claim: We have an equivalence of S-algebras A_∞

$$H\pi_*(BP\langle 2\rangle) \leftarrow S[v_1] \underset{S}{\wedge} S[v_2] \underset{S}{\wedge} H\mathbb{Z}_{(p)} , \quad E_\infty$$

where $S[v_i] := S \vee S_S^{2p^i-2} \vee \left(S_S^{2p^i-2} \underset{S}{\wedge} S_S^{2p^i-2}\right) \vee \cdots$

E_∞ by your paper

proof: $S_S^{2p^i-2} \longrightarrow H\pi_* BP\langle 2\rangle$ induces map of S-algebras

$$S[v_i] \longrightarrow H\pi_* BP\langle 2\rangle$$

$H\mathbb{Z}_{(p)} \longrightarrow H\pi_* BP\langle 2\rangle$ is a map of comm. S-algebras

$\xrightarrow{\simeq} H\pi_0(BP\langle 2\rangle)$

$$S[v_1] \underset{S}{\wedge} S[v_2] \underset{S}{\wedge} H\mathbb{Z}_{(p)} \longrightarrow H\pi_* BP\langle 2\rangle \underset{S}{\wedge} H\pi_* BP\langle 2\rangle \underset{S}{\wedge} H\pi_* BP\langle 2\rangle \xrightarrow{\text{multiplication}} H\pi_*(BP\langle 2\rangle)$$

A_∞

E_∞ map, because $H\pi_* BP\langle 2\rangle$ is E_∞

The map is an equivalence. $\square$

We get $THH(H\pi_* BP\langle 2\rangle) \underset{\text{as spectra}}{\simeq} THH(S[v_1] \underset{S}{\wedge} S[v_2] \underset{S}{\wedge} H\mathbb{Z}_{(p)}) \underset{\text{as spectra}}{\simeq} THH(S[v_1] \underset{S}{\wedge} S[v_2]) \underset{S}{\wedge} THH(H\mathbb{Z}_{(p)})$

$$\underset{\text{as spectra}}{\simeq} THH(S[v_1] \underset{S}{\wedge} S[v_2]) \underset{S}{\wedge} H\mathbb{Z}_{(p)} \underset{H\mathbb{Z}_{(p)}}{\wedge} THH(H\mathbb{Z}_{(p)}) \underset{\text{as spectra}}{=} THH^{H\mathbb{Z}_{(p)}}(S[v_1] \underset{S}{\wedge} S[v_2] \underset{S}{\wedge} H\mathbb{Z}_{(p)}) \underset{H\mathbb{Z}_{(p)}}{\wedge} THH(H\mathbb{Z}_{(p)})$$

$$\underset{\text{as spectra}}{=} THH^{H\mathbb{Z}_{(p)}}(H\pi_* BP\langle 2\rangle) \underset{H\mathbb{Z}_{(p)}}{\wedge} THH(H\mathbb{Z}_{(p)}) \leftarrow \text{has homotopy groups}$$

$$\mathbb{Z}_{(p)}[v_1, v_2] \underset{\mathbb{Z}_{(p)}}{\otimes} E_{\mathbb{Z}_{(p)}}(\sigma v_1, \sigma v_2) \underset{\mathbb{Z}_{(p)}}{\otimes} THH_*(\mathbb{Z}_{(p)})$$

We note that the equivalence is $THH(H\mathbb{Z}_{(p)})$-linear

compatible with the unit $H\pi_* BP\langle 2\rangle \longrightarrow THH(H\pi_* BP\langle 2\rangle)$

$$\cong$$

$$\longrightarrow THH^{H\mathbb{Z}_{(p)}}(H\pi_* BP\langle 2\rangle) \otimes THH(H\mathbb{Z}_{(p)})$$

$H\mathbb{Z}_{(p)}$

and compatible with the map

$$\begin{tikzcd} THH(H\pi_* BP\langle 2\rangle) \arrow[r] & THH^{H\mathbb{Z}_{(p)}}(H\pi_* BP\langle 2\rangle) \\ THH^{H\mathbb{Z}_{(p)}}(H\pi_* BP\langle 2\rangle) \wedge_{H\mathbb{Z}_{(p)}} THH(H\mathbb{Z}_{(p)}) \arrow[u, "\wr"] \arrow[ur] & \end{tikzcd}$$

We denote the equivalence by Φ.

Let σv_i be the class in $THH_*(H\pi_* BP\langle 2\rangle)$ that is mapped to σv_i under Φ

We get an $\mathbb{Z}_{(p)}$-algebra map

$$f: \pi_* BP\langle 2\rangle \otimes_{\mathbb{Z}_{(p)}} E_{\mathbb{Z}_{(p)}}(\sigma v_1, \sigma v_2) \otimes_{\mathbb{Z}_{(p)}} THH_*(\mathbb{Z}_{(p)}) \longrightarrow THH_*(H\pi_* BP\langle 2\rangle)$$

(I believe that we need $p \geq 3$ here, so that 2 is invertible and $E_{\mathbb{Z}_{(p)}}(\sigma v_1, \sigma v_2)$ is the free graded-comm. $\mathbb{Z}_{(p)}$-algebra)

We claim that f is an equivalence

We have $\Phi \circ f$

$v_1^n v_2^m \longmapsto v_1^n v_2^m$

$v_1^n v_2^m \sigma v_1 \longmapsto v_1^n v_2^m \sigma v_1 +$ something in the kernel of $THH_*^{\mathbb{Z}_{(p)}}(H\pi_* BP\langle 2\rangle) \otimes_{\mathbb{Z}_{(p)}} THH_*(\mathbb{Z}_{(p)}) \xrightarrow{p} THH_*^{\mathbb{Z}_{(p)}}(H\pi_* BP\langle 2\rangle)$

$v_1^n v_2^m \sigma v_2 \longmapsto v_1^n v_2^m \sigma v_2 +$ something in the kernel of p

$v_1^n v_2^m \sigma v_1 \sigma v_2 \longmapsto v_1^n v_2^m \sigma v_1 \sigma v_2 +$ something in the kernel of p

Note that kernel of p is generated as abelian group by

$v_1^n v_2^m \sigma v_1^{\varepsilon_1} \sigma v_2^{\varepsilon_2} a \qquad a \in THH_*(\mathbb{Z}_{(p)})$ of positive degree

We get by induction on the degree that all the classes $v_1^n v_2^m \sigma v_1^{\varepsilon_1} \sigma v_2^{\varepsilon_2}$ are in the image of $\phi \circ f$

Hence $\phi \circ f$ is surjective.

We thus have a short exact sequence of $THH(\mathbb{Z}_{(p)})_*$-modules

$$0 \longrightarrow \ker \phi\circ f \longrightarrow \underbrace{\pi_* H\mathbb{F}_p BP\langle 2\rangle \otimes_{\mathbb{Z}_{(p)}} E_{\mathbb{Z}_{(p)}}(\sigma v_1, \sigma v_2) \otimes THH_*(H\mathbb{Z}_{(p)})}_{=: M_*} \xrightarrow{\phi \circ f} \underbrace{THH^{\mathbb{Z}_{(p)}}_*(H\mathbb{F}_p BP\langle 2\rangle) \otimes_{\mathbb{Z}_{(p)}} THH_*(\mathbb{Z}_{(p)})}_{N_* :=}$$

since the right-hand module is a free $THH(\mathbb{Z}_{(p)})_*$-module, the short exact sequence splits.

We thus have $M_n = N_n \oplus (\ker \phi\circ f)_n$ as $\mathbb{Z}_{(p)}$-modules

Since $\mathbb{Z}_{(p)}$ is noetherian $(\ker \phi\circ f)_n$ is a finitely generated $\mathbb{Z}_{(p)}$-module

since M_n and N_n are abstractly isomorphic as finitely generated $\mathbb{Z}_{(p)}$-modules, and since $\mathbb{Z}_{(p)}$ is a principal ideal domain, we get $(\ker \phi\circ f)_n = 0$