# THE SEGAL CONJECTURE FOR TOPOLOGICAL HOCHSCHILD HOMOLOGY OF THE RAVENEL SPECTRA X(n) AND T(n)

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ABSTRACT. In [27], Ravenel introduced sequences of spectra X(n) and T(n) which played an important role in the proof of the Nilpotence Theorem of Devinatz-Hopkins-Smith [11]. In the present paper, we solve the homotopy limit problem for topological Hochschild homology of X(n), which is a generalized version of the Segal Conjecture for the cyclic group of prime order. We prove the same theorem for T(n) under the assumption that T(n) is an  $E_2$ -ring spectrum. This is also a first step towards computing algebraic K-theory of X(n) and T(n) using trace methods.

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### 1. Introduction

In the 1970's, Segal conjectured that after completion at the augmentation ideal, the Burnside ring of a finite group G and the cohomotopy of BG agree [1]. This conjecture inspired an outpouring of exciting research in the early 1980's and was eventually resolved for general finite groups by Carlsson in [8]. When p is a prime and  $G = C_p$  is the cyclic group of order p, the Segal Conjecture may be stated more generally as the question of whether the map  $S^{C_p} \longrightarrow S^{hC_p}$  is a p-adic equivalence, where S is the sphere spectrum. This version of the conjecture was resolved by Lin when p=2 and by Gunawardena when p>2 using an algebraic construction called the Singer construction [20, 2].

The Segal Conjecture for the group  $C_p$  can be rephrased using topological Hochschild homology. Given an S-algebra R, we may construct THH(R) as an  $S^1$ -spectrum which is a genuine C-spectrum for all finite subgroups C of  $S^1$  [18]. When R is the sphere spectrum S, the spectrum THH(S) equipped with the  $C_p$ -action obtained by restriction of the  $S^1$ -action is  $C_p$ -equivariantly equivalent to the sphere spectrum S equipped with the  $C_p$ -action in the statement of the Segal Conjecture [22]. In other words, there is a  $C_p$ -equivariant equivalence  $THH(S) \simeq S$ . Therefore, the Segal Conjecture for  $C_p$  may be rephrased as the question of whether the map

$$THH(S)^{C_p} \xrightarrow{\Gamma} THH(S)^{hC_p}$$

is a p-adic equivalence. For a more general S-algebra R, one may therefore ask whether the map

(1) 
$$THH(R)^{C_p} \xrightarrow{\Gamma} THH(R)^{hC_p}$$

is a p-adic equivalence, which can be seen as a generalization of the Segal Conjecture for the group  $C_p$ . In the present paper, we prove that when R is any of the Ravenel spectra X(n) for  $n \ge 1$  (see Section 2), the map (1) is a p-adic equivalence. Assuming T(n) is an  $E_2$ -ring spectrum, essentially the same proof gives the desired result (see Section 6). The proof uses techniques of [22], where Lunge-Nielsen-Rognes prove that the map (1) is an equivalence for the complex cobordism spectrum MU and the Brown-Peterson spectrum BP. As a consequence of our result and a result of Tsalidis [28, 6], there is also an equivalence

$$(2) THH(R)^{C_{p^k}} \xrightarrow{\Gamma} THH(R)^{hC_{p^k}}$$

for all  $k \geq 1$  when R is either X(n+1) or (assuming T(n) is an  $E_2$ -ring spectrum) T(n) for  $n \geq 0$ . The main motivation for studying the  $C_{p^k}$ -fixed points of topological Hochschild homology is to approximate algebraic K-theory. In particular, topological Hochschild homology of an S-algebra R has the structure of a cyclotomic spectrum, which produces maps

$$THH(R)^{C_{p^k}} \xrightarrow{R \atop F} THH(R)^{C_{p^{k-1}}}$$

where F is the usual inclusion of fixed points and R is a map that is produced using the cyclotomic structure [18]. We may construct p-typical topological cyclic homology as the homotopy limit of these maps

$$TC(R; p) = \underset{F,R}{\text{holim}} THH(R)^{C_{p^k}}$$

which is a close approximation to algebraic K-theory of R in the sense that when R is connective and there is a surjection  $\mathbb{Z}_p \to \pi_0 R$ , there is a weak equivalence  $K(R) \simeq_p \tau_{\geq 0} TC(R;p)$  after taking the connective cover of TC(R;p) and after p-completion [13, Thm. 7.3.1.8]. Generally, the spectra  $THH(R)^{C_{p^k}}$  are used to define TC(R;p) but they are not as computable as  $THH(R)^{hC_{p^k}}$ , so it is desirable to relate  $THH(R)^{C_{p^k}}$  to  $THH(R)^{hC_{p^k}}$ . Our results are therefore a first step towards computing K(X(n)) and K(T(n)).

The algebraic K-theory of the sphere spectrum is of fundamental importance because of its connection to derived algebraic geometry and manifold theory [29]. The program of Dundas-Rognes [12, Sec. 4.5] suggests, in particular, that algebraic K-theory of the sphere spectrum can be computed from the algebraic K-theory of X(n) by descent along the map of ring spectra  $S \to X(n)$ . Our results therefore provide a first step towards computing algebraic K-theory of the sphere spectrum by this approach.

1.1. Outline and Conventions. In Sections 2-4, we compute the continuous homology of the spectrum  $THH(X(n))^{tC_p}$  (see Section 4 for a definition of continuous homology in this setting). In Section 5, we relate  $H_*(THH(X(n))^{tC_p})$  to the homological Singer construction  $R_+(H_*^c(THH(X(n))))$  defined in [23]. This allows us to show that the map

$$THH(X(n)) \xrightarrow{\hat{\Gamma}} THH(X(n))^{tC_p}$$

is a p-adic equivalence. The desired p-adic equivalence for  $C_p$  then follows from considering the norm-restriction diagram. The  $C_{p^k}$ -case follows from the  $C_p$ -case by applying Tsalidis' theorem [28, 6]. In Section 6, we discuss the same computations for T(n) under the assumption that T(n) is an  $E_2$ -ring spectrum. We also give an argument for why it is plausible that T(n) is indeed an  $E_2$ -ring spectrum.

Throughout this paper, homology is always taken with coefficients in  $\mathbb{F}_p$  and  $H_*(-, \mathbb{F}_p)$  will simply be denoted  $H_*(-)$ . We will use the convention of Milnor [24] and write  $\mathcal{A}_* \cong P(\zeta_1, \zeta_2, \ldots)$  for the dual Steenrod algebra at the prime 2 where  $\zeta_i^2 = \xi_i$ . We will write  $\mathcal{A}_* \cong P(\xi_1, \xi_2, \ldots) \otimes E(\tau_0, \tau_1, \ldots)$  for the Steenrod algebra at primes  $p \geq 3$ . We will write  $\mathbb{T} \subset \mathbb{C}$  for the circle group and regard  $C_{p^k} \subseteq \mathbb{T}$  as the subgroup of  $p^k$ -th roots of unity. We will use the notation  $\dot{=}$  for an equality that holds only up to multiplication by a unit in  $\mathbb{F}_p$ .

1.2. **Acknowledgments.** The authors would like to thank Mark Behrens and Andrew Salch for their comments on earlier versions of this paper. The authors also thank anonymous reviewers for their helpful comments. The second author was partially supported by NSF grant DMS-1547292.

2. The spectra 
$$T(n)$$
 and  $X(n)$ 

The spectra T(n) and X(n) were an integral part of the proofs due to Devinatz-Hopkins-Smith [11] of the Ravenel Conjectures [27]. The spectrum X(n) is constructed as the Thom spectrum X(n) := Th(f) of the 2-fold loop map  $f \colon \Omega SU(n) \longrightarrow \Omega SU \simeq BU$ . By work of Lewis [15, Ch. IX], the Thom spectrum of an k-fold loop map is an  $E_k$ -ring spectrum and hence X(n) is an  $E_2$ -ring spectrum. Functoriality of the Thom construction gives a sequence of maps of  $E_2$ -ring spectra

$$S = X(1) \to X(2) \to \cdots \to X(\infty) = MU.$$

Just as MU splits p-locally as a wedge of suspensions of BP, X(k) splits p-locally as a wedge of suspensions of T(n) where n is chosen such that  $p^n \leq k < p^{n+1}$  [26, Thm. 6.5.1]. By [19, pg. 16], the spectra T(n) fit together to form a sequence whose colimit is BP. Due to Basterra-Mandell [4] we know that BP is an  $E_4$  ring spectrum and due to Ravenel [26, Thm. 6.5.1], we know T(n) are homotopy commutative homotopy associative ring spectra, however, it is not known whether T(n) are  $E_2$ -ring spectra (earlier in this project, we believed this to be well known, but we have not found a proof in the literature). We discuss this in more detail in Section 6.

The homology of MU and BP are well-known and the homology of X(k) and T(n) can be found in the work of Ravenel [27, Sec. 3].

**Lemma 2.1.** The homology of BP is given by the isomorphism  $H_*(BP) \cong P(\xi_1, \xi_2, \xi_3, ...)$  induced by the map  $BP \to H\mathbb{F}_p$  where  $|\xi_i| = 2p^i - 2$  for  $i \geq 1$ . The homology of T(n) is the subalgebra of  $H_*(BP)$  given by the isomorphism

$$H_*(T(n)) \cong P(\xi_1, \xi_2, \dots, \xi_n)$$

induced by the map  $T(n) \to BP$ .

**Lemma 2.2.** The homology of MU is given by the isomorphism  $H_*(MU) \cong P(b_1, b_2, b_3, ...)$  where  $|b_i| = 2i$ . The homology of X(k) is the subalgebra of  $H_*(MU)$  given by the isomorphism

$$H_*(X(k)) \cong P(b_1, \ldots, b_k).$$

3. Homology of the topological Hochschild homology of X(n)

Our first step is to calculate the homology of THH(X(n)). For this, we use the Bökstedt spectral sequence [5]. We quickly review the construction of this spectral sequence following [14, Ch. IX]. Let R be an S-algebra. Then THH(R) is the geometric realization of the cyclic bar construction on R, i.e. it is the geometric realization of a simplicial ring spectrum with k-simplices  $R^{\wedge k+1}$ . The Bökstedt spectral sequence arises from applying homology to the simplicial filtration and it has the form

$$E_{*,*}^2(R) \cong HH_*(H_*(R)) \Rightarrow H_*(THH(R))$$

where  $HH_*(-)$  denotes Hochschild homology over  $\mathbb{F}_p$ .

**Proposition 3.1.** The homology of THH(X(n)) is

$$H_*(THH(X(n)) \cong P(b_1, b_2, \dots, b_n) \otimes E(\sigma b_1, \sigma b_2, \dots, \sigma b_n).$$

*Proof.* To compute the  $E^2$ -term of the Bökstedt spectral sequence, we need to know the Hochschild homology  $HH_*(H_*(X(n)))$ . More generally, the Hochschild homology of polynomial algebras can be computed using the Koszul resolution, which shows that

$$HH_*(P(x_1,x_2,\ldots)) \cong P(x_1,x_2,\ldots) \otimes E(\sigma x_1,\sigma x_2,\ldots).$$

Therefore

$$E_{**}^2(X(n)) = P(b_1, b_2, \dots, b_n) \otimes E(\sigma b_1, \sigma b_1, \dots, \sigma b_n)$$

where  $|b_i| = (0, 2i)$  and  $|\sigma b_i| = (1, 2i)$ . Since there are no generators in filtration degree greater than one and the  $d_r$  differentials shift filtration degree by  $r \geq 2$ , there are no possible differentials.

To solve extension problems, we use the map  $X(n) \to MU$ , which is a map of  $E_2$ -ring spectra as discussed in Section 2. This map induces an injective map on homology  $H_*(X(n)) \to H_*(MU)$  which determines an injective map between the Bökstedt spectral sequences converging to  $H_*(THH(X(n)))$  and  $H_*(THH(MU))$ . This is a map of multiplicative spectral sequences by [3, Prop. 4.3] so any multiplicative extensions in  $H_*(THH(MU))$  must occur in  $H_*(THH(X(n)))$ . The multiplicative extensions in the Bökstedt spectral sequence for  $H_*THH(MU)$  were determined by [22, Lem 6.2], completing the proof.

# 4. Homological Tate fixed points of THH(X(n))

The next step is to compute the continuous homology of the  $C_p$ -Tate construction of THH(X(n)). Let X be a genuine  $\mathbb{T}$ -spectrum. Then the  $C_p$ -Tate construction of X, denoted  $X^{tC_p}$ , is defined by

$$X^{tC_p} = \left(\widetilde{E\,\mathbb{T}} \wedge F(E\,\mathbb{T}_+, X)\right)^{C_p}$$

where  $E \mathbb{T} = S(\infty \mathbb{C})$  is the usual model for  $E \mathbb{T}$  and  $\widetilde{E} \mathbb{T}$  is the cofiber of the map  $E \mathbb{T}_+ \to S^0$  which sends the basepoint to the basepoint and collapses everything else to the non-basepoint. Following Greenlees [16], we define  $\widetilde{E} \mathbb{T}_k$  to be the homotopy cofiber of  $(E \mathbb{T}^{(k-1)})_+ \to S^0$  for  $k \geq 0$  where  $E \mathbb{T}^{(k-1)}$  is the k-1-skeleton of  $E \mathbb{T}$  and we define  $\widetilde{E} \mathbb{T}_k = D(\widetilde{E} \mathbb{T}_{-k})$  for k < 0 where  $D(\widetilde{E} \mathbb{T}_{-k})$  is the Spanier-Whitehead dual of  $\widetilde{E} \mathbb{T}_{-k}$ . Note that in this case  $\widetilde{E} \mathbb{T}_{2n} = \widetilde{E} \mathbb{T}_{2n-1}$  can be modeled by  $S^{n\mathbb{C}}$  where  $S^{n\mathbb{C}}$  is the one point compactification of  $\mathbb{C}^n$  with diagonal  $\mathbb{T}$ -action.

We can then define

$$X^{tC_p}[k] = \left(\widetilde{E\,\mathbb{T}}/\widetilde{E\,\mathbb{T}_k} \wedge F(E\,\mathbb{T}_+, X)\right)^{C_p}$$

and we produce a filtration

$$X^{tC_p} \rightarrow \cdots X^{tC_p}[k] \rightarrow X^{tC_p}[k+1] \rightarrow X^{tC_p}[k+2] \rightarrow \cdots$$

of  $X^{tC_p}$  called the *Greenlees filtration* where the filtration quotients are

$$\left(\widetilde{E}\,\mathbb{T}_{2n-1}/\widetilde{E}\,\mathbb{T}_{2n}\wedge F(E\,\mathbb{T}_+,X)\right)^{C_p}\simeq \Sigma^n X$$

when k = 2n - 1 for some integer n and contractible when k is even.

Applying homology produces an exact couple whose associated spectral sequence is called the *homological Tate spectral sequence*. For more details about this spectral sequence, we refer the reader to [7, 17].

We begin by analyzing this spectral sequence for THH(X(n)). The homological Tate spectral sequence has the form

(3) 
$$\hat{E}_{**}^2 = \hat{H}^*(C_p; H_*(THH(X(n))) \Rightarrow H_*^c(THH(X(n))^{tC_p})$$

where the right-hand side is the *continuous homology* of THH(X(n)) defined as the limit

$$H^{c}_{*}(THH(X(n))^{tC_{p}}) := \lim_{h} H_{*}(THH(X(n))^{tC_{p}}[k])$$

of the homology of the spectra in the Greenlees filtration. This terminology originally appeared in the work of Bruner and Rognes [7].

The geometric realization of the cyclic bar construction admits a canonical  $\mathbb{T}$ -action, so in particular THH(X(n)) admits a  $\mathbb{T}$ -action. The  $C_p$ -action on  $H_*(THH(X(n)))$  is just the restriction of this  $\mathbb{T}$ -action which is trivial because  $\mathbb{T}$  is connected and the action of  $\mathbb{T}$  on  $H_*(THH(X(n)))$  is continuous. Therefore the Tate cohomology in the  $E^2$ -page splits as a tensor product

$$\hat{H}^*(C_p; H_*(THH(X(n)))) \cong \hat{H}^*(C_p; \mathbb{F}_p) \otimes H_*(THH(X(n))).$$

Since  $\hat{H}^{-*}(C_2; \mathbb{F}_2) \cong P(t^{\pm 1})$  with |t| = -1 and  $\hat{H}^*(C_p; \mathbb{F}_p) \cong E(h) \otimes P(t^{\pm 1})$  with |h| = -1 and |t| = -2 if p > 2, we can identify the  $E^2$ -page of the spectral sequence as

$$\hat{E}_{**}^2 = \begin{cases} P(t^{\pm 1}) \otimes P(b_1, b_2, \dots, b_n) \otimes E(\sigma b_1, \sigma b_2, \dots, \xi_n), & p = 2, \\ E(h) \otimes P(t^{\pm 1}) \otimes P(b_1, b_2, \dots, b_n) \otimes E(\sigma \xi_1, \sigma b_2, \dots, \sigma \xi_n), & p > 2. \end{cases}$$

For p = 2, the degrees of the generators are |t| = (-1,0),  $|b_i| = (0,2i)$ , and  $|\sigma b_i| = (0,2i+1)$ . For p > 2, the degrees of the generators are |h| = (-1,0), |t| = (-2,0), and the degrees of  $b_i$  and  $\sigma b_i$  are the same as in the case p = 2.

**Proposition 4.1.** In the homological Tate spectral sequence (3), there are  $d^2$ -differentials  $d_2(b_i) = t^2 \sigma b_i$  for p = 2, and  $d^2(b_i) = t \sigma b_i$  for p > 2, for all  $1 \le i \le n$ . There are no further differentials.

Proof. We will understand this spectral sequence by comparison with the Tate spectral sequence converging to  $H^c_*(THH(MU)^{tC_p})$ . The  $d^2$ -differentials for the Tate spectral sequence converging to  $H^c_*(THH(MU)^{tC_p})$  were computed in [22, Prop. 6.3]. They showed that for all  $i \geq 1$ , one has  $d^2(b_i) = t^2 \sigma b_i$  for p = 2, and  $d^2(b_i) = t \sigma b_i$  for p > 2. This can be seen by lifting the differentials from the Tate spectral sequence converging to the continuous homology  $H^c_*(THH(X(n))^{t^T})$  of the T-Tate construction along the inclusion  $C_p \subset \mathbb{T}$ . The T-Tate spectral sequence differentials arise from looking at the skeletal filtration of the model of  $E \mathbb{T}$  given by  $S(\infty \mathbb{C})$  and noting that the attaching maps are given by the T-action [7].

The map  $X(n) \to MU$  induces an injective map of  $E^2$ -pages of homological Tate spectral sequences. In particular, any differential  $d^2(b_i) = t^2 \sigma b_i$  for p = 2 or  $d^2(b_i) = t \sigma b_i$  for p > 2 in the MU case must also occur in the X(n) case when  $i \le n$ . This gives the stated  $d^2$ -differentials. Therefore, the map of  $E^3$ -pages of homological Tate spectral sequences is again injective. Since the homological Tate spectral sequence for MU collapses at  $E^3$  by [22, Prop. 6.3], so must the homological Tate spectral sequence for X(n).

To solve extensions, we note that the map of spectral sequences induced by  $X(n) \to MU$  is multiplicative by naturality of the homological Tate spectral sequence since the map  $X(n) \to MU$  is a map of  $E_2$ -ring spectra. Therefore, the desired extensions follow from [22, Prop. 6.3], as well.  $\square$ 

The continuous homology  $H^c_*(THH(X(n))^{tC_p})$  of the  $C_p$ -Tate construction of THH(X(n)) follows from the above pattern of differentials combined with the Leibniz rule.

Corollary 4.2. The continuous homology of the  $C_p$ -Tate construction of THH(X(n)) is given by

$$H^c_*(THH(X(n))^{tC_p}) \cong \begin{cases} P(t^{\pm 1}) \otimes P(b_1^2, \dots, b_n^2) \otimes E(b_1 \sigma b_1, \dots, b_n \sigma b_n), & p = 2, \\ E(h) \otimes P(t^{\pm 1}) \otimes P(b_1^p, \dots, b_n^p) \otimes E(b_1^{p-1} \sigma b_1, \dots, b_n^{p-1} \sigma b_n), & p > 2. \end{cases}$$

#### 5. Identification with the Singer construction

In this section, we prove the Segal Conjecture for THH(X(n)). Our proof proceeds by modifying the proof of the Segal Conjecture for THH(MU) given by Lunøe-Nielsen-Rognes in [22]. To avoid repeating some of their technical arguments and constructions, we include precise references to their paper where possible.

By the norm-restriction diagram for topological Hochschild homology [18, Prop. 4.1], the map  $THH(R)^{C_p} \xrightarrow{\Gamma} THH(R)^{hC_p}$  in (1) is a *p*-adic equivalence if

$$THH(R) \xrightarrow{\hat{\Gamma}} THH(R)^{tC_p}$$

is a p-adic equivalence. We will show this by exhibiting an Ext-equivalence (see Definition 5.1)

(4) 
$$H_*(THH(R)) \xrightarrow{\epsilon_*} R_+(H_*(THH(R))) \xrightarrow{\Phi_n} H_*^c(THH(R)^{tC_p})$$

where  $R_{+}(-)$  is the homological Singer construction [23, Def. 3.7]. The *p*-adic equivalence then follows from comparing (inverse limit) Adams spectral sequences as in [21, 2, 22]. The map  $\epsilon_*$  is an Ext-equivalence by [21, 2, 23], so we must show that the map  $\Phi_n$  is an Ext-equivalence.

**Definition 5.1.** A homomorphism  $N \to M$  of  $\mathcal{A}_*$ -comodules is an Ext-equivalence if the induced homomorphism

$$Ext_{\mathcal{A}_{*}}^{s,t}(\mathbb{F}_{p},N) \to Ext_{\mathcal{A}_{*}}^{s,t}(\mathbb{F}_{p},M)$$

is an isomorphism for all  $s \geq 0$  and  $t \in \mathbb{Z}$ .

We begin with the case R = X(n) and p > 2. By Corollary 4.2 the continuous homology of the  $C_p$ -Tate construction on THH(X(n)) is given by

$$H^c_*(THH(X(n))^{tC_p}) \cong E(h) \otimes P(t^{\pm 1}) \otimes P(b_1^p, b_2^p, \dots, b_n^p) \otimes E(b_1^{p-1}\sigma b_1, b_2^{p-1}\sigma b_2, \dots, b_n^{p-1}\sigma b_n).$$

On the other hand, one can consider the homological Tate spectral sequence

$$\hat{E}_2^{**} = \hat{H}^{-*}(C_p; H_*(THH(X(n))^{\wedge p})) \Rightarrow H_*(((THH(X(n))^{\wedge p})^{tC_p}) \cong R_+(H_*(THH(X(n))))$$

where the isomorphism on the right-hand side follows from [23, Thm. 5.9]. In this case, the homological Singer construction can be expressed as

$$R_{+}(H_{*}(THH(X(n)))) = E(h) \otimes P(t^{\pm 1}) \otimes P(b_{1}^{\otimes p}, b_{2}^{\otimes p}, \dots, b_{n}^{\otimes p}) \otimes E(\sigma b_{1}^{\otimes p}, \sigma b_{2}^{\otimes p}, \dots, \sigma b_{n}^{\otimes p})$$

which is in bijection with  $H^c_*(THH(X(n))^{tC_p})$  via  $b^p_i \mapsto b^{\otimes p}_i$  and  $b^{p-1}_i \sigma b_i \mapsto t^m \otimes \sigma b^{\otimes p}_i$  where m = (p-1)/2. The goal is to promote this filtration-shifting bijection to an isomorphism of complete  $\mathcal{A}_*$ -comodules.

The homology of X(n) and THH(X(n)) are sub- $\mathcal{A}_*$ -comodules of the homology of MU and THH(MU), respectively, and  $H^c_*(THH(X(n))^{tC_p})$  is a complete sub- $\mathcal{A}_*$ -comodule of  $H^c_*(THH(MU)^{tC_p})$ . Therefore, the formulas and computations leading up to Propositions 7.2 and 7.3 of [22] carry over mutatis mutandis. In particular, we obtain maps

$$R_+(H_*(X(n))) \otimes_{H_*(X(n))} H_*(THH(X(n))) \xrightarrow{f} R_+(H_*(THH(X(n)))$$

$$R_+(H_*(X(n))) \otimes_{H_*(X(n))} H_*(THH(X(n))) \stackrel{g}{\longrightarrow} H_*^c(THH(X(n))^{tC_p})$$

defined by  $f(\alpha \otimes \beta) = R_+(\eta_*)(\alpha) \cdot \epsilon_*(\beta)$  and  $g(\alpha \otimes \beta) = \eta_*^t(\alpha) \cdot \hat{\Gamma}_*(\beta)$ , where  $R_+(\eta_*)$ ,  $\epsilon_*$ ,  $\eta_*^t$  and  $\hat{\Gamma}_*$  are the  $H_*(X(n))$ -linear maps

$$\begin{split} R_{+}(\eta_{*}) \colon & R_{+}(H_{*}(X(n))) \to R_{+}(H_{*}(THH(X(n))) \\ \epsilon_{*} \colon & H_{*}(THH(X(n))) \to R_{+}(H_{*}(THH(X(n))) \\ & \eta_{*}^{t} \colon & R_{+}(H_{*}(X(n)) \to H_{*}^{c}(THH(X(n))^{tC_{p}}) \\ & \hat{\Gamma}_{*} \colon & H_{*}(THH(X(n))) \to H_{*}^{c}(THH(X(n))^{tC_{p}}) \end{split}$$

induced by the usual unit map

$$\eta \colon X(n) \to THH(X(n)),$$

Tate diagonal

$$\epsilon \colon THH(X(n)) \to (THH(X(n))^{\wedge p})^{tC_p},$$

and the usual map

$$\hat{\Gamma} \colon THH(X(n)) \to THH(X(n))^{tC_p}$$

in the norm-restriction diagram [18, Prop. 4.1].

There are filtrations of the above  $A_*$ -comodules defined by

$$F^k H_*(X^{tC_p}) = im \left( H^c_*(X^{tC_p}) \to H_*(X^{tC_p}[k]) \right)$$

where  $H_*(X^{tC_p}[k])$  is the homology of the k-th term in the Greenlees filtration (see Section 4). In particular, this defines a filtration on  $R_+(H_*R)$  and  $R_+(H_*THH(R))$  for an S-algebra R which is bounded below and finite type because, due to [23, Thm. 5.9], there are isomorphisms  $R_+(H_*R) \cong H_*^c((R^{\wedge p})^{tC_p})$  and  $R_+(H_*THH(R)) \cong H_*^c((THH(R)^{\wedge p})^{tC_p})$ . The maps f and g defined above

<sup>&</sup>lt;sup>1</sup>In fact, this result has recently been extended by Nikolaus-Scholze [25, Theorem III.1.7] who show that the map  $X \to (X^{\wedge p})^{tC_p}$  exhibits  $(X^{\wedge p})^{tC_p}$  as the *p*-completion of X for all bounded below spectra without the finite type hypothesis.

induce maps between filtrations  $f_k$  and  $g_k$ , which are  $\mathcal{A}_*$ -comodule homomorphisms for all  $n \in \mathbb{N}$  since the  $\mathcal{A}_*$ -comodule structure on  $H_*(X(n))$  is the restriction of the  $\mathcal{A}_*$ -comodule structure on  $H_*(MU)$ .

**Proposition 5.2.** The sets  $\{f_k\}$  and  $\{g_k\}$  are strict maps of inverse systems which assemble into pro-isomorphisms whose limits  $\hat{f}$  and  $\hat{g}$  are isomorphisms of complete  $\mathcal{A}_*$ -comodules.

*Proof.* Our proof is modified from the proof of [22, Prop. 7.2]. We will only provide the proof for  $\{f_k\}$  since the proof for  $\{g_k\}$  is similar. In each total degree d,  $f_k$  defines a map

$$f_{k,d}: [F^k R_+(H_*(X(n))) \otimes E(\sigma b_1, \sigma b_2, \dots, \sigma b_n)]_d \to F^k R_+(H_*(THH(X(n))))_d.$$

For each k, we would like to define compatible maps

$$\phi_{k,d} \colon [F^N R_+(H_*(THH(X(n))))]_d \to [F^k R_+(H_*(X(n))) \otimes E(\sigma b_1, \sigma b_2, \dots, \sigma b_n)]_d$$

with N = N(k,d) = p(k-d) + d, such that the composition  $\phi_{k,d} \circ f_{N,d}$  is equal to the structural surjection

$$[F^NR_+(H_*(X(n)))\otimes E(\sigma b_1,\sigma b_2\ldots,\sigma b_n)]_d \longrightarrow [F^kR_+(H_*(X(n)))\otimes E(\sigma b_1,\sigma b_2\ldots,\sigma b_n)]_d$$

and such that the composition  $f_{k,d} \circ \phi_{k,d}$ , is equal to the structural surjection

$$[F^{N}R_{+}(H_{*}(THH(X(n)))]_{d} \longrightarrow [F^{k}R_{+}(H_{*}(THH(X(n))))]_{d}.$$

We can then conclude that the collection  $\{f_{k,d}\}_k$  forms a pro-isomorphism with pro-inverse  $\{\phi_{k,d}\}_k$  in each total degree d. These maps therefore assemble into a pro-isomorphism  $\{f_k\}$  with pro-inverse  $\{\phi_k\}$ .

In [22, Proof of Thm. 7.2], Lunøe-Nielsen and Rognes decompose the group  $[R_+(H_*(MU)) \otimes E(\epsilon_*(\sigma m_\ell)|\ell \geq 1)]_d$  into a direct sum indexed by strictly increasing sequences  $L = (\ell_1 < \cdots < \ell_r)$  of natural numbers of length  $r \geq 0$ . Then they define the maps  $\phi_{k,d}$  for MU using this decomposition on [22, Pg. 618-619]. The desired maps  $\phi_{k,d}$  for X(n) follow from exactly the same steps. To decompose  $[R_+(H_*(X(n))) \otimes E(\epsilon_*(\sigma b_1), \ldots, \epsilon_*(\sigma b_n)))]_d$  into a direct sum, we restrict to strictly increasing sequences  $L = (\ell_1 < \cdots < \ell_r)$  where  $0 \leq r \leq n$ . Using the notation  $\epsilon_L = \epsilon_*(\sigma b_{\ell_1}) \cdot \ldots \cdot \epsilon_*(\sigma b_{\ell_r})$ , we obtain homomorphisms

$$[F^{N-s_L}R_+(H_*(X(n)))\otimes \mathbb{F}_p\{\epsilon_L\}]_d\to [F^kR_+(H_*(X(n)))\otimes E(\sigma b_1,\sigma b_2,\ldots,\sigma b_n)]_d$$

defined by  $\epsilon_L \mapsto \sigma b_{\ell_1} \dots \sigma b_{\ell_r}$  where  $s_L = -(p-1)(2\ell_1 + \dots + 2\ell_r + r)$ . The definition of  $\phi_{k,d}$  is then completed by taking the direct sum over L.

The filtration shift estimates from [22, Pg. 618-619] carry over to the X(n) case. Therefore the maps  $f_{k,d}$  and  $\phi_{k,d}$  compose into the desired structural surjections in each total degree d. This implies that the set  $\{f_k\}$  is a strict map of inverse systems which assembles into a pro-isomorphism.

The analogous pro-isomorphisms can be defined for X(n) when p=2 by essentially the same argument. Setting  $\Phi_X := \hat{g} \circ \hat{f}^{-1}$  for the corresponding  $\hat{g} = \lim g_k$  and  $\hat{f} = \lim f_k$  yields the desired isomorphism.

Corollary 5.3. There is an isomorphism of complete  $A_*$ -comodules

$$\Phi_{X(n)} \colon R_+(H_*(THH(X(n)))) \longrightarrow H_*^c(THH(X(n))^{tC_p}).$$

Therefore the composition (4) is an Ext-equivalence for R = X(n) for all  $n \ge 1$ .

Corollary 5.4. The map

$$THH(X(n)) \xrightarrow{\hat{\Gamma}} THH(X(n))^{tC_p}$$

is a p-adic equivalence.

Proof. The map

$$\hat{\Gamma} \colon THH(X(n)) \to THH(X(n))^{tC_p}$$

induces a map between the Adams spectral sequence

(5) 
$$E_2^{s,t} = Ext_{\mathcal{A}_*}^{s,t}(\mathbb{F}_p, H_*(THH(X(n)))) \Rightarrow \pi_{t-s}(THH(X(n)))$$

and the inverse limit of Adams spectral sequences

(6) 
$$E_2^{s,t} = Ext_{\mathcal{A}_*}^{s,t}(\mathbb{F}_p, H_*^c(THH(X(n))^{tC_p})) \Rightarrow \pi_{t-s}(THH(X(n))^{tC_p}).$$

By [23, Thm. 5.9], there is an isomorphism  $H_*(THH(X(n))^{p})^{tC_p} \cong R_+(H_*THH(X(n)))$  and by Lin and Gunawardena [20, 2] the map  $H_*(THH(X(n))) \to R_+(H_*THH(X(n)))$  induces an Ext-equivalence. Since there is an Ext-equivalence between  $R_+(H_*THH(X(n)))$  and  $H_*THH(X(n))^{tC_p}$ , there is an equivalence of  $E_2$ -pages of the spectral sequences (5) and (6). The result then follows from convergence of the inverse limit of Adams spectral sequences [23, Prop. 2.2].

By the norm-restriction diagram [18, Prop. 4.1] and the five lemma, we conclude the main theorem.

Theorem 5.5. The map

$$THH(X(n))^{C_p} \xrightarrow{\Gamma} THH(X(n))^{hC_p}$$

is a p-adic equivalence.

By Tsalidis' theorem [28, 6], we conclude that the same statement holds for  $C_{p^k}$ -fixed points.

Corollary 5.6. The map

$$THH(X(n))^{C_{p^k}} \xrightarrow{\Gamma} THH(X(n))^{hC_{p^k}}$$

is a p-adic equivalence.

### 6. The Segal conjecture for topological Hochschild homology of T(n)

We conclude by briefly describing analogous results for the spectrum T(n) under the assumptions that T(n) is an  $E_2$ -ring spectrum and that the map  $T(n) \to BP$  is a map of  $E_2$ -ring spectra. We begin by discussing the plausibility of these assumptions.

We can express T(n) as the colimit

$$T(n) = \operatorname{colim}_{\epsilon_n} X(n),$$

where  $\epsilon_n$  is the restriction of the Quillen idempotent  $\epsilon\colon MU\to MU$  to X(n) as defined in [19, Lem 1.3.5]. In particular, this implies that T(n) is homotopy commutative and homotopy associative [26, Thm. 6.5.1]. Work of Chadwick-Mandell [10] shows that  $\epsilon\colon MU\to MU$  is a map of  $E_2$ -ring spectra and the map  $MU\to BP$  is a map of  $E_2$ -ring spectra. To prove that T(n) is an  $E_2$ -ring spectrum and the map  $T(n)\to BP$  is a map of  $E_2$ -ring spectra, it would suffice to show that the map  $E_1$  is a map of  $E_2$ -ring spectra. Indeed, this follows from the same argument as [9, Thm. 5.6]; the key point is that the colimit in  $E_2$ -ring spectra is computed as the colimit of underlying spectra.

Let  $\operatorname{HoRing}(X,Y)$  denote the set of homotopy classes of maps of ring spectra  $X \to Y$  in the stable homotopy category, and let  $\operatorname{E_2-Ring}(A,B)$  be the space of  $E_2$ -ring maps  $A \to B$ . We would like to show that  $\epsilon_n \in \operatorname{HoRing}(X(n),X(n))$  pulls back to a class in  $\pi_0(\operatorname{E_2-Ring}(X(n),X(n)))$  along the map

(7) 
$$\pi_0(E_2\text{-Ring}(X(n), X(n))) \to \text{HoRing}(X(n), X(n)).$$

By using methods from [10, Sec. 6], we can identify

$$\pi_0(\mathcal{E}_2\text{-Ring}(X(n), X(n))) \cong \widetilde{sl}_1 X(n)^2 BSU(n)$$

and

$$\operatorname{HoRing}(X(n), X(n)) \cong \widetilde{sl}_1 X(n)^0 (\mathbb{C}P^{n-1}).$$

The map (7) is induced by the map

$$\Sigma^2 \mathbb{C} P^{n-1} \to \Sigma^2 \Omega SU(n) \to B^2 \Omega SU(n) \simeq BSU(n).$$

Therefore it suffices to examine the map of Atiyah-Hirzebruch spectral sequences, which is given on  $E_2$ -pages by

$$H^{s+2}(BSU(n); \pi_{-t}\widetilde{sl_1}X(n)) \to H^s(\mathbb{C}P^{n-1}; \pi_{-t}\widetilde{sl_1}X(n)).$$

We can understand this map with integral coefficients

$$H^{*+2}(BSU(n); \mathbb{Z}) \to H^s(\mathbb{C}P^{n-1}; \mathbb{Z})$$

where  $H^{*+2}(BSU(n); \mathbb{Z}) \cong P(x_2, x_3, \dots, x_n)$  with  $|x_i| = 2i$  and  $H^*(\mathbb{C}P^{n-1}; \mathbb{Z}) \cong P_n(u)$  with |u| = 2. By [10, Prop. 6.3],  $x_i$  maps to  $(-1)^i u^i$  for all i and all decomposables map to zero. Therefore it suffices to understand the map of Atiyah-Hirzebruch spectral sequences modulo decomposables. We may determine the class z detecting  $\epsilon_n$  in the target spectral sequence and a class  $\tilde{z}$  mapping to it from the source spectral sequence. Our goal then is to show that  $\tilde{z}$  is a permanent cycle.

In the work of Chadwick-Mandell, the analogous class is a permanent cycle for bidegree reasons; in particular, there are no possible targets for differentials because all of the spectra they consider have homotopy groups concentrated in even degrees. Since  $\pi_*(X(n))$  is not concentrated in even degrees for \* sufficiently large, we cannot rule out the possibility of  $\tilde{z}$  supporting a long differential. We make the following assumption which implies that T(n) is  $E_2$ .

**Assumption 6.1.** The class  $\tilde{z}$  is a permanent cycle in the Atiyah-Hirzebruch spectral sequence with abutment  $\tilde{sl}_1X(n)^*BSU(n)$ .

Assuming that T(n) is  $E_2$  and the map  $T(n) \to BP$  is a map of  $E_2$ -ring spectra, we can prove the Segal Conjecture for THH(T(n)) by following the same strategy as we did for THH(X(n)). The homology  $H_*(THH(T(n)))$  can be computed using the Bökstedt spectral sequence by comparison with the Bökstedt spectral sequence converging to  $H_*(THH(BP))$ . One obtains

$$H_*(THH(T(n))) \cong P(\xi_1, \xi_2, \dots, \xi_n) \otimes E(\sigma \xi_1, \sigma \xi_2, \dots, \sigma \xi_n).$$

The continuous homology  $H^c_*(THH(T(n))^{tC_p})$  can be computed using the homological Tate spectral sequence by comparison with the homological Tate spectral sequence converging to  $H^c_*(THH(BP)^{tC_p})$  which was computed in [22, Prop. 6.8]. One finds that

$$H^c_*(THH(T(n))^{tC_p}) \cong \begin{cases} P(t^{\pm 1}) \otimes P(\xi_1^2, \dots, \xi_n^2) \otimes E(\xi_1 \sigma \xi_1, \dots, \xi_n \sigma \xi_n), & p = 2, \\ E(h) \otimes P(t^{\pm 1}) \otimes P(\xi_1^p, \dots, \xi_n^p) \otimes E(\xi_1^{p-1} \sigma \xi_1, \dots, \xi_n^{p-1} \sigma \xi_n), & p > 2. \end{cases}$$

For p=2, the degrees of the generators are |t|=(-1,0),  $|\xi_i|=(0,2^{i+1}-2)$ , and  $|\sigma\xi_i|=(0,2^{i+1}-1)$ . For p>2, the degrees of the generators are |h|=(-1,0), |t|=(-2,0),  $|\xi_i|=(0,2p^i-2)$ , and  $|\sigma\xi_i|=(0,2p^i-1)$ . The identification with the Singer construction

$$R_+(H_*(THH(T(n)))) \cong H_+^c(THH(T(n))^{tC_p})$$

follows from a modification of the proof of [22, Thm. 7.2]. This modification is similar to the modification proving the analogous isomorphism for THH(X(n)) in Section 5. One concludes using the (inverse limit) Adams spectral sequence that the map

$$THH(T(n))^{C_p} \xrightarrow{\Gamma} THH(T(n))^{hC_p}$$

is a p-adic equivalence, and by Tsalids' Theorem [28, 6], the map

$$THH(T(n))^{C_{p^k}} \xrightarrow{\Gamma} THH(T(n))^{hC_{p^k}}$$

is a p-adic equivalence for all  $k \geq 1$ .

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