HOCHSCHILD-MAY SPECTRAL SEQUENCE FOR TOPOLOGICAL HOCHSCHILD HOMOLOGY OF THE ADAMS SUMMAND

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Abstract. We give a new computation of topological Hochschild homology the Adams summand, originally due to [?AHL], using the Hochschild-May spectral sequence developed by the first author and Andrew Salch in [?AS18].

Contents

1. Introduction

The goal of this note is to compute $THH_*(\ell)$ using the Hochschild-May spectral sequence developed by the first author and Andrew Salch in [?AS18].

2. THH of ℓ with $H\mathbb{F}_p$, $H\mathbb{Z}_p$, and $k(1)$ coefficients

We first compute the Hochschild-May spectral sequences for $THH_*(\ell; H\mathbb{F}_p)$, $THH_*(\ell; H\mathbb{Z}_p)$ and $THH_*(\ell; k(1))$ using the Whitehead filtration. Throughout we write ℓ for the p-completion of the Adams summand.

We first recall that Bökstedt computed

$$THH_n(\mathbb{Z}_p) \cong \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z}/p^{\nu_p(m)}\{\gamma_m\} & n = 2m-1 \\ 0 & \text{oterwise} \end{cases}$$

where $\nu_p(k)$ is the p-adic valuation of k.

Lemma 2.1. *The Hochschild-May spectral sequence*

$$E_2^{*,*} = THH_*(H\pi_*\ell; H\mathbb{F}_p) \Rightarrow THH_*(\ell; H\mathbb{F}_p)$$

has E_2-page

$$E_2^{*,*} \cong THH_*(H\mathbb{Z}_p; H\mathbb{F}_p) \otimes E(\sigma v_1)$$

where

$$THH_*(H\mathbb{Z}_p; H\mathbb{F}_p) \cong E(\lambda_1) \otimes P(\mu_1)$$

and the differentials are generated by the differential

$$d_1(\mu_1) = \sigma v_1$$

Proof. We first observe that there is an equivalence $H\pi_*\ell \simeq H\mathbb{Z} \wedge S[v_1]$ as A_∞-ring spectra where $S[v_1]$ is the free A_∞-algebra $\bigvee_{i\geq 0} S^{(2p-2)i}$. The first statement then follows by rearranging colimits

$$\begin{aligned} H\mathbb{F}_p \wedge_{H\mathbb{Z}_p \wedge S[v_1]} THH(H\mathbb{Z}_p \wedge S[v_1]) &\simeq \\ H\mathbb{F}_p \wedge_{H\mathbb{Z}_p \wedge S[v_1]} (THH(H\mathbb{Z}_p) \wedge THH(S[v_1])) &\simeq \\ THH(H\mathbb{Z}_p; \mathbb{F}_p) \wedge_{H\mathbb{F}_p} H\mathbb{F}_p \wedge THH(S[v_1], S), & \end{aligned}$$

applying the Künneth isomorphism, and then computing

$$H_*THH(S[v_1], S) \cong E(\sigma v_1)$$

using the Bökstedt spectral sequence, which clearly collapses. We know that $THH_{2p}(\ell, H\mathbb{F}_p) \cong 0$, which forces the differential $d_1(\mu_1) = \sigma v_1$ and this is the only possible differential whose source is a generator in this two line spectral sequence. □

Lemma 2.2. *The Hochschild-May spectral sequence*

$$E_2^{*,*} = THH_*(H\pi_*\ell; H\mathbb{Z}_p) \Rightarrow THH_*(\ell; H\mathbb{Z}_p)$$

has E_2-page

$$E_2^{*,*} = THH_*(\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} E_{\mathbb{Z}_p}(\sigma v_1)$$

and differentials

$$d_1(\gamma_{pk}) = p^{\alpha(k)} \sigma v_1 \gamma_{p(k-1)}$$

where $\alpha(k) = \min\{0, \nu_p(k-1) - \nu_p(k)\}$.

The classes a_i are detected by $p\gamma_{ip^2}$ and the classes b_i are detected by $\gamma_{ip^2}\sigma v_1$, up to multiplication by a unit. There is also a hidden extension

$$p\gamma_p = \sigma v_1.$$

Proof. The identification of the E_2-page is exactly the same argument where the existence of the Bökstedt spectral sequence computing $H\mathbb{Z}_*THH(S[v_1], S)$, in this case, follows because $H\mathbb{Z}_*S[v_1]$ is free over $H\mathbb{Z}_*$. The only possible differentials are the ones stated and these must occur in order to produce the known answer due to [?AHL]. The detection results follow by comparison with the results of [?AHL]. We point out that the hidden extension could be derived explicitly using the formula

$$p\lambda_1 = \sigma v_1$$

due to [?Rog19] since $\gamma_p = \lambda_1$. □

Lemma 2.3. *The Hochschild-May spectral sequence*

$$E_2^{*,*} = THH_*(H\pi_*\ell; H\pi_*k(1)) \Rightarrow THH_*(\ell; k(1))$$

has E_2-page

$$THH_*(H\mathbb{Z}_p; H\mathbb{F}_p) \otimes P(v_1) \otimes E(\sigma v_1).$$

The first nontrivial differential is

$$d_1(\mu_1) = \sigma v_1$$

and the remaining differentials are determined by the differentials in the Bockstein spectral sequence

$$THH_*(\ell; H\mathbb{F}_p)[v_1] \Rightarrow THH_*(\ell; k(1))$$

since the E_3-page of the spectral sequence is isomorphic to the associated graded of a filtration on $THH_*(\ell; H\mathbb{F}_p)$. In particular, the differentials are

$$d_{r'(n)}(\mu_1^{r'(n)}) = v_1^{r'(n)-\epsilon(n)}\lambda_n$$

where $\epsilon(n) = 1$ if n is even and zero otherwise and $r'(n)$ is defined recursively by $r'(0) = 1$, $r'(1) = p$ and

$$r'(n) = p^n + r'(n-2)$$

for $n \geq 0$. The elements λ_n are also defined recursively by $\lambda_n = \lambda_{n-2}\mu_1^{p^{n-2}(p-1)}$ where $\lambda_0 = \sigma v_1$.

Proof. Again, the argument for computing the E_2-page is essentially the same. The first differential can be surmised from the map of Hochschild-May spectral spectral sequences

$$THH_*(H\pi_*\ell; H\pi_*k(1)) \to THH_*(H\pi_*\ell; H\mathbb{F}_p)$$

which sends v_1 to zero and is a bijection on all elements that are not v_1-divisible. We write $E_\infty^{*,*}(\ell; H\mathbb{F}_p)$ for the E_∞-page of the latter spectral sequence. Then we see that there is an isomorphism

$$E_3^{*,*} \cong E_\infty^{*,*}(\ell; H\mathbb{F}_p) \otimes P(v_1).$$

we now recall that, by [?MS], the differentials in the Bockstein spectral sequence are generated by

$$d_{r(n)}(\mu_2^{r(n)}) = v_1^{r(n)}\lambda_n.$$

so by identifying $\mu_2 = \mu_1^p$ and $\lambda_2 = \lambda_1\mu_1^{p-1}$ we get the desired result. □

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