length in the source note that the only possiblility would be a differential with source σv_2 or v_2 and there are no possible differentials of this length on these classes for degree reasons. □

We now note that

$$H_*(E(\lambda_1)\otimes P(v_1)\otimes P(\mu_2); d^3(\mu_2) = v_1^p\lambda_1) \cong \left(P(v_1, \mu_2^3) \otimes \mathbb{F}_3\{1, \lambda_1, \lambda_1\mu_2\} \otimes E(\lambda_1\mu_2^2)\right) / \sim$$

where $\sim$ is the relation

$$\begin{gathered} \lambda_1 \cdot (\lambda_1\mu_2^2) = 0 \\ \lambda_1\mu_2 \cdot (\lambda_1\mu_2^2) = 0 \\ v_1^3 \cdot \lambda_1 = 0 \\ v_1^3 \cdot \lambda_1\mu_2 = 0. \end{gathered}$$

and the classes $\lambda_1\mu_2$ and $\lambda_1\mu_2^2$ are not in the output of either of the Bockstein spectral seqeunces.

$$THH_*(BP\langle 2\rangle, H\mathbb{F}_p)[v_1] \Rightarrow THH_*(BP\langle 2\rangle, k(1))$$

and

$$THH_*(BP\langle 2\rangle, H\mathbb{F}_p)[v_2] \Rightarrow THH_*(BP\langle 2\rangle, k(2))$$

and therefore they cannot survive the $S/3$-based THH-May spectral sequence. (Note that these classes are no longer decomposable). This forces the following differentials.

Lemma 8.16. *There is a differential $d_4(\lambda_1\mu_2) = \lambda_1\sigma v_2$ and $d_4(\lambda_1\mu_2^2) = \lambda_1\sigma v_2\mu_2$ which generates families of differentials by multiplicativity of the $S/3$-THH-May spectral sequence and no further differentials of this length.*

Proof. We know that the elements $\lambda_1\mu_2$ and $\lambda_1\mu_2^2$ must not be cycles by the argument above and the fact that they are not boundaries. We therefore check the possible targets of a differential and the possibilities are $\mu_2 v_1, \lambda_1\lambda_2, \lambda_1\sigma v_2$ for $\lambda_1\mu_2$. We now observe that there is no differential on $\lambda_1\mu_2$ in the $S/3$-THH-May spectral sequence computing $S/3_*THH(BP\langle 1\rangle)$ so if there is a differential on $\lambda_1\mu_2$ in the $S/3$-THH-May spectral sequence computing $S/3_*THH(BP\langle 2\rangle)$ it must hit something that maps to zero under the map of THH-May spectral sequences. The only one of the three classes named above that maps to zero under this map of spectral sequences is $\lambda_1\sigma v_2$. This forces the stated differential.

[Gabe: Add similar argument for the other differential]

[Dom: I think you can obtain a simpler proof by mapping to the $H\wedge V(0)$-based THH-May spectral sequence, wherein these differentials occur.]

□

We conclude that there is an isomorphism

[Gabe: As you pointed out, the formula below is not correct. I forgot $\lambda_1\mu_2\sigma v_2$ and $\lambda_2\mu_2^2 v_2$. I need to look back at this.]

$$E_{p+2} \cong E(\lambda_1, \lambda_2, \sigma v_2, \lambda_1\mu_2^2\sigma v_2) \otimes P(\mu_3, v_1, v_2)/\sim$$

where $v_1^p\lambda_1 \sim 0$, $\lambda_1\sigma v_2 \sim 0$, $\lambda_1\sigma v_2\mu \sim 0$, etc. There is therefore an additive isomorphism

$$E_{p+2} \cong E(\lambda_1, \lambda_2, \lambda_3) \otimes P(\mu_3, v_1, v_2)$$

where we make the additive identifications $\lambda_1 \cdot v_1^p \doteq \sigma v_2$, $\lambda_1 v_1^p \mu^{p-1} \doteq \lambda_3$, $\lambda_1\sigma v_2\mu^{p-1} = \lambda_1\lambda_3$, $\lambda_1\lambda_2\sigma v_2\mu^{p-1} \doteq \lambda_1\lambda_2\lambda_3$.

Note that the class $\lambda_2 v_1^9$ doesn't survive the Bockstein spectral sequence

$$THH_*(BP\langle 2\rangle; H\mathbb{F}_3)[v_1] \Rightarrow THH_*(BP\langle 2\rangle; k(1)),$$

but it does survive the Bockstein spectral sequence

$$THH_*(BP\langle 2\rangle; H\mathbb{F}_3)[v_2] \Rightarrow THH_*(BP\langle 2\rangle; k(2)).$$

Similarly, the class $\lambda_1 v_2^3$ doesn't survive the Bockstein spectral sequence

$$THH_*(BP\langle 2\rangle; H\mathbb{F}_3)[v_2] \Rightarrow THH_*(BP\langle 2\rangle; k(2)),$$

but it does survive the Bockstein spectral sequence

$$THH_*(BP\langle 2\rangle; H\mathbb{F}_3)[v_1] \Rightarrow THH_*(BP\langle 2\rangle; k(1)).$$

One may think that this forces a differential hitting $\lambda_2 v_1^9$ the second Bockstein spectral sequence $THH_*(BP\langle 2\rangle; k(1))[v_2] \Rightarrow THH_*(BP\langle 2\rangle; BP\langle 2\rangle/3)$ and a differential hitting $\lambda_1 v_2^3$ in the Bockstein spectral sequence $THH_*(BP\langle 2\rangle; k(2))[v_1] \Rightarrow THH_*(BP\langle 2\rangle; BP\langle 2\rangle/3)$. However, we conjecture that this doesn't happen.

Conjecture 8.17. In the THH-May spectral sequence, there is a differential

$$d^{12}(\mu_3) = \lambda_1 v_2^3$$

and a hidden multiplicative additive extension $\lambda_1 \cdot v_2^3 \doteq \lambda_2 \cdot v_1^9$.

[Dom: I am not sure about this conjecture... I say this because if you consider the map from the $V(0)$-based May spectral sequence to the $V(1)$-based May spectral sequence, then we can use the fact that the $V(1)$-May SS is a reindexed form of the v_2-BSS. There we have

$$d^3(\mu_3) \doteq v_2^3\lambda_1$$

which translates into a May differential

$$d^{12}(\mu_3) \doteq v_2^3\lambda_1.$$

The differential proposed would actually be a d^{36}-differential.]

[Gabe: There was a typo in the original statement. I expected one of two differentials and proposed one of them (but with an incorrect index). It seems to me that your comment proves that it is the other differential. That still means that there isn't a contradiction!]

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