A MAY-TYPE SPECTRAL SEQUENCE FOR HIGHER TOPOLOGICAL HOCHSCHILD HOMOLOGY

G. ANGELINI-KNOLL AND A. SALCH

ABSTRACT. Given a filtration of a commutative monoid A in a symmetric monoidal stable model category C, we construct a spectral sequence analogous to the May spectral sequence whose input is the higher order topological Hochschild homology of the associated graded commutative monoid of A, and whose output is the higher order topological Hochschild homology of A. We then construct examples of such filtrations and derive some consequences: for example, given a connective commutative graded ring R, we get an upper bound on the size of the THH-groups of E_∞-ring spectra A such that $\pi_*(A) \cong R$.

CONTENTS

1. INTRODUCTION.

Suppose $A = F_0 A \supseteq F_1 A \supseteq F_2 A \supseteq \dots$ is a filtered augmented k-algebra. In J. P. May's 1964 Ph.D. thesis, [24], May sets up a spectral sequence with input $\operatorname{Ext}^{*,*}_{E_0^* A}(k, k)$ and which converges to $\operatorname{Ext}^*_A(k, k)$. Here $E_0^* A = \oplus_{n \geqslant 0} F_n A / F_{n+1} A$ is the associated graded algebra of A.

In the present paper, we do the same thing for topological Hochschild homology and its "higher order" generalizations (as in [28]). Given a filtered E_∞-ring spectrum A, we

1

construct a spectral sequence

$$(1.0.1) \qquad E^1_{*,*} \cong THH_{*,*}(E^*_0 A) \Rightarrow THH_*(A).$$

Here $E^*_0 A$ is the *associated graded E_∞-ring spectrum of A*; part of our work in this paper is to define this "associated graded E_∞-ring spectrum," and prove that it has good formal properties and useful examples (e.g. Whitehead towers; see 1.0.5, below).

More generally: given any generalized homology theory H, and given any simplicial finite set $X_\bullet$, we construct a spectral sequence

$$(1.0.2) \qquad E^1_{*,*} \cong E_{*,*}(X_\bullet \otimes E^*_0 A) \Rightarrow E_*(X_\bullet \otimes A).$$

We recover spectral sequence 1.0.1 as a special case of 1.0.2 by letting $E_* = \pi_*$ and letting $X_\bullet$ be a simplicial model for the circle S^1.

A major part of the work we do in this paper is to formulate a definition (see Definition 3.1.1) of a "filtered E_∞-ring spectrum" which is sufficiently well-behaved that we can actually construct a spectral sequence of the form 1.0.2, identify its E^1- and E^∞-terms and prove its multiplicativity and good convergence properties. Actually our constructions and results work in a somewhat wider level of generality than commutative ring spectra: we fix a symmetric monoidal stable model category C satisfying some mild hypotheses (spelled out in Running Assumptions 2.0.2 and 2.0.3), and we work with filtered commutative monoid objects in C. In the special case where C is the category of symmetric spectra in simplicial sets, in the sense of [15] and [31], the commutative monoid objects are equivalent to E_∞-ring spectra. Our framework is sufficiently general that an interested reader could potentially also apply it to monoidal model categories of equivariant, motivic, and/or parametrized spectra.

In the appendix, we construct a version of spectral sequence 1.0.2 with coefficients in a filtered symmetric A-bimodule M:

$$(1.0.3) \qquad E^1_{*,*} \cong E_{*,*}(X_\bullet \otimes (E^*_0 A, E^*_0 M)) \Rightarrow E_*(X_\bullet \otimes (A, M)),$$

and as a special case,

$$(1.0.4) \qquad E^1_{*,*} \cong E_{*,*} THH(E^*_0 A, E^*_0 M) \Rightarrow E_* THH(A, M).$$

Some of the most important cases of filtered commutative ring spectra, or filtered commutative monoid objects in general, are those which arise from Whitehead towers: given a cofibrant connective commutative monoid in C, we construct a filtered commutative monoid

$$(1.0.5) \qquad A = \tau_{\geqslant 0} A \supseteq \tau_{\geqslant 1} A \supseteq \tau_{\geqslant 2} A \supseteq \dots$$

where the induced map $\pi_n(\tau_{\geqslant m} A) \to \pi_n(\tau_{\geqslant m-1} A)$ is an isomorphism if $n \geqslant m$, and $\pi_n(\tau_{\geqslant m} A) \cong 0$ if $n < m$. While the homotopy type of $\tau_{\geqslant m} A$ is very easy to construct, it takes us some work to construct a sufficiently rigid multiplicative model for the Whitehead tower 1.0.5; this is the content of Theorem 4.1.5.

If C is the category of symmetric spectra in simplicial sets, then the associated graded ring spectrum of the Whitehead tower 1.0.5 is the generalized Eilenberg-Mac Lane ring spectrum $H\pi_*(A)$ of the graded ring $\pi_*(A)$. Consequently we get a spectral sequence

$$(1.0.6) \qquad E^1_{*,*} \cong E_{*,*}(X_\bullet \otimes H\pi_* A) \Rightarrow E_*(X_\bullet \otimes A),$$

and as a special case,

$$(1.0.7) \qquad E^1_{*,*} \cong THH_{*,*}(H\pi_* A) \Rightarrow THH_*(A).$$

Many explicit computations are possible using spectral sequence 1.0.7 and its generalizations with coefficients in a bimodule (defined in Definition 3.4.6, and basic properties proven in Theorems 6.0.11 and 6.0.14). For example, in [1], G. Angelini-Knoll uses these spectral sequences to compute the topological Hochschild homology of the algebraic K-theory spectra of a large class of finite fields.

In lieu of explicit computations using our new spectral sequences, we point out that the mere existence of these spectral sequences implies an upper bound on the size of the topological Hochschild homology groups of a ring spectrum: namely, if R is a graded-commutative ring and $X_\bullet$ is a simplicial finite set and E_* is a generalized homology theory, then for any E_∞-ring spectrum A such that $\pi_*(A) \cong R$, $E_*(X_\bullet \otimes A)$ is a subquotient of $E_*(X_\bullet \otimes HR)$. Here we write HR for the generalized Eilenberg-Maclane spectrum with $\pi_*(HR) \cong R$ as graded rings.

Consequently, in Theorem 5.2.1 we arrive at the slogan: *among all the E_∞-ring spectra A such that $\pi_*(A) \cong R$, the topological Hochschild homology of A is bounded above by the topological Hochschild homology of $H\pi_*(R)$.* This lets us extract a lot of information about the topological Hochschild homology of E_∞-ring spectra A from information depending *only on the ring $\pi_*(A)$ of homotopy groups of A.* We demonstrate how to apply this idea in Theorem 5.2.4 and its corollaries, by working out the special case where $R = \hat{\mathbb{Z}}_p[x]$ for some prime p, with x in positive grading degree $2n$. We get, for example, that for any E_∞-ring spectrum A such that $\pi_*(A) \cong \hat{\mathbb{Z}}_p[x]$, the Poincaré series of the mod p topological Hochschild homology $(S/p)_*(THH(A))$ satisfies the inequality

$$\sum_{i \geqslant 0} \left(\dim_{\mathbb{F}_p} (S/p)_*(THH(A)) \right) t^i \leqslant \frac{(1 + (2p-1)t)(1 + (2n+1)t)}{(1 - 2nt)(1 - 2pt)},$$

and:

- If p does not divide n, then $THH_{2i}(A) \cong 0$ for all i congruent to $-p$ modulo n such that $i \leqslant pn - p - n$, and $THH_{2i}(A) \cong 0$ for all i congruent to $-n$ modulo p such that $i \leqslant pn - p - n$. In particular, $THH_{2(pn-p-n)}(A) \cong 0$.
- If p divides n, then $THH_i(A) \cong 0$, unless i is congruent to $-1, 0$, or 1 modulo $2p$.

There is some precedent for spectral sequence 1.0.1: when A is a filtered commutative ring (rather than a filtered commutative ring spectrum), M. Brun constructed a spectral sequence of the form 1.0.1 in the paper [8]. In Theorem 2.9 of the preprint [3], V. Angeltveit remarks that a version of spectral sequence 1.0.1 exists for commutative ring spectra by virtue of a lemma in [8] on associated graded FSPs of filtered FSPs; filling in the details to make this spectral sequence have the correct E^1-term, E^∞-term, convergence properties, and multiplicativity properties takes a lot of work, and even aside from the substantially greater level of generality of the results in the present paper (allowing $X_\bullet \otimes A$ and not just $S^1 \otimes A$, working with commutative monoids in symmetric monoidal model categories rather than any particular model for ring spectra, working with coefficient bimodules as in 1.0.6), we think it is valuable to add these very nontrivial details to the literature.

We are grateful to C. Ogle and Ohio State University for their hospitality in hosting us during a visit to talk about this project and A. Blumberg for a timely and useful observation (that the Reedy model structure on inverse sequences may not have certain desired properties). The first author would like to thank Ayelet Lindenstrauss, Teena Gerhardt, and Cary Malkiewich for helpful conversations on the material in this paper and for hosting him at their respective universities to discuss this work.

2. CONVENTIONS AND RUNNING ASSUMPTIONS

Conventions 2.0.1. By convention, the "cofiber of $f : X \longrightarrow Y$" will mean that f is a cofibration and we are forming the pushout $Y \coprod_X 0$ in the given pointed model category. By convention we will write Y/X for $Y \coprod_X 0$ when $f : X \longrightarrow Y$ is a cofibration.

We will write $\mathrm{Comm}(C)$ for the category of commutative monoid objects in a symmetric monoidal category C, we will write sC for the category of simplicial objects in C, and we will write $\wedge$ for the symmetric monoidal product in a symmetric monoidal category C since the main example we have in mind is the category of symmetric spectra where the symmetric monoidal product is the smash product.

Running Assumption 2.0.2. Throughout, let C be a complete, co-complete left proper stable model category equipped with the structure of a symmetric monoidal model category in the sense of [32], satisfying the following axioms:

- The unit object $\mathbb{1}$ of C is cofibrant.
- A model structure (necessarily unique) on $\mathrm{Comm}(C)$ exists in which weak equivalences and fibrations are created by the forgetful functor $\mathrm{Comm}(C) \to C$.
- The forgetful functor $\mathrm{Comm}(C) \to C$ commutes with geometric realization of simplicial objects.
- Geometric realization of simplicial cofibrant objects in C commutes with the monoidal product, i.e., if $X_\bullet, Y_\bullet$ are simplicial cofibrant objects of C, then the canonical comparison map
$$|X_\bullet \wedge Y_\bullet| \to |X_\bullet| \wedge |Y_\bullet|$$
  is a weak equivalence in C.

Here are a few immediate consequences of these assumptions about C:

(1) Since being cofibrantly generated is part of the definition of a monoidal model category in [32], C is cofibrantly generated and hence can be equipped with functorial factorization systems. We assume that a choice of functorial factorization has been made and we will use it implicitly whenever a cofibration-acyclic-fibration or acyclic-cofibration-fibration factorization is necessary.

(2) Smashing with any given object preserves colimits. Smashing with any given cofibrant object preserves cofibrations and weak equivalences.

(3) Axioms (TC1)-(TC5) of May's paper [23] are satisfied, so the constructions and conclusions of [23] hold for C. In particular, we have a natural filtration on any finite smash power of a filtered object in C, which we say more about below.

(4) Since C is assumed left proper, a homotopy cofiber of any map $f : X \to Y$ between cofibrant objects in C can be computed by factoring f as $f = f_2 \circ f_1$ with $f_1 : X \to \tilde{Y}$ a cofibration and $f_2 : \tilde{Y} \to Y$ an acyclic fibration, and then taking the pushout of the square

$$
\begin{array}{ccc}
X & \xrightarrow{\ f_1\ } & \tilde{Y} \\
\downarrow & & \\
0. & &
\end{array}
$$

(5) In particular, if f is already a cofibration, the pushout map $Y \to Y \coprod_X 0$ is a homotopy cofiber of f.

Running Assumption 2.0.3. In addition to Running Assumption 2.0.2, we assume our model category C satisfies the following condition: a map $X_\bullet \to Y_\bullet$ in the category of

simplicial objects in C is a Reedy cofibration between Reedy cofibrant objects whenever the following all hold:

(1) Each object X_n and Y_n of C is cofibrant.
(2) Each degeneracy map $s_i : X_n \to X_{n+1}$ and $s_i : Y_n \to Y_{n+1}$ is a cofibration in C
(3) Each map $X_n \to Y_n$ is a cofibration in C.

A consequence of this assumption is that the geometric realization of a map of simplicial objects in C satisfying Item 1, Item 2, and Item 3 is a cofibration.

The main motivating example of such a category C satisfying Running Assumption 2.0.2 is the category of symmetric spectra in a pointed simplicial model category $\mathcal{D}$, denoted $Sp_{\mathcal{D}}$, as in [31]. In the case when C is the category $Sp_{\mathcal{D}}$, then $\mathrm{Comm}(C)$ is the category of commutative ring spectra and it is known to be equivalent to the category of E_∞-ring spectra. The existence of the desired model structure on $\mathrm{Comm}(C)$ is proven in Theorem 4.1 of [32]. We ask that $\mathcal{D}$ admits the mixed Σ-equivariant model structure of [31, Thm. 3.8], so that $Sp_{\mathcal{D}}$ may be equipped with the stable positive flat model structure. The fact that the forgetful functor $\mathrm{Comm}(C) \to C$ commutes with geometric realization in the stable positive flat model structure on C is a consequence of [14, Thm. 1.6].

Under the additional hypothesis that $\mathcal{D}$ is a *graded concrete category* (see [2, Def. 3.1]), the category $Sp_{\mathcal{D}}$ satisfies Running Assumption 2.0.3, as the authors prove in [2]. In fact, in this setting Item 2 of Running Assumption 2.0.3 can be weakened to: each degeneracy map $s_i : X_n \to X_{n+1}$ is levelwise a cofibration in $\mathcal{D}$. The main example of a category $\mathcal{D}$ that satisfies all of these conditions is the category of pointed simplicial sets. *Consequently, the category of symmetric spectra in simplicial sets equipped with the stable positive flat model structure satisfies all of our running assumptions.*

3. Construction of the spectral sequence.

3.1. **Filtered commutative monoids and their associated graded commutative monoids.**

Definition 3.1.1. *By a* cofibrant decreasingly filtered object in C *we mean a sequence of cofibrations in C*

$$\cdots \xrightarrow{f_3} I_2 \xrightarrow{f_2} I_1 \xrightarrow{f_1} I_0,$$

such that each object I_i is cofibrant.

Definition 3.1.2. *By a* cofibrant decreasingly filtered commutative monoid in C *we mean:*

- *a cofibrant decreasingly filtered object*

$$\cdots \xrightarrow{f_3} I_2 \xrightarrow{f_2} I_1 \xrightarrow{f_1} I_0$$

  in C, and
- *for every pair of natural numbers $i, j \in \mathbb{N}$, a map in C*

$$\rho_{i,j} : I_i \wedge I_j \to I_{i+j},$$

  and
- *a map $\eta : \mathbb{1} \to I_0$,*

satisfying the axioms listed below. For the sake of listing the axioms concisely, it will be useful to have the following notation: if $i' \leqslant i$, we will write $f_i^{i'} : I_i \to I_{i'}$ for the composite

$$f_i^{i'} = f_{i'+1} \circ f_{i'+2} \circ \cdots \circ f_{i-1} \circ f_i.$$

Here are the axioms we require:

- **(Compatibility.)** *For all $i, j, i', j' \in \mathbb{N}$ with $i' \leqslant i$ and $j' \leqslant j$, the diagram*

$$
\begin{array}{ccc}
I_i \wedge I_j & \xrightarrow{\rho_{i,j}} & I_{i+j} \\
{\scriptstyle f_i^{i'} \wedge f_j^{j'}} \downarrow & & \downarrow {\scriptstyle f_{i+j}^{i'+j'}} \\
I_{i'} \wedge I_{j'} & \xrightarrow{\rho_{i',j'}} & I_{i'+j'}
\end{array}
$$

  commutes.
- **(Commutativity.)** *For all $i, j \in \mathbb{N}$, the diagram*

$$
\begin{array}{c}
I_i \wedge I_j \\
{\scriptstyle \chi_{I_i,I_j}} \downarrow \searrow {\scriptstyle \rho_{i,j}} \\
I_j \wedge I_i \xrightarrow{\rho_{j,i}} I_{i+j}
\end{array}
$$

  commutes, where $\chi_{I_i,I_j} : I_i \wedge I_j \xrightarrow{\cong} I_j \wedge I_i$ is the symmetry isomorphism in $\mathcal{C}$.
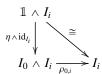- **(Associativity.)** *For all $i, j, k \in \mathbb{N}$, the diagram*

$$
\begin{array}{ccc}
I_i \wedge I_j \wedge I_k & \xrightarrow{\mathrm{id}_{I_i} \wedge \rho_{j,k}} & I_i \wedge I_{j+k} \\
{\scriptstyle \rho_{i,j} \wedge \mathrm{id}_{I_k}} \downarrow & & \downarrow {\scriptstyle \rho_{i,j+k}} \\
I_{i+j} \wedge I_k & \xrightarrow{\rho_{i+j,k}} & I_{i+j+k}
\end{array}
$$

  commutes.
- **(Unitality.)** *For all $i \in \mathbb{N}$, the diagram*

$$
\begin{array}{c}
\mathbb{1} \wedge I_i \\
{\scriptstyle \eta \wedge \mathrm{id}_{I_i}} \downarrow \searrow {\scriptstyle \cong} \\
I_0 \wedge I_i \xrightarrow{\rho_{0,i}} I_i
\end{array}
$$

  commutes, where the map marked $\cong$ is the (left-)unitality isomorphism in $\mathcal{C}$.

We will sometimes write $I_\bullet$ as shorthand for this entire structure.

Note that, if $I_\bullet$ is a cofibrant decreasingly filtered commutative monoid in $\mathcal{C}$, then I_0 really is a commutative monoid in $\mathcal{C}$, with multiplication map $\rho_{0,0} : I_0 \wedge I_0 \to I_0$ and unit map $\eta : \mathbb{1} \to I_0$. The objects I_i for $i > 0$ do not receive commutative monoid structures from the structure of $I_\bullet$, but instead play a role analogous to that of the nested sequence of powers of an ideal in a commutative ring.

Definition 3.1.3. *Suppose $I_\bullet$ is a cofibrant decreasingly filtered commutative monoid in $\mathcal{C}$.*

- *We shall say that $I_\bullet$ is* Hausdorff *if the homotopy limit of the I_n is weakly equivalent to the zero object: $\mathrm{holim}_n I_n \simeq 0$.*
- *We shall say that $I_\bullet$ is* finite *if there exists some $n \in \mathbb{N}$ such that $f_m : I_m \to I_{m-1}$ is a weak equivalence for all $m > n$.*

Remark 3.1.4. Definition 3.1.2 has the advantage of concreteness, but there is an equivalent, more concise definition of a cofibrant decreasingly filtered commutative monoid. Observe that the the data of a decreasingly filtered commutative monoid, without the cofibrancy conditions, is the same as the data of a lax symmetric monoidal functor

$$
I_\bullet : (\mathbb{N}^{\mathrm{op}}, +, 0) \longrightarrow (\mathcal{C}, \wedge, \mathbb{1}).
$$

Recall that due to Day [9], the category of lax symmetric monoidal functors in $\mathcal{C}^{\mathbb{N}^{op}}$ is equivalent to the category Comm $\mathcal{C}^{\mathbb{N}^{op}}$ of commutative monoid objects in the symmetric monoidal category $(\mathcal{C}^{\mathbb{N}^{op}}, \otimes_{\mathrm{Day}}, \mathbb{1}_{\mathrm{Day}})$ where $\otimes_{\mathrm{Day}}$ is the Day convolution symmetric monoidal product also constructed in [9] and $\mathbb{1}_{\mathrm{Day}}$ is a cofibrant replacement for the unit of this symmetric monoidal product. (See [13] for a modern treatment of this in the setting of quasi-categories.)

In particular, a decreasingly filtered commutative monoid is therefore equivalent to an object in Comm $\mathcal{C}^{\mathbb{N}^{op}}$. Now we claim that $\mathcal{C}^{\mathbb{N}^{op}}$ with the projective model structure is cofibrantly generated and it is a monoidal model category satisfying the monoid axiom in the sense of Schwede-Shipley [32]. The fact that a functor category with the projective model structure is cofibrantly generated follows from [20, A.2.8.3]. Therefore, whenever $\mathcal{C}$ is cofibrantly generated, as we assume in Running Assumption 2.0.2, then $\mathcal{C}^{\mathbb{N}^{op}}$ admits the projective model structure and it is cofibrantly generated. The fact that $\mathcal{C}^{\mathbb{N}^{op}}$ is a closed symmetric monoidal model category satisfying the pushout product axiom follows by Propositions 2.2.15 and 2.2.16 of the thesis of Isaacson [16]. To apply the theorem of Isaacson, we need to enrich $\mathbb{N}^{op}$ in $\mathcal{C}$ by letting

$$\mathbb{N}^{op}(n, m) \cong \left\{ \begin{array}{l} \mathbb{1} \text{ if } n \geqslant m \\ 0 \text{ otherwise,} \end{array} \right.$$

where $\mathbb{1}$ is a cofibrant model for the unit of the symmetric monoidal product on $\mathcal{C}$. This ensures that all morphisms are "virtually cofibrant" in the sense of Isaacson [16]. We claim that if $I_\bullet$ is a cofibrant object in the projective model structure on $\mathcal{C}^{\mathbb{N}^{op}}$, then it is a sequence

$$\cdots \xrightarrow{f_3} I_2 \xrightarrow{f_2} I_1 \xrightarrow{f_1} I_0$$

such that each map f_i is a cofibration and each object I_i is cofibrant, and we will prove this in Lemma 3.1.5, which follows. We do not prove the converse statement that all cofibrant objects in the projective model structure on $\mathcal{C}^{\mathbb{N}^{op}}$ are of this form and we make no claim to its validity.

Due to Schwede-Shipley [32, Thm 4.1], if we equip the category Comm $\mathcal{C}^{\mathbb{N}^{op}}$ with the model structure created by the forgetful functor $U : \mathrm{Comm}\ \mathcal{C}^{\mathbb{N}^{op}} \to \mathcal{C}^{\mathbb{N}^{op}}$, then cofibrant objects in Comm $\mathcal{C}^{\mathbb{N}^{op}}$ forget to cofibrant objects in $\mathcal{C}^{\mathbb{N}^{op}}$ since $\mathcal{C}^{\mathbb{N}^{op}}$ is cofibrantly generated, closed symmetric monoidal, and satisfies the pushout product axiom. Therefore cofibrant objects in Comm $\mathcal{C}^{\mathbb{N}^{op}}$ are cofibrant decreasingly filtered commutative monoids in $\mathcal{C}$ as defined in Definition 3.1.1.

Lemma 3.1.5. *Let $\mathcal{D}$ be a cofibrantly generated model category, and let $\mathcal{D}^{\mathbb{N}^{op}}_{proj}$ be the category of inverse sequences in $\mathcal{D}$, i.e., functors $\mathbb{N}^{op} \to \mathcal{D}$, equipped with the projective model structure. (Recall that this is the model structure in which a map $F : X \to Y$ is a weak equivalence, respectively fibration, if $F(n) : X(n) \to Y(n)$ is a weak equivalence, respectively fibration, for all $n \in \mathbb{N}$.) Let P be a cofibrant object in $\mathcal{D}^{\mathbb{N}^{op}}_{proj}$. Then, for all $n \in \mathbb{N}$, the object $P(n)$ of $\mathcal{D}$ is cofibrant, and the morphism $P(n+1) \to P(n)$ is a cofibration in $\mathcal{D}$.*

Proof. First, a quick definition:

- given a morphism $f : Y \to Z$ in $\mathcal{D}$ and a nonnegative integer n, let $b_n f : \mathbb{N}^{op} \to \mathcal{D}$ be the functor given by letting $b_n f(m) = Y$ if $m \geqslant n$, letting $b_n f(n-1) = Z$, and letting $b_n f(m) = 1$ if $m < n - 1$. Here we are writing 1 for the terminal object of $\mathcal{D}$. We let $b_n f(m+1) \to b_n f(m)$ be the identity map on Y if $m \geqslant n$, we let $b_n f(n) \to b_n f(n-1)$ be the map $f : Y \to Z$, and we let $b_n f(m+1) \to b_n f(m)$ be the projection to the terminal object if $m < n - 1$.

- Given an object Y of $\mathcal{D}$ and a nonnegative integer n, we write $c_n Y$ for $b_n \pi$ where π is the projection $Y \to 1$ to the terminal object. It is easy to see that $c_n : \mathcal{D} \to \mathcal{D}^{\mathbb{N}^{op}}$ is a functor by letting $c_n f(m) : c_n Y(m) \to c_n Z(m)$ be $f : Y \to Z$ if $m \geqslant n$ and $c_n f(m) = \mathrm{id}_1$ if $m < n$.

Fix some $n \in \mathbb{N}$, let $\phi : P(n) \to W$ be a map in $\mathcal{D}$, and let $g : V \to W$ be an acyclic fibration in $\mathcal{D}$. We have a commutative diagram in $\mathcal{D}^{\mathbb{N}^{op}}_{proj}$:

(3.1.1)
$$
\begin{array}{ccc}
 & & c_n V \\
 & & \downarrow {\scriptstyle c_n g} \\
P & \xrightarrow{\ h\ } & c_n W
\end{array}
$$

where $h(n) : P(n) \to c_n W(n)$ is ϕ if $m = n$, where $h(m)$ is the projection to the terminal object if $m < n$, and where, if $m > n$, then $h(m)$ is the composite

$$P(m) \to P(m-1) \to \cdots \to P(n+1) \to P(n) \xrightarrow{\ \phi\ } W = c_n W(n).$$

Since ϕ and id_1 are both acyclic fibrations, the map $c_n g$ is an acyclic fibration in the projective model structure on $\mathcal{D}^{\mathbb{N}^{op}}$. So there exists a map $\ell : P \to c_n V$ filling in diagram 3.1.1 and making it commute. Evaluating at n, we get that $\ell(n) : P(n) \to c_n V(n) = V$ is a map satisfying $g \circ \ell(n) = \phi$. So $0 \to P(n)$ lifts over every acyclic fibration in $\mathcal{D}$, so $P(n)$ is cofibrant in $\mathcal{D}$.

Now suppose the map $P(n \leqslant n+1) : P(n+1) \to P(n)$ fits into a commutative diagram

(3.1.2)
$$
\begin{array}{ccc}
P(n+1) & \xrightarrow{\ \psi\ } & V \\
{\scriptstyle P(n\leqslant n+1)} \downarrow & & \downarrow {\scriptstyle t} \\
P(n) & \xrightarrow{\ \phi\ } & W
\end{array}
$$

in $\mathcal{D}$, in which t is an acyclic fibration. We have a commutative diagram in $\mathcal{D}^{\mathbb{N}^{op}}_{proj}$:

(3.1.3)
$$
\begin{array}{ccc}
 & & c_n V \\
 & & \downarrow {\scriptstyle i} \\
P & \xrightarrow{\ j\ } & b_{n+1} t
\end{array}
$$

where $i(n) : V \to W$ is t, where $i(m) = \mathrm{id}_V$ if $m > n$, and where $i(m) = \mathrm{id}_1$ if $m < n$; and where $j(n) : P(n) \to W$ is ϕ, where $j(n+1) : P(n+1) \to V$ is ψ, where $j(m)$ is projection to the terminal object if $m < n$, and where, if $m > n+1$, then $j(m)$ is the composite

$$P(m) \to P(m-1) \to \cdots \to P(n+1) \to P(n) \xrightarrow{\ \phi\ } W = b_n t(n).$$

Since id_V and t and id_1 are all acyclic fibrations, we know that i is an acyclic fibration in the projective model structure on $\mathcal{D}^{\mathbb{N}^{op}}$. So there exists a map $\ell : P \to c_n V$ filling in diagram 3.1.3 and making it commute. Evaluating ℓ at n yields a map

$$\ell(n) : P(n) \to c_n V(n) = V$$

such that

$$
\begin{aligned}
t \circ \ell(n) = i(n) \circ \ell(n) \\
= j(n) \\
= \phi, \quad \text{and} \\
\ell(n+1) = \mathrm{id}_V \circ \ell(n+1) \\
= i(n+1) \circ \ell(n+1) \\
= j(n+1) \\
= \psi, \quad \text{and} \\
\ell(n) \circ P(n \leqslant n+1) = c_n V(n \leqslant n+1) \circ \ell(n+1) \\
= \mathrm{id}_V \circ \psi.
\end{aligned}
$$

(3.1.4)

(3.1.5)

Equations 3.1.4 and 3.1.5 express exactly that the map $\ell(n)$ fills in the diagonal of diagram 3.1.2, making it commute. So $P(n \leqslant n+1)$ lifts over every acyclic fibration. So $P(n \leqslant n+1)$ is a cofibration in $\mathcal{D}$. ∎

Definition 3.1.6. (The associated graded monoid.) *Let $I_\bullet$ be a cofibrant decreasingly filtered commutative monoid in $\mathcal{C}$. By $E_0^* I_\bullet$, the* associated graded commutative monoid *of $I_\bullet$, we mean the graded commutative monoid object in $\mathcal{C}$ defined as follows:*

- *"additively," that is, as an object of $\mathcal{C}$,*

$$
E_0^* I_\bullet \cong \coprod_{n \in \mathbb{N}} I_n / I_{n+1}.
$$

- *The unit map $\mathbb{1} \to E_0^* I_\bullet$ is the composite*

$$
\mathbb{1} \xrightarrow{\eta} I_0 \to I_0 / I_1 \hookrightarrow E_0^* I_\bullet.
$$

(Note that $E_0^* I_\bullet$ is constructed as an I_0/I_1-algebra).

- *The multiplication on $E_0^* I_\bullet$ is given as follows. Since the smash product commutes with colimits, hence with coproducts, to specify a map*

$$
E_0^* I_\bullet \wedge E_0^* I_\bullet \to E_0^* I_\bullet
$$

it suffices to specify a component map

$$
\nabla_{i,j} : I_i / I_{i+1} \wedge I_j / I_{j+1} \to E_0^* I_\bullet
$$

for every $i, j \in \mathbb{N}$. We define such a map $\nabla_{i,j}$ as follows: first, we have the commutative square

$$
\begin{array}{ccc}
I_{i+1} \wedge I_j & \xrightarrow{\rho_{i+1,j}} & I_{i+j+1} \\
{\scriptstyle f_{i+1} \wedge \mathrm{id}_{I_j}} \downarrow & & \downarrow {\scriptstyle f_{i+j+1}} \\
I_i \wedge I_j & \xrightarrow{\rho_{i,j}} & I_{i+j}
\end{array}
$$

so, since the vertical maps are cofibrations by Definition 3.1.2, we can take vertical cofibers to get a map

$$
\tilde{\nabla}_{i,j} : I_i / I_{i+1} \wedge I_j \to I_{i+j} / I_{i+j+1},
$$

which is well-defined by Running Assumption 2.0.2.

Now we have the commutative diagram

$$
\begin{array}{c}
I_{i+1} \wedge I_{j+1} \xrightarrow{\rho_{i+1,j+1}} I_{i+j+2} \\
\end{array}
$$

(commutative cube diagram with vertices $I_{i+1} \wedge I_{j+1}$, I_{i+j+2}, $I_{i+1} \wedge I_j$, I_{i+j+1}, $I_i \wedge I_{j+1}$, I_{i+j+1}, $I_i \wedge I_j$, I_{i+j}, $I_i/I_{i+1} \wedge I_{j+1}$, I_{i+j+1}/I_{i+j+2}, $I_i/I_{i+1} \wedge I_j$, I_{i+j}/I_{i+j+1} with maps $\mathrm{id}_{I_{i+1}} \wedge f_{j+1}$, $f_{i+1} \wedge \mathrm{id}_{I_{j+1}}$, f_{i+j+2}, $\rho_{i+1,j}$, f_{i+j+2}, $\rho_{i,j+1}$, $\mathrm{id}_{I_i} \wedge f_{j+1}$, $f_{i+1} \wedge \mathrm{id}_{I_j}$, f_{i+j+1}, $\rho_{i,j}$, f_{i+j+1}, $\tilde{\nabla}_{i,j+1}$, $\mathrm{id}_{I_i/I_{i+1}} \wedge f_{j+1}$, $\tilde{\nabla}_{i,j}$, 0)

in which the columns are cofiber sequences. So we have a factorization of the composite map $\tilde{\nabla}_{i,j} \circ \left(\mathrm{id}_{I_i/I_{i+1}} \wedge f_{j+1} \right)$ through the zero object by Running Assumption 2.0.2 Item 4. Thus, we have the commutative square

$$
\begin{array}{ccc}
I_i/I_{i+1} \wedge I_{j+1} & \longrightarrow & 0 \\
\downarrow{\scriptstyle \mathrm{id}_{I_i/I_{i+1}} \wedge f_{j+1}} & & \downarrow \\
I_i/I_{i+1} \wedge I_j & \xrightarrow{\tilde{\nabla}_{i,j}} & I_{i+j}/I_{i+j+1}
\end{array}
$$

and, taking vertical cofibers, a map

$$
I_i/I_{i+1} \wedge I_j/I_{j+1} \to I_{i+j}/I_{i+j+1},
$$

which we compose with the inclusion map $I_{i+j}/I_{i+j+1} \hookrightarrow E_0^* I_\bullet$ to produce our desired map $\nabla_{i,j} : I_i/I_{i+1} \wedge I_j/I_{j+1} \to E_0^* I_\bullet$. (Note that all these maps are defined in the model category $\mathcal{C}$, not just in $Ho(\mathcal{C})$.)

3.2. Filtered coefficient bimodules.

Now we lay out all the same definitions as in the previous subsection, but with extra data: we assume that, along with our filtered commutative monoid object $I_\bullet$, we also have a choice of filtered bimodule object $M_\bullet$. These definitions are necessary in order to get a THH-May spectral sequence with coefficients, something that has already proven important in practical computations (e.g. of $THH(K(\mathbb{F}_q))$, in a paper by G. Angelini-Knoll [1]), but these definitions involve some repetition of those in the previous subsection, so we try to present them concisely.

Definition 3.2.1. *Suppose we have a cofibrant decreasingly filtered commutative monoid $I_\bullet$ of a commutative monoid I_0 in $\mathcal{C}$. We therefore have structure maps $\rho_{i,j} : I_i \wedge I_j \longrightarrow I_{i+j}$ and maps $f_i : I_{i+1} \longrightarrow I_i$ for each integer i and j. Let the sequence*

$$
\ldots \xrightarrow{g_{n+1}} M_n \xrightarrow{g_n} \ldots \xrightarrow{g_2} M_1 \xrightarrow{g_1} M_0
$$

be a cofibrant decreasingly filtered object in $\mathcal{C}$ in the sense of 3.1.1. We call the sequence a cofibrant decreasingly filtered $I_\bullet$-bimodule *if we have maps*

$$\psi_{i,j}^r : M_i \wedge I_j \longrightarrow M_{i+j}$$

$$\psi_{i,j}^\ell : I_i \wedge M_j \longrightarrow M_{i+j}$$

satisfying the following axioms. To write the axioms it will be helpful to use the notation,

$$g_n^m = g_{m+1} \circ g_{m+2} \circ \ldots \circ g_n \quad \text{for } n \geqslant m.$$

(1) *(Associativity.) The relations,*

$$\psi_{i+j,k}^r \circ (\psi_{i,j}^r \wedge \mathrm{id}_{I_k}) = \psi_{i,j+k}^r \circ (\mathrm{id}_{M_i} \wedge \rho_{j,k})$$

$$\psi_{i+j,k}^r \circ (\psi_{i,j}^\ell \wedge \mathrm{id}_{I_k}) = \psi_{i,j+k}^\ell (\mathrm{id}_{I_i} \wedge \psi_{j,k}^r)$$

$$\psi_{i,j+k}^\ell \circ (\mathrm{id}_{I_i} \wedge \psi_{j,k}^\ell) = \psi_{i+j,k}^\ell (\rho_{i,j} \wedge \mathrm{id}_{M_k})$$

hold for all integers i, j, and k.

(2) *(Compatibility.) For i, j, i', j' integers such that $i > i'$, $j > j'$, the following relations hold:*

$$g_{i+j}^{i'+j'} \circ \psi_{i,j}^r = \psi_{i',j'}^r \circ (g_i^{i'} \wedge f_j^{j'})$$

$$g_{i+j}^{i'+j'} \circ \psi_{i,j}^\ell = \psi_{i',j'}^\ell \circ (f_i^{i'} \wedge g_j^{j'}).$$

(3) *(Unitality.) The diagrams,*

commute.

We say that the cofibrant decreasingly filtered bimodule $M_\bullet$ is symmetric *if the factor-swap isomorphism*

$$\chi_{i,j} : M_i \wedge I_j \longrightarrow I_j \wedge M_i \quad \text{satisfies} \quad \psi_{i,j}^r = \psi_{j,i}^\ell \circ \chi_{i,j}.$$

Remark 3.2.2. As in Definition 3.1.3, we will say that a cofibrant decreasingly filtered $I_\bullet$-bimodule $M_\bullet$ is *Hausdorff* if $\mathrm{holim}_n M_n$ is weakly equivalent to the zero object, and we will say that $M_\bullet$ is *finite* if there exists $n \in \mathbb{N}$ such that $f_m : M_m \to M_{m-1}$ is a weak equivalence for all $m > n$.

Remark 3.2.3. Just as a cofibrant decreasingly filtered commutative monoid in $\mathcal{C}$ can be considered as a cofibrant object in $\mathrm{Comm}\, \mathcal{C}^{\mathbb{N}^{\mathrm{op}}}$ (See Remark 3.1.4), we can define a cofibrant decreasingly filtered $I_\bullet$-bimodule as an cofibrant symmetric $I_\bullet$-bimodule in the category of functors $\mathcal{C}^{\mathbb{N}^{\mathrm{op}}}$.

Definition 3.2.4. (The associated graded bimodule.) *Let $I_\bullet$ be a cofibrant decreasingly filtered commutative monoid in $\mathcal{C}$, and let $M_\bullet$ be a cofibrant decreasingly filtered $I_\bullet$-bimodule. By $E_0^* M_\bullet$, the* associated graded bimodule of $M_\bullet$*, we mean the graded $E_0^* I_\bullet$-bimodule object in $\mathcal{C}$ defined as follows:*

- *"additively," that is, as an object of $\mathcal{C}$,*

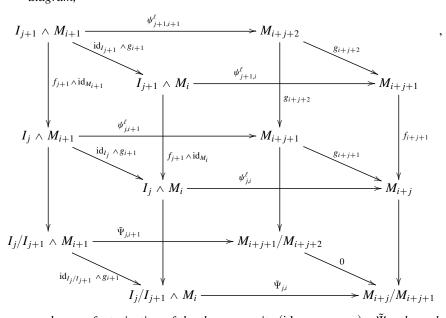$$E_0^* M_\bullet \cong \coprod_{n \in \mathbb{N}} M_n/M_{n+1}.$$

- *The left action map $E_0^* I_\bullet \wedge E_0^* M_\bullet \to E_0^* M_\bullet$ is defined as follows. Using the fact that f_j and g_{i+j+1} are cofibrations in the diagram,*

$$
\begin{array}{ccc}
I_{j+1} \wedge M_i & \xrightarrow{\psi_{j+1,i}^{\ell}} & M_{i+j+1} \\
{\scriptstyle f_j \wedge 1_{M_i}} \downarrow & & \downarrow {\scriptstyle g_{i+j+1}} \\
I_j \wedge M_i & \xrightarrow{\psi_{j,i}^{\ell}} & M_{i+j}
\end{array}
$$

we get a map $\tilde{\Psi}_{i,j} : I_j/I_{j+1} \wedge M_i \longrightarrow M_{i+j}/M_{i+j+1}$. We then observe that the diagram,

produces a factorization of the the composite $(\mathrm{id}_{I_j/I_{j+1}} \wedge g_{i+1}) \circ \tilde{\Psi}_{i,j}$ through the zero object. There is a commutative diagram

$$
\begin{array}{ccc}
I_j/I_{j+1} \wedge M_{i+1} & \longrightarrow & 0 \\
{\scriptstyle \mathrm{id}_{I_j/I_{j+1}} \wedge g_i} \downarrow & & \downarrow \\
I_j/I_{j+1} \wedge M_i & \xrightarrow{\tilde{\Psi}_{i,j}} & M_{i+j}/M_{i+j+1}
\end{array}
$$

which, since the maps f_i and g_i are cofibrations by Definition 3.2.1, produces a map,

$$\bar{\Psi}_{i,j} : I_j/I_{j+1} \wedge M_i/M_{i+1} \longrightarrow M_{i+j}/M_{i+j+1}.$$

We therefore have a module map,

$$\Psi : \coprod_{j \in \mathbb{N}} I_j/I_{j+1} \wedge \coprod_{i \in \mathbb{N}} M_i/M_{i+1} \longrightarrow \coprod_{k \in \mathbb{N}} M_k/M_{k+1},$$

given on each wedge factor by the composite,

$$I_j/I_{j+1} \wedge M_i/M_{i+1} \xrightarrow{\bar{\Psi}_{i,j}} M_{i+j}/M_{i+j+1} \xrightarrow{\iota_{i+j}} \coprod_{k \in \mathbb{N}} M_k/M_{k+1},$$

where ι_{i+j} is the inclusion.

- *The right action map $E_0^* M_\bullet \wedge E_0^* I_\bullet \to E_0^* M_\bullet$ is defined in the same way as above and the symmetry of $M_\bullet$ along with these maps gives $E_0^* M_\bullet$ the structure of a symmetric bimodule over $E_0^* I_\bullet$.*

3.3. **Tensoring and pretensoring over simplicial sets.** We will write $fSet$ for the category of finite sets. First we introduce the *pretensor product*, which is merely a convenient notation for the well-known "Loday construction" of [18]:

Definition 3.3.1. *We define a functor*
$$-\tilde{\otimes}- : sf\,\mathrm{Sets} \times \mathrm{Comm}\,C \to s\,\mathrm{Comm}\,C,$$
which we call the pretensor product, *as follows. If $X_\bullet$ is a simplicial finite set and A a commutative monoid in C, the simplicial commutative monoid $X_\bullet \tilde{\otimes} A$ is given by:*

- *For all $n \in \mathbb{N}$, the n-simplex object*
$$(X_\bullet \tilde{\otimes} A)_n = \coprod_{x \in X_n} A$$
  is a coproduct, taken in $\mathrm{Comm}(C)$, of copies of A, with one copy for each n-simplex $x \in X_n$. Recall that the categorical coproduct in $\mathrm{Comm}(C)$ is the smash product $\wedge$.
- *For all positive $n \in \mathbb{N}$ and all $0 \leqslant i \leqslant n$, the ith face map*
$$d_i : (X_\bullet \tilde{\otimes} A)_n \to (X_\bullet \tilde{\otimes} A)_{n-1}$$
  is given on the component corresponding to an n-simplex $x \in X_n$ by the map
$$A \to \coprod_{y \in X_{n-1}} A$$
  which is inclusion of the coproduct factor corresponding to the $(n-1)$-simplex $\delta_i(x)$.
- *For all positive $n \in \mathbb{N}$ and all $0 \leqslant i \leqslant n$, the ith degeneracy map*
$$s_i : (X_\bullet \tilde{\otimes} A)_n \to (X_\bullet \tilde{\otimes} A)_{n+1}$$
  is given on the component corresponding to an n-simplex $x \in X_n$ by the map
$$A \to \coprod_{y \in X_{n+1}} A$$
  which is inclusion of the coproduct factor corresponding to the $(n+1)$-simplex $\sigma_i(x)$.

We have defined the pretensor product on objects; it is then defined on morphisms in the evident way.

We define the tensor product
$$-\otimes- : sf\,\mathrm{Sets} \times \mathrm{Comm}\,C \to \mathrm{Comm}\,C$$
as the geometric realization of the pretensor product:
$$X_\bullet \otimes A = |X_\bullet \tilde{\otimes} A|.$$

It is easy to check that $X_\bullet \tilde{\otimes} A$ is indeed a simplicial object in $\mathrm{Comm}(C)$.

In the case that C is the category of symmetric spectra, the tensor product $X_\bullet \otimes A$ agrees with the tensoring of commutative ring spectra over simplicial sets. (This is proven in [25], although using (an early incarnation of) S-modules, rather than symmetric spectra; the symmetric monoidal Quillen equivalence of S-modules and symmetric spectra, as

in [30], then gives us the same result in symmetric spectra.) The same is true when E is a commutative S-algebra and C is the category of E-modules. In fact, the tensor product defined in 3.3.1 agrees with the tensoring over simplicial sets in every case of a symmetric monoidal model category whose category of commutative monoids is tensored over simplicial sets that is known to the authors.

In particular, when $X_\bullet$ is the minimal simplicial model for the circle, i.e., $X_\bullet = (\Delta[1]/d\Delta[1])_\bullet$, then $X_\bullet \otimes A$ is the cyclic bar construction on A, and hence (by the main result of [25]) $X_\bullet \otimes A$ agrees with the topological Hochschild homology ring spectrum $THH(A, A)$.

For other simplicial sets, $X_\bullet \otimes A$ is regarded as a generalization of topological Hochschild homology, e.g. as "higher order Hochschild homology" in [28]. For the definition of tensoring a simplicial finite set with a commutative monoid with coefficients in a bimodule, see the appendix.

3.4. **The fundamental theorem of the May filtration.** The fundamental theorem of the May filtration relies on the following lemma. This lemma also occurs as Lemma 4.7 in May's [23]. (May's treatment of this particular lemma addresses the question of the compatibility of cofiber sequence 3.4.1 with other cofiber sequences that arise naturally in this context, but this is omitted from our treatment.)

Lemma 3.4.1. *Suppose I, J are objects of C and $I' \to I$ and $J' \to J$ are cofibrations. Suppose I, J, I', J' are cofibrant. Let P denote the pushout (which, by Running Assumption 2.0.2, is a homotopy pushout) of the diagram*

$$
\begin{array}{ccc}
I' \wedge J' & \longrightarrow & I' \wedge J \\
\downarrow & & \\
I \wedge J' & &
\end{array}
$$

Let $f : P \to I \wedge J$ denote the canonical map given by the universal property of the pushout. Then f is a cofibration by the pushout product axiom in the definition of a monoidal model category, as in [32]. Then the cofiber of f is isomorphic to $(I/I') \wedge (J/J')$, where I/I' and J/J' denote the cofibers of $I' \to I$ and $J' \to J$, respectively. So

(3.4.1) $$ P \xrightarrow{f} I \wedge J \to (I/I') \wedge (J/J') $$

is a cofiber sequence.

Proof. We define three diagrams in C:

$$
X_1 \quad = \quad \left(\begin{array}{ccc}
I' \wedge J' & \longrightarrow & I \wedge J' \\
\downarrow & & \\
I' \wedge J & &
\end{array} \right)
$$

$$
X_2 \quad = \quad \left(\begin{array}{ccc}
I' \wedge J & \longrightarrow & I \wedge J \\
\downarrow & & \\
I' \wedge J & &
\end{array} \right)
$$

$$
X_3 \quad = \quad \left(\begin{array}{ccc}
I' \wedge (J/J') & \longrightarrow & I \wedge (J/J') \\
\downarrow & & \\
0 & &
\end{array} \right).
$$

The obvious maps of diagrams

$$X_1 \to X_2 \to X_3$$

are trivially seen to be levelwise cofiber sequences. Since colimits commute with colimits, we then have a commutative diagram where each row is a cofiber sequence:

$$
\begin{array}{ccccc}
\operatorname{colim} X_1 & \longrightarrow & \operatorname{colim} X_2 & \longrightarrow & \operatorname{colim} X_3 \\
\downarrow \cong & & \downarrow \cong & & \downarrow \cong \\
P & \longrightarrow & I \wedge J & \longrightarrow & (I/I') \wedge (J/J').
\end{array}
$$

$\square$

Definition 3.4.2. (Some important colimit diagrams I.)

- *If S is a finite set, we will equip the set $\mathbb{N}^S$ of functions from S to $\mathbb{N}$ with the strict direct product order, that is, $x \leqslant y$ in $\mathbb{N}^S$ if and only if $x(s) \leqslant y(s)$ for all $s \in S$.*

- *If $T \xrightarrow{f} S$ is a function between finite sets, let $\mathbb{N}^T \xrightarrow{\mathbb{N}^f} \mathbb{N}^S$ be the function of partially-ordered sets defined by*

$$
\left(\mathbb{N}^f(x)\right)(s) = \sum_{\{t \in T : f(t) = s\}} x(t).
$$

  One checks easily that this defines a functor

$$
\mathbb{N}^- : f\,\mathrm{Sets} \to \mathrm{POSets}
$$

  from the category of finite sets to the category of partially-ordered sets.

- *For each finite set S, we also equip $\mathbb{N}^S$ with the L^1-norm:*

$$
\begin{aligned}
|-| : \mathbb{N}^S &\to \mathbb{N} \\
|x| &= \sum_{s \in S} x(s).
\end{aligned}
$$

  One checks easily that, if $T \xrightarrow{f} S$ is a function between finite sets, the induced map $\mathbb{N}^f$ preserves the L^1 norm, that is,

$$
|x| = \left|\mathbb{N}^f(x)\right|
$$

  for all $x \in \mathbb{N}^T$.

Definition 3.4.3. (Some important colimit diagrams II.)

-
  - *If S is a finite set, for each $n \in \mathbb{N}$ we will let $\mathcal{D}_n^S$ be the sub-poset of $\mathbb{N}^S$ consisting of all functions $x \in \mathbb{N}^S$ such that $|x| \geqslant n$.*

  - *If $T \xrightarrow{f} S$ is a function between finite sets, let $\mathcal{D}_n^T \xrightarrow{\mathcal{D}_n^f} \mathcal{D}_n^S$ be the function of partially-ordered sets defined by restricting $\mathbb{N}^f$ to $\mathcal{D}_n^T$. One checks easily that, this assignment $f \longmapsto D_{n;x}^f$ respects composition of functions in the variable f.*

-
  - *For each $x \in \mathbb{N}^S$ and each $n \in \mathbb{N}$, let $\mathcal{D}_{n;x}^S$ denote the following sub-poset of $\mathbb{N}^S$:*

$$
\mathcal{D}_{n;x}^S = \left\{y \in \mathbb{N}^S : y \geqslant x, \text{ and } |y| \geqslant n + |x|\right\}.
$$

  So, for example, $\mathcal{D}_{n;\vec{0}}^S = \mathcal{D}_n^S$, where $\vec{0}$ is the constant zero function.

    – *If $T \xrightarrow{f} S$ is a function between finite sets and $x \in \mathbb{N}^T$ and $n \in \mathbb{N}$, let*
    $\mathcal{D}_{n;x}^T \xrightarrow{\mathcal{D}_{n;x}^f} \mathcal{D}_{n;\mathcal{D}_n^f(x)}^S$ *be the function of partially-ordered sets defined by restrict-*
    ing $\mathbb{N}^f$ to $\mathcal{D}_{n;x}^T$. One checks easily that, for each $n \in \mathbb{N}$ and each $x \in \mathbb{N}^T$, this
    defines a functor

$$\mathcal{D}_{n;x}^- : f \operatorname{Sets} \to \operatorname{POSets}$$

    from the category of finite sets to the category of partially-ordered sets.

Definition 3.4.4. (Some important colimit diagrams III.)

- *Let S be a finite set and let n be a nonnegative integer. We write $\mathcal{E}_n^S$ for the set*

$$\mathcal{E}_{n,k}^S = \left\{ x \in \{0, 1, \dots, n\}^S : \sum_{s \in S} x(s) \geqslant k \right\}$$

    where $k \geqslant n$. We partially-order $\mathcal{E}_{n,k}^S$ by the strict direct product order, i.e., $x' \leqslant x$
    if and only if $x'(s) \leqslant x(s)$ for all $s \in S$. When $n = k$, we simply write $\mathcal{E}_n^S$ for this
    poset.

- *The definition of $\mathcal{E}_n^S$ is natural in S in the following sense: if $T \xrightarrow{f} S$ is a injective*
    map of finite sets, then $\mathbb{N}^f$ naturally restricts to a function $\mathcal{E}_n^T \to \mathcal{E}_n^S$.

Definition 3.4.5. (Some important colimit diagrams IV.)

- *Suppose $I_\bullet$ is a cofibrant decreasingly filtered object in $\mathcal{C}$ and suppose S is a finite*
    set. We will let
$$\mathcal{F}^S(I_\bullet) : \left(\mathbb{N}^S\right)^{\operatorname{op}} \to \mathcal{C}$$
    be the functor sending x to the smash product
$$\wedge_{s \in S} I_{x(s)},$$
    and defined on morphisms in the apparent way, and let
$$\mathcal{F}_n^S(I_\bullet) : \left(\mathcal{D}_n^S\right)^{\operatorname{op}} \to \mathcal{C}$$
    be the functor which is the composite of $\mathcal{F}^S(I_\bullet)$ with the inclusion of $\mathcal{D}_n^S$ into $\mathbb{N}^S$:
$$\left(\mathcal{D}_n^S\right)^{\operatorname{op}} \hookrightarrow \left(\mathbb{N}^S\right)^{\operatorname{op}} \xrightarrow{\mathcal{F}^S(I_\bullet)} \mathcal{C}.$$

- *If $x \in \mathcal{D}_n^S$, we will write $\mathcal{F}_{n;x}^S(I_\bullet)$ for the restriction of the diagram $\mathcal{F}^S(I_\bullet)$ to $\mathcal{D}_{n;x}^S$,*
    i.e., $\mathcal{F}_{n;x}^S(I_\bullet)$ is the composite
$$\left(\mathcal{D}_{n;x}^S\right)^{\operatorname{op}} \hookrightarrow \left(\mathbb{N}^S\right)^{\operatorname{op}} \xrightarrow{\mathcal{F}^S(I_\bullet)} \mathcal{C}.$$

- *Finally, let $\mathcal{M}_n^S(I_\bullet)$ denote the colimit*
$$\mathcal{M}_n^S(I_\bullet) = \operatorname{colim}\left(\mathcal{F}_n^S(I_\bullet)\right)$$
    in $\mathcal{C}$. By the natural inclusion of $\mathcal{D}_n^S$ into $\mathcal{D}_{n-1}^S$, we now have a sequence in $\mathcal{C}$:

(3.4.2) $\cdots \to \mathcal{M}_3^S(I_\bullet) \to \mathcal{M}_2^S(I_\bullet) \to \mathcal{M}_1^S(I_\bullet) \to \mathcal{M}_0^S(I_\bullet) \cong \wedge_{s \in S} I_0.$

    We refer to the functor $\mathbb{N}^{\operatorname{op}} \to \mathcal{C}$ given by sending n to $\mathcal{M}_n^S(I_\bullet)$ as the May filtration
    on $\wedge_{s \in S} I_0$.

- *The May filtration is functorial in S in the following sense: if $T \xrightarrow{f} S$ is a map of finite sets, we have a functor*

$$\mathcal{D}_n^f : \mathcal{D}_n^T \quad \rightarrow \quad \mathcal{D}_n^S$$
$$\left(\mathcal{D}_n^f(x)\right)(s) \quad \mapsto \quad \sum_{\{t \in T : f(t) = s\}} x(t)$$

and a map of diagrams from $\mathcal{F}_n^T(I_\bullet)$ to $\mathcal{F}_n^S(I_\bullet)$ given by sending the object

$$\mathcal{F}_n^T(I_\bullet)(x) = \wedge_{t \in T} I_{x(t)}$$

to the object

$$\mathcal{F}_n^S(I_\bullet)(\mathcal{D}_n^f(x)) = \wedge_{s \in S} I_{\Sigma_{\{t \in T : f(t) = s\}} x(t)}$$

by the map

$$\wedge_{t \in T} I_{x(t)} \rightarrow \wedge_{s \in S} I_{\Sigma_{\{t \in T : f(t) = s\}} x(t)}$$

given as the smash product, across all $s \in S$, of the maps

$$\wedge_{\{t \in T : f(t) = s\}} I_{x(t)} \rightarrow I_{\Sigma_{\{t \in T : f(t) = s\}} x(t)}$$

given by multiplication via the maps ρ of Definition 3.1.2.

To really make Definition 3.4.5 precise, we should say in which order we multiply the factors $I_{x(t)}$ using the maps ρ; but the purpose of the associativity and commutativity axioms in Definition 3.1.2 is that any two such choices commute, so any choice of order of multiplication will do.

Definition 3.4.6. (Definition of the May filtration.) *If $I_\bullet$ is a cofibrant decreasingly filtered commutative monoid in C and $X_\bullet$ a simplicial finite set, by the* May filtration on $X_\bullet \tilde{\otimes} I_0$ *we mean the functor $\mathcal{M}^{X_\bullet}(I_\bullet) : \mathbb{N}^{\mathrm{op}} \rightarrow C^{\Delta^{\mathrm{op}}}$ given by sending a natural number n to the simplicial object of C*

$$\mathcal{M}_n^{X_0}(I_\bullet) \overset{\longrightarrow}{\underset{\longleftarrow}{\rightleftarrows}} \mathcal{M}_n^{X_1}(I_\bullet) \overset{\longrightarrow}{\underset{\longleftarrow}{\overset{\longleftarrow}{\rightleftarrows}}} \mathcal{M}_n^{X_2}(I_\bullet) \overset{\longrightarrow}{\underset{\longleftarrow}{\overset{\longleftarrow}{\rightleftarrows}}} \dots$$

with $\mathcal{M}_n^{X_i}(I_\bullet)$ defined as in Definition 3.4.5, and with face and degeneracy maps defined as follows:

- *The face map*

$$d_i : \mathcal{M}_n^{X_j}(I_\bullet) \rightarrow \mathcal{M}_n^{X_{j-1}}(I_\bullet)$$

is the colimit of the map of diagrams

$$\mathcal{F}_n^{X_j}(I_\bullet) \rightarrow \mathcal{F}_n^{X_{j-1}}(I_\bullet)$$

induced, as in Definition 3.4.5, by $\delta_i : X_j \rightarrow X_{j-1}$.
- *The degeneracy map*

$$s_i : \mathcal{M}_n^{X_j}(I_\bullet) \rightarrow \mathcal{M}_n^{X_{j+1}}(I_\bullet)$$

is the colimit of the map of diagrams

$$\mathcal{F}_n^{X_j}(I_\bullet) \rightarrow \mathcal{F}_n^{X_{j+1}}(I_\bullet)$$

induced, as in Definition 3.4.5, by $\sigma_i : X_j \rightarrow X_{j+1}$.

Remark 3.4.7. Note that the associative, commutative, and unital multiplications on the objects I_i, via the maps ρ of Definition 3.1.2, also yield (by taking smash products of the maps ρ) associative, commutative, and unital multiplication natural transformations

$$(3.4.3) \qquad \mathcal{F}_m^S(I_\bullet) \wedge \mathcal{F}_n^S(I_\bullet) \to \mathcal{F}_{m+n}^S(I_\bullet),$$

hence, on taking colimits, associative, commutative, and unital multiplication maps

$$\mathcal{M}_m^S(I_\bullet) \wedge \mathcal{M}_n^S(I_\bullet) \to \mathcal{M}_{m+n}^S(I_\bullet),$$

i.e., the functor

$$\begin{aligned} \mathbb{N}^{op} &\to & \mathcal{C} \\ n &\mapsto & \mathcal{M}_n^S(I_\bullet) \end{aligned}$$

is a cofibrant decreasingly filtered commutative monoid, in the sense of Definition 3.1.2. Note furthermore that, if $f : T \to S$ is a map of finite sets, then the induced maps $\mathcal{F}_m^T(I_\bullet) \to \mathcal{F}_m^S(I_\bullet)$ commute with the multiplication maps 3.4.3, and so $\mathcal{M}_\bullet^T(I_\bullet) \to \mathcal{M}_\bullet^S(I_\bullet)$ is a map of cofibrant decreasingly filtered commutative monoids.

Consequently, for any simplicial finite set $X_\bullet$, we have that $\mathcal{M}_\bullet^{X_\bullet}(I_\bullet)$ is a simplicial object in the category of cofibrant decreasingly filtered commutative monoids in $\mathcal{C}$. Since geometric realization commutes with the monoidal product in $\mathcal{C}$ by our running assumptions on $\mathcal{C}$, this in turn implies that the geometric realization $\left|\mathcal{M}_\bullet^{X_\bullet}(I_\bullet)\right|$ of $\mathcal{M}_\bullet^{X_\bullet}(I_\bullet)$ is a cofibrant decreasingly filtered commutative monoid in $\mathcal{C}$ by Running Assumption 2.0.3. It can easily be shown that $\mathcal{M}_n^{X_\bullet}(I_\bullet)$ satisfies Running Assumption 2.0.3's Item 2 for each n whenever I_0 is cofibrant as an object in Comm $\mathcal{C}$. Running Assumption 2.0.3's Item 1 and Item 3 are satisfied by definition of $\mathcal{M}_n^{X_\bullet}(I_\bullet)$ and by definition of the maps

$$\mathcal{M}_n^{X_\bullet}(I_\bullet) \to \mathcal{M}_{n-1}^{X_\bullet}(I_\bullet).$$

Therefore, the decreasingly filtered commutative monoid $\left|\mathcal{M}_\bullet^{X_\bullet}(I_\bullet)\right|$ is a cofibrant decreasingly filtered commutative monoid in $\mathcal{C}$. Recall that, by the main theorem of the authors' paper [2], there is a symmetric monoidal model structure on symmetric spectra in simplicial sets in which Running Assumption 2.0.3 holds.

Lemma 3.4.8. *Let $I_\bullet$ be a cofibrant decreasingly filtered object in $\mathcal{C}$ and let S be a finite set. Suppose $n \in \mathbb{N}$. We have the canonical inclusion of categories $\iota : \mathcal{D}_{n+1}^S \to \mathcal{D}_n^S$. Let $\tilde{\mathcal{F}}_{n+1}^S(I_\bullet)$ be the left Kan extension of $\mathcal{F}_{n+1}^S(I_\bullet) : (\mathcal{D}_{n+1}^S)^{op} \to \mathcal{C}$ along ι^{op}, i.e., if we write*

$$Kan : \mathcal{C}^{(\mathcal{D}_{n+1}^S)^{op}} \to \mathcal{C}^{(\mathcal{D}_n^S)^{op}}$$

for the left adjoint of the map $\mathcal{C}^{(\mathcal{D}_n^S)^{op}} \to \mathcal{C}^{(\mathcal{D}_{n+1}^S)^{op}}$ induced by ι, then

$$\tilde{\mathcal{F}}_{n+1}^S(I_\bullet) = Kan(\mathcal{F}_{n+1}^S(I_\bullet)).$$

By the universal property of this Kan extension, we have a canonical map $c : \tilde{\mathcal{F}}_{n+1}^S(I_\bullet) \to \mathcal{F}_n^S(I_\bullet)$.

Then the cofiber of the map

$$\text{colim}\left(\tilde{\mathcal{F}}_{n+1}^S(I_\bullet)\right) \xrightarrow{\text{colim } c} \text{colim}\left(\mathcal{F}_n^S(I_\bullet)\right),$$

computed in $\mathcal{C}$, is isomorphic to the coproduct in $\mathcal{C}$

$$\coprod_{\{x \in \mathbb{N}^S : |x| = n\}} \left(\left(\wedge_{s \in S} I_{x(s)}\right) / \left(\text{colim } \mathcal{F}_{1;x}^S(I_\bullet)\right)\right).$$

This isomorphism is natural in the variable S.

(It is a elementary exercise in combinatorics to show that there are $\binom{n+\#(S)}{n}$ summands in this coproduct, where $\#(S)$ is the cardinality of S.)

Proof. One knows that the left Kan extension of $\mathcal{F}^S_{n+1}(I_\bullet)$ agrees with $\mathcal{F}^S_{n+1}(I_\bullet)$ where both are defined, so

$$
\begin{aligned}
\tilde{\mathcal{F}}^S_{n+1}(I_\bullet)(x) &\cong \mathcal{F}^S_{n+1}(I_\bullet)(x) \\
&= \wedge_{s\in S} I_{x(s)}
\end{aligned}
$$

for all $x \in \mathcal{D}^S_{n+1} \subseteq \mathcal{D}^S_n$. The elements of $\mathcal{D}^S_n$ which are *not* in $\mathcal{D}^S_{n+1}$ are those x such that $|x| = n$, and by the usual basic results (see e.g. [21]) on Kan extensions one knows that, for all x such that $|x| = n$, we have an isomorphism of $\tilde{\mathcal{F}}^S_{n+1}(I_\bullet)(x)$ with the colimit of the values of $\mathcal{F}^S_{n+1}(I_\bullet)$ over those elements of $(\mathcal{D}^S_{n+1})^{\mathrm{op}}$ which map to x, i.e., the colimit of the values of $\mathcal{F}^S_{n+1}(I_\bullet)$ over $(\mathcal{D}^S_{1,x})^{\mathrm{op}} \subseteq (\mathcal{D}^S_{n+1})^{\mathrm{op}}$, i.e., $\mathrm{colim}\left(\mathcal{F}^S_{1,x}(I_\bullet)\right)$.

The map c can be shown to be a cofibration by iterated use of the pushout product axiom, so the cofiber of c is a homotopy cofiber. By the previous paragraph the levelwise cofiber $\mathrm{cof}\, c : (\mathcal{D}^S_n)^{\mathrm{op}} \to C$ of the natural transformation c is given as follows:

$$
(\mathrm{cof}\, c)(x) \cong \begin{cases} 0 & \text{if} \quad |x| > n \\ \left(\mathcal{F}^S_n(I_\bullet)\right) / \left(\mathrm{colim}\left(\mathcal{F}^S_{1,x}(I_\bullet)\right)\right) & \text{if} \quad |x| = n. \end{cases}
$$

Hence, on taking colimits, we have

$$
\begin{aligned}
\mathrm{cof}\, \mathrm{colim}\, c &\cong \mathrm{colim}\, \mathrm{cof}\, c \\
&= \coprod_{\{x\in\mathbb{N}^S : |x|=n\}} \left(\left(\wedge_{s\in S} I_{x(s)}\right) / \left(\mathrm{colim}\, \mathcal{F}^S_{1;x}(I_\bullet)\right)\right),
\end{aligned}
$$

as claimed. $\qquad\qquad\square$

Lemma 3.4.9. *Suppose S is a finite set and suppose $Z_{s,1} \to Z_{s,0}$ is a cofibration for each $s \in S$. Suppose the objects $Z_{s,1}, Z_{s,0}$ are all cofibrant. Let $\mathcal{G}_S : (\mathcal{E}^S_1)^{\mathrm{op}} \to C$ be the functor given on objects by*

$$
\mathcal{G}_S(x) = \wedge_{s\in S} Z_{s,x(s)},
$$

and given on morphisms in the obvious way.

Then the smash product

$$
\wedge_{s\in S} Z_{s,0} \to \wedge_{s\in S} (Z_{s,0}/Z_{s,1})
$$

of the cofiber projections $Z_{s,0} \to Z_{s,0}/Z_{s,1}$ fits into a cofiber sequence:

$$
\mathrm{colim}\, \mathcal{G}_S \to \wedge_{s\in S} Z_{s,0} \to \wedge_{s\in S} (Z_{s,0}/Z_{s,1}).
$$

Proof. If the cardinality of S is one, the statement of the lemma is true by the definition of a cofiber.

The case in which the cardinality of S is two is precisely Lemma 3.4.1, already proven.

For the case in which the cardinality of S is greater than two, we introduce a notation we will need to use: let $\mathcal{PO}$ denote the category indexing pushout diagrams, i.e.,

$$
\mathcal{PO} = \left(\begin{array}{ccc} & [1'] & \\ & \swarrow \quad \searrow & \\ [1] & & [0] \end{array} \right),
$$

the symbols $[1'], [1], [0]$ each representing an object, and the arrows each representing a morphism. We observe that $\mathcal{PO}$ and $\mathcal{E}^S_1$ are not arbitrary small categories but are in fact

partially-ordered sets; this simplifies some of the arguments we will give in the rest of the proof.

Suppose the cardinality of S is greater than two. Choose an element $s_0 \in S$. We will write S' for the complement

$$S' = \{s \in S : s \neq s_0\}$$

of s_0 in S. Define objects X_1', X_2', Y_1', Y_2' in $\mathcal{C}$ as follows:

$$\begin{aligned}
Y_1' &= \text{colim } \mathcal{G}_{S'} \\
X_1' &= \wedge_{s \in S'} Z_{s,0} \\
Y_2' &= Z_{s_0,1} \\
X_2' &= Z_{s_0,0}.
\end{aligned}$$

Now we apply the statement of the lemma, in the (already proven, above) case $S = \{1, 2\}$ and using X_1', X_2', Y_1', Y_2' in place of $Z_{1,0}, Z_{2,0}, Z_{1,1}, Z_{2,1}$ to obtain a cofiber sequence

$$(3.4.4) \qquad \text{colim } \mathcal{B} \to \wedge_{s \in S} Z_{s,0} \to \wedge_{s \in S} (Z_{s,0}/Z_{s,1}),$$

where $\mathcal{B}$ is the functor $\mathcal{PO} \to \mathcal{C}$ given by:

$$\begin{aligned}
\mathcal{B}([1']) &= (\text{colim } \mathcal{G}_{S'}) \wedge Z_{s_0,1} \\
\mathcal{B}([1]) &= (\wedge_{s \in S'} Z_{s_0,0}) \wedge Z_{s_0,1} \\
\mathcal{B}([0]) &= (\text{colim } \mathcal{G}_{S'}) \wedge Z_{s_0,0}.
\end{aligned}$$

By Lemma 3.4.1, we know that the map $\text{colim } \mathcal{G}_{S'} \longrightarrow \wedge_{s \in S'} Z_{s,0}$ is a cofibration in the case $S = \{1, 2\}$. Since $\text{colim } \mathcal{B}$ is constructed as a pushout, the pushout product axiom ensures that the map $\text{colim } \mathcal{B} \longrightarrow \wedge_{s \in S} Z_{s,0}$ is also a cofibration as long as $\text{colim } \mathcal{G}_{S'} \longrightarrow \wedge_{s \in S'} Z_{s,0}$ is a cofibration. It suffices to show that $\text{colim } \mathcal{B} \cong \text{colim } \mathcal{G}_S$. This will show that the map $\text{colim } \mathcal{G}_S \longrightarrow \wedge_{s \in S} Z_{s,0}$ is a cofibration and allow us to identify the cofiber, thus completing the induction on the cardinality of the set S. We can reindex, to describe $\text{colim } \mathcal{B}$ as the colimit of a larger diagram $\mathcal{H}$:

$$\begin{aligned}
\mathcal{H} : (\mathcal{E}_1^{S'})^{\text{op}} \times \mathcal{PO} &\to \mathcal{C} \\
(x, [1']) &\mapsto \left(\wedge_{s \in S'} Z_{s,x(s)} \right) \wedge Z_{s_0,1} \\
(x, [1]) &\mapsto (\wedge_{s \in S'} Z_{s,0}) \wedge Z_{s_0,1} \\
(x, [0]) &\mapsto \left(\wedge_{s \in S'} Z_{s,x(s)} \right) \wedge Z_{s_0,0}
\end{aligned}$$

We have a functor

$$\begin{aligned}
P : (\mathcal{E}_1^{S'})^{\text{op}} \times \mathcal{PO} &\to (\mathcal{E}_1^S)^{\text{op}} \\
(x, [1'])(s) &= \begin{cases} x(s) & \text{if} \quad s \neq s_0 \\ 1 & \text{if} \quad s = s_0 \end{cases} \\
(x, [1])(s) &= \begin{cases} 0 & \text{if} \quad s \neq s_0 \\ 1 & \text{if} \quad s = s_0 \end{cases} \\
(x, [0])(s) &= \begin{cases} x(s) & \text{if} \quad s \neq s_0 \\ 0 & \text{if} \quad s = s_0 \end{cases}.
\end{aligned}$$

Now we claim that the canonical map $\text{colim } \mathcal{H} \to \text{colim } \mathcal{G}_S$ given by P is an isomorphism in $\mathcal{C}$. We define a functor

$$\begin{aligned}
I : (\mathcal{E}_1^S)^{\text{op}} &\to (\mathcal{E}_1^{S'})^{\text{op}} \times \mathcal{PO} \\
x &\mapsto \begin{cases} (I'(x), [1']) & \text{if} \quad x(s_0) = 1 \\ (I'(x), [0]) & \text{if} \quad x(s_0) = 0 \end{cases}
\end{aligned}$$

where $I' : (\mathcal{E}_1^S)^{\mathrm{op}} \to (\mathcal{E}_1^{S'})^{\mathrm{op}}$ is the functor given by restriction, i.e., $I'(x)(s) = x(s)$ for $s \in S'$. Now we observe some convenient identities:

$$(\mathcal{G}_S \circ P)(x, j) = \begin{cases} \left(\wedge_{s \in S'} Z_{s, x(s)}\right) \wedge Z_{s_0, 1} & \text{if} \quad j = [1'] \\ \left(\wedge_{s \in S'} Z_{s, x(s)}\right) \wedge Z_{s_0, 0} & \text{if} \quad j = [0] \\ \left(\wedge_{s \in S'} Z_{s, 0}\right) \wedge Z_{s_0, 1} & \text{if} \quad j = [1] \end{cases}$$

$$= \mathcal{H}(x, j), \text{ and}$$

$$(\mathcal{H} \circ I)(x) = \begin{cases} \left(\wedge_{s'} Z_{s, x(s)}\right) \wedge Z_{s_0, 1} & \text{if} \quad x_{s_0} = 1 \\ \left(\wedge_{s'} Z_{s, x(s)}\right) \wedge Z_{s_0, 0} & \text{if} \quad x_{s_0} = 0 \end{cases}$$

$$= \mathcal{G}_S(x).$$

We conclude that P, I give mutually inverse maps between $\mathrm{colim}\,\mathcal{G}_S$ and $\mathrm{colim}\,\mathcal{H}$, i.e., $\mathrm{colim}\,\mathcal{G}_S \cong \mathrm{colim}\,\mathcal{H}$ and hence $\mathrm{colim}\,\mathcal{G}_S \cong \mathrm{colim}\,\mathcal{B}$, as desired. So from cofiber sequence 3.4.4, we have a cofiber sequence

$$\mathrm{colim}\,\mathcal{G}_S \to \wedge_{s \in S} Z_{s,0} \to \wedge_{s \in S} \left(Z_{s,0}/Z_{s,1}\right),$$

as desired.

From inspection of the colimit diagrams one sees that the cofiber sequence 3.4.4 does not depend on the choice of $s_0 \in S$, and naturality in S follows. $\qquad\square$

Lemma 3.4.10. *Let S be a finite set, let n be a positive integer, and let $x \in \mathbb{N}^S$. Let $\mathcal{E}_n^S$ and $\mathcal{D}_{n;x}^S$ be as in Definition 3.4.3 and Definition 3.4.4. Let $J_{n;x}^S$ be the functor (i.e., morphism of partially-ordered sets) defined by*

$$J_{n;x}^S : \mathcal{E}_n^S \quad \to \quad \mathcal{D}_{n;x}^S$$
$$(J_{n;x}(y))(s) \quad = \quad x(s) + y(s).$$

Then $J_{n;x}$ has a right adjoint. Consequently $J_{n;x}$ is a cofinal functor, i.e., for any functor F defined on $\mathcal{D}_{n;x}^S$ such that the limit $\lim F$ exists, the limit $\lim(F \circ J_{n;x}^S)$ also exists, and the canonical map $\lim(F \circ J_{n;x}^S) \to \lim F$ is an isomorphism.

Proof. We construct the right adjoint explicitly. Let $K_{n;x}^S$ be the functor defined by

$$K_{n;x}^S : \mathcal{D}_{n;x}^S \quad \to \quad \mathcal{E}_n^S$$
$$(K_{n;x}^S(y))(s) \quad = \quad \min\{n, y(s) - x(s)\}.$$

(We remind the reader that every element $y \in \mathcal{D}_{n;x}^S$ has the property that $y(s) \geqslant x(s)$ for all $s \in S$, so $y(s) - x(s)$ will always be nonnegative.)

Now suppose $z \in \mathcal{E}_n^S$ and $y \in \mathcal{D}_{n;x}^S$. Then:

- $z \leqslant K_{n;x}^S(y)$ if and only if $z(s) \leqslant K_{n;x}^S(y)(s)$ for all $s \in S$,
- i.e., $z \leqslant K_{n;x}^S(y)$ if and only if $z(s) \leqslant \min\{n, y(s) - x(s)\}$ for all $s \in S$.
- By the definition of $\mathcal{E}_n^S$, $z(s) \leqslant n$ for all $s \in S$. Hence $z \leqslant K_{n;x}^S(y)$ if and only if $z(s) \leqslant y(s) - x(s)$ for all $s \in S$,
- i.e., $z \leqslant K_{n;x}^S(y)$ if and only if $x(s) + z(s) \leqslant y(s)$ for all $s \in S$,
- i.e., $z \leqslant K_{n;x}^S(y)$ if and only if $J_{n;x}^S(z) \leqslant y$.

Hence $\mathrm{hom}_{\mathcal{E}_n^S}(z, K_{n;x}^S(y))$ is nonempty if and only if $\mathrm{hom}_{\mathcal{D}_{n;x}^S}(J_{n;x}^S(z), y)$ is nonempty. Since $\mathcal{E}_n^S$ and $\mathcal{D}_{n;x}^S$ are partially-ordered sets and hence their hom-sets are either nonempty or have only a single element, we now have a (natural) bijection

$$\mathrm{hom}_{\mathcal{E}_n^S}(z, K_{n;x}^S(y)) \cong \mathrm{hom}_{\mathcal{D}_{n;x}^S}(J_{n;x}^S(z), y)$$

which is exactly what we are looking for: $J_{n;x}^S$ is left adjoint to $K_{n;x}^S$.

For the fact that having a right adjoint implies cofinality, see section IX.3 of Mac Lane's [21]. (Mac Lane handles the equivalent dual case.) □

Theorem 3.4.11. (Fundamental theorem of the May filtration.) *Let $I_\bullet$ be a cofibrant decreasingly filtered commutative monoid in C, and let $X_\bullet$ be a simplicial finite set. Then the associated graded commutative monoid $E_0^* \left| \mathcal{M}^{X_\bullet}(I_\bullet) \right|$ of the geometric realization of the May filtration is weakly equivalent, as a commutative graded monoid, to the tensoring $X_\bullet \otimes E_0^* I_\bullet$ of $X_\bullet$ with the associated graded commutative monoid of $I_\bullet$:*

$$E_0^* \left| \mathcal{M}^{X_\bullet}(I_\bullet) \right| \simeq X_\bullet \otimes E_0^* I_\bullet.$$

Proof. We must compute the filtration quotients

$$\left| \mathcal{M}_n^{X_\bullet}(I_\bullet) \right| / \left| \mathcal{M}_{n+1}^{X_\bullet}(I_\bullet) \right| \cong \left| \mathcal{M}_n^{X_\bullet}(I_\bullet)/\mathcal{M}_{n+1}^{X_\bullet}(I_\bullet) \right|.$$

We handle this as follows. First, we claim that there exists, for any finite set S and for all $n \in \mathbb{N}$, a cofiber sequence

$$(3.4.5) \qquad \operatorname{colim}\left(\mathcal{F}_{n+1}^S(I_\bullet)\right) \to \operatorname{colim}\left(\mathcal{F}_n^S(I_\bullet)\right) \to \coprod_{x \in \mathbb{N}^S : |x|=n} \left(\wedge_{s \in S}\left(I_{x(s)}/I_{1+x(s)}\right)\right)$$

in C, natural in S. We have already defined (in Definition 3.4.5) how $\mathcal{F}_n^S$ is natural, i.e., functorial in S; by taking the obvious coproduct of quotients, this naturality in S induces a naturality in S on the terms $\coprod_{x \in \mathbb{N}^S : |x|=n} \left(\wedge_{s \in S}\left(I_{x(s)}/I_{1+x(s)}\right)\right)$ appearing in 3.4.5. The claim that 3.4.5 is a cofiber sequence implies that

$$(3.4.6) \qquad |\mathcal{M}_n^{X_k}(I_\bullet)|/|\mathcal{M}_{n+1}^{X_k}(I_\bullet)| \cong \coprod_{x \in \mathbb{N}^{X_k} : |x|=n} \left(\wedge_{s \in X_k}\left(I_{x(s)}/I_{1+x(s)}\right)\right),$$

and naturally implies the necessary naturality with respect to the face and degeneracy maps.

We now show that the cofiber sequence 3.4.5 exists. First, by the universal property of the Kan extension, the map of diagrams $\mathcal{F}_{n+1}^S(I_\bullet) \to \mathcal{F}_n^S(I_\bullet)$ factors uniquely through the map to the left Kan extension $\mathcal{F}_{n+1}^S(I_\bullet) \to \tilde{\mathcal{F}}_{n+1}^S(I_\bullet)$ from Lemma 3.4.8, and the cofiber of the map $\operatorname{colim}(\mathcal{F}_{n+1}^S(I_\bullet)) \to \operatorname{colim}(\mathcal{F}_n^S(I_\bullet))$ agrees with the cofiber of the map $\operatorname{colim}(\tilde{\mathcal{F}}_{n+1}^S(I_\bullet)) \to \operatorname{colim}(\mathcal{F}_n^S(I_\bullet))$. By Lemma 3.4.8, this cofiber is the coproduct

$$\coprod_{\{x \in \mathbb{N}^S : |x|=n\}} \left(\left(\wedge_{s \in S} I_{x(s)}\right) / \left(\operatorname{colim} \mathcal{F}_{1;x}^S(I_\bullet)\right)\right).$$

In Lemma 3.4.10, we showed that the functor $J_{1;x}$ is cofinal, hence that the comparison map of colimits

$$\operatorname{colim}\left(\mathcal{F}_{1;x}(I_\bullet) \circ J_{1;x}\right) \to \operatorname{colim}\left(\mathcal{F}_{1;x}(I_\bullet)\right)$$

is an isomorphism. (We here have a colimit, not a limit as in the statement of Lemma 3.4.10, since $\mathcal{F}_{1;x}(I_\bullet)$ is a *contravariant* functor on $\mathcal{D}_{1;x}^S$. Of course Lemma 3.4.10 still holds in this dual form.)

Now Lemma 3.4.9 identifies the cofiber

$$\left(\wedge_{s \in S} I_{x(s)}\right) / \left(\operatorname{colim}\left(\mathcal{F}_{1;x}(I_\bullet) \circ J_x\right)\right)$$

with $\wedge_{s \in S}\left(I_{x(s)}/I_{1+x(s)}\right)$, as desired. So we have our cofiber sequence of the form 3.4.5.
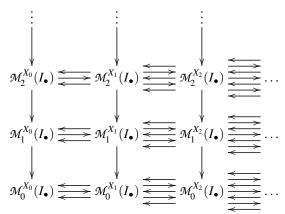
All isomorphisms in the lemmas we have invoked in this proof are natural in S, with the exception of the isomorphisms from Lemma 3.4.10 and Lemma 3.4.9 which directly involve $\mathcal{E}_1^S$, only because we did not specify in Lemma 3.4.9 how $\mathcal{G}_S$ is functorial in S. In

the present proof, $\mathcal{G}_S$ is $\mathcal{F}_{1;x}(I_\bullet) \circ J_{1;x}$, and the cofinality of $J_{1;x}$ together with the fact that $K_{1;x} \circ J_{1;x} = \mathrm{id}_{\mathcal{E}_1^S}$ implies, on inspection of the colimit diagrams, that the isomorphism

$$\begin{aligned} \mathrm{colim}\, \mathcal{G}_S &= \mathrm{colim}\left(\mathcal{F}_{1;x}(I_\bullet) \circ J_x\right) \\ &\cong \mathrm{colim}\left(\mathcal{F}_{1;x}(I_\bullet)\right) \end{aligned}$$

is natural in S; details are routine and left to the reader. We conclude that the cofiber sequence 3.4.5 is indeed natural in S.

Now we have the sequence of simplicial commutative monoids in $\mathcal{C}$:

and geometric realization commuting with cofibers together with the isomorphism 3.4.6 implies that the comparison map

(3.4.7) $$X_\bullet \otimes E_0^* I_\bullet \to E_0^* \left| \mathcal{M}^{X_\bullet}(I_\bullet) \right|$$

of objects in $\mathcal{C}$ is a weak equivalence. Hence the comparison map 3.4.7 in $\mathrm{Comm}(\mathcal{C})$ must also be a weak equivalence, since the weak equivalences in $\mathrm{Comm}(\mathcal{C})$ are created by the forgetful functor $\mathrm{Comm}(\mathcal{C}) \to \mathcal{C}$, by assumption. $\qquad\square$

3.5. **Construction of the topological Hochschild-May spectral sequence.**

Definition 3.5.1. *By a* connective generalized homology theory on $\mathcal{C}$ *we shall mean the following data:*

- *for each integer n, a functor $H_n : \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ab}$, and*
- *for each integer n and each cofiber sequence*

$$X \to Y \to Z$$

in $\mathcal{C}$, a map $\delta_n^{X\to Y\to Z} : H_n(Z) \to H_{n-1}(X)$,

satisfying the axioms:

Exactness: *For each cofiber sequence*

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

in C*, the sequence of abelian groups*

$$\ldots \longrightarrow H_{n+1}(Y) \xrightarrow{H_{n+1}(g)} H_{n+1}(Z)$$
$$\delta_{n+1}^{X \to Y \to Z}$$
$$H_n(X) \xrightarrow{H_n(f)} H_n(Y) \xrightarrow{H_n(g)} H_n(Z)$$
$$\delta_n^{X \to Y \to Z}$$
$$H_{n-1}(X) \xrightarrow{H_{n-1}(f)} H_{n-1}(Y) \xrightarrow{H_{n-1}(g)} \ldots$$

is exact.

Additivity: *For each integer n and each collection of objects* $\{X_i\}_{i \in I}$ *in* $\mathrm{Ho}(C)$*, the canonical map of abelian groups*

$$\coprod_{i \in I} H_n(X_i) \to H_n(\coprod_{i \in I} X_i)$$

is an isomorphism.

Naturality of boundaries: *For each integer n and each map of cofiber sequences*

$$
\begin{array}{ccccc}
X' & \longrightarrow & Y' & \longrightarrow & Z' \\
\downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle h} \\
X & \longrightarrow & Y & \longrightarrow & Z,
\end{array}
$$

the square of abelian groups

$$
\begin{array}{ccc}
H_n(Z') & \xrightarrow{\delta^{X' \to Y' \to Z'}} & H_{n-1}(X') \\
{\scriptstyle H_n(h)}\downarrow & & \downarrow{\scriptstyle H_{n-1}(g)} \\
H_n(Z) & \xrightarrow{\delta^{X \to Y \to Z}} & H_{n-1}(X)
\end{array}
$$

commutes.

Connectivity of the unit object: *We have* $H_n(\mathbb{1}) \cong 0$ *for all* $n < 0$.

Connectivity of smash products: *Suppose that* X, Y *are objects of* C*, and that* A, B *are nonnegative integers such that* $H_n(X) \cong 0$ *for all* $n < A$*, and* $H_n(Y) \cong 0$ *for all* $n < B$*. Then* $H_n(X \wedge Y) \cong 0$ *for all* $n < A + B$.

Clearly Definition 3.5.1 is just a formulation, in a general pointed model category, of the Eilenberg-Steenrod axioms (from [12]) for a generalized homology theory with connective (i.e., vanishing in negative degrees) coefficients. The "connectivity of smash products" axiom is easily proven anytime one has an E-homology Künneth spectral sequence in C, which is the case in (for example) any of the usual models for the stable homotopy category.

Definition 3.5.2. *If* $I_\bullet$ *is a cofibrant decreasingly filtered commutative monoid in* C*,* $X_\bullet$ *is a simplicial finite set, and* H_* *is a connective generalized homology theory on* C*, then by the* topological Hochschild-May spectral sequence for $X_\bullet \tilde{\otimes} I_\bullet$ *we mean the spectral sequence*

in $\mathcal{A}$ obtained by applying H_* to the tower of cofiber sequences in $\mathcal{C}$

(3.5.1)

$$
\begin{array}{ccc}
\vdots & & \\
\downarrow & & \\
\left|\mathcal{M}_2^{X_\bullet}(I_\bullet)\right| & \longrightarrow & \left|\mathcal{M}_2^{X_\bullet}(I_\bullet)\right|/\left|\mathcal{M}_3^{X_\bullet}(I_\bullet)\right| \\
\downarrow & & \\
\left|\mathcal{M}_1^{X_\bullet}(I_\bullet)\right| & \longrightarrow & \left|\mathcal{M}_1^{X_\bullet}(I_\bullet)\right|/\left|\mathcal{M}_2^{X_\bullet}(I_\bullet)\right| \\
\downarrow & & \\
\left|\mathcal{M}_0^{X_\bullet}(I_\bullet)\right| & \longrightarrow & \left|\mathcal{M}_0^{X_\bullet}(I_\bullet)\right|/\left|\mathcal{M}_1^{X_\bullet}(I_\bullet)\right|.
\end{array}
$$

That is, it is the spectral sequence of the exact couple

$$
D_{*,*}^1 \cong \bigoplus_{i,j} H_i\left|\mathcal{M}_j^{X_\bullet}(I_\bullet)\right| \longrightarrow \bigoplus_{i,j} H_i\left|\mathcal{M}_j^{X_\bullet}(I_\bullet)\right| \cong D_{*,*}^1
$$

$$
E_{*,*}^1 \cong \bigoplus_{i,j} H_i\left|\mathcal{M}_j^{X_\bullet}(I_\bullet)\right|/\left|\mathcal{M}_{j+1}^{X_\bullet}(I_\bullet)\right|
$$

Lemma 3.5.3. (Connectivity conditions.) *Let H_* be a connective generalized homology theory on $\mathcal{C}$. Suppose $\mathcal{C}$ is stable, and suppose that there exist objects Z, H of $\mathcal{C}$ such that $H_*(-)$ is naturally isomorphic to $[\Sigma^* Z, H \wedge -]$.*

- *Let*

(3.5.2)
$$
\cdots \to Y_2 \to Y_1 \to Y_0
$$

  be a sequence in $\mathcal{C}$, and suppose that $H_n(Y_i) \cong 0$ for all $n < i$. Then $[\Sigma^n Z, \operatorname{holim}_i(H \wedge Y_i)] \cong 0$ for all n.

- *Suppose that A is a nonnegative integer and that*

(3.5.3)
$$
X_0 \underset{\Longleftarrow}{\overset{\Longrightarrow}{\rightleftarrows}} X_1 \underset{\Longleftarrow}{\overset{\Longrightarrow}{\rightleftarrows}} X_2 \underset{\Longleftarrow}{\overset{\Longrightarrow}{\rightleftarrows}} \cdots
$$

  is a simplicial object of $\mathcal{C}$. Suppose that $H_n(X_i) \cong 0$ for all $n < A$ and all i. Then $H_n\left(|X_\bullet|\right) \cong 0$ for all $n < A$.

Proof.   - Since $\mathcal{C}$ is assumed stable, the homotopy limit $\operatorname{holim}_i Y_i$ is the homotopy fiber of the map

$$
\prod_{n\in\mathbb{N}} Y_n \xrightarrow{\mathrm{id}\,-T} \prod_{n\in\mathbb{N}} Y_n
$$

in $\mathrm{Ho}(\mathcal{C})$, where T is the product of the maps $Y_n \to Y_{n-1}$ occuring in the sequence 3.5.2. For each object Z of $\mathcal{C}$, we then have the long exact sequence

$$\cdots \longrightarrow [\Sigma^j Z, \textstyle\prod_{n \in \mathbb{N}} H \wedge Y_n] \xrightarrow{\mathrm{id}\,-T} [\Sigma^j Z, \textstyle\prod_{n \in \mathbb{N}} H \wedge Y_n]$$

$$[\Sigma^{j-1} Z, \mathrm{holim}_i H \wedge Y_i] \longrightarrow [\Sigma^{j-1} Z, \textstyle\prod_{n \in \mathbb{N}} H \wedge Y_n] \xrightarrow{\mathrm{id}\,-T} [\Sigma^{j-1} Z, \textstyle\prod_{n \in \mathbb{N}} H \wedge Y_n]$$

$$[\Sigma^{j-2} Z, \mathrm{holim}_i H \wedge Y_i] \longrightarrow [\Sigma^{j-2} Z, \textstyle\prod_{n \in \mathbb{N}} H \wedge Y_n] \xrightarrow{\mathrm{id}\,-T} \cdots$$

hence the Milnor exact sequence

$$0 \to R^1 \lim_i [\Sigma^{j+1} Z, H \wedge Y_i] \to [\Sigma^j Z, \mathrm{holim}_i H \wedge Y_i] \to \lim_i [\Sigma^j Z, H \wedge Y_i] \to 0.$$

The assumption that $[\Sigma^j Z, H \wedge Y_i] \cong 0$ for $j < i$ guarantees that the sequence

$$\cdots \to [\Sigma^j Z, H \wedge Y_2] \to [\Sigma^j Z, H \wedge Y_1] \to [\Sigma^j Z, H \wedge Y_0]$$

is eventually constant and zero for all j, hence both its limit and $R^1 \lim$ vanish for all j, hence $[\Sigma^j Z, \mathrm{holim}_i H \wedge Y_i] \cong 0$ for all j.
- The Bousfield-Kan spectral sequence, i.e., the H-homology spectral sequence of the simplicial object 3.5.3, has input $E^1_{s,t} \cong \pi_s(H \wedge X_t)$ and converges to $H_{s+t}(|X_\bullet|)$, since $[\Sigma^* Z, \mathrm{holim}\, H \wedge Y_i]$ vanishes. The differential in this spectral sequence is of the form $d^r : E^r_{s,t} \to E^r_{s-r,t+r-1}$, hence this spectral sequences has a nondecreasing upper vanishing curve at E^1, hence converges strongly. Triviality of $E^1_{s,t}$ for $s < A$ and $t < 0$ then gives us that $H_s(|X_\bullet|)$ vanishes for $s < A$.

$\square$

Lemma 3.5.4. *Suppose H_* is a connective generalized homology theory as defined in Definition 3.5.1, and $\mathcal{M}^S_i(I_\bullet)$ is the i-th degree of the May filtration for a finite set S and a cofibrant decreasingly filtered commutative monoid $I_\bullet$ as defined in Definition 3.1.2. Then, if $H_m(I_i) \cong 0$ for all $m, i \in \mathbb{N}$ such that $m < i$, then*

$$H_m(\mathcal{M}^S_i(I_\bullet)) \cong 0$$

for all $m, i \in \mathbb{N}$ such that $m < i$.

Proof. The proof is inductive on the cardinality of S, which we denote $\#(S)$. First recall that by Definition 3.4.5,

$$\mathcal{M}^S_i(I_\bullet) = \mathrm{colim}\, \mathcal{F}^S_i(I_\bullet)$$

and $\mathcal{F}^S_i(I_\bullet)$ is a functor

$$\mathcal{F}^S_i(I_\bullet) : (\mathcal{D}^S_i)^{\mathrm{op}} \longrightarrow \mathcal{C},$$

where $(\mathcal{D}^S_i)^{\mathrm{op}}$ is the full sub-poset of $(\mathbb{N}^S)^{\mathrm{op}}$ containing exactly the objects $x \in \mathbb{N}^S$ such that $|x| \geq i$. Also, recall the definition

$$\mathcal{E}^S_{n,k} = \{x \in \{0, ..., n\}^S \mid \sum_{s \in S} x(s) \geq k\}$$

where $k \geq n$ and the convention of writing $\mathcal{E}^S_n$ when $n = k$. One can easily see that the diagram $(\mathcal{E}^S_n)^{\mathrm{op}} \longrightarrow \mathcal{C}$ is cofinal in in the diagram $(\mathcal{D}^S_n)^{\mathrm{op}} \longrightarrow \mathcal{C}$.

The case $\#(S) = 1$ is trivial since the constant diagram I_i is cofinal in $\mathcal{F}_i^S(I_\bullet)$. The claim follows by the assumption that

$$H_m(I_i) \cong 0$$

for all $m, i \in \mathbb{N}$ such that $m < i$.

The case $\#(S) \geqslant 2$ and $i = 0$ is trivial as well by the following argument. First, the constant diagram $\bigwedge_{s \in S} I_0$ is cofinal. Second, we assumed that $H_m(I_0) \cong 0$ for all $m, i \in \mathbb{N}$ such that $m < 0$, and, by Definition 3.5.1, a connective generalized homology theory, satisfies $H_m(X \wedge Y) \cong 0$ for $m < m_1 + m_2$ whenever $H_i(X) \cong H_j(Y) \cong 0$ for all $i < m_1$ and all $j < m_2$.

Suppose $\#(S) = 2$. In the case $i = 1$, the diagram $\mathcal{F}_i^S(I_\bullet)$ contains the pushout diagram
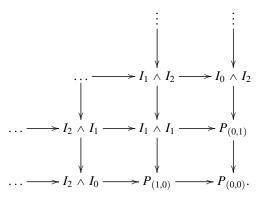
$$
\begin{array}{ccc}
I_1 \wedge I_1 & \longrightarrow & I_0 \wedge I_1 \\
\downarrow & & \\
I_1 \wedge I_0 & &
\end{array}
$$

as a cofinal subdiagram. This colimit, which is $\mathcal{M}_1^S(I_\bullet)$, is a homotopy pushout by Lemma 3.4.1 and can therefore be written as the cofiber in the cofiber sequence

$$I_1 \wedge I_1 \longrightarrow I_0 \wedge I_1 \vee I_1 \wedge I_0 \longrightarrow (I_0 \wedge I_1 \vee I_1 \wedge I_0)/(I_1 \wedge I_1) \cong (I_0/I_1 \wedge I_1) \vee (I_1 \wedge I_0/I_1).$$

Since $H_m(I_1 \wedge I_1) \cong 0$ for $m < 2$ and $H_m((I_0 \wedge I_1) \vee (I_1 \wedge I_0)) \cong 0$ for $m < 1$, by the long exact sequence in H_*, $H_m(\mathrm{colim}\,\mathcal{F}_i^S(I_\bullet)) \cong 0$ for $m < 1$. This proves the claim for $i = 1$. When $i > 1$, we consider the cofinal subdiagram $\mathcal{E}_i^S(I_\bullet) : (\mathcal{E}_i^S)^{\mathrm{op}} \to \mathcal{C}$ of $\mathcal{F}_i^S(I_\bullet)$. By filling in vertices with pushouts, which we denote $P_{(j,k)}$, we can write $\mathrm{colim}\,\mathcal{F}_i^S(I_\bullet) \cong \mathrm{colim}\,\mathcal{E}_i^S(I_\bullet)$ as an iterated pushout; for example, when $n = 2$,

$$
\begin{array}{ccccccc}
& & \vdots & & \vdots & & \\
& & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & I_1 \wedge I_2 & \longrightarrow & I_0 \wedge I_2 & & \\
\downarrow & & \downarrow & & \downarrow & & \\
\cdots \longrightarrow & I_2 \wedge I_1 & \longrightarrow & I_1 \wedge I_1 & \longrightarrow & P_{(0,1)} & \\
\downarrow & & \downarrow & & \downarrow & & \\
\cdots \longrightarrow & I_2 \wedge I_0 & \longrightarrow & P_{(1,0)} & \longrightarrow & P_{(0,0)}.
\end{array}
$$

Note that the colimit of a pushout diagram agrees with the homotopy colimit of that diagram when the maps are all cofibrations and the objects are all cofibrant as is the case here.

On each of the objects $\mathcal{E}_i^S(I_\bullet)(x)$ and the objects $P_{(j,k)}$ where $0 \leqslant j + k < i$, the functor $H_m(-)$ evaluates to zero whenever $m < i$. The same is true about $\mathrm{colim}\,\mathcal{F}_i^S(I_\bullet) \cong P_{(0,0)}$, completing the case $\#(S) = 2$.

Now assume

$$H_m(\mathcal{M}_i^S(I_\bullet)) \cong 0$$

for all $m, i \in \mathbb{N}$ such that $m < i$, whenever $\#(S) < n$. By the same method of filling vertices in cubes that we use in the case $\#(S) = 2$, we just need to prove the case $i = 1$

and the case $i > 1$ will follow by the fact that the colimit $\operatorname{colim} \mathcal{F}_i^S(I_\bullet) = \mathcal{M}_i^S(I_\bullet))$ can be written as an iterated n-cube. It therefore remains to prove that

$$H_m(\mathcal{M}_1^S(I_\bullet)) \cong 0$$

for all $m \in \mathbb{Z}$ such that $m < 1$, whenever $\#(S) = n$. We consider a single directed n-cube missing a single terminal vertex, i.e. a functor

$$\mathcal{E}_1^S(I_\bullet) : (\mathcal{E}_1^S)^{\mathrm{op}} \longrightarrow \mathcal{C}$$

which is a cofinal diagram in $\mathcal{F}_1^S(I_\bullet)$ so that $\operatorname{colim} \mathcal{E}_1^S(I_\bullet) \cong \mathcal{M}_1^S(I_\bullet)$.

We then consider the subdiagrams of $\mathcal{E}_1^S(I_\bullet)$ that are $(n-1)$-cube shaped diagrams of $\mathcal{C}$ containing the object $\mathcal{E}_1^S(I_\bullet)(x)$ such that $x(s) = 1$ for all $s \in S$. We then remove the terminal vertex in each of these $n-1$-cubes and denote the k-th truncated $(n-1)$-cube

$$\mathcal{E}_1^S(I_\bullet)^{(k)}$$

where $k \in \{1, ..., n\}$ runs over all truncated sub $(n-1)$-cubes of this type.

We then construct a diagram, which we call $\mathcal{B}$, which is the same as $\mathcal{E}_1^S$ except that all the vertexes removed as in the process above are replaced with the colimits $\operatorname{colim} \mathcal{E}_1^S(I_\bullet)^{(k)}$. By universality of the colimit we get maps

$$\mathcal{E}_1^S(I_\bullet)^{(k)} \longrightarrow I_0 \wedge ... \wedge I_0 \wedge I_1 \wedge I_0 \wedge ... \wedge I_0$$

for each k, where I_1 is in the k-th position, and by iterated use of the pushout product axiom, this map is a cofibration. We can therefore consider the map of diagrams

$$\mathcal{B} \longrightarrow \mathcal{E}_1^S(I_\bullet)$$

and take levelwise cofibers. We call the resulting diagram $\mathcal{C}of$. By examination of the levelwise cofibers, we see that

$$\operatorname{colim} \mathcal{C}of \cong \bigvee_{k=1}^{n} (I_{\delta_{1k}} \wedge \ldots \wedge I_{\delta_{nk}}) / \mathcal{E}_1^S(I_\bullet)^{(k)}$$

where $\delta_{jk} = 0$ if $j \neq k$ and $\delta_{kk} = 1$. We also observe that $\operatorname{colim} \mathcal{C}of \cong \operatorname{hocolim} \mathcal{C}of$ since it can be written as an iterated pushout of cofibrant objects along cofibrations [10, Prop. 13.10]. The object $\operatorname{colim} \mathcal{C}of$ is therefore a model for the homotopy cofiber of the map

$$\operatorname{colim} \mathcal{B} \longrightarrow \operatorname{colim} \mathcal{E}_1^S(I_\bullet)$$

even though this map is not necessarily a cofibration. We therefore get a long exact sequence in H_* so $H_m(\mathcal{F}_1^S(I_\bullet)) \cong 0$ for $m < \min\{m', m''\}$ where $H_j(\operatorname{colim} \mathcal{B}) \cong 0$ for $j < m'$ and $H_j(\operatorname{colim} \mathcal{C}of) \cong 0$ for $k < m''$. We know $H_j(\operatorname{colim} \mathcal{C}of) \cong 0$ for all $j < 1$ since for each $k \in \{1, \ldots, n\}$ there is an isomorphism $H_j(I_{\delta_{1k}} \wedge \ldots \wedge I_{\delta_{nk}}) \cong 0$ for all $j < 1$, and by the inductive hypothesis there is an isomorphism $H_j(\mathcal{E}_1^S(I_\bullet)^{(k)}) \cong 0$ for all $j < 1$.

To prove $H_j(\operatorname{colim} \mathcal{B}) \cong 0$ for $j < 1$, first note that $\operatorname{colim} \mathcal{B} = \operatorname{colim} \mathcal{E}_{1,2}^S$ as defined. We observe that the object $\operatorname{colim} \mathcal{E}_{1,2}^S$ can be written as a colimit of n truncated $(n-1)$-cubes whose pairwise intersections are $(n-2)$-cubes. We repeat the process and form $\mathcal{B}_1$ by eliminating terminal vertices in each $(n-1)$-cube and replacing each one with the colimit of that $(n-1)$-cube. Observe that $\operatorname{colim} \mathcal{B}_1 \cong \operatorname{colim} \mathcal{E}_{1,3}^S$. We produce another sequence

$$\operatorname{colim} \mathcal{B}_1 \longrightarrow \operatorname{colim} \mathcal{B} \longrightarrow \operatorname{colim} \mathcal{C}of_1$$

and as before $H_j(\mathcal{C}of_1) \cong 0$ for $j < 1$. This begins an inductive procedure that ends with B_{n-2} such that $\operatorname{colim} B_{n-2} \cong \operatorname{colim} \mathcal{E}_{1,n}^S$ and since $\#(S) = n$, $\operatorname{colim} \mathcal{E}_{1,n}^S \cong \wedge_{s \in S} I_1$. Since

$H_j(\wedge_{s \in S} I_1) \cong 0$ for $j < 1$ and $H_j(Cof_m) \cong 0$ for $j < 1$ and all $1 \leqslant m \leqslant n - 2$, we have shown that $H_j(\mathcal{B}) \cong 0$ for all $j < 1$.

The colimit colim $\mathcal{E}_1^S(I_\bullet)$ therefore has the property that $H_j(\text{colim } \mathcal{E}_1^S(I_\bullet)) \cong 0$ for $j < 1$ and, thus, we have proven our claim. $\qquad\square$

Theorem 3.5.5. *Suppose $I_\bullet$ is a Hausdorff cofibrant decreasingly filtered commutative monoid in C, $X_\bullet$ is a simplicial finite set, and H_* is a connective generalized homology theory on C. Suppose $H_*(-) \cong [\Sigma^* Z, - \wedge H]$ for some objects Z and H in C. Suppose the following connectivity axiom:*

- **(Connectivity axiom.)** $H_m(I_n) \cong 0$ *for all $m < n$.*

Then the topological Hochschild-May spectral sequence is strongly convergent, its differential satisfies the graded Leibniz rule, and its input and output and differential are as follows:

$$E_{s,t}^1 \cong H_{s,t}(X_\bullet \otimes E_0^* I_\bullet) \quad \Rightarrow \quad H_s(X_\bullet \otimes I_0)$$
$$d^r : E_{s,t}^r \quad \rightarrow \quad E_{s-1,t+r}^r$$

Proof. It is standard (see e.g. the section on Adams spectral sequences in [6]) that the H-homology spectral sequence of a tower of cofiber sequences of the form 3.5.1 converges to $H_*\left(\left|\mathcal{M}_0^{X_\bullet}(I_\bullet)\right|\right)$ as long as $\left[\Sigma^* Z, \text{holim}_i \left(H \wedge \left|\mathcal{M}_i^{X_\bullet}(I_\bullet)\right|\right)\right]$ is trivial. By Lemma 3.5.4 $H_m\left(\left|\mathcal{M}_i^{X_\bullet}(I_\bullet)\right|\right) \cong 0$ for all $m < i$, so by Lemma 3.5.3,

$$\left[\Sigma^n Z, \text{holim}_i \left(H \wedge \left|\mathcal{M}_i^{X_\bullet}(I_\bullet)\right|\right)\right] \cong 0$$

for all n, as desired. Hence the spectral sequence converges to

$$H_*\left(\left|\mathcal{M}_0^{X_\bullet}(I_\bullet)\right|\right) \cong H_*(X_\bullet \otimes I_0).$$

That the differential has the stated bidegree is a routine and easy computation in the spectral sequence of a tower of cofiber sequences.

The sequence

$$\cdots \longrightarrow \left|\mathcal{M}_2^{X_\bullet}(I_\bullet)\right| \longrightarrow \left|\mathcal{M}_1^{X_\bullet}(I_\bullet)\right| \longrightarrow \left|\mathcal{M}_0^{X_\bullet}(I_\bullet)\right|$$

is a cofibrant decreasingly filtered commutative monoid in C as observed in Remark 3.4.7 and therefore, in particular, it produces a "pairing of towers" in the sense of [11] and therefore by Proposition 5.1 of [11] the differentials in the spectral sequence satisfy a graded Leibniz rule.

The statements about convergence are also standard: the connectivity axiom and the "connectivity of smash products" axiom from Definition 3.5.1 together imply that our spectral sequence has a nondecreasing upper vanishing curve already at the E^1-term, so the spectral sequence converges strongly.

$\qquad\square$

Remark 3.5.6. Another construction of our THH-May spectral sequence

(3.5.4) $$E_{*,*}^1 \cong H_*(X_\bullet \otimes E_0^* I_\bullet) \Rightarrow H_*(X_\bullet \otimes I_0))$$

is possible using the Day convolution product. This construction is conceptually cleaner, but it does not, to our knowledge, simplify the process of proving that the resulting spectral sequence has the correct input term, output term and convergence properties.

Recall from Remark 3.1.4 that a cofibrant decreasingly filtered commutative monoid in C is equivalent to a cofibrant object in Comm $C^{\mathbb{N}^{op}}$ where Comm $C^{\mathbb{N}^{op}}$ has the model

structure created by the forgetful functor to $\mathcal{C}^{\mathbb{N}^{op}}$ and the category $\mathcal{C}^{\mathbb{N}^{op}}$ is equipped with the projective model structure.

Now fix a simplicial finite set $X_\bullet$. A cofibrant commutative monoid object I in $\mathcal{C}^{\mathbb{N}^{op}}$ is a cofibrant decreasingly filtered commutative monoid object $I_\bullet$ in $\mathcal{C}$, and we can form the pretensor product $X_\bullet \tilde{\otimes} I_\bullet$, a simplicial object in Comm $\mathcal{C}^{\mathbb{N}^{op}}$. For example, if $X_\bullet$ is the usual minimal simplicial model $(\Delta[1]/\delta\Delta[1])_\bullet$ for the circle, then $X_\bullet \tilde{\otimes} I_\bullet$ is the cyclic bar construction using the Day convolution as the tensor product:

$$(\Delta[1]/\delta\Delta[1])_\bullet \tilde{\otimes} I = \left(I \rightleftarrows I \otimes_{\text{Day}} I \rightleftarrows I \otimes_{\text{Day}} I \otimes_{\text{Day}} I \rightleftarrows \ldots \right)$$

Since I is a functor $\mathbb{N}^{op} \to \mathcal{C}$, we will write $I(n)$ for the evaluation of this functor at a nonnegative integer n. (If we instead think of I as a decreasingly filtered commutative monoid, as in most of the rest of this paper, we would write I_n instead of $I(n)$.) We write $((\Delta[1]/\delta\Delta[1])_\bullet \tilde{\otimes} I)(i)$ for the the simplicial object in $\mathcal{C}$

$$((\Delta[1]/\delta\Delta[1])_\bullet \tilde{\otimes} I)(i) = \left(I(i) \rightleftarrows (I \otimes_{\text{Day}} I)(i) \rightleftarrows (I \otimes_{\text{Day}} I \otimes_{\text{Day}} I)(i) \rightleftarrows \ldots \right)$$

Applying geometric realization to $((\Delta[1]/\delta\Delta[1])_\bullet \tilde{\otimes} I)(i)$, we get a cofibrant decreasingly filtered object in in $\mathcal{C}$ (assuming Running Assumption 2.0.3)

$$|((\Delta[1]/\delta\Delta[1])_\bullet \tilde{\otimes} I)(0)| \leftarrow |((\Delta[1]/\delta\Delta[1])_\bullet \tilde{\otimes} I)(1)| \leftarrow |((\Delta[1]/\delta\Delta[1])_\bullet \tilde{\otimes} I)(2)| \leftarrow \ldots$$

and the spectral sequence obtained by applying a generalized homology theory E_* to this cofibrant decreasingly filtered object in $\mathcal{C}$ is precisely the spectral sequence 3.5.4, the spectral sequence constructed and considered throughout this paper. (It is an easy exercise in unwinding definitions to check that this spectral sequence agrees with the one constructed in Definition 3.5.2, but to verify that the resulting spectral sequence has the expected input term, output term, and convergence properties amounts to exactly the same proofs already found in this paper which aren't expressed in terms of Day convolution.)

4. DECREASINGLY FILTERED COMMUTATIVE RING SPECTRA.

4.1. **Whitehead towers.** Let R be a cofibrant connective commutative monoid in $\mathcal{C}$. For this section and the next, let $\mathcal{C}$ be a model for the homotopy category of spectra such as symmetric spectra, S-modules, or orthogonal spectra. The goal of this section is to produce a cofibrant decreasingly filtered commutative monoid in $\mathcal{C}$ as a specific multiplicative model for the Whitehead tower of a connective commutative monoid in $\mathcal{C}$. Part of the proof uses a Postnikov tower of a commutative ring spectrum constructed as a tower of square-zero extensions, so first we define square-zero extensions in this context.

Definition 4.1.1. *By a square-zero extension in $\mathcal{C}$, we mean a fiber sequence*

$$I \longrightarrow \tilde{A} \longrightarrow A$$

where $\tilde{A}$ is the pullback in Comm $\mathcal{C}$ *of*

$$\begin{array}{ccc} \tilde{A} & \longrightarrow & A \\ \downarrow & & \downarrow{\scriptstyle \epsilon} \\ A & \xrightarrow{d} & A \ltimes \Sigma I, \end{array}$$

the map ϵ is defined to be the inclusion of A into $A \ltimes \Sigma I$ and d represents a class $[d] \in TAQ_S^0(A, \Sigma I)$. (For a definition of $TAQ_S^*(A, \Sigma I)$, see [5] or [22].) Note that, a priori, A

must be a commutative monoid in C and I must be a A-bimodule. By $A \ltimes \Sigma I$ we mean the trivial square-zero extension of A by ΣI; that is, additively $A \ltimes \Sigma I := A \vee \Sigma I$ and its multiplication is the map

$$\mu : A \wedge A \vee A \wedge I \vee I \wedge A \vee I \wedge I \longrightarrow A \vee I$$

determined by the maps

$$\mu_A : A \wedge A \to A \hookrightarrow A \vee I$$
$$\psi^\ell : A \wedge I \to I \hookrightarrow A \vee I$$
$$\psi^r : I \wedge A \to I \hookrightarrow A \vee I$$
$$sq : I \wedge I \to 0 \hookrightarrow A \vee I$$

where μ_A is the multiplication on A, ψ^r and ψ^ℓ are the right and left action maps of I as an A-bimodule and sq is the usual map $I \wedge I \to I \hookrightarrow A \vee I$, which in this case factors through the zero object.

Definition 4.1.2. *Let R be a connective commutative monoid in C. By a* Postnikov tower of square-zero extensions *associated to R, we mean a tower*

$$\ldots \longrightarrow \tau_{\leqslant 3}R \longrightarrow \tau_{\leqslant 2}R \longrightarrow \tau_{\leqslant 1}R \longrightarrow \tau_{\leqslant 0}R$$

$$\Sigma^3 H\pi_3 R \qquad \Sigma^2 H\pi_2 R \qquad \Sigma^1 H\pi_1 R \qquad H\pi_0 R$$

of fiber sequences where $\pi_k(\tau_{\leqslant n}R) = \pi_k(R)$ for $k \leqslant n$ and $\pi_k(\tau_{\leqslant n}R) = 0$ for $k > n$, such that the fiber sequences

$$\Sigma^n H\pi_n R \longrightarrow \tau_{\leqslant n}R \longrightarrow \tau_{\leqslant n-1}R$$

are square-zero extensions.

As defined it is not clear that such Postnikov towers of square-zero extensions actually exist for a given commutative monoid in C, but it is a theorem that they do.

Theorem 4.1.3. *Let R be a connective commutative monoid in C. Then there exists a model for the Postnikov tower associated to R which is a Postnikov tower of square-zero extensions.*

Proof. See Theorem 4.3 and the comments after in [17] and Theorem 8.1 in [5]. Also, see Lurie's Corollary 3.19 from [19] for the result in the setting of quasi-categories.   □

Recall from Remark 3.1.4 that a cofibrant object in the category $\mathrm{Comm}\, C^{\mathbb{N}^{\mathrm{op}}}$ equipped with the projective model structure is a cofibrant decreasingly filtered commutative monoid. We may define certain n-truncated decreasingly filtered commutative monoids in the following way.

Definition 4.1.4. *Let $J_n \subset \mathbb{N}$ be the sub-poset of the natural numbers consisting of all $i \in \mathbb{N}$ such that $i \leqslant n$. We give this poset the structure of a symmetric monoidal category $(J_n, \dot{+}, 0)$ by letting*

$$i \dot{+} j = min\{i + j, n\}.$$

We may consider lax symmetric monoidal functors in $(C^{J_n^{\mathrm{op}}})$ for each n again as a consequence of [9, Ex. 3.2.2] these are equivalent to the commutative monoids in the functor category under a Day convolution symmetric monoidal product. We may also consider the model structure on $\mathrm{Comm}(C^{J_n^{\mathrm{op}}})$ created by the forgetful functor to $(C^{J_n^{\mathrm{op}}})$, where $(C^{J_n^{\mathrm{op}}})$ has the projective model structure. In this model structure, it is an easy exercise to show that

the cofibrant objects are objects $I^{\leqslant n}$ in $\left(C^{J_n^{\mathrm{op}}} \right)$ such that each $I_i^{\leqslant n}$ is cofibrant in C for $i \leqslant n$ and each map $f_i : I_i^{\leqslant n} \to I_{i-1}^{\leqslant n}$ is a cofibration in C for each $i \leqslant n$.

Theorem 4.1.5. *Let R be a cofibrant connective commutative monoid in C, then there exists a cofibrant decreasingly filtered commutative monoid in C*

$$\ldots \longrightarrow \tau_{\geqslant 2}R \longrightarrow \tau_{\geqslant 1}R \longrightarrow \tau_{\geqslant 0}R$$

with structure maps

$$\rho_{i,j} : \tau_{\geqslant i}R \wedge \tau_{\geqslant j}R \longrightarrow \tau_{\geqslant i+j}R$$

such that $\pi_k(\tau_{\geqslant n}R) \cong \pi_k(R)$ for $k \geqslant n$ and $\pi_k(\tau_{\geqslant n}R) \cong 0$ for $k < n$. This cofibrant decreasingly filtered commutative monoid in C is denoted $\tau_{\geqslant \bullet}R$.

Proof of Theorem 4.1.5. Let R be a cofibrant connective commutative monoid in C and let

$$\ldots \longrightarrow \tau_{\leqslant 2}R \longrightarrow \tau_{\leqslant 1}R \longrightarrow \tau_{\leqslant 0}R$$

$$\Sigma^2 H\pi_2(R) \qquad \Sigma H\pi_1(R) \qquad H\pi_0(R)$$

be a Postnikov tower of square-zero extensions of R in the sense of Definition 4.1.2. To prove the theorem we need to do the following:

(1) Construct $\tau_{\geqslant n}R$.
(2) Construct maps $\rho_{i,j} : \tau_{\geqslant i}R \wedge \tau_{\geqslant j}R \longrightarrow \tau_{\geqslant i+j}R$.
(3) Show that the maps $\rho_{i,j}$ satisfy associativity, commutativity, unitality and compatibility.

The procedure will be inductive. First, define $\tau_{\geqslant 0}R := R$ where R was assumed to be a cofibrant connective commutative monoid in C and is therefore an object in Comm $C^{J_0^{\mathrm{op}}}$. To construct $\tau_{\geqslant 1}R$, we consider the map of commutative ring spectra $R \longrightarrow H\pi_0 R$. We can assume this map is a fibration, since if it wasn't we could factor the map in commutative ring spectra into an acyclic cofibration and a fibration. We then define $\tau_{\geqslant 1}R$ to be the fiber of this map. By design, we have constructed an object $I_\bullet^{\leqslant 1}$ in Comm $C^{J_1^{\mathrm{op}}}$. Commutativity, associativity and unitality follow by the definition of a symmetric R-bimodule action of R on $\tau_{\geqslant 1}R$. This completes the base step in the induction.

Suppose we have a object $I_\bullet^{\leqslant n-1} \in \mathrm{ob}$ Comm $C^{J_{n-1}^{\mathrm{op}}}$ for an arbitrary $n \geqslant 1$. As before, we define $\tau_{\geqslant i}R$ to be $I_i^{\leqslant n-1}$ for all $i \leqslant n-1$. Define

$$P_n := \mathrm{colim}_{\mathcal{D}_n} \tau_{\geqslant i}R \wedge \tau_{\geqslant j}R$$

where $\mathcal{D}_n$ is the full subcategory of $\mathbb{N}^{\mathrm{op}} \times \mathbb{N}^{\mathrm{op}}$ with objects (i, j) such that $0 < i \leqslant j < n$ and $i + j \geqslant n$. Since $I_\bullet^{\leqslant n-1}$ is in Comm $C^{J_{n-1}^{\mathrm{op}}}$, there is a unique map $P_n \longrightarrow \tau_{\geqslant n-1}R$.

The fact that the fiber sequence $\Sigma^k H\pi_k R \longrightarrow \tau_{\leqslant k}R \longrightarrow \tau_{\leqslant k-1}R$ is a square-zero extension for each k implies that the natural maps

$$\Sigma^i H\pi_i R \wedge \Sigma^j H\pi_j R \longrightarrow \Sigma^{n-1} H\pi_{i+j}R,$$

factor through 0 for each $(i, j) \in \mathcal{D}_n$. We get an induced map on fibers by considering the diagrams

$$
\begin{array}{ccccc}
\tau_{\geqslant k}R & \longrightarrow & R & \longrightarrow & \tau_{\leqslant k-1}R \\
\downarrow & & \downarrow & & \downarrow \\
\Sigma^k H\pi_k R & \longrightarrow & \tau_{\leqslant k}R & \longrightarrow & \tau_{\leqslant k-1}R
\end{array}
$$

for $k < n$. There are therefore commutative diagrams

$$
\begin{array}{ccc}
\tau_{\geqslant i}R \wedge \tau_{\geqslant j}R & \longrightarrow & \Sigma^i H\pi_i R \wedge \Sigma^j H\pi_j R \\
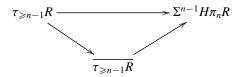\downarrow & & \downarrow {\scriptstyle 0} \\
\tau_{\geqslant n-1}R & \longrightarrow & \Sigma^{n-1} H\pi_n R
\end{array}
$$

for each $(i, j) \in \mathcal{D}_n$, hence, the map

$$
\tau_{\geqslant i}R \wedge \tau_{\geqslant j}R \longrightarrow \tau_{\geqslant n}R \longrightarrow \Sigma^{n-1} H\pi_n R
$$

factors through zero for each $(i, j) \in \mathcal{D}_n$.

We need the map $\tau_{\geqslant n-1}R \to \Sigma^{n-1} H\pi_n R$ to be a fibration, so we use the factorization

$$
\begin{array}{ccc}
\tau_{\geqslant n-1}R & \longrightarrow & \Sigma^{n-1} H\pi_n R \\
& \searrow \quad \nearrow & \\
& \overline{\tau_{\geqslant n-1}R} &
\end{array}
$$

into a trivial cofibration followed by a fibration.

We can define $\tau_{\geqslant n}R$ to be the pullback, in the category of R-modules in $\mathcal{C}$, of the diagram

$$
\begin{array}{ccc}
\tau_{\geqslant n}R & \longrightarrow & 0 \\
\downarrow & & \downarrow \\
\overline{\tau_{\geqslant n-1}R} & \xrightarrow{\ f\ } & \Sigma^{n-1} H\pi_n R.
\end{array}
$$

We then also need to replace P_n by $\overline{P_n}$ where $\overline{P_n}$ is the same colimit as P_n except that each instance of $\tau_{\geqslant n-1}R$ is replaced by $\overline{\tau_{\geqslant n-1}R}$. There is therefore a map $P_n \to \overline{P_n}$ and there is a map $\overline{P_n} \to \Sigma^{n-1} H\pi_{n-1}R$ that factors through the zero map by the same considerations as above.

Therefore, by the universal property of the pullback, there exists a unique map g

$$
\begin{array}{ccc}
\overline{P_n} & & \\
& {\scriptstyle g} \searrow & \\
& \tau_{\geqslant n}R \longrightarrow 0 & \\
& \downarrow \qquad\qquad \downarrow & \\
& \overline{\tau_{\geqslant n-1}R} \longrightarrow \Sigma^{n-1} H\pi_{n-1}R.
\end{array}
$$

By composing the maps $\tau_{\geqslant i}R \wedge \tau_{\geqslant j}R \to \overline{P_n}$ and $\overline{\tau_{\geqslant n-1}R} \wedge \tau_{\geqslant i}R \to \overline{P_n}$ with the map g, we produce the necessary maps $\rho_{i,j} : \tau_{\geqslant i}R \wedge \tau_{\geqslant j}R \longrightarrow \tau_{\geqslant \min\{i+j,n\}}R$ where $0 < i \leqslant j < n$. This also proves, by construction, that they satisfy the compatibility axiom (that is, naturality of the lax symmetric monoidal functor $J_n^{\mathrm{op}} \to \mathcal{C}$). The factor swap map produces all the maps

$$
\rho_{i,j} : \tau_{\geqslant i}R \wedge \tau_{\geqslant j}R \longrightarrow \tau_{\geqslant \min\{i+j,n\}}R
$$

where $i > j$ and the commutativity and compatibility necessary for those maps as well. The maps $\rho_{0,n}$ and $\rho_{n,0}$ are the R-module action maps that we produced by working in the category of R-modules and and again by construction these maps satisfy commutativity

and compatibility with the other maps. Unitality is also easily satisfied for each $\rho_{i,j}$ with $i, j \in \{0, ..., n\}$, since all these maps are R-module maps.

We just need to check associativity. By assumption, we have associativity for all the maps $\rho_{i,j}$ where $i, j < n$, we therefore just need to show that the associativity diagrams involving the maps $\rho_{i,j}$ for i or j equal to n. Since the symmetric monoidal product on R-modules is assumed to be associative, we know that, for $i, j, k \in \{0, n\}$, the diagrams

$$\begin{array}{ccc}
\tau_{\geqslant i}R \wedge \tau_{\geqslant j}R \wedge \tau_{\geqslant k}R & \longrightarrow & \tau_{\geqslant i}R \wedge \tau_{\geqslant j+k}R \\
\downarrow & & \downarrow \\
\tau_{\geqslant i+j}R \wedge \tau_{\geqslant k}R & \longrightarrow & \tau_{\geqslant i+j+k}R
\end{array}$$

commute. We also know, by construction, that the diagram

$$(4.1.1) \qquad \begin{array}{ccc}
& \tau_{\geqslant i}R \wedge \tau_{\geqslant j}R & \\
& \downarrow \quad \searrow & \\
\tau_{\geqslant n-1}R & \longleftarrow & \tau_{\geqslant n}R
\end{array}$$

commutes for all $i + j \geqslant n$. The diagram

$$(4.1.2) \qquad \begin{array}{ccccc}
\tau_{\geqslant i}R \wedge \tau_{\geqslant n}R & \longrightarrow & \tau_{\geqslant n}R & \longleftarrow & \tau_{\geqslant n}R \wedge \tau_{\geqslant i}R \\
\downarrow & & \downarrow & & \downarrow \\
\tau_{\geqslant i}R \wedge \tau_{\geqslant n-1}R & \longrightarrow & \tau_{\geqslant n-1}R & \longleftarrow & \tau_{\geqslant n-1}R \wedge \tau_{\geqslant i}R
\end{array}$$

also commutes by construction.

We need to show that for $i, j, k \in \{0, 1, ..., n\}$ with either i, j, or k equal to n, then

$$(4.1.3) \qquad \begin{array}{ccc}
\tau_{\geqslant i}R \wedge \tau_{\geqslant j}R \wedge \tau_{\geqslant k}R & \longrightarrow & \tau_{\geqslant i}R \wedge \tau_{\geqslant j+k}R \\
\downarrow & & \downarrow \\
\tau_{\geqslant i+j}R \wedge \tau_{\geqslant k}R & \longrightarrow & \tau_{\geqslant n}R
\end{array}$$

commutes. This follows by combining the commutativity of Diagram 4.1.1, Diagram 4.1.2, and the diagrams of the form of Diagram 4.1.3 when $i, j, k < n$, and using the fact that $\tau_{\geqslant n}R \to \tau_{\geqslant n-1}R$ is a monomorphism, since it is the pullback of a monomorphism in C by construction, and hence it is retractile.

We have therefore produced an object in $\mathrm{Comm}\, C^{J_n^{\mathrm{op}}}$. By induction, we can therefore produce an object in $\mathrm{Comm}\, C^{\mathbb{N}^{\mathrm{op}}}$ and then cofibrantly replace it to produce a cofibrant decreasingly filtered commutative monoid in C, denoted $\tau_{\geqslant\bullet}R$, as desired. $\qquad\square$

Remark 4.1.6. Since we have functorial factorizations of maps and functorial cofibrant replacement in our setting, the above theorem is entirely functorial, in other words, a map of connective commutative ring spectra $A \to B$ induces a map of Whitehead towers $\tau_{\geqslant n}A \to \tau_{\geqslant n}B$ compatible with the multiplication maps $\rho_{i,j}^A$ and $\rho_{i,j}^A$. This induces a map of associated graded commutative monoids in C

$$E_0^*(\tau_{\geqslant\bullet}A) \longrightarrow E_0^*(\tau_{\geqslant\bullet}B).$$

and a map of THH-May spectral sequences

$$
\begin{array}{ccc}
H_*(X_\bullet \otimes E_0^*(\tau_{\geqslant\bullet}A)) & \Longrightarrow & H_*(X_\bullet \otimes A) \\
\downarrow & & \downarrow \\
H_*(X_\bullet \otimes E_0^*(\tau_{\geqslant\bullet}B)) & \Longrightarrow & H_*(X_\bullet \otimes B).
\end{array}
$$

Example 4.1.7. Let R be a commutative ring spectrum with homotopy groups $\pi_k(R) \cong \hat{\mathbb{Z}}_p$ for $k = 0, n$ and $\pi_k(R) \cong 0$ otherwise. Then one can build

$$0 \longrightarrow \Sigma^n H\hat{\mathbb{Z}}_p \longrightarrow R$$

as a cofibrant decreasingly filtered commutative ring spectrum using Theorem 4.1.5. Since one can construct a Postnikov truncation of a commutative ring spectrum as a commutative ring spectrum [5], we can produce an example of this type by considering the truncation of the connective p-complete complex K-theory spectrum

$$\Sigma^2 H\pi_2 ku_p \longrightarrow ku_p^{\leqslant 2} \longrightarrow H\pi_0 ku_p.$$

The results of this subsection naturally leads to the question of whether topological Hochschild homology commutes with Postnikov limit; i.e the question of whether the map

$$THH(R) \to \operatorname{holim} THH(R^{\leqslant n})$$

is an equivalence. In the following section, we prove this result in the more general case of tensoring with a simplicial set. One could therefore try to compute $THH(R)$ for some ring spectrum by computing $THH(R^{\leqslant n})$ for each n using the THH-May spectral sequence and then computing the limit. As an example, we carry this out in the case $R = \hat{ku}_p$ and $n = 2$ in the next subsection.

4.2. Tensoring with simplicial sets commutes with the Postnikov limit.

Lemma 4.2.1. *Let $X_\bullet$ be a simplicial pointed finite set. Let E be a spectrum and let R be an E_∞-ring spectrum. Suppose that $E_n(R)$ is finite for all integers n. Then $E_n(X_\bullet \otimes R)$ is finite for all integers n.*

Proof. We will make use of the "pretensor product" $\tilde{\otimes}$ defined in Definition 3.3.1. The Bousfield-Kan-type spectral sequence obtained by applying E_* to the simplicial object $X_\bullet \tilde{\otimes} R$ has E^1-term $E^1_{s,t} \cong \coprod_{X_s} E_t(R)$, differentials $d^r : E^r_{s,t} \to E^r_{s-r,t+r-1}$, and converges to $E_{s+t}(X_\bullet \otimes R)$. Consequently, this spectral sequence is half-plane with exiting differentials, in the sense of [6]. Hence, the spectral sequence is strongly convergent, by Theorem 6.1 of [6], and finiteness of $E^1_{s,t}$ for all s, t such that $s + t = n$ implies finiteness of $E_n(X_\bullet \otimes R)$. $\square$

Definition 4.2.2. *Let R be an E_∞-ring spectrum and let n be an integer. We will write $R^{\leqslant n}$ for the n-th Postnikov truncation of R, that is, $R^{\leqslant n}$ is R with E_∞-cells attached to kill all the homotopy groups of R above dimension n. Consequently we have a map of E_∞-ring spectra $R \to R^{\leqslant n}$ which induces an isomorphism $\pi_i(R) \to \pi_i(R^{\leqslant n})$ for all $i \leqslant n$, and such that $\pi_i(R^{\leqslant n}) \cong 0$ if $i > n$.*

In the statements of Theorems 4.2.3 and 4.2.4, the homotopy limit in 4.2.1 and 4.2.4 can be taken in E_∞-ring spectra or in spectra; since the forgetful functor from E_∞-ring spectra to spectra commutes with homotopy limits, the maps 4.2.1 and 4.2.4 are weak equivalences either way.

Theorem 4.2.3. *Let R be a connective E_∞-ring spectrum. Let p be a prime number such that the i-th mod p homotopy group $(S/p)_i(R)$ is finite for each integer i. Let $X_\bullet$ be a simplicial pointed finite set. Then the natural map of E_∞-ring spectra*

$$(4.2.1) \qquad (X_\bullet \otimes R)\hat{}_p \to \left(\operatorname{holim}_n X_\bullet \otimes (R^{\leqslant n})\right)\hat{}_p$$

is a weak equivalence.

Proof. Since we assumed that $(S/p)_i(R)$ is finite for each i and since we assumed that $X_\bullet$ is a simplicial *finite* set, Lemma 4.2.1 implies that $(S/p)_i(X_\bullet \otimes R^{\leqslant n})$ is finite for all i and all n. So the first right-derived limit $\lim_n{}^1 (S/p)_i(X_\bullet \otimes R^{\leqslant n})$ vanishes, by the well-known vanishing of $\lim^1$ for inverse sequences of finite abelian groups.

For each nonnegative integer n, the map of Bousfield-Kan-type spectral sequences

$$(4.2.2) \qquad \begin{array}{ccc} E^1_{s,t} \cong \coprod_{X_s}(S/p)_t(R) & \Longrightarrow & (S/p)_{s+t}(X_\bullet \otimes R) \\ \downarrow & & \downarrow \\ E^1_{s,t} \cong \coprod_{X_s}(S/p)_t(R^{\leqslant n}) & \Longrightarrow & (S/p)_{s+t}(X_\bullet \otimes R^{\leqslant n}) \end{array}$$

is an isomorphism on the portion of the E^1-page satisfying $t < n - 1$. The differential in these Bousfield-Kan-type spectral sequences is of the form $d^r : E^r_{s,t} \to E^r_{s-r,t+r-1}$, and consequently both of these Bousfield-Kan-type spectral sequences are half-plane spectral sequences with exiting differentials, in the sense of [6]. Consequently both spectral sequence are strongly convergent, by Theorem 6.1 of [6].

Furthermore, since the map of spectral sequences 4.2.2 is an isomorphism at E^1 in bidegrees (s, t) satisfying $t < n - 1$, and since elements in total degree u can only interact, by supporting differentials or being hit by differentials, with elements in bidegrees (s, t) such that $t \leqslant u + 1$, the map of spectral sequences 4.2.2 is an isomorphism of spectral sequences when restricted to total degrees $< n - 1$. Hence the map of abelian groups $(S/p)_u(X_\bullet \otimes R) \to (S/p)_u(X_\bullet \otimes R^{\leqslant n})$ is an isomorphism when $u < n - 1$. Hence the map of graded abelian groups

$$(S/p)_*(X_\bullet \otimes R) \to \lim_n (S/p)_*(X_\bullet \otimes R^{\leqslant n})$$

is an isomorphism. Vanishing of $\lim^1$ then tells us that the map

$$(S/p)_*(X_\bullet \otimes R) \to (S/p)_*(\operatorname{holim}_n X_\bullet \otimes R^{\leqslant n})$$

is an isomorphism, i.e., that

$$(4.2.3) \qquad X_\bullet \otimes R \to \operatorname{holim}_n X_\bullet \otimes R^{\leqslant n}$$

is a S/p-local weak equivalence, i.e., that 4.2.3 is a weak equivalence after p-completion. $\qquad \square$

Theorem 4.2.4. *Let R be a connective E_∞-ring spectrum. Suppose that, for each integer i, the $\mathbb{Q}$-vector space $\pi_i(R) \otimes_{\mathbb{Z}} \mathbb{Q}$ is finite-dimensional. Let $X_\bullet$ be a simplicial pointed finite set. Then the natural map of E_∞-ring spectra*

$$(4.2.4) \qquad L_{H\mathbb{Q}}(X_\bullet \otimes R) \to L_{H\mathbb{Q}}\left(\operatorname{holim}_n X_\bullet \otimes (R^{\leqslant n})\right)$$

is a weak equivalence. (Here we are writing $L_{H\mathbb{Q}}$ for Bousfield localization at the Eilenberg-Mac Lane spectrum $H\mathbb{Q}$, i.e., $L_{H\mathbb{Q}}$ is rationalization.)

Proof. Essentially the same proof as that of Theorem 4.2.3; the only substantial difference is that, rather than $\lim^1$ vanishing being a consequence of finiteness of the mod p homotopy groups, in the present situation we have vanishing of

$$\lim_n{}^1 \left(\pi_*(X_\bullet \otimes R^{\leqslant n}) \right) \otimes_{\mathbb{Z}} \mathbb{Q}$$

due to the fact that $\lim^1$ vanishes on any inverse sequence of finite-dimensional vector spaces over a field; see [27]. $\square$

Corollary 4.2.5. *Let p be a prime, and let R be a p-local connective E_∞-ring spectrum. Suppose that, for each integer i, the $\mathbb{Z}_{(p)}$-module $\pi_i(R)$ is finitely generated. Let $X_\bullet$ be a simplicial pointed finite set. Then the natural map of E_∞-ring spectra*

(4.2.5) $$X_\bullet \otimes R \to \operatorname{holim}_n X_\bullet \otimes (R^{\leqslant n})$$

is a weak equivalence.

Proof. It follows from the pullback square in rings

$$\begin{array}{ccc} \mathbb{Z}_{(p)} & \longrightarrow & \mathbb{Q} \\ \downarrow & & \downarrow \\ \hat{\mathbb{Z}}_p & \longrightarrow & \mathbb{Q}_p \end{array}$$

that a map of connective finite-type p-local spectra which is both a p-complete weak equivalence and a rational weak equivalence is also a weak equivalence. $\square$

Corollary 4.2.6. *Let R be a connective E_∞-ring spectrum. Suppose that, for each integer i, the abelian group $\pi_i(R)$ is finitely generated. Let $X_\bullet$ be a simplicial pointed finite set. Then the natural map of E_∞-ring spectra*

$$X_\bullet \otimes R \to \operatorname{holim}_n X_\bullet \otimes (R^{\leqslant n})$$

is a weak equivalence.

Proof. Again, it is classical that a map of connective finite-type spectra which is a rational equivalence and a p-complete weak equivalence at each prime p is also a weak equivalence. $\square$

5. Applications

We now present two calculations: first, we calculate $(S/p)_* THH(R)$ when R has the property that $\pi_*(R) \cong \hat{\mathbb{Z}}_p[x]/x^2$ where $|x| > 0$; second, we provide a bound on topological Hochschild homology of a connective commutative ring spectrum R in terms of $THH(H\pi_*(R))$ and we give an explicit bound in the case $\pi_*(R) \cong \hat{\mathbb{Z}}_p[x]$ where $|x| = 2n$ for $n > 0$.

5.1. **Topological Hochschild homology of Postnikov truncations.** Let R be a commutative ring spectrum with the property that $\pi_*(R) \cong \hat{\mathbb{Z}}_p[x]/x^2$ with $|x| > 0$. We will consider the THH-May spectral sequence

$$(S/p)_*(THH(H\hat{\mathbb{Z}}_p \ltimes \Sigma^n H\hat{\mathbb{Z}}_p)) \Rightarrow (S/p)_*(THH(R))$$

produced using the short filtration of a commutative ring spectrum R given in Example 4.1.7. First, we compute the input of the S/p-THH-May spectral sequence for this example.

Proposition 5.1.1. *Let p be an odd prime, then*

$$(S/p)_*(THH(H\hat{\mathbb{Z}}_p \ltimes \Sigma^n H\hat{\mathbb{Z}}_p)) \cong E(\lambda_1) \otimes_{\mathbb{F}_p} P(\mu_1) \otimes_{\mathbb{F}_p} HH_*(E(x))$$

where $|x| = n$. The grading of $HH_*(E(x))$ is given by the sum of the internal and homological gradings.

Proof. Due to Bökstedt [7], there is an isomorphism

$$\pi_*(S/p \wedge THH(H\hat{\mathbb{Z}}_p)) \cong E(\lambda_1) \otimes_{\mathbb{F}_p} P(\mu_1).$$

Let $S \ltimes \Sigma^n S$ be the trivial split square-zero extension of S by $\Sigma^n S$. Then $H\mathbb{Z}$ and $S \ltimes \Sigma^n S$ are commutative S-algebras and $H\hat{\mathbb{Z}}_p \ltimes \Sigma^n H\hat{\mathbb{Z}}_p \simeq H\hat{\mathbb{Z}}_p \wedge S \ltimes \Sigma^n S$. By [29, Thm. 3.1], there are equivalences

$$\begin{aligned} THH(H\hat{\mathbb{Z}}_p \ltimes \Sigma^n H\hat{\mathbb{Z}}_p)) &\simeq THH(H\hat{\mathbb{Z}}_p \wedge (S \ltimes \Sigma^n S)) \\ &\simeq THH(H\hat{\mathbb{Z}}_p) \wedge THH(S \ltimes \Sigma^n S) \end{aligned}$$

of commutative ring spectra. Since $S/p \wedge H\hat{\mathbb{Z}}_p \simeq H\mathbb{F}_p$ and the spectrum $THH(H\hat{\mathbb{Z}}_p)$ is a $H\hat{\mathbb{Z}}_p$-algebra, the spectrum $S/p \wedge THH(H\hat{\mathbb{Z}}_p)$ naturally has the structure of a $H\mathbb{F}_p$-module. Hence, there are isomorphisms

$$\begin{aligned} \pi_*(S/p \wedge THH(H\hat{\mathbb{Z}}_p) \wedge THH(S \ltimes \Sigma^n S)) &\cong \\ \pi_*(S/p \wedge THH(H\hat{\mathbb{Z}}_p) \wedge_{H\mathbb{F}_p} H\mathbb{F}_p \wedge THH(S \ltimes \Sigma^n S)) &\cong \\ \pi_*(S/p \wedge THH(H\hat{\mathbb{Z}}_p)) \otimes_{\mathbb{F}_p} H\mathbb{F}_{p*}(THH(S \ltimes \Sigma^n S)). \end{aligned}$$

Now, we apply the Bökstedt spectral sequence

$$HH_*(H\mathbb{F}_{p*}(S \ltimes \Sigma^n S)) \Rightarrow H_*(THH(S \ltimes \Sigma^n S))$$

where the input is $HH_*(E(x))$. If $|x|$ is odd, then $HH_*(E(x)) \cong E(x) \otimes_{\mathbb{F}_p} \Gamma(\sigma x)$, which can be seen from the standard fact that $\operatorname{Tor}_*^{E(x)}(\mathbb{F}_p, \mathbb{F}_p) \cong \Gamma(\sigma x)$ and a change of rings argument, for example see [26]. If $|x|$ is even, then one easily computes

$$HH_n(E(x)) \cong \begin{cases} E(x) & * = 0 \\ \Sigma^{|x|(2i-1)}k\{1\} & n = 2i - 1 \\ \Sigma^{|x|(2i+1)}k\{x\} & n = 2i \end{cases}$$

for $i \geqslant 1$. There is an isomorphism of bigraded rings

$$HH_{*,*}(E(x)) \cong E(x)[x_i, y_j : i \geqslant 1, j \geqslant 0]/\sim$$

where the degrees are given by $|x_i| = (2i, 2|x|i + |x|)$ and $|y_j| = (2j + 1, 2j|x| + |x|)$, and the equivalence relation is the one that makes all products zero. The representatives in the cyclic bar complex for x_i and y_j are $x^{\otimes 2i+1}$ and $1 \otimes x^{\otimes 2j+1}$ respectively. Whether $|x|$ is even or odd, the Bökstedt spectral sequence collapses for bi-degree reasons. (Also see [4, Prop. 3.3] for the more general calculation of $HH_*(\mathbb{F}_p[x]/x^n)$ when $|x| = 2n$ and $n > 0$ and $p \nmid n$.) $\qquad\square$

Corollary 5.1.2. *(Rigidity of $S/p \wedge THH$ for Postnikov truncations) Let R be a connective E_∞-ring spectrum with $\pi_*(R) \cong \hat{\mathbb{Z}}_p[x]/x^2$, $\pi_i(R) \cong 0$ for $i \neq 0, k$. Suppose that*

$$p \not\equiv k + 1 \bmod 2k + 1.$$

Then $\pi_*(S/p \wedge THH(R))$ depends only on $\pi_*(R^{\leqslant 2k})$; i.e only on p and k.

Proof. The THH-May spectral sequence

$$(S/p)_{*,*}(THH(H\hat{\mathbb{Z}}_p \ltimes \Sigma^{2k} H\hat{\mathbb{Z}}_p)) \Rightarrow (S/p)_*(THH(R))$$

collapses since there are no possible differentials for bidegree reasons under the assumptions on k with respect to p. □

Remark 5.1.3. Corollary 5.1.2 can be considered a rigidity theorem in the sense that $S/p \wedge THH$ does not see the first Postnikov k-invariant in the cases given by the congruences above.

Corollary 5.1.4. *Let p be a prime such that $p \not\equiv 2$ mod 3, then*

$$\pi_*(S/p \wedge THH(ku_p^{\leqslant 2})) \cong E(\lambda_1) \otimes_{\mathbb{F}_p} P(\mu_1) \otimes_{\mathbb{F}_p} HH_*(E(x)),$$

where $|x| = 2$ and the degree of $HH_*(E(x))$ in π_* is given by the sum of the internal and homological degree.

5.2. **Upper bounds on the size of THH.** Many explicit computations are possible using the THH-May spectral sequence; for example, G. Angelini-Knoll's computations of topological Hochschild homology of the algebraic K-theory of finite fields, in [1]. These computations are sufficiently lengthy that they merit their own separate paper.

In lieu of explicit computations using the THH-May spectral sequence, we point out that the mere existence of the THH-May spectral sequence implies an upper bound on the size of the topological Hochschild homology groups of a ring spectrum: namely, if R is a graded-commutative ring and $X_\bullet$ is a simplicial finite set and E_* is a generalized homology theory, then for any E_∞-ring spectrum A such that $\pi_*(A) \cong R$, $E_*(X_\bullet \otimes A))$ is a subquotient of $E_*(X_\bullet \otimes HR)$. Here HR is the generalized Eilenberg-Mac Lane spectrum of the graded ring R.

In particular:

Theorem 5.2.1. *For all integers n and all connective E_∞-ring spectra A, the cardinality of $THH_n(A)$ is always less than or equal to the cardinality of $THH_n(H\pi_*(A))$.*

Below are more details in a more restricted class of examples, namely, the E_∞ ring spectra A such that $\pi_*(A) \cong \hat{\mathbb{Z}}_p[x]$.

Definition 5.2.2. *We put a partial ordering on power series with integer coefficients as follows: given $f, g \in \mathbb{Z}[[t]]$, we write $f \leqslant g$ if and only if, for all nonnegative integers n, the coefficient of t^n in f is less than or equal to the coefficient of t^n in g.*

Lemma 5.2.3 is surely not a new result:

Lemma 5.2.3. *Suppose that A is a connective E_∞-ring spectrum such that the abelian group $\pi_n(A)$ is finitely generated for all n. Suppose that $X_\bullet$ is a simplicial finite set. Then $\pi_n(X_\bullet \otimes A)$ is finitely generated for all n.*

Proof. First, a quick induction: if we have already shown that the abelian group $\pi_n(A^{\wedge m})$ is finitely generated for all n, then the Künneth spectral sequence

$$\text{Tor}_{*,*}^{\pi_*(S)}(\pi_* A, \pi_n(A^{\wedge m})) \Rightarrow \pi_*(A^{\wedge m+1})$$

is finitely generated in each bidegree and is a first-quadrant spectral sequence (with differentials according to the Serre convention), hence strongly convergent and has E_∞-page a finitely generated abelian group in each total degree. So the abelian group $\pi_n(A^{\wedge m+1})$ is also finitely generated for each n.

Consequently in the Bousfield-Kan-type spectral sequence

$$E^1_{s,t} \cong \pi_t(A^{\wedge \#(X_s)}) \Rightarrow \pi_*(X_\bullet \otimes A)$$
$$d^r : E^r_{s,t} \to E^r_{s-r,t+r-1}$$

obtained by applying π_* to the simplicial ring spectrum $X_\bullet \tilde{\otimes} A$ (here we are using the pre-tensor product, of Definition 3.3.1), each bidegree is a finitely generated abelian group, and the spectral sequence is half-plane with exiting differentials, hence also strongly convergent by Theorem 6.1 of [6]. Consequently $\pi_n(X_\bullet \otimes A)$ is a finitely generated abelian group for each integer n. □

Theorem 5.2.4. *Let n be a positive integer, p a prime number, and let E be an E_∞-ring spectrum such that $\pi_*(E) \cong \hat{\mathbb{Z}}_p[x]$, with x in grading degree $2n$. Then the Poincaré series of the mod p topological Hochschild homology $(S/p)_*(THH(E))$ satisfies the inequality*

$$\sum_{i \geqslant 0} \left(\dim_{\mathbb{F}_p}(S/p)_*(THH(E)) \right) t^i \leqslant \frac{(1 + (2p-1)t)(1 + (2n+1)t)}{(1 - 2nt)(1 - 2pt)}.$$

Proof. It is a classical computation of Bökstedt (see [7]) that

$$(S/p)_*(THH(H\hat{\mathbb{Z}}_p)) \cong E(\lambda_1) \otimes_{\mathbb{F}_p} P(\mu_1),$$

with λ_1 and μ_1 in grading degrees $2p - 1$ and $2p$ respectively.

Now we use the splitting theorem of Schwänzl, Vogt, and Waldhausen, Lemma 3.1 of [29]: if K is a commutative ring, and W is a q-cofibrant S-algebra (i.e., up to equivalence, an A_∞-ring spectrum), then there exists a weak equivalence of S-modules (not necessarily a weak equivalence of S-algebras!):

$$THH(W \wedge HK) \simeq THH(W) \wedge THH(HK) \simeq THH(W) \wedge HK \wedge_{HK} THH(HK).$$

In our case, W is the free A_∞-algebra on a single $2n$-cell, and $K = \hat{\mathbb{Z}}_p$. Hence $THH(W) \wedge HK$ satisfies

$$(S/p)_*(THH(W) \wedge HK) \cong (H\mathbb{F}_p)_*(THH(W)) \cong P(x) \otimes_{\mathbb{F}_p} E(\sigma x),$$

by collapse of the Bökstedt spectral sequence for bidegree reasons. Hence $(S/p)_*(THH(H\hat{\mathbb{Z}}_p[x]))$ is isomorphic, as a graded $\mathbb{F}_p$-vector space (but not necessarily as an $\mathbb{F}_p$-algebra!), to

$$E(\lambda_1, \sigma x) \otimes_{\mathbb{F}_p} P(\mu_1, x),$$

which has Poincaré series $\frac{(1+(2p-1)t)(1+(2n+1)t)}{(1-2nt)(1-2pt)}$. □

Here are a few amusing consequences:

Corollary 5.2.5. *Let n be a positive integer, p a prime number, and let E be an E_∞-ring spectrum such that $\pi_*(E) \cong \hat{\mathbb{Z}}_p[x]$, with x in grading degree $2n$.*

- *If p does not divide n, then $THH_{2i}(E) \cong 0$ for all i congruent to $-p$ modulo n such that $i \leqslant pn - p - n$, and $THH_{2i}(E) \cong 0$ for all i congruent to $-n$ modulo p such that $i \leqslant pn - p - n$. In particular, $THH_{2(pn-p-n)}(E) \cong 0$.*
- *If p divides n, then $THH_i(E) \cong 0$, unless i is congruent to $-1, 0$, or 1 modulo $2p$.*

Proof. We split the proof into two cases: the case where $p \nmid n$ and the case where $p | n$.

- If p does not divide n, then the largest integer i such that the graded polynomial algebra $P(\mu_1, x)$ is trivial in grading degree $2i$ is $2(pn - p - n)$. (This is a standard exercise in elementary number theory. In schools in the United States it is often presented to students in a form like "What is the largest integer N such that you cannot make exactly $5N$ cents using only dimes and quarters?") Triviality of

$P(\mu_1, x)$ in grading degree $2(pn - p - n)$ also implies triviality of $P(\mu_1, x)$ in grading degree $2(pn - p - n) - 2(p + n)$, hence the triviality of $E(\lambda_1, \sigma x) \otimes_{\mathbb{F}_p} P(\mu_1, x)$ in grading degree $2(pn - p - n)$, hence (multiplying by powers of x or μ_1) the triviality of $E(\lambda_1, \sigma x) \otimes_{\mathbb{F}_p} P(\mu_1, x)$ in all grading degrees $\leqslant 2(pn - p - n)$ which are congruent to $-2p$ modulo $2n$ or congruent to $-2n$ modulo $2p$.

So $(S/p)_{2i}(THH(E))$ vanishes if $i \leqslant pn - p - n$ and $i \equiv -p$ modulo n or $i \equiv -n$ modulo p. The long exact sequence

$$\cdots \to (S/p)_{2i+1}(THH(E)) \to THH_{2i}(E) \xrightarrow{p} THH_{2i}(E) \to (S/p)_{2i}(E) \to \dots$$

then implies that $THH_{2i}(E)$ is p-divisible. By Lemma 5.2.3, $THH_{2i}(E)$ is finitely generated. Since $\pi_0(E) \cong \hat{\mathbb{Z}}_p$, $THH_{2i}(E)$ is a $\hat{\mathbb{Z}}_p$-module. The only finitely generated abelian group which is p-divisible and admits the structure of a $\hat{\mathbb{Z}}_p$-module is the trivial group.

- If p divides n, then $E(\lambda_1, \sigma x) \otimes_{\mathbb{F}_p} P(\mu_1, x)$ is concentrated in grading degrees congruent to $-1, 0$ and 1 modulo $2p$. An argument exactly as in the previous part of this proof then shows that, if i is not congruent to $-1, 0$, or 1 modulo $2p$, then $THH_i(E)$ must be a p-divisible finitely generated abelian group which admits the structure of a $\hat{\mathbb{Z}}_p$-module, hence is trivial.

$\square$

6. Appendix: construction of the spectral sequence with coefficients.

In this appendix, we construct the spectral sequence of Theorem 3.5.5 with coefficients in a symmetric bimodule. This has proven computationally useful in the paper [1] by G. Angelini-Knoll on topological Hochschild homology of $K(\mathbb{F}_q)$.

For clarity of exposition, we gave the construction of the topological Hochschild-May spectral sequence *without* coefficients (i.e., with coefficients in the commutative monoid object itself) in section 3. The essential ideas in the construction of the spectral sequence are clearer in that case. Now we build the more general version of the spectral sequence in which we allow coefficients in a symmetric bimodule object. Since the necessary constructions are all quite similar to those of section 3, and the reader who understands section 3 will find no surprises here, we have relegated this material to an appendix.

We begin by extending Definition 3.3.1 to include coefficients. For the following definitions, let $f\,\text{Sets}_+$ be the category of pointed finite sets and basepoint preserving maps. Let $\mathcal{C}$ be a model category satisfying Running Assumptions 2.0.2.

Definition 6.0.1. *For a cofibrant commutative monoid object A in $\mathcal{C}$, we define a functor*

$$-\tilde{\otimes}(A; -) : sf\,\text{Sets}_+ \times A\text{-mod} \to sA\text{-mod},$$

which we call the pretensor product with coefficients, *as follows. If $Y_\bullet$ is a pointed simplicial finite set with basepoint $\{*_{Y_\bullet}\}$, A is a commutative monoid in $\mathcal{C}$ and M is a symmetric A-bimodule, then the simplicial object in $\mathcal{C}$ is given by:*

- *For all $n \in \mathbb{N}$, the n-simplex object $(Y_\bullet \tilde{\otimes}(A; M))_n$ is defined as*

$$(Y_\bullet \tilde{\otimes}(A; M))_n := M \wedge \bigwedge_{y \in Y_n - \{*_{Y_n}\}} A,$$

i.e, the smash product of M and a copy of A for each n-simplex $y \in Y_n - \{*_{Y_n}\}$.
- *For all positive $n \in \mathbb{N}$ and all $0 \leqslant i \leqslant n$, the i-th face map*

$$d_i : (Y_\bullet \tilde{\otimes}(A; M))_n \to (Y_\bullet \tilde{\otimes}(A; M))_{n-1}$$

is a smash product of two maps. The first map is defined as follows: for each n-simplex $y \in Y_n - \{y \in Y_n : \delta_i(y) \neq *_{Y_{n-1}}\}$, we associate a map

$$A \to \bigwedge_{\{y \in Y_{n-1} - \{*_{Y_{n-1}}\}\}} A$$

which is inclusion into the coproduct in Comm(C) of the smash factor corresponding to the $n - 1$-simplex $\delta_i(y) \in Y_{n-1} - \{*_{Y_{n-1}}\}$. The first map is then defined using the universal property of the coproduct in Comm(C) and then applying the forgetful functor to C. The second map

$$M \wedge \bigwedge_{\{y \in Y_n - \{*_{Y_n}\}: \delta_i(y) = *_{Y_{n-1}}\}} A \to M$$

is given by composing the action map of A on M with itself in the evident way.

- *For all positive $n \in \mathbb{N}$ and all $0 \leqslant i \leqslant n$, the i-th degeneracy is a smash product of two maps. On the component corresponding to a n-simplex*

$$y \in Y_n - \{y \in Y_n : \sigma_i(y) \neq *_{Y_{n+1}}\}$$

we define the map

$$A \to \bigwedge_{\{y \in Y_{n+1} - \{*_{Y_{n+1}}\}\}} A$$

as the inclusion of the smash factor corresponding to the $(n + 1)$-simplex $\sigma_i(y)$ in the coproduct in Comm(C). The first map is then defined using the universal property of the coproduct in Comm(C) and then applying the forgetful functor to C. The second map, corresponding to the i-th degeneracy on $\{*_{Y_n}\}$, is the map

$$M \wedge \bigwedge_{\{y' \in Y_n: \sigma_i(y') = *_{Y_{n+1}}\}} A \to M$$

which is given by composing the action map of A on M with itself in the evident way.

The pretensor product is defined on morphisms in the evident way as in Definition 3.3.1.

For A a commutative monoid in C, we define the tensor product with coefficients

$$- \otimes (A; -) : sf\, \mathrm{Sets}_+ \times A\text{-mod} \to sA\text{-mod}$$

to be the geometric realization of the pretensor product:

$$Y_\bullet \otimes (A; M) = |Y_\bullet \tilde{\otimes}(A; M)|.$$

One can check that when M is a commutative symmetric A-bimodule algebra; i.e., the multiplication map is a map of A-bimodules and the unit $S \to M$ factors through the unit $S \to A$, then $Y_\bullet \otimes (A; M)$ is an object in Comm C.

As one would expect, if $Y_\bullet = (\Delta[1]/\delta\Delta[1])_\bullet$ where the basepoint is $\Delta[0] \subset \Delta[1]$, then $Y_\bullet \otimes (A; M)$ is identified with $THH(A; M)$; i.e. usual topological Hochschild homology with coefficients. We note that $THH(A; M)$ will be a module over $THH(A)$.

Definition 6.0.2. (Some important colimit diagrams with coefficients I.)

- – *Let S and T be finite sets with distinguised basepoints $*_S$ and $*_T$ respectively, and suppose there is a basepoint preserving map $f : T \to S$. We can equip $\mathbb{N}^S$ with the strict direct product order as in Definition 3.4.2 and define a function of partially-ordered sets $\mathbb{N}_+^f : \mathbb{N}^T \to \mathbb{N}^S$ by*

$$(\mathbb{N}_+^f(x))(s) = \sum_{\{t \in T: f(t) = s\}} x(t)$$

as before. (The only way in which this differs from Definition 3.4.2 is that we are assured that $(\mathbb{N}^f_+(x))(*_S)$ has $x(*_T)$ as a summand.) This defines a functor

$$\mathbb{N}^{(-)}_+ : f\,\mathrm{Sets}_+ \to \mathrm{POSets}\,.$$

– *As in Definition 3.4.2, $\mathbb{N}^f_+$ will preserve the evident L^1 norm.*

Definition 6.0.3. (Some important colimit diagrams with coefficients II.)

- – *When S is a pointed set, let $\mathcal{D}^S_n$ be the subposet of $\mathbb{N}^S$ consisting of $x \in \mathbb{N}^S$ such that $|x| \geqslant n$.*
  – *A basepoint preserving function between finite pointed sets, $T \to S$, induces a map $\mathcal{D}^T_n \xrightarrow{\mathcal{D}^f_{n+}} \mathcal{D}^S_n$ of partially-ordered sets by restriction of $\mathbb{N}^f_+$.*
  – *For each $n \in \mathbb{N}$, this defines a functor*

  $$\mathcal{D}^-_{n\ +} : f\,\mathrm{Sets}_+ \to \mathrm{POSets}$$

  from the category of finite pointed sets to the category of partially-ordered sets.

- – *Let S be a pointed finite set. For each $x \in \mathbb{N}^S$ and each $n \in \mathbb{N}$, let $\mathcal{D}^S_{n;x}$ denote the following sub-poset of $\mathbb{N}^S$:*

  $$\mathcal{D}^S_{n;x} = \left\{ y \in \mathbb{N}^S : y \geqslant x,\ and\ |y| \geqslant n + |x| \right\}$$

  as in Definition 3.4.3.
  – *If $T \xrightarrow{f} S$ is a basepoint preserving function between finite pointed sets and $x \in \mathbb{N}^T$ and $n \in \mathbb{N}$, let $\mathcal{D}^T_{n;x} \xrightarrow{\mathcal{D}^f_{n;x+}} \mathcal{D}^S_{n;\mathcal{D}^f_n(x)}$ be the function of partially-ordered sets defined by restricting $\mathbb{N}^f_+$ to $\mathcal{D}^T_{n;x}$.*
  For each $n \in \mathbb{N}$ and each $x \in \mathbb{N}^T$, this defines a functor

  $$\mathcal{D}^-_{n;x+} : f\,\mathrm{Sets}_+ \to \mathrm{POSets}$$

  from the category of finite pointed sets to the category of partially-ordered sets.

Definition 6.0.4. (Some important colimit diagrams with coefficients III.)

- *Let S be a finite pointed set and let n be a nonnegative integer. Write $\mathcal{E}^n_S$ for the set*

  $$\mathcal{E}^n_S = \left\{ x \in \{0, 1, \ldots, n\}^S : \sum_{s \in S} x(s) \geqslant n \right\}.$$

  We partially-order $\mathcal{E}^n_S$ by the strict direct product order, i.e., $x' \leqslant x$ if and only if $x'(s) \leqslant x(s)$ for all $s \in S$.

- *The definition of $\mathcal{E}^n_S$ is natural in S in the following sense: if $T \xrightarrow{f} S$ is a basepoint preserving map of finite pointed sets and $x \in \mathbb{N}^T$, we have a map of partially-ordered sets*

  $$\mathcal{E}^f_{n;x+} : \mathcal{E}^T_n \quad \to \quad \mathcal{E}^S_n$$

  $$\left(\mathcal{E}^f_{n;x}(y) \right)(s) \quad = \quad \min \left\{ n, y(s) - \sum_{\{t \in T, f(t) = s\}} (x(t) + y(t)) \right\}$$

  Note that this functor depends on the choice of $x \in \mathbb{N}^T$. Functoriality follows in the same way as in Definition 3.4.4.

(The only way in which this differs from Definition 3.4.4 is that when we evalu-ate on the basepoint of S,

$$\left(\mathcal{E}_{n;x_+}^f(y)\right)(*_S) = \min\left\{n, y(*_S) - \sum_{\{t \in T_+, f(t) = *_S\}} (x(t) + y(t))\right\}$$

the sum will be nonempty because it contains $x(*_T) + y(*_T)$.)

Definition 6.0.5. (Some important colimit diagrams with coefficients IV.)

- – *Let $(I_\bullet, M_\bullet)$ be a pair with $I_\bullet$ a cofibrant decreasingly filtered commutative monoid in C and $M_\bullet$ a cofibrant decreasingly filtered symmetric $I_\bullet$-module. Let S be a pointed set with basepoint $*_S$. In this case, let*

    $$\mathcal{F}^S(I_\bullet, M_\bullet) : \left(\mathbb{N}^S\right)^{\mathrm{op}} \to C$$

    be the functor sending x to

    $$M_{x(*_S)} \wedge \wedge_{s \in S - \{*_S\}} I_{x(s)},$$

    and let

    $$\mathcal{F}_n^S(I_\bullet, M_\bullet) : \left(\mathcal{D}_n^S\right)^{\mathrm{op}} \to C$$

    be the composite of $\mathcal{F}^S(I_\bullet, M_\bullet)$ with the inclusion of $\mathcal{D}_n^S$ into $\mathbb{N}^S$:

    $$\left(\mathcal{D}_n^S\right)^{\mathrm{op}} \hookrightarrow \left(\mathbb{N}^S\right)^{\mathrm{op}} \xrightarrow{\mathcal{F}^S(I_\bullet, M_\bullet)} C.$$

  - *For $x \in \mathcal{D}_n^S$, we write $\mathcal{F}_{n;x}^S(I_\bullet, M_\bullet)$ for the restriction of the diagram $\mathcal{F}^S(I_\bullet, M_\bullet)$ to $\mathcal{D}_{n;x}^S$, i.e. $\mathcal{F}_{n;x}^S(I_\bullet, M_\bullet)$ is the composite*

    $$\left(\mathcal{D}_{n;x}^S\right)^{\mathrm{op}} \hookrightarrow \left(\mathbb{N}^S\right)^{\mathrm{op}} \xrightarrow{\mathcal{F}^S(I_\bullet, M_\bullet)} C.$$

  - *Finally, let $\mathcal{M}_n^S(I_\bullet, M_\bullet)$ denote the colimit*

    $$\mathcal{M}_n^S(I_\bullet, M_\bullet) = \mathrm{colim}\left(\mathcal{F}_n^S(I_\bullet, M_\bullet)\right)$$

    in C. We get a sequence in C induced by the natural inclusion of $\mathcal{D}_n^S$ into $\mathcal{D}_{n-1}^S$,

(6.0.1) $$\cdots \to \mathcal{M}_2^S(I_\bullet, M_\bullet) \to \mathcal{M}_1^S(I_\bullet, M_\bullet) \to \mathcal{M}_0^S(I_\bullet, M_\bullet) \cong M_0 \wedge \wedge_{s \in S - \{*_S\}} I_0.$$

    We refer to the functor $\mathbb{N}^{\mathrm{op}} \to C$ given by sending n to $\mathcal{M}_n^S(I_\bullet, M_\bullet)$ as the May filtration with coefficients.

  - *The May filtration with coefficients is functorial in S in the following sense: if $T \xrightarrow{f} S$ is a basepoint preserving function of finite pointed sets, we have a functor*

    $$\mathcal{D}_{n_+}^f : \mathcal{D}_n^T \quad \to \quad \mathcal{D}_n^S$$
    $$\left(\mathcal{D}_{n_+}^f(x)\right)(s) \quad \mapsto \quad \sum_{\{t \in T : f(t) = s\}} x(t)$$

    and a map of diagrams from $\mathcal{F}_n^T(I_\bullet, M_\bullet)$ to $\mathcal{F}_n^S(I_\bullet, M_\bullet)$ given by sending the object

    $$\mathcal{F}_n^T(I_\bullet, M_\bullet)(x) = M_{x(*_T)} \wedge \bigwedge_{t \in T - \{*_T\}} I_{x(t)}$$

    by the map

$$M_{x(*_T)} \wedge \bigwedge_{t \in T - \{*_T\}} I_{x(t)} \to M_{\Sigma_{\{t \in T : f(t) = *_S\}} x(t)} \wedge \bigwedge_{s \in S - \{*_S\}} I_{\Sigma_{\{t \in T - \{*_T\} : f(t) = s\}} x(t)}$$

given as the smash product, across all $s \in S$, of the maps

$$\wedge_{\{t \in T - \{*_T\}: f(t) = s\}} I_{x(t)} \rightarrow \wedge I_{\Sigma_{\{t \in T - \{*_T\}: f(t) = s\}} x(t)}$$

given by multiplication via the maps ρ of Definition 3.1.2 and the maps

$$M_{x(*_T)} \wedge \wedge_{\{t \in T - \{*_T\}: f(t) = *_S\}} I_{x(t)} \rightarrow M_{\Sigma_{\{t \in T: f(t) = *_S\}} x(t)}$$

given by module maps ψ of Definition 3.2.1.

To really make Definition 6.0.2 precise, we should say in which order we multiply the factors using the maps ρ and ψ; but the purpose of the associativity and commutativity axioms in Definition 3.1.2 and Definition 3.2.1 is that any two such choices commute, so any choice of order of multiplication will do.

Definition 6.0.6. (Definition of May filtration with coefficients.) *Let $(I_\bullet, M_\bullet)$ be a pair with $I_\bullet$ a cofibrant decreasingly filtered commutative monoid in C and $M_\bullet$ a filtered symmetric $I_\bullet$-bimodule in C. Let $Y_\bullet$ be a pointed simplicial finite set. By the* May filtration with coefficients on $Y_\bullet \tilde{\otimes}(I_0; M_0)$ *we mean the functor*

$$\mathcal{M}^{Y_\bullet}(I_\bullet, M_\bullet) : \mathbb{N}^{\mathrm{op}} \rightarrow sC$$

given by sending a natural number n to the simplicial object of C

$$\mathcal{M}_n^{Y_0}(I_\bullet, M_\bullet) \rightleftarrows \mathcal{M}_n^{Y_1}(I_\bullet, M_\bullet) \rightleftarrows \mathcal{M}_n^{Y_2}(I_\bullet, M_\bullet) \rightleftarrows \cdots$$

with $\mathcal{M}_n^{Y_i}(I_\bullet, M_\bullet)$ defined as in Definition 6.0.2, and with face and degeneracy maps defined as follows:

- *The face map*

$$d_i : \mathcal{M}_n^{Y_j}(I_\bullet, M_\bullet) \rightarrow \mathcal{M}_n^{Y_{j-1}}(I_\bullet, M_\bullet)$$

  is the colimit of the map of diagrams

$$\mathcal{F}_n^{Y_j}(I_\bullet, M_\bullet) \rightarrow \mathcal{F}_n^{Y_{j-1}}(I_\bullet, M_\bullet)$$

  induced, as in Definition 6.0.2, by $\delta_i : Y_j \rightarrow Y_{j-1}$.
- *The degeneracy map*

$$s_i : \mathcal{M}_n^{Y_j}(I_\bullet, M_\bullet) \rightarrow \mathcal{M}_n^{Y_{j+1}}(I_\bullet, M_\bullet)$$

  is the colimit of the map of diagrams

$$\mathcal{F}_n^{Y_j}(I_\bullet, M_\bullet) \rightarrow \mathcal{F}_n^{Y_{j+1}}(I_\bullet, M_\bullet)$$

  induced, as in Definition 6.0.2, by $\sigma_i : Y_j \rightarrow Y_{j+1}$. The only way in which this differs from Definition 3.4.5 is that the maps δ_i and σ_i are now the basepoint preserving structure maps of the pointed simplicial object $Y_\bullet$.

Remark 6.0.7. Note that the maps ρ of Definition 3.1.2 and ψ of Definition 3.2.1 yield, by taking smash products of the maps ρ and ψ associative and symmetric bimodule maps,

$$\mathcal{F}_m^S(I_\bullet) \wedge \mathcal{F}_n^S(I_\bullet, M_\bullet) \rightarrow \mathcal{F}_{n+m}^S(I_\bullet, M_\bullet)$$

hence after taking colimits, we produce maps

$$\mathcal{M}_m^S(I_\bullet) \wedge \mathcal{M}_n^S(I_\bullet, M_\bullet) \rightarrow \mathcal{M}_{n+m}^S(I_\bullet, M_\bullet);$$

i.e. the functor

$$\mathbb{N}^{\mathrm{op}} \rightarrow C$$
$$n \mapsto \mathcal{M}_n^S(I_\bullet, M_\bullet)$$

is a cofibrant decreasingly filtered symmetric $\mathcal{M}_n^S(I_\bullet)$-bimodule in the sense of Definition 3.2.1. The same considerations as in Remark 3.4.7 give $|\mathcal{M}^{X_\bullet}(I_\bullet, M_\bullet)|$ the structure of a cofibrant decreasingly filtered symmetric $|\mathcal{M}^{X_\bullet}(I_\bullet)|$-bimodule in the sense of Definition 3.2.1.

We now need to adapt Lemmas 3.4.8, 3.4.9, and 3.4.10 to our situtation.

Lemma 6.0.8. *Let $I_\bullet$ be a cofibrant decreasingly filtered commutative monoid in C and let $M_\bullet$ be a cofibrant decreasingly filtered symmetric $I_\bullet$ bimodule. Let S be a pointed set and $n \in \mathbb{N}$. There is a monomorphism $\iota : \mathcal{D}_{n+1}^S \to \mathcal{D}_n^S$. We write*

$$Kan : C^{(\mathcal{D}_n^S)^{op}} \to C^{(\mathcal{D}_{n+1}^S)^{op}}$$

for the left Kan extension of $\mathcal{F}_{n+1}^S(I_\bullet, M_\bullet)$ induced by ι^{op} and define

$$\tilde{\mathcal{F}}_{n+1}^S(I_\bullet, M_\bullet) := Kan\left(\mathcal{F}_{n+1}^S(I_\bullet, M_\bullet)\right).$$

The universal property of the Kan extension produces a canonical map

$$c : \tilde{\mathcal{F}}_{n+1}^S(I_\bullet, M_\bullet) \to \mathcal{F}_{n+1}^S(I_\bullet, M_\bullet).$$

With these definitions, the cofiber of the map

$$colim(\tilde{\mathcal{F}}_{n+1}^S(I_\bullet, M_\bullet)) \xrightarrow{colim\, c} colim\mathcal{F}_n^S(I_\bullet, M_\bullet),$$

where colimits are computed in C, is isomorphic to the coproduct in C

$$\coprod_{\{x \in \mathbb{N}^S : |x| = n\}} \left(\left(M_{x(*_S)} \wedge \wedge_{s \in S - \{*_S\}} I_{x(s)}\right) / \left(colim\mathcal{F}_{1;x}^S(I_\bullet, M_\bullet)\right)\right).$$

This isomorphism is natural in the variable S.

Proof. We omit the proof because it follows from an evident generalization of Lemma 3.4.8.
□

Lemma 6.0.9. *Let S be a finite pointed set. Suppose the map $Y_{*_S,1} \to Y_{*_S,0}$ is a cofibration and for $s \in S - \{*_S\}$ the maps $Z_{s,1} \to Z_{s,0}$ are cofibrations. Suppose the objects $Y_{*_S,1}$ and $Y_{*_S,0}$ are cofibrant and that, for $s \in S - \{*_S\}$, $Z_{s,1}$ and $Z_{s,0}$ are cofibrant. Let $\mathcal{G}_S^+ : (\mathcal{E}_1^S)^{op} \to C$ be the functor given on objects by*

$$\mathcal{G}_S^+(x) = Y_{*_S,x(*_s)} \wedge \wedge_{s \in S} Z_{s,x(s)}$$

and given on morphisms in the obvious way. Then the smash product

$$Y_{*_S,0} \wedge \wedge_{s \in S - \{*_S\}} Z_{s,0} \to Y_{*_S,0}/Y_{*_S,1} \wedge \wedge_{s \in S - \{*_S\}} Z_{s,0}/Z_{s,1}$$

of the cofiber projections $Z_{s,0} \to Z_{s,0}/Z_{s,1}$ and $Y_{*_S,0} \to Y_{*_S,0}/Y_{*_S,1}$ fits into a cofiber sequence:

$$colim\mathcal{G}_S^+ \to Y_{*_S,0} \wedge \wedge_{s \in S - \{*_S\}} Z_{s,0} \to Y_{*_S,0}/Y_{*_S,1} \wedge \wedge_{s \in S - \{*_S\}} Z_{s,0}/Z_{s,1}.$$

Proof. Letting $Z_{*_S,i} = Y_{*_S,i}$ for $i = 0, 1$, we can prove this lemma in almost the same way as Lemma 3.4.9. The only difference is that in the proof we need to choose $s_0 \in S - \{*_S\}$ so that $*_S$ is in $S' = S - \{s_0\}$.
□

Lemma 6.0.10. *Suppose S is a pointed finite set and $n \in \mathbb{N}$. Let $x \in \mathbb{N}^S$ and $\mathcal{E}_n^S$ and $\mathcal{D}_{n;x}^S$ be as in Definition 6.0.2 Let $J_{n;x}^S$ be the functor defined by*

$$J_{n;x}^S : \mathcal{E}_n^S \to \mathcal{D}_{n;x}^S$$
$$(J_{n;x}(y))(s) = x(s) + y(s).$$

Then $J_{n;x}$ has a right adjoint. Consequently $J_{n;x}$ is a cofinal functor; i.e. for any functor defined on $\mathcal{D}^S_{n;x}$ such that the limit $\lim F$ exists, the limit $\lim(F \circ J^S_{n;x})$ also exists, and the canonical map $\lim(F \circ J^S_{n;x}) \to \lim F$ is an isomorphism.

Proof. The proof follows easily from the evident generalization of the proof of Lemma 3.4.10.

$\square$

Theorem 6.0.11. (Fundamental theorem of the May filtration with coefficients.) *Let $I_\bullet$ be a cofibrant decreasingly filtered commutative monoid in $\mathcal{C}$, let $M_\bullet$ be a cofibrant decreasingly filtered symmetric $I_\bullet$-bimodule, and let $Y_\bullet$ be a simplicial pointed finite set. Then the associated graded commutative monoid $E_0^* \left| \mathcal{M}^{Y_\bullet}(I_\bullet, M_\bullet) \right|$ of the geometric realization of the May filtration is weakly equivalent to the tensoring $Y_\bullet \otimes (E_0^* I_\bullet; E_0^* M_\bullet)$ of $Y_\bullet$ with the associated graded commutative monoid of $I_\bullet$ with coefficients in the associated graded symmetric $E_0^* I_\bullet$-bimodule:*

$$(6.0.2) \qquad E_0^* \left| \mathcal{M}^{Y_\bullet}(I_\bullet, M_\bullet) \right| \simeq Y_\bullet \otimes (E_0^* I_\bullet; E_0^* M_\bullet).$$

Proof. Since geometric realization commutes with cofibers, there is an equivalence

$$|\mathcal{M}^{Y_\bullet}_n(I_\bullet, M_\bullet)| / |\mathcal{M}^{Y_\bullet}_{n+1}(I_\bullet, M_\bullet)| \simeq |\mathcal{M}^{Y_\bullet}_n(I_\bullet, M_\bullet)/\mathcal{M}^{Y_\bullet}_{n+1}(I_\bullet, M_\bullet)|$$

and we would like to identify this cofiber. Each Y_i is some finite pointed set, so we will compute the cofiber of the map

$$\mathcal{M}^{Y_\bullet}_{n+1}(I_\bullet, M_\bullet) \to \mathcal{M}^{Y_\bullet}_n(I_\bullet, M_\bullet)$$

on each simplicial level as follows. We claim that for any finite pointed set S
$$(6.0.3)$$
$$\mathcal{M}^S_{n+1}(I_\bullet, M_\bullet) \to \mathcal{M}^S_n(I_\bullet, M_\bullet) \to \coprod_{x \in \mathbb{N}^S; |x|=n} M_{x(*_S)}/M_{x(*_S)+1} \wedge \bigwedge_{s \in S - \{*_S\}} I_{x(s)}/I_{x(s)+1}$$

is a cofiber sequence. To prove this claim, first note that $\mathcal{M}^S_{n+1}$ is defined to be $\operatorname{colim} \mathcal{F}^S_{n+1}(I_\bullet, M_\bullet)$. By Lemma 6.0.8, we can identify the cofiber of the left map in Equation 6.0.3 as the cofiber of the left map in the diagram:

$$\operatorname{colim} \widetilde{\mathcal{F}}^S_{n+1}(I_\bullet, M_\bullet) \longrightarrow \operatorname{colim} \mathcal{F}^S_n(I_\bullet, M_\bullet) \longrightarrow \coprod_{\{x \in \mathbb{N}^S; |x|=n\}} \left(M_{*_S} \wedge \bigwedge_{s \in S - \{*_S\}} I_{x(s)} \right) / \left(\operatorname{colim} \mathcal{F}^S_{1;x}(I_\bullet, M_\bullet) \right).$$

Lemma 6.0.8 also demonstrates naturality in the variable S. By Lemma 6.0.10, the functor $J_{1;x}$ is cofinal and hence the map

$$\operatorname{colim} (\mathcal{F}^S_{1,x}(I_\bullet, M_\bullet) \circ J_{1;x}) \to \operatorname{colim} (\mathcal{F}^S_{1;x}(I_\bullet, M_\bullet)$$

is an isomorphism. (We are applying the dual of the statement of Lemma 6.0.10, which also holds.) By Lemma 6.0.9, we identify

$$\left(M_{x(*_S)} \wedge \wedge_{s \in S - \{*_S\}} I_{x(s)} \right) / \operatorname{colim} (\mathcal{F}_{1;x}(I_\bullet; M_\bullet) \circ J_{1;x}) \simeq \left(M_{x(*_S)}/M_{x(*_S)+1} \wedge \wedge_{s \in S - \{*_S\}} I_{x(s)}/I_{x(s)+1} \right)$$

as we needed to prove the cofiber sequence of Equation 6.0.3. The same considerations as in the proof of Theorem 3.4.11 apply, producing naturality in S.

We have a sequence of simplicial objects in $\mathcal{C}$

$$
\begin{array}{cccc}
\vdots & \vdots & \vdots & \\
\downarrow & \downarrow & \downarrow & \\
\mathcal{M}_2^{Y_0}(I_\bullet, M_\bullet) \rightrightarrows \mathcal{M}_2^{Y_1}(I_\bullet, M_\bullet) \rightrightarrows \mathcal{M}_2^{Y_2}(I_\bullet, M_\bullet) \rightrightarrows \cdots \\
\downarrow & \downarrow & \downarrow & \\
\mathcal{M}_1^{Y_0}(I_\bullet, M_\bullet) \rightrightarrows \mathcal{M}_1^{Y_1}(I_\bullet, M_\bullet) \rightrightarrows \mathcal{M}_1^{Y_2}(I_\bullet, M_\bullet) \rightrightarrows \cdots \\
\downarrow & \downarrow & \downarrow & \\
\mathcal{M}_0^{Y_0}(I_\bullet, M_\bullet) \rightrightarrows \mathcal{M}_0^{Y_1}(I_\bullet, M_\bullet) \rightrightarrows \mathcal{M}_0^{Y_2}(I_\bullet, M_\bullet) \rightrightarrows \cdots
\end{array}
$$

and geometric realization commuting with cofibers implies that the comparison map

$$(6.0.4) \qquad Y_\bullet \otimes (E_0^* I_\bullet; E_0^* M_\bullet) \rightarrow E_0^* |\mathcal{M}^{Y_\bullet}(I_\bullet, M_\bullet)|$$

of objects in $\mathcal{C}$ is a weak equivalence. If $M_\bullet = I_\bullet$, then we recover Equation 3.4.7. $\qquad\square$

Remark 6.0.12. One can show that the weak equivalence of Equation 6.0.2 in Theorem 6.0.11 is an equivalence of symmetric $E_0^* |\mathcal{M}^{Y_\bullet}(I_\bullet)|$-bimodules. Also, in the cases when both sides are objects in $\mathrm{Comm}(\mathcal{C})$ and $M_\bullet$ is a cofibrant decreasingly filtered commutative monoid in $\mathcal{C}$ as well as a cofibrant decreasingly filtered symmetric $I_\bullet$-bimodule, one can show that the equivalence is an equivalence of commutative monoids in $\mathcal{C}$.

Definition 6.0.13. *Suppose $I_\bullet$ is a cofibrant decreasingly filtered commutative monoid object in $\mathcal{C}$, $M_\bullet$ is a cofibrant decreasingly filtered symmetric $I_\bullet$-bimodule, and $Y_\bullet$ is a simplicial pointed finite set. Let H_* be a connective generalized homology theory on $\mathcal{C}$ as defined in Definition 3.5.1, then by the topological Hochschild-May spectral sequence for $Y_\bullet \otimes (I_\bullet; M_\bullet)$, we mean the spectral sequence in $\mathcal{C}$ obtained by applying H_* to the tower of cofiber sequences in $\mathcal{C}$*

$$
\begin{array}{ccc}
\vdots & & \\
\downarrow & & \\
\left|\mathcal{M}_2^{Y_\bullet}(I_\bullet, M_\bullet)\right| & \longrightarrow & \left|\mathcal{M}_2^{X_\bullet}(I_\bullet, M_\bullet)\right| / \left|\mathcal{M}_3^{X_\bullet}(I_\bullet, M_\bullet)\right| \\
\downarrow & & \\
\left|\mathcal{M}_1^{Y_\bullet}(I_\bullet, M_\bullet)\right| & \longrightarrow & \left|\mathcal{M}_1^{X_\bullet}(I_\bullet, M_\bullet)\right| / \left|\mathcal{M}_2^{X_\bullet}(I_\bullet, M_\bullet)\right| \\
\downarrow & & \\
\left|\mathcal{M}_0^{Y_\bullet}(I_\bullet, M_\bullet)\right| & \longrightarrow & \left|\mathcal{M}_0^{X_\bullet}(I_\bullet, M_\bullet)\right| / \left|\mathcal{M}_1^{X_\bullet}(I_\bullet, M_\bullet)\right|.
\end{array}
$$

The spectral sequence we refer to is the one associated to the exact couple

$$D^1_{*,*} \cong \bigoplus_{i,j} H_i \left| \mathcal{M}^{Y_\bullet}_j(I_\bullet, M_\bullet) \right| \longrightarrow \bigoplus_{i,j} H_i \left| \mathcal{M}^{Y_\bullet}_j(I_\bullet, M_\bullet) \right| \cong D^1_{*,*}$$

$$E^1_{*,*} \cong \bigoplus_{i,j} H_i \left| \mathcal{M}^{Y_\bullet}_j(I_\bullet, M_\bullet) \right| / \left| \mathcal{M}^{Y_\bullet}_{j+1}(I_\bullet, M_\bullet) \right|$$

.

Theorem 6.0.14. *Given $I_\bullet$, $M_\bullet$, $Y_\bullet$ and H_* as in Definition 6.0.13. Suppose that $I_\bullet$ and $M_\bullet$ are Hausdorff as cofibrant decreasingly filtered objects in C. Suppose $I_\bullet$ and $M_\bullet$ satisfy*

Connectivity axiom: $H_m(I_n) \cong 0$ *for all $m < n$ and $H_m(M_\ell) \cong 0$ for all $m < \ell$.*

Then the topological Hochschild-May spectral sequence is strongly convergent, and its input and output and differential are as follows:

$$E^1_{s,t} \cong H_{s,t}(Y_\bullet \otimes (E^*_0 I_\bullet, E^*_0 M_\bullet)) \Rightarrow H_s(Y_\bullet \otimes (I_0, M_0))$$

$$d^r : E^r_{s,t} \to E^r_{s-1,\, t+r-1}$$

Proof. We need to check that $\left[\Sigma^* Z, \mathrm{holim}_i \left(H \wedge \left| \mathcal{M}^{Y_\bullet}_i(I_\bullet, M_\bullet) \right| \right) \right]$ is trivial, but by an evident generalization of Lemma 3.5.4 $H_m \left(\left| \mathcal{M}^{Y_\bullet}_i(I_\bullet, M_\bullet) \right| \right) \cong 0$ for all $m < i$, so by Lemma 3.5.3, $\left[\Sigma^* Z, \mathrm{holim}_i \left(H \wedge \left| \mathcal{M}^{Y_\bullet}_i(I_\bullet, M_\bullet) \right| \right) \right] \cong 0$ as desired. Thus, the spectral sequence converges to $H_* \left(\left| \mathcal{M}^{Y_\bullet}_0(I_\bullet, M_\bullet) \right| \right) \cong H_*(X_\bullet \otimes I_0)$. A routine computation in the spectral sequence of a tower of cofibrations implies the bidegree of the differential.

The sequence

$$\cdots \longrightarrow \left| \mathcal{M}^{Y_\bullet}_2(I_\bullet, M_\bullet) \right| \longrightarrow \left| \mathcal{M}^{Y_\bullet}_1(I_\bullet, M_\bullet) \right| \longrightarrow \left| \mathcal{M}^{Y_\bullet}_0(I_\bullet, M_\bullet) \right|$$

is a cofibrant decreasingly filtered commutative monoid in C whenever $M_\bullet$ is a cofibrant decreasingly filtered commutative monoid in C compatible with the $I_\bullet$-action as observed in Remark 3.4.7 and therefore, in particular, there is a pairing of towers in the sense of [11] under these conditions. Therefore by Proposition 5.1 of [11] the differentials in the spectral sequence satisfy a graded Leibniz rule when $M_\bullet$ is a cofibrant decreasingly filtered commutative monoid in C compatible with the $I_\bullet$-action.

Convergence is standard and follows as stated in the proof of Theorem 3.5.5. $\square$

We conclude with an example. Suppose $I_\bullet$ is a trivially filtered commutative monoid in C; i.e., $I_n \simeq 0$ for $n \geqslant 1$. Suppose $M_\bullet$ is a cofibrant decreasingly filtered symmetric $I_\bullet$-bimodule object in C with $M_n \simeq 0$ for $n \geqslant 2$. Then the sequence of simplicial commutative monoids becomes

$$\mathcal{M}^{Y_0}_1(I_\bullet, M_\bullet) \rightrightarrows \mathcal{M}^{Y_1}_1(I_\bullet, M_\bullet) \Rrightarrow \mathcal{M}^{Y_2}_1(I_\bullet, M_\bullet) \Rrightarrow \cdots$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$\mathcal{M}^{Y_0}_0(I_\bullet, M_\bullet) \rightrightarrows \mathcal{M}^{Y_1}_0(I_\bullet, M_\bullet) \Rrightarrow \mathcal{M}^{Y_2}_0(I_\bullet, M_\bullet) \Rrightarrow \cdots$$

where the realization of $\mathcal{M}^{Y_\bullet}_0(I_\bullet, M_\bullet)$ is $Y_\bullet \otimes (I_0; M_0)$, the realization of $\mathcal{M}^{Y_\bullet}_1(I_\bullet, M_\bullet)$ is $Y_\bullet \otimes (I_0; M_1)$ and the realization of the quotient $\mathcal{M}^{Y_\bullet}_0(I_\bullet, M_\bullet)/\mathcal{M}^{Y_\bullet}_1(I_\bullet, M_\bullet)$ is $\mathcal{M}^{Y_\bullet}(I_0, M_0/M_1)$. The spectral sequence collapses to produce a long exact sequence coming from the cofiber sequence

$$Y_\bullet \otimes (I_0; M_1) \to Y_\bullet \otimes (I_0; M_0) \to Y_\bullet \otimes (I_0, M_0/M_1).$$

When $Y_\bullet = \Delta[1]/\delta\Delta[1]$ with the obvious basepoint, this specializes to a cofiber sequence,

$$THH(I_0, M_1) \to THH(I_0; M_0) \to THH(I_0, M_0/M_1)$$

which recovers a folklore result; i.e., a cofiber sequence in coefficient bimodules induces a cofiber sequence in topological Hochschild homology. This seems to be well known, but we do not know where it appears explicitly in the literature.

References

[1] G. Angelini-Knoll. On topological Hochschild homology of the $k(1)$-local sphere, Submitted. available on arXiv.

[2] G. Angelini-Knoll and A. Salch. Maps of simplicial spectra whose realizations are cofibrations, Submitted to Homology, Homotopy and Applications. available on arXiv.

[3] Vigleik Angeltveit. On the algebraic K-theory of Witt vectors of finite length. *preprint, available on arXiv*, 2011.

[4] Christian Ausoni. Topological Hochschild homology of connective complex K-theory. *Amer. J. Math.*, 127(6):1261–1313, 2005.

[5] M. Basterra. André-Quillen cohomology of commutative S-algebras. *J. Pure Appl. Algebra*, 144(2):111–143, 1999.

[6] J. Michael Boardman. Conditionally convergent spectral sequences. In *Homotopy invariant algebraic structures (Baltimore, MD, 1998)*, volume 239 of *Contemp. Math.*, pages 49–84. Amer. Math. Soc., Providence, RI, 1999.

[7] M. Bökstedt. The topological Hochschild homology of $\mathbb{Z}$ and of $\mathbb{Z}/p\mathbb{Z}$. preprint, 1987.

[8] M. Brun. Topological Hochschild homology of $\mathbb{Z}/p^n$. *J. Pure Appl. Algebra*, 148(1):29–76, 2000.

[9] Brian J. Day. *Construction of biclosed categories*. PhD thesis, University of New South Wales, 1970.

[10] Daniel Dugger. A primer on homotopy colimits. available online.

[11] Daniel Dugger. Multiplicative structures on homotopy spectral sequences I, 2003. available on arXiv.

[12] Samuel Eilenberg and Norman E. Steenrod. Axiomatic approach to homology theory. *Proc. Nat. Acad. Sci. U. S. A.*, 31:117–120, 1945.

[13] Saul Glasman. *Day convolution and the Hodge filtration on THH*. ProQuest LLC, Ann Arbor, MI, 2015. Thesis (Ph.D.)–Massachusetts Institute of Technology.

[14] John E. Harper. Bar constructions and Quillen homology of modules over operads. *Algebr. Geom. Topol.*, 10(1):87–136, 2010.

[15] Mark Hovey, Brooke Shipley, and Jeff Smith. Symmetric spectra. *J. Amer. Math. Soc.*, 13(1):149–208, 2000.

[16] Samuel Baruch Isaacson. *Cubical homotopy theory and monoidal model categories*. ProQuest LLC, Ann Arbor, MI, 2009. Thesis (Ph.D.)–Harvard University.

[17] A. Lazarev. Cohomology theories for highly structured ring spectra. In *Structured ring spectra*, volume 315 of *London Math. Soc. Lecture Note Ser.*, pages 201–231. Cambridge Univ. Press, Cambridge, 2004.

[18] Jean-Louis Loday. Opérations sur l'homologie cyclique des algèbres commutatives. *Invent. Math.*, 96(1):205–230, 1989.

[19] Jacob Lurie. Derived algebraic geometry IV: Deformation theory, 2007.

[20] Jacob Lurie. *Higher topos theory*, volume 170 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2009.

[21] Saunders Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1998.

[22] Michael A. Mandell. Topological André-Quillen cohomology and E_∞ André-Quillen cohomology. *Adv. Math.*, 177(2):227–279, 2003.

[23] J. P. May. The additivity of traces in triangulated categories. *Adv. Math.*, 163(1):34–73, 2001.

[24] J. Peter May. *THE COHOMOLOGY OF RESTRICTED LIE ALGEBRAS AND OF HOPF ALGEBRAS: APPLICATION TO THE STEENROD ALGEBRA*. ProQuest LLC, Ann Arbor, MI, 1964. Thesis (Ph.D.)–Princeton University.

[25] J. McClure, R. Schwänzl, and R. Vogt. $THH(R) \cong R \otimes S^1$ for E_∞ ring spectra. *J. Pure Appl. Algebra*, 121(2):137–159, 1997.

[26] J. E. McClure and R. E. Staffeldt. On the topological Hochschild homology of bu. I. *Amer. J. Math.*, 115(1):1–45, 1993.

[27] C. A. McGibbon. Phantom maps. In *Handbook of algebraic topology*, pages 1209–1257. North-Holland, Amsterdam, 1995.

[28] Teimuraz Pirashvili. Hodge decomposition for higher order Hochschild homology. *Ann. Sci. École Norm. Sup. (4)*, 33(2):151–179, 2000.

[29] R. Schwänzl, R. M. Vogt, and F. Waldhausen. Topological Hochschild homology. *J. London Math. Soc. (2)*, 62(2):345–356, 2000.

[30] Stefan Schwede. S-modules and symmetric spectra. *Math. Ann.*, 319(3):517–532, 2001.

[31] Stefan Schwede. Symmetric spectra book project. *draft version*, available online.

[32] Stefan Schwede and Brooke E. Shipley. Algebras and modules in monoidal model categories. *Proc. London Math. Soc. (3)*, 80(2):491–511, 2000.