TOPOLOGICAL HOCHSCHILD HOMOLOGY OF THE SECOND TRUNCATED BROWN-PETERSON SPECTRUM I

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Abstract. We compute topological Hochschild homology of the second truncated Brown-Peterson spectrum $BP\langle 2\rangle$ at the primes $2, 3$ with coefficients in $BP\langle 1\rangle$. At the prime $p = 2$ we use the model for $BP\langle 2\rangle$ constructed by Lawson-Naumann [9] using topological modular forms equipped with a $\Gamma_1(3)$-structure and at $p = 3$ we use the model for $BP\langle 2\rangle$ constructed using a Shimura curve of discriminant 14 due to Hill-Lawson [7].

Contents

1. Introduction

Topological Hochschild homology has two main applications: it encodes information about deformations of structured ring spectra and it is the linear approximation to algebraic K-theory in the sense of Goodwillie's calculus of functors.

Algebraic K-theory of ring spectra that arise in chromatic stable homotopy theory are of particular interest because of the program of Ausoni-Rognes [3] which, in a broad sense, suggests that the arithmetic of structured ring spectra is intimately connected to chromatic complexity. One of the most fundamental objects in chromatic stable homotopy theory is the Brown-Peterson spectrum BP, which is a complex oriented cohomology theory associated to the universal p-typical formal group. The coefficients of BP are a polynomial algebra over $\mathbb{Z}_{(p)}$ on generators v_i for $i \geq 1$, and we may form truncated versions of BP, denoted $BP\langle n\rangle$ by coning off a regular sequence $(v_{n+1}, v_{n+2}, \ldots)$.

By convention $BP\langle -1\rangle = H\mathbb{F}_p$ and when $n = 0, 1$, there are known identifications $BP\langle 0\rangle = H\mathbb{Z}_{(p)}$, and $BP\langle 1\rangle = \ell$ where ℓ is the Adams summand of complex topological K-theory ku. Until recently, the previous list exhausted the known examples of $BP\langle n\rangle$ that were known to be E_∞-ring spectra. However, in the last decade, models for $BP\langle 2\rangle$ as an E_∞-ring spectrum were constructed at the prime $p = 2$ by Lawson-Naumann [9] and at the prime $p = 3$ by Hill-Lawson [7]. Lawson-Naumann [9] use the theory of topological Modular forms with a $\Gamma_1(3)$-structure to construct an E_∞ model for $BP\langle 2\rangle$ at the prime 2 and Hill-Lawson [7] use the theory of topological automorphic forms associated to a Shimura curve of discriminant 14 to construct an E_∞ model for $BP\langle 2\rangle$ at the prime $p = 3$. This is especially interesting in view of recent groundbreaking work of Lawson [8], where he proves that at the prime 2 no model for $BP\langle n\rangle$ as an E_∞-ring spectrum exists for $n \geq 4$. This result was also recently extended to all odd primes by Senger [14].

In the present paper, we compute topological Hochschild homology of $BP\langle 2\rangle$ with coefficients in $BP\langle 1\rangle$ at the primes 2 and 3. In future work, we plan to extend these computations to an integral calculation of $THH_*(BP\langle 2\rangle)$.

For small values of n, the calculations of $THH_*(BP\langle n\rangle)$ are known. The first known computations of topological Hochschild homology are Bökstedt's calculations of $THH(BP\langle -1\rangle)$ and $THH(BP\langle 0\rangle)$ [4]. The main result of a paper of McClure-Staffledt [11] is a computation of the Bockstein spectral sequence

$$THH_*(BP\langle 1\rangle; H\mathbb{F}_p)[v_1] \Rightarrow THH_*(BP\langle 1\rangle; k(1)).$$

This result is extended by Angeltveit-Hill-Lawson [1] where they compute the square of spectral sequences

$$\begin{tikzcd} THH(BP\langle 1\rangle; H\mathbb{F}_p)[v_0, v_1] \arrow[r, Rightarrow] \arrow[d, Rightarrow] & THH(BP\langle 1\rangle; H\mathbb{Z}_{(p)})_p[v_1] \arrow[d, Rightarrow] \\ THH(BP\langle 1\rangle; k(1))[v_0] \arrow[r, Rightarrow] & THH(BP\langle 1\rangle; BP\langle 1\rangle)_p. \end{tikzcd}$$

This gives a complete answer for the integral calculation $THH_*(BP\langle 1\rangle)$.

When $n = 2$, the calculation $THH_*(BP\langle 2\rangle; H\mathbb{F}_p)$ follows naturally from [2] as we discuss in Section 2, but no further results towards $THH_*(BP\langle 2\rangle)$ are known.

In the present paper, we compute the square of spectral sequences

$$\begin{tikzcd} THH(BP\langle 2\rangle; H\mathbb{F}_p)[v_0, v_1] \arrow[r, Rightarrow] \arrow[d, Rightarrow] & THH(BP\langle 2\rangle; H\mathbb{Z}_{(p)})_p[v_1] \arrow[d, Rightarrow] \\ THH(BP\langle 2\rangle; k(1))[v_0] \arrow[r, Rightarrow] & THH(BP\langle 2\rangle; BP\langle 1\rangle)_p, \end{tikzcd}$$

which is a similar level of complexity to the result of Angeltveit-Hill-Lawson [1] and many of the techniques developed in their paper carry over.

1.1. **Outline of the strategy.** Beginning with a calculation of

$$\mathrm{THH}_*(BP\langle 2\rangle;\mathbb{F}_p)$$

we then compute the Bockstein spectral sequences

$$\mathrm{THH}_*(BP\langle 2\rangle;\mathbb{F}_p)[v_0] \implies \mathrm{THH}_*(BP\langle 2\rangle;\mathbb{Z}_{(p)})^{\wedge}_p \tag{1.1}$$

$$\mathrm{THH}_*(BP\langle 2\rangle;\mathbb{F}_p)[v_1] \implies \mathrm{THH}_*(BP\langle 2\rangle;k(1)) \tag{1.2}$$

$$\mathrm{THH}_*(BP\langle 2\rangle;H\mathbb{Z}_{(p)})[v_1] \implies \mathrm{THH}_*(BP\langle 2\rangle;BP\langle 1\rangle) \tag{1.3}$$

$$\mathrm{THH}_*(BP\langle 2\rangle;k(1))[v_0] \implies \mathrm{THH}_*(BP\langle 2\rangle;BP\langle 1\rangle)_p. \tag{1.4}$$

The first two Bockstein spectral sequences can be identified with multiplicative Adams spectral spectral sequences

$$Ext^*_{\mathscr{A}_*}(\mathbb{F}_p;H_*(THH(BP\langle 2\rangle;\mathbb{Z}_{(p)})) \implies \mathrm{THH}_*(BP\langle 2\rangle;\mathbb{Z}_{(p)}) \tag{1.5}$$

$$Ext^*_{\mathscr{A}_*}(\mathbb{F}_p;H_*THH(BP\langle 2\rangle;k(1))) \implies \mathrm{THH}_*(BP\langle 2\rangle;k(1)). \tag{1.6}$$

To see this, note that $H_*THH(BP\langle 2\rangle)$ is free over $H_*BP\langle 2\rangle$ and therefore the input becomes

$$Ext^*_{E(\overline{\tau}_i)}(\mathbb{F}_p;E(\lambda_1,\lambda_2,\lambda_3)\otimes P(\mu_2))$$

for $i=0,1$. Since $E(\overline{\tau}_1)$ coacts trivially on $E(\lambda_1,\lambda_2,\lambda_3)\otimes P(\mu_2)$ and

$$Ext^*_{E(\overline{\tau}_1)}(\mathbb{F}_p,\mathbb{F}_p)\cong P(v_1)$$

the spectral sequence (1.6) can be identified with the Adams spectral sequence at $E_1\cong E_2$-pages. For spectral sequence (1.5) we must choose a minimal resolution so that their is an identification of (1.1) with (1.5) at E_1-pages. Therefore these spectral sequences are each multiplicative.

The spectral sequence (1.3) can be identified with the relative Adams spectral sequence

$$Ext^*_{\pi_*(H\mathbb{Z}_{(p}\wedge_{BP\langle 1\rangle}H\mathbb{Z}_{(p)})}(\mathbb{Z}_{(p)};THH_*(BP\langle 2\rangle;H\mathbb{Z}_{(p)}))\Rightarrow THH_*(BP\langle 2\rangle;BP\langle 1\rangle) \tag{1.7}$$

and the spectral sequence (1.4) can be identified with the relative Adams spectral sequence

$$Ext^*_{\pi_*(k(1)\wedge_{BP\langle 1\rangle}k(1))}(k(1)_*;THH_*(BP\langle 2\rangle;k(1)))\Rightarrow THH_*(BP\langle 2\rangle;BP\langle 1\rangle)_p \tag{1.8}$$

and therefore, since $BP\langle 1\rangle$ and $H\mathbb{Z}_{(p)}$ are commutative ring spectra and $k(1)$ is an A_∞ ring spectrum, these spectral sequences are also multiplicative. To identify the E_2-terms of (1.7) and (1.3) note that by the Künneth spectral sequence there is an isomorphism

$$\pi_*H\mathbb{Z}_{(p}\wedge_{BP\langle 1\rangle}H\mathbb{Z}_{(p)}\cong E_{\mathbb{Z}_{(p)}}(\overline{\tau}_1).$$

Then observe that $E_{\mathbb{Z}}(\overline{\tau}_1)$ coacts trivially on $THH_*(BP\langle 2\rangle;H\mathbb{Z}_{(p)})$ and

$$\mathrm{Ext}^*_{E_{\mathbb{Z}_{(p)}}(\overline{\tau}_1)}(\mathbb{Z}_{(p)};\mathbb{Z}_{(p)})\cong P_{\mathbb{Z}_{(p)}}(v_1).$$

We see that (1.7) and (1.3) are therefore isomorphic on $E_1\cong E_2$-terms. To identify the E_1-terms of (1.8) and (1.4), note that

$$\pi_*(k(1)\wedge_{BP\langle 1\rangle}k(1))\cong P(v_1)\otimes E(\overline{\tau}_0)$$

then apply flat base change

$$\mathrm{Ext}^*_{E(\overline{\tau}_0)\otimes k(1)_*}(k(1)_*; THH_*(BP\langle 2\rangle; k(1))) \cong \mathrm{Ext}^*_{E(\overline{\tau}_0)}(\mathbb{F}_p; THH_*(BP\langle 2\rangle; k(1))).$$

to produce an isomorphism of the E_2-page with (1.8)

$$Ext^*_{E(\overline{\tau}_0)}(\mathbb{F}_p; THH_*(BP\langle 2\rangle; k(1))).$$

Then note that we can take a minimal resolution to produce an isomorphism between the E_1-term of (1.8) and the E_1-term of (1.3). Thus, the spectral sequences (1.3) and (1.4) are also multiplicative.

As in [1], the topological Hochschild cohomology of $BP\langle 2\rangle$ will play a role in our calculations. Recall from [6], there there is a universal coefficient spectral sequence (UCSS) of the form

$$\mathrm{Ext}^*_{R_*}(M_*, N_*) \Rightarrow \pi_* F_R(M, N) \tag{1.9}$$

when R is a ring spectrum, and M and N are (left) R-modules. When R is an E_∞-algebra the spectral sequence is a differential graded R_*-algebra spectral sequence.

We will use the Eilenberg-Moore spectral sequence

$$\mathrm{Tor}^*_{E_*(R)}(E_*(M), E_*(N)) \Rightarrow E_*(M \wedge_R N)$$

which exists as long as E_*R is flat as a right R_*-module and E and R are E_∞-ring spectra and M and and N are R-modules.

Conventions. Let $p \in \{2, 3\}$ throughout. We will write $H_*(-)$ for homology with $\mathbb{F}_p$ coefficients, or in other words, the functor $\pi_*(H\mathbb{F}_p \wedge -)$. We write $\doteq$ to mean that an equality holds up to multiplication by a unit. We will write $BP\langle n\rangle$ for the n-th truncated Brown-Peterson spectrum. In particular, $BP\langle 1\rangle$ denotes the E_∞-ring spectrum model for the connective Adams summand [11]. Also, $BP\langle 2\rangle$ will denote the E_∞-model for the second truncated Brown-Peterson spectrum constructed by [9] at $p = 2$ and [7] at $p = 3$. We also note that by coning off v_2 on $BP\langle 2\rangle$ we may construct $BP\langle 1\rangle$ as an E_∞ $BP\langle 2\rangle$-algebra and since the E_∞-ring spectrum structure on $BP\langle 1\rangle$ is unique, this is equivalent to the E_∞ ring spectrum model constructed in [11]. Let $k(n)$ denote an A_∞-ring spectrum model for the connective cover of the Morava K-theory spectrum $K(n)$.

When not otherwise specified, tensor products will be taken over $\mathbb{F}_p$ and $HH_*(A)$ denotes the Hochschild homology of a graded $\mathbb{F}_p$-algebra relative to $\mathbb{F}_p$. We will let $P(x)$, $E(x)$ and $\Gamma(x)$ denote a polynomial algebra, exterior algebra, and divided power algebra over $\mathbb{F}_p$ on a generator x.

The dual Steenrod algebra will be denoted $\mathscr{A}_*$ with coproduct $\Delta\colon \mathscr{A}_* \to \mathscr{A}_* \otimes \mathscr{A}_*$. Given a right $\mathscr{A}_*$-comodule M, its right coaction will be denoted $\nu\colon \mathscr{A} \to \mathscr{A} \otimes M$ where the comodule M is understood from the context.

2. Preliminary results

The homology of topological Hochschild homology of $BP\langle 2\rangle$ is a straightforward application of results of [2, 4, 5] and it appears in [2, Thm. 5.12]. Recall that there is an isomorphism

$$H_*(BP\langle 2\rangle) \cong \begin{cases} P(\bar{\xi}_1, \bar{\xi}_2, \dots) \otimes E(\bar{\tau}_3, \bar{\tau}_4, \dots) \text{ if } p \geq 3 \\ P(\bar{\xi}_1^2, \bar{\xi}_2^2, \bar{\xi}_3^2, \bar{\xi}_4, \dots) \text{ if } p = 2 \end{cases}$$

of $\mathscr{A}_*$-comodules. Then by [2, Thm. 5.12] there is an isomorphism

$$H_*(\mathrm{THH}(BP\langle 2\rangle)) \cong \begin{cases} H_*BP\langle 2\rangle \otimes E(\sigma\bar{\xi}_1, \sigma\bar{\xi}_2, \sigma\bar{\xi}_3) \otimes P(\sigma\bar{\tau}_3) \text{ if } p = 3 \\ H_*BP\langle 2\rangle \otimes E(\sigma\bar{\xi}_1^2, \sigma\bar{\xi}_2^2, \sigma\bar{\xi}_3^2) \otimes P(\sigma\bar{\xi}_4) \text{ if } p = 2 \end{cases} \tag{2.1}$$

of primitively generated $H_*BP\langle 2\rangle$-Hopf algebras. We also note the coaction on $H_*\,\mathrm{THH}(BP\langle 2\rangle)$ as a comodule over $\mathscr{A}_*$ computed in [2, Thm. 5.12]

$$\nu(\sigma\bar{\tau}_m) = 1 \otimes \sigma\bar{\tau}_m + \bar{\tau}_0 \otimes \sigma\bar{\xi}_m \tag{2.2}$$

at $p = 3$ and

$$\nu(\sigma\bar{\xi}_{m+1}) = 1 \otimes \sigma\bar{\xi}_{m+1} + \bar{\xi}_1 \otimes \sigma\bar{\xi}_2^2. \tag{2.3}$$

at $p = 2$. These both follow from the formula

$$\nu \circ \sigma = (1 \otimes \sigma) \circ \nu \tag{2.4}$$

in [2, Eq. 5.11] and the well known $\mathscr{A}_*$-coaction on $H_*BP\langle 2\rangle$. By the same argument, $\sigma\xi_i$ is primitive at $p = 3$ and $\sigma\xi_i^2$ is primitive at $p = 2$ for $i = 1, 2, 3$.

2.1. THH of $BP\langle 2\rangle$ modulo (p, v_1, v_2).

We now compute

$$\mathrm{THH}_*(BP\langle 2\rangle; H\mathbb{F}_p).$$

By [2, Lem. 4.1], it suffices to compute the sub-algebra of co-mododule primitives in $H_*(\mathrm{THH}(BP\langle 2\rangle; H\mathbb{F}_p))$ since $\mathrm{THH}(BP\langle 2\rangle; H\mathbb{F}_p)$ is an $H\mathbb{F}_p$-algebra. Since $BP\langle 2\rangle$ and $H\mathbb{F}_p$ are commutative ring spectra there is a weak equivalence of commutative ring spectra

$$\mathrm{THH}(BP\langle 2\rangle; H\mathbb{F}_p) \simeq \mathrm{THH}(BP\langle 2\rangle) \wedge_{BP\langle 2\rangle} H\mathbb{F}_p.$$

Since $H_*(\mathrm{THH}(BP\langle 2\rangle))$ is free over $H_*BP\langle 2\rangle$ by (2.1), the Eilenberg-Moore spectral sequence and [2, Cor. 5.13] immediately implies

$$H_*(\mathrm{THH}(BP\langle 2\rangle; H\mathbb{F}_p)) \cong \begin{cases} \mathscr{A}_* \otimes E(\sigma\bar{\xi}_1, \sigma\bar{\xi}_2, \sigma\bar{\xi}_3) \otimes P(\sigma\bar{\tau}_3) \text{ if } p = 3 \\ \mathscr{A}_* \otimes E(\sigma\bar{\xi}_1^2, \sigma\bar{\xi}_2^2, \sigma\bar{\xi}_3^2) \otimes P(\sigma\bar{\xi}_3) \text{ if } p = 2. \end{cases} \tag{2.5}$$

The $\mathscr{A}_*$ coaction on elements in $\mathscr{A}_*$ is given by the coproduct and the remaining coactions are determined by the formula (2.4) and are therefore the same as in (2.2) and (2.3). We write $\lambda_i = \sigma\bar{\xi}_i$ at $p = 3$ and $\lambda_i = \sigma\bar{\xi}_i^2$ are $p = 2$. We also define

$$\mu_3 = \begin{cases} \sigma\bar{\tau}_3 - \bar{\tau}_0 \otimes \sigma\bar{\xi}_3 \text{ if } p = 3 \\ \sigma\bar{\xi}_4 - \bar{\xi}_1 \otimes \sigma\bar{\xi}_3^2 \text{ if } p = 2 \end{cases}.$$

Then it is clear that the algebra of comodule primitives in $H_*(\mathrm{THH}(BP\langle 2\rangle; H\mathbb{F}_p))$ is generated by μ_3 and λ_i for $1 \leq i \leq 3$. We therefore produce the following isomorphism of graded $\mathbb{F}_p$-algebras

$$\mathrm{THH}_*(BP\langle 2\rangle; H\mathbb{F}_p) \cong E(\lambda_1, \lambda_2, \lambda_3) \otimes P(\mu_3). \tag{2.6}$$

The degrees of the algebra generators are $|\lambda_i| = 2p^i - 1$ for $1 \leq i \leq 3$ and $|\mu_3| = 2p^3$.

2.2. **Rational homology.** Next, we compute the rational homology of $\mathrm{THH}(BP\langle 2\rangle)$ to locate the torsion free component of $\mathrm{THH}_*(BP\langle 2\rangle)$. Towards this end, we will use the $H\mathbb{Q}$-based Bökstedt spectral sequence. This is a spectral sequence of the form

$$E_2^{**} = \mathrm{HH}_*^{\mathbb{Q}}(H\mathbb{Q}_* BP\langle 2\rangle) \implies H\mathbb{Q}_* \, \mathrm{THH}(BP\langle 2\rangle).$$

Recall that the rational homology of $BP\langle 2\rangle$ is

$$H\mathbb{Q}_* BP\langle 2\rangle \cong P_{\mathbb{Q}}(v_1, v_2).$$

Thus the E_2-term of the Bökstedt spectral sequence is

$$P_{\mathbb{Q}}(v_1, v_2) \otimes E_{\mathbb{Q}}(\sigma v_1, \sigma v_2)$$

and the bidegree of σv_i is $(1, 2(p^i - 1))$. Note that $BP\langle 2\rangle$ is a commutative ring spectrum, so by [2, Prop. 4.3] the Bökstedt spectral sequence is multiplicative. All the algebra generators are in Bökstedt filtration 0 and 1 and the d^2 differential shifts Bökstedt filtration by two, so there is no room for differentials. Thus, the E_2-term is isomorphic to the E_∞-term as graded $\mathbb{Q}$-algebras. There are also no hidden extensions. Thus, there is an isomorphism of graded $\mathbb{Q}$-algebras

$$\mathrm{THH}_*(BP\langle 2\rangle) \otimes \mathbb{Q} \cong P_{\mathbb{Q}}(v_1, v_2) \otimes_{\mathbb{Q}} E_{\mathbb{Q}}(\sigma v_1, \sigma v_2)$$

where $|\sigma v_i| = 2p^i - 1$. By the same method as [?Rog19, Thm. 1.1], one can prove that

$$\sigma v_1 = p\lambda_1$$
$$\sigma v_2 = p\lambda_2 - v_1^p \lambda_1 - v_1^p \sigma v_1.$$

Consequently, up to a change of basis,

$$THH_*(BP\langle 2\rangle; H\mathbb{Q}) \cong E_{\mathbb{Q}}(\lambda_1, \lambda_2). \tag{2.7}$$

Also, we conclude that

$$L_0 \, \mathrm{THH}(BP\langle 2\rangle) \simeq L_0 BP\langle 2\rangle \wedge \Sigma^{2p-1} L_0 BP\langle 2\rangle \vee \Sigma^{2p^2-1} L_0 BP\langle 2\rangle \vee \Sigma^{2p^2+2p-2} L_0 BP\langle 2\rangle$$

where $L_0 = L_{H\mathbb{Q}}$, since L_0 is a smashing localization and $L_0 S = H\mathbb{Q}$.

3. The $H\mathbb{Z}$-Bockstein spectral sequence

Recall that there is an isomorphism of $\mathcal{A}_*$-comodules

$$H_*(S/p \wedge THH(BP\langle 2\rangle)) \cong \begin{cases} E(\bar{\tau}_0) \otimes H_*(BP\langle 2\rangle) \otimes E(\sigma\bar{\xi}_1, \sigma\bar{\xi}_2, \sigma\bar{\xi}_3) \otimes P(\sigma\bar{\tau}_3) & \text{if } p = 3 \\ E(\bar{\xi}_1) \otimes H_*(BP\langle 2\rangle) \otimes E(\sigma\bar{\xi}_1^2, \sigma\bar{\xi}_2^2, \sigma\bar{\xi}_3^2) \otimes P(\sigma\bar{\xi}_4) & \text{if } p = 2 \end{cases}$$

where the coaction on $x \in \mathcal{A}_*$ is $\nu(x) = \Delta(x)$ and the remaining coactions follow from (2.4). In this section, we compute the Bockstein spectral sequence

$$E^1_{*,*} = THH_*(BP\langle 2\rangle; \mathbb{F}_p)[v_0] \Rightarrow THH(BP\langle 2\rangle; \mathbb{Z}_{(p)})_p. \tag{3.1}$$

As a direct consequence the coaction on $\sigma\bar{\tau}_3$, there is a differential

$$d_1(\mu_3) = v_0 \lambda_3. \tag{3.2}$$

in the $H\mathbb{Z}$-Bockstein spectral sequence (3.1).

The following lemma follows from [10, Prop. 6.8] by translating to the E_∞-context, see the proof of [1, Lem. 3.2].

Lemma 3.3. *If $d_j(x) \neq 0$ in the $H\mathbb{Z}$-Bockstein spectral sequence (3.1) then*

$$d_{j+1}(x^p) = v_0 x^{p-1} d_j(x)$$

if $p > 2$ or if $p = 2$ and $j \geq 2$. If $p = 2$ and $j = 1$ then

$$d_{j+1}(x^p) = v_0 x^{p-1} d_j(x) + Q^{|x|}(d_1(x))$$

When $p = 2$, we have the differential

$$d_1(\mu) = v_0 \lambda_3.$$

Therefore, the error term for $d_2(\mu_3^2)$ is

$$Q^{16}\lambda_3 = Q^{16}(\sigma\bar{\xi}_3^2) = \sigma(Q^{16}(\bar{\xi}_3^2)) = \sigma((Q^8\bar{\xi}_3)^2) = \sigma(\bar{\xi}_4^2) = 0.$$

The first equality holds by definition of λ_3, the second equality holds because σ commutes with Dyer-Lashoff operations by [4], the third equality holds by [5], and the last equality holds because σ is a derivation [2].

Corollary 3.4. *When $p = 2, 3$, there are differentials*

$$d_{i+1}(\mu^{p^i}) = v_0^{i+1}\mu_3^{p^i-1}\lambda_3.$$

Consequently, there are differentials

$$d_{\nu_p(k)+1}(\mu^k) \doteq v_0^{\nu_p(k)+1}\mu^{k-1}\lambda_3$$

where $\nu_p(k)$ denotes the p-adic valuation of k.

Proof. Let $\alpha = \nu_p(k)$. We have that $k = p^\alpha j$ where p does not divide j. So by the Leibniz rule

$$d_{\alpha+1}(\mu_3^k) = d_{\alpha+1}((\mu_3^{p^\alpha})^k) = k\mu_3^{p^\alpha(k-1)} d_{\alpha+1}(\mu_3^\alpha) = kv_0^{\alpha+1}\mu^{p^\alpha(k-1)}\mu^{p^\alpha-1}\lambda_3 = kv_0^{\alpha+1}\mu^{k-1}\lambda_3.$$

Since k is not divisible by p, it is a unit mod p. □

Now recall from (2.7) that $THH_*(BP\langle 2\rangle; H\mathbb{Q}) \cong E_{\mathbb{Q}}(\lambda_1, \lambda_2)$. In fact the map, $THH_*(B; H\mathbb{Z}_{(p)}) \to THH_*(B; H\mathbb{Q})$ sends λ_i to λ_i for $i = 1, 2$. Therefore, the elements λ_1, λ_2 are p-torsion free and there are no further differentials in the $H\mathbb{Z}$-Bockstein spectral sequence. We rename the following classes as follows

$$\begin{aligned} a_i^{(1)} &:= \lambda_3\mu_3^{i-1}, & b_i^{(1)} &:= \lambda_2 a_i^{(1)}, \\ a_i^{(2)} &:= \lambda_1 a_i^{(1)}, & b_i^{(2)} &:= \lambda_1 b_i^{(1)}. \end{aligned} \tag{3.5}$$

Thus we have the following

Corollary 3.6. *There is an isomorphism of $\mathbb{Z}_{(p)}$-algebras*

$$THH_*(BP\langle 2\rangle; H\mathbb{Z}_{(p)}) \cong E_{\mathbb{Z}_{(p)}}(\lambda_1, \lambda_2) \oplus T_0$$

where T_0 is a torsion $\mathbb{Z}_{(p)}$-module defined by

$$T_0 = \left(\mathbb{Z}_{(p)}\{a_i^{(k)}, b_i^{(k)} \mid i \geq 1, 1 \leq k \leq 2\}\right) / (p^j a_i^{(k)}, p^j b_i^{(k)} \mid j = \nu_p(i))$$

where the products on the elements $a_i^{(k)}$, b_i^k are specified by Formula (3.5) and by letting all other products be zero.

Remark 3.7. *In particular, we note that*

$$THH_*(BP\langle 2\rangle; H\mathbb{Z}_{(p)}) \cong E(\lambda_1) \otimes M$$

where M is a free $\mathbb{Z}_{(p)}$ module generated by $\{1, \lambda_2\}$ tensored with a torsion $\mathbb{Z}_{(p)}$-module generated by $a_i^{(1)}, b_i^{(1)}$ with the same p-torsion as described above. We will write

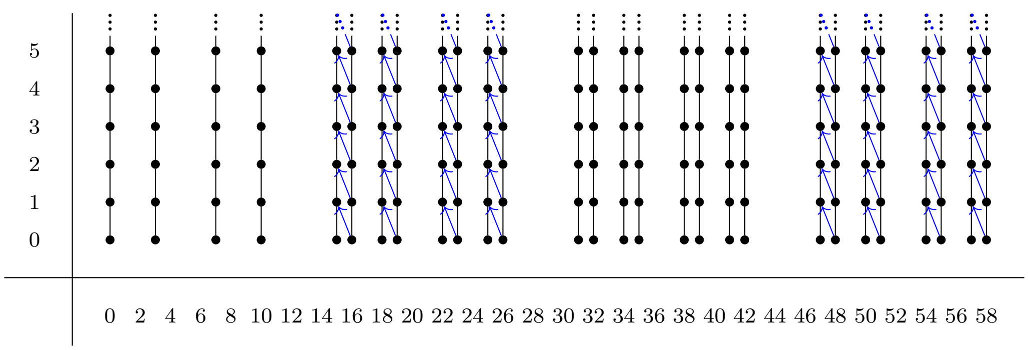
$$THH_*(BP\langle 2\rangle; H\mathbb{Z}_{(p)})/(\lambda_1) := M$$

and

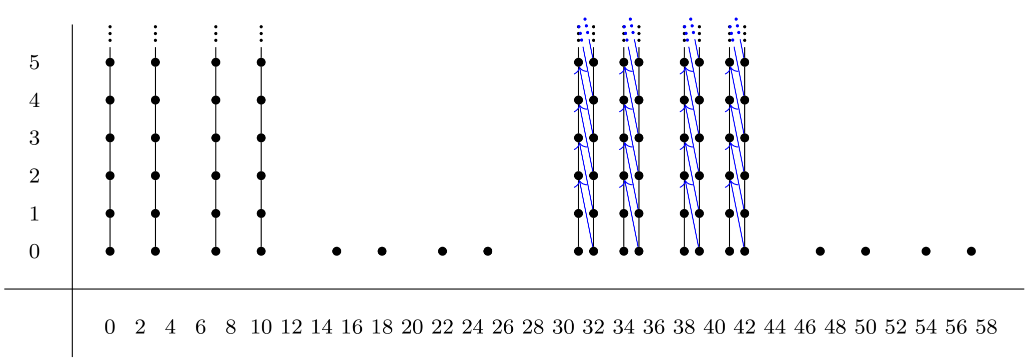
$$\lambda_1 \cdot THH_*(BP\langle 2\rangle; H\mathbb{Z}_{(p)})$$

for the elements in $THH_*(BP\langle 2\rangle; H\mathbb{Z}_{(p)})$ that are λ_1-divisible, i.e. $M\{\lambda_1\}$.

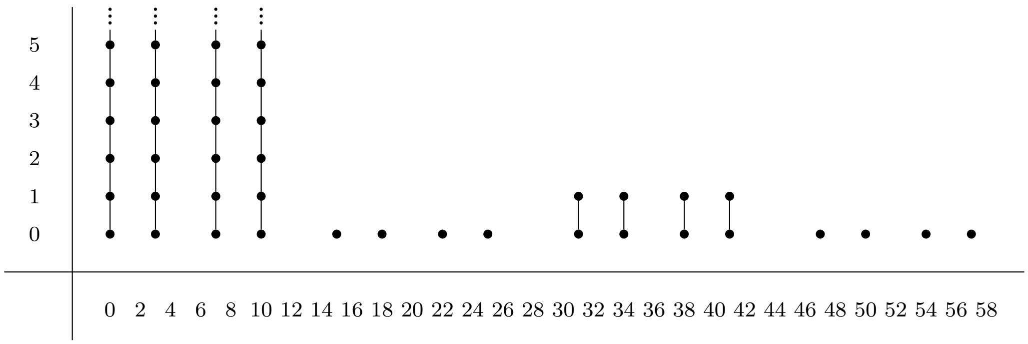
E_1-page of v_0-Bockstein Spectral Sequence at $p = 2$



E_2-page of v_0-Bockstein Spectral Sequence at $p = 2$



E_3-page of v_0-Bockstein Spectral Sequence at $p = 2$



4. The v_1-Bockstein spectral sequence

[Gabe: This section was taken directly from the other paper so at the moment it doesn't blend all that well with the rest of the paper.]

In this section, we begin our analysis of the v_1-Bockstein spectral sequence (1.2) for computing the homotopy of $\mathrm{THH}(BP\langle 2\rangle; k(1))$. To start, we need to compute $K(1)_*BP\langle 2\rangle$. This requires determining $\eta_R(v_{2+n})$ in $K(1)_*BP$ modulo the ideal generated by $(\eta_R(v_3), \ldots, \eta_R(v_{1+n}))$. We will need the following.

Lemma 4.1. [13, Lemma A.2.2.5] *Let v_n denote the Araki generators. Then there is the following equality in BP_*BP*

$$\sum_{i,j\geq 0}^{F} t_i\eta_R(v_j)^{p^i} = \sum_{i,j\geq 0}^{F} v_i t_j^{p^i}$$

In our context, the distinction between Hazewinkel generators and Araki generators is unimportant, as the two sets of generators coincide modulo p. In $K(1)_*BP$, we have killed all v_i's except v_1, which gives us the following equation

$$\sum_{i,j\geq 0}^{F} t_i\eta_R(v_j)^{p^i} = \sum_{k\geq 0}^{F} v_1 t_k^{p}$$

Note that the following degrees of the terms:

$$|v_1 t_j^p| = 2(p^{j+1} - 1)$$
$$|t_i\eta_R(v_j)^{p^i}| = 2(p^{i+j} - 1)$$

Since we are interested in the term $\eta_R(v_{2+n})$, we collect all the terms on the left of degree $2(p^{2+n} - 1)$. Thus we are summing over the ordered pairs (i, j) such that $i + j = 2 + n$. Since we only care about $\eta_R(v_{2+n})$ modulo $\eta_R(v_3), \ldots, \eta_R(v_{1+n})$ we only need to collect the terms where $j = 1, 2$, or $2 + n$. This shows that

$$t_{1+n}\eta_R(v_1)^{p^{n+1}} + t_n\eta_R(v_2)^{p^n} + \eta_R(v_{n+2}) = v_1 t_{n+1}^p$$

The value of η_R on v_1 and v_2 can also be computed by Lemma 4.1. One obtains, in $K(1)_*BP$, the following

$$\begin{aligned}\eta_R(v_1) &= v_1\\ \eta_R(v_2) &= v_1t_1^p - t_1v_1^p.\end{aligned}$$

Combining these observations, we obtain

Lemma 4.2. *In $K(1)_*BP$, the following congruence is satisfied*

$$\eta_R(v_{2+n}) \equiv v_1t_{n+1}^p - v_1^{p^n}t_1^{p^{n+1}}t_n + v_1^{p^{n+1}}(t_1^{p^n}t_n - t_{n+1}) \mod (\eta_R(v_3),\dots,\eta_R(v_{1+n}))$$

for $n \geq 1$.

Consequently, we have the following corollary.

Corollary 4.3. *There is an isomorphism of $K(1)_*$-algebras*

$$K(1)_*BP\langle 2\rangle \cong K(1)_*BP/(v_1t_{n+1}^p - v_1^{p^n}t_1^{p^{n+1}}t_n + v_1^{p^{n+1}}(t_1^{p^n}t_n - t_{n+1}) \mid n \geq 1)$$

Define elements

$$u_n := v_1^{\frac{p^n-1}{p-1}}t_n.$$

These elements are in degree 0, and therefore there is an isomorphism of $K(1)_*$-algebras

$$K(1)_*BP\langle 2\rangle \cong K(1)_* \otimes_{\mathbb{F}_p} K(1)_0BP\langle 2\rangle.$$

The calculations above imply the following corollary.

Corollary 4.4. *There is an isomorphism of $\mathbb{F}_p$-algebras*

$$K(1)_0BP\langle 2\rangle \cong P(u_i \mid i \geq 1)/(u_{n+1}^p - u_1^{p^{n+1}}u_n + u_1^{p^n}u_n - u_{n+1} \mid n \geq 1).$$

Our goal is to use this and the $K(1)$-based Bökstedt spectral sequence to compute the $K(1)$-homology of $\mathrm{THH}(BP\langle 2\rangle)$. This is a spectral sequence of the form

$$E^2_{s,t} = \mathrm{HH}_s^{K(1)_*}(K(1)_*BP\langle 2\rangle) \implies K(1)_{s+t}\,\mathrm{THH}(BP\langle 2\rangle).$$

The above considerations imply

$$E^2_{*,*} \cong K(1)_* \otimes \mathrm{HH}_*^{\mathbb{F}_p}(K(1)_0BP\langle 2\rangle).$$

The following results will be useful for our calculation.

Lemma 4.5 ([12])**.** *Let $V = \mathrm{Spec}(A)$ be a nonsingular affine variety over a field k. Let W be the subvariety of $V \times \mathbb{A}^n$ defined by equations*

$$g_i(Y_1,\dots,Y_n) = 0,\ g_i \in A[Y_1,\dots,Y_n],\ i = 1,\dots,n.$$

Then the projection map $W \to V$ is étale at a point $(P; b_1,\dots,b_n)$ of W if and only if the Jacobian matrix $\left(\frac{\partial g_i}{\partial Y_j}\right)$ is a nonsingular matrix at $(P; b_1,\dots,b_n)$.

Theorem 4.6 (Étale Descent, [15])**.** *Let $A \hookrightarrow B$ be an étale extension of commutative k-algebras. Then there is an isomorphism*

$$\mathrm{HH}_*(B) \cong \mathrm{HH}_*(A) \otimes_A B$$

Example 4.7. Consider the subalgebra
$$\mathbb{F}_p[u_1,u_2]/(u_2^p - u_1^{p^2+1} + u_1^{p+1} - u_2 = f_1).$$
We will regard this as a $\mathbb{F}_p[u_1]$-algebra. The partial derivative $\partial_{u_2} f_1$ is $-1 \pmod p$, and therefore a unit at every point. Then Lemma 4.5 tells us that this algebra is then étale over $\mathbb{F}_p[u_1]$.

By the same argument given above, we claim that there are a sequence of subalgebras A_n of
$$K(1)_0 BP\langle 2\rangle \cong \mathbb{F}_p[u_i \mid i\geq 1]/(u_{n+1}^p - u_1^{p^{n+1}} u_n + u_1^{p^n} u_n - u_{n+1} \mid n \geq 1) =: A.$$
such that each map $A_i \hookrightarrow A_{i+1}$ is an étale extension. Here
$$A_n := \mathbb{F}_p[u_1,u_2,\dots u_n]/(u_{k+1}^p - u_1^{p^{k+1}} u_k + u_1^{p^k} u_k - u_{k+1} = f_k \mid 1\leq k\leq n)$$
and the partial derivative $\partial_{u_k} f_k = -1 \pmod p$ for all $1 < k \leq n$ and therefore a unit at each point. The claim then follows by Lemma 4.5.

By the étale base change formula for Hochschild homology in Theorem 4.6, there is an isomorphism
$$\mathrm{HH}_*^{\mathbb{F}_p}(A_{i+1}) \cong \mathrm{HH}_*^{\mathbb{F}_p}(A_i)\otimes_{A_i} A_{i+1}$$
and since the functors $\mathrm{HH}_*(-)$ and $\mathrm{HH}_*^{\mathbb{F}_p}(A_1)\otimes_{A_1}(-)$ commute with filtered colimits of $\mathbb{F}_p$-algebras, there are isomorphisms
$$\begin{aligned}\mathrm{HH}_*^{\mathbb{F}_p}(A) &\cong \mathrm{HH}_*^{\mathbb{F}_p}(\operatorname{colim} A_n)\\ &\cong \operatorname{colim}\mathrm{HH}_*^{\mathbb{F}_p}(A_n)\\ &\cong \operatorname{colim}\mathrm{HH}_*^{\mathbb{F}_p}(A_1)\otimes_{A_1} A_n\\ &\cong \mathrm{HH}_*^{\mathbb{F}_p}(A_1)\otimes_{A_1} A.\end{aligned}$$
Consequently,
$$\mathrm{HH}_*(K(1)_* BP\langle 2\rangle) \cong K(1)_* \otimes E(\sigma t_1)\otimes K_0(R)$$
and therefore, since $\sigma t_1 \doteq \lambda_1$,
$$K(1)_*\,\mathrm{THH}(BP\langle 2\rangle) \cong K(1)_* BP\langle 2\rangle \otimes E(\lambda_1)$$
and
$$\mathrm{THH}_*(BP\langle 2\rangle; K(1)) \cong K(1)_*\otimes E(\lambda_1).$$
In other words,
$$THH_*(BP\langle 2\rangle; k(1)) \cong F\oplus T$$
where F is a free $P(v_1)$-module generated by 1 and λ_1 and T is a torsion $P(v_1)$-module. In summary, we have proven the following theorem.

Theorem 4.8.

1. *The $K(1)$-homology of $\mathrm{THH}(BP\langle 2\rangle; K(1))$ is $K(1)_*K(1)\otimes E(\lambda_1)$*
2. *Their is a weak equivalence*
$$K(1)\vee \Sigma^{2p-1}K(1) \simeq \mathrm{THH}(BP\langle 2\rangle; K(1)).$$
3. *The v_1-torsion free part of $\mathrm{THH}(BP\langle 2\rangle; k(1))$ is generated by 1 and λ_1.*

[Gabe: Double check that all of these results actually follow without need for proof. If not, include a short proof.]

4.1. **Differentials in the v_1-BSS.** We now analyze the v_1-BSS (1.2). Recall that this spectral sequence is of the form

$$\mathrm{THH}(BP\langle 2\rangle;\mathbb{F}_p)[v_1] \implies \mathrm{THH}(B;k(1)).$$

Thus the E_1-page is

$$E(\lambda_1,\lambda_2,\lambda_3)\otimes P(\mu_3)\otimes P(v_1). \tag{4.9}$$

Since the λ_i are all in odd total degree and since v_1^k are known to be survive to the E_∞-term, the λ_i are all permanent cycles. If μ_3 were a permanent cycle, then by multiplicativity, the spectral sequence would collapse. This would contradict Theorem 4.8. Therefore, the element μ_3 must support a differential. The only possibility for bidegree reasons is

$$d_{p^2}(\mu_3)\doteq v_1^{p^2}\lambda_2.$$

Thus, we obtain

$$v_1^{-1}E_{p^2+1}\cong K(1)_*\otimes E(\lambda_1,\lambda_3,\lambda_4')\otimes P(\mu_3^p)$$

where

$$\lambda_4' := \lambda_2\mu_3^{p-1}.$$

So the bidegree of λ_4' is given by

$$|\lambda_4'| = (2p^4-2p^3+2p^2-1,0).$$

For analogous reasons, the class λ_4' is a permanent cycle, and μ_3^p cannot be a permanent cycle. Based on degree considerations, there are two possible differentials,

$$d_{p^2}(\mu_3^p)\doteq v_1^{p^2}\lambda_4' \quad \text{or} \quad d_{p^3}(\mu_3^p)\doteq v_1^{p^3}\lambda_3.$$

The first would contradict the Leibniz rule for d_{p^2} and the fact that $d_{p^2}(\mu_3)\doteq v_1^{p^2}\lambda_2$. Thus,

$$v_1^{-1}E_{p^3+1}\cong K(1)_*\otimes E(\lambda_1,\lambda_4',\lambda_5')\otimes P(\mu_3^{p^2})$$

where

$$\lambda_5' := \lambda_3\mu_3^{p(p-1)}.$$

The bidegree of λ_5' is

$$|\lambda_5'| = (2p^5-2p^4+2p^3-1,0).$$

For degree reasons, the class λ_5' is a permanent cycle. As before, the class $\mu_3^{p^2}$ must support a differential. Degree considerations, again, give two possibilities

$$d_{p^3}(\mu_3^{p^2})\doteq v_1^{p^3}\lambda_5'$$

or

$$d_{p^4+p^2}(\mu_3^{p^2})\doteq v_1^{p^4+p^2}\lambda_4'.$$

The former would contradict the Leibniz rule, leaving the latter as the only possibility. This gives us

$$v_1^{-1}E_{p^4+p^2+1}\cong K(1)_*\otimes E(\lambda_1,\lambda_5',\lambda_6')\otimes P(\mu_3^{p^3})$$

where $\lambda_6' := \lambda_4'\mu_3^{p^2(p-1)}$. We will continue via induction. First we need some notation. We will recursively define classes λ_n' by

$$\lambda_n' := \begin{cases} \lambda_n & n = 1, 2, 3 \\ \lambda_{n-2}'\mu_3^{p^{n-4}(p-1)} & n \geq 4 \end{cases}$$

We let $d'(n)$ denote the topological degree of λ_n'. Then this function is given recursively by

$$d'(n) = \begin{cases} 2p^n - 1 & n = 1, 2, 3 \\ 2p^n - 2p^{n-1} + d(n-2) & n > 3 \end{cases}$$

Thus, by a simple induction, one has

$$d'(n) = \begin{cases} 2p^n - 1 & n = 1, 2, 3 \\ 2p^n - 2p^{n-1} + 2p^{n-2} - 2p^{n-3} + \cdots + 2p^2 - 1 & n \equiv 0 \mod 2,\ n > 3 \\ 2p^n - 2p^{n-1} + 2p^{n-2} - 2p^{n-3} + \cdots + 2p^3 - 1 & n \equiv 1 \mod 2,\ n > 3 \end{cases}.$$

Observe that the integers $2p^{n+2} - d(n+1) - 1$ and $2p^{n+2} - d(n+2) - 1$ are divisible by $|v_1|$. Let $r'(n)$ denote the integer

$$r'(n) := |v_1|^{-1}(|\mu_3^{p^{n-1}}| - |\lambda_{n+1}'| - 1) = |v_1|^{-1}(2p^{n+2} - d'(n+1) - 1).$$

Then a simple induction shows that

$$r'(n) = \begin{cases} p^{n+1} + p^{n-1} + p^{n-3} + \cdots + p^2 & n \equiv 1 \mod 2 \\ p^{n+1} + p^{n-1} + p^{n-3} + \cdots + p^3 & n \equiv 0 \mod 2 \end{cases}.$$

We can now describe the differentials in the v_1-BSS.

Theorem 4.10. *In the v_1-BSS, the following hold:*

(1) *The only nonzero differentials are in $v_1^{-1}E_{r'(n)}$.*
(2) *The $r'(n)$-th page is given by*

$$v_1^{-1}E_{r'(n)} \cong K(1)_* \otimes E(\lambda_1, \lambda_{n+1}', \lambda_{n+2}') \otimes P(\mu_3^{p^{n-1}})$$

and the classes $\lambda_{n+1}', \lambda_{n+2}'$ are permanent cycles.
(3) *The differential $d_{r'(n)}$ is uniquely determined by multiplicativity of the BSS and the differential*

$$d_{r'(n)}(\mu_3^{p^{n-1}}) \doteq v_1^{r'(n)}\lambda_{n+1}'.$$

for $n \geq 1$.

Proof. We proceed by induction. We have already shown the theorem for $n \leq 4$. Assume that

$$v_1^{-1}E_{r'(n)} \cong K(1)_* \otimes E(\lambda_1, \lambda_n', \lambda_{n+1}') \otimes P(\mu_3^{p^{n-1}}).$$

and λ_n' is an infinite cycle.

Since $\lambda_n', \lambda_{n+1}'$ are both in odd topological degree, λ_{n+1}' cannot support a differential hitting the v_1-towers on λ_i' for $i < n+1$. Thus, the only possibility is that λ_{n+1}' supports a differential into the v_1-tower on 1 or λ_1. But this would contradict Theorem 4.8. Therefore, the class λ_{n+1}' is a permanent cycle.

The class $\mu_3^{p^{n-1}}$ must support a differential, for if it did not, then the spectral sequence would collapse. This would lead to a contradiction of Theorem 4.8. Degree considerations show that the following differentials are possible

$$d_{k(n)}(\mu_3^{p^{n-1}}) \dot{=} v_1^{k(n)}\lambda'_{n+2}$$

and

$$d_{r(n)}(\mu_3^{p^{n-1}}) \dot{=} v_1^{r'(n)}\lambda'_{n+1}$$

where

$$\ell(n) = |v_1^{-1}|(2p^{n+2} - |\lambda'_{n+2}|).$$

An elementary inductive computation shows that

$$\ell(n) = r'(n-1).$$

We claim that the former differential cannot occur. This follows because, by the inductive hypothesis,

$$d_{r'(n-1)}(\mu_3^{p^{n-2}}) \dot{=} \lambda'_n,$$

and the former differential would contradict the Leibniz rule. So we must conclude the latter differential occurs. This concludes proof. $\square$

We now state the main result of this section.

Theorem 4.11. *For each $n \geq 2$ and each nonnegative integer m with $m \not\equiv p-1$* mod p *there are elements $z_{n,m}$ and $z'_{n,m}$ in $THH_*(BP\langle 2\rangle; k(1))$ such that*

1. *$z_{n,m}$ projects to $\lambda'_n\mu_3^{mp^{n-2}}$ in $E_\infty^{*,0}$*
2. *$z'_{n,m}$ projects to $\lambda'_n\lambda'_{n+1}\mu_3^{mp^{n-2}}$ in $E_\infty^{*,0}$*

As a $P(v_1)$-module, $THH_*(BP\langle 2\rangle; k(1))$ is generated by the unit element 1, λ_1, and the elements $\lambda_1^{\epsilon}z_{n,m}, \lambda_1^{\epsilon}z'_{n,m}$ where $\epsilon \in \{0,1\}$. The only relations are

$$v_1^{r'(n-1)}\lambda_1^{\epsilon}z_{n,m} = v_1^{r'(n-1)}\lambda_1^{\epsilon}z'_{n,m} = 0.$$

To prove this, we first need to prove a couple lemmas. We first introduce notation. Let $P(m)$ denote a free rank one $P(v_m)$-module and let $P(m)_i$ denote the $P(v_m)$-module $P(m)/v_m^i$. Let X be a $BP\langle n\rangle$-module such that $H_*X \cong H_*BP\langle n\rangle \otimes H_*(\overline{X})$ as a $H_*BP\langle n\rangle$-module and consider the Adams spectral sequence

$$E_2^{*,*}(X) = Ext_{E(Q_m)}^{*,*}(\mathbb{F}_p, H_*(\overline{X})) \Rightarrow \pi_*(X \wedge_{BP\langle n\rangle} k(m))_p \tag{4.12}$$

and the v_n-inverted Adams spectral sequence

$$v_m^{-1}E_2^{*,*}(X) = v_m^{-1}Ext_{E(Q_m)}^{*,*}(\mathbb{F}_p, H_*(\overline{X})) \Rightarrow \pi_*(X \wedge_{BP\langle n\rangle} K(m))_p \tag{4.13}$$

There is a map of spectral sequences

$$E_2^{*,*}(X) \longrightarrow v_m^{-1}E_2^{*,*}(X)$$

induced by the localization map $k(m) \to v_m^{-1}k(m) = K(m)$.

Lemma 4.14. *Let $r \geq 2$. Suppose the $E_r(X)$-page of the Adams spectral sequence (4.12) is generated by elements in filtration 0 as a $P(k)$-module and $E_r^{*,*}(X)$ is a direct sum of copies of $P(k)$ and $P(k)_i$ with $i \leq r$ as a $P(k)$-module. Then*

(1) *the map of E_r-pages*
$$E_r^{s,t}(X) \to v_k^{-1}E_r^{s,t}(X)$$
is a monomorphism when $t \geq r+1 \geq 3$.

(2) *Also, the differentials in $E_{r+1}^{*,*}$ are the same as their image in $v_k^{-1}E_{r+1}^{*,*}$.*

Proof. Statement (1) is a consequence of our assumptions since elements in filtration $r+1$ are v_k-torsion free. To prove statement (2) it suffices to prove the following: if $x \in E_r(X)$ maps to a cycle $\bar{x} \in v_k^{-1}E_r(X)$,then x is a cycle. By our assumption, there is an $a \in E_r^{*,0}$ such that $x = v_k^m a$. Statement (1) then implies $d_{r+1}(a) = 0$ so since the differentials are v_k-linear the result follows. □

[Gabe: In MS, they claim that the proof works for $t \geq r-1 \geq 1$ and $E_r^{*,*}$, but I don't see why they get indices instead of the ones I have here.]

Remark 4.15. *The Lemma above is a generalization of part (a) and (b) of Theorem 7.1 [11]. Note that $H_*THH(BP\langle 2\rangle)$ is a free $H_*BP\langle 2\rangle$-module.*

Lemma 4.16. *For $r \geq 2$, the E_r-page of the Adams spectral sequence (4.12) for $X = THH(BP\langle 2\rangle)$ and $m = 1$ is generated by elements in filtration 0 as a $P(1)$-module and $E_r^{*,*}$ is a direct sum of copies of $P(1)$ and $P(1)_i$ for $i \leq r$.*

Proof. We will begin by proving the first statement by induction. Note that (4.9) implies the base case in the induction when $r = 2$. Suppose the statement holds for some r. Choose a basis y_i for the $\mathbb{F}_p$-vector space V_r such that
$$V_r = \{x \in E_r^{*,0} \mid v_1^{r-1}x = 0\}.$$
Then $d_r(y_i)$ is in filtration r and since the differentials are v_1-linear, $v_1^{r-1}d_r(y_i) = 0$. However, this contradicts the induction hypothesis because the induction hypothesis implies that all elements in filtration r are v_1-torsion-free. Thus, each basis element y_i is a d_r-cycle. Next choose a set of elements $\{y_j'\} \subset E_r^{*,0}$ such that $\{d_r(y_j')\}$ is a basis for $\mathrm{im}(d_r\colon\ E_r^{*,0} \to E_r^{*,r})$. Choose $y_j'' \in E_r^{*,0}$ such that $v_1^r y_j'' = d_r(y_j')$. Then y_j'' are d_r-cycles and y_j'' and y_j are linearly independent. We can therefore choose d_r-cycles y_j''' such that $\{y_j\} \cup \{y_j''\} \cup \{y_j'''\}$ are a basis for the d_r-cycles in $E_r^{*,0}$. Then $\{y_j\} \cup \{y_j'\} \cup \{y_j''\} \cup \{y_j'''\}$ are a basis for $E_r^{*,0}$ and the differential is completely determined by the formulas
$$d_r(y_i) = 0, \quad d_r(y_j') = v_1^r y_j'', \quad d_r(y_j'') = 0, \text{ and } \quad d_r(y_j''') = 0.$$
Thus, $E_r^{*,*}$ is generated as a $P(1)$-module by y_i, y_i'', and y_i''' where $v_1^{r-1}y_i = 0$ and $v_1^r y_i'' = 0$ and y_i''' is v_1-torsion free. □

Proof of Theorem 4.11. For brevity, we will let $\delta_{n,m}$ denote $\lambda_n'\mu_3^{mp^{n-2}}$ and we will let $\delta_{n,m}'$ denote $\lambda_n'\lambda_{n+1}'\mu_3^{mp^{n-2}}$. By Lemma 4.14 and Lemma 4.16 it suffices to prove that the elements $\delta_{n,m}$, and $\delta_{n,m}'$ are infinite cycles that, together with 1 and λ_1, form a basis for $E_\infty^{*,0}$ as an $\mathbb{F}_p$-vector space, and that each of $\delta_{n,m}$, $\delta_{n,m}'$ are killed by $v_1^{r'(n)}$. By induction on n, we will prove
$$E_{r(n)}(THH(BP\langle 2\rangle)) \cong M_n \oplus E(\lambda_1, \lambda_n', \lambda_{n+1}') \otimes P(\mu_3^{p^{n-1}})$$

where M_n is generated by $\{\delta_{k,m}, \delta'_{k,m} \mid k < n\}$ modulo the relations

$$v_2^{r(k)}\delta_{k,m} = v_2^{r(k)}\delta'_{k,m} = 0.$$

This statement holds for $n = 1$ by (4.9). Assume the statement holds for all integers less than or equal to some $N \geq 1$. Lemma 4.14, Lemma 4.16, and Theorem 4.10 imply that the only nontrivial differentials with source in $E_{r(N)}^{*,0}$ are the differentials

$$d_{r(N)}(\mu_3^{(m+1)r'(N)}) = (m+1)v_1^{r'(N)}\lambda'_N\mu_3^{mp^{N-1}} \doteq \delta_{N,m},$$

and the differentials

$$d_{r'(N)}(\lambda'_{N+1}\mu_3^{(m+1)r'(N)}) = (m+1)v_1^{r'(N)}\lambda'_n\lambda'_{N+1}\mu_3^{mp^{N-2}} \doteq \delta'_{N,m}$$

where $m \not\equiv p-1 \mod p$. Combining this with Lemma 4.16 and Lemma 4.14, this implies that

$$E_{r(N)+1}(\mathrm{THH}(BP\langle 2\rangle)) \cong M_n \oplus V_{N+1} \oplus \Big(P(2) \otimes E(\lambda_1, \lambda'_N, \lambda'_N\mu_3^{(p-1)p^{N-2}}) \otimes P(\mu_3^{p^N})\Big)$$

where V_{N+1} has generators $\delta_{N,m}$ and $\delta'_{N,m}$ and relations

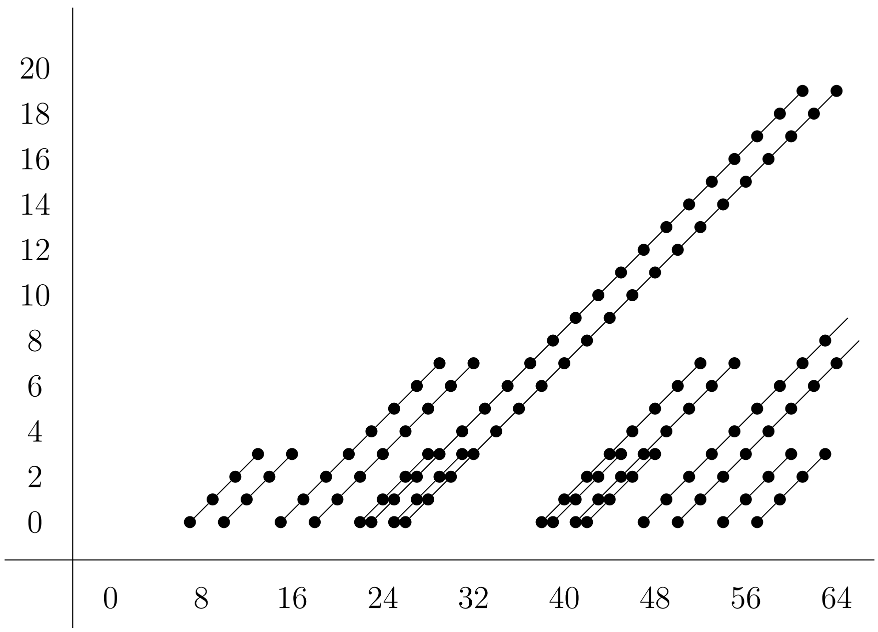
$$v_2^{r(N)}\delta_{N,m} = v_2^{r(N)}\delta'_{N,m} = 0.$$

By Lemma 4.14, Lemma 4.16, and Theorem 4.10 there is an isomorphism

$$E_{r(N)+1}(\mathrm{THH}(BP\langle 2\rangle)) \cong E_{r(N+1)}(\mathrm{THH}(BP\langle 2\rangle)).$$

Also, note that $M_N \oplus V_{N+1} = M_{N+1}$ and $\lambda'_N\mu_3^{(p-1)p^{N-2}} = \lambda'_{N+2}$ by definition. This completes the inductive step and consequently the proof. $\square$

v_1-torsion in the E_∞-page of v_1-Bockstein Spectral Sequence for $0 \leq x \leq 64$



5. Topological Hochschild cohomology of $BP\langle 2\rangle$

We will write $THC(BP\langle 2\rangle)$ for topological Hochschild cohomology of $BP\langle 2\rangle$, which is defined to be

$$THC(BP\langle 2\rangle) := F_{BP\langle 2\rangle^e}(BP\langle 2\rangle, BP\langle 2\rangle)$$

where $BP\langle 2\rangle^e := BP\langle 2\rangle \wedge BP\langle 2\rangle^{\mathrm{op}}$. We recall that there is a universal coefficient spectral sequence (UCSS) computing the homotopy groups of $F_{BP\langle 2\rangle^e}(BP\langle 2\rangle, BP\langle 2\rangle)$

$$Ext^{*,*}_{\pi_*(BP\langle 2\rangle^e)}(BP\langle 2\rangle_*, BP\langle 2\rangle_*) \Rightarrow THC^*(BP\langle 2\rangle),$$

but this is usually not computable. With coefficients in $H\mathbb{F}_p$, however, we can compute $THC^*(BP\langle 2\rangle; H\mathbb{F}_p)$ by a different means. First, note that

$$THH_*(BP\langle 2\rangle; H\mathbb{F}_p) \cong E(\lambda_1, \lambda_2, \lambda_3) \otimes P(\mu_3)$$

is a finite type graded $\mathbb{F}_p$-algebra and $THH(BP\langle 2\rangle; H\mathbb{F}_p)$ is an $H\mathbb{F}_p$-algebra. By adjunction, there is an equivalence

$$F_{H\mathbb{F}_p}(THH(BP\langle 2\rangle; H\mathbb{F}_p), H\mathbb{F}_p) \simeq THC(BP\langle 2\rangle, H\mathbb{F}_p).$$

Consequently, the UCSS

$$\mathrm{Ext}^{*,*}_{\mathbb{F}_p}(\pi_* THH(BP\langle 2\rangle; H\mathbb{F}_p), \mathbb{F}_p) \Rightarrow \pi_* THC(BP\langle 2\rangle; H\mathbb{F}_p)$$

collapses and

(5.1) $THC^*(BP\langle 2\rangle, H\mathbb{F}_p) \cong Hom_{\mathbb{F}_p}(THH_*(BP\langle 2\rangle; H\mathbb{F}_p), \mathbb{F}_p) \cong E(x_1, x_2, x_3) \otimes \Gamma(c_1)$

where $|x_i| = 2p^i - 1$ and $|c_1| = 2p^3$. The classes x_i are dual to λ_i and the class $c_i = \gamma_i(c_1)$ is dual to μ_3^i.

5.1. **Relative topological Hochschild cohomology of $BP\langle 2\rangle$.** Recall that

$$H_*MU \cong P(b_k \mid k \geq 1)$$

and the map

$$H_*MU \to H_*BP$$

sends b_{p^k-1} to $\bar{\xi}_k$ for $k \geq 1$.

Lemma 5.2. *There is an isomorphism of rings*

$$\pi_*H\mathbb{F}_p \wedge_{MU} BP\langle 2\rangle \cong E(\tau_3, \tau_4, \dots) \otimes E(\delta b_i \mid i \not\equiv 0 \pmod{p^k - 1}, k \geq 1)$$

and the map from $H_*BP\langle 2\rangle$ is given by the canonical quotient

$$H_*BP\langle 2\rangle \cong P(\xi_1, \xi_2, \dots) \otimes E(\tau_3, \tau_4, \dots) \to E(\tau_3, \tau_4, \dots)$$

tensored with the unit map

$$\mathbb{F}_p \to E(\delta b_i \mid i \not\equiv 0 \pmod{p^k - 1}, k \geq 1).$$

Here $|\delta b_i| = 1 + |b_i|$

Proof. First note that there is an equivalence of ring spectra

$$H\mathbb{F}_p \wedge_{MU} BP\langle 2\rangle \simeq H\mathbb{F}_p \wedge_{H\mathbb{F}_p \wedge MU} H\mathbb{F}_p \wedge BP\langle 2\rangle.$$

The Künneth spectral sequence has input

$$\begin{aligned} \mathrm{Tor}_*^{H_*MU}(\mathbb{F}_p, H_*BP\langle 2\rangle) &\cong \mathrm{Tor}_*^{P(\bar{\xi}_1, \bar{\xi}_2, \dots)}(\mathbb{F}_p, H_*BP\langle 2\rangle) \otimes \mathrm{Tor}^{P(b_i \mid i \not\equiv 0 \pmod{p^k-1}, k\geq 1)}(\mathbb{F}_p, \mathbb{F}_p) \\ &= E(\tau_3, \tau_4, \dots) \otimes E(\delta b_i \mid i \not\equiv 0 \pmod{p^k - 1}, k \geq 1). \end{aligned}$$

The Künneth spectral sequence collapses because all the generators are in filtration $0, 1$ and the differential shifts filtration by 2. By factoring the relevant map as

$$H_*BP\langle 2\rangle \to \pi_*H\mathbb{F}_p \wedge_{BP} BP\langle 2\rangle \to \pi_*H\mathbb{F}_p \wedge_{MU} BP\langle 2\rangle$$

and computing $\pi_*H\mathbb{F}_p \wedge_{BP} BP\langle 2\rangle \cong E(\tau_3, \tau_4, \dots)$ by the same argument, we see that the map is the composite of the canonical quotient with the identity tensored with the unit map as desired. □

Recall from Lemma 2.4 [1] that when $R \to Q$ is a map of E-algebras and M is a Q-R-bimodule, with an R-R-bimodule structure by pullback, then

$$THC_E(R; M) \simeq F_{Q \wedge_E R^{\mathrm{op}}}(Q, M).$$

Lemma 5.3. *The following hold:*

(1) *There is an isomorphism of rings*

$$THC^*_{MU}(BP\langle 2\rangle; H\mathbb{F}_p) \cong P(\sigma\tau_i \mid i \geq 3) \otimes P(\sigma\delta b_i \mid i \not\equiv 0 \pmod{p^k - 1}, k \geq 1).$$

(2) *Consequently, $THC^*_{MU}(BP\langle 2\rangle)$ is isomorphic to*

$$BP\langle 2\rangle_*(\sigma\tau_i \mid i \geq 3) \otimes_{BP\langle 2\rangle_*} P_{BP\langle 2\rangle_*}(\sigma\delta b_i \mid i \not\equiv 0 \pmod{p^k - 1}, k \geq 1).$$

(3) *The map*

$$THC^*_{MU}(BP\langle 2\rangle) \to THC^*_{MU}(BP\langle 2\rangle; H\mathbb{F}_p)$$

is induced by the quotient by (p, v_1, v_2).

(4) *The map*

$$THC^*_{MU}(BP\langle 2\rangle; H\mathbb{F}_p) \to THC^*(BP\langle 2\rangle; H\mathbb{F}_p)$$

sends $\sigma\tau_i$ to $c_{p^{i-3}}$ for $i \geq 3$.

(5) *Consequently, the elements $c_{p^{i-3}}$ pull back to elements in $THC^*(BP\langle 2\rangle)$ and $THC^*(BP\langle 2\rangle; BP\langle 1\rangle)$.*

Proof. The first statement follows by the universal coefficient spectral sequence computing

$$THC_E(R; M) \simeq F_{Q\wedge_E R^{\mathrm{op}}}(Q, M)$$

where $E = MU$, $R = BP\langle 2\rangle$ and $M = Q = H\mathbb{F}_p$. The UCSS computing

$$THC^*_{MU}(BP\langle 2\rangle; H\mathbb{F}_p)$$

has input

$$\mathrm{Ext}^*_{\pi_* H\mathbb{F}_p\wedge_{MU} BP\langle 2\rangle}(\mathbb{F}_p, \mathbb{F}_p) \cong P(\sigma\tau_i \mid i \geq 3) \otimes P(b_i \mid i \not\equiv 0 \pmod{p^k - 1}, k \geq 1)$$

and since all elements are in even total degree there is no room for differentials. (Note that by Koszul duality $\mathrm{Tor}^{P(b)}_*(\mathbb{F}_p, \mathbb{F}_p) \cong E(\delta b_i)$ and $\mathrm{Ext}^*_{E(\delta b_i)}(\mathbb{F}_p, \mathbb{F}_p) \cong P(b_i)$ for all i). Consequently, the spectral sequence collapses. This proves the first statement.

There are three Bockstein spectral sequences to go from $THH^*_{MU}(BP\langle 2\rangle; H\mathbb{F}_p)$ to $THH^*_{MU}(BP\langle 2\rangle)$, but in each case all elements are in even columns and the spectral sequences collapse since there is an Adams style differential convention. This proves the the second statement and the third statement.

Now, by the commutative diagram

$$\begin{tikzcd} THC^*_{MU}(BP\langle 2\rangle) \arrow[r] \arrow[d] & THH^*_{MU}(BP\langle 2\rangle; H\mathbb{F}_p) \arrow[d] \\ THC^*(BP\langle 2\rangle) \arrow[r] & THC^*(BP\langle 2\rangle; H\mathbb{F}_p) \end{tikzcd}$$

the last statement follows by the statement preceding it. It therefore remains to show that the map

$$THC^*_{MU}(BP\langle 2\rangle; H\mathbb{F}_p) \to THC^*(BP\langle 2\rangle; H\mathbb{F}_p)$$

sends $\sigma\tau_i$ to $c_{p^{i-3}}$ for $i \geq 3$. Recall the map $H_*BP\langle 2\rangle \to \pi_* H\mathbb{F}_p \wedge_{MU} BP\langle 2\rangle$ sends τ_i to τ_i for $i \geq 3$. Tracing this through the induced map of universal coefficient spectral sequences produces the desired result. $\square$

Next we determine whether the elements $c_{p^{i-3}}$ are torsion by computing Hopf-algebra

$$THC^*(BP\langle 2\rangle; H\mathbb{Z}_{(p)}).$$

Lemma 5.4. *There is an isomorphism of Hopf algebras*

$$THC^*(BP\langle 2\rangle; H\mathbb{Z}_{(p)}) \cong E_{\mathbb{Z}_{(p)}}(x_1, x_2) \otimes \Gamma_{\mathbb{Z}_{(p)}}(c_1)/(pc_1)$$

where x_1, x_2 and c_1 are primitive.

Proof. In general, if R is a commutative ring spectrum and $H\mathbb{Z}_{(p)}$ is a commutative R-algebra, then $THH_*(R, H\mathbb{Z}_{(p)})$ is a $\mathbb{Z}_{(p)}$ Hopf-algebra spectrum whenever $THH_k(R; H\mathbb{Z}_{(p)})$ is a finitely generated $\mathbb{Z}_{(p)}$-algebra for all k. By Corollary 3.6, $THH_*(BP\langle 2\rangle; H\mathbb{Z}_{(p)})$ is a finitely generated $\mathbb{Z}_{(p)}$-algebra in each degree. Consequently, $THC^*(BP\langle 2\rangle; H\mathbb{Z}_{(p)})$ is the the $\mathbb{Z}_{(p)}$-dual of $THH_*(BP\langle 2\rangle; H\mathbb{Z}_{(p)})$ and it is also finitely generated in each degree and the Bockstein spectral sequence

$$THC^*(BP\langle 2\rangle; H\mathbb{F}_p)[v_0] \Rightarrow THC^*(BP\langle 2\rangle; H\mathbb{Z}_{(p)})_p$$

converges. Since multiplication by p commutes with the coproduct this is a spectral sequence of Hopf algebras. In order for $THC^*(BP\langle 2\rangle; H\mathbb{Z}_{(p)})$ to be $\mathbb{Z}_{(p)}$-dual to $THH_*(BP\langle 2\rangle; H\mathbb{Z}_{(p)})$ the differentials

$$d_{i+1}(c_{p^i-1}x_3) \doteq v_0^{i+1}c_{p^i}$$

are forced for $i \geq 0$ where $c_0 = 1$. □

Recall that there is a cap product

$$THC^k(BP\langle 2\rangle; H\mathbb{Z}_{(p)}) \wedge THH_m(BP\langle 2\rangle; H\mathbb{Z}_{(p)}) \to THH_{m-k}(BP\langle 2\rangle; H\mathbb{Z}_{(p)}).$$

Corollary 5.5. *For $k < n$, the cap product satisfies the following formulae*

$$c_k \cap a_n^{(m)} \doteq \binom{n-1}{k} a_{n-k}^{(m)}$$

$$c_k \cap b_n^{(m)} \doteq \binom{n-1}{k} b_{n-k}^{(m)}$$

for $1 \leq m \leq 2$.

Since c_{p^k} is p^{k+1} torsion in $THH^*(BP\langle 2\rangle; H\mathbb{Z}_{(p)})$ we see that $p^k c_{p^k} \neq 0$. However, in the Adams spectral sequence for $THH_*(BP\langle 2\rangle; BP\langle 1\rangle)$ and $THH_*(BP\langle 2\rangle)$ these classes appear to be torsion free. The cap product is also natural and therefore it commutes with Bockstein spectral sequence differentials.

6. Computation of $THH(BP\langle 2\rangle; BP\langle 1\rangle)$

We now begin the final computation. We will compare the two spectral sequences

$$THH_*(BP\langle 2\rangle; H\mathbb{Z}_{(p)})[v_1] \Rightarrow THH_*(BP\langle 2\rangle; BP\langle 1\rangle) \tag{6.1}$$

$$THH_*(BP\langle 2\rangle; k(1))[v_0] \Rightarrow THH_*(BP\langle 2\rangle; BP\langle 1\rangle)_p. \tag{6.2}$$

We begin with results about spectral sequence (6.2) and then use them to compute (6.1).

6.1. **The v_0-Bockstein spectral sequence computing $THH_*(BP\langle 2\rangle; BP\langle 1\rangle)_p$.** In the v_0-Bockstein spectral sequence, the v_0-towers are vertical which allows us to draw conclusions about vanishing of certain topological degrees. of $THH_*(BP\langle 2\rangle; BP\langle 1\rangle)_p$. We include a figure to highlight the degrees where these gaps occur.

[Gabe: Here is would help to work with $\overline{THH}_*(BP\langle 2\rangle; k(1))/(\lambda_1)$, is there a way to define this as the homotopy groups of a spectrum? Do we need to define it as the homotopy groups of a spectrum for the computations?]

Recall from Theorem 4.11 that

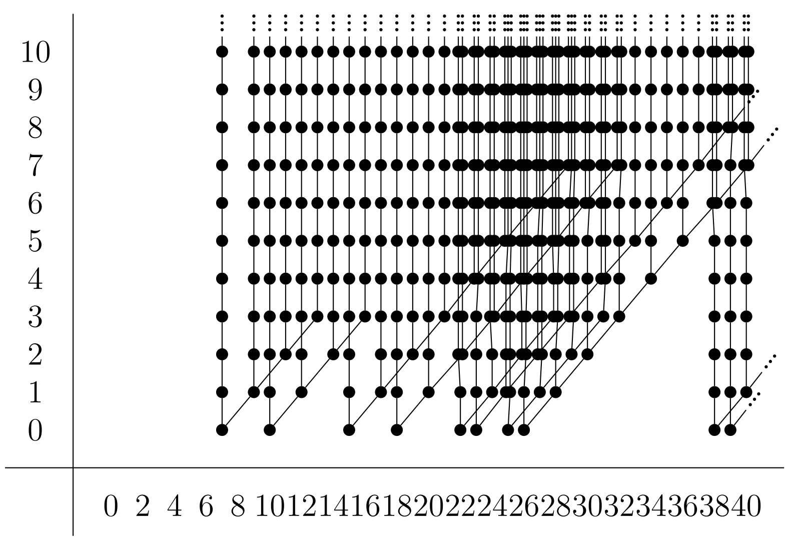
$$THH_*(BP\langle 2\rangle; k(1)) \cong \left(P(v_1) \otimes E(\lambda_1) \otimes \mathbb{F}_p\{1, z_{n,m}, z'_{n,m}\}\right) / (v_1^{r'(n)} \lambda_1^{\epsilon} z_{n,m} = v_1^{r'(n)} \lambda_1 z'_{n,m} = 0).$$

We can form the quotient $THH_*(BP\langle 2\rangle; k(1))/(\lambda_1)$ algebraically and we will consider the v_1-torsion summand and denote it $\overline{THH}_*(BP\langle 2\rangle; k(1))/(\lambda_1)$. After these reductions, the input of the Bockstein spectral sequence

$$THH_*(BP\langle 2\rangle; k(1))[v_0] \Rightarrow THH_*(BP\langle 2\rangle; BP\langle 1\rangle)$$

is described in Figure 6.1. The element v_1 is really in filtration 0, but here we draw it in filtration 1 to present a simpler picture, following the convention in [1].

v_1-torsion in the E_1-page of the of v_0-Bockstein Spectral Sequence for $0 \leq x \leq 40$ modulo λ_1



Lemma 6.3. *The following hold:*

(1) *The groups $\overline{THH}_{2p^{n+3}-2p^2}(BP\langle 2\rangle; BP\langle 1\rangle)/(\lambda_1)$ are cyclic for $n \geq 1$.*
(2) *The groups $\overline{THH}_{2p^{n+3}-2p^2+(2p-2)\ell}(BP\langle 2\rangle; BP\langle 1\rangle)/(\lambda_1)$, are trivial for $n \geq 0$ and $p+1 \leq \ell \leq 2p+2$*
(3) *The groups $\overline{THH}_{2p^{n+3}+2p^2-2}(BP\langle 2\rangle; BP\langle 1\rangle)/(\lambda_1)$ are cyclic for $n \geq 1$.*

Proof. **[Gabe: Fix this proof]**

We will determine that there is exactly one v_0-tower in degrees $2p^{n+3} - 2p^2$ and $2p^{n+2} + 2p^2 - 2p$ for $n \geq 1$ and therefore after resolving additive extensions the groups are cyclic in these degrees. We will also determine there are elements in the columns $2p^{n+3}-2$, and $2p^{n+3}$ for $n \geq 0$ by a degree argument.

By Theorem 4.11, the even generators (of $\overline{THH}_*(BP\langle 2\rangle; k(1))/(\lambda_1)$ as a $P(v_1)$-module) are the elements of the form $z'_{n,m}$. The degrees of these generators are

$$|z'_{n,m}| \quad = \quad |\lambda'_n| + |\lambda'_{n+1}| + |\mu_3^{mp^{n-2}}|$$

where by a simple induction

$$|\lambda'_n| + |\lambda'_{n+1}| = 2p^{n+1} + 2p^2 - 2$$

and $|\mu_3^{mp^{n-2}}| = 2mp^{n+1}$. Also, recall that the v_1-tower on $z'_{n,m}$ is truncated at $r(n-1)'$

We now consider elements of the form $z'_{n,m}v_1^k$, which is in degree

$$|z'_{n,m}v_1^k| = 2p^{n+1}(1+m) + (2p-2)(p+1+k).$$

We want to find triples of integers (j,k,m) such that

$$2p^{n+3} - 2p^2 = 2p^{j+1}(1+m) + (2p-2)(p+1+k)$$

$j \geq 2$ and $0 \geq k < r'(n-1)$. Since $2p-2 \mid 2p^{n+3}-2p^2$, we observe that $2p-2 \mid 2p^{j+1}(1+m)+(2p-2)(p+1+k)$ and consequently, there exists an integer m' such that $m+1 = m'(p-1)$ and

$$p^{n+1} + p^n + \ldots p^2 = p^{j+1}m' + p + 1 + k.$$

This implies that

$$k \geq p^{j+1} + p^j + \ldots + p^2 - p - 1$$

so by a simple induction $k > r(j-1)$ when $j \geq 3$. When $j = 2$, there is a unique solution with $j = p^{n-1} + p^{n-2} + \ldots + 1$ and $k = p^2 - p - 1$. Thus, the element $z'_{2,p^n-2}v_1^{p^2-p-1}$ is the only generator in degree $2p^{n+3} - 2p^2$ and it generates a v_0-tower, proving the first assertion.

We then observe that $|v_1^{p+1}| = 2p^2 - 2$ and $z'_{2,p^n-2}v_1^{p^2-p-1} \cdot v_1^{p+1} = z_{2,p^n-2}v_1^{r'(1)} = 0$. Also, the next smallest even class is $|z_{2,p^n-1}|$ which are is in degree $2p^3(1+p^n-1) + 2p^2 - 2 = 2p^{n+3} + 2p^2 - 2$, which is strictly greater that $2p^{n+3} - 2$. This proves the first part of the second claim. The second part of the second claim follows for the same reason.

The last claim follows by a similar argument to the first claim. As already observed, the element z_{2,p^n-1} is in degree $2p^{n+3} + 2p^2 - 2$ and this is the only generator in this degree. Thus there is a single v_0-tower in the column corresponding to this degree. $\square$

6.2. **Differentials.** We now prove the first differential in the spectral sequence

$$THH(BP\langle 2\rangle, H\mathbb{Z}_{(p)})_p[v_1] \Rightarrow THH(BP\langle 2\rangle, BP\langle 1\rangle)_p. \tag{6.4}$$

Lemma 6.5. *The first nontrivial differential in spectral sequence 6.4 is a differential*

$$d_4(a_2) \doteq v_1^4 b_1^{(1)}.$$

Proof. We will prove that there is a differential $d_4(a_2)\doteq v_1^4 b_1^{(1)}$ and along the way we will show that there are no previous differentials in spectral sequence (6.4). There are three possible elements that could be possible targets of a differential on a_2

$$\{v_1^4 b_1^{(1)}, v_1^6 a_1^{(2)}, v_1^{10}\lambda_1\lambda_2\}$$

so we need to rule out the other two possibilities. To do this, we need to rule out all possible differentials in lower degree. There is clearly no nontrivial differential on $\lambda_1\lambda_2$, by the Leibniz rule. We observe that $v_1^4 z_{2,0}$ is zero in the companion spectral sequence

$$THH(BP\langle 2\rangle, k(1))_p[v_0] \Rightarrow THH(BP\langle 2\rangle, BP\langle 1\rangle)_p. \tag{6.6}$$

There are no possible differentials in spectral sequence (6.4) that could hit $v_1^4\lambda_2$ and consequently, there must be a hidden extension $v_1 \cdot v_1^3 z_{2,0} = v_0 \cdot z_{3,0}$ in the spectral sequence (6.6). This also forces the hidden extension $v_1 \cdot v_1^3\lambda_2 = 2 \cdot a_1$ in the other spectral sequence. Additionally, this forces a hidden extension $v_1 \cdot v_1^3\lambda_1 z_{2,0} = v_0 \cdot \lambda_1 z_{3,0}$ and $v_1 \cdot v_1^3\lambda_1\lambda_2 = 2 \cdot \lambda_1 a_1$. By examination of the spectral sequence (6.4), there is no possible differential hitting $v_1^4\lambda_1 z_{2,0}$ so this is a nontrivial permanent cycle. Consequenetly, $v_1^4\lambda_1 z_{2,0}$ is also a nontrivial permanent cycle. The remaining differentials in the spectral sequence (6.4) in degrees less than 31 can be ruled out by the Leibniz rule. We also observe, by comparing p-torsion in the two spectral sequences that there are differentials

$$d_1(z_{4,0}) = 2z'_{2,0}$$

and

$$d_1(\lambda_1 z_{4,0}) = 2\lambda_1 z'_{2,0}$$

in spectral sequence (6.6). There are no further differentials in either spectral sequence in the range less than 31. We would now like to rule out the possible differential on a_2 hitting $v_1^{10}\lambda_1\lambda_2$. If we did have this differential, then we would also have to have a differential killing $v_0\lambda_1 z_{3,0} v_1^6$ in spectral sequence (6.6), since $v_0 \cdot \lambda_1 z_{3,0} v_1^6 = v_1 \cdot v_1^9 \lambda_1 z_{2,0}$. The only possibilities are

$$d_2(z_{4,0}v_1^4) = \alpha v_0\lambda_1 z_{3,0} v_1^6,$$
$$d_3(\lambda_1 z'_{2,0})v_1^3) = \beta v_0 \lambda_1 z_{3,0} v_1^6.$$

We observe that $2v_1^8 z_{3,0} = v_1^{12} z_{2,0}$ is zero in spectral sequence (6.4) and $v_1^{12}\lambda_2$ survives spectral sequence (6.4) by the Leibniz rule and a bidegree argument. We conclude that there must by a hidden extension

$$v_1 \cdot v_1^7 v_0 z_{3,0} = v_0^2 \cdot v_1^4 z_{4,0}.$$

This implies that $\alpha = 0$ above. We also know $\lambda_1 z_{2,0}$ corresponds to $\lambda_1\lambda_2$ in spectral sequence (6.6), so by the Leibniz rule in that spectral sequence there is no differential on this element. This implies that there cannot be a nontrivial differential on $\lambda_1 z_{2,0}$ as well, so $\beta = 0$. Thus, $v_1^{10}\lambda_1\lambda_2$ survives and there is not the boundary of a differential on a_2.

We then consider $v_1^6 a_1^{(2)}$ in spectral sequence (6.6). This element corresponds to $v_1^6\lambda_1 z_{3,0}$ in spectral sequence (6.4). In that spectral sequence, the only possible elements hitting it are $\lambda_1 z_{3,0} v_1^5$ and $v_1^3 z'_{2,0}$, since we already determined the differential $d_1(v_1\lambda_1 z_{4,0}) = v_0\lambda_1 z'_{2,0}$. We can rule each of these out because these classes must survive the other spectral sequence. Therefore, $v_1^6\lambda_1 z_{3,0}$ must survive and consequently $v_1^6 a_1^{(2)}$ is not a boundary. Therefore, the only remaining possibility is that $d_4(a_2) = v_1^4 b_1^{(1)}$. □

Remark 6.7. *In Angeltveit-Hill-Lawson [1], the argument for a differential is simpler because there are vanishing columns in one spectral sequence that imply differentials in the other. The presence of λ_1-divisible classes makes this more complicated. In order to argue for a differential in degree $2p^4-2$, we needed to know about all differentials and hidden extensions in lower degrees. For each differential, we will therefore inductively compute all information in the range $2np^3-1 \leq t \leq 2(n+1)p^3-1$ for $n \geq 1$ in order to compute the next differential. This forces us to combine arguments about hidden extensions with arguments about differentials. This is a bit delicate since hidden extensions occur after the E_∞-page whereas differentials occur on some fixed page.*

Lemma 6.8. *The next nontrivial differential, besides those implied by the Leibniz rule and the differential of Lemma 6.5 in spectral sequence 6.4 is a differential*

$$d_4(a_3^{(1)}) \doteq 2v_1^4 b_2^{(1)}.$$

Proof. To compute this differential, we also need to compute the spectral sequence in the range $4p^3-1 \leq t \leq 6p^3-1$. The Leibniz rule implies that $d_4(a_2^{(2)}) = v_1^4 b_1^{(2)}$ and $d_4(b_2^{(k)}) = 0$ for $k = 1,2$. We claim that $b_2^{(k)}$ is a permanent cycle for $k = 1,2$. There are possible differentials

$$d_{11}(b_2^{(1)}) = \alpha v_1^{11} a_1,$$

$$d_{15}(b_2^{(1)}) = \beta v_1^{15} \lambda_2,$$

$$d_{15}(b_2^{(1)}) = \gamma v_1^{17} \lambda_1.$$

Since λ_1 and λ_2 are torsion free and $b_2^{(1)}$ is p-torsion, we know that $\beta = \gamma = 0$. Suffices to show $\alpha = 0$. □

[Gabe: Here is my guess about the differential pattern:

$$d_{r(2n+1)}(a^{(j)}_{kp^{n-1}} p^{n-1}) = (k-1) v_1^{r'(2n+1)} b^{(j)}_{kp^{n-1}-1}$$

for $j = 1,2$.]

[Gabe: As of July 30th, we now believe this to be wrong. Perhaps instead the differential pattern is something like

$$d_{\ell(n)}(a^{(j)}_{kp^{n-1}} p^{n-1}) = (k-1) v_1^{r^{\ell(n)}(2n+1)} a^{(j)}_{kp^{n-1}-1}$$

$$d_{\ell(n)}(b^{(j)}_{kp^{n-1}} p^{n-1}) = (k-1) v_1^{\ell(n)} b^{(j)}_{kp^{n-1}-1}$$

]

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