We consider the Brun spectral sequence

$$E^2(BP\langle 2\rangle, BP\langle 0\rangle, S) = \mathrm{THH}_*(H\mathbb{Z}_p)\otimes_{\mathbb{Z}_p} E_{\mathbb{Z}_p}(\sigma v_1, \sigma v_2) \Longrightarrow \mathrm{THH}_*(BP\langle 2\rangle; H\mathbb{Z}_p).$$

First note that $E_{\mathbb{Z}_p}(\sigma v_1, \sigma v_2)$ consists of permanent cycles because all classes with positive filtration degree are p-torsion. Also note that the only possibly nonzero differentials are d^{2p} and d^{2p^2}.

Lemma 0.1. *We have:*

1. *The class σv_1 maps under the edge homomorphism*

$$E^2_{0,2p-1} = E^\infty_{0,2p-1} \cong (F_0)_{2p-1} \subseteq \mathrm{THH}_{2p-1}(BP\langle 2\rangle; H\mathbb{Z}_p)$$

   to $p\alpha\lambda_1$ for a unit $\alpha \in \mathbb{Z}_p$. The class γ_p is a permanent cycle. Up to a unit it is the image of λ_1 under $(F_{2p-1})_{2p-1} \to (F_{2p-1}/F_0)_{2p-1} = E^\infty_{2p-1,0}$.

2. *The class σv_2 maps under the edge homomorphism*

$$E^2_{0,2p^2-1} = E^\infty_{0,2p^2-1} \cong (F_0)_{2p^2-1} \subseteq \mathrm{THH}_{2p^2-1}(BP\langle 2\rangle; H\mathbb{Z}_p)$$

   to $p\beta\lambda_2$ for a unit $\beta \in \mathbb{Z}_p$. The class $p\gamma_{p^2}$ is a permanent cycle. Up to a unit it is the image of λ_2 under $(F_{2p^2-1})_{2p^2-1} \to (F_{2p^2-1}/F_0)_{2p^2-1} = E^\infty_{2p^2-1,0}$.

3. *We have a differential*

$$d^{2p}(\gamma_{p(p+1)}) = pn\gamma_{p^2}\sigma v_1$$

   for an $n \in \mathbb{Z}$ with $p \nmid n$.

Proof. Item 1 is clear. For item 2 note that the image of γ_{p^2} in the Brun spectral sequence

$$E^*_{*,*}(BP\langle 2\rangle, BP\langle 0\rangle, V(0))$$

is $\lambda_1\mu_1^{p-1}$ which has a nontrivial d^{2p} differential. Thus,

$$d^{2p} : \mathbb{Z}/p^2\{\gamma_{p^2}\} \to \mathbb{Z}/p\{\sigma v_1 \gamma_{p(p-1)}\}$$

is nontrivial and we get $E^\infty_{2p^2-1,0} = \mathbb{Z}/p\{p\gamma_{p^2}\}$. This shows claim 2. For item 3 note that in total degree $2p^2+2p-2$ the E^2-page is given by

$$\mathbb{Z}_p\{\sigma v_1\sigma v_2\} \oplus \mathbb{Z}/p\{\sigma v_2\gamma_p\} \oplus \mathbb{Z}/p^2\{\sigma v_1\gamma_{p^2}\}.$$

By item 1 and 2 the edge homomorphism

$$\mathbb{Z}_p\{\sigma v_1\sigma v_2\} = E^\infty_{0,2p^2+2p-2} \to \mathrm{THH}_{2p^2+2p-2}(BP\langle 2\rangle; H\mathbb{Z}_p) = \mathbb{Z}_p\{\lambda_1\lambda_2\}$$

maps $\sigma v_1\sigma v_2$ to $p^2\alpha\beta\lambda_1\lambda_2$. This implies that

$$(F_{2p^2-1}/F_0)_{2p^2+2p-2} = \mathbb{Z}/p^2.$$

It also implies that the element $p\lambda_1\lambda_2$ does not lie in F_0. Since λ_1 lies in F_{2p-1} and $p\lambda_2$ lies in F_0 by item 2, we have that $p\lambda_1\lambda_2$ lies in F_{2p-1}. Thus, $(F_{2p-1}/F_0)_{2p^2+2p-2}$ is not zero. We get that $\mathbb{Z}/p\{\sigma v_2\gamma_p\}$ consists of permanent cycles and that we have a nontrivial differential

$$d^{2p} : \mathbb{Z}/p\{\gamma_{p(p+1)}\} \to \mathbb{Z}/p^2\{\sigma v_1\gamma_{p^2}\}.$$

□

Lemma 0.2. *For $k > 1$ we have $d^{2p}(\gamma_{pk}) \neq 0$. We get an additive isomorphism*

$$\begin{aligned} E^{2p+1}_{*,*} &= \mathbb{Z}_p\{1\} \oplus \mathbb{Z}/p\{\gamma_p\} \oplus \bigoplus_{p|k} \mathbb{Z}/p^{v_p(k)}\{p\gamma_{pk}\} \\ &\oplus \mathbb{Z}_p\{\sigma v_1\} \oplus \bigoplus_{p|k} \mathbb{Z}/p^{v_p(k)}\{\sigma v_1\gamma_{pk}\} \\ &\oplus \mathbb{Z}_p\{\sigma v_2\} \oplus \mathbb{Z}/p\{\sigma v_2\gamma_p\} \oplus \bigoplus_{p|k} \mathbb{Z}/p^{v_p(k)}\{p\sigma v_2\gamma_{pk}\} \\ &\oplus \mathbb{Z}_p\{\sigma v_1\sigma v_2\} \oplus \bigoplus_{p|k} \mathbb{Z}/p^{v_p(k)}\{\sigma v_1\sigma v_2\gamma_{pk}\}. \end{aligned}$$

Proof. The image of γ_{pk} in the Brun spectral sequence

$$E^*_{*,*}(BP\langle 2\rangle, BP\langle 0\rangle, V(0))$$

is $\lambda_1\mu_1^{k-1}$ which has d^{2p}-differential

$$d^{2p}(\lambda_1\mu_1^{k-1}) = (k-1)\lambda_1\sigma v_1\mu_1^{k-2}.$$

If $p \nmid k-1$ (so in particular if $p \mid k$) then this is nonzero and we get that

$$d^{2p} : \mathbb{Z}/p^{1+v_p(k)}\{\gamma_{pk}\} \to \mathbb{Z}/p\{\sigma v_1\gamma_{p(k-1)}\}$$

is nonzero. For $p \mid k-1$ we proceed by induction. The first case, $\gamma_{p(p+1)}$, has been treated in Lemma 0.1. Now suppose that $k = pn+1$ for an $n > 1$ and that we have proven the statement for $k' = p(n-1)+1$. Since $\mathrm{THH}_{2pk-1}(BP\langle 2\rangle, H\mathbb{Z}_p)$ is zero, $\mathbb{Z}/p\{\gamma_{pk}\}$ cannot survive to the E^∞-page. By induction hypothesis we know that the differential

$$d^{2p} : \mathbb{Z}/p\{\sigma v_2\gamma_{pk'}\} \to \mathbb{Z}/p^{1+v_p(k'-1)}\{\sigma v_1\sigma v_2\gamma_{p(k'-1)}\}$$

is nontrivial. This implies that the target in

$$d^{2p^2} : E^{2p^2}_{2pk-1,0} \to E^{2p^2}_{2pk'-1,2p^2-1}$$

is trivial. Thus, we have to have a nontrivial differential

$$d^{2p} : \mathbb{Z}/p\{\gamma_{pk}\} \to \mathbb{Z}/p^{1+v_p(k-1)}\{\sigma v_1 \gamma_{p(k-1)}\}.$$

$\square$

Lemma 0.3. *For $p \mid k$ we have differentials*

$$d^{2p^2} : \mathbb{Z}/p^{v_p(k+p)}\{p\gamma_{p(k+p)}\} \xrightarrow{p^{v_p(k)-1}n_k} \mathbb{Z}/p^{v_p(k)}\{p\sigma v_2 \gamma_{pk}\}$$

and

$$d^{2p^2} : \mathbb{Z}/p^{v_p(k+p)}\{\sigma v_1 \gamma_{p(k+p)}\} \xrightarrow{p^{v_p(k)-1}m_k} \mathbb{Z}/p^{v_p(k)}\{\sigma v_1 \sigma v_2 \gamma_{pk}\}$$

for certain $n_k, m_k \in \mathbb{Z}$ with $p \mid n_k, m_k$. Thus, we have

$$\begin{aligned} E^{\infty}_{*,*} &= \mathbb{Z}_p\{1\} \oplus \mathbb{Z}/p\{\gamma_p\} \oplus \mathbb{Z}/p\{p\gamma_{p^2}\} \oplus \bigoplus_{i>0} \mathbb{Z}/p^{v_p(i)+1}\{p^2\gamma_{p^3 i}\} \\ &\oplus \mathbb{Z}_p\{\sigma v_1\} \oplus \mathbb{Z}/p\{\sigma v_1 \gamma_{p^2}\} \oplus \bigoplus_{i>0} \mathbb{Z}/p^{v_p(i)+1}\{p\sigma v_1 \gamma_{p^3 i}\} \\ &\oplus \mathbb{Z}_p\{\sigma v_2\} \oplus \mathbb{Z}/p\{\sigma v_2 \gamma_p\} \oplus \bigoplus_{i>0} \mathbb{Z}/p^{v_p(i)+1}\{p\sigma v_2 \gamma_{p^3 i}\} \\ &\oplus \mathbb{Z}_p\{\sigma v_1 \sigma v_2\} \oplus \bigoplus_{i>0} \mathbb{Z}/p^{v_p(i)+1}\{\sigma v_1 \sigma v_2 \gamma_{p^3 i}\}. \end{aligned}$$

Proof. For $k \geq 1$ with $p \mid k$ we know that

$$\begin{aligned} \mathrm{THH}_{2p(k+p)-1}(BP\langle 2\rangle; H\mathbb{Z}_p) &= \mathbb{Z}/p^{v_p(k+p)-1} \\ \mathrm{THH}_{2p^2+2p+2pk-3}(BP\langle 2\rangle; H\mathbb{Z}_p) &= \mathbb{Z}/p^{v_p(k)-1}. \end{aligned}$$

This implies the statement about the differentials. The statement about the E^{∞}-page now follows easily.

$\square$