Notes on $THH(tmf_1(3); k(2))$

1. In McClure-Staffeldt and Angeltveit-Rognes, the first step in computation of the Bockstein spectral sequence

$$THH_*(R; H\mathbb{F}_2)[v_n] \Rightarrow THH_*(R; k(n))$$

is to show that the unit map

$$R \to THH(R)$$

is a $K(n)_*$-equivalence and hence the map

$$L_{K(n)}R \to L_{K(n)}THH(R)$$

is an equivalence. The first question is then, is this true for $R = tmf_1(3)$ and $n = 2$? We have a sketch of a proof that we should write up rigorously (or at least we should write up the sketch before we forget it). [Also note that we may want to be careful about choosing the generators v_n for $tmf_1(3)$ vs. $BP < 2 >$ and whether that has an effect on the computation].

2. The next step is to show that the first item implies that

$$v_n^{-1}V \wedge R \to v_n^{-1}V \wedge THH(R)$$

is an en equivalence for some finite type n complex V. This is an easy consequence in McClure-Staffeldt, and in Angeltveit-Rognes they are able to do this using the notion of a μ-spectrum, which is weaker than a ring spectrum. The problem in our situation is that there doesn't exist a finite type n complex V such that $tmf_1(3) \wedge V \simeq k(2)$ (as is the case in all the other examples that McClure-Staffeldt and Angeltveit-Rognes consider).

We are therefore left with the following analogue of the same question: given item (1), is the map

$$K(2) \to THH(tmf; K(2))$$

an equivalence? (Note that for ℓ the spectrum $v_1^{-1}S/p \wedge \ell$ is exactly $K(1)$ and

$$THH(\ell; K(1)) \simeq THH(\ell) \wedge_\ell K(1) \simeq THH(\ell) \wedge_\ell \ell \wedge v_1^{-1}S/p$$

so this is really the same question).

3. We have a sketch that item (2) forces all the differentials in the Bockstein spectral sequence $THH_*(tmf_1(3); H\mathbb{F}_2)[v_2] \Rightarrow THH_*(tmf_1(3); k(2))$. Prove that it does and prove the pattern of differentials.
4. Compute the Bockstein spectral sequence

$$THH_*(tmf_1(3); H\mathbb{F}_2)[2] \to THH_*(tmf_1(3); H\mathbb{Z}_2)$$

5. Another miscellaneous question: As noted above, there is not a spectrum such that when we smash it with $tmf_1(3)$, we get $H\mathbb{F}_2$, but is there a spectrum (finite or maybe infinite complex) such that $V \wedge tmf_1(3)$ is as small as possible and $V_*THH(tmf_1(3))$ is reasonably computable? If so, then an alternate project that we could work on is computing this and then working on computing homotopy fixed points in order to detect v_3-periodic stuff.

(Feel free to add to or edit this list)