

TOPOLOGICAL HOCHSCHILD HOMOLOGY OF THE SECOND TRUNCATED BROWN-PETERSON SPECTRUM I

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ABSTRACT. We compute topological Hochschild homology of the second truncated Brown-Peterson spectrum $BP\langle 2 \rangle$ at the primes 2, 3 with coefficients in $BP\langle 1 \rangle$. At the prime $p = 2$ we use the model for $BP\langle 2 \rangle$ constructed by Lawson-Naumann using topological modular forms equipped with a $\Gamma_1(3)$ -structure and at $p = 3$ we use the model for $BP\langle 2 \rangle$ constructed using a Shimura curve of discriminant 14 due to Hill-Lawson.

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1. INTRODUCTION

Topological Hochschild cohomology encodes information about deformations of structured ring spectra and the topological Hochschild homology of a structured ring spectrum is the linear approximation to algebraic K-theory in the sense of Goodwillie's calculus of functors. Algebraic K-theory of ring spectra that arise in chromatic stable homotopy theory are of particular interest because of the program of Ausoni-Rognes [4] which, in a broad sense, suggests that the arithmetic of structured ring spectra is intimately connected to chromatic complexity. One of the most fundamental objects in chromatic stable homotopy theory is the Brown-Peterson spectrum BP , which is a complex oriented cohomology theory associated to the universal p -typical formal group. The coefficients of BP are a polynomial algebra over $\mathbb{Z}_{(p)}$ on generators v_i for $i \geq 1$,

and we may form truncated versions of BP , denoted $BP\langle n \rangle$ by coning off a regular sequence $(v_{n+1}, v_{n+2}, \dots)$.

By convention $BP\langle -1 \rangle = H\mathbb{F}_p$ and when $n = 0, 1$, there are known identifications $BP\langle 0 \rangle = H\mathbb{Z}_{(p)}$, and $BP\langle 1 \rangle = \ell$ where ℓ is the Adams summand of complex topological K-theory ku . Until recently, the previous list exhausted the known examples of $BP\langle n \rangle$ that were known to have models as E_∞ -ring spectra. However, in the last decade, models for $BP\langle 2 \rangle$ as an E_∞ -ring spectrum were constructed at the prime $p = 2$ by Lawson-Naumann [11] and at the prime $p = 3$ by Hill-Lawson [8]. Lawson-Naumann [11] use the theory of topological Modular forms with a $\Gamma_1(3)$ -structure to construct an E_∞ model for $BP\langle 2 \rangle$ at the prime 2 and Hill-Lawson [8] use the theory of topological automorphic forms associated to a Shimura curve of discriminant 14 to construct an E_∞ model for $BP\langle 2 \rangle$ at the prime $p = 3$. This is especially interesting in view of recent work of Lawson [10] and Senger [17], where they prove that no model for $BP\langle n \rangle$ as an E_∞ -ring spectrum exists for $n \geq 4$ at any prime.

In the present paper, we compute topological Hochschild homology of $BP\langle 2 \rangle$ with coefficients in $BP\langle 1 \rangle$ at the primes 2 and 3. For small values of n , the calculations of $THH_*(BP\langle n \rangle)$ are known and of fundamental importance. The first known computations of topological Hochschild homology are Bökstedt's calculations of $THH_*(BP\langle -1 \rangle)$ and $THH_*(BP\langle 0 \rangle)$ [5] and the computation

$$THH_*(BP\langle -1 \rangle) \cong P(\mu_0)$$

where $|\mu_0| = 2$ may be used to give a new proof of Bott periodicity [?HN20].

In McClure-Staffeldt [13], they compute the Bockstein spectral sequence

$$THH_*(BP\langle 1 \rangle; H\mathbb{F}_p)[v_1] \Rightarrow THH_*(BP\langle 1 \rangle; k(1)).$$

This result is extended by Angeltveit-Hill-Lawson [2] where they compute the square of spectral sequences

$$\begin{array}{ccc} THH_*(BP\langle 1 \rangle; H\mathbb{F}_p)[v_0, v_1] & \Longrightarrow & THH_*(BP\langle 1 \rangle; H\mathbb{Z}_{(p)})_p[v_1] \\ \Downarrow & & \Downarrow \\ THH_*(BP\langle 1 \rangle; k(1))[v_0] & \Longrightarrow & THH_*(BP\langle 1 \rangle; BP\langle 1 \rangle)_p. \end{array}$$

This gives a complete answer for the “integral” calculation $THH_*(BP\langle 1 \rangle)$.

When $n = 2$, the calculation $THH_*(BP\langle 2 \rangle; H\mathbb{F}_p)$ follows naturally from [3] as we discuss in Section 2.1, but no further results towards $THH_*(BP\langle 2 \rangle)$ are known.

In the present paper, we compute the square of spectral sequences

$$\begin{array}{ccc} THH_*(BP\langle 2 \rangle; H\mathbb{F}_p)[v_0, v_1] & \Longrightarrow & THH_*(BP\langle 2 \rangle; H\mathbb{Z}_p)[v_1] \\ \Downarrow & & \Downarrow \\ THH_*(BP\langle 2 \rangle; k(1))[v_0] & \Longrightarrow & THH_*(BP\langle 2 \rangle; BP\langle 1 \rangle)_p, \end{array}$$

which is slightly more complex computationally than the result of Angeltveit-Hill-Lawson [2], though many of the techniques developed in [2] and [13] carry over.

[Gabe: Fix below in light of current method of proof]

We apply a new tool, however, introduced by the first author and Salch, called the topological Hochschild-May spectral sequence [1]. This allows one to compute $THH_*(BP\langle 2 \rangle, BP\langle 1 \rangle)$ directly. This will not replace the Bockstein spectral sequence, however, and instead we think of it as computing the diagonal of the square. This is actually the case at some later page in the topological Hochschild-May spectral sequence up to associated graded.

We therefore use all three ways of computing the output in order to figure out all the differentials and hidden extensions.

[Gabe: Include statements of main results.]

1.1. Outline of the strategy.

[Gabe: Rewrite this section to reflect current strategy.]

Conventions. Fix $p \in \{2, 3\}$ throughout. We will write $H_*(-)$ for homology with \mathbb{F}_p coefficients, or in other words, the functor $\pi_*(H\mathbb{F}_p \wedge -)$. We write \doteq to mean that an equality holds up to multiplication by a unit. We will write $BP\langle n \rangle$ for the n -th truncated Brown-Peterson spectrum. In particular, $BP\langle 1 \rangle$ denotes the E_∞ -ring spectrum model for the connective Adams summand [13]. Also, $BP\langle 2 \rangle$ will denote the E_∞ -model for the second truncated Brown-Peterson spectrum constructed by [11] at $p = 2$ and [8] at $p = 3$. We also note that by coning off v_2 on $BP\langle 2 \rangle$ we may construct $BP\langle 1 \rangle$ as an E_∞ - $BP\langle 2 \rangle$ -algebra and since the E_∞ -ring spectrum structure on $BP\langle 1 \rangle$ is unique, this is equivalent to the E_∞ ring spectrum model constructed in [13].

[Gabe: As I understand it, uniqueness of $BP\langle n \rangle$ only holds after p -completion so we need to be careful about the sentence above if we don't p -complete implicitly thoughtout. We should also include a citation for this fact.]

Let $k(n)$ denote an A_∞ -ring spectrum model for the connective cover of the Morava K-theory spectrum $K(n)$.

When not otherwise specified, tensor products will be taken over \mathbb{F}_p and $HH_*(A)$ denotes the Hochschild homology of a graded \mathbb{F}_p -algebra relative to \mathbb{F}_p . We will let $P(x)$, $E(x)$ and $\Gamma(x)$ denote a polynomial algebra, exterior algebra, and divided power algebra over \mathbb{F}_p on a generator x .

The dual Steenrod algebra will be denoted \mathcal{A}_* with coproduct $\Delta: \mathcal{A}_* \rightarrow \mathcal{A}_* \otimes \mathcal{A}_*$. Given a right \mathcal{A}_* -comodule M , its right coaction will be denoted $\nu: \mathcal{A} \rightarrow \mathcal{A} \otimes M$ where the comodule M is understood from the context. The antipode $\chi: \mathcal{A}_* \rightarrow \mathcal{A}_*$, will not play a role except that we will write $\bar{\xi}_i := \chi(\xi_i)$ and $\bar{\tau}_i := \chi(\tau_i)$.

2. FIRST TWO BOCKSTEIN SPECTRAL SEQUENCES

2.1. Preliminary results. The homology of topological Hochschild homology of $BP\langle 2 \rangle$ is a straightforward application of results of [3, 5, 6] and it appears in [3, Thm. 5.12]. Recall that there is an isomorphism

$$H_*(BP\langle 2 \rangle) \cong \begin{cases} P(\bar{\xi}_1, \bar{\xi}_2, \dots) \otimes E(\bar{\tau}_3, \bar{\tau}_4, \dots) & \text{if } p \geq 3 \\ P(\bar{\xi}_1^2, \bar{\xi}_2^2, \bar{\xi}_3^2, \bar{\xi}_4, \dots) & \text{if } p = 2 \end{cases}$$

of \mathcal{A}_* -comodules. Then by [3, Thm. 5.12, Cor. 5.12] there is an isomorphism

$$(2.1) \quad H_*(\mathrm{THH}(BP\langle 2 \rangle)) \cong \begin{cases} H_*BP\langle 2 \rangle \otimes E(\sigma\bar{\xi}_1, \sigma\bar{\xi}_2, \sigma\bar{\xi}_3) \otimes P(\sigma\bar{\tau}_3) & \text{if } p = 3 \\ H_*BP\langle 2 \rangle \otimes E(\sigma\bar{\xi}_1^2, \sigma\bar{\xi}_2^2, \sigma\bar{\xi}_3^2) \otimes P(\sigma\bar{\xi}_4) & \text{if } p = 2 \end{cases}$$

of primitively generated $H_*BP\langle 2 \rangle$ -Hopf algebras and \mathcal{A}_* -comodules. We also note the coaction on $H_*(\mathrm{THH}(BP\langle 2 \rangle))$ as a comodule over \mathcal{A}_* computed in [3, Thm. 5.12]

$$(2.2) \quad \nu(\sigma\bar{\tau}_3) = 1 \otimes \sigma\bar{\tau}_3 + \bar{\tau}_0 \otimes \sigma\bar{\xi}_3$$

at $p = 3$ and

$$(2.3) \quad \nu(\sigma\bar{\xi}_4) = 1 \otimes \sigma\bar{\xi}_4 + \bar{\xi}_1 \otimes \sigma\bar{\xi}_3^2.$$

at $p = 2$. These both follow from the formula

$$(2.4) \quad \nu \circ \sigma = (1 \otimes \sigma) \circ \nu$$

in [3, Eq. 5.11] and the well known \mathcal{A}_* -coaction on $H_*(BP\langle 2 \rangle)$. By the same argument, $\sigma\xi_i$ is primitive at $p = 3$ and $\sigma\xi_i^2$ is primitive at $p = 2$ for $i = 1, 2, 3$.

2.1.1. *THH of $BP\langle 2 \rangle$ modulo (p, v_1, v_2) .* We now compute

$$\mathrm{THH}_*(BP\langle 2 \rangle; H\mathbb{F}_p).$$

By [3, Lem. 4.1], it suffices to compute the sub-algebra of co-mododule primitives in

$$H_*(\mathrm{THH}(BP\langle 2 \rangle; H\mathbb{F}_p))$$

since $\mathrm{THH}(BP\langle 2 \rangle; H\mathbb{F}_p)$ is an $H\mathbb{F}_p$ -algebra. Since $BP\langle 2 \rangle$ and $H\mathbb{F}_p$ are commutative ring spectra there is a weak equivalence of commutative ring spectra

$$\mathrm{THH}(BP\langle 2 \rangle; H\mathbb{F}_p) \simeq \mathrm{THH}(BP\langle 2 \rangle) \wedge_{BP\langle 2 \rangle} H\mathbb{F}_p.$$

Since $H_*(\mathrm{THH}(BP\langle 2 \rangle))$ is free over $H_*BP\langle 2 \rangle$ by (2.1), the Eilenberg-Moore spectral sequence

$$\mathrm{Tor}_{*,*}^{H_*(BP\langle 2 \rangle)}(H_*(\mathrm{THH}(BP\langle 2 \rangle)), H_*(H\mathbb{F}_p)) \Rightarrow H_*(\mathrm{THH}(BP\langle 2 \rangle; H\mathbb{F}_p))$$

collapses immediately implying

$$(2.5) \quad H_*(\mathrm{THH}(BP\langle 2 \rangle; H\mathbb{F}_p)) \cong \begin{cases} \mathcal{A}_* \otimes E(\sigma\bar{\xi}_1, \sigma\bar{\xi}_2, \sigma\bar{\xi}_3) \otimes P(\sigma\bar{\tau}_3) & \text{if } p = 3 \\ \mathcal{A}_* \otimes E(\sigma\bar{\xi}_1^2, \sigma\bar{\xi}_2^2, \sigma\bar{\xi}_3^2) \otimes P(\sigma\bar{\xi}_4) & \text{if } p = 2. \end{cases}$$

The \mathcal{A}_* coaction on elements in \mathcal{A}_* is given by the coproduct and the remaining coactions are determined by the formula (2.4) and are therefore the same as in (2.2) and (2.3). We write $\lambda_i = \sigma\bar{\xi}_i$ at $p = 3$ and $\lambda_i = \sigma\bar{\xi}_i^2$ at $p = 2$. We also define

$$\mu_3 = \begin{cases} \sigma\bar{\tau}_3 - \bar{\tau}_0 \otimes \sigma\bar{\xi}_3 & \text{if } p = 3 \\ \sigma\bar{\xi}_4 - \bar{\xi}_1 \otimes \sigma\bar{\xi}_3^2 & \text{if } p = 2 \end{cases}.$$

Then it is clear that the algebra of comodule primitives in $H_*(\mathrm{THH}(BP\langle 2 \rangle; H\mathbb{F}_p))$ is generated by μ_3 and λ_i for $1 \leq i \leq 3$. We therefore produce the following isomorphism of graded \mathbb{F}_p -algebras

$$(2.6) \quad \mathrm{THH}_*(BP\langle 2 \rangle; H\mathbb{F}_p) \cong E(\lambda_1, \lambda_2, \lambda_3) \otimes P(\mu_3).$$

The degrees of the algebra generators are $|\lambda_i| = 2p^i - 1$ for $1 \leq i \leq 3$ and $|\mu_3| = 2p^3$.

2.1.2. Rational homology.

[Gabe: Change $E_2^{*,*}$ to $E_{*,*}^2$ for Bökstedt spectral sequence throughout.]

Next, we compute the rational homology of $\mathrm{THH}(BP\langle 2 \rangle)$ to locate the torsion free component of $\mathrm{THH}_*(BP\langle 2 \rangle)$. Towards this end, we will use the $H\mathbb{Q}$ -based Bökstedt spectral sequence. This is a spectral sequence of the form

$$E_2^{**} = \mathrm{HH}_*^{\mathbb{Q}}(H\mathbb{Q}_*(BP\langle 2 \rangle)) \implies H\mathbb{Q}_* \mathrm{THH}(BP\langle 2 \rangle).$$

Recall that the rational homology of $BP\langle 2 \rangle$ is

$$H\mathbb{Q}_*(BP\langle 2 \rangle) \cong P_{\mathbb{Q}}(v_1, v_2).$$

Thus the E_2 -term of the Bökstedt spectral sequence is

$$P_{\mathbb{Q}}(v_1, v_2) \otimes E_{\mathbb{Q}}(\sigma v_1, \sigma v_2)$$

and the bidegree of σv_i is $(1, 2(p^i - 1))$. Note that $BP\langle 2 \rangle$ is a commutative ring spectrum, so by [3, Prop. 4.3] the Bökstedt spectral sequence is multiplicative. All the algebra generators are in Bökstedt filtration 0 and 1 and the d^2 differential shifts Bökstedt filtration by two, so there is no room for differentials. Thus, the E_2 -term is isomorphic to the E_{∞} -term as graded \mathbb{Q} -algebras. There are clearly no additive extensions since the abutment is a \mathbb{Q} -algebra. There are no multiplicative extensions for bidegree reasons. Thus, there is an isomorphism of graded \mathbb{Q} -algebras

$$\mathrm{THH}_*(BP\langle 2 \rangle) \otimes \mathbb{Q} \cong P_{\mathbb{Q}}(v_1, v_2) \otimes_{\mathbb{Q}} E_{\mathbb{Q}}(\sigma v_1, \sigma v_2)$$

where $|\sigma v_i| = 2p^i - 1$. At $p = 2, 3$, there is an E_2 -ring map

$$BP \rightarrow BP\langle 2 \rangle.$$

To see this, we note that our E_{∞} ring spectrum models for $BP\langle 2 \rangle$ are clearly complex oriented and therefore come equipped with formal groups. It is also clear that these formal groups are p -typical. There is therefore an associated E_1 ring map

$$BP \rightarrow BP\langle 2 \rangle$$

and then by [7, Thm. 1.2] this E_1 -ring map can be lifted to an E_2 -ring map. Rationally, this map

$$H\mathbb{Q}_*(BP) \rightarrow H\mathbb{Q}_*(BP\langle 2 \rangle)$$

sends v_1 and v_2 to the generators of the same name. We therefore produce a multiplicative map of rational Bökstedt spectral sequences

$$\begin{array}{ccc} HH_*^{\mathbb{Q}}(H\mathbb{Q}_*(BP)) & \implies & H\mathbb{Q}_*(\mathrm{THH}(BP)) \\ \downarrow & & \downarrow \\ HH_*^{\mathbb{Q}}(H\mathbb{Q}_*(BP\langle 2 \rangle)) & \implies & H\mathbb{Q}_*(\mathrm{THH}(BP\langle 2 \rangle)) \end{array}$$

where on E_2 pages the map

$$P_{\mathbb{Q}}(v_i \mid i \geq 1) \otimes E_{\mathbb{Q}}(\sigma v_i \mid i \geq 1) \rightarrow P_{\mathbb{Q}}(v_1, v_2) \otimes E_{\mathbb{Q}}(\sigma v_1, \sigma v_2)$$

sends v_i to v_i and σv_i to σv_i for $i = 1, 2$. By [16, Thm. 1.1],

$$\sigma v_1 = p\lambda_1$$

$$\sigma v_2 = p\lambda_2 - v_1^p \lambda_1 - v_1^p \sigma v_1$$

in $THH_*(BP)$. Since the map

$$THH_*(BP) \rightarrow THH_*(BP\langle 2 \rangle)$$

sends λ_1 and λ_2 to classes of the same name, we have the same relations in $THH_*(BP\langle 2 \rangle)$.

[Gabe: This part isn't proven yet. I think we can prove it, but maybe this part belongs later.]

Consequently, up to a change of basis,

$$(2.7) \quad THH_*(BP\langle 2 \rangle; H\mathbb{Q}) \cong E_{\mathbb{Q}}(\lambda_1, \lambda_2).$$

Also, we conclude that

$$L_0 THH(BP\langle 2 \rangle) \simeq L_0 BP\langle 2 \rangle \vee \Sigma^{2p-1} L_0 BP\langle 2 \rangle \vee \Sigma^{2p^2-1} L_0 BP\langle 2 \rangle \vee \Sigma^{2p^2+2p-2} L_0 BP\langle 2 \rangle$$

where $L_0 = L_{H\mathbb{Q}}$, since L_0 is a smashing localization and $L_0 S = H\mathbb{Q}$.

2.2. The $H\mathbb{Z}$ -Bockstein spectral sequence. Recall that there is an isomorphism of \mathcal{A}_* -comodules

$$H_*(S/p \wedge THH(BP\langle 2 \rangle)) \cong \begin{cases} E(\bar{\tau}_0) \otimes H_*(BP\langle 2 \rangle) \otimes E(\sigma \bar{\xi}_1, \sigma \bar{\xi}_2, \sigma \bar{\xi}_3) \otimes P(\sigma \bar{\tau}_3) & \text{if } p = 3 \\ E(\bar{\xi}_1) \otimes H_*(BP\langle 2 \rangle) \otimes E(\sigma \bar{\xi}_1^2, \sigma \bar{\xi}_2^2, \sigma \bar{\xi}_3^2) \otimes P(\sigma \bar{\xi}_4) & \text{if } p = 2 \end{cases}$$

where the coaction on $x \in \mathcal{A}_*$ is $\nu(x)$ is given by the restriction of the coproduct Δ of the dual Steenrod algebra to $H_*(S/p \wedge BP\langle 2 \rangle) \subset \mathcal{A}_*$ and the remaining coactions follow from (2.4). In this section, we compute the Bockstein spectral sequence

$$(2.8) \quad E_{*,*}^1 = THH_*(BP\langle 2 \rangle; \mathbb{F}_p)[v_0] \Rightarrow THH(BP\langle 2 \rangle; \mathbb{Z}_{(p)})_p.$$

As a direct consequence the coaction on $\sigma \bar{\tau}_3$, there is a differential

$$(2.9) \quad d_1(\mu_3) = v_0 \lambda_3.$$

in the $H\mathbb{Z}$ -Bockstein spectral sequence (2.8).

[Gabe: Should we add more details here?]

The following lemma follows from [12, Prop. 6.8] by translating to the E_∞ -context (cf. the proof of [2, Lem. 3.2]).

Lemma 2.10. *If $d_j(x) \neq 0$ in the $H\mathbb{Z}$ -Bockstein spectral sequence (2.8) then*

$$d_{j+1}(x^p) = v_0 x^{p-1} d_j(x)$$

if $p > 2$ or if $p = 2$ and $j \geq 2$. If $p = 2$ and $j = 1$ then

$$d_{j+1}(x^p) = v_0 x^{p-1} d_j(x) + Q^{|x|}(d_1(x))$$

When $p = 2$, we have the differential

$$d_1(\mu) = v_0 \lambda_3.$$

Therefore, the error term for $d_2(\mu_3^2)$ is

$$Q^{16} \lambda_3 = Q^{16}(\sigma \bar{\xi}_3^2) = \sigma(Q^{16}(\bar{\xi}_3^2)) = \sigma((Q^8 \bar{\xi}_3)^2) = \sigma(\bar{\xi}_4^2) = 0.$$

The first equality holds by definition of λ_3 , the second equality holds because σ commutes with Dyer-Lashoff operations by [5], the third equality holds by [6], and the last equality holds because σ is a derivation [3].

Corollary 2.11. *When $p = 2, 3$, there are differentials*

$$d_{i+1}(\mu^{p^i}) = v_0^{i+1} \mu_3^{p^i-1} \lambda_3.$$

Consequently, there are differentials

$$d_{\nu_p(k)+1}(\mu^k) = v_0^{\nu_p(k)+1} \mu^{k-1} \lambda_3$$

where $\nu_p(k)$ denotes the p -adic valuation of k .

Proof. Let $\alpha = \nu_p(k)$. We have that $k = p^\alpha j$ where p does not divide j . So by the Leibniz rule

$$d_{\alpha+1}(\mu_3^k) = d_{\alpha+1}((\mu_3^{p^\alpha})^j) = j \mu_3^{p^\alpha(j-1)} d_{\alpha+1}(\mu_3^{p^\alpha}) = j v_0^{\alpha+1} \mu^{p^\alpha(j-1)} \mu^{p^\alpha-1} \lambda_3 = j v_0^{\alpha+1} \mu^{j-1} \lambda_3.$$

Since j is not divisible by p , it is a unit mod p . \square

Now recall from (2.7) that $THH_*(BP\langle 2 \rangle; H\mathbb{Q}) \cong E_{\mathbb{Q}}(\lambda_1, \lambda_2)$. In fact the map, $THH_*(B; H\mathbb{Z}_{(p)}) \rightarrow THH_*(B; H\mathbb{Q})$ sends λ_i to λ_i for $i = 1, 2$. Therefore, the elements λ_1, λ_2 are p -torsion free and there are no further differentials in the $H\mathbb{Z}$ -Bockstein spectral sequence. We rename the following classes as follows

$$(2.12) \quad \begin{aligned} c_i^{(1)} &:= \lambda_3 \mu_3^{i-1}, & d_i^{(1)} &:= \lambda_1 c_i^{(1)}, \\ c_i^{(2)} &:= \lambda_2 c_i^{(1)}, & d_i^{(2)} &:= \lambda_2 d_i^{(1)}. \end{aligned}$$

Thus we have the following

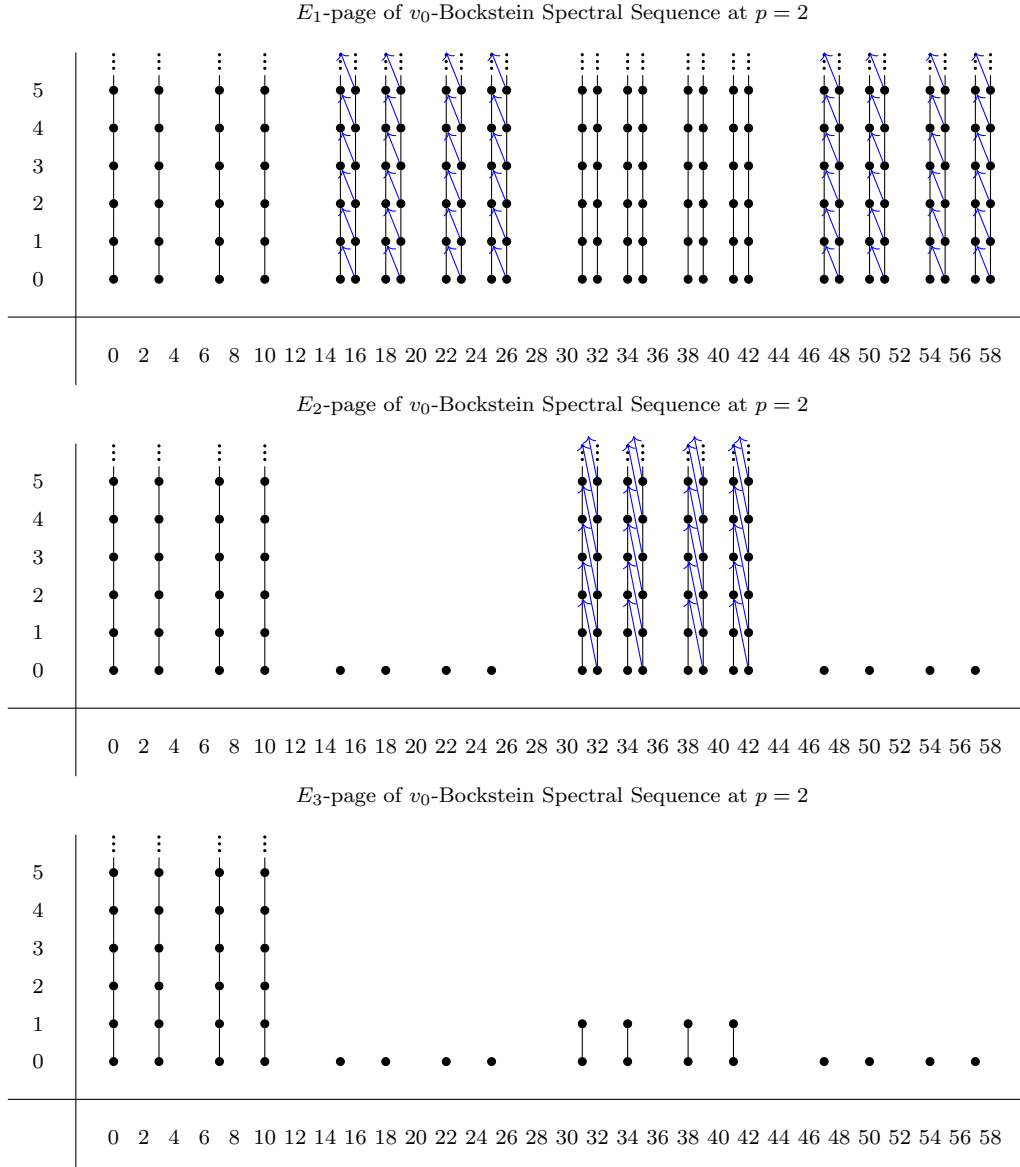
Corollary 2.13. *There is an isomorphism of $\mathbb{Z}_{(p)}$ -algebras*

$$THH_*(BP\langle 2 \rangle; H\mathbb{Z}_{(p)}) \cong E_{\mathbb{Z}_{(p)}}(\lambda_1, \lambda_2) \oplus T_0$$

where T_0 is a torsion $\mathbb{Z}_{(p)}$ -module defined by

$$T_0 = \left(\mathbb{Z}_{(p)} \{ c_i^{(k)}, d_i^{(k)} \mid i \geq 1, 1 \leq k \leq 2 \} \right) / (p^j c_i^{(k)}, p^j d_i^{(k)} \mid j = \nu_p(i) + 1, i \geq 1, 1 \leq k \leq 2)$$

where the products on the elements $c_i^{(k)}, d_i^{(k)}$ are specified by Formula (2.12) and by letting all other products be zero.



2.3. The v_1 -Bockstein spectral sequence. In this section, we begin our analysis of the v_1 -Bockstein spectral sequence

$$\mathrm{THH}_*(BP\langle 2 \rangle; H\mathbb{F}_p)[v_1] \Rightarrow \mathrm{THH}_*(BP\langle 2 \rangle; k(1)).$$

To start, we need to compute $K(1)_*(BP\langle 2 \rangle)$. This requires determining $\eta_R(v_{2+n})$ in $K(1)_*(BP)$ modulo the ideal generated by $(\eta_R(v_3), \dots, \eta_R(v_{1+n}))$. We will need the following.

[Gabe: Double check reference]

Lemma 2.14. [15, Lemma A.2.2.5] *Let v_n denote the Araki generators. Then there is the following equality in $BP_*(BP)$*

$$\sum_{i,j \geq 0}^F t_i \eta_R(v_j)^{p^i} = \sum_{i,j \geq 0}^F v_i t_j^{p^i}$$

In this section, the distinction between Hazewinkel generators and Araki generators is unimportant, as the two sets of generators coincide modulo p . In $K(1)_*BP$, we have killed all v_i 's except v_1 , which gives us the following equation

$$\sum_{i,j \geq 0}^F t_i \eta_R(v_j)^{p^i} = \sum_{k \geq 0}^F v_1 t_k^p$$

Note that the following degrees of the terms:

$$\begin{aligned} |v_1 t_j^p| &= 2(p^{j+1} - 1) \\ |t_i \eta_R(v_j)^{p^i}| &= 2(p^{i+j} - 1) \end{aligned}$$

Since we are interested in the term $\eta_R(v_{2+n})$, we collect all the terms on the left of degree $2(p^{2+n} - 1)$. Thus we are summing over the ordered pairs (i, j) such that $i + j = 2 + n$. Since we only care about $\eta_R(v_{2+n})$ modulo $\eta_R(v_3), \dots, \eta_R(v_{1+n})$ we only need to collect the terms where $j = 1, 2$, or $2 + n$. This shows that

$$t_{1+n} \eta_R(v_1)^{p^{n+1}} + t_n \eta_R(v_2)^{p^n} + \eta_R(v_{n+2}) = v_1 t_{n+1}^p$$

The value of η_R on v_1 and v_2 can also be computed by Lemma 2.14. One obtains, in $K(1)_*(BP)$, the following

[Gabe: Double check these formulas.]

$$\begin{aligned} \eta_R(v_1) &= v_1 \\ \eta_R(v_2) &= v_1 t_1^p - t_1 v_1^p. \end{aligned}$$

Combining these observations, we obtain

Lemma 2.15. *In $K(1)_*(BP)$, the following congruence is satisfied*

$$\eta_R(v_{2+n}) \equiv v_1 t_{n+1}^p - v_1^{p^n} t_1^{p^{n+1}} t_n + v_1^{p^{n+1}} (t_1^{p^n} t_n - t_{n+1}) \pmod{(\eta_R(v_3), \dots, \eta_R(v_{1+n}))}$$

for $n \geq 1$.

Consequently, we have the following corollary.

Corollary 2.16. *There is an isomorphism of $K(1)_*$ -algebras*

$$K(1)_*(BP\langle 2 \rangle) \cong K(1)_*(BP) / (v_1 t_{n+1}^p - v_1^{p^n} t_1^{p^{n+1}} t_n + v_1^{p^{n+1}} (t_1^{p^n} t_n - t_{n+1}) \mid n \geq 1)$$

Define elements

$$u_n := v_1^{\frac{1-p^n}{p-1}} t_n.$$

These elements are in degree 0, and therefore there is an isomorphism of $K(1)_*$ -algebras

$$K(1)_*(BP\langle 2 \rangle) \cong K(1)_* \otimes_{\mathbb{F}_p} K(1)_0(BP\langle 2 \rangle).$$

The calculations above imply the following corollary.

Corollary 2.17. *There is an isomorphism of \mathbb{F}_p -algebras*

$$K(1)_0(BP\langle 2 \rangle) \cong P(u_i \mid i \geq 1) / (u_{n+1}^p - u_1^{p^{n+1}} u_n + u_1^{p^n} u_n - u_{n+1} \mid n \geq 1).$$

Our goal is to use this and the $K(1)$ -based Bökstedt spectral sequence to compute the $K(1)$ -homology of $\mathrm{THH}(BP\langle 2 \rangle)$. This is a spectral sequence of the form

$$E_{s,t}^2 = \mathrm{HH}_s^{K(1)*}(K(1)_*(BP\langle 2 \rangle)) \implies K(1)_{s+t}(\mathrm{THH}(BP\langle 2 \rangle)).$$

The above considerations imply

$$E_{*,*}^2 \cong K(1)_* \otimes \mathrm{HH}_{*}^{\mathbb{F}_p}(K(1)_0 BP\langle 2 \rangle).$$

The following results will be useful for our calculation.

Lemma 2.18 ([14]). *Let $V = \mathrm{Spec}(A)$ be a nonsingular affine variety over a field k . Let W be the subvariety of $V \times \mathbb{A}^n$ defined by equations*

$$g_i(Y_1, \dots, Y_n) = 0, \quad g_i \in A[Y_1, \dots, Y_n], \quad i = 1, \dots, n.$$

Then the projection map $W \rightarrow V$ is étale at a point $(P; b_1, \dots, b_n)$ of W if and only if the Jacobian matrix $\left(\frac{\partial g_i}{\partial Y_j} \right)$ is a nonsingular matrix at $(P; b_1, \dots, b_n)$.

Theorem 2.19 (Étale Descent, [18]). *Let $A \hookrightarrow B$ be an étale extension of commutative k -algebras. Then there is an isomorphism*

$$\mathrm{HH}_*(B) \cong \mathrm{HH}_*(A) \otimes_A B$$

Example 2.20. Consider the subalgebra

$$\mathbb{F}_p[u_1, u_2] / (u_2^p - u_1^{p^2+1} + u_1^{p+1} - u_2 = f_1).$$

We will regard this as a $\mathbb{F}_p[u_1]$ -algebra. The partial derivative $\partial_{u_2} f_1$ is $-1 \pmod{p}$, and therefore a unit at every point. Then Lemma 2.18 tells us that this algebra is then étale over $\mathbb{F}_p[u_1]$.

By the same argument given above, we claim that there are a sequence of subalgebras A_n of

$$A := K(1)_0(BP\langle 2 \rangle) \cong \mathbb{F}_p[u_i \mid i \geq 1] / (u_{n+1}^p - u_1^{p^{n+1}} u_n + u_1^{p^n} u_n - u_{n+1} \mid n \geq 1)$$

such that each map $A_i \hookrightarrow A_{i+1}$ is an étale extension. Here

$$A_0 := \mathbb{F}_p[u_1]$$

$$A_n := \mathbb{F}_p[u_1, u_2, \dots, u_n, u_{n+1}] / (u_{k+1}^p - u_1^{p^{k+1}} u_k + u_1^{p^k} u_k - u_{k+1} = f_k \mid 1 \leq k \leq n)$$

and the partial derivative

$$\partial_{u_k} f_k = -1 \pmod{p}$$

for all $1 < k \leq n$ and therefore a unit at each point. The claim then follows by Lemma 2.18.

By the étale base change formula for Hochschild homology in Theorem 2.19, there is an isomorphism

$$\mathrm{HH}_{*}^{\mathbb{F}_p}(A_{i+1}) \cong \mathrm{HH}_{*}^{\mathbb{F}_p}(A_i) \otimes_{A_i} A_{i+1}$$

and since the functors $\mathrm{HH}_*(-)$ and $\mathrm{HH}_*^{\mathbb{F}_p}(A_1) \otimes_{A_1} (-)$ commute with filtered colimits of \mathbb{F}_p -algebras, there are isomorphisms

$$\begin{aligned} \mathrm{HH}_*^{\mathbb{F}_p}(A) &\cong \mathrm{HH}_*^{\mathbb{F}_p}(\mathrm{colim} A_n) \\ &\cong \mathrm{colim} \mathrm{HH}_*^{\mathbb{F}_p}(A_n) \\ &\cong \mathrm{colim} \mathrm{HH}_*^{\mathbb{F}_p}(A_0) \otimes_{A_0} A_n \\ &\cong \mathrm{HH}_*^{\mathbb{F}_p}(A_0) \otimes_{A_0} A. \end{aligned}$$

Consequently,

$$\mathrm{HH}_*^{K(1)*}(K(1)_*(BP\langle 2 \rangle)) \cong K(1)_* \otimes E(\sigma t_1) \otimes K_0(BP\langle 2 \rangle)$$

and therefore, since $\sigma t_1 \doteq \lambda_1 \pmod{p}$,

$$K(1)_*(\mathrm{THH}(BP\langle 2 \rangle)) \cong K(1)_*(BP\langle 2 \rangle) \otimes E(\lambda_1)$$

and

$$\mathrm{THH}_*(BP\langle 2 \rangle; K(1)) \cong K(1)_* \otimes E(\lambda_1).$$

In other words,

$$THH_*(BP\langle 2 \rangle; k(1)) \cong F \oplus T$$

where F is a free $P(v_1)$ -module generated by 1 and λ_1 and T is a torsion $P(v_1)$ -module.

In summary, we have proven the following theorem.

Theorem 2.21. *The following hold:*

- (1) *The $K(1)$ -homology of $\mathrm{THH}(BP\langle 2 \rangle; K(1))$ is $K(1)_*K(1) \otimes E(\lambda_1)$*
- (2) *There is a weak equivalence*

$$K(1) \vee \Sigma^{2p-1}K(1) \simeq \mathrm{THH}(BP\langle 2 \rangle; K(1)).$$

- (3) *The v_1 -torsion free part of $\mathrm{THH}(BP\langle 2 \rangle; k(1))$ is generated by 1 and λ_1 .*

2.3.1. *Differentials in the v_1 -BSS.* We now analyze the v_1 -BSS. Recall that this spectral sequence is of the form

$$\mathrm{THH}(BP\langle 2 \rangle; \mathbb{F}_p)[v_1] \implies \mathrm{THH}(BP\langle 2 \rangle; k(1)).$$

Thus the E_1 -page is

$$(2.22) \quad E(\lambda_1, \lambda_2, \lambda_3) \otimes P(\mu_3) \otimes P(v_1).$$

Since the λ_i are all in odd total degree and since v_1^k are known to survive to the E_∞ -term, the λ_i are all permanent cycles. If μ_3 were a permanent cycle, then by multiplicativity, the spectral sequence would collapse. This would contradict Theorem 2.21. Therefore, the element μ_3 must support a differential. The only possibility for bidegree reasons is

$$d_{p^2}(\mu_3) \doteq v_1^{p^2} \lambda_2.$$

Thus, we obtain

$$v_1^{-1}E_{p^2+1} \cong K(1)_* \otimes E(\lambda_1, \lambda_3, \lambda'_4) \otimes P(\mu_3^p)$$

where

$$\lambda'_4 := \lambda_2 \mu_3^{p-1}.$$

So the bidegree of λ'_4 is given by

$$|\lambda'_4| = (2p^4 - 2p^3 + 2p^2 - 1, 0).$$

For analogous reasons, the class λ'_4 is a permanent cycle, and μ_3^p cannot be a permanent cycle. Based on degree considerations, there are two possible differentials,

$$d_{p^2}(\mu_3^p) \doteq v_1^{p^2} \lambda'_4 \quad \text{or} \quad d_{p^3}(\mu_3^p) \doteq v_1^{p^3} \lambda_3.$$

The first would contradict the Leibniz rule for d_{p^2} and the fact that $d_{p^2}(\mu_3) \doteq v_1^{p^2} \lambda_2$. Thus,

$$v_1^{-1} E_{p^3+1} \cong K(1)_* \otimes E(\lambda_1, \lambda'_4, \lambda'_5) \otimes P(\mu_3^{p^2})$$

where

$$\lambda'_5 := \lambda_3 \mu_3^{p(p-1)}.$$

The bidegree of λ'_5 is

$$|\lambda'_5| = (2p^5 - 2p^4 + 2p^3 - 1, 0).$$

For degree reasons, the class λ'_5 is a permanent cycle. As before, the class $\mu_3^{p^2}$ must support a differential. Degree considerations, again, give two possibilities

$$d_{p^3}(\mu_3^{p^2}) \doteq v_1^{p^3} \lambda'_5$$

or

$$d_{p^4+p^2}(\mu_3^{p^2}) \doteq v_1^{p^4+p^2} \lambda'_4.$$

The former would contradict the Leibniz rule, leaving the latter as the only possibility. This gives us

$$v_1^{-1} E_{p^4+p^2+1} \cong K(1)_* \otimes E(\lambda_1, \lambda'_5, \lambda'_6) \otimes P(\mu_3^{p^3})$$

where $\lambda'_6 := \lambda'_4 \mu_3^{p^2(p-1)}$. We will continue via induction. First we need some notation. We will recursively define classes λ'_n by

$$\lambda'_n := \begin{cases} \lambda_n & n = 1, 2, 3 \\ \lambda'_{n-2} \mu_3^{p^{n-4}(p-1)} & n \geq 4 \end{cases}$$

We let $d'(n)$ denote the topological degree of λ'_n . Then this function is given recursively by

$$d'(n) = \begin{cases} 2p^n - 1 & n = 1, 2, 3 \\ 2p^n - 2p^{n-1} + d'(n-2) & n > 3 \end{cases}$$

Thus, by a simple induction, one has

$$d'(n) = \begin{cases} 2p^n - 1 & n = 1, 2, 3 \\ 2p^n - 2p^{n-1} + 2p^{n-2} - 2p^{n-3} + \dots + 2p^2 - 1 & n \equiv 0 \pmod{2}, n > 3 \\ 2p^n - 2p^{n-1} + 2p^{n-2} - 2p^{n-3} + \dots + 2p^3 - 1 & n \equiv 1 \pmod{2}, n > 3 \end{cases}$$

Observe that the integers $2p^{n+2} - d(n+1) - 1$ and $2p^{n+2} - d(n+2) - 1$ are divisible by $|v_1|$. Let $r'(n)$ denote the integer

$$r'(n) := |v_1|^{-1} (|\mu_3^{p^{n-1}}| - |\lambda'_{n+1}| - 1) = |v_1|^{-1} (2p^{n+2} - d'(n+1) - 1).$$

Then a simple induction shows that

$$r'(n) = \begin{cases} p^{n+1} + p^{n-1} + p^{n-3} + \cdots + p^2 & n \equiv 1 \pmod{2} \\ p^{n+1} + p^{n-1} + p^{n-3} + \cdots + p^3 & n \equiv 0 \pmod{2} \end{cases}.$$

We can now describe the differentials in the v_1 -BSS.

Theorem 2.23. *In the v_1 -BSS, the following hold:*

- (1) *The only nonzero differentials are in $v_1^{-1}E_{r'(n)}$.*
- (2) *The $r'(n)$ -th page is given by*

$$v_1^{-1}E_{r'(n)} \cong K(1)_* \otimes E(\lambda_1, \lambda'_{n+1}, \lambda'_{n+2}) \otimes P(\mu_3^{p^{n-1}})$$

and the classes $\lambda'_{n+1}, \lambda'_{n+2}$ are permanent cycles.

- (3) *The differential $d_{r'(n)}$ is uniquely determined by multiplicativity of the BSS and the differential*

$$d_{r'(n)}(\mu_3^{p^{n-1}}) \doteq v_1^{r'(n)} \lambda'_{n+1}.$$

for $n \geq 1$.

Proof. We proceed by induction. We have already shown the theorem for $n \leq 4$. Assume that

$$v_1^{-1}E_{r'(n-1)} \cong K(1)_* \otimes E(\lambda_1, \lambda'_n, \lambda'_{n+1}) \otimes P(\mu_3^{p^{n-2}}).$$

and λ'_n is an infinite cycle.

Since $\lambda'_n, \lambda'_{n+1}$ are both in odd topological degree, λ'_{n+1} cannot support a differential hitting the v_1 -towers on λ'_i for $i < n+1$. Thus, the only possibility is that λ'_{n+1} supports a differential into the v_1 -tower on 1 or λ_1 . But this would contradict Theorem 2.21. Therefore, the class λ'_{n+1} is a permanent cycle.

The class $\mu_3^{p^{n-1}}$ must support a differential, for if it did not, then the spectral sequence would collapse. This would lead to a contradiction of Theorem 2.21. Degree considerations show that the following differentials are possible

$$d_{\ell(n)}(\mu_3^{p^{n-1}}) \doteq v_1^{\ell(n)} \lambda'_{n+2}$$

and

$$d_{r(n)}(\mu_3^{p^{n-1}}) \doteq v_1^{r'(n)} \lambda'_{n+1}$$

where

$$\ell(n) = |v_1^{-1}|(2p^{n+2} - |\lambda'_{n+2}|).$$

An elementary inductive computation shows that

$$\ell(n) = r'(n-1).$$

We claim that the former differential cannot occur. This follows because, by the inductive hypothesis,

$$d_{r'(n-1)}(\mu_3^{p^{n-2}}) \doteq \lambda'_n,$$

and the former differential would contradict the Leibniz rule. So we must conclude the latter differential occurs. \square

We now state the main result of this section.

Theorem 2.24. *For each $n \geq 2$ and each nonnegative integer m with $m \not\equiv p-1 \pmod p$ there are elements $z_{n,m}$ and $z'_{n,m}$ in $THH_*(BP\langle 2 \rangle; k(1))$ such that*

- (1) $z_{n,m}$ projects to $\lambda'_n \mu_3^{mp^{n-2}}$ in $E_\infty^{*,0}$
- (2) $z'_{n,m}$ projects to $\lambda'_n \lambda'_{n+1} \mu_3^{mp^{n-2}}$ in $E_\infty^{*,0}$

As a $P(v_1)$ -module, $THH_(BP\langle 2 \rangle; k(1))$ is generated by the unit element 1, λ_1 , and the elements $\lambda_1^\epsilon z_{n,m}$, $\lambda_1^\epsilon z'_{n,m}$ where $\epsilon \in \{0, 1\}$. The only relations are*

$$v_1^{r'(n-1)} \lambda_1^\epsilon z_{n,m} = v_1^{r'(n-1)} \lambda_1^\epsilon z'_{n,m} = 0.$$

To prove this, we first need to prove a couple lemmas. We first introduce notation. Let $P(m)$ denote a free rank one $P(v_m)$ -module and let $P(m)_i$ denote the $P(v_m)$ -module $P(m)/v_m^i$. Let X be a $BP\langle 2 \rangle$ -module such that

$$H_*(X) \cong H_*(BP\langle 2 \rangle) \otimes H_*(\overline{X})$$

as a $H_*(BP\langle 2 \rangle)$ -module and consider the Adams spectral sequence

$$(2.25) \quad E_2^{*,*}(X) = Ext_{E(Q_m)_*}^{*,*}(\mathbb{F}_p, H_*(\overline{X})) \Rightarrow \pi_*(X \wedge_{BP\langle 2 \rangle} k(m))_p$$

and the v_m -inverted Adams spectral sequence

$$(2.26) \quad v_m^{-1} E_2^{*,*}(X) = v_m^{-1} Ext_{E(Q_m)_*}^{*,*}(\mathbb{F}_p, H_*(\overline{X})) \Rightarrow \pi_*(X \wedge_{BP\langle 2 \rangle} K(m))_p.$$

Consider the map of spectral sequences

$$E_2^{*,*}(X) \longrightarrow v_m^{-1} E_2^{*,*}(X)$$

induced by the localization map $k(m) \rightarrow v_m^{-1} k(m) = K(m)$.

Lemma 2.27. *For $r \geq 2$, the E_r -page of the Adams spectral sequence (2.25) for $X = THH(BP\langle 2 \rangle; k(1))$ and $m = 1$ is generated by elements in filtration 0 as a $P(1)$ -module and $E_r^{*,*}$ is a direct sum of copies of $P(1)$ and $P(1)_i$ for $i \leq r$.*

Proof. We will begin by proving the first statement by induction. Note that (2.22) implies the base case in the induction when $r = 2$, since the E_2 -page of the Adams spectral sequence with signature

$$E_2(THH(BP\langle 2 \rangle; k(1))) \Rightarrow THH_*(BP\langle 2 \rangle; k(1))$$

is isomorphic to the E_1 -page of the Bockstein spectral sequence

$$P(v_1) \otimes E(\lambda_1, \lambda_2, \lambda_3) \otimes P(\mu_1)$$

which is finitely generated in each degree and therefore can be written as direct sum of (suspensions) of copies of $P(v_1) = P(1)$. Suppose the statement holds for some r . Choose a basis y_i for the \mathbb{F}_p -vector space V_r such that

$$V_r = \{x \in E_r^{*,0} \mid v_1^{r-1} x = 0\}.$$

Then $d_r(y_i)$ is in filtration r and since the differentials are v_1 -linear, $v_1^{r-1} d_r(y_i) = 0$. However, this contradicts the induction hypothesis because the induction hypothesis implies that all elements in filtration r are v_1 -torsion-free. Thus, each basis element y_i is a d_r -cycle. Next choose a set of elements $\{y'_j\} \subset E_r^{*,0}$ such that $\{d_r(y'_j)\}$ is a basis for $\text{im}(d_r: E_r^{*,0} \rightarrow E_r^{*,r})$. Choose $y''_j \in E_r^{*,0}$ such that $v_1^r y''_j = d_r(y'_j)$. Then y''_j are d_r -cycles

and y_j'' and y_j are linearly independent. We can therefore choose d_r -cycles y_j''' such that $\{y_j\} \cup \{y_j''\} \cup \{y_j'''\}$ are a basis for the d_r -cycles in $E_r^{*,0}$. Then $\{y_j\} \cup \{y_j''\} \cup \{y_j'''\}$ are a basis for $E_r^{*,0}$ and the differential is completely determined by the formulas

$$d_r(y_i) = 0, \quad d_r(y_j') = v_1^r y_j'', \quad d_r(y_j'') = 0, \quad \text{and} \quad d_r(y_j''') = 0.$$

Thus, $E_r^{*,*}$ is generated as a $P(1)$ -module by y_i , y_i'' , and y_i''' where $v_1^{r-1} y_i = 0$ and $v_1^r y_i'' = 0$ and y_i''' is v_1 -torsion free. \square

Corollary 2.28. *For each $r \geq 2$ the localization map*

$$E_r(THH(BP\langle 2 \rangle; k(1))) \rightarrow v_1^{-1} E_r^{s,t}(THH(BP\langle 2 \rangle; k(1)))$$

is a monomorphism in filtration t for $t \geq r - 1$. Consequently, for each $r \geq 2$, the differentials in the source spectral sequence are determined by those in the target.

Proof. This follows by applying Lemma 2.27 and [13, Thm. 7.1] as in the remark after the proof of loc. cit. \square

Proof of Theorem 2.24. For brevity, we will let $\delta_{n,m}$ denote $\lambda'_n \mu_3^{mp^{n-2}}$ and we will let $\delta'_{n,m}$ denote $\lambda'_n \lambda'_{n+1} \mu_3^{mp^{n-2}}$. By Corollary 2.28, to prove that the elements $\delta_{n,m}$, $\delta'_{n,m}$, $\lambda_1 \delta_{n,m}$, and $\lambda_1 \delta'_{n,m}$ are infinite cycles, it suffices to show that

- (1) the elements $\lambda_1^\epsilon \delta_{n,m}$, and $\lambda_1^\epsilon \delta'_{n,m}$ for $\epsilon = 0, 1$ together with 1, form a basis for $E_\infty^{*,0}$ as an \mathbb{F}_p -vector space, and
- (2) that each of $\delta_{n,m}$, $\delta'_{n,m}$, $\lambda_1 \delta_{n,m}$, and $\lambda_1 \delta'_{n,m}$ are killed by $v_1^{r'(n-1)}$.

By induction on n , we will prove

$$E_{r(n-1)}(THH(BP\langle 2 \rangle; k(1))) \cong M_n \oplus E(\lambda_1, \lambda'_n, \lambda'_{n+1}) \otimes P(\mu_3^{p^{n-2}})$$

where M_n is generated by $\{\lambda_1^\epsilon \delta_{k,m}, \lambda_1^\epsilon \delta'_{k,m} \mid k < n, \epsilon = 0, 1\}$ modulo the relations

$$v_1^{r'(k-1)} \lambda_1^\epsilon \delta_{k,m} = v_1^{r'(k-1)} \lambda_1^\epsilon \delta'_{k,m} = 0.$$

This statement holds for $n = 2$ by (2.22). Assume the statement holds for all integers less than or equal to some $N \geq 2$. Lemma 2.28, Lemma 2.27, and Theorem 2.23 imply that the only nontrivial differentials with source in $E_{r(N)}^{*,0}$ are the differentials

$$d_{r(N)}(\lambda_1^\epsilon \mu_3^{(m+1)p^{N-1}}) = (m+1) v_1^{r'(N)} \lambda_1^\epsilon \lambda'_N \mu_3^{mp^{N-2}} \doteq v_1^{r'(N)} \lambda_1^\epsilon \delta_{N,m},$$

and the differentials

$$d_{r'(N)}(\lambda'_{N+1} \lambda_1^\epsilon \mu_3^{(m+1)p^{N-1}}) = (m+1) v_1^{r'(N)} \lambda_1^\epsilon \lambda'_N \lambda'_{N+1} \mu_3^{mp^{N-2}} \doteq v_1^{r'(N)} \lambda_1^\epsilon \delta'_{N,m}$$

where $m \not\equiv p-1 \pmod{p}$.

[Gabe: Need to give an argument for why λ_1 is a permanent cycle using previous results.]

Combining this with Lemma 2.27 and Lemma 2.28, this implies that

$$E_{r'(N)+1}(THH(BP\langle 2 \rangle)) \cong M_n \oplus V_{N+1} \oplus \left(P(1) \otimes E(\lambda_1, \lambda'_N, \lambda'_N \mu_3^{(p-1)p^{N-2}}) \otimes P(\mu_3^{p^N}) \right)$$

where V_{N+1} has generators $\delta_{N,m}$ and $\delta'_{N,m}$ and relations

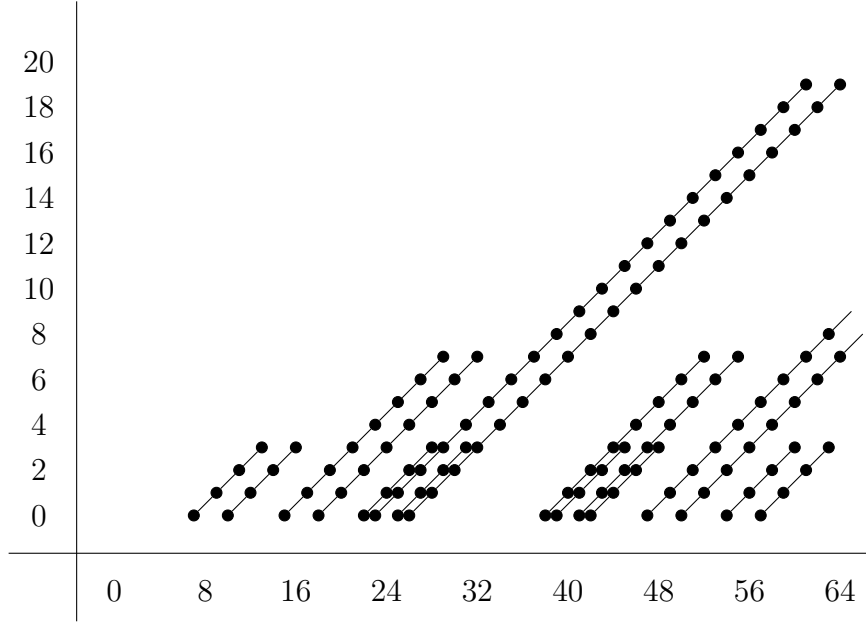
$$v_2^{r'(N)} \delta_{N,m} = v_2^{r'(N)} \delta'_{N,m} = 0.$$

By Lemma 2.28, Lemma 2.27, and Theorem 2.23 there is an isomorphism

$$E_{r(N)+1}(\mathrm{THH}(BP\langle 2 \rangle)) \cong E_{r(N+1)}(\mathrm{THH}(BP\langle 2 \rangle)).$$

Also, note that $M_N \oplus V_{N+1} = M_{N+1}$ and $\lambda'_N \mu_3^{(p-1)p^{N-2}} = \lambda'_{N+2}$ by definition. This completes the inductive step and consequently the proof. \square

v_1 -torsion in the E_∞ -page of v_1 -Bockstein Spectral Sequence for $0 \leq x \leq 64$



Remark 2.29. One may attempt to run the same arguments as in [2], but then one runs into the issue that the E_2 -pages of the remaining Bockstein spectral sequences are more dense and therefore the “vanishing column” arguments that are essential to completing the results by their methods do not apply here. We therefore use the Brun spectral sequence in the next section instead to determine the first family of differentials. This can also be done using the Whitehead filtration and the topological Hochschild-May spectral sequence and this was the first approach of the authors, but the Brun spectral sequence is quite similar computationally and avoids some noise early on in the spectral sequences.

3. BRUN SPECTRAL SEQUENCES

[Gabe: Fix some conventions. Write $THH(R)$ for $THH(HR)$ for a discrete ring? I’m fine with either as long as it is clear. I’d like to try to prove everything without implicitly p -completing if we can, so I started to adjust this, but right now this may be inconsistent.]

We first recall the main theorem from [9].

Theorem 3.1 (Thm. 1.1 [9]). *Let A be a cofibrant commutative S -algebra and let B be a connective cofibrant commutative A -algebra. Let E be a ring spectrum. Then, there is a strongly convergent, multiplicative spectral sequence of the form*

$$E_{n,m}^2 = \pi_n THH(B; HE_m(B \wedge_A B)) \Rightarrow E_{n+m} THH(A; B).$$

If $E_m(B \wedge_A B)$ is an \mathbb{F}_p -vector space for all m and if $\pi_0(B)/p\pi_0(B) = \mathbb{F}_p$ as rings, we have

$$E_{n,m}^2 = E_m(B \wedge_A B) \otimes_{\mathbb{F}_p} \pi_n(THH(B; H\mathbb{F}_p)).$$

Notation. We introduce notation $E^r(B, A, E)$ for the r -th page of the Brun spectral sequence for the triple (B, A, E) , where

$$E^2(B, A, E) := \pi_* THH(B; HE_*(B \wedge_A B)).$$

The main examples we will be interested in are the triples $(BP\langle i \rangle, BP\langle j \rangle, V(k))$ for $i, j, k \in \{-1, 0, 1, 2\}$ where $V(-1) = S$ and

$$V(k) = S/(p, v_1, \dots, v_k),$$

when this spectrum exists. We begin by computing relative cooperations for some spectra that will be needed in our later computations.

Lemma 3.2. *There is an isomorphism*

$$\pi_*(H\mathbb{Z}_p \wedge_{BP\langle 2 \rangle} H\mathbb{Z}_p) \cong E_{\mathbb{Z}_p}(\sigma v_1, \sigma v_2)$$

of graded \mathbb{Z}_p -algebras and there is an isomorphism

$$V(0)_*(H\mathbb{Z}_p \wedge_{BP\langle 2 \rangle} H\mathbb{Z}_p) \cong E(\sigma v_1, \sigma v_2)$$

Proof. The first result follows by the multiplicative Künneth spectral sequence

$$\mathrm{Tor}_{*,*}^{BP\langle 2 \rangle*}(\mathbb{Z}_p, \mathbb{Z}_p) \Rightarrow \pi_*(H\mathbb{Z}_p \wedge_{BP\langle 2 \rangle} H\mathbb{Z}_p).$$

by computing the input

$$\mathrm{Tor}_{*,*}^{BP\langle 2 \rangle*}(\mathbb{Z}_p, \mathbb{Z}_p) \cong E_{\mathbb{Z}_p}(\sigma v_1, \sigma v_2)$$

using Tor-duality and observing that there is the algebra generators are all infinite cycles for bidegree reasons.

The second result follows by noting that

$$V(0)_*(H\mathbb{Z}_p \wedge_{BP\langle 2 \rangle} H\mathbb{Z}_p) \cong \pi_*(H\mathbb{F}_p \wedge_{BP\langle 2 \rangle} H\mathbb{Z}_p) \cong \pi_* H\mathbb{F}_p \wedge_{\mathbb{Z}_p} (H\mathbb{Z}_p \wedge_{BP\langle 2 \rangle} H\mathbb{Z}_p)$$

so by the previous argument we know that $\pi_*(H\mathbb{Z}_p \wedge_{BP\langle 2 \rangle} H\mathbb{Z}_p) \cong E_{\mathbb{Z}_p}(\sigma v_1, \sigma v_2)$, which is, in particular, a free graded \mathbb{Z}_p -algebra. So again we apply a multiplicative Künneth spectral sequence, this time with signature

$$\mathrm{Tor}_{*,*}^{\mathbb{Z}_p}(\mathbb{F}_p, E_{\mathbb{Z}_p}(\sigma v_1, \sigma v_2)) \Rightarrow V(0)_*(H\mathbb{Z}_p \wedge_{BP\langle 2 \rangle} H\mathbb{Z}_p).$$

The E_2 -page is given by the isomorphism

$$\mathrm{Tor}_{*,*}^{\mathbb{Z}_p}(\mathbb{F}_p, E_{\mathbb{Z}_p}(\sigma v_1, \sigma v_2)) \cong \mathbb{F}_p \otimes_{\mathbb{Z}_p} E_{\mathbb{Z}_p}(\sigma v_1, \sigma v_2)$$

which in particular, collapses to the zero line so the Künneth spectral sequence again collapses and the desired answer follows by the isomorphism

$$\mathbb{F}_p \otimes_{\mathbb{Z}_p} E_{\mathbb{Z}_p}(\sigma v_1, \sigma v_2) \cong E(\sigma v_1, \sigma v_2).$$

□

We begin by computing the Brun spectral sequence for the triple $(BP\langle 2 \rangle, BP\langle 0 \rangle, S)$ where $BP\langle 0 \rangle = H\mathbb{Z}_p$. First, we need the following result of Bökstedt.

Theorem 3.3 (Bökstedt). *There is an isomorphism of graded \mathbb{F}_p -algebras*

$$THH_*(\mathbb{Z}; \mathbb{F}_p) \cong E(\lambda_1) \otimes P(\mu_1),$$

there are isomorphisms of groups

$$\pi_t THH(\mathbb{Z}) \cong \begin{cases} \mathbb{Z}\{1\} & t = 0 \\ \mathbb{Z}/n\{\gamma_n\} & t = 2n - 1 > 0 \\ 0 & \text{else} \end{cases}$$

and the map

$$THH_*(\mathbb{Z}) \rightarrow THH_*(\mathbb{Z}; H\mathbb{F}_p)$$

sends γ_n to $\lambda_1 \mu_1^{k-1}$ when $n = pk$ for some integer $k \geq 1$ and to 0 otherwise. This is also a map of graded rings where the former has a graded ring structure by letting $\gamma_i \cdot \gamma_j = 0$ for all i, j .

Corollary 3.4. *There is an isomorphism*

$$\pi_t THH(\mathbb{Z}_{(p)})_p \cong \begin{cases} \mathbb{Z}_p & t = 0 \\ \mathbb{Z}/p^{\nu_p(n)}\{\gamma_n\} & t = 2n - 1 > 0 \\ 0 & \text{else} \end{cases}$$

where ν_p denotes the p -adic valuation and the map $THH_*(\mathbb{Z}) \rightarrow THH_*(\mathbb{Z}_{(p)})_p$ sends γ_n to γ_n if $p \mid n$ and zero otherwise, so the map of graded \mathbb{Z}_p -algebras

$$THH_*(\mathbb{Z}_{(p)})_p \rightarrow THH_*(\mathbb{Z}_{(p)}; \mathbb{F}_p)$$

sends γ_{pk} to $\lambda_1 \mu_1^{k-1}$ as before with $\gamma_i \cdot \gamma_j = 0$ for all i, j as before.

Proof. It is clear that

$$THH_*(\mathbb{Z}_{(p)}; \mathbb{F}_p) \cong THH_*(\mathbb{Z}_{(p)}; \mathbb{F}_p) \cong E(\lambda_1) \otimes P(\mu_1)$$

so the computation of $THH_*(H\mathbb{Z}_{(p)})$ is clear from the Bockstein spectral sequence

$$THH_*(\mathbb{Z}_{(p)}; \mathbb{F}_p)[v_0] \Rightarrow THH_*(\mathbb{Z}_{(p)})_p$$

with differentials

$$d_k(\mu_1^{p^{k-1}}) = v_0^k \lambda_1,$$

all possible additive extensions, and no possible multiplicative extensions. □

Do we need to know that $THH(\mathbb{Z}_{(p)})$ is a finitely generated \mathbb{Z}_p -module? Is this clear?

It is maybe easier to see directly $THH(\mathbb{Z}_{(p)}) \cong THH(\mathbb{Z}_{(p)})_p$?

3.1. Brun spectral sequences with known abutment. We now compute the Brun spectral sequence for the triple $(BP\langle 2 \rangle, BP\langle 0 \rangle, V(0))$ using the known answer for the abutment. An argument intrinsic to this approach should also be possible, but in the interest of brevity we do not take this approach.

Lemma 3.5. *The Brun spectral sequence with signature*

$$E^2(BP\langle 2 \rangle, BP\langle 0 \rangle, V(0)) \Rightarrow \pi_*(THH(BP\langle 2 \rangle, H\mathbb{F}_p))$$

$p=3$?

has E^2 -page

$$E^2(BP\langle 2 \rangle, BP\langle 0 \rangle, V(0)) \cong THH_*(H\mathbb{Z}_p; \mathbb{F}_p) \otimes E(\sigma v_1, \sigma v_2),$$

differentials

$$\begin{array}{c} 2p \quad u_1 \\ \curvearrowright \\ d^1(\sigma \cancel{\tau}_1) = \sigma v_1, \\ 2p^2 \quad \quad d^1((\sigma \cancel{\tau}_1)^p) = \sigma v_2 \\ \quad \quad \quad u_1^p \end{array}$$

and E^2 -page isomorphic to the abutment, which is in turn isomorphic to

$$E(\lambda_1, \lambda_2, \lambda_2) \otimes P(\mu_3)$$

by an isomorphism mapping λ_1 to λ_1 , $\sigma v_1(\sigma \cancel{\tau}_1)^{p-1}$ to λ_2 , $\sigma v_2 \sigma \cancel{\tau}_1^{p^2-p}$ to λ_3 , and $(\sigma \cancel{\tau}_1)^{p^2}$ to μ_3 .

Proof. The input is determined by Theorem 3.1, Theorem 3.3, and Lemma 3.2. We know that the abutment has no elements in degrees k where $k \equiv 0 \pmod{2p}$ and $k < 2p^3$. This immediately implies the two differentials and the rest is determined by the Leibniz rule and the fact that any further differentials would result in a smaller output than the known abutment. \square

Lemma 3.6. *The map*

$$E^2(BP\langle 2 \rangle, BP\langle 0 \rangle, S) \rightarrow E^2(BP\langle 2 \rangle, BP\langle 0 \rangle, V(0))$$

of E^2 -pages of Brun spectral sequences is determined by the map

$$THH_*(\mathbb{Z}_{(p)}) \otimes_{\mathbb{Z}_{(p)}} E_{\mathbb{Z}_{(p)}}(\sigma v_1, \sigma v_2) \rightarrow THH_*(\mathbb{Z}_{(p)}; H\mathbb{F}_p) \otimes E(\sigma v_1, \sigma v_2)$$

given by tensoring the map

$$THH_*(\mathbb{Z}_{(p)}) \rightarrow THH_*(\mathbb{Z}_{(p)}; H\mathbb{F}_p)$$

of Corollary 3.4 with the mod p -reduction map

$$E_{\mathbb{Z}_p}(\sigma v_1, \sigma v_2) \rightarrow E(\sigma v_1, \sigma v_2)$$

over the canonical quotient map $\mathbb{Z}_p \rightarrow \mathbb{F}_p$.

Proof. This follows easily by functoriality of the identification of the E^2 -page in Theorem 3.1 and Corollary 3.4. \square

We will now use the previous result as well as the known abutment from a previous section to determine the differentials in the Brun spectral sequence for the triple $(BP\langle 2 \rangle, BP\langle 0 \rangle, S)$. This will then be used to import key differentials into the main Brun spectral sequence of interest.

I think this is correct, but we can't refer to 3.1

Lemma 3.7. *The Brun spectral sequence with signature*

$$E^2(BP\langle 2 \rangle, BP\langle 0 \rangle, S) \Rightarrow \pi_* THH(BP\langle 2 \rangle, BP\langle 0 \rangle)$$

has E^2 -page

$$E^2(BP\langle 2 \rangle, BP\langle 0 \rangle, S) \cong THH_*(H\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} E(\sigma v_1, \sigma v_2)$$

and differentials

$$\begin{array}{l} d^{2p}(\gamma_{pk}) = (k-1)\sigma v_1 \gamma_{p(k-1)}, \\ d^{p+1}(a_k) = (k-1)\sigma v_2 a_{k-1}, \\ d^{p+1}(b_k) = (k-1)\sigma v_2 b_{k-1} \end{array}$$

d^{2p^2} \rightarrow d^{2p}

where $a_k = p\gamma_{p^2k}$ and $b_k = \cancel{p\gamma_{p^2k}\sigma v_1}$ and additive extensions

$$p\gamma_p = \sigma v_1$$

$$pa_1 = \sigma v_2.$$

Proof. We consider the map of spectral sequences

$$E^2(BP\langle 2 \rangle, BP\langle 0 \rangle, S) \rightarrow E^2(BP\langle 2 \rangle, BP\langle 0 \rangle, V(0)).$$

This is completely described by Lemma 3.6 and therefore we can determine the d^{2p} -differentials

$$d^{2p}(\gamma_{pk}) = (k-1)\sigma v_1 \gamma_{p(k-1)}$$

Do we need an additional argument? (see my notes)

directly. This accounts for all d^{2p} -differentials. Using the translation $a_k = p\gamma_{p^2k}$ and $b_k = \cancel{p\gamma_{p^2k}\sigma v_1}$ we then observe that the E^2 -page is isomorphic to the associated graded of a filtration of

$$THH_*(\ell, H\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} E_{\mathbb{Z}_p}(\sigma v_2)$$

the differential pattern

$$\begin{array}{l} d^{2p+1}(a_k) = (k-1)\sigma v_2 a_{k-1}, \\ d^{p+1}(b_k) = (k-1)\sigma v_2 b_{k-1} \end{array}$$

is then forced in order to get the desired answer in the abutment and so are the hidden additive extensions. \square

3.2. Brun spectral sequence for the triple $(BP\langle 2 \rangle, BP\langle 1 \rangle, S)$. We now consider the main example of interest, the Brun spectral sequence for the triple $(BP\langle 2 \rangle, BP\langle 1 \rangle, S)$. Note that we are choosing $BP\langle 1 \rangle$ to be a commutative $BP\langle 2 \rangle$ -algebra model for $BP\langle 1 \rangle$ and both $BP\langle 2 \rangle$ and $BP\langle 1 \rangle$ are implicitly p -completed. We first begin with a necessary lemma.

Lemma 3.8. *There is an isomorphism of graded \mathbb{Z}_p -algebras*

$$\pi_*(BP\langle 1 \rangle \wedge_{BP\langle 2 \rangle} BP\langle 1 \rangle) \cong \pi_*(BP\langle 1 \rangle) \otimes_{\mathbb{Z}_p} E_{\mathbb{Z}_p}(\sigma v_2)$$

Proof. We compute the Künneth spectral sequence

$$\mathrm{Tor}_*^{BP\langle 2 \rangle}(BP\langle 1 \rangle_*, BP\langle 1 \rangle_*) \Rightarrow \pi_*(BP\langle 1 \rangle \wedge_{BP\langle 2 \rangle} BP\langle 1 \rangle).$$

The input is $\pi_*(BP\langle 1 \rangle)_* \otimes E(\sigma v_2)$ by Tor-duality. The spectral sequence then collapses by multiplicativity and the because all algebra generators are infinite cycles for bidegree reasons. \square

Corollary 3.9. *The E_2 -page of the Brun spectral sequence for the triple $(BP\langle 2 \rangle, BP\langle 1 \rangle, S)$ is*

$$E_2^{BP\langle 2 \rangle, BP\langle 1 \rangle, S} = \pi_*(\mathrm{THH}(BP\langle 1 \rangle; H\mathbb{Z}_p)) \otimes_{\mathbb{Z}_p} \pi_*(BP\langle 1 \rangle) \otimes_{\mathbb{Z}_p} E_{\mathbb{Z}_p}(\sigma v_2).$$

In addition the E_2 -page of the Brun spectral sequence for the triple $(BP\langle 1 \rangle, BP\langle 1 \rangle, S)$ is

$$E_2^{BP\langle 1 \rangle, BP\langle 1 \rangle, S} = \pi_n(\mathrm{THH}(BP\langle 1 \rangle; H\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \pi_*(BP\langle 1 \rangle))$$

and the map

$$E_2^{BP\langle 2 \rangle, BP\langle 1 \rangle, S} \rightarrow E_2^{BP\langle 1 \rangle, BP\langle 1 \rangle, S}$$

is the identity map tensored with the usual counit map $E_{\mathbb{Z}_p}(\sigma v_2) \rightarrow \mathbb{Z}_p$ of the Hopf algebra $E_{\mathbb{Z}_p}(\sigma v_2)$.

Since the Brun spectral sequence for the triple $(BP\langle 1 \rangle, BP\langle 1 \rangle, S)$ can be identified with the Bockstein spectral sequence

$$THH_*(BP\langle 1 \rangle, H\mathbb{Z}_{(p)})[v_1] \Rightarrow THH_*(BP\langle 1 \rangle)$$

up to a shift in filtration, we know that all the differentials are determined by the formulas

$$(3.10) \quad d_{f(n)}(p^{n-1}a_{kp^{n-1}}) \doteq (k-1)v_1^{p^n+\dots+p}b_{(k-1)p^{n-1}}$$

where $f(n) = |v_1^{p^n+\dots+p}|$ by [2].

Since we determined the map of Brun spectral sequences

$$E_2^{BP\langle 2 \rangle, BP\langle 1 \rangle, S} \rightarrow E_2^{BP\langle 1 \rangle, BP\langle 1 \rangle, S}$$

we may then import these differentials into the spectral sequence

$$E_2^{BP\langle 2 \rangle, BP\langle 1 \rangle, S} \Rightarrow \pi_*\mathrm{THH}(BP\langle 2 \rangle; BP\langle 1 \rangle)$$

though some care must be taken when doing this. In particular, a priori, there could be a differential hitting a σv_2 -divisible element in $E_2^{BP\langle 2 \rangle, BP\langle 1 \rangle, S}$ that interrupts the other differential pattern. This cannot happen for the first differential because $|\sigma v_2| = 2p^2 - 1$ and the length of the first family of differentials is $2p^2 - 2p$, which is clearly smaller. We therefore immediately determine the first differential pattern.

Lemma 3.11. *There is a family of differentials*

$$(3.12) \quad d_{2p^2-2p+1}(a_k) \doteq (k-1)v_1^p b_{(k-1)}$$

in the Brun spectral sequence with signature

$$E^2(BP\langle 2 \rangle, BP\langle 1 \rangle, S) \Rightarrow \pi_*\mathrm{THH}(BP\langle 2 \rangle, BP\langle 1 \rangle).$$

Do the two ways of constructing the Atiyah-Hirzebruch spectral sequence give the same spectral sequence?

We observe that

$$E^{2p^2-2p+1}(BP\langle 2 \rangle, BP\langle 2 \rangle, S) \cong E^{2p^2-2p+1}(BP\langle 1 \rangle, BP\langle 2 \rangle, S) \otimes E(\sigma v_2).$$

In particular, note that

$$\{pa_{pi}, v_1^p b_{pk} : i \geq 1, k \geq 1\}$$

survive where $p^j pa_{pi} = 0$ when $j = \nu_p(i) + 1$ and $pkv_1^p b_{pk} = 0$.

We also can determine the map of Brun spectral sequences

$$E_2(BP\langle 2 \rangle, BP\langle 1 \rangle, S) \rightarrow E_2(BP\langle 2 \rangle, BP\langle 0 \rangle, S),$$

which is isomorphic to

$$\pi_*(THH(BP\langle 1 \rangle; H\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \pi_*(BP\langle 1 \rangle) \otimes_{\mathbb{Z}_p} E_{\mathbb{Z}_p}(\sigma v_2)) \rightarrow THH_*(H\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} E(\sigma v_1, \sigma v_2),$$

is given by tensoring the map

$$\pi_*(THH(BP\langle 1 \rangle; H\mathbb{Z}_p) \rightarrow THH_*(H\mathbb{Z}_p))$$

induced by the map $BP\langle 1 \rangle \rightarrow H\mathbb{Z}_p$ with the canonical quotient

$$\pi_*(BP\langle 1 \rangle) \rightarrow \mathbb{Z}_p$$

the identity

$$E_{\mathbb{Z}_p}(\sigma v_2) \rightarrow E_{\mathbb{Z}_p}(\sigma v_2)$$

and the unit map

$$\mathbb{Z}_p \rightarrow E_{\mathbb{Z}_p}(\sigma v_1)$$

of the \mathbb{Z}_p -algebra $E_{\mathbb{Z}_p}(\sigma v_1)$.

We determined a differential hitting a σv_2 -divisible element in the Brun spectral sequence with signature

$$E_2(BP\langle 2 \rangle, BP\langle 0 \rangle, S) \Rightarrow THH_*(BP\langle 2 \rangle, BP\langle 0 \rangle)$$

given by

$$d_{2p^2-1}(b_k) \doteq (k-1)\sigma v_2 b_{k-1}.$$

and this differential lifts to the same differential in the Brun spectral sequence with signature

$$E_2(BP\langle 2 \rangle, BP\langle 1 \rangle, S) \Rightarrow THH_*(BP\langle 2 \rangle, BP\langle 1 \rangle).$$

We summarize this in the following lemma.

Lemma 3.13. *There is a family of differentials*

$$(3.14) \quad d_{2p^2-1}(b_k) \doteq (k-1)\sigma v_2 b_{k-1}.$$

in the Brun spectral sequence for the triple $(BP\langle 2 \rangle, BP\langle 1 \rangle, S)$.

Consequently, we have the identification

$$E^{2p^2}(BP\langle 2 \rangle, BP\langle 2 \rangle, S) \cong H(E^{2p^2-2p+1}(BP\langle 1 \rangle, BP\langle 2 \rangle, S) \otimes E(\sigma v_2); d)$$

where the differential we simply denote by d here is determined by (3.14) and the Leibniz rule, and we write $H(M, d)$ for the homology of a differential bigraded algebra with respect to a differential. We observe that, in particular the elements

$$\{pb_{pj}, \sigma v_2 b_{pj} : j \geq 1\}$$

survive where pb_{pj} is indecomposable, $p \cdot (p^{i-1} \cdot pb_{pj}) = 0$ for $i = \nu_p(j) + 1$, and $v_1 \cdot (v_1^{p-1}(pb_{pj})) = 0$. Also, $v_1 \cdot (v_1^{p-1}\sigma v_2 b_{pj}) = 0$ and $pj(\sigma v_2 b_{pj}) = 0$.

We claim that this E_{2p^2-2p+1} -page

$$E^{2p^2-2p+1}(BP\langle 2 \rangle, BP\langle 2 \rangle, S)$$

and the E_1 -page

$$THH_*(BP\langle 2 \rangle, H\mathbb{Z}_p)[v_1]$$

of the Bockstein spectral sequence

$$THH_*(BP\langle 2 \rangle, H\mathbb{Z}_p)[v_1] \Rightarrow THH_*(BP\langle 2 \rangle, BP\langle 1 \rangle)$$

are two different associated graded algebras of two different filtrations of the same bigraded algebra.

Proposition 3.15. *There is a family of differentials*

$$d_{f(2)}(pa_{pk}) \doteq (k-1)v_1^{p^2+p}b_{p(k-1)}$$

where $f(2) = |v_1^{p^2+p}|$.

Proof. This is a sketch. Since we know the differential $d_{2p^2-1}(b_k) \doteq (k-1)\sigma v_2 b_{k-1}$ and there are no other possibly differentials of length between this one and $f(2)$, the next possibility is the differential of length $f(2)$. We then know that this spectral sequence maps to the one for the triple $(BP\langle 1 \rangle, BP\langle 1 \rangle, S)$ and there we have the differential $f(2)$. We can therefore lift this differential. We should probably be more careful about what exactly it lifts to. Anyways, this should give us the analogue of the first nontrivial differential in the Bökstein spectral sequence. We can then use the cap product argument to propagate this differential to some later differentials. \square

4. TOPOLOGICAL HOCHSCHILD COHOMOLOGY OF $BP\langle 2 \rangle$

We will write $THH_S^*(BP\langle 2 \rangle M)$ for topological Hochschild cohomology of $BP\langle 2 \rangle$ with coefficients in a $BP\langle 2 \rangle \wedge BP\langle 2 \rangle^{\text{op}}$ -module M , which is defined to be

$$THH_S^*(BP\langle 2 \rangle; M) := \pi_* (F_{BP\langle 2 \rangle^e}(BP\langle 2 \rangle, M))$$

where $BP\langle 2 \rangle^e := BP\langle 2 \rangle \wedge BP\langle 2 \rangle^{\text{op}}$. We recall that there is a universal coefficient spectral sequence (UCSS) computing the homotopy groups of $F_{BP\langle 2 \rangle^e}(BP\langle 2 \rangle, BP\langle 2 \rangle)$

$$Ext_{\pi_*(BP\langle 2 \rangle^e)}^{*,*}(BP\langle 2 \rangle_*, BP\langle 2 \rangle_*) \Rightarrow THC^*(BP\langle 2 \rangle),$$

but this is usually not computable. With coefficients in $H\mathbb{F}_p$, however, we can compute $THC^*(BP\langle 2 \rangle; H\mathbb{F}_p)$ by a different means. First, note that

$$THH_*(BP\langle 2 \rangle; H\mathbb{F}_p) \cong E(\lambda_1, \lambda_2, \lambda_3) \otimes P(\mu_3)$$

is a finite type graded \mathbb{F}_p -algebra and $THH(BP\langle 2 \rangle; H\mathbb{F}_p)$ is an $H\mathbb{F}_p$ -algebra. Given a map of commutative ring spectra $f: R \rightarrow S$ there is an associated adjunction $f_! \dashv f^*$ where $f_!(M) = M \wedge_R S$ is extension of scalars and f^* is restriction. By the adjunction $f_! \dashv f^*$ associated to the map of commutative ring spectra $f: BP\langle 2 \rangle^e \rightarrow H\mathbb{F}_p$, there is an equivalence

$$F_{H\mathbb{F}_p}(THH(BP\langle 2 \rangle; H\mathbb{F}_p), H\mathbb{F}_p) \simeq F_{BP\langle 2 \rangle^e}(BP\langle 2 \rangle, H\mathbb{F}_p).$$

The UCSS

$$\mathrm{Ext}_{\mathbb{F}_p}^{*,*}(\pi_*(\mathrm{THH}(BP\langle 2 \rangle; H\mathbb{F}_p)), \mathbb{F}_p) \Rightarrow \mathrm{THH}_S^*(BP\langle 2 \rangle; H\mathbb{F}_p)$$

collapses and

$$(4.1) \quad \mathrm{THH}_S^*(BP\langle 2 \rangle, H\mathbb{F}_p) \cong \mathrm{Hom}_{\mathbb{F}_p}(\mathrm{THH}_*(BP\langle 2 \rangle; H\mathbb{F}_p), \mathbb{F}_p)$$

where

$$\mathrm{Hom}_{\mathbb{F}_p}(\mathrm{THH}_*(BP\langle 2 \rangle; H\mathbb{F}_p), \mathbb{F}_p) \cong E(x_1, x_2, x_3) \otimes \Gamma(c_3)$$

and $|x_i| = 2p^i - 1$ and $|c_3| = 2p^3$. The classes x_i are dual to λ_i and the class $c_{3i} = \gamma_i(c_3)$ is dual to μ_3^i .

4.1. Relative topological Hochschild cohomology of $BP\langle 2 \rangle$. Recall that there is an isomorphism

$$H_*(MU) \cong P(b_k \mid k \geq 1)$$

and the map

$$H_*(MU) \rightarrow H_*(BP) \cong P(\bar{\xi}_k \mid k \geq 1)$$

sends b_j to $\bar{\xi}_k$ for $k \geq 1$ if $j = p^k - 1$ and zero otherwise at $p = 3$.

Lemma 4.2. *There is an isomorphism of rings*

$$\pi_*(H\mathbb{F}_p \wedge_{MU} BP\langle 2 \rangle) \cong E(\tau_3, \tau_4, \dots) \otimes E(\delta b_i \mid i \not\equiv 0 \pmod{p^k - 1}, k \geq 1)$$

and the map from $H_*(BP\langle 2 \rangle)$ is given by the canonical quotient

$$H_*(BP\langle 2 \rangle) \cong P(\xi_1, \xi_2, \dots) \otimes E(\tau_3, \tau_4, \dots) \rightarrow E(\tau_3, \tau_4, \dots)$$

tensoed with the unit map

$$\mathbb{F}_p \rightarrow E(\delta b_i \mid i \not\equiv 0 \pmod{p^k - 1}, k \geq 1).$$

Here $|\delta b_i| = 1 + |b_i|$.

Proof. First note that there is an equivalence of commutative $H\mathbb{F}_p$ -algebras

$$H\mathbb{F}_p \wedge_{MU} BP\langle 2 \rangle \simeq H\mathbb{F}_p \wedge_{H\mathbb{F}_p \wedge MU} H\mathbb{F}_p \wedge BP\langle 2 \rangle.$$

The Künneth spectral sequence has input

$$\begin{aligned} \mathrm{Tor}_*^{H_*MU}(\mathbb{F}_p, H_*BP\langle 2 \rangle) &\cong \mathrm{Tor}_*^{P(\bar{\xi}_1, \bar{\xi}_2, \dots)}(\mathbb{F}_p, H_*BP\langle 2 \rangle) \otimes \mathrm{Tor}^{P(b_i \mid i \not\equiv 0 \pmod{p^k - 1}, k \geq 1)}(\mathbb{F}_p, \mathbb{F}_p) \\ &= E(\tau_3, \tau_4, \dots) \otimes E(\delta b_i \mid i \not\equiv 0 \pmod{p^k - 1}, k \geq 1) \end{aligned}$$

and since MU is a commutative ring spectrum, it is a multiplicative spectral sequence. The Künneth spectral sequence collapses because all the algebra generators are in filtration 0, 1 and the differentials shift filtration by at least 2. By factoring the relevant map as

$$H_*(BP\langle 2 \rangle) \rightarrow \pi_*(H\mathbb{F}_p \wedge_{BP} BP\langle 2 \rangle) \rightarrow \pi_*(H\mathbb{F}_p \wedge_{MU} BP\langle 2 \rangle)$$

and computing $\pi_*(H\mathbb{F}_p \wedge_{BP} BP\langle 2 \rangle) \cong E(\tau_3, \tau_4, \dots)$ by the same argument, we see that the map is the composite of the canonical quotient with the identity tensoed with the unit map as desired. \square

Recall from Lemma 2.4 [2] that when $R \rightarrow Q$ is a map of E -algebras and M is a Q - R -bimodule, with an R - R -bimodule structure by pullback, then

$$THH_E(R; M) \simeq F_{Q \wedge_E R^{\text{op}}}(Q, M).$$

Lemma 4.3. *The following hold:*

(1) *There is an isomorphism of rings*

$$THH_{MU}^*(BP\langle 2 \rangle; H\mathbb{F}_p) \cong P(e_i \mid i \geq 3) \otimes P(g_i \mid i \not\equiv 0 \pmod{p^k - 1}, k \geq 1)$$

where $|e_i| = |\tau_i| + 1$ and $|g_i| = 2i + 2$.

(2) *Consequently, $THC_{MU}^*(BP\langle 2 \rangle)$ is isomorphic to*

$$BP\langle 2 \rangle_*(e_i \mid i \geq 3) \otimes P(g_i \mid i \not\equiv 0 \pmod{p^k - 1}, k \geq 1).$$

(3) *The map*

$$THH_{MU}^*(BP\langle 2 \rangle) \rightarrow THH_{MU}^*(BP\langle 2 \rangle; H\mathbb{F}_p)$$

is induced by the quotient by (p, v_1, v_2) .

(4) *The map*

$$THH_{MU}^*(BP\langle 2 \rangle; H\mathbb{F}_p) \rightarrow THH_S^*(BP\langle 2 \rangle; H\mathbb{F}_p)$$

sends e_i to c_{3p^i-3} for $i \geq 3$.

(5) *Consequently, the elements c_{3p^i-3} pull back to elements in $THC^*(BP\langle 2 \rangle)$.*

Proof. From the setup before this lemma, we may consider the UCSS computing

$$THH_E(R; M) \simeq F_{Q \wedge_E R^{\text{op}}}(Q, M)$$

with input

$$\text{Ext}_{\pi_*(Q \wedge_E R^{\text{op}})}^{*,*}(Q_*, M_{*\cdot})$$

When $E = MU$, $R = BP\langle 2 \rangle$ and $M = Q = H\mathbb{F}_p$. The UCSS computing

$$THH_{MU}^*(BP\langle 2 \rangle; H\mathbb{F}_p)$$

has input

$$\text{Ext}_{\pi_* H\mathbb{F}_p \wedge_{MU} BP\langle 2 \rangle}^*(\mathbb{F}_p, \mathbb{F}_p) \cong P(e_i \mid i \geq 3) \otimes P(g_i \mid i \not\equiv 0 \pmod{p^k - 1}, k \geq 1)$$

where $|e_i| = 2p^i$ and $|g_i| = 2i + 2$. Note that by Tor duality and Koszul duality

$$\text{Tor}_*^{P(b)}(\mathbb{F}_p, \mathbb{F}_p) \cong E(\delta b_i)$$

where $|\delta b_i| = |b_i| + 1$ and

$$\text{Ext}_{E(\delta b_i)}^*(\mathbb{F}_p, \mathbb{F}_p) \cong P(g_i)$$

where $|g_i| = |b_i| + 2$ for all i . Since all elements are in even total degree there is no room for differentials and the spectral sequence collapses. This proves the first statement.

There are three Bockstein spectral sequences to go from $THH_{MU}^*(BP\langle 2 \rangle; H\mathbb{F}_p)$ to $THH_{MU}^*(BP\langle 2 \rangle)$, but in each case all elements are in even columns and the spectral sequences collapse since there is an Adams style differential convention. This proves the second statement and the third statement.

Now, by the commutative diagram

$$\begin{array}{ccc} THH_{MU}^*(BP\langle 2 \rangle) & \longrightarrow & THH_{MU}^*(BP\langle 2 \rangle; H\mathbb{F}_p) \\ \downarrow & & \downarrow \\ THH_S^*(BP\langle 2 \rangle) & \longrightarrow & THH_S^*(BP\langle 2 \rangle; H\mathbb{F}_p) \end{array}$$

the fifth statement follows from the fourth statement. It therefore remains to show that the map

$$THH_{MU}^*(BP\langle 2 \rangle; H\mathbb{F}_p) \rightarrow THH_S^*(BP\langle 2 \rangle; H\mathbb{F}_p)$$

sends e_i to c_{p^i-3} for $i \geq 3$. Recall the map $H_*(BP\langle 2 \rangle) \rightarrow \pi_*(H\mathbb{F}_p \wedge_{MU} BP\langle 2 \rangle)$ sends τ_i to τ_i for $i \geq 3$. Tracing this through the induced map of universal coefficient spectral sequences produces the desired result. \square

Next we determine whether the elements c_{p^i-3} are torsion in the Hopf algebra

$$THH_S^*(BP\langle 2 \rangle; H\mathbb{Z}_{(p)}).$$

Lemma 4.4. *There is an isomorphism of Hopf algebras*

$$THH_S^*(BP\langle 2 \rangle; H\mathbb{Z}_{(p)}) \cong E_{\mathbb{Z}_{(p)}}(x_1, x_2) \otimes \Gamma_{\mathbb{Z}_{(p)}}(c_1)/(pc_1)$$

where x_1, x_2 and c_1 are primitive.

Proof. In general, if R is a commutative ring spectrum and $H\mathbb{Z}_{(p)}$ is a commutative R -algebra, then $THH_*(R, H\mathbb{Z}_{(p)})$ is a $\mathbb{Z}_{(p)}$ Hopf-algebra spectrum whenever the $\mathbb{Z}_{(p)}$ -modules $THH_k(R; H\mathbb{Z}_{(p)})$ are finitely generated for all k . By Corollary 2.13, $THH_S^*(BP\langle 2 \rangle; H\mathbb{Z}_{(p)})$ is a finitely generated $\mathbb{Z}_{(p)}$ -algebra in each degree. Consequently, $THC^*(BP\langle 2 \rangle; H\mathbb{Z}_{(p)})$ is the $\mathbb{Z}_{(p)}$ -dual of $THH_*(BP\langle 2 \rangle; H\mathbb{Z}_{(p)})$ and it is also finitely generated in each degree and the Bockstein spectral sequence

$$THH_S^*(BP\langle 2 \rangle; H\mathbb{F}_p)[v_0] \Rightarrow THH_S^*(BP\langle 2 \rangle; H\mathbb{Z}_{(p)})_p$$

converges. Since multiplication by p commutes with the coproduct this is a spectral sequence of Hopf algebras. In order for $THC^*(BP\langle 2 \rangle; H\mathbb{Z}_{(p)})$ to be $\mathbb{Z}_{(p)}$ -dual to $THH_*(BP\langle 2 \rangle; H\mathbb{Z}_{(p)})$ the differentials

$$d_{i+1}(c_{3(p^i-1)}x_3) \doteq v_0^{i+1}c_{3p^i}$$

are forced for $i \geq 0$ where $c_0 = 1$ by convention. \square

Recall that there is a cap product

$$THH_S^k(BP\langle 2 \rangle) \otimes THH_m^S(BP\langle 2 \rangle; H\mathbb{Z}_{(p)}) \rightarrow THH_{m-k}^S(BP\langle 2 \rangle; H\mathbb{Z}_{(p)})$$

and we showed that the elements c_k lift to elements in $THH_S^*(BP\langle 2 \rangle)$.

Corollary 4.5. *For $k < n$, the cap product satisfies the following formulae*

$$\begin{aligned} c_{3k} \cap c_n^{(m)} &\doteq \binom{n-1}{k} c_{n-k}^{(m)} \\ c_{3k} \cap d_n^{(m)} &\doteq \binom{n-1}{k} d_{n-k}^{(m)} \end{aligned}$$

for $1 \leq m \leq 2$.

[Gabe: My reasoning for the corollary above is that capping with c_{3k} “undoes” multiplication by μ_3^k and then the relation is clear except for the binomial coefficient. I still have to double check that the binomial coefficient is correct.]

Since c_{p^k} is p^{k+1} torsion in $THH_S^*(BP\langle 2 \rangle; H\mathbb{Z}_{(p)})$ we see that $p^k c_{p^k} \neq 0$. However, in the Adams spectral sequence for $THH_S^*(BP\langle 2 \rangle)$ these classes are torsion free.

[Gabe: Why is this?]

The cap product is also natural and therefore it commutes with Bockstein spectral sequence differentials.

5. REMAINING TWO BOCKSTEIN SPECTRAL SEQUENCES

We now compute the remaining two Bockstein spectral sequences

$$(5.1) \quad THH_*(BP\langle 2 \rangle; H\mathbb{Z}_p)[v_1] \Rightarrow THH_*(BP\langle 2 \rangle; BP\langle 1 \rangle)$$

and

$$(5.2) \quad THH_*(BP\langle 2 \rangle; k(1))[v_0] \Rightarrow THH_*(BP\langle 2 \rangle; BP\langle 1 \rangle).$$

First, we claim that the spectral sequence (5.1) is multiplicative because it is equivalent to a multiplicative relative Adams spectral sequence. To see this, recall that

$$\pi_*(H\mathbb{Z}_p \wedge_{BP\langle 1 \rangle} H\mathbb{Z}_p) \cong E_{\mathbb{Z}_p}(\tau_1)$$

and

$$\pi_*(H\mathbb{Z}_p \wedge_{BP\langle 1 \rangle} BP\langle 1 \rangle \wedge_{BP\langle 2 \rangle} THH(BP\langle 2 \rangle)) \cong \pi_* THH(BP\langle 2 \rangle; \mathbb{Z}_p)$$

so the relative Adams spectral sequence with signature

$$\text{Ext}_{\pi_*(\mathbb{Z}_p \wedge_{BP\langle 1 \rangle} \mathbb{Z}_p)}^{*,*}(\mathbb{Z}_p, \pi_*(THH(BP\langle 2 \rangle; H\mathbb{Z}_p))) \Rightarrow THH_*(BP\langle 2 \rangle, BP\langle 1 \rangle)$$

is isomorphic to the spectral sequence (5.1). The spectral sequence (5.2), however is not known to be multiplicative so we will just use it as a comparison tool.

First, we note that the first two families of differentials that we computed in the Brun spectral sequence in (3.12) and (3.14) simply do not appear because it is taken into account in the previous Bockstein spectral sequence.

Lemma 5.3. *There are differentials*

$$d_{p^2+p}(c_k^{(j)}) \doteq (k-1)v_1^{p^2+p}d_{k-1}^{(j)}$$

for $k \geq 1$ and $j = 1, 2$.

Proof. [Gabe: This is somewhat speculative at this point. I think we can get the first differentials in this spectral sequence from the Brun spectral sequence, but I just did the translation in my head and I haven't checked these differentials carefully in the Brun spectral sequence yet either.]

□

We then use the cap product from section (insert internal ref qx) to propagate these differentials. By the same considerations as [2] we know that the cap product is compatible with the Bockstein differentials in (5.2).

[Gabe: Add theorem that proves propagation of the first family of differentials that we know using the cap product. For any argument using vanishing columns replace with dimension considerations in (5.2) and Brun SS.]

1

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