$$\begin{tikzcd}
\mathrm{THH}_*(H\mathbb{Z}_{(p)}) \otimes_{\mathbb{Z}_{(p)}} E(\sigma v_1, \sigma v_2) \arrow[d, Rightarrow] \arrow[dr] & & p\, b_{pk} \arrow[r, mapsto] & p\, b_{pk} \\
 & \mathrm{THH}_*(H\mathbb{Z}_{(p)}) \otimes E_{\mathbb{Z}_{(p)}}(\sigma v_1) \arrow[d, Rightarrow] & & \\
\mathrm{THH}_*(BP\langle 2\rangle; \mathbb{Z}_{(p)}) \arrow[r] & \mathrm{THH}_*(\ell; \mathbb{Z}_{(p)}) & &
\end{tikzcd}$$

$$a_i = p\gamma_{p^2 i}$$

$$\mathrm{THH}_*(\mathbb{Z}_{(p)}) \cong \begin{cases} \mathbb{Z}_{(p)} & * = 0 \\ \mathbb{Z}_{(p)}\{\gamma_{pk}\} / (p^{\nu_p(pk)}) & * = 2pk-1 \\ 0 & \text{o.w.} \end{cases}$$

$$\gamma_{pk} = \text{"}\lambda_1 \mu_1^{k-1}\text{"} \qquad d^1(\mu_1) = \sigma v_1$$

$$d^1(\gamma_{pk}) \doteq (k-1)\gamma_{p(k-1)}\sigma v_1$$

$$\nu_p(p(pk-1)) = 1$$
$$\nu_p(p^2 k) = 2$$

$$d^1(p\gamma_{p(pk)}) \doteq p(pk-1)\gamma_{p(pk-1)}\sigma v_1$$
$$= 0$$

$$\to p\gamma_{p^2 k} \sim a_k \qquad d^1(\gamma_{p(pk+1)}) \doteq pk\gamma_{p^2 k}\sigma v_1$$

$$p^{\nu_p(pk)}(\underline{p\gamma_p}{}^2 k) = 0 \qquad b_k = \frac{\gamma_{p^2k}\,\sigma v_1}{p^{\nu_p(pk)}\, b_k = 0}$$

$$THH_{\otimes}(\ell; \mathbb{Z}_{(p)}) \otimes_{\mathbb{Z}_{(p)}} P_{\mathbb{Z}_{(p)}}(v_1) \cong$$

$$\Downarrow \qquad \text{Bockstein}$$

$$THH_{\otimes}(\ell)$$

$$E^2 = \Big(E(\sigma v_1) \oplus \mathbb{Z}_{(p)}\{\gamma_p, \underbrace{p\gamma_{p^2k}}_{}, \underbrace{\gamma_{p^2k}\sigma v_1}_{}\}\Big) \otimes E(\sigma v_2)$$

$$p\gamma_{p^2k} = a_k, \qquad \gamma_{p^2k}\sigma v_1 = b_k$$

$$\left(\begin{array}{l}\text{Mult ext}\\ \gamma_p \cdot a_k = b_k\end{array}\right) \qquad \left(p\gamma_p = 0,\ p^{\nu_p(pk)}(p\gamma_{p^2k}) = p^{\nu_p(pk)}(\gamma_{p^2k}\sigma v_1) = 0\right)$$

$$d^{p+1}(a_k) \doteq (k-1)\, a_{k-1}\sigma v_2$$

$$d^{p+1}(b_k) \doteq (k-1)\, b_{k-1}\sigma v_2$$

$$c_k^{(1)} = p\,a_{pk} \qquad c_k^{(2)} = a_{pk}\sigma v_2$$

$$d_k^{(1)} = p\,b_{pk} \qquad d_k^{(2)} = b_{pk}\sigma v_2$$

$$p\,b_{pk} \longmapsto p\,b_{pk}$$

$$\shortparallel \qquad\qquad \shortparallel$$

$$
\begin{array}{ccc}
\Vert & & \Vert \\
p\gamma_p^2 k\sigma v_1 & & p\gamma_p^2 k\sigma v_1 \quad THH_*(H\mathbb{Z}_{(p)}) \otimes_{\mathbb{Z}_{(p)}} P_{\mathbb{Z}_{(p)}}(v_1) \\
 & & \cong \qquad \otimes E_{\mathbb{Z}_{(p)}}(\sigma v_1)
\end{array}
$$

$$
\begin{array}{ccc}
THH_*(H\pi_* BP\langle 2\rangle; H\pi_* \ell) & \longrightarrow & THH_*(H\pi_*(\ell)) \\
\Downarrow & & \Downarrow \\
THH_*(BP\langle 2\rangle; \ell) & \longrightarrow & THH_*(\ell)
\end{array}
$$

$$E^2 \sim THH_*(\ell; \mathbb{Z}_{(p)}) \otimes P_{\mathbb{Z}_{(p)}}(v_1)$$

$$E^2 \sim THH_*(\ell; \mathbb{Z}_{(p)}) \otimes_{\mathbb{Z}_{(p)}} P_{\mathbb{Z}_{(p)}}(v_1) \otimes E(\sigma v_2)$$

$$d^{p+1}(b_k) \doteq (k-1)\, b_{k-1}\sigma v_2 \qquad p b_{pk}, \; b_{pk}\sigma v_2$$

$$d^{p+1}(a_k) \doteq \boxed{(k-1)\, a_{k-1}\sigma v_2 + (k-1) v_1^p b_{k-1}}$$

$$\alpha_k^1, \beta_k^1 \in \mathbb{Z}_p^\times \qquad a_{pk}\sigma v_2 + v_1^p b_{pk}$$

$$\Vert$$

$$c_k^{(2)}$$

a_{pk}

Conj:

$$\left(d(p b_{pk}) \doteq (k-1) v_1^{p^2} b_{p(k-1)} \sigma v_2 \right)$$

$$d(p a_{pk}) \doteq \alpha_k^2 (k-1) v_1^{p^2} a_{p(k-1)} \sigma v_2 + \beta_k^2 (k-1) v_1^{p^2+p} (b_{p(k-1)})$$

$$\left(\begin{array}{l} \text{AHL} \\ d(p a_{pk}) \doteq (k-1) v_1^{p^2+p} b_{p(k-1)} \end{array} \right)$$

$$d(p a_{pk}) = \alpha_k^2 (k-1) v_1^{p^2} a_{p(k-1)} \sigma v_2$$

$$\begin{tikzcd}
(k-1)\sigma v b_{k-1} \arrow[rr, mapsto] & & 0 & \\
& b_k \arrow[ul] \arrow[rr, mapsto] & & b_k \arrow[ul] \\
\ & & (k-1) v_1^{p^2+p} b_k & \\
& p a_{pk} \arrow[ul] \arrow[rr, mapsto] & & p a_{pk} \arrow[ul]
\end{tikzcd}$$

$$d\left(\alpha'_k (k-1) a_{k-1} \sigma v_2 + \beta'_k (k-1) v_1^{p} b_{k-1}\right) = 0$$

$$d(b_k) = (k-1) b_{k-1} \sigma v_2$$

$$d\left(\alpha'_k (k-1) a_{k-1} \sigma v_2 + \beta'_k (k-1) v_1^{p} b_{k-1}\right)$$

$$\alpha_k^1 (k-1)\left(\beta_{k-1}^1 (k-2) v_1^p b_{k-2} \sigma v_2\right.$$

$$\left. + \beta_k^1 \beta_{k-1}^1 (k-1)(k-2) v_1^p b_{k-2} \sigma v_2\right)$$

$$\alpha_k^1 \cdot \beta_{k-1}^1 = \beta_k^1 \beta_{k-1}^1$$

$$\alpha_k^1 = \beta_k^1$$

$$d(p^{n-1} a_{p^{n-1}k}) \doteq (k-1) v_1^{p^n + \dots + p} b_{p^{n-1}(k-1)}$$

<u>Conj:</u>

$$d(p^{n-1} c_k^{(1)}) \doteq (k-1) v_1^{p^n + \dots + p^2} c_{k-1}^{(2)}$$

$$d(p^{n-1} d_k^{(1)}) = (k-1) v_1^{p^n + \dots + p^2} d_{k-1}^{(2)} \quad ??$$

$$a_{pk} = c_k^{(1)}$$

$$a_{pk} \sigma v_2 + v_1^p b_{pk}$$