TOPOLOGICAL HOCHSCHILD HOMOLOGY OF THE SECOND TRUNCATED BROWN-PETERSON SPECTRUM II

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ABSTRACT. We compute mod p topological Hochschild homology of the second truncated Brown-Peterson spectrum $BP\langle 2\rangle$ at the prime 3 where $BP\langle 2\rangle$ is constructed using Shimura curves of small discriminant by Hill-Lawson [9]. This is as a step towards a program of studying potential chromatic height three information in mod p algebraic K-theory of $BP\langle 2\rangle$.

CONTENTS

1. INTRODUCTION

The Brown-Peterson spectrum BP is a complex oriented cohomology theory and it is associated to the universal p-typical formal group law. The cohomology of the associated Hopf algebroid (BP_*, BP_*BP) is the input for an Adams spectral sequence computing π_*S_p. This spectral sequence led to significant new computations of the homotopy groups of spheres and a better understanding of periodic phenomena in the homotopy groups of spheres. The coefficients of BP are a polynomial algebra over $\mathbb{Z}_{(p)}$ on generators v_i for $i \geq 1$. By coning off the regular sequence $(v_{n+1}, v_{n+2}, \ldots)$, one can construct the n-th truncated Brown-Peterson spectrum $BP\langle n\rangle$ where $BP\langle 0\rangle = H\mathbb{F}_p$, $BP\langle 0\rangle = H\mathbb{Z}_{(p)}$, and $BP\langle 1\rangle$ is the Adams summand ℓ of p-local complex K-theory

$ku_{(p)}$. Until the last ten years, no analogous interpretation of $BP\langle 2\rangle$ was known, but then in [11] Lawson-Nauman showed that there is an E_∞-model for $BP\langle 2\rangle$ at the prime 2 using topological modular forms with level structure. More recently in [9], Hill-Lawson also give an E_∞-model for $BP\langle 2\rangle$ at the prime 3 using spectra associated to Shimura curves of small discriminant. This is especially interesting in view of recent groundbreaking work of [10], where Lawson proves that at the prime 2 no such E_∞-model for $BP\langle n\rangle$ exists for $n \geq 4$, which was extended to odd primes in [15].

[Gabe: At the moment this introduction is here to fill space. We should discuss how we want to pitch our results.]

We compute mod p topological Hochschild homology of $BP\langle 2\rangle$ at the primes 2 and 3 where, by work of [9, 11], it is known that an E_∞-ring spectrum model for $BP\langle 2\rangle$ exists.

1.1. **Outline of the strategy.** Beginning with a calculation of

$$\mathrm{THH}_*(BP\langle 2\rangle; \mathbb{F}_3)$$

we compute the square of Bockstein spectral sequences

$$\begin{tikzcd}
THH_*(BP\langle 2\rangle;\mathbb{F}_3)[v_1,v_2] \arrow[r, Rightarrow] \arrow[d, Rightarrow] & THH_*(BP\langle 2\rangle;k(2))[v_1] \arrow[d, Rightarrow] \\
THH_*(BP\langle 2\rangle;k(1))[v_2] \arrow[r, Rightarrow] & THH_*(BP\langle 2\rangle;BP\langle 2\rangle/3).
\end{tikzcd}$$

and compare to the THH-May spectral sequence

$$S/3_*THH_*(H\pi_*BP\langle 2\rangle; H\pi_*BP\langle 2\rangle) \Rightarrow THH_*(BP\langle 2\rangle; BP\langle 2\rangle/3$$

which packages the the diagonal of the square above into a single spectral sequence (in a sense we make precise). We view the THH-May spectral sequence in this setting as an organizational tool that allows us to rule out certain possible differentials and detect hidden multiplicative extensions in the Bockstein spectral sequences rather than a replacement for the square of Bockstein spectral sequences itself.

We begin by computing the Bockstein spectral sequences

$$\mathrm{THH}_*(BP\langle 2\rangle; \mathbb{F}_3)[v_1] \implies \mathrm{THH}_*(BP\langle 2\rangle; k(1)) \tag{1.1}$$

$$\mathrm{THH}_*(BP\langle 2\rangle; \mathbb{F}_3)[v_2] \implies \mathrm{THH}_*(BP\langle 2\rangle; k(2)). \tag{1.2}$$

These can be identified with multiplicative Adams spectral spectral sequences

$$Ext^*_{\mathscr{A}_*}(\mathbb{F}_p; H_*THH(BP\langle 2\rangle; k(1)) \implies \mathrm{THH}_*(BP\langle 2\rangle; k(1)) \tag{1.3}$$

$$Ext^*_{\mathscr{A}_*}(\mathbb{F}_p; H_*THH(BP\langle 2\rangle; k(2))) \implies \mathrm{THH}_*(BP\langle 2\rangle; k(2)) \tag{1.4}$$

since $H_*THH(BP\langle 2\rangle)$ is free over $H_*BP\langle 2\rangle$ and therefore the input becomes

$$Ext^*_{E(\tau_i)}(\mathbb{F}_p; E(\lambda_1,\lambda_2,\lambda_3)\otimes P(\mu_2))$$

for $i = 1, 2$ and since τ_i coacts trivially on $E(\lambda_1,\lambda_2,\lambda_3)\otimes P(\mu_2)$ and

$$Ext^*_{E(\tau_i)}(\mathbb{F}_p, \mathbb{F}_p) \cong P(v_i)$$

these each are identified with the input of the Bockstein spectral sequences (1.1) and (1.2) respecively. Each of these two spectral sequences contains a similar level of complexity to the Bockstein spectral sequence of [12] and our proofs are inspired by those of McClure-Staffeldt.

We will then compute the Bockstein spectral sequences

$$\mathrm{THH}_*(BP\langle 2\rangle; k(1))[v_2] \implies \mathrm{THH}_*(BP\langle 2\rangle; BP\langle 2\rangle/3) \tag{1.5}$$

$$\mathrm{THH}_*(BP\langle 2\rangle; k(2))[v_2] \implies \mathrm{THH}_*(BP\langle 2\rangle; BP\langle 2\rangle/3). \tag{1.6}$$

This second part of the story is the analogue of the work of Angeltveit-Hill-Lawson [3] on $\mathrm{THH}(BP\langle 1\rangle)$. In Angeltveit-Hill-Lawson [3] they use certain elementary number theory lemmas about divisibility by a prime. In our case, this is replaced by divisibility by v_1 which shifts topological degree making the computation a bit less tractable. This is overcome by combining this approach with another approach.

In parallel, we also compute the THH-May spectral sequences

$$V(0)_{*,*}\,\mathrm{THH}(H\pi_*BP\langle 2\rangle; H\pi_*BP\langle 1\rangle) \Rightarrow \mathrm{THH}_*(BP\langle 2\rangle; k(1)) \tag{1.7}$$

$$V(1)_{*,*}\,\mathrm{THH}(H\pi_*BP\langle 2\rangle) \Rightarrow \mathrm{THH}(BP\langle 2\rangle; k(2)) \tag{1.8}$$

$$V(0)_{*,*}\,\mathrm{THH}(H\pi_*BP\langle 2\rangle; H\pi_*BP\langle 2\rangle) \Rightarrow \mathrm{THH}_*(BP\langle 2\rangle; BP\langle 2\rangle). \tag{1.9}$$

The spectral sequences (1.7) and (1.9) are multiplicative because $BP\langle 2\rangle$ is a commutative ring spectrum, $BP\langle 1\rangle$ is a commutative $BP\langle 2\rangle$-algebra, and $V(0)$ is a ring spectrum. The spectral sequence (1.8) is not known to be multiplicative, but the output is the same as the Bockstein spectral sequence (1.2) and we simply argue that at some page the two spectral sequences can be identified. This allows us to import Bockstein spectral sequence differentials into the THH-May spectral sequence.

Conventions. Let $p = 3$ throughout. We will write $H_*(-)$ for homology with $\mathbb{F}_p$ coefficients, or in other words, the functor $\pi_*(H\mathbb{F}_p \wedge -)$. We write $\doteq$ to mean that an equality holds up to multiplication by a unit. We will write $BP\langle n\rangle$ for the n-th truncated Brown-Peterson spectrum. In particular, $BP\langle 1\rangle$ denotes the E_∞-ring spectrum model for the connective Adams summand [12]. Also, $BP\langle 2\rangle$ will denote the E_∞-model for the second truncated Brown-Peterson spectrum constructed in [9] at $p = 3$. We also note once and for all that $BP\langle 1\rangle$ can be constructed as a commutative $BP\langle 2\rangle$-algbra by quotienting by the regular element v_2 in positive degree by [8]. We let $k(n)$ denote an A_∞-ring spectrum model for the connective cover of the Morava K-theory spectrum $K(n)$.

When not otherwise specified, tensor products will be taken over $\mathbb{F}_p$ and $HH_*(A)$ denotes the Hochschild homology of a graded $\mathbb{F}_p$-algebra relative to $\mathbb{F}_p$. We will let $P(x)$, $E(x)$ and $\Gamma(x)$ denote a polynomial algebra, exterior algebra, and divided power algebra over $\mathbb{F}_p$ on a generator x.

The dual Steenrod algebra will be denoted $\mathscr{A}_*$ with coproduct $\Delta\colon \mathscr{A}_* \to \mathscr{A}_* \otimes \mathscr{A}_*$. Given a (right) $\mathscr{A}_*$-comodule M, its coaction will be denoted $\nu\colon \mathscr{A} \to \mathscr{A} \otimes M$ where the comodule M is understood from the context.

2. Preliminary results

Recall that at a prime $p \geq 3$ there is an isomorphism

$$H_*(BP\langle 2\rangle) \cong P(\bar{\xi}_1, \bar{\xi}_2, \ldots) \otimes E(\bar{\tau}_3, \bar{\tau}_4, \ldots) \subset \mathscr{A}_*$$

of $\mathscr{A}_*$-comodules. By [4, Thm. 5.12], there is an isomorphism

$$H_*(\mathrm{THH}(BP\langle 2\rangle)) \cong H_*BP\langle 2\rangle \otimes E(\sigma\bar{\xi}_1, \sigma\bar{\xi}_2, \sigma\bar{\xi}_3) \otimes P(\sigma\bar{\tau}_3) \tag{2.1}$$

of primitively generated $H_*BP\langle 2\rangle$-Hopf algebras when $p = 3$. The $\mathscr{A}_*$-comodule coaction on $H_*\overline{\mathrm{THH}}(BP\langle 2\rangle)$ was also computed in [4, Thm. 5.12]. It is determined by the $\mathscr{A}_*$-coaction on $H_*BP\langle 2\rangle$ and the formulas

$$\nu(\sigma\bar{\tau}_m) = 1 \otimes \sigma\bar{\tau}_m + \bar{\tau}_0 \otimes \sigma\bar{\xi}_m \tag{2.2}$$

$$\nu(\sigma\bar{\xi}_i) = 1 \otimes \sigma\bar{\xi}_i \tag{2.3}$$

for $i = 1, 2, 3$. These formulas follow from the general formula

$$\nu \circ \sigma = (1 \otimes \sigma) \circ \nu \tag{2.4}$$

in [4, Eq. 5.11] and the $\mathscr{A}_*$-coaction on $H_*BP\langle 2\rangle$.

2.1. THH of $BP\langle 2\rangle$ modulo (p, v_1, v_2).

[Gabe: Adjust this based on other paper.]

We now compute

$$\mathrm{THH}_*(BP\langle 2\rangle; H\mathbb{F}_p).$$

By [4, Lem. 4.1], it suffices to compute the sub-algebra of comodule primitives in $H_*(\mathrm{THH}(BP\langle 2\rangle; H\mathbb{F}_p))$ since $\mathrm{THH}(BP\langle 2\rangle; H\mathbb{F}_p)$ is an $H\mathbb{F}_p$-algebra. Since $BP\langle 2\rangle$ and $H\mathbb{F}_p$ are commutative there is a weak equivalence of commutative ring spectra

$$\mathrm{THH}(BP\langle 2\rangle; H\mathbb{F}_p) \simeq \mathrm{THH}(BP\langle 2\rangle) \wedge_{BP\langle 2\rangle} H\mathbb{F}_p.$$

Since $H_*(\mathrm{THH}(BP\langle 2\rangle))$ is free over $H_*BP\langle 2\rangle$ by (2.1), the Eilenberg-Moore spectral sequence and [4, Cor. 5.13] immediately implies

$$H_*(\mathrm{THH}(BP\langle 2\rangle; H\mathbb{F}_p)) \cong \mathscr{A}_* \otimes E(\sigma\bar{\xi}_1, \sigma\bar{\xi}_2, \sigma\bar{\xi}_3) \otimes P(\sigma\bar{\tau}_3) \tag{2.5}$$

The $\mathscr{A}_*$ coaction on elements in $\mathscr{A}_*$ is given by the coproduct and the remaining coactions are determined by the formula (2.4) and are therefore the same as in (2.2). We write $\lambda_i = \sigma\bar{\xi}_i$ and define $\mu_3 := \sigma\bar{\tau}_3 - \bar{\tau}_0\sigma\bar{\xi}_3$ Then it is clear that the algebra of comodule primitives in $H_*(\mathrm{THH}(BP\langle 2\rangle; H\mathbb{F}_p))$ is generated by μ_3 and λ_i for $1 \leq i \leq 3$. We therefore produce the following isomorphism of graded $\mathbb{F}_p$-algebras

$$\mathrm{THH}_*(BP\langle 2\rangle; H\mathbb{F}_p) \cong E(\lambda_1, \lambda_2, \lambda_3) \otimes P(\mu_3). \tag{2.6}$$

The degrees are $|\lambda_i| = 2p^i - 1$ for $1 \leq i \leq 3$ and $|\mu_3| = 2p^3$.

3. Computation of $THH(BP\langle 2\rangle; k(2))$

The goal of this section is to compute the homotopy groups of $\mathrm{THH}(BP\langle 2\rangle; k(2))$. We achieve this through an analysis of the v_2-Bockstein spectral sequence (1.2). We first outline our strategy.

In [12], McClure-Staffeldt they compute $\mathrm{THH}_*(\ell, k(1))$ by first arguing that upon inverting v_1, there is an isomorphism

$$v_1^{-1}\,\mathrm{THH}_*(\ell; k(1)) \cong K(1)_* \tag{3.1}$$

when $p \geq 3$. This implies that in the abutment of the v_1-Bockstein spectral sequence

$$\mathrm{THH}_*(\ell; \mathbb{F}_p)[v_1] \implies \mathrm{THH}_*(\ell; k(1))$$

all classes are v_1-torsion besides the powers of v_1. It turns out that there is only one of pattern of differentials that makes this possible, which gives a complete description of this spectral sequence.

In this section we will use a similar method to compute $THH(BP\langle 2\rangle; k(2))$. In particular, we will prove that in the abutment of the v_2-Bockstein spectral sequence

$$\mathrm{THH}_*(BP\langle 2\rangle; \mathbb{F}_p)[v_2] \implies \mathrm{THH}_*(BP\langle 2\rangle; k(2))$$

all elements in the abutment are v_2-torsion except the powers of v_2. Generalizing naively, we would replace S/p with $V(1)$, however, at $p = 2$ the spectrum $V(1)$ does not exist and at $p = 3$ the spectrum $V(1)$ is not a ring spectrum. This potential issue can be easily avoided using the fact that

$$S/p_* THH(\ell) \cong THH(\ell; k(1))$$

and in our case we therefore compute $THH_*(BP\langle 2\rangle; k(2))$.

Since $k(2)$ has a v_2-self map, there is an induced v_2-self map of $THH(BP\langle 2\rangle; k(2)$ and we define $v_2^{-1} THH(BP\langle 2\rangle; k(2))$ to be the colimit of iterations of this self-map. It is therefore clear that

$$v_2^{-1}\,\mathrm{THH}(BP\langle 2\rangle; k(2)) \simeq \mathrm{THH}(BP\langle 2\rangle; K(2)).$$

and there is a canonical unit morphism

$$K(2) \to \mathrm{THH}(BP\langle 2\rangle; K(2)).$$

We will therefore argue that this unit map is a $K(2)$-equivalence. Since the source and target are both $K(2)$-modules, and hence $K(2)$-local, this will show that the map is in fact an equivalence of spectra.

To establish this, we just need to argue that the unit map induces an isomorphism

$$K(2)_* K(2) \cong K(2)_*\,\mathrm{THH}(BP\langle 2\rangle; K(2)).$$

Note that

$$\mathrm{THH}(BP\langle 2\rangle; K(2)) \simeq K(2) \wedge_{BP\langle 2\rangle \wedge BP\langle 2\rangle} BP\langle 2\rangle$$

so their is an Eilenberg-Moore spectral sequence

$$\mathrm{Tor}_{s,t}^{(K(2)_* BP\langle 2\rangle)^e}(K(2)_* K(2), K(2)_* BP\langle 2\rangle) \implies K(2)_{s+t}(\mathrm{THH}(BP\langle 2\rangle; K(2))). \tag{3.2}$$

where

$$(K(2)_* BP\langle 2\rangle)^e \cong (K(2)_* BP\langle 2\rangle) \wedge_{K(2)_*} (K(2)_* BP\langle 2\rangle).$$

Here we are using the fact that $K(2)_*BP\langle 2\rangle$ is flat over $K(2)_*$, which follows because all $K(2)$-modules are free.

[Dom: Convergence issues in E-M spectral sequence?]

3.1. **The $K(2)$-homology of $\mathrm{THH}(BP\langle 2\rangle; K(2))$.** To begin, we need to compute $K(2)_*BP\langle 2\rangle$. Since the Johnson-Wilson theory $E(2)$ is Landweber exact, one has

$$E(2)_*BP\langle 2\rangle \cong E(2)_* \otimes_{BP_*} BP_*BP \otimes_{BP_*} BP\langle 2\rangle_*.$$

It is known that

$$BP_*BP \otimes_{BP_*} BP\langle 2\rangle_* \cong BP_*[t_1, t_2, \ldots]/(\eta_R(v_i) \mid i \geq 3)$$

where $\eta_R : BP_* \to BP_*BP$ denotes the right unit. Thus,

$$E(2)_*BP\langle 2\rangle \cong E(2)_*[t_i \mid i \geq 1]/(\eta_R(v_i) \mid i \geq 3).$$

Since $K(2)$ is obtained from $E(2)$ by coning off p and v_1, we find that

$$K(2)_*BP\langle 2\rangle \cong K(2)_*[t_i \mid i \geq 1]/(\eta_R(v_i) \mid i \geq 3).$$

We have the following congruences

$$\eta_R(v_{2+k}) \equiv v_2 t_k^{p^2} - v_2^{p^k} t_k \mod (\eta_R(v_3), \ldots, \eta_R(v_{k+1})).$$

in $K(2)_*BP$ for all $k \geq 1$ (cf. formula 6.1.13 of [14]). Thus, the following lemma follows.

Lemma 3.3. *There is an isomorphism of graded rings*

$$K(2)_*BP\langle 2\rangle \cong K(2)_*[t_1, t_2, \ldots]/(v_2 t_k^{p^2} - v_2^{p^k} t_k \mid k \geq 1) \tag{3.4}$$

We proceed by analyzing the Eilenberg-Moore spectral sequence (3.2). First, we note that the E^2-page is

$$E^2_{*,*} \cong \mathrm{HH}_*^{K(2)_*}(K(2)_*BP\langle 2\rangle; K(2)_*K(2)).$$

We compute this E^2-page in the following proposition.

Theorem 3.5. *The unit map induces an isomorphism of $K(2)_*$-modules*

$$K(2)_*K(2) \cong \mathrm{HH}_*^{K(2)_*}(K(2)_*BP\langle 2\rangle; K(2)_*K(2)).$$

Proof. First, we note that the input of (3.2) is isomorphic to

$$K(2)_*K(2) \otimes_{K(2)_*} \mathrm{Tor}^{(K(2)_*BP\langle 2\rangle)}(K(2)_*; K(2)_*)$$

by [6] and the fact that $K(2)_*K(2)$ is a symmetric $K(2)_*BP\langle 2\rangle$-module. We will now argue that

$$\mathrm{Tor}^{(K(2)_*BP\langle 2\rangle)}(K(2)_*; K(2)_*) \cong K(2)_*.$$

Recall that the topological degree of t_k is $2(p^k - 1)$, and that the degree of v_2 is $2(p^2 - 1)$. Thus $|v_2|$ divides $|t_k|$ if and only if k is even. We observe that there is an isomorphism of $K(2)_*$-algebras

$$K(2)_*BP\langle 2\rangle \cong \bigotimes_k K(2)_*[t_k \mid k \geq 1]/(v_2 t_k^{p^2} - v_2^{p^k} t_k).$$

Let $u_k = v_2^{m(k)} t_k$ where
$$m(k) = -p^{k-2} - p^{k-4} - \cdots - p^2 - 1$$
when $2 \mid k$ and and let $u_k = v_2^{\ell(k)} t_k$ where
$$\ell(k) = -p^{k-2} - p^{k-4} - \cdots - p$$
when k is odd. Thus
$$|u_k| = \begin{cases} 0 & k \equiv 0 \mod 2 \\ 2(p-1) & k \equiv 1 \mod 2 \end{cases}.$$
Define A_n to be the subalgebra of $K(2)_*BP\langle 2\rangle$ generated by $t_1, \ldots, t_n$, and let $A(t_k)$ denote the subalgebra generated by t_k so that there is an isomorphism of $K(2)_*$-algebras
$$A_n \cong \bigotimes_{k=1}^{n} A(t_k).$$
Then there is an isomorphism of $K(2)_*$-algebras
$$\mathrm{Tor}_*^{A_n}(K(2)_*; K(2)_*) \cong \bigotimes_{k=1}^{n} \mathrm{Tor}_*^{A(t_k)}(K(2)_*; K(2)_*).$$
by the Künneth formula for Tor. Since the functor $\mathrm{Tor}_*^{(-)}(K(2)_*; K(2)_*)$ commutes with filtered colimits of $K(2)_*$-algebras and $K(2)_*BP\langle 2\rangle = \operatorname{colim} A_n$, it follows that there is an isomorphism of $K(2)_*$-algebras
$$\mathrm{Tor}^{K(2)_*BP\langle 2\rangle}(K(2)_*, K(2)_*) \cong \operatorname{colim} \mathrm{Tor}_*^{A_n}(K(2)_*; K(2)_*) \cong \bigotimes_{k=1}^{\infty} \mathrm{Tor}_*^{A(t_k)}(K(2)_*; K(2)_*)$$
Thus, it suffices to compute $\mathrm{Tor}_*^{A(t_k)}(K(2)_*, K(2)_*)$. When k is even, we have
$$A(t_k) = K(2)_* \otimes \mathbb{F}_p[u_k]/(u_k^{p^2} - u_k),$$
in which case there is an isomorphism of $K(2)_*$-algebras
$$\mathrm{Tor}^{A(t_k)}(K(2)_*, K(2)_*) \cong K(2)_* \otimes \mathrm{Tor}^{\mathbb{F}_p[u_k]/(u_k^{p^2} - u_k)}(\mathbb{F}_p, \mathbb{F}_p)$$
by the base-change formula for Tor. Since the $\mathbb{F}_p$-algebra
$$\mathbb{F}_p[u_k]/(u_k^{p^2} - u_k)$$
is étale over $\mathbb{F}_p$, it follows that
$$\mathrm{HH}^{\mathbb{F}_p}(\mathbb{F}_p[u_k]/(u_k^{p^2} - u_k)) \cong \mathbb{F}_p[u_k]/(u_k^{p^2} - u_k)$$
by [16] and since
$$\mathrm{HH}^{\mathbb{F}_p}(\mathbb{F}_p[u_k]/(u_k^{p^2} - u_k)) \cong \mathbb{F}_p[u_k]/(u_k^{p^2} - u_k) \otimes \mathrm{Tor}_*^{\mathbb{F}_p[u_k]/(u_k^{p^2} - u_k)}(\mathbb{F}_p, \mathbb{F}_p)$$
we see that
$$\mathrm{Tor}_*^{\mathbb{F}_p[u_k]/(u_k^{p^2} - u_k)}(\mathbb{F}_p, \mathbb{F}_p) \cong \mathbb{F}_p.$$

When k is odd, there is again an isomorphism

$$\mathrm{Tor}_*^{\mathbb{F}_p[u_k]/(u_k^{p^2}-u_k)}(\mathbb{F}_p,\mathbb{F}_p)\cong \mathbb{F}_p$$

because $\mathbb{F}_p[u_k]/(u_k^{p^2}-u_k)$ is also étale over $\mathbb{F}_p$.

Consequently, there is an isomorphism of $K(2)_*$-algebras

$$\mathrm{Tor}_*^{K(2)_*BP\langle 2\rangle}(K(2)_*,K(2)_*)\cong K(2)_*$$

completing the proof. □

The following corollary is immediate.

Corollary 3.6. *The map*

$$K(2)_*K(2)\to K(2)_*\,\mathrm{THH}(BP\langle 2\rangle;K(2))$$

induced by the unit map

$$\eta\colon\ K(2)\to \mathrm{THH}(BP\langle 2\rangle;K(2))$$

is an isomorphism. Since $\mathrm{THH}(BP\langle 2\rangle;K(2))$ *is a free* $K(2)$*-module and hence* $K(2)$*-local, it follows that the unit morphism* η *is a weak equivalence.*

3.2. **Differentials in the v_2-BSS.** We now turn to analyzing the v_2-BSS (1.2). In particular, we will argue that Corollary 3.6 implies a unique pattern of differentials in the spectral sequence. We adapt the proof of [12] to our setting.

Recall that the E_2-term of the v_2-BSS is

$$\mathrm{THH}_*(BP\langle 2\rangle;\mathbb{F}_p)[v_2]\cong E(\lambda_1,\lambda_2,\lambda_3)\otimes P(\mu_3)\otimes P(v_2),$$

where $|\lambda_i|=(2p^i-1,0)$ and $|\mu_3|=(2p^3,0)$. It will be more convenient to work in the v_2-localized Bockstein spectral sequence. Since the elements λ_i are in odd total degree and 1 is v_2-torsion free, they cannot support differentials. If μ_3 were a infinite cycle as well, then by multiplicativity of the Bockstein spectral sequence, it would follow that the spectral sequence collapses at the E_1-page. However, this would contradict Corollary 3.6. Therefore, μ_3 supports a differential, the only possible differential for bi-degree reasons is

$$d_p(\mu_3)\doteq v_2^p\lambda_1.$$

Thus,

$$v_2^{-1}E_{p+1}^{*,*}\cong K(2)_*\otimes E(\lambda_2,\lambda_3,\lambda_4)\otimes P(\mu_3^p),$$

where $\lambda_4:=\lambda_1\mu_3^{p-1}$. Note that the bidegree of λ_4 is

$$|\lambda_4|=(2p^4-2p^3+2p-1,0).$$

In particular, its total degree is odd. So this class cannot support a differential which truncates the the v_2-tower on λ_2 or λ_3. So this class is an infinite cycle. By multiplicativity again, if μ_3^p were an infinite cycle, then the spectral sequence would collapse at E_{p+1}, which would contradict Corollary 3.6. So μ_3^p supports a differential. The only possibility is

$$d_{p^2}(\mu_3^p)\doteq v_2^{p^2}\lambda_2.$$

Thus, there is an isomorphism

$$v_2^{-1}E_{p^2+1}^{*,*} \cong K(2)_* \otimes E(\lambda_3, \lambda_4, \lambda_5) \otimes P(\mu_3^{p^2})$$

where

$$\lambda_5 := \lambda_2\mu_3^{p^2-p}.$$

The bidegree of this class is

$$|\lambda_5| = (2p^5 - 2p^4 + 2p^2 - 1, 0).$$

Since $\lambda_3, \lambda_4, \lambda_5$ all have odd total degree, they are necessarily infinite cycles. As before, the class $\mu_3^{p^2}$ must support a differential. The only possibility is

$$d_{p^3}(\mu_3^{p^2}) \doteq v_2^{p^3}\lambda_3.$$

This shows that

$$v_2^{-1}E_{p^3+1}^{*,*} \cong K(2)_* \otimes E(\lambda_4, \lambda_5, \lambda_6) \otimes P(\mu_3^{p^4})$$

where

$$\lambda_6 := \lambda_3\mu_3^{p^2(p-1)} = \lambda_3\mu_3^{p^3-p^2},$$

so that the bidegree of λ_6 is

$$|\lambda_6| = (2p^6 - 2p^5 + 2p^3 - 1, 0).$$

Consequently, as we saw before, the class λ_6 cannot support a differential, and hence is an infinite cycle. As before, the class $\mu_3^{p^3}$ must support a differential. An elementary calculation shows the only possibility is

$$d_{p^4+p}\mu^{p^3} \doteq v_2^{p^4+p}\lambda_4$$

Recursively define a function $d(n)$ by

$$d(n) := \begin{cases} 2p^n - 1 & \text{if } 1 \le n \le 3 \\ 2p^3(p^{n-3} - p^{n-4}) + d(n-3) & \text{if } n > 3 \end{cases}$$

and recursively define classes λ_n by

$$\lambda_n := \begin{cases} \lambda_n & 1 \le n \le 3 \\ \lambda_{n-3}\mu^{p^{n-4}(p-1)} & n > 3 \end{cases}.$$

Then a simple inductive argument shows that the bidegree of λ_n is given by

$$|\lambda_n| = (d(n), 0). \tag{3.7}$$

Notice that $d(n)$ is always odd, and so λ_n is always in odd total degree. A simple induction shows that

$$d(n) = \begin{cases} 2p^n - 2p^{n-1} + 2p^{n-2} - 2p^{n-3} + \cdots + 2p - 1 & n \equiv 1 \mod 3 \\ 2p^n - 2p^{n-1} + 2p^{n-2} - 2p^{n-3} + \cdots + 2p^2 - 1 & n \equiv 2 \mod 3 \\ 2p^n - 2p^{n-1} + 2p^{n-2} - 2p^{n-3} + \cdots + 2p^3 - 1 & n \equiv 0 \mod 3. \end{cases}$$

Lemma 3.8. *The integer $2p^{n+2} - d(n) - 1$ is divisible by $|v_2|$.*

Proof. We proceed by induction. The base case easily holds because $2p^3-2p$ is divisible by $2p^2-2$. Since

$$(2p^{n+3}-1)-d(n+1)=(2p^2-2)p^n+(2p^n-d(n-2)-1),$$

the induction hypothesis implies $(2p^{n+3}-1)-d(n+1)$ is divisible by $2p^2-2$. □

Now let $r(n)$ be the function given by

$$r(n):=|v_2|^{-1}(2p^{n+2}-d(n)-1).$$

Then we obtain as a corollary to the lemma,

Corollary 3.9. *The function $r(n)$ is given by*

$$r(n)=\begin{cases} p^n+p^{n-3}+\cdots+p^4+p & n\equiv 1 \mod 3 \\ p^n+p^{n-3}+\cdots+p^5+p^2 & n\equiv 2 \mod 3 \\ p^n+p^{n-3}+\cdots+p^6+p^3 & n\equiv 0 \mod 3. \end{cases}$$

We are now in a position to determine the differentials in the spectral sequence.

Theorem 3.10. *In the v_2-BSS, one has*

(1) *The only nonzero differentials are in $v_2^{-1}E_{r(n)}$.*
(2) *The page $v_2^{-1}E_{r(n)}$ is given by*

$$v_2^{-1}E_{r(n)}\cong K(2)_*\otimes E(\lambda_n,\lambda_{n+1},\lambda_{n+2})\otimes P(\mu_3^{p^{n-1}}).$$

Moreover, $\lambda_n,\lambda_{n+1},\lambda_{n+2}$ are infinite cycles.
(3) *The differential $d_{r(n)}$ is determined by the multiplicativity of the BSS and*

$$d_{r(n)}\mu_3^{p^{n-1}}=v_2^{r(n)}\lambda_n.$$

Proof. We proceed by induction, having already shown the theorem for $n\le 4$. Assume inductively that

$$v_2^{-1}E_{r(n)}^{*,*}\cong K(2)_*\otimes E(\lambda_n,\lambda_{n+1},\lambda_{n+2})\otimes P(\mu_3^{p^{n-1}}).$$

By the inductive hypothesis, λ_n,λ_{n+1} are infinite cycles. Since $\lambda_n,\lambda_{n+1},\lambda_{n+2}$ all have odd total degree, it follows that a differential on λ_{n+2} cannot truncate the v_2-towers on λ_n or λ_{n+1}. Therefore, the only possibility is that λ_n supports a differential hitting v_2^j for some positive integer j. But that would contradict Corollary 3.6. So λ_{n+2} must also be a cycle.

If the class $\mu_3^{p^{n-1}}$ does not support a differential then by multiplicativity the spectral sequence would collapse at $E_{r(n)}$, and this would contradict Corollary 3.6. Thus $\mu_3^{p^{n-1}}$ supports a differential. Lemma 3.8 and a simple modular arithmetic argument shows that the only possibility is

$$d_{r(n)}\doteq v_2^{r(n)}\lambda_n.$$

Since the differential satisfies the Leibniz rule, this gives

$$v_2^{-1}E_{r(n)+1}\cong K(2)_*\otimes E(\lambda_{n+1},\lambda_{n+2},\lambda_{n+3})\otimes P(\mu_3^{p^n}).$$

This completes the inductive step, proving the theorem. □

We now state the main theorem of this section.

Theorem 3.11. *For each $n \geq 2$ and each nonnegative integer m with $m \not\equiv p-1 \mod p$ there are elements $y_{n,m}$ and $y'_{n,m}$ and $y''_{n,m}$ in $THH_*(BP\langle 2\rangle; k(2))$ such that*

(1) *$y_{n,m}$ projects to $\lambda_n\mu^{mp^{n-1}}$ in $E_\infty^{*,0}$*
(2) *$y'_{n,m}$ projects to $\lambda_n\lambda_{n+1}\mu^{mp^{n-1}}$ in $E_\infty^{*,0}$*
(3) *$y''_{n,m}$ projects to $\lambda_n\lambda_{n+1}\lambda_{n+2}\mu^{mp^{n-1}}$ in $E_\infty^{*,0}$*

As a $P(v_2)$-module, $THH_*(BP\langle 2\rangle; k(2))$ is generated by the unit element 1 and the elements $y_{n,m}, y'_{n,m}, y''_{n,m}$. The only relations are

$$v_n^{r(n)}y_{n,m} = v_n^{r(n)}y'_{n,m} = v_n^{r(n)}y''_{n,m} = 0.$$

This theorem will follow from the previous results and two additional lemmas. Let $P(m)$ denote a free rank one $P(v_m)$-module and let $P(m)_i$ denote the $P(v_m)$-module $P(m)/v_m^i$. Let X be a $BP\langle n\rangle$-module such that $H_*X \cong H_*BP\langle n\rangle \otimes H_*(\overline{X})$ as a $H_*BP\langle n\rangle$-module and consider the Adams spectral sequence

$$E_2^{*,*}(X) = Ext^{*,*}_{E(Q_m)}(\mathbb{F}_p, H_*(\overline{X})) \Rightarrow \pi_*(X \wedge_{BP\langle n\rangle} k(m))_p \tag{3.12}$$

and the v_n-inverted Adams spectral sequence

$$v_m^{-1}E_2^{*,*}(X) = v_m^{-1}Ext^{*,*}_{E(Q_m)}(\mathbb{F}_p, H_*(\overline{X})) \Rightarrow \pi_*(X \wedge_{BP\langle n\rangle} K(m))_p \tag{3.13}$$

There is a map of spectral sequences

$$E_2^{*,*}(X) \longrightarrow v_m^{-1}E_2^{*,*}(X)$$

induced by the localization map $k(m) \to v_m^{-1}k(m) = K(m)$.

Lemma 3.14. *Let $r \geq 2$. Suppose the $E_r(X)$-page of the Adams spectral sequence (3.12) is generated by elements in filtration 0 as a $P(k)$-module and $E_r^{*,*}(X)$ is a direct sum of copies of $P(k)$ and $P(k)_i$ with $i \leq r$ as a $P(k)$-module. Then*

(1) *the map of E_r-pages*
$$E_r^{s,t}(X) \to v_k^{-1}E_r^{s,t}(X)$$
is a monomorphism when $t \geq r+1 \geq 3$.
(2) *Also, the differentials in $E_{r+1}^{*,*}$ are the same as their image in $v_k^{-1}E_{r+1}^{*,*}$.*

Proof. Statement (1) is a consequence of our assumptions since elements in filtration $r+1$ are v_k-torsion free. To prove statement (2) it suffices to prove the following: if $x \in E_r(X)$ maps to a cycle $\bar{x} \in v_k^{-1}E_r(X)$,then x is a cycle. By our assumption, there is an $a \in E_r^{*,0}$ such that $x = v_k^m a$. Statement (1) then implies $d_{r+1}(a) = 0$ so since the differentials are v_k-linear the result follows. ☐

[Gabe: In MS, they claim that the proof works for $t \geq r-1 \geq 1$ and $E_r^{*,*}$, but I don't see why they get indices instead of the ones I have here.]

Remark 3.15. *The Lemma above is a generalization of part (a) and (b) of Theorem 7.1 [12]. We believe this level of generality was known to the authors.*

Lemma 3.16. *For $r \geq 2$ and $n = 2$, the $E_r(THH(BP\langle 2\rangle))$-page of the Adams spectral sequence is generated by elements in filtration 0 as a $P(2)$-module and $E_r^{*,*}$ is a direct sum of copies of $P(2)$ and $P(2)_i$ for $i \leq r$.*

Proof. We will begin by proving the first statement by induction. Note that (2.5) implies the base case in the induction when $r = 2$. Suppose the statement holds for some r. Choose a basis z_i for the $\mathbb{F}_p$-vector space V_r such that

$$V_r = \{x \in E_r^{*,0} \mid v_2^{r-1}x = 0\}.$$

Then $d_r(z_i)$ is in filtration r and since the differentials are v_2-linear, $v_2^{r-1}d_r(z_i) = 0$. However, this contradicts the induction hypothesis because the induction hypothesis implies that all elements in filtration r are v_2-torsion-free. Thus, each basis element z_i is a d_r-cycle. Next choose a set of elements $\{z_j'\} \subset E_r^{*,0}$ such that $\{d_r(z_j')\}$ is a basis for $\operatorname{im}(d_r\colon E_r^{*,0} \to E_r^{*,r})$. Choose $z_j'' \in E_r^{*,0}$ such that $v_2^r z_j'' = d_r(z_j')$. Then z_j'' are d_r-cycles and z_j'' and z_j are linearly independent. We can therefore choose d_r-cycles z_j''' such that $\{z_j\} \cup \{z_j''\} \cup \{z_j'''\}$ are a basis for the d_r-cycles in $E_r^{*,0}$. Then $\{z_j\} \cup \{z_j'\} \cup \{z_j''\} \cup \{z_j'''\}$ are a basis for $E_r^{*,0}$ and the differential is completely determined by the formulas

$$d_r(z_i) = 0, \quad d_r(z_j') = v_2^r z_j'', \quad d_r(z_j'') = 0, \text{ and } \quad d_r(z_j''') = 0.$$

Thus, $E_r^{*,*}$ is generated as a P-module by z_i, z_i'', and z_i''' where $v_2^{r-1}z_i = 0$ and $v_2^r z_i'' = 0$ and z_i''' is v_2-torsion free. □

Proof of Theorem 3.11. For brevity, we will write $\gamma_{n,m} = \lambda_n\mu^{mp^{n-1}}$, $\gamma_{n,m}' = \lambda_n\lambda_{n+1}\mu^{mp^{n-1}}$ and $\gamma_{n,m}'' = \lambda_n\lambda_{n+1}\lambda_{n+2}\mu^{mp^{n-1}}$. By Lemma 3.16 and Lemma 3.14 it suffices to prove that the elements $\gamma_{n,m}$, $\gamma_{n,m}'$, and $\gamma_{n,m}''$ are infinite cycles, that, together with 1, form a basis for $E_\infty^{*,0}$ as an $\mathbb{F}_p$-vector space, and that each of $\gamma_{n,m}$, $\gamma_{n,m}'$ and $\gamma_{n,m}''$ are killed by $v_2^{r(n)}$. By induction on n, we will prove

$$E_{r(n)}(THH(BP\langle 2\rangle)) \cong M_n \oplus E(\lambda_n, \lambda_{n+1}, \lambda_{n+2}) \otimes P(\mu^{p^{n-1}})$$

where M_n is generated by $\{\gamma_{k,m}, \gamma_{k,m}'\gamma_{k,m}'' \mid k < n\}$ modulo the relations

$$v_2^{r(k)}\gamma_{k,m} = v_2^{r(k)}\gamma_{k,m}' = v_2^{r(k)}\gamma_{k,m}'' = 0.$$

This statement holds for $n = 1$ by (2.5). Assume the statement holds for all integers less than or equal to some $N > 1$. Lemma 3.16, Lemma 3.14 and Theorem 3.10 imply that the only nontrivial differentials with source in $E_{r(N)}^{*,0}$ are the differentials

$$d_{r(N)}(\mu^{(m+1)r(N)} = (m+1)v_2^{r(N)}\lambda_N\mu^{mp^{N-1}} \doteq \gamma_{N,m},$$

the differentials

$$d_{r(N)}(\lambda_{N+1}\mu^{(m+1)r(N)} = (m+1)v_2^{r(N)}\lambda_n\lambda_{N+1}\mu^{mp^{N-1}} \doteq \gamma_{N,m}'$$

and the differentials

$$d_{r(N)}(\lambda_{N+2}\lambda_{N+1}\mu^{(m+1)r(N)} = (m+1)v_2^{r(N)}\lambda_N\lambda_{N+1}\lambda_{N+2}\mu^{mp^{N-1}} \doteq \gamma_{N,m}''$$

where $m \not\equiv p-1 \mod p$. Combining this with Lemma 3.16 and Lemma 3.14, this implies that

$$E_{r(N)+1}(\mathrm{THH}(BP\langle 2\rangle)) \cong M_n \oplus V_{N+1} \oplus \Big(P(2) \otimes E(\lambda_{N+1}, \lambda_{N+2}, \lambda_N \mu_3^{(p-1)p^N-1}) \otimes P(\mu^{p^N})\Big)$$

where V_{N+1} has generators $\gamma_{N,m}$, $\gamma'_{N,m}$, and $\gamma''_{N,m}$ and relations

$$v_2^{r(N)}\gamma_{N,m} = v_2^{r(N)}\gamma'_{N,m} = v_2^{r(N)}\gamma''_{N,m} = 0.$$

By Lemma 3.16, Lemma 3.14 and Theorem 3.10 there is an isomorphism

$$E_{r(N)+1}(\mathrm{THH}(BP\langle 2\rangle)) \cong E_{r(N+1)}(\mathrm{THH}(BP\langle 2\rangle)).$$

Also, note that $M_N \oplus V_{N+1} = M_{N+1}$ and $\lambda_N \mu_3^{(p-1)p^N-1} = \lambda_{N+3}$ by definition. This completes the inductive step and consequently the proof. □

4. The v_1-Bockstein spectral sequence

In this section we will begin our analysis of the v_1-Bockstein spectral sequence (1.1) for computing the homotopy of $\mathrm{THH}(BP\langle 2\rangle; k(1))$. We will take a similar approach to the previous section. To start, we need to compute $K(1)_*BP\langle 2\rangle$. This requires determining $\eta_R(v_{2+n})$ in $K(1)_*BP$ modulo the ideal generated by $(\eta_R(v_3), \ldots, \eta_R(v_{1+n}))$. We will need the following.

Lemma 4.1. [14, Lemma A.2.2.5] *Let v_n denote the Araki generators. Then there is the following equality in BP_*BP*

$$\sum_{i,j\geq 0}^F t_i \eta_R(v_j)^{p^i} = \sum_{i,j\geq 0}^F v_i t_j^{p^i}$$

In our context, the distinction between Hazewinkel generators and Araki generators is unimportant, as the two sets of generators coincide modulo p. In $K(1)_*BP$, we have killed all v_i's except v_1, which gives us the following equation

$$\sum_{i,j\geq 0}^F t_i \eta_R(v_j)^{p^i} = \sum_{k\geq 0}^F v_1 t_k^p$$

Note that the following degrees of the terms:

$$|v_1 t_j^p| = 2(p^{j+1}-1)$$
$$|t_i \eta_R(v_j)^{p^i}| = 2(p^{i+j}-1)$$

Since we are interested in the term $\eta_R(v_{2+n})$, we collect all the terms on the left of degree $2(p^{2+n}-1)$. Thus we are summing over the ordered pairs (i,j) such that $i+j = 2+n$. Since we only care about $\eta_R(v_{2+n})$ modulo $\eta_R(v_3), \ldots, \eta_R(v_{1+n})$ we only need to collect the terms where $j = 1, 2$, or $2+n$. This shows that

$$t_{1+n}\eta_R(v_1)^{p^{n+1}} + t_n \eta_R(v_2)^{p^n} + \eta_R(v_{n+2}) = v_1 t_{n+1}^p$$

The value of η_R on v_1 and v_2 can also be computed by Lemma 4.1. One obtains, in $K(1)_*BP$, the following

$$\begin{aligned}\eta_R(v_1) &= v_1\\ \eta_R(v_2) &= v_1t_1^p - t_1v_1^p.\end{aligned}$$

Combining these observations, we obtain

Lemma 4.2. *In $K(1)_*BP$, the following congruence is satisfied*

$$\eta_R(v_{2+n}) \equiv v_1t_{n+1}^p - v_1^{p^n}t_1^{p^{n+1}}t_n + v_1^{p^{n+1}}(t_1^{p^n}t_n - t_{n+1}) \mod (\eta_R(v_3), \ldots, \eta_R(v_{1+n}))$$

for $n \geq 1$.

Consequently, we have the following corollary.

Corollary 4.3. *There is an isomorphism of $K(1)_*$-algebras*

$$K(1)_*BP\langle 2\rangle \cong K(1)_*BP/(v_1t_{n+1}^p - v_1^{p^n}t_1^{p^{n+1}}t_n + v_1^{p^{n+1}}(t_1^{p^n}t_n - t_{n+1}) \mid n \geq 1)$$

Define elements

$$u_n := v_1^{\frac{p^n-1}{p-1}}t_n.$$

These elements are in degree 0, and therefore there is an isomorphism

$$K(1)_*BP\langle 2\rangle \cong_{K(1)_*} K(1)_* \otimes_{\mathbb{F}_p} K(1)_0BP\langle 2\rangle.$$

The calculations above imply the following corollary.

Corollary 4.4. *There is an isomorphism of $\mathbb{F}_p$-algebras*

$$K(1)_0R \cong \mathbb{F}_p[u_i \mid i \geq 1]/(u_{n+1}^p - u_1^{p^{n+1}}u_n + u_1^{p^n}u_n - u_{n+1} \mid n \geq 1).$$

Our goal is to use this and the $K(1)$-based Bökstedt spectral sequence to compute the $K(1)$-homology of $\mathrm{THH}(R)$. This is a spectral sequence of the form

$$E^2_{s,t} = \mathrm{HH}_s^{K(1)_*}(K(1)_*R) \implies K(1)_{s+t}\,\mathrm{THH}(R).$$

The above considerations tell us that the E^2-page is

$$E^2 \cong K(1)_* \otimes \mathrm{HH}^{\mathbb{F}_p}(K(1)_0R).$$

The following will be useful for our calculation.

Lemma 4.5 ([13])**.** *Let $V = \mathrm{Spec}(A)$ be a nonsingular affine variety over a field k. Let W be the subvariety of $V \times \mathbb{A}^n$ defined by equations*

$$g_i(Y_1, \ldots, Y_n) = 0,\ g_i \in A[Y_1, \ldots, Y_n],\ i = 1, \ldots, n.$$

Then the projection map $W \to V$ is étale at a point $(P; b_1, \ldots, b_n)$ of W if and only if the Jacobian matrix $\left(\frac{\partial g_i}{\partial Y_j}\right)$ is a nonsingular matrix at $(P; b_1, \ldots, b_n)$.

Theorem 4.6 (Étale Descent, [16])**.** *Let $A \hookrightarrow B$ be an étale extension of commutative k-algebras. Then there is an isomorphism*

$$\mathrm{HH}_*(B) \cong \mathrm{HH}_*(A) \otimes_A B$$

Example 4.7. Consider the subalgebra

$$\mathbb{F}_p[u_1,u_2]/(u_2^p - u_1^{p^2+1} + u_1^{p+1} - u_2 = f_1).$$

We will regard this as a $\mathbb{F}_p[u_1]$-algebra. The partial derivative $\partial_{u_2} f_1$ is $-1 \pmod p$, and therefore a unit at every point. Then Lemma 4.5 tells us that this algebra is then étale over $\mathbb{F}_p[u_1]$.

By the same argument given above, we claim that there are a sequence of subalgebras A_n of

$$K(1)_0R \cong \mathbb{F}_p[u_i \mid i \geq 1]/(u_{n+1}^p - u_1^{p^{n+1}}u_n + u_1^{p^n}u_n - u_{n+1} \mid n \geq 1) =: A.$$

such that each map $A_i \hookrightarrow A_{i+1}$ is an étale extension. Here

$$A_n := \mathbb{F}_p[u_1,u_2,\dots u_n]/(u_{k+1}^p - u_1^{p^{k+1}}u_k + u_1^{p^k}u_k - u_{k+1} = f_k \mid 1 \leq k \leq n)$$

and the partial derivative $\partial_{u_k} f_k = -1 \pmod p$ for all $1 < k \leq n$ and therefore a unit at each point. The claim then follows by Lemma 4.5.

By the étale base change formula for Hochschild homology in Theorem 4.6, there is an isomorphism

$$\mathrm{HH}_*^{\mathbb{F}_p}(A_{i+1}) \cong \mathrm{HH}_*^{\mathbb{F}_p}(A_i) \otimes_{A_i} A_{i+1}$$

and since the functors $HH_*(-)$ and $\mathrm{HH}_*^{\mathbb{F}_p}(A_1) \otimes_{A_1} (-)$ commute with filtered colimits of $\mathbb{F}_p$-algebras, there are isomorphisms

$$\begin{aligned} \mathrm{HH}_*^{\mathbb{F}_p}(A) &\cong \mathrm{HH}_*^{\mathbb{F}_p}(\operatorname{colim} A_n) \\ &\cong \operatorname{colim} \mathrm{HH}_*^{\mathbb{F}_p}(A_n) \\ &\cong \operatorname{colim} \mathrm{HH}_*^{\mathbb{F}_p}(A_1) \otimes_{A_1} A_n \\ &\cong \mathrm{HH}_*^{\mathbb{F}_p}(A_1) \otimes_{A_1} A. \end{aligned}$$

Consequently,

$$\mathrm{HH}_*(K(1)_*BP\langle 2\rangle) \cong K(1)_* \otimes E(\sigma t_1) \otimes K_0(R)$$

and therefore, since $\sigma t_1 \doteq \lambda_1$,

$$K(1)_*\,\mathrm{THH}(BP\langle 2\rangle) \cong K(1)_*BP\langle 2\rangle \otimes E(\lambda_1)$$

and

$$\mathrm{THH}_*(BP\langle 2\rangle; K(1)) \cong K(1)_* \otimes E(\lambda_1).$$

In other words,

$$THH_*(BP\langle 2\rangle; k(1)) \cong F \oplus T$$

where F is a free $P(v_1)$-module generated by 1 and λ_1 and T is a torsion $P(v_1)$-module. In summary, we have proven the following theorem.

Theorem 4.8.

(1) *The $K(1)$-homology of* $\mathrm{THH}(R;K(1))$ *is* $K(1)_*K(1) \otimes E(\lambda_1)$
(2) *Their is a weak equivalence*

$$K(1) \vee \Sigma^{2p-1}K(1) \simeq \mathrm{THH}(R;K(1)).$$

(3) *The only v_1-torsion free classes in* $\mathrm{THH}(R;k(1))$ *are v_1^k and $\lambda_1 v_1^k$ for $k \geq 0$.*

4.1. **Differentials in the v_1-BSS.** We now analyze the v_1-BSS (1.1). Recall that this spectral sequence is of the form

$$\mathrm{THH}(BP\langle 2\rangle;\mathbb{F}_p)[v_1] \implies \mathrm{THH}(B;k(1)).$$

Thus the E_1-page is

$$E(\lambda_1,\lambda_2,\lambda_3)\otimes P(\mu_3)\otimes P(v_1). \tag{4.9}$$

Since the λ_i are all in odd total degree and since v_1^k are known to be v_1-torsion free for all k, the λ_i are all permanent cycles. If μ_3 were a permanent cycle, then by multiplicativity, the spectral sequence would collapse. This would contradict Theorem 4.8. Therefore, the element μ_3 must support a differential. The only possibility for bidegree reasons is

$$d_{p^2}(\mu_3)\doteq v_1^{p^2}\lambda_2.$$

Thus, we obtain

$$v_1^{-1}E_{p^2+1}\cong K(1)_*\otimes E(\lambda_1,\lambda_3,\lambda_4')\otimes P(\mu_3^p)$$

where

$$\lambda_4' := \lambda_2\mu_3^{p-1}.$$

So the bidegree of λ_4' is given by

$$|\lambda_4'| = (2p^4-2p^3+2p^2-1,0).$$

For analogous reasons, the class λ_4' is a permanent cycle, and μ_3^p cannot be a permanent cycle. Based on degree considerations, there are two possible differentials,

$$d_{p^2}(\mu_3^p)\doteq v_1^{p^2}\lambda_4' \quad \text{or} \quad d_{p^3}(\mu_3^p)\doteq v_1^{p^3}\lambda_3.$$

The first would contradict the Leibniz rule for d_{p^2} and the fact that $d_{p^2}(\mu_3)\doteq v_1^{p^2}\lambda_2$. Thus,

$$v_1^{-1}E_{p^3+1}\cong K(1)_*\otimes E(\lambda_1,\lambda_4',\lambda_5')\otimes P(\mu_3^{p^2})$$

where

$$\lambda_5' := \lambda_3\mu_3^{p(p-1)}.$$

The bidegree of λ_5' is

$$|\lambda_5'| = (2p^5-2p^4+2p^3-1,0).$$

For degree reasons, the class λ_5' is a permanent cycle. As before, the class $\mu_3^{p^2}$ must support a differential. Degree considerations, again, give two possibilities

$$d_{p^3}(\mu_3^{p^2})\doteq v_1^{p^3}\lambda_5'$$

or

$$d_{p^4+p^2}(\mu_3^{p^2})\doteq v_1^{p^4+p^2}\lambda_4'.$$

The former would contradict the Leibniz rule, leaving the latter as the only possibility. This gives us

$$v_1^{-1}E_{p^4+p^2+1}\cong K(1)_*\otimes E(\lambda_1,\lambda_5',\lambda_6')\otimes P(\mu_3^{p^3})$$

where $\lambda_6' := \lambda_4'\mu_3^{p^2(p-1)}$. We will continue via induction. First we need some notation. We will recursively define classes λ_n' by

$$\lambda_n' := \begin{cases} \lambda_n & n = 1,2,3 \\ \lambda_{n-2}'\mu_3^{p^{n-4}(p-1)} & n \geq 4 \end{cases}$$

We let $d'(n)$ denote the topological degree of λ_n'. Then this function is given recursively by

$$d'(n) = \begin{cases} 2p^n - 1 & n = 1,2,3 \\ 2p^n - 2p^{n-1} + d(n-2) & n > 3 \end{cases}$$

Thus, by a simple induction, one has

$$d'(n) = \begin{cases} 2p^n - 1 & n = 1,2,3 \\ 2p^n - 2p^{n-1} + 2p^{n-2} - 2p^{n-3} + \cdots + 2p^2 - 1 & n \equiv 0 \mod 2,\ n > 3 \\ 2p^n - 2p^{n-1} + 2p^{n-2} - 2p^{n-3} + \cdots + 2p^3 - 1 & n \equiv 1 \mod 2,\ n > 3 \end{cases}.$$

Observe that the integers $2p^{n+1} - d(n) - 1$ and $2p^{n+1} - d(n+1) - 1$ are both divisible by $|v_1|$. Let $r'(n)$ denote the integer

$$r'(n) := |v_1|^{-1}(|\mu_3^{p^{n-1}} - |\lambda_n'| - 1) = |v_1|^{-1}(2p^{n+2} - d'(n+1) - 1).$$

Then a simple induction shows that

$$r'(n) = \begin{cases} p^{n+1} + p^{n-1} + p^{n-3} + \cdots + p^2 & n \equiv 1 \mod 2 \\ p^{n+1} + p^{n-1} + p^{n-3} + \cdots + p^3 & n \equiv 0 \mod 2 \end{cases}.$$

We can now describe the differentials in the v_1-BSS.

Theorem 4.10. *In the v_1-BSS, the following hold:*

(1) *The only nonzero differentials are in $v_1^{-1}E_{r'(n)}$.*
(2) *The $r'(n)$th page is given by*

$$v_1^{-1}E_{r'(n)} \cong K(1)_* \otimes E(\lambda_1, \lambda_n', \lambda_{n+1}') \otimes P(\mu_3^{p^{n-1}})$$

and the classes $\lambda_n', \lambda_{n+1}'$ are permanent cycles.
(3) *The differential $d_{r'(n)}$ is uniquely determined by multiplicativity of the BSS and the differential*

$$d_{r'(n)}(\mu_3^{p^{n-1}}) \dot{=} v_1^{r'(n)}\lambda_n'.$$

Proof. We proceed by induction. We have already shown the theorem for $n \leq 4$. Assume inductively that

$$v_1^{-1}E_{r'(n)} \cong K(1)_* \otimes E(\lambda_1, \lambda_n', \lambda_{n+1}') \otimes P(\mu_3^{p^{n-1}}).$$

By the inductive hypothesis, λ_n' is an infinite cycle.

Since $\lambda_n', \lambda_{n+1}'$ are both in odd topological degree, λ_{n+1}' cannot support a differential hitting the v_1-towers on λ_i' for $i < n+1$. Thus, the only possibility is that λ_{n+1}' supports a differential into the v_1-tower on 1 or λ_1. But this would contradict Theorem 4.8. Therefore, the class λ_{n+1}' is a permanent cycle.

The class $\mu_3^{p^{n-1}}$ must support a differential, for if it did not, then the spectral sequence would collapse. This would lead to a contradiction of Theorem 4.8. Degree considerations show that the following differentials are possible

$$d_{k(n)}(\mu_3^{p^{n-1}})\doteq v_1^{k(n)}\lambda'_{n+1}$$

and

$$d_{r(n)}(\mu_3^{p^{n-1}})\doteq v_1^{r'(n)}\lambda'_{n}$$

where

$$\ell(n) = |v_1^{-1}|(2p^{n+2} - |\lambda'_{n+1}|).$$

An elementary inductive computation shows that

$$\ell(n) = r'(n-1).$$

The former differential cannot occur, for by the inductive hypothesis,

$$d_{r'(n-1)}(\mu_3^{p^{n-2}})\doteq\lambda'_{n-1},$$

and the former differential would contradict the Leibniz rule. So we must conclude the latter differential occurs. This concludes proof. □

We now state the main result of this section.

Theorem 4.11. *For each $n \geq 2$ and each nonnegative integer m with $m \not\equiv p-1$ mod p there are elements $z_{n,m}$ and $z'_{n,m}$ in $THH_*(BP\langle 2\rangle; k(1))$ such that*

1. *$z_{n,m}$ projects to $\lambda'_n\mu_3^{mp^{n-2}}$ in $E_\infty^{*,0}$*
2. *$z'_{n,m}$ projects to $\lambda'_n\lambda'_{n+1}\mu_3^{mp^{n-2}}$ in $E_\infty^{*,0}$*

As a $P(v_1)$-module, $THH_*(BP\langle 2\rangle; k(1))$ is generated by the unit element 1, λ_1, and the elements $z_{n,m}, z'_{n,m}$. The only relations are

$$v_1^{r'(n)}z_{n,m} = v_1^{r'(n)}z'_{n,m} = 0.$$

To prove this, we first need to prove a lemma.

Lemma 4.12. *For $r \geq 2$ with $n = 2$ and $k = 1$, the $E_r(THH(BP\langle 2\rangle))$-page of the Adams spectral sequence* (**??**) *is generated by elements in filtration 0 as a $P(1)$-module and $E_r^{*,*}$ is a direct sum of copies of $P(1)$ and $P(1)_i$ for $i \leq r$.*

Proof. We will begin by proving the first statement by induction. Note that (4.9) implies the base case in the induction when $r = 2$. Suppose the statement holds for some r. Choose a basis y_i for the $\mathbb{F}_p$-vector space V_r such that $V_r = \{x \in E_r^{*,0} \mid v_1^{r-1}x = 0\}$. Then $d_r(y_i)$ is in filtration r and since the differentials are v_1-linear, $v_1^{r-1}d_r(y_i) = 0$. However, this contradicts the induction hypothesis because the induction hypothesis implies that all elements in filtration r are v_1-torsion-free. Thus, each basis element y_i is a d_r-cycle. Next choose a set of elements $\{y'_j\} \subset E_r^{*,0}$ such that $\{d_r(y'_j)\}$ is a basis for $\mathrm{im}(d_r\colon\ E_r^{*,0} \to E_r^{*,r})$. Choose $y''_j \in E_r^{*,0}$ such that $v_1^r y''_j = d_r(y'_j)$. Then y''_j are d_r-cycles and y''_j and y_j are linearly independent. We can therefore choose d_r-cycles y'''_j such that

$\{y_j\}\cup\{y''_j\}\cup\{y'''_j\}$ are a basis for the d_r-cycles in $E_r^{*,0}$. Then $\{y_j\}\cup\{y'_j\}\cup\{y''_j\}\cup\{y'''_j\}$ are a basis for $E_r^{*,0}$ and the differential is completely determined by the formulas

$$d_r(y_i)=0,\quad d_r(y'_j)=v_1^r y''_j,\quad d_r(y''_j)=0,\ \text{and}\quad d_r(y'''_j)=0.$$

Thus, $E_r^{*,*}$ is generated as a $P(1)$-module by y_i, y''_i, and y'''_i where $v_1^{r-1}y_i=0$ and $v_1^r y''_i=0$ and y'''_i is v_1-torsion free. □

Proof of Theorem 4.11. For brevity, we will let $\delta_{n,m}$ denote $\lambda'_n\mu^{mp^{n-2}}$ and we will let $\delta'_{n,m}$ denote $\lambda'_n\lambda'_{n+1}\mu^{mp^{n-2}}$. By Lemma 4.12 and Lemma 3.14 it suffices to prove that the elements $\delta_{n,m}$, and $\delta'_{n,m}$ are infinite cycles that, together with 1 and λ_1, form a basis for $E_\infty^{*,0}$ as an $\mathbb{F}_p$-vector space, and that each of $\delta_{n,m}$, $\delta'_{n,m}$ are killed by $v_1^{r'(n)}$. By induction on n, we will prove

$$E_{r(n)}(THH(BP\langle 2\rangle))\cong M_n\oplus E(\lambda_1,\lambda'_n,\lambda'_{n+1})\otimes P(\mu_3^{p^{n-1}})$$

where M_n is generated by $\{\delta_{k,m},\delta'_{k,m}\mid k<n\}$ modulo the relations

$$v_2^{r(k)}\delta_{k,m}=v_2^{r(k)}\delta'_{k,m}=0.$$

This statement holds for $n=1$ by (4.9). Assume the statement holds for all integers less than or equal to some $N\geq 1$. Lemma 4.12, Lemma 3.14 and Theorem 3.10 imply that the only nontrivial differentials with source in $E_{r(N)}^{*,0}$ are the differentials

$$d_{r(N)}(\mu^{(m+1)r'(N)})=(m+1)v_1^{r'(N)}\lambda'_N\mu^{mp^{N-1}}\doteq\delta_{N,m},$$

and the differentials

$$d_{r'(N)}(\lambda'_{N+1}\mu^{(m+1)r'(N)})=(m+1)v_1^{r'(N)}\lambda'_n\lambda'_{N+1}\mu^{mp^{N-2}}\doteq\delta'_{N,m}$$

where $m\not\equiv p-1 \mod p$. Combining this with Lemma 4.12 and Lemma 3.14, this implies that

$$E_{r(N)+1}(\mathrm{THH}(BP\langle 2\rangle))\cong M_n\oplus V_{N+1}\oplus\Big(P(2)\otimes E(\lambda_1,\lambda'_N,\lambda'_N\mu_3^{(p-1)p^{N-2}})\otimes P(\mu^{p^N})\Big)$$

where V_{N+1} has generators $\delta_{N,m}$ and $\delta'_{N,m}$ and relations

$$v_2^{r(N)}\delta_{N,m}=v_2^{r(N)}\delta'_{N,m}=0.$$

By Lemma 4.12, Lemma 3.14 and Theorem 4.10 there is an isomorphism

$$E_{r(N)+1}(\mathrm{THH}(BP\langle 2\rangle))\cong E_{r(N+1)}(\mathrm{THH}(BP\langle 2\rangle)).$$

Also, note that $M_N\oplus V_{N+1}=M_{N+1}$ and $\lambda'_N\mu_3^{(p-1)p^{N-2}}=\lambda'_{N+2}$ by definition. This completes the inductive step and consequently the proof. □

5. The v_2-inverted homotopy of $\mathrm{THH}(BP\langle 2\rangle)$

In this section, we briefly describe the calculation of $\mathrm{THH}(BP\langle 2\rangle; E(2))$ based on recent work of [5]. In [5], they assume that an E_∞-model for $E(2)$ exists. At $p = 3$, we may construct an E_∞-model for $E(2)$ as the spectrum $E(2) := v_2^{-1}BP\langle 2\rangle$ by [8]. There is a map

$$\mathrm{THH}(BP\langle 2\rangle; E(2)) \to \mathrm{THH}(E(2))$$

induced by the localization map

$$BP\langle 2\rangle \to L_2^f BP\langle 2\rangle =: E(2).$$

We first recall the main theorem of [5]

Theorem 5.1 (Ausoni-Richter [5])**.** *There is an equivalence*

$$\mathrm{THH}(E(2)) \simeq E(2) \vee \Sigma^{2p-1} L_1 E(2) \vee \Sigma^{2p^2-1} L_0 E(2) \vee \Sigma^{2p^2+2p-2} L_0 E(2).$$

Since there is an equivalence $THH(BP\langle 2\rangle) \wedge_{BP\langle 2\rangle} E(2) \simeq THH(E(2))$, we get the following corollary.

Corollary 5.2. *There is an equivalence*

$$THH(BP\langle 2\rangle; E(2)) \simeq E(2) \vee \Sigma^{2p-1} L_1 E(2) \vee \Sigma^{2p^2-1} L_0 E(2) \vee \Sigma^{2p^2+2p-2} L_0(E(2)$$

Note that this is consistent with our independent computations of $L_0 THH(BP\langle 2\rangle)$, $THH_*(BP\langle 2\rangle; E(2)/(p, v_1))$, and

$$THH_*(BP\langle 2\rangle; BP\langle 1\rangle).$$

Note that $\pi_*(v_1^{-1}S/p \wedge E(2)) \cong P(v_1^{\pm 1}, v_2^{\pm 1})$. In particular, this also implies the following.

Corollary 5.3. *There is an equivalence*

$$THH(BP\langle 2\rangle; E(2)/p) \simeq E(2)/p \vee \Sigma^{2p-1}(L_1 E(2))/p$$

and consequently, an equivalence

$$v_1^{-1}S/p \wedge THH(BP\langle 2\rangle; E(2)) \simeq v_1^{-1}E(2)/p \vee \Sigma^{2p-1} v_1^{-1}(L_1 E(2))/p$$

and therefore the torsion free part of $THH_*(BP\langle 2\rangle; BP\langle 2\rangle/p)$ as a $P(v_1, v_2)$-module is generated by 1 *and* λ_1*.*

6. The mod p homotopy of $\mathrm{THH}(BP\langle 2\rangle)$

In this section, we begin our study of the mod p homotopy of $\mathrm{THH}(BP\langle 2\rangle)$. We will assume that $p = 3$, since in this case the mod 3 Moore spectrum $V(0)$ is a ring spectrum. Our approach to this computation will be to make use of the *THH-May spectral sequence*, which was developed by the first author and Andrew Salch in [2] and applied by the first author in [1].

Let us briefly describe the strategy we will employ. The mod 3 homotopy of $\mathrm{THH}(BP\langle 2\rangle)$ is exactly the homotopy groups

$$\pi_*(\mathrm{THH}(BP\langle 2\rangle); \mathbb{Z}/3) := \pi_*(V(0) \wedge \mathrm{THH}(BP\langle 2\rangle)) = V(0)_* \mathrm{THH}(BP\langle 2\rangle).$$

To compute this, we will use the $V(0)$-based THH-May spectral sequence. Using the Whitehead filtration for $BP\langle 2\rangle$ as developed in [2], the THH-May spectral sequence based on $V(0)$ takes the form

$$\pi_*(\mathrm{THH}(H\pi_*BP\langle 2\rangle);\mathbb{Z}/3) \implies \pi_*(\mathrm{THH}(BP\langle 2\rangle);\mathbb{Z}/3) \tag{6.1}$$

where $H\pi_*BP\langle 2\rangle$ is a commutative ring spectrum whose underlying spectrum is the generalized Eilenberg-MacLane spectrum

$$\bigvee_{i\geq 0}\Sigma^i H\pi_i BP\langle 2\rangle.$$

In order to obtain the first several differentials in this spectral sequence, we will consider the $H\mathbb{F}_p\wedge V(0)$-based THH-May spectral sequence. This takes the form

$$H_*(V(0)\wedge \mathrm{THH}(H\pi_*BP\langle 2\rangle)) \implies H_*(V(0)\wedge \mathrm{THH}(BP\langle 2\rangle)). \tag{6.2}$$

The abutment of this spectral sequence is known from which we will determine the differentials in (6.2).

The morphism $V(0)\to H\mathbb{F}_p\wedge V(0)$ of spectra induces a morphism of THH-May spectral sequences

$$\begin{tikzcd} \pi_*(\mathrm{THH}(H\pi_*BP\langle 2\rangle);\mathbb{Z}/3) \arrow[r, Rightarrow] \arrow[d, hook] & \pi_*(\mathrm{THH}(BP\langle 2\rangle);\mathbb{Z}/3) \arrow[d] \\ H_*(V(0)\wedge \mathrm{THH}(H\pi_*BP\langle 2\rangle)) \arrow[r, Rightarrow] & H_*(V(0)\wedge \mathrm{THH}(BP\langle 2\rangle)) \end{tikzcd} \tag{6.3}$$

We will argue that the map on E^1-terms is injective, which will allow us to determine the first several pages of the $V(0)$-based THH-May spectral sequence. We will argue that there is an (additive) isomorphism

$$E^5_{*,*}\cong E(\lambda_1,\lambda_2,\lambda_3)\otimes P(\mu_3)\otimes P(v_1,v_2)$$

which is isomorphic to the input of the square of Bockstein spectral sequences computing $THH_*(BP\langle 2\rangle;BP\langle 2\rangle/3)$ and therefore allows us to pass along the diagonal of the square of spectral sequences.

6.1. **Review of the THH-May spectral sequence.** The THH-May spectral sequence takes as input a cofibrant decreasingly filtered commutative monoid I in spectra (specifically symmetric spectra of pointed simplicial sets with the positive stable flat model structure denoted $\mathcal{S}$) and produces a spectral sequence

$$E_1^{*,*}=E_*THH(E_0^*I)\Rightarrow E_*THH(I_0)$$

for any connective generalized homology theory E. First, recall the definition of a cofibrant decreasingly filtered commutative monoid in spectra. Let $\mathbb{N}^{\mathrm{op}}$ be the opposite of the category $\mathbb{N}$ viewed as a partially ordered set. The category $\mathbb{N}^{\mathrm{op}}$ is a symmetric monoidal category via $+$ with symmetric monoidal unit 0. Recall that due to Day [7] there is an equivalence of categories

$$\mathsf{CommFun}(\mathbb{N}^{\mathrm{op}},\mathcal{S})\simeq \mathsf{Fun}^{\otimes}(\mathbb{N}^{\mathrm{op}},\mathcal{S})$$

where the right hand side is the category of symmetric monoidal functors $\mathbb{N}^{\mathrm{op}}\to\mathcal{S}$.

Definition 6.4. A cofibrant decreasingly filtered commutative monoid in spectra I is a lax symmetric monoidal functor $\mathbb{N}^{\text{op}} \to \mathcal{S}$, which is cofibrant in model structure on $\mathsf{CommFun}(\mathbb{N}^{\text{op}}, \mathcal{S})$ created by the forgetful functor to $\mathsf{Fun}(\mathbb{N}^{\text{op}}, \mathcal{S})$ (where $\mathsf{Fun}(\mathbb{N}^{\text{op}}, \mathcal{S})$ is equipped with the projective model structure).

To a cofibrant decreasingly filtered commutative monoid I we can associate its associated graded commutative ring spectrum $E_0^* I$. It is constructed as a commutative ring spectrum in [2, Def. 3.16] so that its underlying spectrum is

$$E_0^* I = \vee_{i \geq 0} I_i / I_{i+1}$$

where I_i is our decreasingly filtered commutative monoid evaluated at a natural number i and I_i/I_{i+1} is the cofiber of the cofibration $I_{i+1} \to I_i$. Given an object in $\mathsf{Fun}(\mathbb{N}^{\text{op}}, \mathcal{S})$ one may easily produce an object $\mathsf{Fun}(d\mathbb{N}^{\text{op}}, \mathcal{S})$, where $d\mathbb{N}^{\text{op}}$ is the discrete category of natural numbers, and then take the colimit to produce $E_0^* I$ additively. The main result needed to construct the THH-May spectral sequence is the identification of the E^1-page as $E^1_{*,*} = E_{*,*} THH(E_0^* I)$. This can easily be extended to include coefficients in a cofibrant I-module in $\mathsf{Fun}(\mathbb{N}^{\text{op}}, \mathcal{S})$ as well and we therefore describe the most general form.

Theorem 6.5 (Angelini-Knoll-Salch [2])**.** *Let I be a cofibrant decreasingly filtered commutative monoid in spectra and let M be a cofibrant I-module, then there is a (strongly convergent) spectral sequence*

$$E^1(BP\langle 2\rangle; M) = E_{*,*} THH(E_0^* I; E_0^* M) \Rightarrow E_* THH(I_0; M_0), \tag{6.6}$$

which is multiplicative when M is also a cofibrant I-algebra, for any connective homology theory E_*. We will refer to this spectral sequence as the E-THH-May spectral sequence when $I = M$ is understood from the context or the E-THH-May spectral sequence with coefficients when $I \neq M$ but both are understood from the context. s

In order to make use of this spectral sequence, one would like a large supply of cofibrant decreasingly filtered commutative ring spectra, and this is provided by [2, Thm. 4.2.1]. In other words, there is a model for the Whitehead tower of a connective cofibrant commutative ring spectrum A, written

$$\to \tau_{\geq 3} A \to \tau_{\geq 2} A \to \tau_{\geq 1} A \to \tau_{\geq 0} A$$

which is a cofibrant decreasingly filtered commutative monoid in spectra. In other words, there are structure maps

$$\rho_{i,j} \colon \ \tau_{\geq i} A \wedge \tau_{\geq j} A \longrightarrow \tau_{\geq i+j} A$$

satisfying associativity, commutativity, compatibility, and unitality axioms. The associated graded of this filtration can be identified with $H\pi_* A$, the generalized Eilenberg-MacLane spectrum of the differential graded algebra $\pi_* A$.

6.2. **The $V(1)$-THH-May spectral sequence.** In this section, we analyze $V(1)$-THH-May spectral sequence. At the prime 3, this spectral sequence is unfortunately not multiplicative because $V(1)$ is not a ring spectrum, but we will be able to skirt this issue. In particular, we will show that the E^5-term is a reindexed version of the v_2-Bockstein spectral sequence converging to $V(1)_* \operatorname{THH}(BP\langle 2\rangle)$ and this will imply the remaining differentials in this spectral sequence. There is a map of THH-May spectral sequences

$$\begin{array}{ccc} V(0)_*THH(H\pi_*BP\langle 2\rangle) & \Longrightarrow & \mathrm{THH}(BP\langle 2\rangle; BP\langle 2\rangle/p) \\ \downarrow & & \downarrow \\ V(1)_*THH(H\pi_*BP\langle 2\rangle) & \Longrightarrow & \mathrm{THH}(BP\langle 2\rangle; k(2)) \end{array}$$

induced by the usual map $j\colon V(0) \to V(1)$. This will use to import differentials into the $V(0)$-THH-May spectral sequence.

Specifically, our goal is to show that

$$E^5 \cong E(\lambda_1, \lambda_2, \lambda_3) \otimes P(\mu_3) \otimes P(v_2).$$

We begin by analyzing the multiplicative $H\mathbb{F}_p$-THH-May spectral sequence:

$$H_*(\mathrm{THH}(H\pi_*BP\langle 2\rangle)) \implies H_*(THH(BP\langle 2\rangle)). \tag{6.7}$$

The abutment is known by [4]

$$H_*(\mathrm{THH}(BP\langle 2\rangle; k(2))) \cong E(\widetilde{\tau}_0, \widetilde{\tau}_1) \otimes A/\!\!/E(2)_* \otimes E(\lambda_1, \lambda_2, \lambda_3) \otimes P(\mu_3).$$

We first determine the E^1-page using the Bökstedt spectral sequence

$$\mathrm{HH}_*(H_*(H\pi_*BP\langle 2\rangle)) \implies H_*(\mathrm{THH}(H\pi_*BP\langle 2\rangle)).$$

We need to determine the E^1-term of this Bökstedt spectral sequence. In the following lemma, we will use the notation $R[x]$ for the ring spectrum $\bigvee_{i\geq 0} \Sigma^{|x|i}R$ and $R[x,y]$ for $R[x] \wedge_R R[y]$ where $|x| \geq 0$, $|y| \geq 0$ and R is a ring spectrum.

Lemma 6.8. *The mod 3 homology of $H\pi_*BP\langle 2\rangle$ is $A/\!\!/E(0)_* \otimes P(v_1, v_2)$ where v_1 and v_2 are comodule primitives. Consequently, $HH_*(H_*(H\pi_*BP\langle 2\rangle))$ is isomorphic to*

$$H_*(H\pi_*BP\langle 2\rangle) \otimes E(\sigma\bar{\xi}_n \mid n \geq 1) \otimes \Gamma(\sigma\overline{\tau}_k \mid k \geq 1) \otimes E(\sigma v_1, \sigma v_2).$$

as $\mathscr{A}_*$-comodule $H_*BP\langle 2\rangle$-Hopf algebras.

Proof. There is an equivalence of $H\mathbb{Z}_{(p)}$-algebras

$$H\pi_*BP\langle 2\rangle \simeq H\mathbb{Z}_{(p)}[v_1, v_2]$$

where $|v_1| = 2p-2$ and $|v_2| = 2p^2-2$. Thus, there is an isomorphism of rings

$$H_*H\pi_*BP\langle 2\rangle \cong \pi_*(H\mathbb{F}_p \wedge H\mathbb{Z}_{(3)}[v_1, v_2]) \cong \pi_*((H\mathbb{F}_p \wedge H\mathbb{Z}_{(3)} \wedge_{H\mathbb{F}_p} H\mathbb{F}_p[v_1, v_2])$$

and the result follows by the Künneth isomorphism. The element v_1 in homology arises from the inclusion of the summand indexed by v_1. Applying homology to this map takes 1 to v_1. As this is a map of comodules, v_1 is necessarily primitive. A similar argument shows that v_2 is primitive.

The Hochschild homology of an exterior algebra $E(x)$ with a generator in x odd degree is $E(x) \otimes \Gamma(\sigma x)$ where $\Gamma(\sigma x)$ is a divided power algebra on a generator σx with $|\sigma x| = 1 + |x|$, by Koszul duality. The Hochschild homology of a polynomial algebra on a class y in even degree is $P(y) \otimes E(\sigma y)$ where $|\sigma y| = 1 + |y|$. (See [12] for details). Therefore, the result follows by the Künneth isomorphism for Hochschild homology. $\square$

Proposition 6.9. *There is an isomorphism*

$$H_* \operatorname{THH}(H\pi_* BP\langle 2\rangle) \cong H_*(H\pi_* BP\langle 2\rangle) \otimes E(\lambda_1) \otimes P(\mu_1) \otimes E(\sigma v_1, \sigma v_2)$$

of $\mathscr{A}_*$-comodule $H_*H\pi_*BP\langle 2\rangle$-Hopf algebras where $\lambda_1 = \sigma\bar{\xi}_1$ and $\mu_1 = \sigma\tau_1 - \bar{\tau}_0\sigma\bar{\xi}_1$ are co-module primitives as well as v_1, v_2, σv_1, and σv_2.

Proof. The map of commutative rings $H\mathbb{Z}_p \to H\pi_*BP\langle 2\rangle$ induces a multiplicative map of Bökstedt spectral sequences. This completely determines the differentials and hidden extensions in the Bökstedt spectral sequence

$$HH_*(H_*(H\pi_*BP\langle 2\rangle)) \Rightarrow H_*THH(H\pi_*BP\langle 2\rangle)$$

□

Remark 6.10. *Note that the May filtration of σv_1 and σv_2 are $2(p-1)$ and $2(p^2-1)$ respectively. Since the May filtration is always divisible by $2(p-1)$, we reindex to give σv_1 and σv_2 May filtration 1 and 4 respectively. We will continue to use this convention throughout this section.*

We can then establish

Proposition 6.11. *In the spectral sequence (6.7), we have the differentials*

(1) $d^1\overline{\tau}_1 \doteq v_1$,
(2) $d^1\mu_1 \doteq \sigma v_1$,
(3) $d^4\overline{\tau}_2 \doteq v_2$
(4) $d^4\mu_1^3 \doteq \sigma v_2$

and the classes λ_2, λ_3 are detected by $\mu_1^2\sigma v_1$ and $\mu_1^6\sigma v_2$ respectively[1]. The remaining d^1 and d^4 differentials are determined by these differentials and the Leibniz rule.

Proof. This follows by comparing the rank of the abutment as an $\mathbb{F}_p$-vector space to the rank of E^1-page as an $\mathbb{F}_p$-vector spaces and determining all possible differentials in the pairs of columns corresponding to the differentials stated. The differentials as stated are the only possible answer. □

Since $V(1) \wedge H\pi_*BP\langle 2\rangle$ is an $H\mathbb{F}_p$-algbera, $V(1) \wedge THH(H\pi_*BP\langle 2\rangle$ is also an $H\mathbb{F}_p$-algebra. We recall the following well known result.

Lemma 6.12 (cf. [1])**.** *Let M be an $H\mathbb{F}_p$-module. Then M is equivalent to a wedge of suspensions of $H\mathbb{F}_p$, and the Hurewicz map*

$$\pi_*M \to H_*M$$

is an injection whose image is sub-comodule of $\mathscr{A}_*$-comodule primitives.

Corollary 6.13. *The E^1-page of the $V(1)$-THH-May spectral sequence is*

$$E^1 \cong E(\lambda_1, \sigma v_1, \sigma v_2) \otimes P(\widetilde{\mu}_1, v_2).$$

There is a differential

$$d^1(\widetilde{\mu}_1) \doteq \sigma v_1$$

and multiplicativity of the spectral sequence implies all remaining d^1 differentials.

[1] We needed the class μ_1 as opposed to $\sigma\tau_1$ since λ_2, λ_3 are comodule primitives.

From this we can determine the E^2-page,

Corollary 6.14. *Consequnently, there is an isomorphism*

$$E^2(BP\langle 2\rangle; k(2)) \cong E(\lambda_1, \sigma v_2, \mu_1^2\sigma v_1) \otimes P(\mu_1^3, v_2).$$

Furthermore, the map on the E^2-term induced by the Hurewicz map is again injective. Since the $H\mathbb{F}_p$-THH-May spectral sequence has no d^3 or d^4 differentials, the map on E^3 and E^4-terms is also injective. The next differential is

$$d^4(\mu_1^3) \doteq \sigma v_2$$

and there are no further d^4 differentials except those implied by the Leibniz rule in target spectral sequence 6.7. This results in

$$E^5(BP\langle 2\rangle; k(2)) \cong E(\lambda_1, \widetilde{\mu_1}^2\sigma v_1, \widetilde{\mu_1}^6\sigma v_2) \otimes P(\widetilde{\mu_1}^9).$$

Note that

$$\mu_1^9 = \mu_3.$$

Thus we can rename μ_1^9 as μ_3. Renaming classes, the E^5-page is

$$E^5 \cong E(\lambda_1, \lambda_2, \lambda_2) \otimes P(\mu_3, v_2)$$

where v_2 is in May filtration 4. Thus the E^5-page is a reindexed form of the v_2-BSS, and this determines the rest of the $V(1)$-May spectral sequence.

[Dom: Here is an idea: to make things easier for the reader, maybe we should have a table somewhere in the paper with all the names of the various elements, their representatives, their coactions, etc.]

6.3. **The THH-May spectral sequence for $BP\langle 2\rangle$ with coefficients in $BP\langle 2\rangle/p$.** We now study the THH-May spectral sequence (6.6) for $M = BP\langle 2\rangle/p$. We begin by determining the E^1-term.

Proposition 6.15. *The E^1-term of (6.1) is isomorphic to*

$$E(\lambda_1) \otimes P(\mu_1) \otimes P(v_1, v_2) \otimes E(\sigma v_1, \sigma v_2),$$

and the spectral sequence has only the d^1 differential, $d^1\mu_1 \doteq \sigma\, v_1$. Consequently, the E^2-term of (6.1) is

$$E(\lambda_1, \mu_1^2\,\sigma v_1) \otimes P(\mu_1^3) \otimes P(v_1, v_2) \otimes E(\sigma v_2).$$

Proof. The description of the E^1-term follows directly from the lemma and Proposition 6.9. Because the map on E^1-terms is injective, we can pull back differentials, which provides the stated d^1 differential. (Note that that the Hurewicz map is not injective at the E^2-page so we cannot continue in this manner as in the previous section.) ☐

We now use the fact that $\mu_1^2\,\sigma v_1$ detects λ_2 to rename this class. We also rename the class μ_1^p by μ_2.

Proposition 6.16. *There is a d^3 differential*

$$d^3(\mu_2) = v_1^3\lambda_1$$

and no further differentials of this length. The E^4 term of (6.1) is

$$H_*(E(\lambda_1) \otimes P(v_1) \otimes P(\mu_2); d^3(\mu_2) = v_1^3\lambda_1) \otimes E(\lambda_2) \otimes P(v_2) \otimes E(\sigma v_2)$$

The reader may be concerned at this point that $v_1^3\lambda_1$ dies and yet it survived in the first Bockstein spectral sequence computing $THH_*(BP\langle 2\rangle; k(1))$. However, note that in the THH-May spectral sequence the names of classes often change and there is still a class, namely σv_2, which survives in the degree of $v_1^3\lambda_1$.

Proof. Note that there is a map of THH-May spectral sequences with abutment

$$V(0)_*THH(BP\langle 2\rangle) \to V(0)_*THH(BP\langle 1\rangle)$$

and with input

$$P(\lambda_1) \otimes P(\mu_1) \otimes P(v_1, v_2) \otimes E(\sigma v_1, \sigma v_2) \to P(\lambda_1) \otimes P(\mu_1) \otimes P(v_1) \otimes E(\sigma v_1)$$

and by inspection all classes map to classes of the same name except v_2 and σv_2, which map to zero. In the target spectral sequence, we compute the differential $d_1(\mu_1) = \sigma v_1$ by the same means as we did before. Therefore, the map of E^2-terms is

$$P(\lambda_1, \lambda_2) \otimes P(\mu_2) \otimes P(v_1, v_2) \otimes E(\sigma v_2) \to P(\lambda_1, \lambda_2) \otimes P(\mu_2) \otimes P(v_1)$$

and again the classes all map to classes of the same name except v_2 and σv_2, which map to zero. Note that this verifies that the renaming of λ_2 and μ_2 is reasonable. The target of this map is exactly the same as the input of the Bockstein spectral sequence computing $THH_*(BP\langle 1\rangle; k(1))$ and therefore we know what the remaining differentials have to be by McClure-Staffeldt [12]. In particular, there is a differential $d^3(\mu_2) = v_1^3\lambda_1$ and this is the only differential of this length. This implies that the same differential takes place in the source spectral sequence. To see that there are no further differentials of this length in the source note that the only possibility would be a differential with source σv_2 or v_2 and there are no possible differentials of this length on these classes for degree reasons. □

We now note that

$$H_*(E(\lambda_1) \otimes P(v_1) \otimes P(\mu_2); d^3(\mu_2) = v_1^3\lambda_1) \cong \left(P(v_1, \mu_2^3) \otimes E(\lambda_1, \lambda_1\mu_2, \lambda_1\mu_2^2)\right) / \sim$$

where $\sim$ is the relation

$$\begin{aligned} \lambda_1 \cdot (\lambda_1\mu_2) &= 0 \\ \lambda_1 \cdot (\lambda_1\mu_2^2) &= 0 \\ \lambda_1\mu_2 \cdot (\lambda_1\mu_2^2) &= 0 \\ v_1^3 \cdot \lambda_1 &= 0 \\ v_1^3 \cdot \lambda_1\mu_2 &= 0. \end{aligned}$$

and the classes $\lambda_1\mu_2$ and $\lambda_1\mu_2^2$ are not in the output of either of the Bockstein spectral sequences.

$$THH_*(BP\langle 2\rangle, H\mathbb{F}_p)[v_1] \Rightarrow THH_*(BP\langle 2\rangle, k(1))$$

and

$$THH_*(BP\langle 2\rangle, H\mathbb{F}_p)[v_2] \Rightarrow THH_*(BP\langle 2\rangle, k(2))$$

and therefore they cannot survive the $V(0)$-based THH-May spectral sequence. (Note that these classes are no longer decomposable). This forces the following differentials.

Lemma 6.17. *There is a differential $d_4(\lambda_1\mu_2) = \lambda_1\sigma v_2$ and $d_4(\lambda_1\mu_2^2) = \lambda_1\sigma v_2\mu_2$ which generates families of differentials by multiplicativity of the $V(0)$-THH-May spectral sequence and no further differentials of this length.*

Proof. We know that the elements $\lambda_1\mu_2$ and $\lambda_1\mu_2^2$ must not be cycles by the argument above and the fact that they are not boundaries. We therefore check the possible targets of a differential and the possibilities are $\mu_2 v_1, \lambda_1\lambda_2, \lambda_1\sigma v_2$ for $\lambda_1\mu_2$. We now observe that there is no differential on $\lambda_1\mu_2$ in the $V(0)$-THH-May spectral sequence computing $V(0)_*THH(BP\langle 1\rangle)$ so if there is a differential on $\lambda_1\mu_2$ in the $V(0)$-THH-May spectral sequence computing $V(0)_*THH(BP\langle 2\rangle)$ it must hit something that maps to zero under the map of THH-May spectral sequences. The only one of the three classes named above that maps to zero under this map of spectral sequences is $\lambda_1\sigma v_2$. This forces the stated differential. □

We conclude that there is an isomorphism

$$E_{p+2} \cong E(\lambda_1, \lambda_2, \sigma v_2, \lambda_1\mu_2^2\sigma v_2) \otimes P(\mu_3, v_1, v_2)/\sim$$

where $v_1^3\lambda_1 \sim 0$, $\lambda_1\sigma v_2 \sim 0$, $\lambda_1\sigma v_2\mu \sim 0$, etc. There is therefore an additive isomorphism

$$E_5 \cong E(\lambda_1, \lambda_2, \lambda_3) \otimes P(\mu_3, v_1, v_2)$$

where we make the additive identifications $\lambda_1 \cdot v_1^3 \doteq \sigma v_2$, $\lambda_1 v_1^3\mu^2 \doteq \lambda_3$, $\lambda_1\sigma v_2\mu_2^2 = \lambda_1\lambda_3$, $\lambda_2 \cdot (\lambda_1\sigma v_2\mu_2^2) \doteq \lambda_1\lambda_2\lambda_3$.

Note that the class $\lambda_2 v_1^9$ doesn't survive the Bockstein spectral sequence

$$THH_*(BP\langle 2\rangle; H\mathbb{F}_3)[v_1] \Rightarrow THH_*(BP\langle 2\rangle; k(1)),$$

but it does survive the Bockstein spectral sequence

$$THH_*(BP\langle 2\rangle; H\mathbb{F}_3)[v_2] \Rightarrow THH_*(BP\langle 2\rangle; k(2)).$$

Similarly, the class $\lambda_1 v_2^3$ doesn't survive the Bockstein spectral sequence

$$THH_*(BP\langle 2\rangle; H\mathbb{F}_3)[v_2] \Rightarrow THH_*(BP\langle 2\rangle; k(2)),$$

but it does survive the Bockstein spectral sequence

$$THH_*(BP\langle 2\rangle; H\mathbb{F}_3)[v_1] \Rightarrow THH_*(BP\langle 2\rangle; k(1)).$$

One may think that this forces a differential hitting $\lambda_2 v_1^9$ the second Bockstein spectral sequence $THH_*(BP\langle 2\rangle; k(1))[v_2] \Rightarrow THH_*(BP\langle 2\rangle; BP\langle 2\rangle/3)$ and a differential hitting $\lambda_1 v_2^3$ in the Bockstein spectral sequence $THH_*(BP\langle 2\rangle; k(2))[v_1] \Rightarrow THH_*(BP\langle 2\rangle; BP\langle 2\rangle/3)$. However, we will show that in fact neither differential occurs and instead there is a hidden multiplicative extension that resolves the conflict.

[Gabe: I think I should add a section before this one where the spectral sequence

$$V(0)_*THH(H\pi_*BP\langle 2\rangle; H\pi_*BP\langle 1\rangle) \Rightarrow V(0)_*THH(BP\langle 2\rangle; BP\langle 1\rangle)$$

is discussed.]

Lemma 6.18. *In the $V(0)$-THH-May spectral sequence with coefficients computing $V(0)_*THH(BP\langle 2\rangle; BP\langle 1\rangle)$, there is a differential*

$$d^9(\mu_3) = \lambda_2 v_1^9.$$

There is also an additive isomorphism

$$E^{p+2} \cong E(\lambda_1, \lambda_2, \lambda_3) \otimes P(\mu_3) \otimes P(v_1)$$

and the remaining differentials are the same as the differentials in the Bockstein spectral sequence (**??**).

Proof. Consider the map of multiplicative THH-May spectral sequences

$$\begin{tikzcd} E^1(BP\langle 2\rangle) = V(0)_*THH(H\pi_*BP\langle 2\rangle; H\pi_*BP\langle 1\rangle) \arrow[r, Rightarrow] \arrow[d] & S/3_*THH(BP\langle 2\rangle; BP\langle 1\rangle) \arrow[d] \\ E^1(BP\langle 1\rangle) = V(0)_*THH(H\pi_*BP\langle 1\rangle, H\pi_*BP\langle 1\rangle) \arrow[r, Rightarrow] & S/3_*THH(BP\langle 1\rangle) \end{tikzcd}$$

induced by a map $BP\langle 2\rangle \to BP\langle 1\rangle$ of commutative ring spectra. The input is easily computed to be

$$E^1(BP\langle 2\rangle) = E(\lambda_1) \otimes P(\mu_1) \otimes P(v_1) \otimes E(\sigma v_1, \sigma v_2)$$

where we divide the May filtration degree by $2p-2$. There is then a differential $d_1(\mu_1) = \sigma v_1$ by the same considerations as before. Thus,

$$E^2(BP\langle 2\rangle) = E(\lambda_1, \lambda_2) \otimes P(\mu_2) \otimes P(v_1) \otimes E(\sigma v_2)$$

where $\lambda_2 = \lambda_1 \mu_1^{p-1}$ and $\mu_2 = \mu_1^p$ and similarly

$$E^2(BP\langle 1\rangle) = E(\lambda_1, \lambda_2) \otimes P(\mu_2) \otimes P(v_1)$$

for the same reason. The map on E^2-terms is exactly the surjection that sends each element to the element of the same name besides σv_2, which maps to zero. We know the remaining differentials in the THH-May spectral sequence with E^2-term $E^2(BP\langle 1\rangle)$ since they correspond exactly to differentials in the Bockstein spectral sequence of [12]. Therefore, there are differentials $d_p(\mu_2) = v_1^3\lambda_1$ and $d_{p^2}(\mu_3) = v_1^9\lambda_2$ in that spectral sequence. This implies the same differentials in the THH-May spectral sequence with E^2-term $E^2(BP\langle 2\rangle)$ and consequently the THH-May spectral sequence

$$S/3_*THH(H\pi_*BP\langle 2\rangle) \Rightarrow S/3_*THH(BP\langle 2\rangle).$$

This also implies E^{p+2}-term of this spectral sequence is additively isomorphic to

$$E(\lambda_1, \lambda_2, \lambda_3) \otimes P(\mu_3, v_1, v_2)$$

and that the next differential in the spectral sequence is $d_9(\mu_3) = v_1^9\lambda_2$. By comparing to the two Bockstein spectral sequences, the only possibility is then that there is a hidden multiplicative extension $v_1^9 \cdot \lambda_2 = v_2^3 \cdot \lambda_1$. $\square$

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