THH OF $BP\langle 2\rangle$

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Abstract. Various spectral sequences for getting the calculation of $THH_*BP\langle 2\rangle$ started.

Contents

Throughout, we will probably let $p=2$. Also, I don't feel like writing $BP\langle 2\rangle$ all the time, so we will set R to be $BP\langle 2\rangle$ or $\mathrm{tmf}_1(3)$ or whatever...

1. Bökstedt spectral sequence for $H_*\,\mathrm{THH}(R)$

The Bökstedt spectral sequence for R takes the form

$$HH_s(H_tR) \implies H_{s+t}\,\mathrm{THH}(R).$$

Since

$$H_*(R) = A/\!\!/E(2)_* = P(\zeta_1^2, \zeta_2^2, \zeta_3^2, \zeta_4, \zeta_5, \ldots).$$

So by standard theorems about Hochschild homology,

$$HH_*R \cong A/\!\!/E(2)_* \otimes E(\sigma\zeta_1^2, \sigma\zeta_2^2, \sigma\zeta_3^2, \sigma\zeta_4, \ldots)$$

For degree reasons, this spectral sequence collapses immediately (or by comparison with the Bökstedt spectral sequence for $\mathrm{THH}(\mathbb{F}_2)$, which is known to collapse immediately). However, there are hidden multiplicative extensions arising from power operations. We get

$$H_*\,\mathrm{THH}(R) = A/\!\!/E(2)_* \otimes E(\lambda_1, \lambda_2, \lambda_3) \otimes P(\mu)$$

where $|\lambda_1| = 3$, $|\lambda_2| = 7$, and $|\lambda_3| = 15$ and $|\mu| = 16$.

2. $THH(BP\langle 2\rangle;\mathbb{F}_2)$

Recall that

$$\mathrm{THH}(BP\langle 2\rangle;\mathbb{F}_2) := H\mathbb{F}_2 \wedge_{R\wedge R^{\mathrm{op}}} R$$

It turns out that there is an equivalence

$$\mathrm{THH}(R;\mathbb{F}_2) \simeq H\mathbb{F}_2 \wedge_R \mathrm{THH}(R).$$

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The first way we will compute $\mathrm{THH}_*(R;\mathbb{F}_2)$ is by The Bökstedt spectral sequence for this then becomes

$$HH_*(H_*R;A_*) \implies H_*(\mathrm{THH}(R;\mathbb{F}_2))$$

I am pretty sure the Bökstedt spectral sequence with coefficients take this form... I think it does, I think this is because we can think of $THH(R;M)$ as the realization of the diagram simplicial module

$$[n] \mapsto M \wedge R^{\wedge n}$$

and then we just take the skeleton spectral sequence. Think about applying homology to this simplicial object, then the E_1-page is the alternating sign chain complex of this simplicial graded abelian group and it happens to be the same as the chain complex computing $HH_*(H_*(R);H_*(M))$.

As in the topological case[1], we should have

$$HH_*(H_*R;A_*) \cong \mathrm{Tor}^{A/\!\!/E(2)_*}(A_*, H_*\,\mathrm{THH}(R)).$$

I don't understand this statement. It is true that $HH_*(H_*(R);H_*HF_2) \cong Tor^{H_*(R)\otimes H_*(R)^{op}}(A_*, H_*(R))$ and it is also true that the RHS is the input for a Künneth spectral sequence computing $H_*THH(R;HF_2)$, but the RHS and LHS will only agree when the Bökstedt spectral sequence collapses and there are no hidden multiplicative extensions. Since A_* is free over $A/\!\!/E(2)_*$[2], we have that this Tor-group is concentrated in Tor_0, and so the E^2-page is

$$\mathrm{Tor}_0^{A/\!\!/E(2)_*}(A_*, A/\!\!/E(2)_* \otimes E(\lambda_1,\lambda_2,\lambda_3)\otimes P(\mu)) = A_* \otimes E(\lambda_1,\lambda_2,\lambda_3)\otimes P(\mu).$$

Feeding this into the Adams spectral sequence yields

$$\mathrm{THH}_*(R;\mathbb{F}_2) \cong E(\lambda_1,\lambda_2,\lambda_3)\otimes P(\mu)$$

As a check that this is correct, we will also calculate via the Künneth spectral sequence. This is a spectral sequence of the form

$$\mathrm{Tor}_{s,t}^{H_*R}(\mathbb{F}_2,\mathbb{F}_2) \implies \mathrm{THH}_{s+t}(R;\mathbb{F}_2).$$

I don't understand this either. I think of the Künneth spectral sequence as a spectral sequence for computing E_* of a relative smash product $A \wedge_R B$ and then the input is

[1] Lemma 2.1 of [1] which states that when R is a commutative S-algebra and M an R-module, then

$$\mathrm{THH}(R;M) \simeq M \wedge_R \mathrm{THH}(R).$$

[2] Indeed, as an $A/\!\!/E(2)_*$-module it is $A/\!\!/E(2)_*\{\zeta_1,\zeta_2,\zeta_1\zeta_2\}$

$Tor^{E_*(R)}(E_*(A); E_*(B))$ so the LHS above looks like the input for a Künneth spectral sequence computing $\pi_*(HF_2 \wedge_{HF_2 \wedge R} HF_2)$, whereas the the RHS is $\pi_*(HF_2 \wedge_{R \wedge R^{op}} R)$. The E^2-term is

$$\mathrm{Tor}^{A/\!\!/E(2)_*}(\mathbb{F}_2, \mathbb{F}_2) \cong E(\sigma\zeta_1^2, \sigma\zeta_2^2, \sigma\zeta_3^2, \sigma\zeta_4, \ldots)$$

with the degree of $\sigma\zeta_n$ being $(1, 2^n - 1)$. This spectral sequence collapses at E_2 and also plays well with power operations. Thus giving the desired answer.

3. $\mathrm{THH}(R; k(2))$

Let $k(2)$ denote the connective second Morava K-theory. We will follow the analogous proof for $THH(\ell; k(1))$ which can be found in [3] and [2].

Recall that

$$\mathrm{THH}(R; k(2)) := k(2) \wedge_{R \wedge R^{\mathrm{op}}} R \simeq k(2) \wedge_R \mathrm{THH}(R)$$

In the case of ku, one could note that

$$V(0) \wedge \mathrm{THH}(ku) = V(0) \wedge (\mathrm{ku} \wedge_{\mathrm{ku} \wedge \mathrm{ku}^{\mathrm{op}}} \mathrm{ku}) \simeq (V(0) \wedge \mathrm{ku}) \wedge_{\mathrm{ku} \wedge \mathrm{ku}^{\mathrm{op}}} \mathrm{ku} = k(1) \wedge_{\mathrm{ku} \wedge \mathrm{ku}^{\mathrm{op}}} \mathrm{ku}$$

$$V(0) \wedge R \simeq R/2$$

I just want to see if there is unique pattern of differentials.... So let's just assume for now that the only v_2-torsion free class in $\mathrm{THH}(R; k(2))$ is 1. Let's see if this uniquely determines the Adams differentials.

First we need to calculate the mod 2 homology of $\mathrm{THH}(R; k(2))$.

There is an isomorphism

$$H_*(THH(R; k(2)) \cong H_*(THH(R) \wedge_R k(2))$$

Since $H_* THH(R)$ is free over $H_*(R)$ the Künneth spectral sequence collapses and therefore

$$H_*(THH(R; k(2)) \cong H_*(k(2)) \otimes E(\lambda_1, \lambda_2, \lambda_3) \otimes P(\mu)$$

We then want to compute the Adams spectral sequence

$$Ext_A^{*,*}(H\mathbb{F}_p; H_*(k(2)) \otimes E(\lambda_1, \lambda_2, \lambda_3) \otimes P(\mu)) \Rightarrow \pi_*(HH(R; k(2))$$

and the input is isomorphic to

$$Ext_{E(Q_2)}(H\mathbb{F}_p; E(\lambda_1, \lambda_2, \lambda_3) \otimes P(\mu))$$

by a change of rings isomorphism. Since $E(\lambda_1, \lambda_2, \lambda_3) \otimes P(\mu)$ has trivial $E(Q_2)$-coaction, this in in turn is isomorphic to

$$E(\lambda_1, \lambda_2, \lambda_3) \otimes P(\mu) \otimes P(v_2)$$

where v_2 is in bidegree $(2p - 2, 1)$. In other words, this spectral sequence has isomorphic E_2-page to the Bockstein spectral sequence.

We then want to compute differentials in this spectral sequence. To do this, it will be useful to consider the v_2^{-1} Adams spectral sequence

$$v_2^{-1}Ext_{E(Q_2)}(H\mathbb{F}_p; E(\lambda_1, \lambda_2, \lambda_3) \otimes P(\mu))$$

converging to $v_2^{-1}\pi_* THH(R; k(2))$.

Question: Is $v_2^{-1}\pi_* THH(R; k(2)) \cong \pi_*(L_{K(2)} THH(R; k(2)))$? This is definitely not true in general (conjecturally), but I think it is true for $tmf_1(3)$, and $k(2)$ and $THH(tmf_1(3); k(2))$ is a $tmf_1(3)$-module and a $k(2)$-module.

In McClure-Staffeldt, they prove that the unit map $\ell \to THH(\ell)$ is a $K(1)_*$-equivalence, so $K(1)$-locally all the classes of the form σx must die. Since the telescope conjecture is true at height 1, this implies that all the classes of the form σx are v_2-torsion and that forces all the differentials (note we also know in the case of ℓ that all powers of v_1 itself survive so there can be no differentials on the λ_i for $i = 0, 1$ hitting powers of v_1.

References

[1] Vigleik Angeltveit, Michael Hill, and Tyler Lawson, *Topological hochschild homology of l and ko* (200710), available at `0710.4368`.

[2] Vigleik Angeltveit and John Rognes, *Hopf algebra structure on topological hochschild homology*, available at `math/0502195`.

[3] J.E. McClure and R. E. Staffeldt, *On the topological Hochschild homology of bu, i*, American Journal of Mathematics **115** (1993Feb.), no. 1, 1–45.

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