THH-MAY SPECTRAL SEQUENCE FOR $\mathrm{THH}(BP\langle 2\rangle;\mathbb{Z}_p)$

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CONTENTS

The goal of this note is to prove the results that Gabe and I talked about when he visited Champaing. In particular, if you put the usual Whitehead filtrations on $BP\langle 2\rangle$ and $H\mathbb{Z}_p$, then we get a THH-May spectral sequence of the form

$$E^2_{**} = \mathrm{THH}(H\pi_*BP\langle 2\rangle; H\mathbb{Z}_p) \implies \mathrm{THH}(BP\langle 2\rangle;\mathbb{Z}_p)$$

So I don't have to keep writing shit, let B denote $BP\langle 2\rangle$.

1. THE E^2-PAGE

First we need to compute the E^2-page. Observe that the E^2-term can be expressed as

$$H\mathbb{Z}_p \wedge_{H\pi_*B} \mathrm{THH}(H\pi_*B).$$

Now observe that, there is an equivalence of E_1-algebras,

$$H\pi_*B \cong H\mathbb{Z}_p \wedge \mathbb{S}[v_1, v_2]$$

where $\mathbb{S}[x]$ denotes the free E_1-algebra on a generator x in some degree. Reference??. Since $\mathrm{THH}(R) = S^1 \otimes R$, it follows that THH is a left adjoint, and so commutes with colimits. In particular, it commutes with smash products, and so

$$\mathrm{THH}(H\pi_*B) \simeq \mathrm{THH}(H\mathbb{Z}_p \wedge \mathbb{S}[v_1, v_2]) \simeq \mathrm{THH}(\mathbb{Z}_p) \wedge \mathrm{THH}(\mathbb{S}[v_1, v_2]).$$

Thus, we can rewrite the E^2-term as

$$H\mathbb{Z}_p \wedge_{H\mathbb{Z}_p \wedge \mathbb{S}[v_1,v_2]} \Big(\mathrm{THH}(\mathbb{Z}_p) \wedge \mathrm{THH}(\mathbb{S}[v_1, v_2])\Big).$$

Noting that $H\mathbb{Z}_p \simeq H\mathbb{Z}_p \wedge \mathbb{S}$, we have (e.g. EKMM Proposition 3.10) that this the E^2-term is equivalent to

$$(H\mathbb{Z}_p \wedge_{H\mathbb{Z}_p} \mathrm{THH}(\mathbb{Z}_p)) \wedge (\mathbb{S} \wedge_{\mathbb{S}[v_1,v_2]} \mathrm{THH}(\mathbb{S}[v_1,v_2]))$$

which itself is equivalent to

$$\mathrm{THH}(\mathbb{Z}_p) \wedge \mathrm{THH}(\mathbb{S}[v_1,v_2];\mathbb{S}) \simeq \mathrm{THH}(\mathbb{Z}_p) \wedge_{H\mathbb{Z}_p} (H\mathbb{Z}_p \wedge \mathrm{THH}(\mathbb{S}[v_1,v_2];\mathbb{S}))$$

So we need to compute $H\mathbb{Z}_p \wedge \mathrm{THH}(\mathbb{S}[v_1,v_2];\mathbb{S})$. Since the spectra involved have torsion free p-adic homology, we have Bökstedt spectral sequence

$$HH_*^{\mathbb{Z}_p}((H\mathbb{Z}_p)_*\mathbb{S}[v_1,v_2];(H\mathbb{Z}_p)_*\mathbb{S}) \cong HH^{\mathbb{Z}_p}(\mathbb{Z}_p[v_1,v_2];\mathbb{Z}_p) = \mathbb{Z}_p \otimes_{\mathbb{Z}_p[v_1,v_2]} (\mathbb{Z}_p[v_1,v_2] \otimes E(\sigma v_1, \sigma v_2))$$

Thus

$$HH_*^{\mathbb{Z}_p}((H\mathbb{Z}_p)_*\mathbb{S}[v_1,v_2];(H\mathbb{Z}_p)_*\mathbb{S}) = \Lambda_{\mathbb{Z}_p}(\sigma v_1, \sigma v_2).$$

Note that the May filtration of an element is where it appears in the Whitehead filtration. So the May filtration of $v_1, \sigma v_1$ is $2(p-1)$ and of $v_2, \sigma v_2$ is $2(p^2-1)$. We reindex by dividing by $2(p-1)$.

To get to the E^2-term, we need to use the Künneth spectral sequence:

$$\mathrm{Tor}^{\mathbb{Z}_p}(\mathrm{THH}_*(\mathbb{Z}_p), \Lambda_{\mathbb{Z}_p}(\sigma v_1, \sigma v_2)) \implies {}^{May}E^1_{**}(B;\mathbb{Z}_p).$$

As $\Lambda_{\mathbb{Z}_p}(\sigma v_1, \sigma v_2)$ is torsion free, the spectral sequence collapses and yields

$$E^2 \cong \pi_*(\mathrm{THH}(\mathbb{Z}_p)) \otimes_{\mathbb{Z}_p} \Lambda_{\mathbb{Z}_p}(\sigma v_1, \sigma v_2).$$

Note that the classes of $\mathrm{THH}_*(\mathbb{Z}_p)$ are of May filtration 0. With the reindexed form, $|\sigma v_1| = (2p-1, 1)$ and $|\sigma v_2| = (2p^2-1, p+1)$. Thus, we have shown the following.

Proposition 1.1. *The E^1-term of the* THH*-May spectral sequence for* $\mathrm{THH}(B;\mathbb{Z}_p)$ *is given by*

$$E^2_{*,*} = \mathrm{THH}_*(\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \Lambda_{\mathbb{Z}_p} \sigma v_1, \sigma v_2,$$

where the classes in $\mathrm{THH}_*(\mathbb{Z}_p)$ *are in May filtration 0 and where the bidegree of σv_i is $(2p^i - 1, (i-1)p + 1)$.*

We also need the following result of Bökstedt.

Theorem 1.2. *(Bökstedt) The homotopy groups of* $\mathrm{THH}(\mathbb{Z})$ *are given by the following,*

$$\pi_t \mathrm{THH}(\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & t = 0 \\ \mathbb{Z}/n & t = 2n-1 > 0 \\ 0 & else \end{cases}$$

Corollary 1.3. *Taking the p-completion yields*

$$\pi_t \,\mathrm{THH}(\mathbb{Z}_p)^\wedge_p \cong \begin{cases} \mathbb{Z} & t=0 \\ \mathbb{Z}/p^{\nu_p(n)} & t=2n-1>0 \\ 0 & else \end{cases}$$

where ν_p denotes the p-adic valuation.

It will be helpful to first compute the THH-May spectral sequence for $\mathrm{THH}(\ell;\mathbb{Z}_p)^\wedge_p$. We will then use the reduction map

$$\mathrm{THH}(B;\mathbb{Z}_p)\to \mathrm{THH}(\ell;\mathbb{Z}_p)$$

in order to lift d_1-May differentials.

A similar argument to the above shows that the THH-May spectral sequence for $\mathrm{THH}(\ell;\mathbb{Z}_p)$ has E^1-page

$$E^1_{**}\cong \mathrm{THH}(\mathbb{Z}_p)\otimes_{\mathbb{Z}_p}\Lambda_{\mathbb{Z}_p}(\sigma v_1).$$

In particular, this spectral sequence will collapse at the E^2-page. Note that the bidegree of σv_1 is $(2p-1,1)$ (again we are reindexing).

Let γ_n denote the generator in $\mathrm{THH}_{2n-1}(\mathbb{Z}_p)$.

Lemma 1.4. *For $n\not\equiv 0 \mod p$, the groups $\mathrm{THH}_{2n-1}(\mathbb{Z})^\wedge_p$ are trivial.*

Thus the only generators we need to worry about is γ_{kp} for natural numbers k. This shows that, on the 0-line, the only nontrivial groups are in degrees $2pk-1$ for natural numbers k. These are spaced out every $2p$ spaces. Also, on the 1-line, there are the classes $\gamma_{pk}\sigma v_1$. Note that this class is in degree $2p(k+1)-2$, and so is the potential target of a d_1-differential on $\gamma_{p(k+1)}$. In fact these differentials must occur. Indeed, we have

Theorem 1.5. *(Angeltveit-Hill-Lawson) The homotopy groups of $\mathrm{THH}(\ell;\mathbb{Z}_p)$ is given additively by the following $\mathbb{Z}_p$-module,*

$$\Lambda_{\mathbb{Z}_p}\lambda_1\oplus\left(\mathbb{Z}_p\{a_i,b_i\mid i\geq 1\}\right)/(p^{\nu_p(i)+1}a_i,p^{\nu_p(i)+1}b_i)$$

where $|a_i|=2p^2i-1$ and $|b_i|=2p^2i+2(p-1)$. As a ring, we have $\lambda_1 a_i=b_i$ and all other products are trivial.

This forces a unique pattern of differentials.

Proposition 1.6. *In the THH-May spectral sequence for $\mathrm{THH}(\ell;\mathbb{Z}_p)^\wedge_p$, for* double check
$k>1$, we have the following differentials

$$d_1(\gamma_{pk})\doteq p^{\max\{0,\nu_p(k-1)-\nu_p(k)\}}\gamma_{(k-1)p}\sigma v_1.$$

Moreover, the classes a_i are detected, up to a unit, by the class $p\gamma_{p^2i}$, and b_i is detected up to a unit by the class $\gamma_{p^2i}\sigma v_1$. Finally, there is a hidden extension $p\gamma_p = \sigma v_1$.

Proof. For degree reasons, we know that the only possible differentials are of the form
$$d_1(\gamma_{pk}) = \lambda \gamma_{(k-1)p}\sigma v_1$$
for some integer λ. Moreover, we also know that λ must be divisible by $p^{\max\{0,\nu_p(k-1)-\nu_p(k)\}}$. The only classes which could detect the classes a_i are multiples of γ_{p^2i}. Since the order of γ_{p^2i} is $p^{\nu_p(p^2i)} = p^{\nu_p(i)+2}$, and since the order of $\gamma_{p(pi-1)}$ is p, we have that
$$d_1(\gamma_{p^2i}) = \gamma_{p(pi-1)}\sigma v_1.$$
This also shows that $p\gamma_{p^2i}$ detects a_i.

The only classes which could detect the b_i are $\gamma_{p^2i}\sigma v_1$. There are potential d_1-differentials
$$d_1(\gamma_{p(pi+1)}) = \lambda \gamma_{p^2i}\sigma v_1$$
for some integer λ. Since the order of. $\gamma_{p(pi+1)}$ is p, it follows that $\lambda = p$. For degree reasons, all of the other classes wipe themselves out, and this makes sense because the other classes are of the form $\gamma_{pk}\sigma v_1^{\varepsilon}$ where $(p,k)=1$. □

Now we have the following square of spectral sequences,

$$\begin{array}{ccc} \mathrm{THH}(H\pi_* B, \mathbb{Z}_p) & \Longrightarrow & \mathrm{THH}(B;\mathbb{Z}_p) \\ \downarrow & & \downarrow \\ \mathrm{THH}(H\pi_* \ell, \mathbb{Z}_p) & \Longrightarrow & \mathrm{THH}(\ell;\mathbb{Z}_p) \end{array}$$

and the map of E^1-terms is
$$\mathrm{THH}(\mathbb{Z}_p)\otimes_{\mathbb{Z}_p}\Lambda_{\mathbb{Z}_p}(\sigma v_1, \sigma v_2) \to \mathrm{THH}(\mathbb{Z}_p)\otimes_{\mathbb{Z}_p}\Lambda_{\mathbb{Z}_p}(\sigma v_1)$$
which is given in the obvious way. From this, we obtain the following,

Corollary 1.7. *In the* THH*-May spectral sequence for* $\mathrm{THH}(B;\mathbb{Z}_p)$*, we have the following differentials*
$$d_1(\gamma_{pk}) \doteq p^{\max\{0,\nu_p(k-1)-\nu_p(k)\}}\gamma_{(k-1)p}\sigma v_1.$$
Thus we also have the differentials
$$d_1(\gamma_{pk}\sigma v_2) \doteq p^{\max\{0,\nu_p(k-1)-\nu_p(k)\}}\gamma_{(k-1)p}\sigma v_1 \sigma v_2.$$

Proof. □

Since σv_2 is a d^1-cycle, the Künneth theorem implies that we have the following as the E^2-term.

$$E^2 \cong \Lambda_{\mathbb{Z}_p}\sigma v_2 \otimes_{\mathbb{Z}_p}\Big(\Lambda_{\mathbb{Z}_p}\sigma v_1 \oplus \Big(\mathbb{Z}_p\{\gamma_p, a_i, b_i \mid i\geq 1\}\Big)/(p\gamma_p, p^{\nu_p(i)+1}a_i, p^{\nu_p(i)+1}b_i)\Big)$$

In the THH-May spectral sequence the bidegrees are $|a_i| = (2p^2 i-1,0)$ and $|b_i| = (2p^2 i + 2(p-1),1)$, and recall that $|\sigma v_2| = (2p^2-1, p+1)$.

keep in mind that we don't necessarily have the desired product structure at the level of E^2, since $\gamma_p a_i = 0$. We do get part of though, using multiplication by σv_1.

Now in the May spectral sequence for THH$(B;\mathbb{Z}_p)$, there is still a possibility for d^{p+1}-differentials. Note that the source and target of any d^{p+1}-differential originating on the 0-line is a_i and a multiple of $a_{i-1}\sigma v_2$. Recall the following

Theorem 1.8. *(Angelini-Knoll-Culver) The homotopy groups of* THH$(B;\mathbb{Z}_p)$ *are given by*

$$\Lambda_{\mathbb{Z}_p}(\lambda_1,\lambda_2)\oplus\Big(\mathbb{Z}_p\{c_i^{(k)}, d_i^{(k)} \mid i\geq 1, k=1,2\}/p^{\nu_p(i)+1}c_i^{(k)}, p^{\nu_p(i)+1}d_i^{(k)}\Big)$$

with degrees

(1) $|c_i^{(1)}| = 2ip^3-1$
(2) $|c_i^{(2)}| = 2ip^3+2p-2$
(3) $|d_i^{(1)}| = 2ip^3+2p^2-2$
(4) $|d_i^{(2)}| = 2ip^3+2p^2+2p-3$

This forces a unique pattern of differentials and hidden extensions.

Proposition 1.9. *The E^{p+1}-page of the* THH*-May spectral sequence for* THH$(B;\mathbb{Z}_p)$ *has differentials given by*

$$d^{p+1}(a_i) \doteq p^{\max(0,\nu_p(i-1)-\nu_p(i))}\sigma v_2\cdot a_{i-1},$$

and

$$d^{p+1}(b_i) \doteq p^{\max(0,\nu_p(i-1)-\nu_p(i))}\sigma v_2\cdot b_{i-1}$$

for $i>1$, and there are no other differentials. Moreover, there are no rooms for longer differentials for degree reasons, so $E^{p+2}\cong E^\infty$. Furthermore, pa_{pn} detects $c_n^{(1)}$ and pb_{pn} detects $c_n^{(2)}$, and also $\sigma v_2 a_{p(n-1)}$ detects $d_n^{(1)}$ and $\sigma v_2 b_{n-1}$ detects $d_n^{(2)}$; for $n>0$. This also implies the necessary family of hidden extensions.

2. The THH-May spectral sequence with $\mathbb{F}_p$-coefficients

Before getting into the $k(1)$-coefficient May spectral sequence, we first say some things about the $\mathbb{F}_p$-coefficients May spectral sequence. The reason we do this is so that we can import differentials in to the $k(1)$-coefficient May spectral sequence.

Thus, we are considering the spectral sequence which takes the form

$$\mathrm{THH}_*(H\pi_*B;\mathbb{F}_p) \Longrightarrow \mathrm{THH}_*(B;\mathbb{F}_p)$$

Proposition 2.1. *The E^1-page of the May spectral sequence is given by*

$$E^1 \cong \mathrm{THH}(\mathbb{Z}_p;\mathbb{F}_p)\otimes E(\sigma v_1, \sigma v_2)$$

where the bidegree of σv_i is $(2p^i-1, p^{i-1}+1)$. The May filtration of $\mathrm{THH}_*(\mathbb{Z}_p;\mathbb{F}_p)$ is entirely in degree 0.

Proof. Proved in the way we computed the E^1-page of the May spectral sequence in the previous section. □

Recall that since $\mathrm{THH}(\mathbb{Z}_p;\mathbb{F}_p)$ is an $H\mathbb{F}_p$-algebra, its homotopy is given by the comodule primitives in the mod p homology of $\mathrm{THH}(\mathbb{Z}_p;\mathbb{F}_p)$. Moreover, there is an equivalence (as ring spectra)

$$\mathrm{THH}(\mathbb{Z}_p;\mathbb{F}_p)\simeq \mathbb{F}_p\otimes_{\mathbb{Z}_p}\mathrm{THH}(\mathbb{Z}_p).$$

Since $H_*(\mathrm{THH}(\mathbb{Z}_p)$ is free over $H_*H\mathbb{Z}_p$, we find that the Künneth spectral sequence immediately collapses. This yields

$$(H\mathbb{F}_p)_*\mathrm{THH}(\mathbb{Z}_p;\mathbb{F}_p)\cong A_*\otimes_{A/\!\!/A(0)_*} H_*\mathrm{THH}(\mathbb{Z}_p).$$

Thus, we need the following,

Theorem 2.2 (Bökstedt)**.** *The mod p homology of $\mathrm{THH}(\mathbb{Z}_p)$ is given by*

$$(H\mathbb{F}_p)_*(\mathrm{THH}(\mathbb{Z}_p))\cong A/\!\!/A(0)_*\otimes E(\lambda_1)\otimes P(\mu_1)$$

where λ_1 is detected by

$$\lambda_1 = \begin{cases}\sigma\zeta_1^2 & p=2\\ \sigma\zeta_1 & p>2\end{cases}$$

and μ_1 is detected by

$$\mu_1 = \begin{cases}\sigma\zeta_2 & p=2\\ \sigma\overline{\tau}_1 & p>2\end{cases}.$$

From this we obtain the following.

Corollary 2.3. *The mod p homology of $\mathrm{THH}(\mathbb{Z}_p;\mathbb{F}_p)$ is given by*

$$A_*\otimes E(\lambda_1)\otimes P(\mu_1).$$

In fact, this isomorphism is an isomorphism of Hopf-algebras.

Since $\mathrm{THH}(\mathbb{Z}_p)$ is actually an E_∞-ring spectrum, the mod p homology is a Hopf algebra, and the Bökstedt spectral sequence is a spectral sequence of Hopf algebras. One sees immediately that λ_1 is a comodule primitive. One also finds that

$$\alpha(\mu_1) = \begin{cases} 1 \otimes \mu_1 & p = 2 \\ \overline{\tau}_0 \otimes \lambda_1 + 1 \otimes \mu_1 & p > 2 \end{cases}.$$

Define

$$\tilde{\mu}_1 := \begin{cases} \mu_1 & p = 2 \\ \mu_1 - \overline{\tau}_0 \otimes \lambda_1. \end{cases}$$

Then $\alpha(\tilde{\mu}_1) = 1 \otimes \tilde{\mu}_1$. Thus we have the following.

Theorem 2.4. *The homotopy of* $\mathrm{THH}(\mathbb{Z}_p;\mathbb{F}_p)$ *is given by,*

$$\pi_* \mathrm{THH}(\mathbb{Z}_p;\mathbb{F}_p) = E(\lambda_1) \otimes P(\tilde{\mu}_1).$$

Thus, we derive

Corollary 2.5. *The E^1-page of the May spectral sequence for* $\mathrm{THH}_*(B;\mathbb{F}_p)$ *is isomorphic to*

$$P(u_1) \otimes E(\lambda_1, \sigma v_1, \sigma v_2).$$

The bidegrees of u_1 and λ_1 are $(2p, 0)$ and $(2p-1, 0)$ respectively.

Since we know that

$$\mathrm{THH}_*(B;\mathbb{F}_p) \cong P(u_3) \otimes E(\lambda_1, \lambda_2, \lambda_3)$$

where $|\lambda_i| = 2p^i - 1$ and $|u_3| = 2p^3$, this allows us to compute the May spectral sequence.

Proposition 2.6. *In the May spectral sequence*

$$P(u_1) \otimes E(\lambda_1, \sigma v_1, \sigma v_2) \Longrightarrow \mathrm{THH}_*(B;\mathbb{F}_p)$$

the differentials are uniquely determined by multiplicativity and the differentials

$$d^1(u_1) = \sigma v_1$$

and

$$d^{p+1}(u_1^p) = \sigma v_2.$$

The classes λ_2 and λ_3 are detected by $u_1^{p-1} \cdot \sigma v_1$ and $u_1^{p(p-1)} \sigma v_2$, respectively. There are no hidden extensions.

3. THE THH-MAY SPECTRAL SEQUENCE WITH $k(1)$-COEFFICIENTS

We also need to write down the THH-May spectral sequence with $k(1)$-coefficients. Let's begin by determining the E^2-term. The E^2-term is given by

$$E^2 \cong \mathrm{THH}_*(H\pi_*B; H\pi_*k(1)).$$

There is an equivalence

$$\mathrm{THH}(H\pi_*B; H\pi_*k(1)) \simeq H\pi_*k(1) \wedge_{H\pi_*B} \mathrm{THH}(H\pi_*B).$$

Let $\mathbb{S}[v_1]$ denote the free E_1-algebra generated by a class in degree $2p-2$. Then

$$H\pi_*k(1) \simeq H\mathbb{F}_p \wedge \mathbb{S}[v_1].$$

Similarly, we have an equivalence

$$H\pi_*B \simeq H\mathbb{Z}_p \wedge \mathbb{S}[v_1, v_2].$$

Thus, we have an equivalence

$$\mathrm{THH}(H\pi_*B; H\pi_*) \simeq (H\mathbb{F}_p \wedge \S[v_1]) \wedge_{H\mathbb{Z}_p \wedge \mathbb{S}[v_1,v_2]} (\mathrm{THH}(\mathbb{Z}_p) \wedge \mathrm{THH}(\S[v_1, v_2])),$$

and this is equivalent to

$$(H\mathbb{F}_p \wedge_{H\mathbb{Z}_p} \mathrm{THH}(\mathbb{Z}_p)) \wedge (\mathbb{S}[v_1] \wedge_{\mathbb{S}[v_1,v_2]} \mathrm{THH}(\mathbb{S}[v_1, v_2])) \simeq \mathrm{THH}(\mathbb{Z}_p; \mathbb{F}_p) \wedge_{H\mathbb{Z}_p} (H\mathbb{Z}_p \wedge \mathrm{THH}(\mathbb{S}[v_1, v_2]$$

Thus we need to compute the $H\mathbb{Z}_p$-homology of $\mathrm{THH}(\mathbb{S}[v_1, v_2]; \mathbb{S}[v_1])$. For this, we can use the Bökstedt spectral sequence

$$HH^{\mathbb{Z}_p}(\mathbb{Z}[v_1, v_2]; \mathbb{Z}[v_1]) \implies (H\mathbb{Z}_p)_* \mathrm{THH}(\mathbb{S}[v_1, v_2]; \mathbb{S}[v_1])$$

The E^2-term of the Bökstedt spectral sequence is concentrated on the 0-line and is given by

$$\mathbb{Z}[v_1] \otimes_{\mathbb{Z}_p[v_1,v_2]} (\mathbb{Z}_p[v_1, v_2] \otimes \Lambda_{\mathbb{Z}_p}(\sigma v_1, \sigma v_2)) \cong \mathbb{Z}_p[v_1] \otimes_{\mathbb{Z}_p} \Lambda_{\mathbb{Z}_p}(\sigma v_1, \sigma v_2).$$

Since $(H\mathbb{Z}_p)_*(\mathrm{THH}(\mathbb{S}[v_1, v_2]; \mathbb{S}[v_1]))$ is torsion free, we find that the May E^2-term is given by

$$^{May}E^1(B; k(1)) \cong \mathrm{THH}(\mathbb{Z}_p; \mathbb{F}_p) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[v_1] \otimes_{\mathbb{Z}_p} \Lambda_{\mathbb{Z}_p} \sigma v_1, \sigma v_2.$$

Thus, we have derived

Corollary 3.1. *We have an isomorphism*

$$^{May}E^1(B; k(1)) \cong P(u_1, v_1) \otimes E(\lambda_1, \sigma v_1, \sigma v_2)$$

where the bidegrees are given by

- $|u_1| = (2p, 0)$,
- $|v_1| = (2p-2, 1)$, *and*
- $|\sigma v_i| = (2p^i - 1, p^{i-1} + 1)$

Remark 3.2. Note that ${}^{May}E^1(B;k(1)) \cong {}^{May}E^1(B;\mathbb{F}_p)\otimes P(v_1)$, and that the map of spectral sequences induced by the map $k(1)\to \mathbb{F}_p$ is the projection map sending v_1 to 0. This allows us to lift differentials.

We will now argue that the May spectral sequence for $\mathrm{THH}(B;k(1))$ is isomorphic to a reindexed version of the v_1-Bockstein spectral sequence at the

Proposition 3.3. *We can lift the d^1 and d^{p+1}-differentials from the $\mathbb{F}_p$-coefficient May spectral sequence. We have that*

$$ {}^{May}E^{p+2}(B;k(1)) \cong P(u_3)\otimes P(v_1)\otimes E(\lambda_1,\lambda_2,\lambda_3) $$

where $\lambda_2 = u_1\sigma v_1$, $\lambda_3 = u_1^p\sigma v_2$, and $u_3 = u_1^{p^2}$.

Proof. We clearly can lift the d^1-differentials, which shows that

$$ {}^{May}E^2(B;k(1)) \cong P(u_2,v_1)\otimes E(\lambda_1,\lambda_2,\sigma v_2) $$

where $\lambda_2 = u_1^{p-1}\sigma v_1$. We would like to lift the d^{p+1}-differentials, so we must exclude the possibility of an earlier differential.

Observe that for bidegree reasons that v_1,λ_1,λ_2 and σv_2 are all infinite cycles. For bidegree reasons, the first class that could be a target of a differential supported by u_2 is σv_2. Thus we can lift the d^{p+1}-differential from the $\mathbb{F}_p$-May spectral sequence. □

Corollary 3.4. *The May spectral sequence for* $\mathrm{THH}(B;k(1))$ *is a reindexed version of the v_1-Bockstein spectral sequence from the E^{p+2}-page onward.*

4. $\mathrm{THH}(B;L)$

In this section we calculate the homotopy of $\mathrm{THH}(B;L)$. We mimic the argument found in McClure-Staffeldt. In particular, we are going to study the homotopy pull-back diagram

$$\begin{tikzcd}
\mathrm{THH}(B;L) \arrow[r] \arrow[d] & \prod_q L_{H\mathbb{F}_q}\mathrm{THH}(B;L) \arrow[d] \\
\mathrm{THH}(B;L)_{\mathbb{Q}} \arrow[r] & \left(\prod_q L_{H\mathbb{F}_q}\mathrm{THH}(B;L)\right)
\end{tikzcd}.$$

Here q ranges over all primes. Note that since $H\mathbb{F}_q\wedge L\simeq *$ for $q\neq p$, we have that the upper right hand corner is $\mathrm{THH}(B;L)^\wedge_p$. We now identify the homotopy type of $\mathrm{THH}(B;L)^\wedge_p$. First, note that the class λ_1 survives to $\mathrm{THH}(B;\ell)$. Since $\mathrm{THH}(B;L)$ is an L-module, we have a morphism of L-modules

$$ L\vee\Sigma^{2p-1}L\to \mathrm{THH}(B;L). $$

Proposition 4.1. *The map above induces an isomorphism in $K(1)$-homology.*

Proof. Recall the equivalence
$$\mathrm{THH}(B;L) \simeq L \wedge_B \mathrm{THH}(B).$$
The EMSS thus collapses at E_2 and gives an isomorphism
$$K(1)_*(\mathrm{THH}(B;L)) \cong K(1)_* L \otimes_{K(1)_* B} K(1)_* \mathrm{THH}(B).$$
We have previously seen that $K(1)_* \mathrm{THH}(B) \cong K(1)_* B \otimes_{K(1)_*} E(\lambda_1)$, and so we have
$$K(1)_* \mathrm{THH}(B;L) \cong K(1)_* L \otimes_{K(1)_*} E(\lambda_1).$$
This implies the map is a $K(1)$-isomorphism. □

Corollary 4.2. *The map above induces an equivalence*
$$(L \vee \Sigma^{2p-1} L)_{K(1)} \to \mathrm{THH}(B;L)_{K(1)}.$$

Remark 4.3. *Recall that (cf. Ravenel "localization...") that the Bousfield class of $v_1^{-1}B$ is the same as the Bousfield class of L, and that the Bousfield class of L is the Bousfield class of $H\mathbb{Q} \vee K(1)$. So we also need to check this map induces an isomorphism on $H\mathbb{Q}$-homology.*

Note the following string of equivalences
$$\mathrm{THH}(B;L) \simeq L \wedge_B \mathrm{THH}(B) \simeq L \wedge_{v_1^{-1}B} \mathrm{THH}(v_1^{-1}B)$$
prove this

and that $v_1^{-1}B$ is L-local. Thus (I think) it follows that $\mathrm{THH}(B;L)$ is L-local. (Alternatively, and more easily, this follows from the fact that $\mathrm{THH}(B;L)$ is an L-module, and so L-local.)

We know from Prop 2.11 of Bousfield that
$$L_{K(1)} \simeq L_{S\mathbb{Z}/p} L_L.$$
Thus we can write the above equivalence as
$$((L \vee \Sigma^{2p-1} L)_L)_{S\mathbb{Z}/p} \xrightarrow{\simeq} (\mathrm{THH}(B;L)_L)_{S\mathbb{Z}/p}.$$
But both $L \vee \Sigma^{2p-1} L$ and $\mathrm{THH}(B;L)$ are L-local. Thus we conclude the following.

Corollary 4.4. *There is an equivalence*
$$(L \vee \Sigma^{2p-1} L)_{S\mathbb{Z}/p} \to ((B;L)_{K(1)})_{S\mathbb{Z}/p}$$
.

I actually think this result is enough for our purposes.

5. The May spectral sequence with ℓ-coefficients

We now move towards finding differentials in the May spectral sequence for $\mathrm{THH}_*(B;\ell)$. We need to know the E^1-page. The same sort of argument we have been giving yields the following.

Proposition 5.1. *The E^1-page of the* THH*-May spectral sequence for* $\mathrm{THH}(B;\ell)$ *is given by*

$$^{May}E^1(B;\ell) \cong \mathrm{THH}_*(\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[v_1] \otimes \Lambda_{\mathbb{Z}_p}(\sigma v_1, \sigma v_2)$$

We will start by computing the maps

$$^{May}E^1(B;\ell) \to {}^{May}E^1(B;\mathbb{Z}_p)$$

and

$$^{May}E^1(B;\ell) \to {}^{May}E^1(B;k(1))$$

with the aim of lifting some differentials.

Proposition 5.2. *The map*

$$^{May}E^1(B;\ell) \to {}^{May}E^1(B;\mathbb{Z}_p)$$

is the projection map induced by sending v_1 to 0.

Proof. $\square$

actually prove this.

Proposition 5.3. *The map*

$$^{May}E^1(B;\ell) \to {}^{May}E^1(B;k(1))$$

is induced by modding out by p and the map $\mathrm{THH}(\mathbb{Z}_p) \to \mathrm{THH}(\mathbb{Z}_p;\mathbb{F}_p)$.

Identify this map

I think the map

$$\mathrm{THH}(\mathbb{Z}_p) \to \mathrm{THH}(\mathbb{Z}_p;\mathbb{F}_p)$$

is induced by projecting γ_{pk} to $u_1^{k-1}\lambda_1$. Indeed, this map is the edge homomorphism for the v_0-BSS, and the Bockstein spectral sequence takes the form

$$\mathrm{THH}_*(\mathbb{Z}_p;\mathbb{F}_p)[v_0] \implies \mathrm{THH}_*(\mathbb{F}_p).$$

Since the only classes in filtration 0 which are in the correct degree are $u_1^{k-1}\lambda_1$, it follows that γ_{pk} projects onto $u_1^{k-1}\lambda_1$.

Remark 5.4. Since we have the identification

$$^{May}E^{p+2}(\ell) \cong {}^{May}E^{p+2}(\ell;\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[v_1]$$

and recall that ${}^{May}E^{p+2}(\ell;\mathbb{Z}_p)$ is an associated graded of $\mathrm{THH}(\ell;\mathbb{Z}_p)$. In particular we have

$${}^{May}E^{p+2}(\ell)\cong E(\gamma_1)\otimes\Lambda_{\mathbb{Z}_p}(\sigma v_1)\otimes\Lambda_{\mathbb{Z}_p}(a_i,b_i\mid i\geq 1)/(p^{\nu_p(i)+1}a_i,p^{\nu_p(i)+1}b_i,\lambda_1 a_i,\lambda_1 b_i),$$

in the answer there are hidden extensions $p\gamma_1=\sigma v_1$ and $\gamma_1 a_i=b_i$. In [AHL], they determine the differentials for the spectral sequence

$$\mathrm{THH}_*(\ell;\mathbb{Z}_p)[v_1]\implies \mathrm{THH}_*(\ell).$$

They found that the b_i are permanent cycles and that all the differentials are derived from the following

$$d_{p^n+p^{n-1}+\cdots+p}(p^{n-1}a_{kp^{n-1}})=(k-1)v_1^{p^n+\cdots+p}b_{(k-1)p^{n-1}}.$$

These uniquely correspond to the following differentials in the May spectral sequence

$$d_{p^n+p^{n-1}+\cdots+p+1}(p^{n-1}a_{kp^{n-1}})=(k-1)v_1^{p^n+\cdots+p}b_{(k-1)p^{n-1}}.$$

The $+1$ follows from the fact that the b_i are in May filtration 1.

We will now find an infinite family of d^{p+1}-differentials in the May spectral sequence for $\mathrm{THH}(B;\ell)$.

Proposition 5.5. *We have the following differentials in the May spectral sequence for* $\mathrm{THH}(B;\ell)$,

$$d_{p+1}(p^{n-1}a_i)\doteq p^{\nu_p(i-1)}v_1^p b_{i-1}+\varepsilon p^{\max(0,\nu_p(i-1)-\nu_p(i))}\sigma v_2\cdot a_{i-1}$$

where $\varepsilon\in\mathbb{Z}_p^\times$. *We also have the differentials*

$$d_{p+1}(b_i)=p^{\max(0,\nu_p(i-1)-\nu_p(i))}\sigma v_2 b_{i-1}.$$

Proof. Note that $\sigma v_2 a_{i-1}$ and $v_1^p b_{i-1}$ are the only two classes in the appropriate bidegree. So the d_{p+1}-differential is necessarily a linear combination of these classes. The result follows by projecting on to the May spectral sequences for $\mathrm{THH}(B;\mathbb{Z}_p)$ and $\mathrm{THH}(\ell)$. The other differential also is deduced from projecting to these two spectral sequences and using that b_i is a permanent cycle in the May spectral sequence for $\mathrm{THH}(\ell)$. $\square$

This allows us to deduce the following.

Corollary 5.6. *In* ${}^{May}E^{p+2}(B;\ell)$ *we have the relations*

$$p^{\nu_p(i-1)}v_1^p b_{i-1}\doteq p^{\max\{0,\nu_p(i-1)-\nu_p(i)\}}\sigma v_2\cdot a_{i-1}.$$

and

$$p^{\max\{0,\nu_p(i-1)-\nu_p(i)\}}\sigma v_2 b_{i-1}=0.$$

Remark 5.7. We also find that

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