TOPOLOGICAL HOCHSCHILD HOMOLOGY OF TRUNCATED BROWN-PETERSON SPECTRA II

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Abstract. We compute topological Hochschild homology of the second truncated Brown-Peterson spectrum at primes $p \geq 3$ with Adams summand coefficients.

Contents

1. Introduction

1.1. **Conventions.** We write L_E for Bousfield localization at a spectrum E. We write $BP\langle n\rangle$ for a family of $\mathbb{E}_3$-MU-algebra forms of $BP\langle n\rangle$ such that

$$\mathrm{MU} \to \cdots \to \mathrm{BP}\langle n\rangle \to \mathrm{BP}\langle n-1\rangle \to \cdots \to H\mathbb{Z}_{(p)} \to H\mathbb{F}_p$$

and therefore we fix classes v_i such that on graded commutative rings

$$\mathrm{MU}_* \to \mathrm{BP}\langle n\rangle_*$$

is given by sending x_{p^i-1} to v_i for $0 \leq i \leq n$ (with $v_0 = p$) and $x_j \mapsto 0$ otherwise. This also fixes the map of graded commutative rings

$$\mathrm{BP}\langle n\rangle_* \to \mathrm{BP}\langle n-1\rangle_*$$

sending v_i to v_i for $0 \leq i \leq n-1$ and $v_n \mapsto 0$. Such a family exists by [5].

2. Recollections

In this section, we recall the necessery results from [1].

Proposition 2.1. *There is an isomorphism of* $\pi_* L_{H\mathbb{Q}}\mathrm{BP}\langle n\rangle = \mathbb{Q}[v_1, \ldots, v_n]$*-algebras*

$$\pi_* L_{H\mathbb{Q}} \,\mathrm{THH}(\mathrm{BP}\langle n\rangle) \cong \mathbb{Q}[v_1, \ldots, v_n] \otimes_{\mathbb{Q}} \Lambda_{\mathbb{Q}}(\sigma v_1, \ldots, \sigma v_n)\,.$$

Proof. The authors computed

$$\pi_* \,\mathrm{THH}(\mathrm{BP}\langle n\rangle; H\mathbb{Q}) = \Lambda_{\mathbb{Q}}(\sigma v_1, \ldots, \sigma v_n)\,.$$

We then observe that

$$L_{H\mathbb{Q}}\,\mathrm{THH}(\mathrm{BP}\langle n\rangle) \simeq \mathrm{THH}(\mathrm{BP}\langle n\rangle; L_{H\mathbb{Q}}\mathrm{BP}\langle n\rangle)$$

because $L_{H\mathbb{Q}}$ is a smashing localization. We then consider the spectral sequence

$$\mathrm{THH}_*(\mathrm{BP}\langle n\rangle;\mathbb{Q}) \otimes_{\mathbb{Q}} \pi_* L_{H\mathbb{Q}}\mathrm{BP}\langle n\rangle \implies \mathrm{THH}_*(\mathrm{BP}\langle n\rangle; L_{H\mathbb{Q}}\mathrm{BP}\langle n\rangle)$$

associated to the multiplicative complete filtration $\tau_{\geq \bullet}\,\mathrm{THH}(\mathrm{BP}\langle n\rangle;\mathbb{Q})$ in $H\mathbb{Q}$-modules. This spectral sequence has input

$$\pi_* L_{H\mathbb{Q}}\mathrm{BP}\langle n\rangle \otimes_{\mathbb{Q}} \Lambda_{\mathbb{Q}}(\sigma v_1, \ldots, \sigma v_n)$$

by [?AKCH22, Proposition 3.7]. It collapses because the targets of all differentials are zero groups. The abutment and the E_∞-term are a free $\pi_* L_{H\mathbb{Q}}\mathrm{BP}\langle n\rangle$-modules and consequently there are no $\pi_* L_{H\mathbb{Q}}\mathrm{BP}\langle n\rangle)$-module extensions. There isn't room for algebra extensions. □

Proposition 2.2. *The groups*

$$\mathrm{THH}_s(\mathrm{BP}\langle n\rangle)$$

are finitely generated for all integers s. Consequently, we have

$$|\,\mathrm{THH}_s(\mathrm{BP}\langle n\rangle)| < \infty$$

for $s \neq 2p^i - 1 \mod 2p^j - 2$ for $1 \leq i, j \leq n$.

Proof. Since $\pi_a\mathbb{S}$ and $\pi_b\mathrm{BP}\langle n\rangle$ are finitely generated abelian groups for all integers a, b, the strongly convergent Künneth spectral sequence computing $\pi_*(\mathrm{BP}\langle n\rangle \wedge \mathrm{BP}\langle n\rangle)$ is finitely generated in each bidegree and has a vanishing line of postive slope so $\pi_c(\mathrm{BP}\langle n\rangle \wedge \mathrm{BP}\langle n\rangle)$ is finitely generated for each integer c. The same argument implies that $\mathrm{THH}_s(\mathrm{BP}\langle n\rangle)$ is finitely generated for each integer s. The second statement then follows from Proposition 2.1 and the classification of finitely generated $\mathbb{Z}_{(p)}$-modules. □

3. Bounding Hochschild homology of $\mathrm{BP}\langle n\rangle$

The goal of this section is to use the cosimplicial descent spectral sequence from work of [3] to produce a useful upper bound on $\mathrm{THH}_*(\mathrm{BP}\langle n\rangle)$.

Definition 3.1. Let $C^\bullet(A/B)$ denote the cosimplicial cobar complex with q-simplices $C^q(A/B) = A^{\otimes_B q+1}$.

First, we need a lemma.

Lemma 3.2. *Let $n \geq 1$. There is an isomorphism rings*

$$\mathrm{E}_2^{*,*} \cong \mathrm{Tor}^{\pi_*\mathrm{BP}\langle n-1\rangle\wedge\mathrm{BP}\langle n-1\rangle}(\mathrm{BP}\langle n-1\rangle, \mathrm{BP}\langle n-1\rangle) \otimes \Gamma\{\sigma^2 v_n^{(j)} : 1 \leq j \leq q\} \otimes \Lambda(\sigma v_1^{(j)} : 1 \leq j \leq q)$$

where

$$\mathrm{E}_2^{*,*} = \pi_*\left(\pi_* H\pi_*\,\mathrm{THH}(\mathrm{BP}\langle n-1\rangle)^{\wedge_{H\pi_*\,\mathrm{THH}(\mathrm{BP}\langle n\rangle)} q+1}\right)$$

is the E_2-term of the multiplicative Künneth spectral sequence

$$\mathrm{E}_2^{*,*} \implies \mathrm{THH}_*(\mathrm{BP}\langle n-1\rangle^{\wedge q+1}).$$

Proof. When $q = 0$, then $\mathrm{THH}(\mathrm{BP}\langle n-1\rangle^{\wedge_{\mathrm{BP}\langle n\rangle} q+1}) = \mathrm{THH}(\mathrm{BP}\langle n-1\rangle)$. We first compute $\pi_*(\mathrm{BP}\langle n-1\rangle \otimes_{\mathrm{BP}\langle n\rangle} \mathrm{BP}\langle n-1\rangle)$ by a Künneth spectral sequence. The E_2-term is $\mathrm{BP}\langle n-1\rangle_* \otimes \Lambda(\sigma v_n)$ so it is concentrated in Künneth filtration $[0,1]$ and therefore the spectral sequence collapses because the targets of all differentials are zero groups. We then use the equivalence

$$A \wedge_B A \wedge_B A \simeq (A \wedge_B A) \wedge_A (A \wedge_B A)$$

where $A = \mathrm{BP}\langle n-1\rangle$ and $B = \mathrm{BP}\langle n\rangle$ and the fact that $\pi_*(\mathrm{BP}\langle n-1\rangle \wedge_{\mathrm{BP}\langle n\rangle} \mathrm{BP}\langle n-1\rangle)$ is free as a $\mathrm{BP}\langle n-1\rangle_*$-module to inductively determine from the Künneth spectral sequence that

$$\pi_*(\mathrm{BP}\langle n-1\rangle^{\wedge_{\mathrm{BP}\langle n\rangle} q+1}) \cong \mathrm{BP}\langle n-1\rangle_* \otimes \Lambda(\sigma v_n^{(1)}, \dots, \sigma v_n^{(q)})\,.$$

By obstruction theory, we determine that $\mathrm{BP}\langle n-1\rangle^{\wedge_{\mathrm{BP}\langle n\rangle} q+1}$ is the smash product of square zero extensions

$$\left(BP\langle n-1\rangle \vee \Sigma^{2p-1} BP\langle n-1\rangle\right)^{\wedge_{BP\langle n-1\rangle} q}\,.$$

Consequently, we determine that

$$\pi_* \left(\mathrm{BP}\langle n-1\rangle^{\wedge_{\mathrm{BP}\langle n\rangle} q+1}\right)^{\wedge 2} \cong \pi_*(\mathrm{BP}\langle n-1\rangle^{\wedge 2}) \otimes \Lambda(\sigma v_n^{(1)}, \dots, \sigma v_n^{(q)})^{\wedge 2}\,.$$

We then note that the Künneth spectral sequence sequence computing

$$\mathrm{THH}_*(\mathrm{BP}\langle n-1\rangle^{\wedge_{\mathrm{BP}\langle n\rangle} q+1})$$

has E_2-term

$$\mathrm{Tor}^{\pi_*\mathrm{BP}\langle n-1\rangle \wedge \mathrm{BP}\langle n-1}(\mathrm{BP}\langle n-1\rangle, \mathrm{BP}\langle n-1\rangle) \otimes \Gamma(\sigma^2 v_n^{(1)}, \dots, \sigma^2 v_n^{(q)}) \otimes \Lambda(\sigma v_n^{(1)}, \dots, \sigma v_n^{(q)})$$

and that is exactly what we describe in the statement of the lemma. □

Proposition 3.3. *There is an equivalence*

$$\mathrm{THH}(\mathrm{BP}\langle n\rangle) \simeq \mathrm{Tot}\left(\mathrm{THH}(C^\bullet(\mathrm{BP}\langle n-1\rangle/\mathrm{BP}\langle n\rangle))\right).$$

Consequently, there is a spectral sequence

$$\pi_{t-s} \lim_{\Delta} \mathrm{Tot}\, H\pi_s\, \mathrm{THH}(C^\bullet(\mathrm{BP}\langle n-1\rangle/\mathrm{BP}\langle n\rangle)) \implies \pi_{t-s}\, \mathrm{THH}(\mathrm{BP}\langle n\rangle)$$

associated to the filtration

$$\lim \mathrm{Tot}\, \tau_{\geq s}\, \mathrm{THH}(C^\bullet(\mathrm{BP}\langle n-1\rangle/\mathrm{BP}\langle n\rangle))\,. \tag{3.4}$$

The E_2-term is

$$\mathrm{THH}_*(BP\langle n-1\rangle) \otimes_{\mathbb{Z}_{(p)}} \Lambda_{\mathbb{Z}_{(p)}}(\sigma v_n)\,.$$

Consequently,

$$|\,\mathrm{THH}_t(BP\langle n\rangle)| \leq |\,\mathrm{THH}_t(BP\langle n-1\rangle)| + |\,\mathrm{THH}_{t-2p^n+1}(BP\langle n-1\rangle)|$$

Proof. Since $\mathrm{BP}\langle n\rangle \to \mathrm{BP}\langle n-1\rangle$ is an isomorphism on π_i for $i = 0, 1$ the first statement follows directly from [3, Theorem 3.7]. The second statement follows from [4, Remark 3.7] which identifies the filtration (3.4) with the décalage (cf. [2, pp. 21]) of the filtration whose associated graded is the E_1-term of the Bousfield–Kan spectral sequence.

It therefore suffices to compute the E_2-term, which is the cohomology of the Hopf algebroid $(\mathrm{THH}_*(\mathrm{BP}\langle n-1\rangle), \mathrm{THH}_*(\mathrm{BP}\langle n-1\rangle) \otimes \Gamma\{\sigma^2 v_n\})$ by Lemma 3.2. We note from the proof of Lemma 3.2 that this Hopf algebroid is the tensor product of the Hopf algebroids $(\mathrm{THH}_*(\mathrm{BP}\langle n{-}1\rangle), \mathrm{THH}_*(\mathrm{BP}\langle n{-}1\rangle))$ and $(\mathbb{Z}_{(p)}, \Gamma_{\mathbb{Z}_{(p)}}\{\sigma^2 v_n\}\}$. Consequently, the cohomology of this Hopf algebroid is $\mathrm{THH}_*(\mathrm{BP}\langle n-1\rangle) \otimes \Lambda(\sigma v_n)$ as desired. $\square$

Example 3.5. We consider the case $n = 1$. Then there is a spectral sequence

$$\mathrm{THH}_*(H\mathbb{Z}) \otimes \Lambda(\sigma v_2) \implies \mathrm{THH}_*(\ell)\,.$$

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