THE TOPOLOGICAL HOCHSCHILD HOMOLOGY OF THE SECOND TRUNCATED BROWN-PETERSON SPECTRUM

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Contents

1. Introduction

We compute...

Occasionally, we will need to use an E_∞-model of $BP\langle 2\rangle$. When this is the case, we will assume that $p = 2$ or $p = 3$, since these are the only primes for which it is known there are E_∞-models. See [7] and [8].

Conventions.

2. Preliminary results

We begin by computing the topological Hochschild homology of R with coefficients in $\mathbb{F}_p$, $\mathrm{THH}(R; H\mathbb{F}_p)$. It suffices to compute the sub-algebra of co-mododule primitives in $H_* \operatorname{THH}(R; H\mathbb{F}_p)$ by [5, Lem. 4.1] for example. By the Künneth spectral sequence and [4, Cor. 5.13] we obtain

$$H_* \operatorname{THH}(R; \mathbb{F}_p) \cong \mathcal{A}_* \otimes E(\lambda_1, \lambda_2, \lambda_3) \otimes P(\mu_3). \tag{2.1}$$

where $\lambda_i = \sigma\bar{\xi}_i$, $\mu_3 = \sigma\bar{\tau}_2$, and the co-action is determined by the formula

$$\nu(\sigma x) = ((1 \otimes \sigma) \circ \nu)(x))$$

and the coaction on elements in $\mathcal{A}_*$ are given by the coproduct [4, Equation 5.11]. We see that

$$\nu(\lambda_i) = \lambda_i$$

because $\sigma\bar{\xi}_i^{p^k} = 0$ for $k \geq 1$ and $i \geq 1$ and

$$\nu(\mu_3) = \mu_3 + \bar{\tau}_0 \otimes \lambda_3.$$

Since there are no other comodule primitives, we get an isomorphism,

$$\operatorname{THH}_*(R; \mathbb{F}_p) \cong E(\lambda_1, \lambda_2, \lambda_3) \otimes P(\tilde{\mu}_3). \tag{2.2}$$

The degrees are $|\lambda_i| = 2p^i - 1$ and $|\mu_3| = 2p^3$ where $\tilde{\mu}_3 = \mu_3 - \bar{\tau}_0\lambda_3$

2.1. **Rational homology.** Next, we compute the rational homology of $\operatorname{THH}(R)$, as this locates the torsion free component inside $\operatorname{THH}_*(R)$. Towards this end, we will use the $H\mathbb{Q}$-based Bökstedt spectral sequence. This is a spectral sequence of the form

$$\operatorname{HH}_*^{\mathbb{Q}}(H\mathbb{Q}_* R) \implies H\mathbb{Q}_* \operatorname{THH}(R).$$

Recall that the rational homology of R is given by

$$H\mathbb{Q}_* R \cong P_{\mathbb{Q}}[v_1, v_2].$$

Thus the E_2-term of the Bökstedt spectral sequence is

$$P_{\mathbb{Q}}(v_1, v_2) \otimes E_{\mathbb{Q}}(\sigma v_1, \sigma v_2)$$

and the bidegree of σv_i is $(1, 2(p^i - 1))$. For degree reasons, there can be no differentials, showing that the E_∞-term is this $\mathbb{Q}$-algebra. There are also no hidden extensions. Thus, we have that

$$\operatorname{THH}_*(R) \otimes \mathbb{Q} \cong P_{\mathbb{Q}}(v_1, v_2) \otimes_{\mathbb{Q}} E_{\mathbb{Q}}(\sigma v_1, \sigma v_2)$$

where $|\sigma v_i| = 2p^i - 1$.

3. The $H\mathbb{Z}$-Bockstein spectral sequences

Working from our calculation of $\operatorname{THH}(R; \mathbb{F}_p)$ we will analyze three Bockstein spectral sequences to obtain the topological Hochschild homology of R with coefficients in the connective Morava K-theories $k(0), k(1), k(2)$. Of course, $k(0)$ is $H\mathbb{Z}_{(p)}$. These spectral sequences take the following forms:

$$\operatorname{THH}_*(R; \mathbb{F}_p)[v_0] \implies \operatorname{THH}_*(R; \mathbb{Z}_{(p)})_p^\wedge \tag{3.1}$$

$$\operatorname{THH}_*(R; \mathbb{F}_p)[v_1] \implies \operatorname{THH}_*(R; k(1)) \tag{3.2}$$

$$\operatorname{THH}_*(R; \mathbb{F}_p)[v_2] \implies \operatorname{THH}_*(R; k(2)) \tag{3.3}$$

The methods for these calculations are inspired by [9],[4], and [3].

3.1. **The $H\mathbb{Z}$-Bockstein spectral sequence.** This calculation is based on the $H\mathbb{Z}$-BSS found in [3]. For this calculation, we need to know the comodule structure of the mod p homology of $\mathrm{THH}(R)$. This was computed in [4] (as Theorem 5.12). They showed that

$$H_* \,\mathrm{THH}(R) \cong H_*R \otimes E(\lambda_1, \lambda_2, \lambda_3) \otimes P(\mu_3) \tag{3.4}$$

and that λ_i are A_*-primitives and

$$\alpha(\mu) = 1 \otimes \mu_3 + \overline{\tau}_0 \otimes \lambda_3.$$

This translates into the following differential in the spectral sequence (3.1):

$$d_1(\mu) = \lambda_3$$

In [3], they use the following fact from May's "General algebraic approach to Steenrod Operations"

Lemma 3.5. *If x supports a d_j differential in the Bockstein spectral sequence then*

$$d_{j+1}(x^p) = v_0 x^{p-1} d_j(x)$$

if $p > 2$ or if $p = 2$ and $j \geq 2$. If $p = 2$ and $j = 1$ then there is an error term of $\mathcal{P}_4(d_1(x))$.

[Dom: I am actually not sure if this is the correct error term. Quite honestly, I am not sure how [3] get this result for the E_∞-context.]

When $p = 2$, we have the differential

$$d_1(\mu) = v_0\lambda_3.$$

So the error term for $d_2(\mu^2)$ is

$$Q^{16}\lambda_3 = Q^{16}(\sigma\zeta_3^2) = \sigma(Q^{16}\zeta_3^2) = \sigma(Q^8\zeta_3)^2 = \sigma(\zeta_4^2) = 0.$$

From this we derive

Proposition 3.6. *When the prime is 2 or 3, then we have*

$$d_{i+1}(\mu^{p^i}) = v_0^{i+1}\mu^{p^i-1}\lambda_3.$$

Consequently, we have

$$d_{\nu_p(k)+1}(\mu^k) \doteq v_0^{\nu_p(k)+1}\mu^{k-1}\lambda_3$$

where $\nu_p(k)$ denotes the p-adic valuation of k.

Proof. Let $\alpha = \nu_p(k)$. We have that $k = p^\alpha j$ where p does not divide j. So by Leibniz

$$d_{\alpha+1}(\mu^k) = d_{\alpha+1}((\mu^{p^\alpha})^k) = k\mu^{p^\alpha(k-1)}d_{\alpha+1}(\mu^\alpha) = kv_0^{\alpha+1}\mu^{p^\alpha(k-1)}\mu^{p^\alpha-1}\lambda_3 = kv_0^{\alpha+1}\mu^{k-1}\lambda_3.$$

Since k is not divisible by p, it is a unit mod p. □

Thus we have the following,

Corollary 3.7. *The E_∞ page of of the $H\mathbb{Z}$-Bockstein spectral sequence for R is the algebra*

$$P(v_0) \otimes E(\lambda_1, \lambda_2, \lambda_3) \otimes P(\mu)/(v_0^{\nu_p(k)+1}\mu^k\lambda_3 \mid i \geq 1).$$

[Dom: I think we should put a picture here]

[Gabe: We need to include an argument for why there are no other differentials. For d_1 differentials, seems to follow from the fact that the generators λ_i are in odd degrees which are smaller than μ, but we have to rule later differentials on classes that become indecomposable on later pages too. Whatever argument they use in AHL should work here as well though.]

4. Computation of $THH(BP\langle 2\rangle; k(2))$

The goal is to compute $THH(R; k(2))$ via the v_2-Bockstein spectral sequence (3.3),

$$THH_*(R; H\mathbb{F}_p)[v_2] \Rightarrow THH(R; k(2)).$$

The first goal is to show

$$K(2)_* \cong THH_*(BP\langle 2\rangle; K(2)) \tag{4.1}$$

which will imply that all the classes except the classes in the subalgebra $P(v_2) \subset THH_*(BP\langle 2\rangle; H\mathbb{F}_p)[v_2]$ are v_2-torsion and this will force differentials in the spectral sequence. Our approach is entirely analogous to the calculation of McClure-Staffeldt except for one minor difference, which we will point out.

To compute $THH_*(BP\langle 2\rangle; K(2))$, we can first compute

$$K(2)_*THH(BP\langle 2\rangle; K(2))$$

and then use the fact that $THH(BP\langle 2\rangle; K(2))$ is a free $K(2)$-module (since $K(2)$ is a field spectrum) and the collapse of the $K(2)$-based Adams spectral sequence to finish the computation.

We use the $K(2)$-based Bökstedt spectral sequence to compute $K(2)_*THH(BP\langle 2\rangle; K(2))$; i.e. the spectral sequence

$$HH_*^{K(2)_*}(K(2)_*BP\langle 2\rangle; K(2)) \Rightarrow K(2)_*THH(BP\langle 2\rangle; K(2)).$$

the first goal will be to compute the input.

Lemma 4.2. *There is an isomorphism of graded rings*

$$K(n)_*BP\langle n\rangle \cong K(n)_*[t_1, t_2, \dots]/(v_n t_k^{p^n} - v_n^{p^k} t_k | k \geq 1).$$

Proof. We adapt the proof in McClure-Staffeldt. First $K(n)_*BP \cong K(n)_* \otimes_{BP_*} BP_*BP$ because BP is Landweber exact. Furthermore, $K(n)_* \otimes_{BP_*} BP_*BP \cong K(n)_*[t_1, t_2, \dots]$ and we can restrict $\eta_R : BP_* \to BP_*BP$ to $K(n)_* \otimes_{BP_*} BP_*BP$ to produce the map $\bar{\eta}_R$ and by Ravenel

$$\bar{\eta}_R(v_{n+k}) = v_n t_k^{p^n} - v_n^{p^k} t_k \text{ mod } (\bar{\eta}_R(v_{n+1}), \bar{\eta}_R(v_{n+2}), \dots \bar{\eta}_R(v_{n+k-1})) \tag{4.3}$$

We can then construct $BP\langle n\rangle$ using Baas-Sullivan theory and the effect is that

$$K(n)_*BP\langle n\rangle \cong K(n)_* \otimes_{BP_*} BP_*BP/(\bar{\eta}_R(v_{n+1}), \bar{\eta}_R(v_{n+2}), \dots)$$

or in other words, by (4.3)

$$K(n)_*BP\langle n\rangle \cong K(n)_*[t_1, t_2, \dots]/(v_n t_k^{p^n} - v_n^{p^k} t_k | k \geq 1)$$

as desired. □

Remark 4.4. *Note that the model of $BP\langle 2\rangle$ that we used in this lemma is not the same model that gives you an E_∞-structure, but since there is a weak equivalence $BP\langle 2\rangle \simeq tmf_1(3)$ the map $K(n)_*BP\langle 2\rangle \cong K(n)_*tmf_1(3)$ as $K(n)_*$-modules.*

[Gabe: Is the equivalence $BP\langle 2\rangle \simeq tmf_1(3)$ known to be an equivalence of E_2-algebras? Does this reasoning make sense to you?]

[Dom: I think that the calculation of $K(2)_*\mathrm{tmf}_1(3)$ does not rely on the E_∞-structure, but rather only on the $\mathrm{tmf}_1(3)$ as a ring object in the stable homotopy category. In this case, $BP\langle 2\rangle$ and $\mathrm{tmf}_1(3)$ are isomorphic *after* completion at 2 (this is work of Angeltveit-Lind). We may have to be careful about the generators though; the generators of $\pi_*\mathrm{tmf}_1(3)$ are not the same as the Araki generators. We should discuss this more.]

We now describe the structure of $K(2)_*[t_1, t_2, \dots]/(v_2t_k^{p^2} - v_2^{p^k}t_k|k \geq 1)$. Note that it can be written as

$$K(2)_*[t_1, t_2, \dots]/(v_2t_k^{p^2} - v_2^{p^k}t_k|k \geq 1) \cong \bigotimes_{k\geq 1} K(2)_*[t_k]/(v_2t_k^{p^2} - v_2^{p^k}t_k) \tag{4.5}$$

where the tensor is taken over $K(2)_*$. Note that $|t_k| = 2p^k - 2$ and $2(p^2-1)|2(p^k-1)$ when k is even and $2(p-1)|2(p^k-1)$ for all k. Therefore,

$$K(2)_*[t_k]/(v_2t_k^{p^2} - v_2^{p^k}t_k) \cong K(2)_* \otimes \mathbb{F}_p[u_k]/(u_k^{p^2} - v_2^{p^k-1}u_k]$$

where $u_k = t_kv_2^{m(k)}$ where $m(k) = -p^{k-2} - p^{k-4} - \dots p^2 - 1$ when $2|k$ and

$$K(2)_*[t_k]/(v_2t_k^{p^2} - v_2^{p^k}t_k) \cong K(2)_* \otimes \mathbb{F}_p[w_k]/(w_k^{p^2} - v_2^{p^k-1}w_k]$$

where $w_k = t_kv_2^{\ell(k)}$ where $\ell(k) = -p^{k-2} - p^{k-4} - \dots - p$ and k is odd so that $|w_k| = 2p-2$.

Lemma 4.6. *There is an isomorphism*

$$K(2)_*K(2) \cong HH_*^{K(2)_*}(K(2)_*BP\langle 2\rangle; K(2)_*K(2))$$

and hence the $K(2)_*$-based Adams spectral sequence collapses with no room for hidden extensions and the natural map

$$K(2)_*K(2) \to K(2)_*THH(BP\langle 2\rangle; K(2))$$

is an isomorphism

Proof. Since $K(2)_*BP\langle 2\rangle$ is flat over $K(2)_*$

$$HH_*^{K(2)_*}(K(2)_*BP\langle 2\rangle) \cong Tor_*^{K(2)_*BP\langle 2\rangle \otimes_{K(2)_*} K(2)_*BP\langle 2\rangle}(K(2)_*BP\langle 2\rangle; K(2)_*BP\langle 2\rangle).$$

Also, by (4.5),

$$\begin{aligned}
&Tor_*^{(K(2)_*BP\langle 2\rangle)^e}(K(2)_*BP\langle 2\rangle; K(2)_*BP\langle 2\rangle) \cong \\
&\textstyle\bigotimes_{k\geq 1} Tor_*^{K(2)_*[t_k]/(v_2t_k^{p^2} - v_2^{p^k}t_k))^e}(K(2)_*[t_k]/(v_2t_k^{p^2} - v_2^{p^k}t_k); K(2)_*[t_k]/(v_2t_k^{p^2} - v_2^{p^k}t_k)) \cong \\
&\textstyle\bigotimes_{k\geq 1; k|2} K(2)_* \otimes HH_*^{K(2)_*}(K(2)_*[u_k]/(v_2u_k^{p^2} - v_2^{p^k}u_k)) \otimes \\
&\textstyle\bigotimes_{k\geq 1; (k+1)|2} K(2)_* \otimes HH_*^{K(2)_*}(K(2)_*[w_k]/(v_2w_k^{p^2} - v_2^{p^k}w_k))
\end{aligned}$$

By Cartan-Eilenberg, for $k \geq 0$ an odd integer

$$HH_*^{K(2)_*}(K(2)_*[w_k]/(v_2w_k^{p^2}-v_2^{p^k}w_k) \cong K(2)_*[w_k]/(v_2w_k^{p^2}-v_2^{p^k}w_k)\otimes_{K(2)_*}Tor^{K(2)_*K(2)_*[w_k]/(v_2w_k^{p^2}-v_2^{p^k}w_k}(K(2$$

and by an elementary calculation,

$$Tor^{K(2)_*K(2)_*[w_k]/(v_2w_k^{p^2}-v_2^{p^k}w_k}(K(2)_*, K(2)_*) \cong K(2)_*$$

and therefore

$$HH_*^{K(2)_*}(K(2)_*[w_k]/(v_2w_k^{p^2} - v_2^{p^k}w_k) \cong K(2)_*[w_k]/(v_2w_k^{p^2} - v_2^{p^k}w_k).$$

Also, there is an isomorphism

$$HH_*^{K(2)_*}(K(2)_*[u_k]/(v_2u_k^{p^2} - v_2^{p^k}u_k)) \cong K(2)_* \otimes HH_*(\mathbb{F}_p[u_k]/(u_k^{p^2} - u_k))$$

and since

$$\mathbb{F}_p[u_k]/(u_k^{p^2} - u_k$$

is isomorphic as a $\mathbb{F}_p$-algebra to a product of finite field extensions of $\mathbb{F}_p$

[Gabe: We should be more precise here.]

and since Hochschild homology commutes with limits and $HH_*(\mathbb{F}_{p^n}) \cong \mathbb{F}_{p^n})$,

$$HH_*(\mathbb{F}_p[u_k]/(u_k^{p^2} - u_k)) \cong \mathbb{F}_p[u_k]/(u_k^{p^2} - u_k).$$

Putting this all together, we produce an isomorphism

$$K(2)_*BP\langle 2\rangle \cong HH_*^{K(2)_*}(K(2)_*BP\langle 2\rangle)$$

and since

$$HH_*^{K(2)_*}(K(2)_*BP\langle 2\rangle; K(2)_*K(2)) \cong K(2)_*K(2) \otimes_{K(2)_*BP\langle 2\rangle} HH_*^{K(2)_*}(K(2)_*BP\langle 2\rangle)$$

we produce the desired isomorphism

$$K(2)_*K(2) \cong HH_*^{K(2)_*}(K(2)_*BP\langle 2\rangle; K(2)_*K(2))$$

The Bökstedt spectral sequence

$$HH_*^{K(2)_*}(K(2)_*BP\langle 2\rangle; K(2)_*K(2)) \Rightarrow K(2)_*THH(BP\langle 2\rangle; K(2))$$

therefore collapses with no room for hidden extensions and hence the map

$$K(2) \to THH(BP\langle 2\rangle; K(2))$$

induces a $K(2)_*$-equivalence. □

Corollary 4.7. *The map $K(2) \to THH(BP\langle 2\rangle; K(2))$ is a weak equivalence and therefore*

$$THH_*(BP\langle 2\rangle; k(2)) \cong P(v_2) \otimes T$$

where T is a v_2-torsion $P(v_2)$-module.

Proof. Since the map $K(2) \to THH(BP\langle 2\rangle; K(2))$ induces an isomorphism $K(2)_*K(2) \cong K(2)_*THH(BP\langle 2\rangle; K(2))$, the $K(2)$-based Adams spectral sequence for $THH(BP\langle 2\rangle; K(2))$ converges and collapses to the zero line and the map of $K(2)$-based Adams spectral sequences induces an isomorphism

$$K(2)_* \to THH_*(BP\langle 2\rangle; K(2)).$$

Since we have a map that induces an isomorphism on homotopy groups the Whitehead theorem for spectra implies that the map $K(2) \to THH(BP\langle 2\rangle; K(2))$ is a weak equivalence.

Alternatively, we could compute $THH_*(BP\langle 2\rangle; K(2))$ using the v_2-inverted classical Adams spectral sequence, which is equivalent to the Bockstein spectral sequence

$$THH_*(BP\langle 2\rangle; H\mathbb{F}_p)[v_2^{\pm 1}] \Rightarrow THH_*(BP\langle 2\rangle; K(2))$$

and by the computation we just did, we know that all the classes must die except those in $P(v_2^{\pm 1})$. There is also a map of spectral sequences

$$\begin{tikzcd} THH_*(BP\langle 2\rangle; H\mathbb{F}_p)[v_2^{\pm 1}] \arrow[r, Rightarrow] & THH_*(BP\langle 2\rangle; K(2)) \\ THH_*(BP\langle 2\rangle; H\mathbb{F}_p)[v_2] \arrow[r, Rightarrow] \arrow[u] & THH_*(BP\langle 2\rangle; k(2)) \arrow[u] \end{tikzcd}$$

because $v_2^{-1}(-)$ is a localization. This implies that

$$THH_*(BP\langle 2\rangle; k(2)) \cong P(v_2) \otimes T$$

and forces differentials in the bottom spectral sequence above. $\square$

Corollary 4.8. *There are differentials $d_{r(n)}(\mu^{r(n)}) = \lambda_{[n]}v_2^{r(n)}$ where $r(n)$ is ..., and $\lambda_{[n]}$ is ....*

[Gabe: Finish the corollary above.]

[Dom: I am writing a slightly different way of approaching this computation, mostly for my benefit. We can decide how to merge these two together later...]

The goal of this section is to compute the homotopy groups of $\mathrm{THH}(R; k(2))$. We achieve this through an analysis of of the v_2-Bockstein spectral sequence (3.3). We first outline our strategy.

In [4] and [9], to compute $\mathrm{THH}_*(\ell, k(1))$, the authors first argue that upon inverting v_1, there is an isomorphism

$$v_1^{-1}\,\mathrm{THH}_*(\ell; k(1)) \cong K(1)_*. \tag{4.9}$$

This implies that in the v_1-Bockstein spectral sequence

$$\mathrm{THH}_*(\ell; \mathbb{F}_p)[v_1] \implies \mathrm{THH}_*(\ell; k(1))$$

all classes except 1 are v_1-torsion. It turns out that there is only one of pattern of differentials that makes this possible, which gives a complete description of this spectral sequence.

Here, we proceed in much the same way. However, some adaptions need to be made. More specifically, in establishing (4.9), [4] and [9] smash $\mathrm{THH}(\ell)$ with the mod p Moore spectrum and then take the v_1-telescope. It is difficult to adjust this proof to the case at hand, because it would require smashing the spectrum $\mathrm{THH}(R)$ with the Smith-Toda complex $V(1)$, which does not exist at $p = 2$ and is not a ring spectrum at $p = 3$.

Instead, we opt for a different approach. It is based upon the following observations. Inverting v_2 in spectra

[Gabe: Here and elsewhere we should be a bit more precise about what we mean by inverting v_2 in spectra.]

provides an equivalence

$$v_2^{-1}\,\mathrm{THH}(R; k(2)) \simeq \mathrm{THH}(R; K(2)).$$

There is a canonical unit morphism

$$K(2) \to \mathrm{THH}(R; K(2))$$

which we will argue induces an isomorphism in $K(2)$-homology. Since the source and target are both $K(2)$-modules, and hence $K(2)$-local, this will show that the map is in fact an equivalence of spectra.

To establish this, we just need to argue that

$$K(2)_*\,\mathrm{THH}(R; K(2)) \cong K(2)_*K(2).$$

To do this, we proceed as follows. First, recall that the Morava K-theories possess a Künneth isomorphism, which gives us a Bökstedt spectral sequence

$$\mathrm{HH}_*^{K(2)_*}(K(2)_*R) \implies K(2)_*\,\mathrm{THH}(R). \tag{4.10}$$

We will analyze this spectral sequence below to show that $K(2)_*\,\mathrm{THH}(R)$ is isomorphic to $K(2)_*R$. This part of our analysis is a modification of the calculation of $K(1)_*\,\mathrm{THH}(\ell)$ found in [9]. There is also an equivalence

$$\mathrm{THH}(R; K(2)) \simeq_{S^0} K(2) \wedge_R \mathrm{THH}(R).$$

To compute the $K(2)$-homology of this spectrum, we apply a Eilenberg-Moore type spectral sequence (cf. [6, IV, 6.4]), which takes the form

$$\mathrm{Tor}_{s,t}^{K(2)_*R}(K(2)_*K(2), K(2)_*\,\mathrm{THH}(R)) \implies K(2)_{s+t}(\mathrm{THH}(R; K(2))).$$

Since $K(2)_*\,\mathrm{THH}(R)$ is just $K(2)_*R$, the E_2-term is concentrated in $s = 0$, resulting in the collapsing of this spectral sequence. This will show that

$$K(2)_*\,\mathrm{THH}(R; K(2)) \cong K(2)_*K(2)$$

from which we can conclude that every class but 1 in $\mathrm{THH}(R; k(2))$ is v_2-torsion. From this we will deduce the differentials in the v_2-Bockstein spectral sequence.

[Dom: Here is a technical point: In either of the approaches we are taking, we will make use of the Eilenberg-Moore spectral sequence. All that is stated in EKMM is the abutment of this spectral sequence. We should take a look at convergence issues, just to be sure.]

4.1. **The $K(2)$-homology of** $\mathrm{THH}(R;K(2))$**.** To begin, we need to compute $K(2)_*R$. Since the Johnson-Wilson theory $E(2)$ is Landweber exact, one has

$$E(2)_*R \cong E(2)_* \otimes_{BP_*} BP_*BP \otimes_{BP_*} R_*.$$

It is known that

$$BP_*BP \otimes_{BP_*} R_* \cong BP_*[t_1, t_2, \dots]/(\eta_R(v_i) \mid i \geq 3)$$

where $\eta_R : BP_* \to BP_*BP$ denotes the right unit. Thus,

$$E(2)_*BP\langle 2\rangle \cong E(2)_*[t_1, t_2, \dots]/(\eta_R(v_i) \mid i \geq 3).$$

Since $K(2)$ is obtained from $E(2)$ by coning off p and v_1, we find that

$$K(2)_*R \cong K(2)_*[t_1, t_2, \dots]/(\eta_R(v_i) \mid i \geq 3).$$

We have the following congruences

$$\eta_R(v_{2+k}) \equiv v_2 t_k^{p^2} - v_2^{p^k} t_k \mod (\eta_R(v_3), \dots, \eta_R(v_{k+1})).$$

in $K(2)_*BP$ for all $k \geq 1$ (cf. formula 6.1.13 of [11]). Thus

Lemma 4.11. *There is an isomorphism of graded rings*

$$K(2)_*R \cong K(2)_*[t_1, t_2, \dots]/(v_2 t_k^{p^2} - v_2^{p^k} t_k \mid k \geq 1)$$

[Gabe: Nice! I like this proof of this lemma a lot better than the sketch I gave.]

We proceed to analyze the $K(2)$-Bökstedt spectral sequence for $\mathrm{THH}(R)$, (4.10). We begin by determining the E^2-page

$$E^2_{*,*} \cong \mathrm{HH}^{K(2)_*}(K(2)_*R).$$

Recall that the topological degree of t_k is $2(p^k-1)$, and that the degree of v_2 is $2(p^2-1)$. Thus $|v_2|$ divides $|t_k|$ if and only if k is even. Observe that

$$K(2)_*R \cong_{K(2)_*} \bigotimes_k K(2)_*[t_k]/(v_2 t_k^{p^2} - v_2^{p^k} t_k)$$

Let $u_k = v_2^{m(k)} t_k$ where

$$m(k) = -p^{k-2} - p^{k-4} - \dots p^2 - 1$$

when $2|k$ and and let $u_k = v_2^{\ell(k)} t_k$ where

$$\ell(k) = -p^{k-2} - p^{k-4} - \dots - p$$

when k is odd. Thus

$$|u_k| = \begin{cases} 0 & k \equiv 0 \mod 2 \\ 2(p-1) & k \equiv 1 \mod 2 \end{cases}.$$

Define A_n to be the subalgebra of $K(2)_*R$ generated by $t_1, \dots, t_n$, and let $A(t_k)$ denote the subalgebra generated by t_k. Then elementary properties of Hochschild homology give

$$\mathrm{HH}^{K(2)_*}(A_n) \cong_{K(2)_*} \bigotimes_{k=1}^n \mathrm{HH}^{K(2)_*}(A(t_k)).$$

Since Hochschild homology commutes with colimits, it follows that

$$\mathrm{HH}^{K(2)_*}(K(2)_*R) \cong_{K(2)_*} \bigotimes_{k=1}^{\infty} \mathrm{HH}^{K(2)_*}(A(t_k)),$$

so we are reduced to computing the Hochschild homology of the subalgebras generated by a single t_k. When k is even, we have

$$A(t_k) = K(2)_* \otimes \mathbb{F}_p[u_k]/(u_k^{p^2} - u_k),$$

in which case one has

$$\mathrm{HH}^{K(2)_*}(A(t_k)) \cong_{K(2)_*} K(2)_* \otimes \mathrm{HH}^{\mathbb{F}_p}(\mathbb{F}_p[u_k]/(u_k^{p^2} - u_k)).$$

Since the $\mathbb{F}_p$-algebra

$$\mathbb{F}_p[u_k]/(u_k^{p^2} - u_k)$$

ut reference ere...

is étale over $\mathbb{F}_p$, it follows that

$$\mathrm{HH}^{\mathbb{F}_p}(\mathbb{F}_p[u_k]/(u_k^{p^2} - u_k)) \cong \mathbb{F}_p[u_k]/(u_k^{p^2} - u_k).$$

hat's the ex- t proposi- on in Cartan- ilenberg??

Moving on to the case when k is odd, it follows from Cartan-Eilenberg that

$$\mathrm{HH}^{K(2)_*}(A(t_k)) \cong A(t_k) \otimes_{K(2)_*} \mathrm{Tor}^{A(t_k)}(K(2)_*, K(2)_*)$$

Lemma 4.12. *For odd k, we have*

$$\mathrm{Tor}^{A(t_k)}(K(2)_*, K(2)_*) \cong K(2)_*$$

Proof. .... $\square$

Thus, we find that when k is odd,

$$\mathrm{HH}^{K(2)_*}(A(t_k)) \cong_{K(2)_*} A(t_k).$$

Combining all these observations together, we have proven,

Theorem 4.13. *The Hochschild homology of $K(2)_*R$ is isomorphic to $K(2)_*R$:*

$$\mathrm{HH}^{K(2)_*}(K(2)_*R) \cong K(2)_*R.$$

We now proceed as how we sketched above. Consider the Eilenberg-Moore spectral sequence

$$\mathrm{Tor}_{s,t}^{K(2)_*R}(K(2)_*K(2), K(2)_*\,\mathrm{THH}(R)) \implies K(2)_{s+t}(\mathrm{THH}(R; K(2))).$$

By the theorem, the E^2-term is

$$\mathrm{Tor}^{K(2)_*R}(K(2)_*K(2), K(2)_*R).$$

Thus, the the E^2-term is concentrated in the $s = 0$-line, where it is exactly $K(2)_*K(2)$. Consequently the spectral sequence collapses, showing the following.

Corollary 4.14. *The $K(2)$-homology of $\mathrm{THH}(R; K(2))$ is isomorphic to $K(2)_*K(2)$. Since $\mathrm{THH}(R; K(2))$ is a $K(2)$-module, it follows that the unit morphism*

$$K(2) \to \mathrm{THH}(R; K(2))$$

is a weak equivalence.

Corollary 4.15. *In $\mathrm{THH}_*(R; k(2))$ all classes other than 1 are v_2-torsion.*

4.2. **Differentials in the v_2-BSS.** We now turn to analyzing the v_2-BSS (3.3). In particular, we will argue that Corollary 4.15 allows for a single pattern of differentials in the spectral sequence. Our argument is an adaptation of the one found in [9].

Recall that the E_2-term of the v_2-BSS is

$$\mathrm{THH}(R;\mathbb{F}_p)[v_2] \cong P(v_2)\otimes E(\lambda_1,\lambda_2,\lambda_3)\otimes P(\mu_3),$$

where

$$|\lambda_i| = (2p^i-1,0)$$

and

$$|\mu_3| = (2p^3,0).$$

It will be more convenient to work in the v_2-localized Bockstein spectral sequence. Since the λ_i are in odd total degree and 1 is v_2-torsion free, they cannot support a differential. If μ_3 is a permanent cycle as well, then by multiplicativity of the Bockstein spectral sequence, it follows that it will collapse at E_1. But this would contradict Corollary 4.15. Thus μ_3 supports a differential, the only possibility is

$$d_p(\mu_3)\doteq v_2^p\lambda_1.$$

Thus

$$v_2^{-1}E_{p+1}^{*,*}\cong K(2)_*\otimes E(\lambda_2,\lambda_3,\lambda_4)\otimes P(\mu_3^p),$$

where $\lambda_4 := \lambda_1\mu_3^{p-1}$. Note that the bidegree of λ_4 is

$$|\lambda_4| = (2p^4-2p^3+2p-1,0).$$

In particular, its total degree is odd. So this class cannot support a differential which truncates the the v_2-tower on λ_2 or λ_3. So this class is a permanent cycle. By multiplicativity again, if μ_3^p were an infinite cycle, then the spectral sequence would collapse at E_{p+1}, which would contradict Corollary 4.15. So μ_3^p supports a differential. The only possibility is

$$d_{p^2}(\mu_3^p)\doteq v_2^{p^2}\lambda_2.$$

Thus one has

$$v_2^{-1}E_{p^2+1}^{*,*}\cong K(2)_*\otimes E(\lambda_3,\lambda_4,\lambda_5)\otimes P(\mu_3^{p^2})$$

where

$$\lambda_5 := \lambda_2\mu_3^{p^2-p}.$$

The bidegree of this class is

$$|\lambda_5| = (2p^5-2p^4+2p^2-1,0).$$

Since $\lambda_3,\lambda_4,\lambda_5$ all have odd total degree, they are necessarily permanent cycles. As before, the class $\mu_3^{p^2}$ must support a differential. The only possibility is

$$d_{p^3}(\mu_3^{p^2})\doteq v_2^{p^3}\lambda_3.$$

This shows that

$$v_2^{-1}E_{p^3+1}^{*,*}\cong K(2)_*\otimes E(\lambda_4,\lambda_5,\lambda_6)\otimes P(\mu_3^{p^4})$$

where

$$\lambda_6 := \lambda_3\mu_3^{p^2(p-1)} = \lambda_3\mu_3^{p^3-p^2},$$

so that the bidegree of λ_6 is

$$|\lambda_6| = (2p^6 - 2p^5 + 2p^3 - 1, 0).$$

Consequently, as we saw before, the class λ_6 cannot support a differential, and hence is a permanent cycle. As before, we must have that $\mu_3^{p^3}$ must support a differential. An elementary calculation shows the only possibility is

$$d_{p^4+p}\mu^{p^3} \doteq v_2^{p^4+p}\lambda_4$$

Recursively define a function $d(n)$ by

$$d(n) := \begin{cases} 2p^n - 1 & \text{if } 1 \leq n \leq 3 \\ 2p^3(p^{n-3} - p^{n-4}) + d(n-3) \end{cases}$$

and recursively define classes λ_n by

$$\lambda_n := \begin{cases} \lambda_n & 1 \leq n \leq 3 \\ \lambda_{n-3}\mu^{p^{n-4}(p-1)} & n > 3 \end{cases}.$$

Then a simple inductive argument shows that the bidegree of λ_n is given by

$$|\lambda_n| = (d(n), 0). \tag{4.16}$$

Notice that $d(n)$ is always odd, and so λ_n is always in odd total degree. A simple induction shows that

$$d(n) = \begin{cases} 2p^n - 2p^{n-1} + 2p^{n-2} - 2p^{n-3} + \cdots + 2p - 1 & n \equiv 1 \mod 3 \\ 2p^n - 2p^{n-1} + 2p^{n-2} - 2p^{n-3} + \cdots + 2p^2 - 1 & n \equiv 2 \mod 3 \\ 2p^n - 2p^{n-1} + 2p^{n-2} - 2p^{n-3} + \cdots + 2p^3 - 1 & n \equiv 0 \mod 3 \end{cases}.$$

> **[Gabe: Since, for example, $d(4) = 2p^3(p-1) + 2p - 1$, $d(7) = 2p^3(p^4 - p^3) + 2p^3(p-1) + 2p - 1 = 2p^7 - 2p^6 + 2p^4 - 2p^3 + 2p - 1$ so the formulas above aren't exactly right.]**

Lemma 4.17. *The integer $2p^{n+2} - d(n) - 1$ is divisible by $|v_2|$.*

Proof. Via induction. One has

$$(2p^{n+3} - 1) - d(n+1) = (2p^{n+3} - 2p^{n+1}) + (2p^n - d(n-2) - 1)$$

and the induction hypothesis shows the second term is divisible by $|v_2|$. □

> **[Gabe: You didn't prove the base step in the induction above. Also, why does that equality hold? What is the convention for $d(n)$ when $n < 0$, that makes this hold for $n < 3$ or is this only true when $n \geq 3$?]**

Now let $r(n)$ be the function given by

$$r(n) := |v_2|^{-1}(2p^{n+2} - d(n) - 1).$$

Then we obtain as a corollary to the lemma,

Corollary 4.18. *The function $r(n)$ is given by*

$$r(n) = \begin{cases} p^n + p^{n-3} + \cdots + p^4 + p & n \equiv 1 \mod 3 \\ p^n + p^{n-3} + \cdots + p^5 + p^2 & n \equiv 2 \mod 3 \\ p^n + p^{n-3} + \cdots + p^6 + p^3 & n \equiv 0 \mod 3 \end{cases}.$$

We are now in a position to determine the differentials in the spectral sequence.

Theorem 4.19. *In the v_2-BSS, one has*

(1) *The only nonzero differentials are in $v_2^{-1}E_{r(n)}$.*
(2) *The page $v_2^{-1}E_{r(n)}$ is given by*

$$v_2^{-1}E_{r(n)} \cong K(2)_* \otimes E(\lambda_n, \lambda_{n+1}, \lambda_{n+2}) \otimes P(\mu_3^{p^{n-1}}).$$

Moreover, $\lambda_n, \lambda_{n+1}, \lambda_{n+2}$ are permanent cycles.
[Gabe: Wait, but λ_n cannot be a permanent cycle in the v_2^{-1} Bockstein spectral sequence because it is v_2-torsion. I believe you mean it will be a permanent cycle before inverting v_2?]

(3) *The differential $d_{r(n)}$ is determined by the multiplicativity of the BSS and*

$$d_{r(n)}\mu_3^{p^{n-1}} = v_2^{r(n)}\lambda_n.$$

[Gabe: We need to discuss why this BSS is multiplicative at some point earlier]

Proof. We proceed by induction, having already shown the theorem for $n \leq 4$. Assume inductively that

$$v_2^{-1}E_{r(n)}^{*,*} \cong K(2)_* \otimes E(\lambda_n, \lambda_{n+1}, \lambda_{n+2}) \otimes P(\mu_3^{p^{n-1}}).$$

By the inductive hypothesis, λ_n, λ_{n+1} are permanent cycles. Since $\lambda_n, \lambda_{n+1}, \lambda_{n+2}$ all have odd total degree, it follows that λ_{n+2} cannot truncate the v_2-towers on λ_n or λ_{n+1}. Therefore, the only possibility is that λ_n supports a differential hitting v_2^j for some $j \in \mathbb{Z}$. But that would contradict Corollary 4.14. So λ_{n+2} must also be a cycle.

If the class $\mu_3^{p^{n-1}}$ does not support a differential, then by multiplicativity the spectral sequence would collapse at $E_{r(n)}$, and this would contradict Corollary 4.14. Thus $\mu_3^{p^{n-1}}$ supports a differential. Lemma 4.17 and a simple modular arithmetic argument shows that the only possibility is

$$d_{r(n)} \doteq v_2^{r(n)}\lambda_n.$$

Since the differential satisfies the Leibniz rule, this gives

$$v_2^{-1}E_{r(n)+1} \cong K(2)_* \otimes E(\lambda_{n+1}, \lambda_{n+2}, \lambda_{n+3}) \otimes P(\mu_3^{p^n}).$$

This completes the inductive step, proving the theorem. ∎

prove the analogue of Theorem 7.1 of [9]

5. The v_1-Bockstein spectral sequence

In this section we will begin our analysis of the v_1-Bockstein spectral sequence for computing the homotopy of $\mathrm{THH}(R; k(1))$, i.e. the spectral sequence (3.2). Our approach is very similar to the one outlined in the previous section.

To start, we need to compute $K(1)_*BP\langle 2\rangle$. This requires determining $\eta_R(v_{2+n})$ in $K(1)_*BP$ modulo the ideal generated by $(\eta_R(v_3), \dots, \eta_R(v_{1+n}))$. We will need the following.

Lemma 5.1. [11, Lemma A.2.2.5] *Let v_n denote the Araki generators. Then there is the following equality in BP_*BP*

$$\sum_{i,j\geq 0}^F t_i \eta_R(v_j)^{p^i} = \sum_{i,j\geq 0}^F v_i t_j^{p^i}$$

In our context, the distinction between Hazewinkel generators and Araki generators is unimportant, as the two sets of generators coincide modulo p. In $K(1)_*BP$, we have killed all v_i's except v_1, which gives us the following equation

$$\sum_{i,j\geq 0}^F t_i \eta_R(v_j)^{p^i} = \sum_{k\geq 0}^F v_1 t_k^p$$

Note that the following degrees of the terms:

$$|v_1 t_j^p| = 2(p^{j+1} - 1)$$
$$|t_i \eta_R(v_j)^{p^i}| = 2(p^{i+j} - 1)$$

Since we are interested in the term $\eta_R(v_{2+n})$, we collect all the terms on the left of degree $2(p^{2+n} - 1)$. Thus we are summing over the ordered pairs (i, j) such that $i + j = 2 + n$. Since we only care about $\eta_R(v_{2+n})$ modulo $\eta_R(v_3), \dots, \eta_R(v_{1+n})$ we only need to collect the terms where $j = 1, 2$, or $2 + n$. This shows that

$$t_{1+n}\eta_R(v_1)^{p^{n+1}} + t_n \eta_R(v_2)^{p^n} + \eta_R(v_{n+2}) = v_1 t_{n+1}^p$$

The value of η_R on v_1 and v_2 can also be computed by Lemma 5.1. One obtains, in $K(1)_*BP$, the following

$$\eta_R(v_1) = v_1$$
$$\eta_R(v_2) = v_1 t_1^p - t_1 v_1^p.$$

Combining these observations, we obtain

Lemma 5.2. *In $K(1)_*BP$, the following congruence is satisfied*

$$\eta_R(v_{2+n}) \equiv v_1 t_{n+1}^p - v_1^{p^n} t_1^{p^{n+1}} t_n + v_1^{p^{n+1}}(t_1^{p^n} t_n - t_{n+1}) \mod (\eta_R(v_3), \dots, \eta_R(v_{1+n}))$$

for $n \geq 1$.

[Dom: URG!! that is not too pretty.]

[Gabe: Perhaps leaving it as

$$\eta_R(v_{1+k}) = v_1 t_k^p - t_k v_1^{p^k} - t_{k-1}\eta_R(v_2)^{p^{k-1}} \mod (\eta_R(v_3), \dots, \eta_R(v_{1+k-1}))$$

for $k \geq 2$ is simpler?]

Consequently, we have

Corollary 5.3. *There is an isomorphism of $K(1)_*$-algebras*

$$K(1)_*R \cong K(1)_*BP/(v_1 t_{n+1}^p - v_1^{p^n} t_1^{p^{n+1}} t_n + v_1^{p^{n+1}}(t_1^{p^n} t_n - t_{n+1}) \mid n \geq 1)$$

Let $u_n := v_1^{\frac{p^n-1}{p-1}} t_n$. These elements are in degree 0, giving us an isomorphism

$$K(1)_*R \cong_{K(1)_*} K(1)_* \otimes_{\mathbb{F}_p} K(1)_0 R.$$

The calculations above tell us

Corollary 5.4. *There is an isomorphism of $\mathbb{F}_p$-algebras*

$$K(1)_0 R \cong \mathbb{F}_p[t_i \mid i \geq 1]/(u_{n+1}^p - u_1^{p^{n+1}} u_n + u_1^{p^n} u_n - u_{n+1} \mid n \geq 1).$$

Our goal is to use this and the $K(1)$-based Bökstedt spectral sequence to compute the $K(1)$-homology of $\mathrm{THH}(R)$. This is a spectral sequence of the form

$$E^2_{s,t} = \mathrm{HH}_s^{K(1)_*}(K(1)_*R) \implies K(1)_{s+t}\,\mathrm{THH}(R).$$

The above considerations tell us that the E^2-page is

$$E^2 \cong K(1)_* \otimes \mathrm{HH}^{\mathbb{F}_p}(K(1)_0 R).$$

The following will be useful for our calculation.

Lemma 5.5 ([10])**.** *Let $V = \mathrm{Spec}(A)$ be a nonsingular affine variety over a field k. Let W be the subvariety of $V \times \mathbb{A}^n$ defined by equations*

$$g_i(Y_1, \ldots, Y_n) = 0,\ g_i \in A[Y_1, \ldots, Y_n],\ i = 1, \ldots, n.$$

Then the projection map $W \to V$ is étale at a point $(P; b_1, \ldots, b_n)$ of W if and only if the Jacobian matrix $\left(\frac{\partial g_i}{\partial Y_j}\right)$ is a nonsingular matrix at $(P; b_1, \ldots, b_n)$.

Theorem 5.6 (Étale Descent, [12])**.** *Let $A \hookrightarrow B$ be an étale extension of commutative k-algebras. Then there is an isomorphism*

$$\mathrm{HH}_*(B) \cong \mathrm{HH}_*(A) \otimes_A B$$

[Dom: here is an idea for how to use this isomorphism. Hochschild homology behaves nicely for etale extensions, can we show that the finitely generated subalgebras of the above are etale over $\mathbb{F}_p$. I think this may be a good avenue since this is the essential point in the calculation of McClure and Staffeldt.]

Example 5.7. Consider the subalgebra

$$\mathbb{F}_p[u_1, u_2]/(u_2^p - u_1^{p^2+1} + u_1^{p+1} - u_2 = f_1).$$

We will regard this as a $\mathbb{F}_p[u_1]$-algebra. The derivative $\partial_{u_2} f_1$ is -1, and therefore a unit at every point. Then Lemma 5.5 tells us that this algebra is then étale over $\mathbb{F}_p[u_1]$.

[Dom: but we really want étale over $\mathbb{F}_p$. I have a feeling its not etale over $\mathbb{F}_p$.]

[Gabe: I like this idea! Let me run with it a bit]

By the same argument given above, there are a sequence of sub-algebras A_n of

$$K(1)_0R \cong \mathbb{F}_p[t_i \mid i \geq 1]/(u_{n+1}^p - u_1^{p^{n+1}}u_n + u_1^{p^n}u_n - u_{n+1} \mid n \geq 1) =: A.$$

such that each map $A_i \to A_{i+1}$ is an étale map. By the étale base change formula for Hochschild homology,

$$\mathrm{HH}_*^{\mathbb{F}_p}(A_{i+1}) \cong \mathrm{HH}_*^{\mathbb{F}_p}(A_i) \otimes_{A_i} A_{i+1}$$

and since HH_* commutes with colimits of k-algebras, we have the following:

$$\begin{array}{rcl} \mathrm{HH}_*^{\mathbb{F}_p}(A) & \cong & \mathrm{HH}_*^{\mathbb{F}_p}(\operatorname{colim} A_n) \\ & \cong & \operatorname{colim} \mathrm{HH}_*^{\mathbb{F}_p}(A_n) \\ & \cong & \operatorname{colim} \mathrm{HH}_*^{\mathbb{F}_p}(A_1) \otimes_{A_1} A_n \\ & \cong & \mathrm{HH}_*^{\mathbb{F}_p}(A_1) \otimes_{A_1} A \end{array}$$

where we have the second to last isomorphism by an easy induction and the last isomorphism because colimits of k-algebras commute with $\otimes$. This shows that

$$\mathrm{HH}_*(K(1)_*R) \cong K(1)_* \otimes E(\sigma t_1) \otimes K_0(R)$$

and therefore, since $\sigma t_1 = \lambda_1$,

$$K(1)_* \mathrm{THH}(R) \cong K(1)_*R \otimes E(\lambda_1)$$

and

$$\mathrm{THH}_*(R; K(1)) \cong K(1)_* \otimes E(\lambda_1)$$

or in other words λ_1 is $P(v_1)$-free and the differentials will just go between the other two generators.

[Gabe: I still have to check a lot of parts of this argument, but it seems promising.]

We have the input needed to compute the following Eilenberg-Moore spectral sequence,

$$\mathrm{Tor}^{K(1)_*R}(K(1)_*K(1), K(1)_* \mathrm{THH}(R)) \implies K(1)_* \mathrm{THH}(R; K(1))$$

From the previous computation, the E^2-term is concentrated in Tor_0 and is

$$K(1)_*K(1) \otimes E(\lambda_1).$$

Thus, every class besides 1 and λ_1 is v_1-torsion in $\mathrm{THH}_*(R; k(1))$. Since $\mathrm{THH}(R; K(1))$ is a $K(1)$-module, this implies that

$$\mathrm{THH}(R; K(1)) \simeq K(1) \vee \Sigma^{2p-1}K(1).$$

In summary, we have shown

Theorem 5.8.

(1) *The $K(1)$-homology of $\mathrm{THH}(R; K(1))$ is $K(1)_*K(1) \otimes E(\lambda_1)$.*
(2) *The only v_1-torsion free classes in $\mathrm{THH}(R; k(1))$ are 1 and λ_1.*

5.1. **Differentials in the v_1-BSS.** We now analyze the v_1-BSS (3.2). Recall that this spectral sequence is of the form

$$\mathrm{THH}(R;\mathbb{F}_p)[v_1] \implies \mathrm{THH}(R;k(1)).$$

Thus the E_1-page is

$$K(1)_* \otimes E(\lambda_1,\lambda_2,\lambda_3)\otimes P(\mu_3).$$

Since the λ_i are all in odd total degree and since 1 is to be v_1-torsion free, the λ_i are all permanent cycles. If μ_3 were a permanent cycle, then by multiplicativity, the spectral sequence would collapse, and this would contradict Theorem 5.8. So we must have that μ_3 supports a differential. The only possibility is

$$d_{p^2}(\mu_3)\doteq v_1^{p^2}\lambda_2.$$

Thus, one obtains

$$v_1^{-1}E_{p^2+1}\cong K(1)_*\otimes E(\lambda_1,\lambda_3,\lambda_4)\otimes P(\mu_3^p)$$

where

$$\lambda_4' := \lambda_2\mu_3^{p-1}.$$

So the bidegree of λ_4' is given by

$$|\lambda_4'| = (2p^4-2p^3+2p^2-1,0).$$

For analogous reasons, the class λ_4' is a permanent cycle, and μ_3^p cannot be a permanent cycle. Based on degree considerations, there are two possible differentials,

$$d_{p^2}(\mu_3^p)\doteq v_1^{p^2}\lambda_4'$$

or

$$d_{p^3}(\mu_3^p)\doteq v_1^{p^3}\lambda_3.$$

The first would contradict the Leibniz rule for d_{p^2} and the fact that $d_{p^2}(\mu_3)\doteq v_1^{p^2}\lambda_2$. This leaves the second as the only possibility. Thus

$$v_1^{-1}E_{p^3+1}\cong K(1)_*\otimes E(\lambda_1,\lambda_4',\lambda_5')\otimes P(\mu_3^{p^2})$$

where

$$\lambda_5 := \lambda_3\mu_3^{p(p-1)}.$$

The bidegree of λ_5' is

$$|\lambda_5'| = (2p^5-2p^4+2p^3-1,0).$$

For degree reasons, the class λ_5' is a permanent cycle. As before, the class $\mu_3^{p^2}$ must support a differential. Degree considerations, again, give two possibilities

$$d_{p^3}(\mu_3^{p^2})\doteq v_1^{p^3}\lambda_5'$$

or

$$d_{p^4+p^2}(\mu_3^{p^2})\doteq v_1^{p^4+p^2}\lambda_4'.$$

The former would contradict the Leibniz rule, leaving the latter as the only possibility. This gives us

$$v_1^{-1}E_{p^4+p^2+1}\cong K(1)_*\otimes E(\lambda_1,\lambda_5',\lambda_6')\otimes P(\mu_3^{p^3})$$

where $\lambda_6' := \lambda_4'\mu_3^{p^2(p-1)}$. We will continue via induction. First we need some notation. We will recursively define classes λ_n' by

$$\lambda_n' := \begin{cases} \lambda_n & n = 1,2,3 \\ \lambda_{n-2}'\mu_3^{p^{n-4}(p-1)} & n \geq 4 \end{cases}$$

We let $d'(n)$ denote the topological degree of λ_n'. Then this function is given recursively by

$$d'(n) = \begin{cases} 2p^n - 1 & n = 1,2,3 \\ 2p^n - 2p^{n-1} + d(n-2) & n > 3 \end{cases}$$

Thus, by a simple induction, one has

$$d'(n) = \begin{cases} 2p^n - 1 & n = 1,2,3 \\ 2p^n - 2p^{n-1} + 2p^{n-2} - 2p^{n-3} + \cdots + 2p^2 - 1 & n \equiv 0 \mod 2,\ n > 3 \\ 2p^n - 2p^{n-1} + 2p^{n-2} - 2p^{n-3} + \cdots + 2p^3 - 1 & n \equiv 1 \mod 2,\ n > 3 \end{cases}.$$

Observe that the integers $2p^{n+1} - d(n) - 1$ and $2p^{n+1} - d(n+1) - 1$ are both divisible by $|v_1|$. Let $r'(n)$ denote the integer

$$r'(n) := |v_1|^{-1}(|\mu_3^{p^{n-1}} - |\lambda_n'| - 1) = |v_1|^{-1}(2p^{n+2} - d'(n+1) - 1).$$

Then a simple induction shows that

$$r'(n) = \begin{cases} p^{n+1} + p^{n-1} + p^{n-3} + \cdots + p^2 & n \equiv 1 \mod 2 \\ p^{n+1} + p^{n-1} + p^{n-3} + \cdots + p^3 & n \equiv 0 \mod 2 \end{cases}.$$

We can now describe the differentials in the v_1-BSS.

Theorem 5.9. *In the v_1-BSS, one has*

(1) *The only nonzero differentials are in $v_1^{-1}E_{r'(n)}$.*
(2) *The $r'(n)$th page is given by*

$$v_1^{-1}E_{r'(n)} \cong K(1)_* \otimes E(\lambda_1, \lambda_n', \lambda_{n+1}') \otimes P(\mu_3^{p^{n-1}})$$

and the classes $\lambda_n', \lambda_{n+1}'$ are permanent cycles.
(3) *The differential $d_{r'(n)}$ is uniquely determined by multiplicativity of the BSS and the differential*

$$d_{r'(n)}(\mu_3^{p^{n-1}}) \doteq v_1^{r'(n)}\lambda_n'.$$

Proof. We proceed by induction. We have already shown the theorem for $n \leq 4$. Assume inductively that

$$v_1^{-1}E_{r'(n)} \cong K(1)_* \otimes E(\lambda_1, \lambda_n', \lambda_{n+1}') \otimes P(\mu_3^{p^{n-1}}).$$

By inductive hypothesis, λ_n' is a permanent cycle.

[Gabe: Again, you are saying λ_n' is a permanent cycle in the v_1-inverted spectral sequence, but this can't be the case. I think you mean that it is a permanent cycle before inverting v_1.]

Since $\lambda'_n, \lambda'_{n+1}$ are both in odd topological degree, λ'_{n+1} cannot support a differential into the v_1-towers on λ'_n. Thus the only possibility is that λ'_{n+1} supports a differential into the v_1-tower on 1 or λ_1. But this would contradict Theorem 5.8. So λ'_{n+1} is a permanent cycle.

The class $\mu_3^{p^{n-1}}$ must support a differential, for if it did not, then the spectral sequence would collapse. This would lead to a contradiction of Theorem 5.8. Degree considerations show that the following differentials are possible

$$d_{k(n)}(\mu_3^{p^{n-1}}) \doteq v_1^{k(n)}\lambda'_{n+1}$$

and

$$d_{r(n)}(\mu_3^{p^{n-1}}) \doteq v_1^{r'(n)}\lambda'_n$$

where

$$k(n) = |v_1^{-1}|(2p^{n+2} - |\lambda'_{n+1}|).$$

An elementary inductive computation shows that

$$k(n) = r'(n-1).$$

The former differential cannot occur, for by the inductive hypothesis,

$$d_{r'(n-1)}(\mu_3^{p^{n-2}}) \doteq \lambda'_{n-1},$$

and the former differential would contradict the Leibniz rule. So we must conclude the latter differential occurs. This concludes proof. □

Prove the analogue of Theorem 7.1 of [9]

6. The v_2-inverted homotopy of $\mathrm{THH}(BP\langle 2\rangle)$

In this section, we briefly describe the calculation of $v_2^{-1}\,\mathrm{THH}(BP\langle 2\rangle)$ based on recent work in progress of Ausoni-Richter. First, we note that the localization $v_2^{-1}\,\mathrm{THH}(BP\langle 2\rangle)$ agrees with the smashing localization $L_2^f\,\mathrm{THH}(BP\langle 2\rangle)$ (H. Miller)[?qx]. Also, note that by construction the functor $\mathrm{THH}(-)$ commutes with smashing localizations. Therefore, there are equivalences $v_2^{-1}THH(BP\langle 2\rangle; BP\langle 2\rangle) \simeq THH(BP\langle 2\rangle; E(2)) \simeq THH(E(2)$ where $E(2)$ is Johnson-Wilson E-theory, which has coefficients $\pi_*E(2) \cong \mathbb{Z}_{(p)}[v_1, v_2^{\pm 1}]$. We now recall pre-theorem of Ausoni-Richter, which depends on the following assumption: the spectrum $E(2)$ can be constructed as an E_∞-ring spectrum.

Theorem 6.1. *There is an equivalence*

$$THH(E(2)) \simeq E(2) \vee \Sigma^{2p^2-1}L_1E(2) \vee \Sigma^{2p-1}L_{H\mathbb{Q}}E(2) \vee \Sigma^{2p^2+2p-2}L_{H\mathbb{Q}}E(2).$$

Consequently,

$$v_2^{-1}THH(BP\langle 2\rangle; BP\langle 2\rangle) \simeq E(2) \vee \Sigma^{2p^2-1}L_1E(2) \vee \Sigma^{2p-1}L_{H\mathbb{Q}}E(2) \vee \Sigma^{2p^2+2p-2}L_{H\mathbb{Q}}E(2)$$

and therefore

$$v_2^{-1}THH(BP\langle 2\rangle; BP\langle 2\rangle/3) \simeq E(2)/3 \vee \Sigma^{2p^2-1}L_1E(2)/3$$

since smashing localizations commute with cofiber sequences.

7. Topological Hochschild comology of $BP\langle 2\rangle$ with coefficients

We will write $THC(BP\langle 2\rangle)$ for topological Hochschild cohomology of $BP\langle 2\rangle$, which may be defined as

$$THC(BP\langle 2\rangle) = F_{BP\langle 2\rangle^e}(BP\langle 2\rangle, BP\langle 2\rangle)$$

where $BP\langle 2\rangle^e := BP\langle 2\rangle \wedge BP\langle 2\rangle^{\mathrm{op}}$. We recall that there is a UCT spectral sequence computing $F_{BP\langle 2\rangle^e}(BP\langle 2\rangle, BP\langle 2\rangle)$ with input

$$Ext_{\pi_*(BP\langle 2\rangle^e)}(\pi_*(BP\langle 2\rangle), \pi_*(BP\langle 2\rangle)) \Rightarrow THC^*(BP\langle 2\rangle),$$

but this is usually not computable. With coefficients in $H\mathbb{F}_p$, however, it is easily computable in this case. First, note that

$$THH_*(BP\langle 2\rangle; H\mathbb{F}_p) \cong E(\lambda_1, \lambda_2, \lambda_3) \otimes P(\mu_3)$$

is a finitely generated $\mathbb{F}_p$-algebra and $THH(BP\langle 2\rangle; H\mathbb{F}_p))$ is an $H\mathbb{F}_p$-algebra. Also, by adjunction

$$F_{H\mathbb{F}_p}(THH(BP\langle 2\rangle), H\mathbb{F}_p) \simeq THC(BP\langle 2\rangle, H\mathbb{F}_p).$$

Consequently, the UCT spectral sequence collapses and

$$THC^*(THH(BP\langle 2\rangle, H\mathbb{F}_p) \cong Hom_{\mathbb{F}_p}(THH_*(BP\langle 2\rangle; H\mathbb{F}_p), \mathbb{F}_p) \cong E(x_1, x_2, x_3) \otimes \Gamma(c_1)$$

where $|x_i| = 2p^i - 1$ and $|c_1| = 2p^3$. The classes x_i are dual to λ_i and the class $c_i = \gamma_i(c_1)$ is dual to μ_3^i.

8. The mod p homotopy of THH(R)

In this section, we begin our study of the mod p homotopy of THH(R). At first, we will assume that $p = 3$, since in this case the mod 3 Moore spectrum $V(0)$ is a ring spectrum. Our approach to this computation will be to make use of the *THH-May spectral sequence*, which was developed by the first author and Andrew Salch in [2] and applied by the first author in [1].

Let us briefly describe the strategy we will employ. The mod 3 homotopy of THH(R) is exactly the homotopy groups

$$\pi_*(\mathrm{THH}(R); \mathbb{Z}/3) := \pi_*(V(0) \wedge \mathrm{THH}(R)) = V(0)_* \mathrm{THH}(R).$$

To compute this, we will use the $V(0)$-based THH-May spectral sequence. Using the Whitehead filtration for R as developed in [2], the THH-May spectral sequence based on $V(0)$ takes the form

$$\pi_*(\mathrm{THH}(E_0^* R); \mathbb{Z}/3) \implies \pi_*(\mathrm{THH}(R); \mathbb{Z}/3) \tag{8.1}$$

In order to obtain the first several differentials in this spectral sequence, we will consider the $H \wedge V(0)$-based THH-May spectral sequence. This takes the form

$$H_*(V(0) \wedge \mathrm{THH}(E_0^* R)) \implies H_*(V(0) \wedge \mathrm{THH}(R)). \tag{8.2}$$

The abutment of this spectral sequence is known from which we will determine the differentials in (8.2).

The morphism $V(0) \to H \wedge V(0)$ of spectra gives us a morphism of THH-May spectral sequences

$$\begin{tikzcd} \pi_*(\mathrm{THH}(E_0^*R);\mathbb{Z}/3) \arrow[r, Rightarrow] \arrow[d, hook] & \pi_*(\mathrm{THH}(R);\mathbb{Z}/3) \arrow[d] \\ H_*(V(0)\wedge \mathrm{THH}(E_0^*R)) \arrow[r, Rightarrow] & H_*(V(0)\wedge \mathrm{THH}(R)) \end{tikzcd} \tag{8.3}$$

We will argue that the map on E^1-terms is injective, which will allow us to determine the first several pages of the $V(0)$-based THH-May spectral sequence.

8.1. **Review of the THH-May spectral sequence.** The THH-May spectral sequence takes as input a cofibrant decreasingly filtered commutative monoid I in spectra (specifically symmetric spectra of pointed simplicial sets with the positive stable flat model structure) and produces a spectral sequence

$$E_1^{*,*} = E_*THH(E_0^*I) \Rightarrow E_*THH(I_0)$$

for any connective generalized homology theory E First, recall the definition of a cofibrant decreasingly filtered commutative monoid in spectra.

Definition 8.4. A cofibrant decreasingly filtered commutative monoid in spectra I is a lax symmetric monoidal functor $\mathbb{N}^{\mathrm{op}} \to \mathcal{S}$, which is cofibrant in the projective model structure on the functor category.

Remark 8.5. Note that this differs slightly from the original definition in [2], every cofibrant decreasingly filtered commutative monoid in spectra in the sense of 8.4 is in particular a cofibrant decreasingly filtered commutative monoid in spectra in the sense of [2, Def. qx]. Also, in the final version of [2] the authors added a couple assumptions to the definition, but these turn out to be redundant in symmetric spectra of pointed simplicial sets with the positive stable flat model structure, so we do not include them here.

To a cofibrant decreasingly filtered commutative monoid I we can associate its associated graded commutative ring spectrum E_0^*I. It is constructed as a commutative monoid in spectra in **??** and as a spectrum it is defined to be $E_0^*I = \vee I_i/I_{i+1}$ where I_i is our decreasingly filtered commutative monoid evaluated at a natural number i, I_i/I_{i+1} is the cofiber of the cofibration $I_{i+1} \to I_i$ and the wedge is taken over all natural numbers. Given an object in $F(\mathbb{N}^{\mathrm{op}}, \mathcal{S})$ one may easily produce an object $F(d\mathbb{N}^{\mathrm{op}}, \mathcal{S})$, where $d\mathbb{N}^{\mathrm{op}}$ is the discrete category of natural numbers, and then take the colimit to produce E_0^*I additively. In [2], we explicitly describe how to make this construction multiplicative as well. The main result needed to construct the THH-May spectral sequence is the identification of the E_1-page as $E^1_{*,*} = E_*THH(E_0^*I)$. This is a reduction of the level of generality that the theorem is proven in the paper, and at this reduced level of generality the identification also follows from work of Brun in [`?qx`], though he works with FSP's in his paper and does not discuss this identification explicitly.

In order to make use of this spectral sequence, one would like a large supply of cofibrant decreasingly filtered commutative ring spectra, and this is provided by [2, Thm. qx]. In other words, there is a model for the Whitehead tower of a connective cofibrant commutative ring spectrum A, written

$$\to \tau_{\geq 3}A \to \tau_{\geq 2}A \to \tau_{\geq 1}A \to \tau_{\geq 0}A$$

which is a cofibrant decreasingly filtered commutative monoid in spectra. The associated grade of this filtration can be identified with $H\pi_*A$, the Eilenberg-Mac Lane spectrum of the differential graded algebra π_*A in the sense of Schwede [?qx].

8.2. **the $H \wedge V(0)$-based THH-May spectral sequence.** We begin our analysis of the THH-May spectral sequence based on $H \wedge V(0)$, which is the spectral sequence (8.2). The main point is that we know the abutment of the spectral sequence. It follows from (3.4) that

$$H_*(V(0) \wedge \mathrm{THH}(R)) \cong E(\overline{\tau}_0) \otimes A/\!\!/E(2)_* \otimes E(\lambda_1, \lambda_2, \lambda_3) \otimes P(\mu_3).$$

To deduce differentials in this spectral sequence, we need to determine the E^1-page. We have that the associated E_∞-ring for the Whitehead filtration on R (cf. [2]) is

$$E_0^*R \simeq H\pi_*R$$

where $H\pi_*R$ denotes the generalized Eilenberg-MacLane spectrum on π_*R.

To compute the E^1-page, we will utilize the Bökstedt spectral sequence:

$$\mathrm{HH}_*(H_*(\mathrm{THH}(H\pi_*R))) \implies H_*(\mathrm{THH}(H\pi_*R)).$$

We need to determine the E^1-term of this

Lemma 8.6. *The mod 3 homology of $H\pi_*R$ is $A/\!\!/E(0)_* \otimes P(v_1, v_2)$ where v_1 and v_2 are comodule primitives. Consequently, the the E^2-page of the Bökstedt spectral sequence computing $H_*THH(R)$ is*

$$E(\overline{\tau}_0) \otimes A/\!\!/E(0)_* \otimes E(\sigma\zeta_n \mid n \geq 1) \otimes \Gamma(\sigma\overline{\tau}_k \mid k \geq 1) \otimes P(v_1, v_2) \otimes E(\sigma v_1, \sigma v_2)$$

Proof. Additively, we have that the underlying $H\mathbb{Z}_{(3)}$-module of $H\pi_*R$ is

$$H\pi_*R \simeq \bigvee_{m \in \pi_*R} \Sigma^{|m|}H\mathbb{Z}_{(3)}$$

where the index m is varying over all of the monomials in π_*R. Thus,

$$H_*H\pi_*R \cong \pi_*(H\mathbb{F}_p \wedge (\bigvee_{m \in \pi_*R} \Sigma^{|m|}H\mathbb{Z}_{(3)}) \cong \pi_*(\bigvee_{m \in \pi_*R} H\mathbb{F}_p \wedge H\mathbb{Z}_{(3)})$$

and since multiplicatively $\pi_*H\pi_*R \cong \pi_*R$, we have $H_*H\pi_*R \cong A/\!\!/E(0)_* \otimes P(v_1, v_2)$ as desired. The element v_1 in homology is arising from the inclusion of the summand indexed by v_1. Applying homology to this map takes 1 to v_1. As this is a map of comodules, v_1 is necessarily primitive. A similar argument shows that v_2 is primitive.

To compute the E^2-page of the Bökstedt spectral sequence we must compute $HH_*(H_*H\pi_*R)$. By the calculation above, this is $HH_*(A/\!\!/E(0)_* \otimes P(v_1, v_2))$. The Hochschild homology of an exterior algebra $E(x)$ with a generator in x odd degree is $E(x) \otimes \Gamma(\sigma x)$ where $\Gamma(\sigma x)$ is a divided power algebra on a generator σx with $|\sigma x| = 1 + |x|$, by Koszul duality. The Hochschild homology of an polynomial algebra on a class y in even degree is $P(y) \otimes E(\sigma y)$ where $|\sigma y| = 1 + |y|$. (See [McClure-Staffeldt [?qx]] for details about these calculations). Therefore, by the Künneth isomorphism for Hochschild homology, the result follows. □

The morphism

$$H\mathbb{Z}_p \to H\pi_* R$$

allows us to deduce differentials and hidden extensions in the Bökstedt spectral sequence, resulting in the following.

Proposition 8.7. *We have an isomorphism*

$$H_* \operatorname{THH}(H\pi_* R) \cong A/\!\!/E(0)_* \otimes E(\lambda_1) \otimes P(\mu_1) \otimes P(v_1, v_2) \otimes E(\sigma v_1, \sigma v_2).$$

Note that the May filtration of σv_1 and σv_2 are $2(p-1)$ and $2(p^2-1)$ respectively. Since the May filtration is always divisible by $2(p-1)$, we reindex to give σv_1 and σv_2 May filtration 1 and 4 respectively.

The abutment of the $H \wedge V(0)$-based THH-May spectral sequence is known to be

$$E(\overline{\tau}_0) \otimes A/\!\!/E(2)_* \otimes E(\lambda_1, \lambda_2, \lambda_3) \otimes P(\mu_3)$$

by (3.4).

Recall that the degree of λ_i is $2p^i - 1$ and the degree of μ_3 is $2p^3$.

Proposition 8.8. *We have the following d_1-differentials:*

(1) $d_1 \overline{\tau}_1 \doteq v_1$,
(2) $d_1 \mu_1 \doteq \sigma v_1$.

Thus the E^2-page of the $H \wedge V(0)$-based THH-May spectral sequence is given by

$$E^2 \cong E(\overline{\tau}_0) \otimes A/\!\!/E(1)_* \otimes E(\lambda_1, \mu_1^3 \sigma v_1) \otimes P(\mu_1^3) \otimes P(v_2) \otimes E(\sigma v_2).$$

.

Proof. The abutment has dimension 1 in degree $2(p-1)$ with generator the class ζ_1. As v_1 is in degree $2(p-1)$ as well, it follows that it must be hit by a differential. It follows that

$$d_1(\overline{\tau}_1) \doteq v_1.$$

Also the class μ_1 is in degree $2p$ and the abutment is one dimensional in this degree. In the abutment, the generator is $\overline{\tau}_0 \lambda_1$, which is detected by $\overline{\tau}_0 \lambda_1$ in the E^1-page. Thus, as μ_1 is in May filtration 0, it must support a differential. The only possibility is

$$d_1(\mu_1) \doteq \sigma v_1$$

The multiplicative structure of the THH-May spectral sequence accounts for all other d_1-differentials on the E^1-page. □

We now determine the next differentials.

Proposition 8.9. *The next differentials in the $H \wedge V(0)$-based THH-May spectral sequence for* $\operatorname{THH}(R)$ *are*

(1) $d^4 \overline{\tau}_2 \doteq v_2$
(2) $d^4 \mu_1^3 \doteq \sigma v_2$

and the class $\mu_1^2\sigma v_1$ detects λ_2. Moreover, we obtain

$$E^5 \cong E(\overline{\tau}_0) \otimes A /\!\!/ E(2)_* \otimes E(\lambda_1, \mu_1^2\sigma v_1, \mu_1^6\sigma v_2)$$

Proof. Note that the class $\mu_1^2\sigma v_1$ is in degree 17. Thus, the E^2-term is of dimension 3 in degree 17, with generators $\overline{\tau}_2, \mu_1^3\sigma v_1$, and σv_2. On the other hand, the abutment has dimension 1 in degree 17, with generator λ_2.

Note that λ_2 is in the kernel of the natural map in homology induced by

$$\mathrm{THH}(R) \to \mathrm{THH}(\mathbb{Z}_p).$$

Thus λ_2 is in positive May filtration. Thus, $\overline{\tau}_2$ must support a differential. The only possibility is the the following differential

$$d^4\overline{\tau}_2 \doteq v_2.$$

As we have already computed the E^2-page, the class $\mu_1^2\sigma v_1$ cannot be hit by a differential, and there are no classes for it to hit. Thus, this class will represent a non-zero permanent cycle in the E^∞-term and detects λ_2. Consequently, σv_2 must be the target of a differential. This results in the differential

$$d^4\mu_1^3 \doteq \sigma v_2.$$

These and the multiplicative structure of the spectral sequence accounts for all d^{p+1}-differentials. This results in

$$E^5 \cong E(\overline{\tau}_0) \otimes A /\!\!/ E(2)_* \otimes E(\lambda_1, \mu_1^3\sigma v_1, \mu_1^6\sigma v_2) \otimes P(\mu_1^9).$$

□

We can infer from the description of the E^{p+2} that the $H \wedge V(0)$-based THH-May spectral collapses at this page.

We have already shown that $\mu_1^3\sigma v_1$ detects λ_2. We also have,

Proposition 8.10. *The class $\mu_1^6\sigma v_2$ detects λ_3 and the class μ_1^9 detects μ_3.*

Proof. A direct computation with Bökstedt spectral sequence for the morphism

$$\mathrm{THH}(R) \to \mathrm{THH}(\mathbb{Z}_p)$$

shows that μ_3 is mapped to μ_1^9 in homology. Thus μ_1^9 detects μ_3 in the May spectral sequence.

ish this
roof

[Gabe: We won't be able to prove that $\mu_1^6\sigma v_2$ detects λ_3 using this map because λ_3 maps to zero. Can't you prove both of these detection results by using the Hurewicz map? That's how I have done it previously.]

□

8.3. **the $V(0)$-based THH-May spectral sequence.** We now study the $V(0)$-based THH-May spectral sequence (8.1). We begin by determining the E^1-term. We recall the following standard fact,

Lemma 8.11 (cf. [1])**.** *Let M be an $H\mathbb{F}_p$-module. Then M is equivalent to a wedge of suspensions of $H\mathbb{F}_p$, and the Hurewicz map*

$$\pi_* M \to H_* M$$

is an injection onto the A_*-comodule primitives.

Proposition 8.12. *The E^1-term of* (8.1) *is isomorphic to*

$$E(\lambda_1) \otimes P(\mu_1) \otimes P(v_1, v_2) \otimes E(\sigma v_1, \sigma v_2),$$

and the spectral sequence has only the d^1 differential, $d^1\mu_1 \doteq \sigma\, v_1$. Consequently, the E^2-term of (8.1) *is*

$$E(\lambda_1, \mu_1^2 \sigma v_1) \otimes P(\mu_1^3) \otimes P(v_1, v_2) \otimes E(\sigma v_2).$$

Proof. The description of the E^1-term follows directly from the lemma and Proposition 8.7. Because the map on E^1-terms is injective, we can pull back differentials, which provides the state d_1 differential. [Note that that the Hurewicz map is NOT injective at the E^2-page so the same argument doesn't work for the later differential! This was my mistake in my initial calculation. In particular, there could be differentials hitting v_1^k times some element for some k. I'm starting to think that one of these differentials may actually occur.] ☐

We now use the fact that $\mu_1^2\, \sigma v_1$ detects λ_2 to rename this class. We also rename the class μ_1^p by μ_2.

Proposition 8.13. *There is a d^3 differential*

$$d^3(\mu_1^p) = v_1^3 \lambda_1$$

and no further differentials of this length. The E^4 term of (8.1) *is*

$$H_*(E(\lambda_1) \otimes P(v_1) \otimes P(\mu_2); d^3(\mu_2) = v_1^3\lambda_1) \otimes E(\lambda_2) \otimes P(v_2) \otimes E(\sigma v_2)$$

The reader may be concerned at this point that $v_1^3\lambda_1$ dies and yet it survived in the first Bökstein spectral sequence computing $THH_*(BP\langle 2\rangle; k(1))$. However, note that in the THH-May spectral sequence the names of classes often change and there is still a class, namely σv_2, which survives in the degree of $v_1^3\lambda_1$.

[Dom: Are there more simple calculations with the THH-May spectral sequence where a similar phenomenon occurs? If we can provide one, or point to one, I think this would be helpful for the reader.]

Proof. Note that there is a map of THH-May spectral sequences with abutment $S/3_*THH(BP\langle 2\rangle) \to S/3_*THH(BP\langle 1\rangle)$ and with input

$$P(\lambda_1) \otimes P(\mu_1) \otimes P(v_1, v_2) \otimes E(\sigma v_1, \sigma v_2) \to P(\lambda_1) \otimes P(\mu_1) \otimes P(v_1) \otimes E(\sigma v_1)$$

and by inspection all classes map to classes of the same name except v_2 and σv_2, which map to zero. In the target spectral sequence, we compute the differential $d_1(\mu_1) = \sigma v_1$ by the same means as we did before. Therefore, the map of E^2-terms is

$$P(\lambda_1, \lambda_2) \otimes P(\mu_2) \otimes P(v_1, v_2) \otimes E(\sigma v_2) \to P(\lambda_1, \lambda_2) \otimes P(\mu_2) \otimes P(v_1)$$

and again the classes all map to classes of the same name except v_2 and σv_2, which map to zero. Note that this verfies that the renaming of λ_2 and μ_2 is reasonable. The target of this map is exactly the same as the input of the Bökstein spectral seqeunce computing $THH_*(BP\langle 1\rangle; k(1))$ and therefore we know what the remaining differentials have to be by McClure-Staffeldt [9]. In particular, there is a differential $d^3(\mu_2) = v_1^3\lambda_1$ and this is the only differential of this length. This implies that the same differential takes place in the source spectral sequence. To see that there are no further differentials of this length in the source note that the only possiblility would be a differential with source σv_2 or v_2 and there are no possible differentials of this length on these classes for degree reasons. □

We now note that

$$H_*(E(\lambda_1) \otimes P(v_1) \otimes P(\mu_2); d^3(\mu_2) = v_1^p\lambda_1) \cong \left(P(v_1, \mu_2^3) \otimes \mathbb{F}_3\{1, \lambda_1, \lambda_1\mu_2\} \otimes E(\lambda_1\mu_2^2)\right) / \sim$$

where $\sim$ is the relation

$$\begin{aligned} \lambda_1 \cdot (\lambda_1\mu_2^2) &= 0 \\ \lambda_1\mu_2 \cdot (\lambda_1\mu_2^2) &= 0 \\ v_1^3 \cdot \lambda_1 &= 0 \\ v_1^3 \cdot \lambda_1\mu_2 &= 0. \end{aligned}$$

and the classes $\lambda_1\mu_2$ and $\lambda_1\mu_2^2$ are not in the output of either of the Bockstein spectral seqeunces.

$$THH_*(BP\langle 2\rangle, H\mathbb{F}_p)[v_1] \Rightarrow THH_*(BP\langle 2\rangle, k(1))$$

and

$$THH_*(BP\langle 2\rangle, H\mathbb{F}_p)[v_2] \Rightarrow THH_*(BP\langle 2\rangle, k(2))$$

and therefore they cannot survive the $S/3$-based THH-May spectral sequence. (Note that these classes are no longer decomposable). This forces the following differentials.

Lemma 8.14. *There is a differential $d_4(\lambda_1\mu_2) = \lambda_1\sigma v_2$ and $d_4(\lambda_1\mu_2^2) = \lambda_1\sigma v_2\mu_2$ which generates families of differentials by multiplicativity of the $S/3$-THH-May spectral sequence and no further differentials of this length.*

Proof. We know that the elements $\lambda_1\mu_2$ and $\lambda_1\mu_2^2$ must not be cycles by the argument above and the fact that they are not boundaries. We therefore check the possible targets of a differential and the possibilities are $\mu_2 v_1, \lambda_1\lambda_2, \lambda_1\sigma v_2$ for $\lambda_1\mu_2$. We now observe that there is no differential on $\lambda_1\mu_2$ in the $S/3$-THH-May spectral sequence computing $S/3_*THH(BP\langle 1\rangle)$ so if there is a differential on $\lambda_1\mu_2$ in the $S/3$-THH-May spectral sequence computing $S/3_*THH(BP\langle 2\rangle)$ it must hit something that maps to zero under the map of THH-May spectral sequences. The only one of the three classes named above that maps to zero under this map of spectral sequences is $\lambda_1\sigma v_2$. This forces the stated differential.

[Gabe: Add similar argument for the other differential]

[Dom: I think you can obtain a simpler proof by mapping to the $H \wedge V(0)$-based THH-May spectral sequence, wherein these differentials occur.]

□

We conclude that there is an isomorphism

$$E_{p+2} \cong E(\lambda_1, \lambda_2, \sigma v_2) \otimes P(\mu_3, v_1, v_2)/\sim$$

where $v_1^p\lambda_1 \sim 0$, $\lambda_1\sigma v_2 \sim 0$, $\lambda_1\sigma v_2\mu \sim 0$, etc. There is therefore an additive isomorphism

I think you forg
the class $\lambda_1 u_2^2\sigma$

$$E_{p+2} \cong E(\lambda_1, \lambda_2, \lambda_3) \otimes P(\mu_3, v_1, v_2)$$

where we make the additive identifications $\lambda_1 \cdot v_1^p \doteq \sigma v_2$, $\lambda_1 v_1^p \mu^{p-1} \doteq \lambda_3$, $\lambda_1\sigma v_2\mu^{p-1} = \lambda_1\lambda_3$, $\lambda_1\lambda_2\sigma v_2\mu^{p-1} \doteq \lambda_1\lambda_2\lambda_3$.

Note that the class $\lambda_2 v_1^9$ doesn't survive the Bockstein spectral sequence $THH_*(BP\langle 2\rangle; H\mathbb{F}_3)[v_1] \Rightarrow THH_*(BP\langle 2\rangle; k(1))$, but it does survive the Bockstein spectral sequence $THH_*(BP\langle 2\rangle; H\mathbb{F}_3)[v_2] \Rightarrow THH_*(BP\langle 2\rangle; k(2))$. Similarly, the class $\lambda_1 v_2^3$ doesn't survive the Bockstein spectral sequence $THH_*(BP\langle 2\rangle; H\mathbb{F}_3)[v_2] \Rightarrow THH_*(BP\langle 2\rangle; k(2))$, but it does survive the Bockstein spectral sequence $THH_*(BP\langle 2\rangle; H\mathbb{F}_3)[v_1] \Rightarrow THH_*(BP\langle 2\rangle; k(1))$. One may think that this forces a differential hitting $\lambda_2 v_1^9$ the second Bockstein spectral sequence $THH_*(BP\langle 2\rangle; k(1))[v_2] \Rightarrow THH_*(BP\langle 2\rangle; BP\langle 2\rangle/3)$ and a differential hitting $\lambda_1 v_2^3$ in the Bockstein spectral sequence $THH_*(BP\langle 2\rangle; k(2))[v_1] \Rightarrow THH_*(BP\langle 2\rangle; BP\langle 2\rangle/3)$. However, we conjecture that this doesn't happen.

Conjecture 8.15. In the THH-May spectral sequence, there is a differential $d_{p^2}(\mu_3) = \lambda_2 v_2^9$ and a hidden additive extension $\lambda_1 \cdot v_2^3 \doteq \lambda_2 \cdot v_1^9$.

[Gabe: We should try to prove this.]

[Dom: I am not sure about this conjecture... I say this because if you consider the map from the $V(0)$-based May spectral sequence to the $V(1)$-based May spectral sequence, then we can use the fact that the $V(1)$-May SS is a reindexed form of the v_2-BSS. There we have

$$d^3(\mu_3) \doteq v_2^3\lambda_1$$

which translates into a May differential

$$d^{12}(\mu_3) \doteq v_2^3\lambda_1.$$

The differential proposed would actually be a d^{36}-differential.]

8.4. **the $V(1)$-based THH-May spectral sequence.** In this section, we analyze the $V(1)$-based THH-May spectral sequence, much as we did for the $V(0)$-THH May spectral sequence. We will show that the E^5-page of the $V(1)$-THH May spectral sequence is a reindexed version of the v_2-Bockstein spectral sequence converging to $\mathrm{THH}_*(R; k(2))$. Recall that the Smith-Toda complex $V(1)$ fits in a cofibre sequence

$$\Sigma^4 V(0) \xrightarrow{v_1} V(0) \longrightarrow V(1),$$

the second map provides a map of May spectral sequences

$$\begin{array}{ccc} E^1(R; V(0)) & \Longrightarrow & V(0)_* \operatorname{THH}(R) \\ \downarrow & & \downarrow \\ E^1(R; V(1)) & \Longrightarrow & V(1)_* \operatorname{THH}(R) \end{array} .$$

Because the $V(1)$-May spectral sequence is a reindexed version of the v_2-BSS spectral sequence, this means that we know all the differentials arising in the bottom. Our first goal is to show that

$$E^5(R; V(1)) \cong P(v_2) \otimes E(\lambda_1, \lambda_2, \lambda_3) \otimes P(\mu_3).$$

Remark 8.16. The obstruction to $V(1)$ being a ring spectrum at the prime 3 is given by an element in stem 10. One has that $\pi_*(V(1) \wedge R) \cong \mathbb{F}_3[v_2]$, which is trivial in degree 10. Thus the obstruction vanishes after smashing with R. So the $V(1)$-May spectral sequence is a spectral sequence of algebras.

We begin by analyzing the $H \wedge V(1)$-May spectral sequence:

$$H_*(V(1) \wedge \operatorname{THH}(E^0 R)) \implies H_*(V(1) \wedge THH(R)).$$

The abutment is computable by the Bökstedt spectral sequence, and it is given by (cf. [4])

$$H_*(V(1) \wedge \operatorname{THH}(R)) \cong E(\widetilde{\tau}_0, \widetilde{\tau}_1) \otimes A /\!\!/ E(2)_* \otimes E(\lambda_1, \lambda_2, \lambda_3) \otimes P(\mu_3)$$

On the other hand, the E^1-term is

$$H_*(V(1) \wedge \operatorname{THH}(E^0 R)) \cong E(\widetilde{\tau}_0, \widetilde{\tau}_1) \otimes A /\!\!/ E(0)_* \otimes E(\lambda_1, \sigma v_1, \sigma v_2) \otimes P(v_1, v_2, \mu_1).$$

We use the tildes to distinguish from the corresponding classes in the dual Steenrod algebra. Note that the coaction on μ_1 is

$$\alpha(\mu_1) = 1 \otimes \mu_1 + \overline{\tau}_0 \otimes \lambda_1.$$

Thus the class

$$\widetilde{\mu_1} := \mu_1 - \overline{\tau}_0 \lambda_1$$

is a comodule primitive. We will need this class later.

As in the previous section, we can establish

Proposition 8.17. *In the $H \wedge V(1)$-May spectral sequence, we have the differentials*

1. $d_1 \overline{\tau}_1 \doteq v_1$,
2. $d_1 \mu_1 \doteq \sigma v_1$,
3. $d^{p+1} \overline{\tau}_2 \doteq v_2$
4. $d^{p+1} \mu_1^3 \doteq \sigma v_2$

and the classes λ_2, λ_3 are detected by $\widetilde{\mu_1}^2 \sigma v_1$ and $\widetilde{\mu_1}^6 \sigma v_2$ respectively[1].

[1]We needed the class $\widetilde{\mu_1}$ as opposed to μ_1 since λ_2, λ_3 are comodule primitives.

Smashing the cofibre sequence for $V(1)$ with $H\mathbb{Z}$ shows that

$$V(1) \wedge H\mathbb{Z} \simeq_{H\mathbb{F}_p} H\mathbb{F}_3 \vee \Sigma^5 H\mathbb{F}_3, \tag{8.18}$$

in particular, $V(1)\wedge H\mathbb{Z}$ is an $H\mathbb{F}_p$-module. Thus $V(1)\wedge \mathrm{THH}(E^0R)$ is an $H\mathbb{F}_p$-module. So the Hurewicz map

$$\pi_*(V(1) \wedge \mathrm{THH}(E^0R)) \to H_*(V(1) \wedge \mathrm{THH}(E^0R))$$

is injective onto the comodule primitives. Similarly to the $V(0)$-May spectral sequence, we pull-back differentials to understand the $V(1)$-May spectral sequence. There is one fundamental distinction from the previous section, however. Namely, there will be a differential which kills v_1 in the $V(1)$-May spectral sequence. This will keep the map on the E^2, E^3, and E^4-pages induced by the Hurewicz map injective, bypassing the difficulties of the previous section. We proceed to show this below.

Observe that

$$H_*(V(1) \wedge H\mathbb{Z}) \cong E(\widetilde{\tau}_0, \widetilde{\tau}_1) \otimes A/\!\!/E(0)_*.$$

From (8.18), we conclude that

$$H_*(V(1) \wedge H\mathbb{Z}) \cong A_* \oplus A_*\{\varepsilon_1\}$$

on some comodule primitive ε_1. We will identify this element ε_1 and show that $d^1\varepsilon_1 = v_1$.

Proposition 8.19. *The class*

$$\varepsilon_1 := \overline{\tau}_1 + \widetilde{\tau}_1 - \xi_1\widetilde{\tau}_0$$

is a comodule primitive. Furthermore, $d^1\varepsilon_1 = v_1$.

Proof. We need to recall formulas in the Steenrod algebra relating ξ_i and τ_i to their conjugates. Namely, we have the relations

$$\sum_{i+j=n} \xi_j^{p^i}\overline{\xi_i} = 0 \qquad\qquad \tau_n + \sum_{i+j=n} \xi_j^{p^i}\overline{\tau}_n = 0$$

In particular, letting n be 1 or 0, we find

$$\zeta_1 = -\xi_1 \qquad\qquad \overline{\tau}_0 = -\tau_0 \qquad\qquad \tau_1 + \xi_1\overline{\tau}_0 + \overline{\tau}_1 = 0$$

These relations will show that ϵ_1 is a comodule primitive. Indeed, we obtain

$$\alpha(\varepsilon_1) = \overline{\tau}_1 \otimes 1 + \overline{\tau}_0 \otimes \zeta_1 + 1 \otimes \overline{\tau}_1 + \tau_1 \otimes 1 + \xi_1 \otimes \widetilde{\tau}_0 + 1 \otimes \widetilde{\tau}_1 - (\tau_0 \otimes 1 + 1 \otimes \widetilde{\tau}_0)(\xi_1 \otimes 1 + 1 \otimes \xi_1)$$

Observe that $\overline{\tau}_0 \otimes \zeta_1 = \tau_0 \otimes \xi_1$. After expanding the last term and cancelling, we obtain

$$\alpha(\varepsilon_1) = (\overline{\tau}_1 + \tau_1) \otimes -\tau_0\xi_1 \otimes 1 + 1 \otimes \epsilon_1.$$

From the above relations, we get

$$\overline{\tau}_1 + \tau_1 = \xi_1\tau_0,$$

which shows that $\alpha(\varepsilon_1) = \varepsilon_1$. ∎

Corollary 8.20. *The E^1-page of the $V(1)$-May spectral sequence is*

$$E^1(R;V(1)) \cong E(\varepsilon_1, \lambda_1, \sigma v_1, \sigma v_2) \otimes P(\widetilde{\mu}_1, v_1, v_2).$$

The d^1-differential is obtained from

$$d^1(\varepsilon_1) \doteq v_1 \qquad d^1(\widetilde{\mu}_1) \doteq \sigma v_1$$

and multiplicativity of the spectral sequence.

From this we can determine the E^2-page,

Corollary 8.21. *One has*

$$E^2(R;V(1)) \cong E(\lambda_1, \sigma v_2, \widetilde{u}_1^2 \sigma v_1) \otimes P(\widetilde{\mu}_1^3, v_2).$$

Furthermore, the map on the E^2-term induced by the Hurewicz map is injective, and hence is also on E^3 and E^4. So the next differentials are

$$d^4(\mu_1^3) \doteq \sigma v_2.$$

This results in

$$E^5(R;V(1)) \cong E(\lambda_1, \widetilde{\mu_1}^2 \sigma v_1, \widetilde{\mu_1}^6 \sigma v_2) \otimes P(\widetilde{\mu_1}^9).$$

Note that

$$\widetilde{\mu_1}^9 = \mu_1^9 = \mu_3.$$

Thus we can rename $\widetilde{\mu_1}^9$ as μ_3. Renaming classes, the E^5-page is

$$E^5 \cong E(\lambda_1, \lambda_2, \lambda_2) \otimes P(\mu_3, v_2)$$

where v_2 is in May filtration 4. Thus the E^5-page is a reindexed form of the v_2-BSS, and this determines the rest of the $V(1)$-May spectral sequence.

[Dom: Here is an idea: to make things easier for the reader, maybe we should have a table somewhere in the paper with all the names of the various elements, their representatives, their coactions, etc.]

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