TOPOLOGICAL HOCHSCHILD HOMOLOGY OF THE SECOND TRUNCATED BROWN-PETERSON SPECTRUM I

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Abstract. We compute topological Hochschild homology of the second truncated Brown-Peterson spectrum $BP\langle 2\rangle$ at the primes $2,3$ with coefficients in $BP\langle 1\rangle$. At the prime $p=2$ we use the model for $BP\langle 2\rangle$ constructed by Lawson-Naumann using topological modular forms equipped with a $\Gamma_1(3)$-structure and at $p=3$ we use the model for $BP\langle 2\rangle$ constructed using a Shimura curve of discriminant 14 due to Hill-Lawson.

Contents

1. Introduction

Topological Hochschild (co)homology encodes information about deformations of structured ring spectra and the topological Hochschild homology of a structured ring spectrum is also the linear approximation to algebraic K-theory in the sense of Goodwillie's calculus of functors.

Algebraic K-theory of ring spectra that arise in chromatic stable homotopy theory are of particular interest because of the program of Ausoni-Rognes [4] which, in a broad sense, suggests that the arithmetic of structured ring spectra is intimately connected to chromatic complexity. One of the most fundamental objects in chromatic stable homotopy theory is the Brown-Peterson spectrum BP, which is a complex oriented cohomology theory associated to the universal p-typical formal group. The coefficients of BP are a polynomial algebra over $\mathbb{Z}_{(p)}$ on generators v_i for $i\geq 1$, and we may form truncated versions of BP, denoted $BP\langle n\rangle$ by coning off a regular sequence $(v_{n+1},v_{n+2},\ldots)$.

By convention $BP\langle -1\rangle = H\mathbb{F}_p$ and when $n=0,1$, there are known identifications $BP\langle 0\rangle = H\mathbb{Z}_{(p)}$, and $BP\langle 1\rangle=\ell$ where ℓ is the Adams summand of complex topological

K-theory ku. Until recently, the previous list exhausted the known examples of $BP\langle n\rangle$ that were known to have models as E_∞-ring spectra. However, in the last decade, models for $BP\langle 2\rangle$ as an E_∞-ring spectrum were constructed at the prime $p=2$ by Lawson-Naumann [10] and at the prime $p=3$ by Hill-Lawson [7]. Lawson-Naumann [10] use the theory of topological Modular forms with a $\Gamma_1(3)$-structure to construct an E_∞ model for $BP\langle 2\rangle$ at the prime 2 and Hill-Lawson [7] use the theory of topological automorphic forms associated to a Shimura curve of discriminant 14 to construct an E_∞ model for $BP\langle 2\rangle$ at the prime $p=3$. This is especially interesting in view of recent groundbreaking work of Lawson [9] and Senger [15], where they proves that no model for $BP\langle n\rangle$ as an E_∞-ring spectrum exists for $n\geq 4$ and any prime.

In the present paper, we compute topological Hochschild homology of $BP\langle 2\rangle$ with coefficients in $BP\langle 1\rangle$ at the primes 2 and 3. In future work, we plan to extend these computations to an integral calculation of $THH_*(BP\langle 2\rangle)$.

For small values of n, the calculations of $THH_*(BP\langle n\rangle)$ are known. The first known computations of topological Hochschild homology are Bökstedt's calculations of $THH_*(BP\langle -1\rangle)$ and $THH_*(BP\langle 0\rangle)$ [5]. In McClure-Staffledt [12], they compute the Bockstein spectral sequence

$$THH_*(BP\langle 1\rangle; H\mathbb{F}_p)[v_1] \Rightarrow THH_*(BP\langle 1\rangle; k(1)).$$

This result is extended by Angeltveit-Hill-Lawson [2] where they compute the square of spectral sequences

$$\begin{tikzcd} THH_*(BP\langle 1\rangle; H\mathbb{F}_p)[v_0,v_1] \arrow[r, Rightarrow] \arrow[d, Rightarrow] & THH_*(BP\langle 1\rangle; H\mathbb{Z}_{(p)})_p[v_1] \arrow[d, Rightarrow] \\ THH_*(BP\langle 1\rangle; k(1))[v_0] \arrow[r, Rightarrow] & THH_*(BP\langle 1\rangle; BP\langle 1\rangle)_p. \end{tikzcd}$$

This gives a complete answer for the "integral" calculation $THH_*(BP\langle 1\rangle)$.

When $n=2$, the calculation $THH_*(BP\langle 2\rangle; H\mathbb{F}_p)$ follows naturally from [3] as we discuss in Section 2.1, but no further results towards $THH_*(BP\langle 2\rangle)$ are known.

In the present paper, we compute the square of spectral sequences

$$\begin{tikzcd} THH_*(BP\langle 2\rangle; H\mathbb{F}_p)[v_0,v_1] \arrow[r, Rightarrow] \arrow[d, Rightarrow] & THH_*(BP\langle 2\rangle; H\mathbb{Z}_p)[v_1] \arrow[d, Rightarrow] \\ THH_*(BP\langle 2\rangle; k(1))[v_0] \arrow[r, Rightarrow] & THH_*(BP\langle 2\rangle; BP\langle 1\rangle)_p, \end{tikzcd}$$

which is only slightly more complex than the result of Angeltveit-Hill-Lawson [2] and therefore many of the techniques developed in[2] and [12] carry over.

We apply a new tool, however, introduced by the first author and Salch, called the topological Hochschild-May spectral sequence [1]. This allows one to compute $THH_*(BP\langle 2\rangle, BP\langle 1\rangle)$ directly. This will not replace the Bockstein spectral sequence, however, and instead we think of it as computing the diagonal of the square. This is actually the case at some later page in the topological Hochschild-May spectral sequence up to associated graded. We therefore use all three ways of computing the output in order to figure out all the differentials and hidden extensions.

1.1. Outline of the strategy.

[Gabe: Rewrite this section to reflect current strategy.]

Conventions. Let $p \in \{2, 3\}$ throughout. We will write $H_*(-)$ for homology with $\mathbb{F}_p$ coefficients, or in other words, the functor $\pi_*(H\mathbb{F}_p \wedge -)$. We write $\doteq$ to mean that an equality holds up to multiplication by a unit. We will write $BP\langle n\rangle$ for the n-th truncated Brown-Peterson spectrum. In particular, $BP\langle 1\rangle$ denotes the E_∞-ring spectrum model for the connective Adams summand [12]. Also, $BP\langle 2\rangle$ will denote the E_∞-model for the second truncated Brown-Peterson spectrum constructed by [10] at $p = 2$ and [7] at $p = 3$. We also note that by coning off v_2 on $BP\langle 2\rangle$ we may construct $BP\langle 1\rangle$ as an E_∞-$BP\langle 2\rangle$-algebra and since the E_∞-ring spectrum structure on $BP\langle 1\rangle$ is unique, this is equivalent to the E_∞ ring spectrum model constructed in [12]. Let $k(n)$ denote an A_∞-ring spectrum model for the connective cover of the Morava K-theory spectrum $K(n)$.

When not otherwise specified, tensor products will be taken over $\mathbb{F}_p$ and $HH_*(A)$ denotes the Hochschild homology of a graded $\mathbb{F}_p$-algebra relative to $\mathbb{F}_p$. We will let $P(x)$, $E(x)$ and $\Gamma(x)$ denote a polynomial algebra, exterior algebra, and divided power algebra over $\mathbb{F}_p$ on a generator x.

The dual Steenrod algebra will be denoted $\mathscr{A}_*$ with coproduct $\Delta\colon \mathscr{A}_* \to \mathscr{A}_* \otimes \mathscr{A}_*$. Given a right $\mathscr{A}_*$-comodule M, its right coaction will be denoted $\nu\colon \mathscr{A} \to \mathscr{A} \otimes M$ where the comodule M is understood from the context. The antipode $\chi\colon \mathscr{A}_* \to \mathscr{A}_*$, will not play a role except that we will write $\bar{\xi}_i := \chi(\xi_i)$ and $\bar{\tau}_i := \chi(\tau_i)$.

2. First two Bockstein spectral sequences

2.1. Preliminary results. The homology of topological Hochschild homology of $BP\langle 2\rangle$ is a straightforward application of results of [3, 5, 6] and it appears in [3, Thm. 5.12]. Recall that there is an isomorphism

$$H_*(BP\langle 2\rangle) \cong \begin{cases} P(\bar{\xi}_1, \bar{\xi}_2, \dots) \otimes E(\bar{\tau}_3, \bar{\tau}_4, \dots) \text{ if } p \geq 3 \\ P(\bar{\xi}_1^2, \bar{\xi}_2^2, \bar{\xi}_3^2, \bar{\xi}_4, \dots) \text{ if } p = 2 \end{cases}$$

of $\mathscr{A}_*$-comodules. Then by [3, Thm. 5.12, Cor. 5.12] there is an isomorphism

$$H_*(\mathrm{THH}(BP\langle 2\rangle)) \cong \begin{cases} H_*BP\langle 2\rangle \otimes E(\sigma\bar{\xi}_1, \sigma\bar{\xi}_2, \sigma\bar{\xi}_3) \otimes P(\sigma\bar{\tau}_3) \text{ if } p = 3 \\ H_*BP\langle 2\rangle \otimes E(\sigma\bar{\xi}_1^2, \sigma\bar{\xi}_2^2, \sigma\bar{\xi}_3^2) \otimes P(\sigma\bar{\xi}_4) \text{ if } p = 2 \end{cases} \tag{2.1}$$

of primitively generated $H_*BP\langle 2\rangle$-Hopf algebras and $\mathscr{A}_*$-comodules. We also note the coaction on $H_*(\mathrm{THH}(BP\langle 2\rangle))$ as a comodule over $\mathscr{A}_*$ computed in [3, Thm. 5.12]

$$\nu(\sigma\bar{\tau}_{\underset{3}{\cancel{m}}}) = 1 \otimes \sigma\bar{\tau}_{\underset{3}{\cancel{m}}} + \bar{\tau}_0 \otimes \sigma\bar{\xi}_{\underset{3}{\cancel{m}}} \tag{2.2}$$

at $p = 3$ and

$$\nu(\sigma\bar{\xi}_{\underset{4}{\cancel{m+1}}}) = 1 \otimes \sigma\bar{\xi}_{\underset{4}{\cancel{m+1}}} + \bar{\xi}_1 \otimes \sigma\bar{\xi}^2_{\underset{3}{\cancel{2}}}. \tag{2.3}$$

at $p = 2$. These both follow from the formula

$$\nu \circ \sigma = (1 \otimes \sigma) \circ \nu \tag{2.4}$$

in [3, Eq. 5.11] and the well known $\mathscr{A}_*$-coaction on $H_*(BP\langle 2\rangle)$. By the same argument, $\sigma\xi_i$ is primitive at $p=3$ and $\sigma\xi_i^2$ is primitive at $p=2$ for $i=1,2,3$.

2.1.1. *THH of $BP\langle 2\rangle$ modulo (p,v_1,v_2).* We now compute

$$\mathrm{THH}_*(BP\langle 2\rangle; H\mathbb{F}_p).$$

By [3, Lem. 4.1], it suffices to compute the sub-algebra of co-mododule primitives in

$$H_*(\mathrm{THH}(BP\langle 2\rangle; H\mathbb{F}_p))$$

since $\mathrm{THH}(BP\langle 2\rangle; H\mathbb{F}_p)$ is an $H\mathbb{F}_p$-algebra. Since $BP\langle 2\rangle$ and $H\mathbb{F}_p$ are commutative ring spectra there is a weak equivalence of commutative ring spectra

$$\mathrm{THH}(BP\langle 2\rangle; H\mathbb{F}_p) \simeq \mathrm{THH}(BP\langle 2\rangle) \wedge_{BP\langle 2\rangle} H\mathbb{F}_p.$$

Since $H_*(\mathrm{THH}(BP\langle 2\rangle))$ is free over $H_*BP\langle 2\rangle$ by (2.1), the Eilenberg-Moore spectral sequence

$$\mathrm{Tor}_{*,*}^{H_*(BP\langle 2\rangle)}(H_*(THH(BP\langle 2\rangle)), H_*(H\mathbb{F}_p)) \Rightarrow H_*(THH(BP\langle 2\rangle; H\mathbb{F}_p))$$

collapses immediately implying

$$H_*(\mathrm{THH}(BP\langle 2\rangle; H\mathbb{F}_p)) \cong \begin{cases} \mathscr{A}_* \otimes E(\sigma\bar{\xi}_1, \sigma\bar{\xi}_2, \sigma\bar{\xi}_3) \otimes P(\sigma\bar{\tau}_3) \text{ if } p=3 \\ \mathscr{A}_* \otimes E(\sigma\bar{\xi}_1^2, \sigma\bar{\xi}_2^2, \sigma\bar{\xi}_3^2) \otimes P(\sigma\bar{\xi}_{\cancel{3}\,4}) \text{ if } p=2. \end{cases} \tag{2.5}$$

The $\mathscr{A}_*$ coaction on elements in $\mathscr{A}_*$ is given by the coproduct and the remaining coactions are determined by the formula (2.4) and are therefore the same as in (2.2) and (2.3). We write $\lambda_i = \sigma\bar{\xi}_i$ at $p=3$ and $\lambda_i = \sigma\bar{\xi}_i^2$ are $p=2$. We also define

$$\mu_3 = \begin{cases} \sigma\bar{\tau}_3 - \bar{\tau}_0 \otimes \sigma\bar{\xi}_3 \text{ if } p=3 \\ \sigma\bar{\xi}_4 - \bar{\xi}_1 \otimes \sigma\bar{\xi}_3^2 \text{ if } p=2 \end{cases}.$$

Then it is clear that the algebra of comodule primitives in $H_*(\mathrm{THH}(BP\langle 2\rangle; H\mathbb{F}_p))$ is generated by μ_3 and λ_i for $1 \le i \le 3$. We therefore produce the following isomorphism of graded $\mathbb{F}_p$-algebras

$$\mathrm{THH}_*(BP\langle 2\rangle; H\mathbb{F}_p) \cong E(\lambda_1, \lambda_2, \lambda_3) \otimes P(\mu_3). \tag{2.6}$$

The degrees of the algebra generators are $|\lambda_i| = 2p^i - 1$ for $1 \le i \le 3$ and $|\mu_3| = 2p^3$.

2.1.2. *Rational homology.* Next, we compute the rational homology of $\mathrm{THH}(BP\langle 2\rangle)$ to locate the torsion free component of $\mathrm{THH}_*(BP\langle 2\rangle)$. Towards this end, we will use the $H\mathbb{Q}$-based Bökstedt spectral sequence. This is a spectral sequence of the form

$E^{2,?}$ $$\cancel{E_2^{*,*}} = \mathrm{HH}_*^{\mathbb{Q}}(H\mathbb{Q}_*(BP\langle 2\rangle)) \implies H\mathbb{Q}_(* \mathrm{THH}(BP\langle 2\rangle)).$$

Recall that the rational homology of $BP\langle 2\rangle$ is

$$H\mathbb{Q}_*(BP\langle 2\rangle) \cong P_{\mathbb{Q}}(v_1, v_2).$$

Thus the E_2-term of the Bökstedt spectral sequence is

$$P_{\mathbb{Q}}(v_1, v_2) \otimes E_{\mathbb{Q}}(\sigma v_1, \sigma v_2)$$

and the bidegree of σv_i is $(1, 2(p^i - 1))$. Note that $BP\langle 2\rangle$ is a commutative ring spectrum, so by [3, Prop. 4.3] the Bökstedt spectral sequence is multiplicative. All the algebra generators are in Bökstedt filtration 0 and 1 and the d^2 differential shifts Bökstedt filtration by two, so there is no room for differentials. Thus, the E_2-term is isomorphic to the E_∞-term as graded $\mathbb{Q}$-algebras. There are clearly no additive extensions since the abutment is a $\mathbb{Q}$-algebra. There are no multiplicative extensions for bidegree reasons. Thus, there is an isomorphism of graded $\mathbb{Q}$-algebras

$$\mathrm{THH}_*(BP\langle 2\rangle)\otimes\mathbb{Q}\cong P_{\mathbb{Q}}(v_1,v_2)\otimes_{\mathbb{Q}} E_{\mathbb{Q}}(\sigma v_1,\sigma v_2)$$

where $|\sigma v_i| = 2p^i-1$. At $p=2,3$, there is an E_2-ring map

$$BP\to BP\langle 2\rangle.$$

To see this, we note that our E_∞ ring spectrum models for $BP\langle 2\rangle$ are clearly complex oriented and therefore come equipped with formal groups. It is also clear that these formal groups are p-typical. There is therefore an associated E_1 ring map

$$BP\to BP\langle 2\rangle$$

and then by [?qx, Chadwick-Mandell] this E_1-ring map can be lifted to an E_2-ring map. Rationally, this map

$$H\mathbb{Q}_*(BP)\to H\mathbb{Q}_*(BP\langle 2\rangle)$$

sends v_1 and v_2 to the generators of the same name. We therefore produce a multiplicative map of rational Bökstedt spectral sequences

$$\begin{tikzcd} HH^{\mathbb{Q}}_*(H\mathbb{Q}_*(BP)) \arrow[r, Rightarrow] \arrow[d] & H\mathbb{Q}_*(THH(BP)) \arrow[d] \\ HH^{\mathbb{Q}}_*(H\mathbb{Q}_*(BP\langle 2\rangle)) \arrow[r, Rightarrow] & H\mathbb{Q}_*(THH(BP\langle 2\rangle)) \end{tikzcd}$$

where on E_2 pages the map

$$P_{\mathbb{Q}}(v_i\mid i\geq 1)\otimes E_{\mathbb{Q}}(\sigma v_i\mid\geq 1)\to P_{\mathbb{Q}}(v_1,v_2)\otimes E_{\mathbb{Q}}(\sigma v_1\sigma v_2)$$

sends v_i to v_i and σv_i to σv_i for $i=1,2$. By [?Rog19, Thm. 1.1],

$$\sigma v_1 = p\lambda_1$$
$$\sigma v_2 = p\lambda_2 - v_1^p\lambda_1 - v_1^p\sigma v_1$$

in $THH_*(BP)$. Since the map

$$THH_*(BP)\to THH_*(BP\langle 2\rangle)$$

sends λ_1 and λ_2 to classes of the same name, we have the same relations in $THH_*(BP\langle 2\rangle)$.

[Gabe: This part isn't proven yet. I think we can prove it, but maybe this part belongs later.]

Consequently, up to a change of basis,

$$THH_*(BP\langle 2\rangle; H\mathbb{Q})\cong E_{\mathbb{Q}}(\lambda_1,\lambda_2). \tag{2.7}$$

Also, we conclude that

$$L_0\,\mathrm{THH}(BP\langle 2\rangle)\simeq L_0BP\langle 2\rangle\vee\Sigma^{2p-1}L_0BP\langle 2\rangle\vee\Sigma^{2p^2-1}L_0BP\langle 2\rangle\vee\Sigma^{2p^2+2p-2}L_0BP\langle 2\rangle$$

where $L_0 = L_{H\mathbb{Q}}$, since L_0 is a smashing localization and $L_0S = H\mathbb{Q}$.

2.2. **The $H\mathbb{Z}$-Bockstein spectral sequence.** Recall that there is an isomorphism of $\mathcal{A}_*$-comodules

$$H_*(S/p \wedge THH(BP\langle 2\rangle)) \cong \begin{cases} E(\bar{\tau}_0) \otimes H_*(BP\langle 2\rangle) \otimes E(\sigma\bar{\xi}_1, \sigma\bar{\xi}_2, \sigma\bar{\xi}_3) \otimes P(\sigma\bar{\tau}_3) & \text{if } p = 3 \\ E(\bar{\xi}_1) \otimes H_*(BP\langle 2\rangle) \otimes E(\sigma\bar{\xi}_1^2, \sigma\bar{\xi}_2^2, \sigma\bar{\xi}_3^2) \otimes P(\sigma\bar{\xi}_4) & \text{if } p = 2 \end{cases}$$

where the coaction on $x \in \mathcal{A}_*$ is $\nu(x)$ is given by the restriction of the coproduct Δ of the dual Steenrod algebra to $H_*(S/p \wedge BP\langle 2\rangle) \subset \mathcal{A}_*$ and the remaining coactions follow from (2.4). In this section, we compute the Bockstein spectral sequence

$$E^1_{*,*} = THH_*(BP\langle 2\rangle; \mathbb{F}_p)[v_0] \Rightarrow THH(BP\langle 2\rangle; \mathbb{Z}_{(p)})_p. \tag{2.8}$$

As a direct consequence the coaction on $\sigma\bar{\tau}_3$, there is a differential

Can you explain that to me?

$$d_1(\mu_3) = v_0\lambda_3. \tag{2.9}$$

in the $H\mathbb{Z}$-Bockstein spectral sequence (2.8).

The following lemma follows from [11, Prop. 6.8] by translating to the E_∞-context (cf. the proof of [2, Lem. 3.2]).

Lemma 2.10. *If $d_j(x) \neq 0$ in the $H\mathbb{Z}$-Bockstein spectral sequence (2.8) then*

$$d_{j+1}(x^p) = v_0 x^{p-1} d_j(x)$$

if $p > 2$ or if $p = 2$ and $j \geq 2$. If $p = 2$ and $j = 1$ then

$$d_{j+1}(x^p) = v_0 x^{p-1} d_j(x) + Q^{|x|}(d_1(x))$$

When $p = 2$, we have the differential

$$d_1(\mu) = v_0\lambda_3.$$

Therefore, the error term for $d_2(\mu_3^2)$ is

$$Q^{16}\lambda_3 = Q^{16}(\sigma\bar{\xi}_3^2) = \sigma(Q^{16}(\bar{\xi}_3^2)) = \sigma((Q^8\bar{\xi}_3)^2) = \sigma(\bar{\xi}_4^2) = 0.$$

The first equality holds by definition of λ_3, the second equality holds because σ commutes with Dyer-Lashoff operations by [5], the third equality holds by [6], and the last equality holds because σ is a derivation [3].

Corollary 2.11. *When $p = 2, 3$, there are differentials*

$$d_{i+1}(\mu^{p^i}) = v_0^{i+1}\mu_3^{p^i-1}\lambda_3.$$

Consequently, there are differentials

$$d_{\nu_p(k)+1}(\mu^k) \doteq v_0^{\nu_p(k)+1}\mu^{k-1}\lambda_3$$

where $\nu_p(k)$ denotes the p-adic valuation of k.

Proof. Let $\alpha = \nu_p(k)$. We have that $k = p^\alpha j$ where p does not divide j. So by the Leibniz rule

$$d_{\alpha+1}(\mu_3^k) = d_{\alpha+1}((\mu_3^{p^\alpha})^k) = k\mu_3^{p^\alpha(k-1)} d_{\alpha+1}(\mu_3^\alpha) = kv_0^{\alpha+1}\mu^{p^\alpha(k-1)}\mu^{p^\alpha-1}\lambda_3 = kv_0^{\alpha+1}\mu^{k-1}\lambda_3.$$

Since k is not divisible by p, it is a unit mod p. $\square$

Now recall from (2.7) that $THH_*(BP\langle 2\rangle; H\mathbb{Q}) \cong E_{\mathbb{Q}}(\lambda_1, \lambda_2)$. In fact the map, $THH_*(B; H\mathbb{Z}_{(p)}) \to THH_*(B; H\mathbb{Q})$ sends λ_i to λ_i for $i = 1, 2$. Therefore, the elements λ_1, λ_2 are p-torsion free and there are no further differentials in the $H\mathbb{Z}$-Bockstein spectral sequence. We rename the following classes as follows

$$\begin{aligned} c_i^{(1)} &:= \lambda_3 \mu_3^{i-1}, \quad d_i^{(1)} := \lambda_1 c_i^{(1)}, \\ c_i^{(2)} &:= \lambda_2 c_i^{(1)}, \quad d_i^{(2)} := \lambda_2 d_i^{(1)}. \end{aligned} \tag{2.12}$$

Thus we have the following

Corollary 2.13. *There is an isomorphism of $\mathbb{Z}_{(p)}$-algebras*

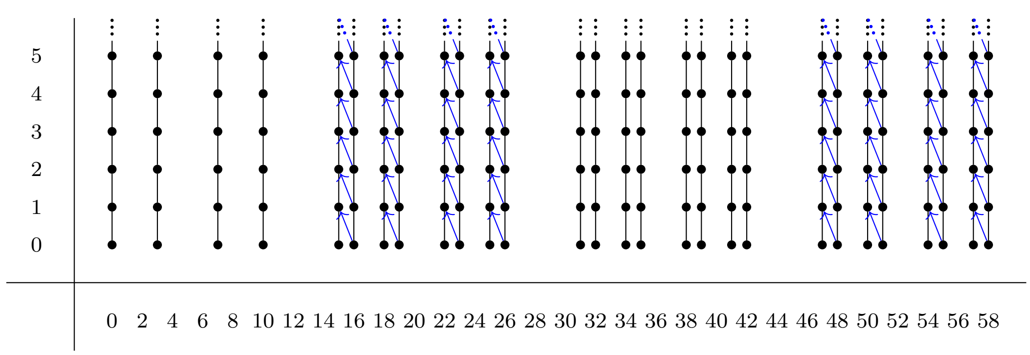
$$THH_*(BP\langle 2\rangle; H\mathbb{Z}_{(p)}) \cong E_{\mathbb{Z}_{(p)}}(\lambda_1, \lambda_2) \oplus T_0$$

where T_0 is a torsion $\mathbb{Z}_{(p)}$-module defined by

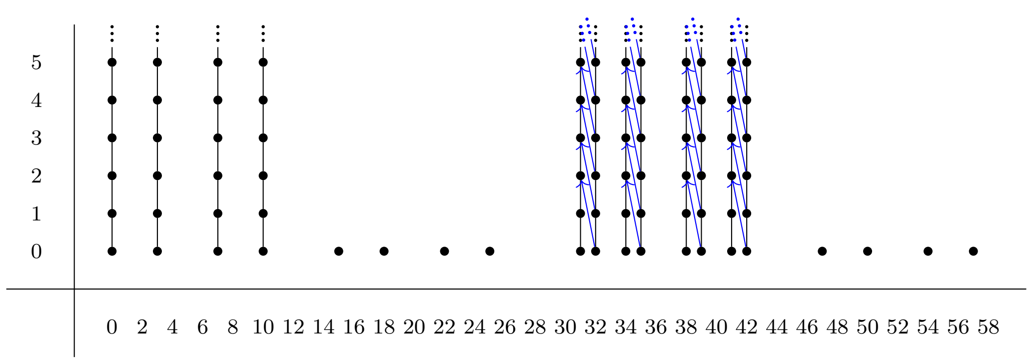
$$T_0 = \left(\mathbb{Z}_{(p)}\{c_i^{(k)}, d_i^{(k)} \mid i \geq 1, 1 \leq k \leq 2\}\right) / (p^j c_i^{(k)}, p^j d_i^{(k)} \mid j = \nu_p(i)+1, i \geq 1, 1 \leq k \leq 2)$$

where the products on the elements $c_i^{(k)}$, $d_i^{(k)}$ are specified by Formula (2.12) and by letting all other products be zero.

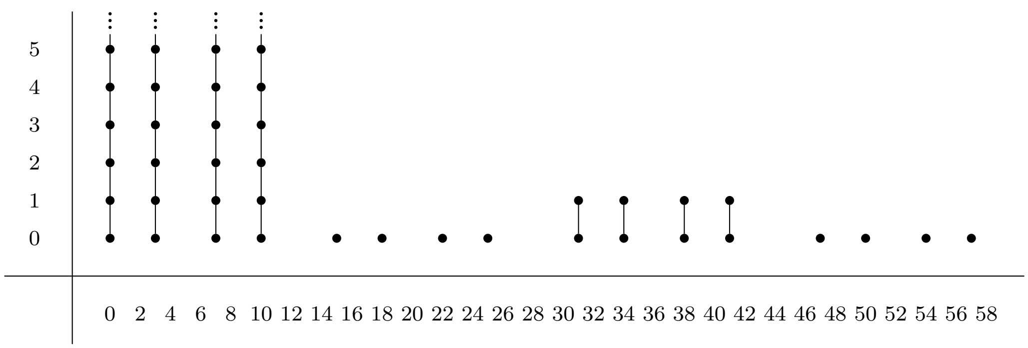
E_1-page of v_0-Bockstein Spectral Sequence at $p = 2$



E_2-page of v_0-Bockstein Spectral Sequence at $p = 2$



E_3-page of v_0-Bockstein Spectral Sequence at $p = 2$



2.3. **The v_1-Bockstein spectral sequence.** In this section, we begin our analysis of the v_1-Bockstein spectral sequence

$$\mathrm{THH}_*(BP\langle 2\rangle; H\mathbb{F}_p)[v_1] \Rightarrow \mathrm{THH}_*(BP\langle 2\rangle; k(1)).$$

To start, we need to compute $K(1)_*(BP\langle 2\rangle)$. This requires determining $\eta_R(v_{2+n})$ in $K(1)_*(BP)$ modulo the ideal generated by $(\eta_R(v_3), \ldots, \eta_R(v_{1+n}))$. We will need the following.

6 ?

Lemma 2.14. [14, Lemma A.2.2.~~5~~] *Let v_n denote the Araki generators. Then there is the following equality in $BP_*(BP)$*

$$\sum_{i,j\geq 0}^{F} t_i \eta_R(v_j)^{p^i} = \sum_{i,j\geq 0}^{F} v_i t_j^{p^i}$$

In this section, the distinction between Hazewinkel generators and Araki generators is unimportant, as the two sets of generators coincide modulo p. In $K(1)_*BP$, we have killed all v_i's except v_1, which gives us the following equation

$$\sum_{i,j\geq 0}^{F} t_i \eta_R(v_j)^{p^i} = \sum_{k\geq 0}^{F} v_1 t_k^{p}$$

Note that the following degrees of the terms:

$$|v_1 t_j^p| = 2(p^{j+1} - 1)$$
$$|t_i \eta_R(v_j)^{p^i}| = 2(p^{i+j} - 1)$$

Since we are interested in the term $\eta_R(v_{2+n})$, we collect all the terms on the left of degree $2(p^{2+n} - 1)$. Thus we are summing over the ordered pairs (i, j) such that $i + j = 2 + n$. Since we only care about $\eta_R(v_{2+n})$ modulo $\eta_R(v_3), \ldots, \eta_R(v_{1+n})$ we only need to collect the terms where $j = 1, 2,$ or $2 + n$. This shows that

$$t_{1+n}\eta_R(v_1)^{p^{n+1}} + t_n \eta_R(v_2)^{p^n} + \eta_R(v_{n+2}) = v_1 t_{n+1}^p$$

The value of η_R on v_1 and v_2 can also be computed by Lemma 2.14. One obtains, in $K(1)_*(BP)$, the following

$$\eta_R(v_1) = v_1 \quad - t_1 \qquad ?$$
$$\eta_R(v_2) = v_1 t_1^p - t_1 v_1^p. \quad - t_2$$

Combining these observations, we obtain

Lemma 2.15. *In $K(1)_*(BP)$, the following congruence is satisfied*

$$\eta_R(v_{2+n}) \equiv v_1 t_{n+1}^p - v_1^{p^n} t_1^{p^{n+1}} t_n + v_1^{p^{n+1}}(t_1^{p^n} t_n - t_{n+1}) \mod (\eta_R(v_3), \dots, \eta_R(v_{1+n}))$$

for $n \geq 1$.

Consequently, we have the following corollary.

Corollary 2.16. *There is an isomorphism of $K(1)_*$-algebras*

$$K(1)_*(BP\langle 2\rangle) \cong K(1)_*(BP)/(v_1 t_{n+1}^p - v_1^{p^n} t_1^{p^{n+1}} t_n + v_1^{p^{n+1}}(t_1^{p^n} t_n - t_{n+1}) \mid n \geq 1)$$

Define elements

?

$$u_n := v_1^{\frac{p^n-1}{p-1}} t_n.$$

? These elements are in degree 0, and therefore there is an isomorphism of $K(1)_*$-algebras

$$K(1)_*(BP\langle 2\rangle) \cong K(1)_* \otimes_{\mathbb{F}_p} K(1)_0(BP\langle 2\rangle).$$

The calculations above imply the following corollary.

Corollary 2.17. *There is an isomorphism of $\mathbb{F}_p$-algebras*

$$K(1)_0(BP\langle 2\rangle) \cong P(u_i \mid i \geq 1)/(u_{n+1}^p - u_1^{p^{n+1}} u_n + u_1^{p^n} u_n - u_{n+1} \mid n \geq 1).$$

Our goal is to use this and the $K(1)$-based Bökstedt spectral sequence to compute the $K(1)$-homology of $\mathrm{THH}(BP\langle 2\rangle)$. This is a spectral sequence of the form

$$E^2_{s,t} = \mathrm{HH}_s^{K(1)_*}(K(1)_*(BP\langle 2\rangle)) \implies K(1)_{s+t}(\mathrm{THH}(BP\langle 2\rangle)).$$

The above considerations imply

$$E^2_{*,*} \cong K(1)_* \otimes \mathrm{HH}_*^{\mathbb{F}_p}(K(1)_0 BP\langle 2\rangle).$$

The following results will be useful for our calculation.

Lemma 2.18 ([13])**.** *Let $V = \mathrm{Spec}(A)$ be a nonsingular affine variety over a field k. Let W be the subvariety of $V \times \mathbb{A}^n$ defined by equations*

$$g_i(Y_1, \dots, Y_n) = 0,\ g_i \in A[Y_1, \dots, Y_n],\ i = 1, \dots, n.$$

Then the projection map $W \to V$ is étale at a point $(P; b_1, \dots, b_n)$ of W if and only if the Jacobian matrix $\left(\frac{\partial g_i}{\partial Y_j}\right)$ is a nonsingular matrix at $(P; b_1, \dots, b_n)$.

Theorem 2.19 (Étale Descent, [16])**.** *Let $A \hookrightarrow B$ be an étale extension of commutative k-algebras. Then there is an isomorphism*

$$\mathrm{HH}_*(B) \cong \mathrm{HH}_*(A) \otimes_A B$$

Example 2.20. Consider the subalgebra

$$\mathbb{F}_p[u_1, u_2]/(u_2^p - u_1^{p^2+1} + u_1^{p+1} - u_2 = f_1).$$

We will regard this as a $\mathbb{F}_p[u_1]$-algebra. The partial derivative $\partial_{u_2} f_1$ is $-1 \pmod p$, and therefore a unit at every point. Then Lemma 2.18 tells us that this algebra is then étale over $\mathbb{F}_p[u_1]$.

By the same argument given above, we claim that there are a sequence of sub-algebras A_n of

$$A := K(1)_0(BP\langle 2\rangle) \cong \mathbb{F}_p[u_i \mid i \geq 1]/(u_{n+1}^p - u_1^{p^{n+1}} u_n + u_1^{p^n} u_n - u_{n+1} \mid n \geq 1)$$

such that each map $A_i \hookrightarrow A_{i+1}$ is an étale extension. Here

$$A_n := \mathbb{F}_p[u_1, u_2, \dots u_n]/(u_{k+1}^p - u_1^{p^{k+1}} u_k + u_1^{p^k} u_k - u_{k+1} = f_k \mid 1 \leq k \leq n)$$

u_{n+1}

and the partial derivative

$$\partial_{u_k} f_k = -1 \pmod p$$

for all $1 < k \leq n$ and therefore a unit at each point. The claim then follows by Lemma 2.18.

By the étale base change formula for Hochschild homology in Theorem 2.19, there is an isomorphism

$$\mathrm{HH}_*^{\mathbb{F}_p}(A_{i+1}) \cong \mathrm{HH}_*^{\mathbb{F}_p}(A_i) \otimes_{A_i} A_{i+1}$$

and since the functors $\mathrm{HH}_*(-)$ and $\mathrm{HH}_*^{\mathbb{F}_p}(A_1) \otimes_{A_1} (-)$ commute with filtered colimits of $\mathbb{F}_p$-algebras, there are isomorphisms

$$\begin{aligned} \mathrm{HH}_*^{\mathbb{F}_p}(A) &\cong \mathrm{HH}_*^{\mathbb{F}_p}(\operatorname{colim} A_n) \\ &\cong \operatorname{colim} \mathrm{HH}_*^{\mathbb{F}_p}(A_n) \\ &\cong \operatorname{colim} \mathrm{HH}_*^{\mathbb{F}_p}(A_1) \otimes_{A_1} A_n \\ &\cong \mathrm{HH}_*^{\mathbb{F}_p}(A_1) \otimes_{A_1} A. \end{aligned}$$

Consequently,

$$\mathrm{HH}_*^{K(1)_*}(K(1)_*(BP\langle 2\rangle)) \cong K(1)_* \otimes E(\sigma t_1) \otimes K_0(BP\langle 2\rangle)$$

and therefore, since $\sigma t_1 \doteq \lambda_1 \bmod p$,

$$K(1)_*(\mathrm{THH}(BP\langle 2\rangle)) \cong K(1)_*(BP\langle 2\rangle) \otimes E(\lambda_1)$$

and

$$\mathrm{THH}_*(BP\langle 2\rangle; K(1)) \cong K(1)_* \otimes E(\lambda_1).$$

In other words,

$$THH_*(BP\langle 2\rangle; k(1)) \cong F \oplus T$$

where F is a free $P(v_1)$-module generated by 1 and λ_1 and T is a torsion $P(v_1)$-module. In summary, we have proven the following theorem.

Theorem 2.21.

(1) *The $K(1)$-homology of* $\mathrm{THH}(BP\langle 2\rangle; K(1))$ *is* $K(1)_*K(1) \otimes E(\lambda_1)$

(2) *Their is a weak equivalence*

$$K(1) \vee \Sigma^{2p-1} K(1) \simeq \mathrm{THH}(BP\langle 2\rangle; K(1)).$$

(3) *The v_1-torsion free part of* $\mathrm{THH}(BP\langle 2\rangle; k(1))$ *is generated by* 1 *and* λ_1.

2.3.1. *Differentials in the v_1-BSS.* We now analyze the v_1-BSS. Recall that this spectral sequence is of the form

$BP\langle 2 \rangle$

$$\mathrm{THH}(BP\langle 2\rangle;\mathbb{F}_p)[v_1] \implies \mathrm{THH}(B;k(1)).$$

Thus the E_1-page is

$$E(\lambda_1,\lambda_2,\lambda_3)\otimes P(\mu_3)\otimes P(v_1). \tag{2.22}$$

Since the λ_i are all in odd total degree and since v_1^k are known to be survive to the E_∞-term, the λ_i are all permanent cycles. If μ_3 were a permanent cycle, then by multiplicativity, the spectral sequence would collapse. This would contradict Theorem 2.21. Therefore, the element μ_3 must support a differential. The only possibility for bidegree reasons is

$$d_{p^2}(\mu_3)\doteq v_1^{p^2}\lambda_2.$$

Thus, we obtain

$$v_1^{-1}E_{p^2+1}\cong K(1)_*\otimes E(\lambda_1,\lambda_3,\lambda_4')\otimes P(\mu_3^p)$$

where

$$\lambda_4' := \lambda_2\mu_3^{p-1}.$$

So the bidegree of λ_4' is given by

$$|\lambda_4'| = (2p^4-2p^3+2p^2-1,0).$$

For analogous reasons, the class λ_4' is a permanent cycle, and μ_3^p cannot be a permanent cycle. Based on degree considerations, there are two possible differentials,

$$d_{p^2}(\mu_3^p)\doteq v_1^{p^2}\lambda_4' \quad \text{or} \quad d_{p^3}(\mu_3^p)\doteq v_1^{p^3}\lambda_3.$$

The first would contradict the Leibniz rule for d_{p^2} and the fact that $d_{p^2}(\mu_3)\doteq v_1^{p^2}\lambda_2$. Thus,

$$v_1^{-1}E_{p^3+1}\cong K(1)_*\otimes E(\lambda_1,\lambda_4',\lambda_5')\otimes P(\mu_3^{p^2})$$

where

$$\lambda_5' := \lambda_3\mu_3^{p(p-1)}.$$

The bidegree of λ_5' is

$$|\lambda_5'| = (2p^5-2p^4+2p^3-1,0).$$

For degree reasons, the class λ_5' is a permanent cycle. As before, the class $\mu_3^{p^2}$ must support a differential. Degree considerations, again, give two possibilities

$$d_{p^3}(\mu_3^{p^2})\doteq v_1^{p^3}\lambda_5'$$

or

$$d_{p^4+p^2}(\mu_3^{p^2})\doteq v_1^{p^4+p^2}\lambda_4'.$$

The former would contradict the Leibniz rule, leaving the latter as the only possibility. This gives us

$$v_1^{-1}E_{p^4+p^2+1}\cong K(1)_*\otimes E(\lambda_1,\lambda_5',\lambda_6')\otimes P(\mu_3^{p^3})$$

where $\lambda_6' := \lambda_4'\mu_3^{p^2(p-1)}$. We will continue via induction. First we need some notation. We will recursively define classes λ_n' by

$$\lambda_n' := \begin{cases} \lambda_n & n = 1,2,3 \\ \lambda_{n-2}'\mu_3^{p^{n-4}(p-1)} & n \geq 4 \end{cases}$$

We let $d'(n)$ denote the topological degree of λ_n'. Then this function is given recursively by

$$d'(n) = \begin{cases} 2p^n - 1 & n = 1,2,3 \\ 2p^n - 2p^{n-1} + d(n-2) & n > 3 \end{cases}$$

Thus, by a simple induction, one has

$$d'(n) = \begin{cases} 2p^n - 1 & n = 1,2,3 \\ 2p^n - 2p^{n-1} + 2p^{n-2} - 2p^{n-3} + \cdots + 2p^2 - 1 & n \equiv 0 \mod 2,\ n > 3 \\ 2p^n - 2p^{n-1} + 2p^{n-2} - 2p^{n-3} + \cdots + 2p^3 - 1 & n \equiv 1 \mod 2,\ n > 3 \end{cases}.$$

Observe that the integers $2p^{n+2} - d(n+1) - 1$ and $2p^{n+2} - d(n+2) - 1$ are divisible by $|v_1|$. Let $r'(n)$ denote the integer

$$r'(n) := |v_1|^{-1}(|\mu_3^{p^{n-1}}| - |\lambda_{n+1}'| - 1) = |v_1|^{-1}(2p^{n+2} - d'(n+1) - 1).$$

Then a simple induction shows that

$$r'(n) = \begin{cases} p^{n+1} + p^{n-1} + p^{n-3} + \cdots + p^2 & n \equiv 1 \mod 2 \\ p^{n+1} + p^{n-1} + p^{n-3} + \cdots + p^3 & n \equiv 0 \mod 2 \end{cases}.$$

We can now describe the differentials in the v_1-BSS.

Theorem 2.23. *In the v_1-BSS, the following hold:*

(1) *The only nonzero differentials are in $v_1^{-1}E_{r'(n)}$.*

(2) *The $r'(n)$-th page is given by*

$$v_1^{-1}E_{r'(n)} \cong K(1)_* \otimes E(\lambda_1, \lambda_{n+1}', \lambda_{n+2}') \otimes P(\mu_3^{p^{n-1}})$$

and the classes $\lambda_{n+1}', \lambda_{n+2}'$ are permanent cycles.

(3) *The differential $d_{r'(n)}$ is uniquely determined by multiplicativity of the BSS and the differential*

$$d_{r'(n)}(\mu_3^{p^{n-1}}) \doteq v_1^{r'(n)}\lambda_{n+1}'.$$

for $n \geq 1$.

Proof. We proceed by induction. We have already shown the theorem for $n \leq 4$. Assume that

$$v_1^{-1}E_{r'(n)} \cong K(1)_* \otimes E(\lambda_1, \lambda_n', \lambda_{n+2}') \otimes P(\mu_3^{p^{n-1}}).$$

and λ_n' is an infinite cycle.

Since $\lambda_n', \lambda_{n+1}'$ are both in odd topological degree, λ_{n+1}' cannot support a differential hitting the v_1-towers on λ_i' for $i < n+1$. Thus, the only possibility is that λ_{n+1}' supports a differential into the v_1-tower on 1 or λ_1. But this would contradict Theorem 2.21. Therefore, the class λ_{n+1}' is a permanent cycle.

Shouldn't one also explain why $d(\lambda'_{n+2}) = \lambda_1 \lambda'_{n+1} v_1^{\ell}$ is not possible? degree reasons?

The class $\mu_3^{p^{n-1}}$ must support a differential, for if it did not, then the spectral sequence would collapse. This would lead to a contradiction of Theorem 2.21. Degree considerations show that the following differentials are possible

$$d_{k(n)}(\mu_3^{p^{n-1}})\doteq v_1^{k(n)}\lambda'_{n+2}$$

and

$$d_{r(n)}(\mu_3^{p^{n-1}})\doteq v_1^{r'(n)}\lambda'_{n+1}$$

where

$$\ell(n) = |v_1^{-1}|(2p^{n+2} - |\lambda'_{n+2}|).$$

An elementary inductive computation shows that

$k(n) = \ell(n)$?

$$\ell(n) = r'(n-1).$$

We claim that the former differential cannot occur. This follows because, by the inductive hypothesis,

$$d_{r'(n-1)}(\mu_3^{p^{n-2}})\doteq\lambda'_n,$$

and the former differential would contradict the Leibniz rule. So we must conclude the latter differential occurs. $\square$

We now state the main result of this section.

Theorem 2.24. *For each $n \geq 2$ and each nonnegative integer m with $m \not\equiv p-1$ mod p there are elements $z_{n,m}$ and $z'_{n,m}$ in $THH_*(BP\langle 2\rangle; k(1))$ such that*

(1) *$z_{n,m}$ projects to $\lambda'_n\mu_3^{mp^{n-2}}$ in $E_\infty^{*,0}$*
(2) *$z'_{n,m}$ projects to $\lambda'_n\lambda'_{n+1}\mu_3^{mp^{n-2}}$ in $E_\infty^{*,0}$*

As a $P(v_1)$-module, $THH_*(BP\langle 2\rangle; k(1))$ is generated by the unit element 1, λ_1, and the elements $\lambda_1^\epsilon z_{n,m}, \lambda_1^\epsilon z'_{n,m}$ where $\epsilon \in \{0,1\}$. The only relations are

$$v_1^{r'(n-1)}\lambda_1^\epsilon z_{n,m} = v_1^{r'(n-1)}\lambda_1^\epsilon z'_{n,m} = 0.$$

To prove this, we first need to prove a couple lemmas. We first introduce notation. Let $P(m)$ denote a free rank one $P(v_m)$-module and let $P(m)_i$ denote the $P(v_m)$-module $P(m)/v_m^i$. Let X be a free $BP\langle n\rangle$-module such that

$$H_*(X) \cong H_*(BP\langle n\rangle) \otimes H_*(\overline{X})$$

as a $H_*(BP\langle n\rangle)$-module and consider the Adams spectral sequence

$$E_2^{*,*}(X) = Ext^{*,*}_{E(Q_m)_*}(\mathbb{F}_p, H_*(\overline{X})) \Rightarrow \pi_*(X \wedge_{BP\langle n\rangle} k(m))_p \tag{2.25}$$

and the v_m-inverted Adams spectral sequence

$$v_m^{-1}E_2^{*,*}(X) = v_m^{-1}Ext^{*,*}_{E(Q_m)_*}(\mathbb{F}_p, H_*(\overline{X})) \Rightarrow \pi_*(X \wedge_{BP\langle n\rangle} K(m))_p. \tag{2.26}$$

Also, assume that $BP\langle n\rangle$ is sufficiently multiplicative that this Adams spectral sequence is multiplicative. In our case, $BP\langle 2\rangle$ is E_∞ so this will certainly be true. Consider the map of spectral sequences

$$E_2^{*,*}(X) \longrightarrow v_m^{-1}E_2^{*,*}(X)$$

induced by the localization map $k(m) \to v_m^{-1}k(m) = K(m)$.

Lemma 2.27. *Let $r \geq 2$. Suppose the $E_r(X)$-page of the Adams spectral sequence (2.25) is generated by elements in filtration 0 as a $P(k)$-module and $E_r^{*,*}(X)$ is a direct sum of copies of $P(k)$ and $P(k)_i$ with $i \leq r$ as a $P(k)$-module. Then*

(1) *the map of E_r-pages*

$$E_r^{s,t}(X) \to v_k^{-1} E_r^{s,t}(X)$$

r ?

is a monomorphism when $t \geq r-1 \geq 1$.

(2) *Also, the differentials in $E_{r\cancel{+1}}^{*,*}$ are the same as their image in $v_k^{-1}E_{r\cancel{+1}}^{*,*}$.*

Proof. Statement (1) is a consequence of our assumptions since elements in filtration $r-1$ are v_k-torsion free. To prove statement (2) it suffices to prove the following: if $x \in E_r(X)$ maps to a cycle $\bar{x} \in v_k^{-1}E_r(X)$,then x is a cycle. By our assumption, there is an $a \in E_r^{*,0}$ such that $x = v_k^m a$. Statement (1) then implies $d_{\cancel{r+1}}(a) = 0$ so since the differentials are v_k-linear the result follows. r ? ∎

The Lemma above is a generalization of part (a) and (b) of Theorem 7.1 [12], which must have also been known to the authors.

Lemma 2.28. *For $r \geq 2$, the E_r-page of the Adams spectral sequence (2.25) for $X = THH(BP\langle 2 \rangle)$ and $m = 1$ is generated by elements in filtration 0 as a $P(1)$-module and $E_r^{*,*}$ is a direct sum of copies of $P(1)$ and $P(1)_i$ for $i \leq r$.* r-1 ?

Proof. We will begin by proving the first statement by induction. Note that (2.22) 2 implies the base case in the induction when $r = 2$. Suppose the statement holds for some r. Choose a basis y_i for the $\mathbb{F}_p$-vector space V_r such that

$$V_r = \{x \in E_r^{*,0} \mid v_1^{r-1}x = 0\}.$$

Then $d_r(y_i)$ is in filtration r and since the differentials are v_1-linear, $v_1^{r-1}d_r(y_i) = 0$. However, this contradicts the induction hypothesis because the induction hypothesis implies that all elements in filtration r are v_1-torsion-free. Thus, each basis element y_i is a d_r-cycle. Next choose a set of elements $\{y_j'\} \subset E_r^{*,0}$ such that $\{d_r(y_j')\}$ is a basis for $\mathrm{im}(d_r\colon E_r^{*,0} \to E_r^{*,r})$. Choose $y_j'' \in E_r^{*,0}$ such that $v_1^r y_j'' = d_r(y_j')$. Then y_j'' are d_r-cycles and y_j'' and y_j are linearly independent. We can therefore choose d_r-cycles y_j''' such that $\{y_j\} \cup \{y_j''\} \cup \{y_j'''\}$ are a basis for the d_r-cycles in $E_r^{*,0}$. Then $\{y_j\} \cup \{y_j'\} \cup \{y_j''\} \cup \{y_j'''\}$ are a basis for $E_r^{*,0}$ and the differential is completely determined by the formulas

$$d_r(y_i) = 0, \quad d_r(y_j') = v_1^r y_j'', \quad d_r(y_j'') = 0, \text{ and } \quad d_r(y_j''') = 0.$$

Thus, $E_{r+1}^{*,*}$ is generated as a $P(1)$-module by y_i, y_i'', and y_i''' where $v_1^{r-1}y_i = 0$ and $v_1^r y_i'' = 0$ and y_i''' is v_1-torsion free. Why? ∎

Proof of Theorem 2.24. For brevity, we will let $\delta_{n,m}$ denote $\lambda_n'\mu_3^{mp^{n-2}}$ and we will let $\delta_{n,m}'$ denote $\lambda_n'\lambda_{n+1}'\mu_3^{mp^{n-2}}$. By Lemma 2.27 and Lemma 2.28 it suffices to prove that the elements $\delta_{n,m}$, and $\delta_{n,m}'$ are infinite cycles that, together with 1 and λ_1, form a basis for $E_\infty^{*,0}$ as an $\mathbb{F}_p$-vector space, and that each of $\delta_{n,m}$, $\delta_{n,m}'$ are killed by $v_1^{r'(n)}$. By induction on n, we will prove

$$E_{r(n)}(THH(BP\langle 2\rangle)) \cong M_n \oplus E(\lambda_1, \lambda_n', \lambda_{n+1}') \otimes P(\mu_3^{p^{n-1}})$$

$\lambda_1 \delta_{n,m}$?
$\lambda_1 \delta_{n,m}'$?

$\otimes P(v_1)$?

$r'(n-1)$?

$\lambda'_{n+1}, \lambda'_{n+2}$?

where M_n is generated by $\{\delta_{k,m}, \delta'_{k,m} \mid k < n\}$ modulo the relations

$$v_2^{r(k)}\delta_{k,m} = v_2^{r(k)}\delta'_{k,m} = 0.$$

Do you identify the Bockstein and the Adams-ss?

This statement holds for $n = 1$ by (2.22). Assume the statement holds for all integers less than or equal to some $N \geq 1$. Lemma 2.27, Lemma 2.28, and Theorem 2.23 imply that the only nontrivial differentials with source in $E_{r(N)}^{*,0}$ are the differentials

p^{N-1} ?

$$d_{r(N)}(\mu_3^{(m+1)r'(\cancel{N})}) = (m+1)v_1^{r'(N)}\lambda'_N\mu_3^{mp^{N-1}} \doteq \delta_{N,m}, \quad \cdot v_1^{r'(N)} \text{ ?}$$

and the differentials

$$d_{r'(N)}(\lambda'_{N+1}\mu_3^{(m+1)r'(N)}) = (m+1)v_1^{r'(N)}\lambda'_n\lambda'_{N+1}\mu_3^{mp^{N-2}} \doteq \delta'_{N,m}$$

where $m \not\equiv p-1 \mod p$. Combining this with Lemma 2.28 and Lemma 2.27, this implies that

$$E_{r(N)+1}(\mathrm{THH}(BP\langle 2\rangle)) \cong M_n \oplus V_{N+1} \oplus \Big(P(2) \otimes E(\lambda_1, \lambda'_N, \lambda'_N\mu_3^{(p-1)p^{N-2}}) \otimes P(\mu_3^{p^N})\Big)$$

$P(1)$?

where V_{N+1} has generators $\delta_{N,m}$ and $\delta'_{N,m}$ and relations

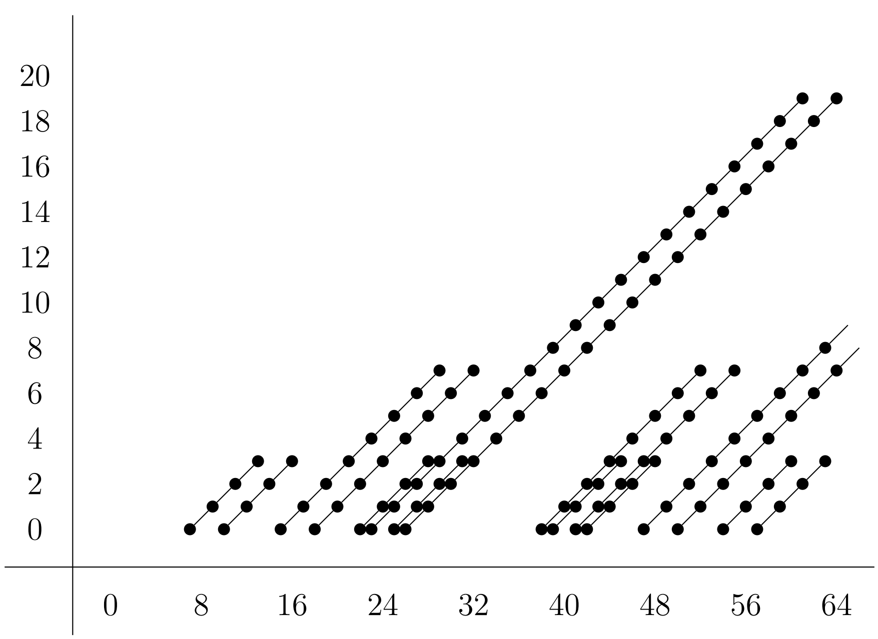
$$v_2^{r(N)}\delta_{N,m} = v_2^{r(N)}\delta'_{N,m} = 0.$$

By Lemma 2.27, Lemma 2.28, and Theorem 2.23 there is an isomorphism

$$E_{r(N)+1}(\mathrm{THH}(BP\langle 2\rangle)) \cong E_{r(N+1)}(\mathrm{THH}(BP\langle 2\rangle)).$$

Also, note that $M_N \oplus V_{N+1} = M_{N+1}$ and $\lambda'_N\mu_3^{(p-1)p^{N-2}} = \lambda'_{N+2}$ by definition. This completes the inductive step and consequently the proof. □

v_1-torsion in the E_∞-page of v_1-Bockstein Spectral Sequence for $0 \leq x \leq 64$



Remark 2.29. One may attempt to run the same arguments as in [2], but then one runs into the issue that the E_2-pages of the remaining Bockstein spectral sequences are more dense and therefore the “vanishing column” arguments that are essential to completing the results by their methods do not apply here. We therefore use the Brun spectral sequence in the next section instead to determine the first family of differentials. This can also be done using the Whitehead filtration and the topological Hochschild-May spectral sequence and this was the first approach of the authors, but the Brun spectral sequence is quite similar computationally and avoids some noise early on in the spectral sequences.

3. Brun spectral sequences

3.1. Preliminaries.

We first recall the main theorem from [8].

Theorem 3.1 (Thm. 1.1 [8]). *Let A be a cofibrant commutative S-algebra and let B be a connective cofibrant commutative A-algebra. Let E be a ring spectrum. Then, there is a strongly convergent, multiplicative spectral sequence of the form*

$$E^2_{n,m} = \pi_n THH(B; HE_m(B \wedge_A B)) \Rightarrow E_{n+m} THH(A; B).$$

If $E_m(B \wedge_A B)$ is an $\mathbb{F}_p$-vector space for all m and if $\pi_0(B)/p\pi_0(B) = \mathbb{F}_p$ as rings, we have

$$E^2_{n,m} = E_m(B \wedge_A B) \otimes_{\mathbb{F}_p} \pi_n(THH(B; H\mathbb{F}_p).$$

Notation. *We introduce notation $E^r(B, A, E)$ for the r-th page of the Brun spectral sequence for the triple (B, A, E), where*

$$E^2(B, A, E) := \pi_* THH(B; HE_*(B \wedge_A B)).$$

The main examples we will be interested in are the triples $(BP\langle i\rangle, BP\langle j\rangle, V(k))$ for $i, j, k \in \{-1, 0, 1, 2\}$ where $V(-1) = S$,

$$V(k) = S/(p, v_1, \dots v_k),$$

when this spectrum exists. We begin by computing relative cooperations for some spectra that will be needed in our later computations.

Lemma 3.2. *There is an isomorphism*

$$\pi_*(H\mathbb{Z}_p \wedge_{BP\langle 2\rangle} H\mathbb{Z}_p) \cong E_{\mathbb{Z}_p}(\sigma v_1, \sigma v_2)$$

of graded $\mathbb{Z}_p$-algebras and there is an isomorphism

$$V(0)_*(H\mathbb{Z}_p \wedge_{BP\langle 2\rangle} H\mathbb{Z}_p) \cong E(\sigma v_1, \sigma v_2)$$

Proof. The first result follows by the multiplicative Künneth spectral sequence

$$\mathrm{Tor}^{BP\langle 2\rangle_*}_{*,*}(\mathbb{Z}_p, \mathbb{Z}_p) \Rightarrow \pi_*(H\mathbb{Z}_p \wedge_{BP\langle 2\rangle} H\mathbb{Z}_p).$$

by computing the input

$$\mathrm{Tor}^{BP\langle 2\rangle_*}_{*,*}(\mathbb{Z}_p, \mathbb{Z}_p) \cong E_{\mathbb{Z}_p}(\sigma v_1, \sigma v_2)$$

using Tor-duality and observing that there is the algebra generators are all infinite cycles for bidegree reasons.

The second result follows by noting that

$$V(0)_*(H\mathbb{Z}_p \wedge_{BP\langle 2\rangle} H\mathbb{Z}_p) \cong \pi_*(H\mathbb{F}_p \wedge_{BP\langle 2\rangle} H\mathbb{Z}_p) \cong \pi_* H\mathbb{F}_p \wedge_{\mathbb{Z}_p} (H\mathbb{Z}_p \wedge_{BP\langle 2\rangle} H\mathbb{Z}_p)$$

so by the previous argument we know that $\pi_*(H\mathbb{Z}_p \wedge_{BP\langle 2\rangle} H\mathbb{Z}_p) \cong E_{\mathbb{Z}_p}(\sigma v_1, \sigma v_2)$, which is, in particular, a free graded $\mathbb{Z}_p$-algebra. So again we apply a multiplicative Künneth spectral sequence, this time with signature

$$\mathrm{Tor}_{*,*}^{\mathbb{Z}_p}(\mathbb{F}_p, E_{\mathbb{Z}_p}(\sigma v_1, \sigma v_2) \Rightarrow V(0)_*(H\mathbb{Z}_p \wedge_{BP\langle 2\rangle} H\mathbb{Z}_p).$$

The E_2-page is given by the isomorphism

$$\mathrm{Tor}_{*,*}^{\mathbb{Z}_p}(\mathbb{F}_p, E_{\mathbb{Z}_p}(\sigma v_1, \sigma v_2) \cong \mathbb{F}_p \otimes_{\mathbb{Z}_p} E_{\mathbb{Z}_p}(\sigma v_1, \sigma v_2)$$

which in particular, collapses to the zero line so the Künneth spectral sequence again collapses and the desired answer follows by the isomorphsim

$$\mathbb{F}_p \otimes_{\mathbb{Z}_p} E_{\mathbb{Z}_p}(\sigma v_1, \sigma v_2) \cong E(\sigma v_1, \sigma v_2).$$

□

We begin by computing the Brun spectral sequence for the triple $(BP\langle 2\rangle, BP\langle 0\rangle, S)$ where $BP\langle 0\rangle = H\mathbb{Z}_p$. First, we need the following result of Bökstedt.

Theorem 3.3 (Bökstedt)**.** *There is an isomorphism of graded $\mathbb{F}_p$-algebras*

$$\mathrm{THH}_*(H\mathbb{Z}_p; H\mathbb{F}_p) \cong E(\lambda_1) \otimes P(\mu_1),$$

there are isomorphisms of groups

$$\pi_t \,\mathrm{THH}(\mathbb{Z}) \cong \begin{cases} \mathbb{Z}\{1\} & t = 0 \\ \mathbb{Z}/n\{\gamma_n\} & t = 2n-1 > 0 \\ 0 & \textit{else} \end{cases}$$

and the map

$$THH_*(H\mathbb{Z}) \to THH_*(H\mathbb{Z}; H\mathbb{F}_p)$$

sends γ_n to $\lambda_1 \mu_1^{k-1}$ when $n = pk$ for some integer $k \geq 1$ and to 0 otherwise. This is also a map of graded rings where the former has a graded ring structure by letting $\gamma_i \cdot \gamma_j = 0$ for all i, j.

Corollary 3.4. *There is an isomorphism*

$$\pi_t \,\mathrm{THH}(\mathbb{Z}_p) \cong \begin{cases} \mathbb{Z}_p & t = 0 \\ \mathbb{Z}/p^{\nu_p(n)}\{\gamma_n\} & t = 2n-1 > 0 \\ 0 & \textit{else} \end{cases}$$

where ν_p denotes the p-adic valuation and the map $THH_*(\mathbb{Z}) \to THH_*(\mathbb{Z}_p)$ sends γ_n to γ_n if $p \mid n$ and zero otherwise, so the map of graded $\mathbb{Z}_p$-algebras

$$THH_*(H\mathbb{Z}_p) \to THH_*(H\mathbb{Z}_p; H\mathbb{F}_p)$$

sends γ_{pk} to $\lambda_1 \mu_1^{k-1}$ as before with $\gamma_i \cdot \gamma_j = 0$ for all i, j as before.

Proof. It is clear that

$$THH_*(H\mathbb{Z}_p; H\mathbb{F}_p) \simeq THH_*(H\mathbb{Z}; H\mathbb{F}_p) \cong E(\lambda_1, \cancel{\mu_1}) \otimes P(\mu_1)$$

so the computation of $THH_*(H\mathbb{Z}_p)$ is clear from the Bockstein spectral sequence

$$THH_*(H\mathbb{Z}; H\mathbb{F}_p)[v_0] \Rightarrow THH_*(\mathrm{H}\mathbb{Z}_p)$$

with differentials

$$d_k(\mu_1^{p^{k-1}}) = \cancel{v_0^k} \lambda_1, \quad \cdot\, v_0^a \mu_1^{p^{a-1}-1} \lambda_1 \quad ?$$

all possible additive extensions, and no possible multiplicative extensions. ☐

We then compute the Brun spectral sequence for the triple $(BP\langle 2\rangle, BP\langle 0\rangle, V(0))$ using the known answer for the abutment. An argument intrinsic to this approach should also be possible, but in the interest of brevity we do not take this approach.

Lemma 3.5. *The Brun spectral sequence with signature*

$$E^2(BP\langle 2\rangle, BP\langle 0\rangle, V(0)) \Rightarrow \pi_*(THH(BP\langle 2\rangle, H\mathbb{F}_p))$$

has E^2-page

p ≥ 2?

$$E^2(BP\langle 2\rangle, BP\langle 0\rangle, V(0)) \cong THH_*(H\mathbb{Z}_p; \mathbb{F}_p) \otimes E(\sigma v_1, \sigma v_2),$$

differentials

name of generator changed

$$2p \to d^1(\sigma\tau_1) = \sigma v_1,$$
$$2p^2 \to d^p((\sigma\tau_1)^p) = \sigma v_2$$

and E_∞-page isomorphic to the abutment, which is in turn isomorphic to

$$E(\lambda_1, \lambda_2, \cancel{\lambda_2}\,3) \otimes P(\mu_3)$$

by an isomorphism mapping λ_1 to λ_1, $\sigma v_1(\sigma\tau_1)^{p-1}$ to λ_2, $\sigma v_2(\sigma\tau_1)^{p^2-p}$ to λ_3, and $(\sigma\tau_1)^{p^2}$ to μ_3.

Proof. The input is determined by Theorem 3.1, Theorem 3.3, and Lemma 3.2. We know that the abutment has no elements in degrees k where $k \equiv 0 \pmod{2p}$ and $k < 2p^3$. This immediately implies the two differentials and the rest is determined by the Leibniz rule and the fact that any further differentials would result in a smaller output than the known abutment. ☐

Lemma 3.6. *The map*

$$E^2(BP\langle 2\rangle, BP\langle 0\rangle, S) \to E^2(BP\langle 2\rangle, BP\langle 0\rangle, V(0))$$

of E^2-pages of Brun spectral sequences is determined by the map

$$THH_*(H\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} E_{\mathbb{Z}_p}(\sigma v_1, \sigma v_2) \to THH_*(H\mathbb{Z}_p; H\mathbb{F}_p) \otimes E(\sigma v_1, \sigma v_2)$$

given by tensoring the map

$$THH_*(H\mathbb{Z}_p) \to THH_*(H\mathbb{Z}_p; H\mathbb{F}_p)$$

of Corollary 3.4 with the mod p-reduction map

$$E_{\mathbb{Z}_p}(\sigma v_1, \sigma v_2) \to E(\sigma v_1, \sigma v_2)$$

over the canonical quotient map $\mathbb{Z}_p \to \mathbb{F}_p$.

Proof. This follows easily by functoriality of the identification of the E^2-page in Theorem 3.1 and Corollary 3.4. $\square$

We will now use the previous result as well as the known abutment from a previous section to determine the differentials in the Brun spectral sequence for the triple $(BP\langle 2\rangle, BP\langle 0\rangle, S)$. This will then be used to import key differentials into the main Brun spectral sequence of interest.

Lemma 3.7. *The Brun spectral sequence with signature*

$$E^2(BP\langle 2\rangle, BP\langle 0\rangle, S) \Rightarrow \pi_* THH(BP\langle 2\rangle, BP\langle 0\rangle)$$

has E^2-page

$$E^2(BP\langle 2\rangle, BP\langle 0\rangle, S) \cong THH_*(H\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} E(\sigma v_1, \sigma v_2)$$

and differentials

$$\begin{aligned} d^1(\gamma_{pk}) &= (k-1)\sigma v_1 \gamma_{p(k-1)}, \\ d^{p+1}(a_k) &= (k-1)\sigma v_2 a_{k-1}, \\ d^{p+1}(b_k) &= (k-1)\sigma v_2 b_{k-1} \end{aligned}$$

γ_{pa} $p \nmid k$

where $a_k = p\gamma_{p^2k}$ and $b_k = p^2 i \gamma_{p^2k}\sigma v_1$ and additive extensions

$$\begin{aligned} p\gamma_p &= \sigma v_1 \\ pa_1 &= \sigma v_2. \end{aligned}$$

What does the i mean?

why are the differentials determined?

Proof. We consider the map of spectral sequences

$$E^2(BP\langle 2\rangle, BP\langle 0\rangle, S) \to E^2(BP\langle 2\rangle, BP\langle 0\rangle, V(0)).$$

This is completely described by Lemma 3.6 and therefore we can determine the d^1-differentials

Can be zero or not zero

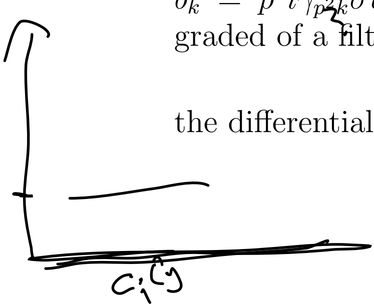
$$d^1(\gamma_{pk}) = (k-1)\sigma v_1 \gamma_{p(k-1)}$$

directly. This accounts for all d_1-differentials. Using the translation $a_k = p\gamma_{p^2k}$ and $b_k = p^2 i\gamma_{p^2k}\sigma v_1$ we then observe that the E^2-page is isomorphic to the associated graded of a filtration of

$$THH_*(\ell, H\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} E_{\mathbb{Z}_p}(\sigma v_2)$$

$$\begin{array}{ccc} \mathbb{Z}/p\{\gamma_{p(p+1)}\} & \xrightarrow[0 \quad p]{d^{2p}} & \mathbb{Z}/p^2\{\sigma v_1 \gamma_{p^2}\} \\ \downarrow & & \downarrow \\ \mathbb{Z}/p\{\lambda_1 \mu_1^p\} & \xrightarrow[=0 \quad \cdot p]{d^{2p}} & \mathbb{Z}/p\{\sigma v_1 \lambda_1 \mu_1^{p-1}\} \end{array}$$

the differential pattern



$$\begin{aligned} d^{p+1}(a_k) &= (k-1)\sigma v_2 a_{k-1}, \\ d^{p+1}(b_k) &= (k-1)\sigma v_2 b_{k-1} \end{aligned}$$

is then forced in order to get the desired answer in the abutment and so are the hidden additive extensions. $\square$

Let us now consider the main example of interest, for the triple $(BP\langle 2\rangle, BP\langle 1\rangle, S)$. Note that we are choosing $BP\langle 1\rangle$ to be a commutative $BP\langle 2\rangle$-algebra model for $BP\langle 1\rangle$ and both $BP\langle 2\rangle$ and $BP\langle 1\rangle$ are implicitly p-completed. We first begin with a necessary lemma.

Lemma 3.8. *There is an isomorphism of graded $\mathbb{Z}_p$-algebras*

$$\pi_*(BP\langle 1\rangle \wedge_{BP\langle 2\rangle} BP\langle 1\rangle) \cong \pi_*(BP\langle 1\rangle) \otimes_{\mathbb{Z}_p} E_{\mathbb{Z}_p}(\sigma v_2)$$

Proof. We compute the Künneth spectral sequence

$$\mathrm{Tor}_*^{BP\langle 2\rangle_*}(BP\langle 1\rangle_*, BP\langle 1\rangle_*) \Rightarrow \pi_*(BP\langle 1\rangle \wedge_{BP\langle 2\rangle} BP\langle 1\rangle).$$

The input is $\pi_*(BP\langle 1\rangle)_* \otimes E(\sigma v_2)$ by Tor-duality. The spectral sequence then collapses by multiplicativity and the because all algebra generators are infinite cycles for bidegree reasons. □

Corollary 3.9. *The E_2-page of the Brun spectral sequence for the triple $(BP\langle 2\rangle, BP\langle 1\rangle, S)$ is*

$$E_2(BP\langle 2\rangle, BP\langle 1\rangle, S) = \pi_n(\mathrm{THH}(BP\langle 1\rangle; H\mathbb{Z}_p)) \otimes_{\mathbb{Z}_p} \pi_*(BP\langle 1\rangle) \otimes_{\mathbb{Z}_p} E_{\mathbb{Z}_p}(\sigma v_2).$$

In addition the E_2-page of the Brun spectral sequence for the triple $(BP\langle 1\rangle, BP\langle 1\rangle, S)$ is

$$E_2(BP\langle 1\rangle, BP\langle 1\rangle, S) = \pi_n(\mathrm{THH}(BP\langle 1\rangle; H\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \pi_*(BP\langle 1\rangle)$$

and the map

$$E_2(BP\langle 2\rangle, BP\langle 1\rangle, S) \to E_2(BP\langle 1\rangle, BP\langle 1\rangle, S)$$

is the identity map tensored with the usual counit map $E_{\mathbb{Z}_p}(\sigma v_2) \to \mathbb{Z}_p$ of the Hopf algebra $E_{\mathbb{Z}_p}(\sigma v_2)$.

By [2], we know that all the differentials are determined by the formulas

$$d_{f(n)}(p^{n-1}a_{kp^{n-1}}) \doteq (k-1)v_1^{p^n+\cdots+p}b_{(k-1)p^{n-1}} \tag{3.10}$$

where $f(n) = |v_1^{p^n+\cdots+p}|$ and the Leibniz rule in the spectral sequence

$$E_2(BP\langle 1\rangle, BP\langle 1\rangle, S) \Rightarrow \pi_*(THH(BP\langle 1\rangle))$$

since this spectral sequence can be identified with the v_1-Bockstein spectral sequence already at the E_2-page, up to a shift in filtration.

Since we determined the map of Brun spectral sequences

$$E_2(BP\langle 2\rangle, BP\langle 1\rangle, S) \to E_2(BP\langle 1\rangle, BP\langle 1\rangle, S)$$

we may then import these differentials into the spectral sequence

$$E_2(BP\langle 2\rangle, BP\langle 1\rangle, S) \Rightarrow \pi_*THH(BP\langle 2\rangle; BP\langle 1\rangle)$$

though some care must be taken when doing this. In particular, a priori, there could be a differential hitting a σv_2-divisible element in $E_2(BP\langle 2\rangle, BP\langle 1\rangle, S)$ that interrupts the other differential pattern. This cannot happen for the first differential because $|\sigma v_2| = 2p^2 - 1$ and the length of the first family of differentials is $2p^2 - 2p$, which is clearly smaller. We therefore immediately determine the first differential pattern.

Lemma 3.11. *There is a family of differentials*

$$d_{2p^2-2p}(a_k)\doteq(k-1)v_1^p b_{(k-1)} \tag{3.12}$$

in the Brun spectral sequence with signature

$$E^2(BP\langle 2\rangle, BP\langle 1\rangle, S) \Rightarrow \pi_* THH(BP\langle 2\rangle, BP\langle 1\rangle).$$

We observe that

$$E^{2p^2-2p+1}(BP\langle 2\rangle, BP\langle \cancel{2}\rangle, S) \cong E^{2p^2-2p+1}(BP\langle 1\rangle, BP\langle \cancel{2}\rangle, S) \otimes E(\sigma v_2).$$

In particular, note that

$$\{pa_{pi}, v_1^p b_{pk} : j \geq 1, k \geq 1\}$$

survive where $p^j pa_{pi} = 0$ when $j = \nu_p(i) + 1$ and $pkv_1^p b_{pk} = 0$.

We also can determine the map of Brun spectral sequences

$$E_2(BP\langle 2\rangle, BP\langle 1\rangle, S) \to E_2(BP\langle 2\rangle, BP\langle 0\rangle, S),$$

which is isomorphic to

$$\pi_*(\mathrm{THH}(BP\langle 1\rangle; H\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \pi_*(BP\langle 1\rangle) \otimes_{\mathbb{Z}_p} E_{\mathbb{Z}_p}(\sigma v_2) \to THH_*(H\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} E(\sigma v_1, \sigma v_2),$$

is given by tensoring the map

$$\pi_*(\mathrm{THH}(BP\langle 1\rangle; H\mathbb{Z}_p) \to THH_*(H\mathbb{Z}_p)$$

induced by the map $BP\langle 1\rangle \to H\mathbb{Z}_p$ with the canonical quotient

$$\pi_*(BP\langle 1\rangle) \to \mathbb{Z}_p$$

the identity

$$E_{\mathbb{Z}_p}(\sigma v_2) \to E_{\mathbb{Z}_p}(\sigma v_2)$$

and the unit map

$$\mathbb{Z}_p \to E_{\mathbb{Z}_p}(\sigma v_1).$$

We determined a differential hitting a σv_2-divisible element in the Brun spectral sequence with signature

$$E_2(BP\langle 2\rangle, BP\langle 0\rangle, S) \Rightarrow THH_*(BP\langle 2\rangle, BP\langle 0\rangle)$$

given by

$$d_{2p^2-1}(b_k)\doteq(k-1)\sigma v_2 b_{k-1}. \tag{3.13}$$

and this differential lefts to the same differential in the Brun spectral sequence with signature

$$E_2(BP\langle 2\rangle, BP\langle 1\rangle, S) \Rightarrow THH_*(BP\langle 2\rangle, BP\langle 1\rangle).$$

Consequently, we have the identification

$$E^{2p^2}(BP\langle 2\rangle, BP\langle 2\rangle, S) \cong H(E^{2p^2-2p+1}(BP\langle 1\rangle, BP\langle 2\rangle, S) \otimes E(\sigma v_2); d)$$

where the differential we simply denote by d here is determined by (3.13) and the Leibniz rule, and we write $H(M, d)$ for the homology of a differential bigraded algebra with respect to a differential. We observe that, in particular the elements

$$\{pb_{pj}, \sigma v_2 b_{pj} : j \geq 1\}$$

survive where pb_{pj} is indecomposable, $p \cdot ((p^{i-1} \cdot pb_{pj})) = 0$ for $i = \nu_p(j) + 1$, and $v_1 \cdot (v_1^{p-1}(pb_{pj})) = 0$. Also, $v_1 \cdot (v_1^{p-1}\sigma v_2 b_{pj}) = 0$ and $pj(\sigma v_2 b_{pj}) = 0$.

We claim that this E_{2p^2-2p+1}-page

$$E^{2p^2}(BP\langle 2\rangle, BP\langle 2\rangle, S)$$

and the E_1-page

$$THH_*(BP\langle 2\rangle, H\mathbb{Z}_p)[v_1]$$

of the Bockstein spectral sequence

$$THH_*(BP\langle 2\rangle, H\mathbb{Z}_p)[v_1] \Rightarrow THH_*(BP\langle 2\rangle, BP\langle 1\rangle)$$

are two different associated graded algebras of two different filtrations of the same bigraded algebra.

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