TOPOLOGICAL HOCHSCHILD HOMOLOGY OF TRUNCATED BROWN-PETERSON SPECTRA II

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Abstract. We compute topological Hochschild homology of the second truncated Brown-Peterson spectrum at primes $p \geq 3$ with Adams summand coefficients.

Contents

1. Introduction

1.1. **Conventions.** We write L_E for Bousfield localization at a spectrum E. We write $BP\langle n\rangle$ for a family of $\mathbb{E}_2$-MU-algebra forms of $BP\langle n\rangle$ such that

$$\mathrm{MU} \to \cdots \to \mathrm{BP}\langle n\rangle \to \mathrm{BP}\langle n-1\rangle \to \cdots \to H\mathbb{Z}_{(p)} \to H\mathbb{F}_p$$

and therefore we fix classes v_i such that on graded commutative rings

$$\mathrm{MU}_* \to \mathrm{BP}\langle n\rangle_*$$

is given by sending x_{p^i-1} to v_i for $0 \leq i \leq n$ (with $v_0 = p$) and $x_j \mapsto 0$ otherwise. This also fixes the map of graded commutative rings

$$\mathrm{BP}\langle n\rangle_* \to \mathrm{BP}\langle n-1\rangle_*$$

sending v_i to v_i for $0 \leq i \leq n-1$ and $v_n \mapsto 0$. Such a family exists by [6]. Alternatively, when $n \leq 2$ and $p \in \{2,3\}$, there are $\mathbb{E}_2$-MU-algebra forms of $\mathrm{BP}\langle 2\rangle$ denoted $\mathrm{tmf}_1(3)$ and taf^D respectively, which have the extra property that their $\mathbb{E}_2$-algebra structures lift to $\mathbb{E}_\infty$-algebra structures by [2, 7–9].

2. Recollections

In this section, we recall the necessery results from [1].

Proposition 2.1. *There is an isomorphism of $\pi_* L_{H\mathbb{Q}}\mathrm{BP}\langle n\rangle = \mathbb{Q}[v_1, \ldots, v_n]$-algebras*

$$\pi_* L_{H\mathbb{Q}}\,\mathrm{THH}(\mathrm{BP}\langle n\rangle) \cong \mathbb{Q}[v_1, \ldots, v_n] \otimes_{\mathbb{Q}} \Lambda_{\mathbb{Q}}(\sigma v_1, \ldots, \sigma v_n)\,.$$

Proof. The authors computed

$$\pi_* \operatorname{THH}(\mathrm{BP}\langle n\rangle; H\mathbb{Q}) = \Lambda_{\mathbb{Q}}(\sigma v_1, \ldots, \sigma v_n)\,.$$

We then observe that

$$L_{H\mathbb{Q}} \operatorname{THH}(\mathrm{BP}\langle n\rangle) \simeq \operatorname{THH}(\mathrm{BP}\langle n\rangle; L_{H\mathbb{Q}}\mathrm{BP}\langle n\rangle)$$

because $L_{H\mathbb{Q}}$ is a smashing localization. We then consider the spectral sequence

$$\operatorname{THH}_*(\mathrm{BP}\langle n\rangle; \mathbb{Q}) \otimes_{\mathbb{Q}} \pi_* L_{H\mathbb{Q}}\mathrm{BP}\langle n\rangle \implies \operatorname{THH}_*(\mathrm{BP}\langle n\rangle; L_{H\mathbb{Q}}\mathrm{BP}\langle n\rangle)$$

associated to the multiplicative complete filtration $\tau_{\geq \bullet} \operatorname{THH}(\mathrm{BP}\langle n\rangle; \mathbb{Q})$ in $H\mathbb{Q}$-modules. This spectral sequence has input

$$\pi_* L_{H\mathbb{Q}}\mathrm{BP}\langle n\rangle \otimes_{\mathbb{Q}} \Lambda_{\mathbb{Q}}(\sigma v_1, \ldots, \sigma v_n)$$

by [1, Proposition 3.7]. It collapses because the targets of all differentials whose source is an indecomposable algebra generator land in zero groups. The abutment and the E_∞-term are a free $\pi_* L_{H\mathbb{Q}}\mathrm{BP}\langle n\rangle$-modules and consequently there are no $\pi_* L_{H\mathbb{Q}}\mathrm{BP}\langle n\rangle)$-module extensions. There isn't room for algebra extensions. □

Proposition 2.2. *The groups*

$$\operatorname{THH}_s(\mathrm{BP}\langle n\rangle)$$

are finitely generated for all integers s. Consequently, we have

$$|\operatorname{THH}_s(\mathrm{BP}\langle n\rangle)| < \infty$$

for $s \neq 2p^i - 1 \mod 2p^j - 2$ for $1 \leq i, j \leq n$.

Proof. Since $\pi_a\mathbb{S}$ and $\pi_b\mathrm{BP}\langle n\rangle$ are finitely generated abelian groups for all integers a, b, the strongly convergent Künneth spectral sequence computing $\pi_*(\mathrm{BP}\langle n\rangle \wedge \mathrm{BP}\langle n\rangle)$ is finitely generated in each bidegree and has a vanishing line of postive slope so $\pi_c(\mathrm{BP}\langle n\rangle \wedge \mathrm{BP}\langle n\rangle)$ is finitely generated for each integer c. The same argument implies that $\operatorname{THH}_s(\mathrm{BP}\langle n\rangle)$ is finitely generated for each integer s. The second statement then follows from Proposition 2.1 and the classification of finitely generated $\mathbb{Z}_{(p)}$-modules. □

3. Bounding Hochschild homology of $\mathrm{BP}\langle n\rangle$

The goal of this section is to use the cosimplicial descent spectral sequence from work of [4] to produce a useful upper bound on $\operatorname{THH}_*(\mathrm{BP}\langle n\rangle)$.

Definition 3.1. Let $C^\bullet(A/B)$ denote the cosimplicial cobar complex with q-simplices $C^q(A/B) = A^{\otimes_B q+1}$.

First, we need a lemma.

Lemma 3.2. *Let $n \geq 1$. There is an isomorphism rings*

$$\mathrm{E}_2^{*,*} \cong \operatorname{Tor}^{\pi_*\mathrm{BP}\langle n-1\rangle \wedge \mathrm{BP}\langle n-1\rangle}(\mathrm{BP}\langle n-1\rangle, \mathrm{BP}\langle n-1\rangle) \otimes \Gamma\{\sigma^2 v_n^{(j)} : 1 \leq j \leq q\} \otimes \Lambda(\sigma v_1^{(j)} : 1 \leq j \leq q)$$

where

$$\mathrm{E}_2^{*,*} = \pi_* \left(\pi_* H\pi_* \operatorname{THH}(\mathrm{BP}\langle n-1\rangle)^{\wedge_{H\pi_* \operatorname{THH}(\mathrm{BP}\langle n\rangle)} q+1}\right)$$

is the E_2*-term of the multiplicative Künneth spectral sequence*

$$\mathrm{E}_2^{*,*} \implies \mathrm{THH}_*(\mathrm{BP}\langle n-1\rangle^{\wedge q+1}).$$

Proof. When $q = 0$, then $\mathrm{THH}(\mathrm{BP}\langle n-1\rangle^{\wedge_{\mathrm{BP}\langle n\rangle} q+1}) = \mathrm{THH}(\mathrm{BP}\langle n-1\rangle)$. We first compute $\pi_*(\mathrm{BP}\langle n-1\rangle \otimes_{\mathrm{BP}\langle n\rangle} \mathrm{BP}\langle n-1\rangle)$ by a Künneth spectral sequence. The E_2-term is $\mathrm{BP}\langle n-1\rangle_* \otimes \Lambda(\sigma v_n)$ so it is concentrated in Künneth filtration $[0,1]$ and therefore the spectral sequence collapses because the targets of all differentials are zero groups. We then use the equivalence

$$A \wedge_B A \wedge_B A \simeq (A \wedge_B A) \wedge_A (A \wedge_B A)$$

where $A = \mathrm{BP}\langle n-1\rangle$ and $B = \mathrm{BP}\langle n\rangle$ and the fact that $\pi_*(\mathrm{BP}\langle n-1\rangle \wedge_{\mathrm{BP}\langle n\rangle} \mathrm{BP}\langle n-1\rangle)$ is free as a $\mathrm{BP}\langle n-1\rangle_*$-module to inductively determine from the Künneth spectral sequence that

$$\pi_*(\mathrm{BP}\langle n-1\rangle^{\wedge_{\mathrm{BP}\langle n\rangle} q+1}) \cong \mathrm{BP}\langle n-1\rangle_* \otimes \Lambda(\sigma v_n^{(1)}, \dots, \sigma v_n^{(q)}).$$

By obstruction theory, we determine that $\mathrm{BP}\langle n-1\rangle^{\wedge_{\mathrm{BP}\langle n\rangle} q+1}$ is the smash product of square zero extensions

$$\left(BP\langle n-1\rangle \vee \Sigma^{2p-1} BP\langle n-1\rangle\right)^{\wedge_{BP\langle n-1\rangle} q}.$$

Consequently, we determine that

$$\pi_* \left(\mathrm{BP}\langle n-1\rangle^{\wedge_{\mathrm{BP}\langle n\rangle} q+1}\right)^{\wedge 2} \cong \pi_*(\mathrm{BP}\langle n-1\rangle^{\wedge 2}) \otimes \Lambda(\sigma v_n^{(1)}, \dots, \sigma v_n^{(q)})^{\wedge 2}.$$

We then note that the Künneth spectral sequence sequence computing

$$\mathrm{THH}_*(\mathrm{BP}\langle n-1\rangle^{\wedge_{\mathrm{BP}\langle n\rangle} q+1})$$

has E_2-term

$$\mathrm{Tor}^{\pi_*\mathrm{BP}\langle n-1\rangle\wedge\mathrm{BP}\langle n-1}(\mathrm{BP}\langle n-1\rangle, \mathrm{BP}\langle n-1\rangle) \otimes \Gamma(\sigma^2 v_n^{(1)}, \dots, \sigma^2 v_n^{(q)}) \otimes \Lambda(\sigma v_n^{(1)}, \dots, \sigma v_n^{(q)})$$

and that is exactly what we describe in the statement of the lemma. ☐

Proposition 3.3. *There is an equivalence*

$$\mathrm{THH}(\mathrm{BP}\langle n\rangle, \mathrm{BP}\langle n-1\rangle) \simeq \mathrm{Tot}\left(\mathrm{THH}(C^\bullet(\mathrm{BP}\langle n-1\rangle/\mathrm{BP}\langle n\rangle, \mathrm{BP}\langle n-1\rangle))\right).$$

Consequently, there is a spectral sequence

$$\pi_{t-s} \lim_{\Delta} \mathrm{Tot}\, H\pi_s\, \mathrm{THH}(C^\bullet(\mathrm{BP}\langle n-1\rangle/\mathrm{BP}\langle n\rangle, \mathrm{BP}\langle n-1\rangle)) \implies \pi_{t-s}\, \mathrm{THH}(\mathrm{BP}\langle n\rangle; \mathrm{BP}\langle n-1\rangle)$$

associated to the filtration

$$\lim \mathrm{Tot}\, \tau_{\geq s}\, \mathrm{THH}(C^\bullet(\mathrm{BP}\langle n-1\rangle/\mathrm{BP}\langle n\rangle; \mathrm{BP}\langle n-1\rangle)). \tag{3.4}$$

The E_2*-term is*

$$\mathrm{THH}_*(BP\langle n-1\rangle) \otimes_{\mathbb{Z}_{(p)}} \Lambda_{\mathbb{Z}_{(p)}}(\sigma v_n).$$

Consequently,

$$|\,\mathrm{THH}_t(BP\langle n\rangle; \mathrm{BP}\langle n-1)| \leq |\,\mathrm{THH}_t(BP\langle n-1\rangle)| + |\,\mathrm{THH}_{t-2p^n+1}(BP\langle n-1\rangle)|$$

Proof. Since $\mathrm{BP}\langle n\rangle \to \mathrm{BP}\langle n-1\rangle$ is an isomorphism on π_i for $i = 0, 1$ the first statement follows directly from [4, Theorem 3.7]. The second statement follows from [5, Remark 3.7] which identifies the filtration (3.4) with the décalage (cf. [3, pp. 21]) of the filtration whose associated graded is the E_1-term of the Bousfield–Kan spectral sequence.

It therefore suffices to compute the E_2-term, which is the cohomology of the Hopf algebroid $(\mathrm{THH}_*(\mathrm{BP}\langle n-1\rangle), \mathrm{THH}_*(\mathrm{BP}\langle n-1\rangle) \otimes \Gamma\{\sigma^2 v_n\})$ by Lemma 3.2. We note from the proof of Lemma 3.2 that this Hopf algebroid is the tensor product of the Hopf algebroids $(\mathrm{THH}_*(\mathrm{BP}\langle n-1\rangle), \mathrm{THH}_*(\mathrm{BP}\langle n-1\rangle))$ and $(\mathbb{Z}_{(p)}, \Gamma_{\mathbb{Z}_{(p)}}\{\sigma^2 v_n)\}$. Consequently, the cohomology of this Hopf algebroid is $\mathrm{THH}_*(\mathrm{BP}\langle n-1\rangle) \otimes \Lambda(\sigma v_n)$ as desired. □

Example 3.5. We consider the case $n = 0$. Then there is a spectral sequence

$$\mathrm{THH}_*(H\mathbb{F}_p) \otimes \Lambda(\sigma v_0) \implies \mathrm{THH}_*(\mathbb{Z}_{(p}; \mathbb{F}_p)\,.$$

This spectral sequence has a differential $d_1(\mu) = \sigma v_0$ and no further differentials except those generated by the Leibniz rule yielding the known answer $\mathbb{F}_p[\mu^p]\langle \sigma v_0 \mu^{p-1}\rangle$.

3.1. **The Bockstein spectral sequences.** Associated to the square

$$\begin{tikzcd}
\mathrm{BP}\langle 1\rangle \arrow[r] \arrow[d] & H\mathbb{Z}_{(p)} \arrow[d] \\
k(1) \arrow[r] & H\mathbb{F}_p
\end{tikzcd}$$

there is a square of Bockstein spectral sequences

$$\begin{tikzcd}
\mathrm{THH}(\mathrm{BP}\langle 2\rangle; \mathbb{F}_p)[v_0, v_1] \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \mathrm{THH}(\mathrm{BP}\langle 2\rangle; \mathbb{Z}_{(p)})[v_1] \arrow[d, Rightarrow] \\
\mathrm{THH}(\mathrm{BP}\langle 2\rangle; k(1))[v_0] \arrow[r, Rightarrow] & \mathrm{THH}(\mathrm{BP}\langle 2\rangle; \mathrm{BP}\langle 1\rangle)\,.
\end{tikzcd}$$

In [1], the authors computed the first Bockstein spectral sequence in each composite of spectral sequences above.

Theorem 3.6 ([1, Theorem 3.8, Theorem 4.6])**.** *Let* $\mathrm{B}\langle 2\rangle$ *be an arbitrary* E_3-MU-*algebra form of* $\mathrm{BP}\langle 2\rangle$ *at* $p > 2$ *and let* $\mathrm{B}\langle 2\rangle := \mathrm{tmf}_1(3)$ *at* $p = 2$.

(1) *There is an isomorphism*

$$\mathrm{THH}_*(\mathrm{B}\langle 2\rangle; \mathbb{Z}_{(p)}) = \mathbb{F}_p\langle \lambda_1, \lambda_2\rangle \otimes \left(\mathbb{Z}_{(p)} \oplus T_0^2\right)$$

where

$$T_0^2 = \oplus_{s\geq 1} \mathbb{Z}/p^s \otimes \mathbb{Z}_{(p)}[\mu_3^{p^s}] \otimes \mathbb{Z}_{(p)}\{\lambda_{s+2}\mu_3^{jp^{s-1}} \mid 0 \leq j \leq p-2\}\,.$$

(2) *There is an isomorphism*

$$\mathrm{THH}_*(\mathrm{B}\langle 2\rangle; k(1)) = \mathbb{F}_p\langle \lambda_1\rangle \otimes \left(\mathbb{F}_p[v_1] \oplus T_1^2\right)$$

where

$$T_1^2 = \bigoplus_{s\geq 1} \mathbb{F}_p[v_1]/(v_1^{r(s,1)}) \otimes \mathbb{F}_p[\mu_3^{p^s}] \otimes \mathbb{F}_p\langle \lambda_{s+2}\rangle \otimes \mathbb{F}_p\{\lambda_{s+1}\mu_3^{jp^{s-1}} \mid 0 \leq j \leq p-2\}\,.$$

Note that the map

$$\mathrm{THH}_*(\mathrm{B}\langle 2\rangle, k(1)) \longrightarrow \mathrm{THH}_*(\mathrm{B}\langle 2\rangle, \mathbb{F}_p)$$

is injective modulo v_1.

Theorem 3.7. *The*

From this, we produce a second bound on $\mathrm{THH}_*(\mathrm{B}\langle 2\rangle; \mathrm{B}\langle 1\rangle)$.

Corollary 3.8. *There is an inequality*

$$|\,\mathrm{THH}_k(\mathrm{B}\langle 2\rangle| \leq \mathrm{THH}_k(\mathrm{B}\langle 2\rangle, k(1))[v_0]\,.$$

In particular,

$$\mathrm{THH}_k(\mathrm{B}\langle 2\rangle, \mathrm{B}\langle 1\rangle) = 0$$

when $k \not\equiv 0, -1 \mod 2p$.

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