Lecture 4 :

Simplicial Methods

I. Simplicial objects.

<u>Def</u>: Let Ord be the category of finite totally ordered sets and orderpreserving maps. Let $\Delta = \mathrm{sk}\,\mathrm{Ord}$.

Then $\mathrm{ob}\,\Delta = \{[n] : n \geq 0\}$

Note: $\Delta \subseteq \mathrm{Cat}$ = category of small categories

$$[n] \longmapsto 0 \to 1 \to \dots \to n$$

So a map $[n] \to [m]$ in Δ is a functor. All morphisms in Δ are generated by functors

$$\delta_n^i : [n] \longrightarrow [n+1] \quad 0 \leq i \leq n$$

$$\sigma_n^j : [n+1] \to [n] \quad 0 \leq j \leq n$$

$$\delta_n^i\left(0 \to \dots \to i-1 \to i \to i+1 \to \dots \to n\right)$$

$$\parallel$$

$$0 \to 1 \to \dots \to \dots \to i-1 \to i+1 \to \dots \to n$$

(i.e. compose $i-1 \to i \to i+1$
to produce $i-1 \to i+1$)

Ex: δ_2^1:

$$\begin{tikzcd} 0 \arrow[r] & 0 \\ 1 \arrow[rd] & 1 \\ & 2 \end{tikzcd}$$

$$\sigma_n^j\left(0 \to 1 \to \dots \to j \to \dots \to n+1\right)$$

$$\parallel$$

$$0 \to 1 \to \dots \to j \xrightarrow{1} j \to \dots \to n$$

(insert the identity in j-th spot.)

Exercise: Show that these satisfy the identities

1) $\delta_n^j \circ \delta_{n-1}^i = \delta_n^i \circ \delta_{n-1}^{j-1}$ if $i<j$

2) $$\sigma_n^j \delta_n^i = \begin{cases} \delta_n^i \sigma_n^{j-1} & \text{if } i<j \\ id_{[n]} & \text{if } i=j,j+1 \\ \delta_n^{i-1} \sigma_n^j & \text{if } i>j+1 \end{cases}$$

$s^j \circ s^i = s^i \circ s^{j+1}$ if $i \leq j$

3) $s_{n-1}^j \circ s_n^i = s_{n-1}^i \circ s_n^{j+1}$ if $i \leq j$

for all $n, i, j \geq 0$

Such the formulas are sensible.

<u>Def</u>: Let $\mathcal{C}$ be a category.

A simplicial object in $\mathcal{C}$ is a functor

$$X_\bullet : \Delta^{op} \longrightarrow \mathcal{C}.$$

A map of simplicial sets

$$f : X_\bullet \to Y_\bullet$$

is a natural transformation.

A cosimplicial object in $\mathcal{C}$ is a functor

$$\Delta \longrightarrow \mathcal{C}.$$

A map of cosimplicial objects in $\mathcal{C}$ is a natural transformation.

Unpacking this, a simplicial object in $\mathscr{C}$ consists of a set $\{X_n : n \geq 0\}$ of objects in $\mathscr{C}$ and morphisms

$$
\begin{tikzcd}
X_0 & X_1 \arrow[l, shift left=2, "d_0"'] \arrow[l, shift right=2, "d_1"] & X_2 \arrow[l, shift left=4, "d_0"'] \arrow[l, "d_0"] \arrow[l, shift right=4, "d_1"] & \cdots \arrow[l, shift left=6] \arrow[l, shift left=2] \arrow[l, shift right=2] \arrow[l, shift right=6]
\arrow[from=1-1, to=1-2, "s_0"]
\arrow[from=1-2, to=1-3, shift left=2, "s_0"]
\arrow[from=1-2, to=1-3, shift right=2, "s_1"]
\arrow[from=1-3, to=1-4, shift left=4]
\arrow[from=1-3, to=1-4]
\arrow[from=1-3, to=1-4, shift right=4]
\end{tikzcd}
$$

where we write

$$d_i = X_\bullet(\delta_i)$$

$$s_i = X_\bullet(\sigma_i).$$

These satisfy the simplicial identities.

6

1) $d_i \circ d_j = d_{j-1} \circ d_i$ if $i<j$

2) $d_i \circ s_j = \begin{cases} s_{j-1} d_i & \text{if } i<j \\ 1 & \text{if } i=j,j+1 \\ s_j \circ d_{i-1} & \text{if } i>j+1 \end{cases}$

3) $s_i \circ s_j = s_{j+1} \circ s_i$ if $i \leq j$

which are the obvious duals of the identities in Δ.

Ex 1:

$\{[n] : n \geq 0\}$ forms a cosimplicial object in Cat

$$\Delta \longrightarrow \mathrm{Cat}$$

$$[n] \longmapsto 0 \to 1 \to 2 \to \cdots \to n$$

via the embedding $\Delta \subseteq \mathrm{Cat}$.

Ex 2:

$$\mathrm{Hom}_{\Delta}(-,[n]) : \Delta^{op} \longrightarrow \mathrm{Set}$$

is a simplicial set, which we call

$$\Delta^n = \mathrm{Hom}_{\Delta}(-,[n]).$$

(In fact, $\Delta^{\bullet}$ is a cosimplicial simplicial set.)

The topological n-simplex

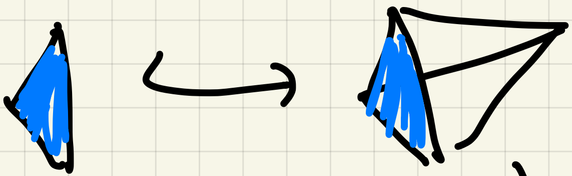
$$|\Delta^{n+1}| := \{(t_0, t_1, \dots, t_n) \in [0,1]^{n+1} : \sum_{i=0}^{n} t_i = 1\}$$

This forms a cosimplicial topological space with

$$d_i : |\Delta^n| \longrightarrow |\Delta^{n+1}|$$

$$(t_0, \dots, t_n) \longmapsto (t_0, \dots, t_{i-1}, 1, t_{i+1}, \dots, t_n$$

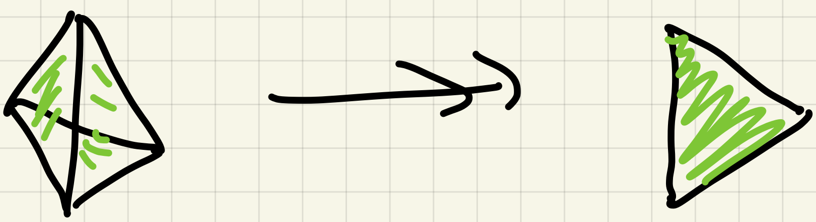
Ex:



$$\sigma_i : |\Delta^n| \longrightarrow |\Delta^{n-1}|$$

$$(t_0, \dots, t_{n-1}) \longmapsto (t_0, \dots, t_i + t_{i+1}, \dots, t_{n-1}$$

Ex:



Def: : Let X be a topological space Then

$$\mathrm{Sing}(X) : \Delta^{op} \longrightarrow \mathrm{Set}$$

$$\mathrm{Sin}_n(X) = \mathrm{Hom}_{\mathrm{Top}}(\Delta^n, X)$$

Given a simplicial set $Y_\bullet$, we can form $\mathbb{Z}[Y_\bullet]$ by functoriality.

$$\Delta^{op} \longrightarrow \mathrm{Set} \xrightarrow{\mathbb{Z}[-]} \mathrm{Ab}.$$

Then define

$$\mathbb{Z}[X_0] \xleftarrow{\partial_0} \mathbb{Z}[X_1] \xleftarrow{\partial_1} \mathbb{Z}[X_2] \leftarrow \cdots$$

by $\partial_i = \sum_{i=0}^{n} (-1)^i d_i$.

Ex:

$$H_*(\mathbb{Z}[\mathrm{Sing}_\bullet(X)]) = H_*^{\mathrm{sing}}(X;\mathbb{Z})$$

<u>Construction</u>: Let $\mathscr{C}$ be a category. We can consider functors

$$[n] \to \mathscr{C}$$

i.e. strings of composable morphisms

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \to \cdots \xrightarrow{f_n} X_n$$

in $\mathscr{C}$.

By Ex 1, we can form simplicial set w/ n-simplices

$$N_n \mathscr{C} := \mathrm{Fun}([n], \mathscr{C}).$$

Ex: Let G be a discrete group. We can regard it as a category with one object $*$ and morphism set

$$G(*,*) = G.$$

The identity is a map

$$\eta : * \to G$$

and the group operation corresponds to composition

$$\begin{array}{ccc} G(*,*) \times G(*,*) & \longrightarrow & G(*,*) \\ \parallel & & \parallel \\ \mu : G \times G & & G \end{array}.$$

In this case, we can be very explicit.

$$N_\bullet G = \begin{tikzcd} * \arrow[r, "\eta"] & G \arrow[l, shift right=3, "\varepsilon"'] \arrow[l, shift left=3, "\varepsilon"] \arrow[r, shift left=2, "\eta\times 1_G"] \arrow[r, shift right=2, "1_G\times\eta"'] & G\times G \arrow[l, shift right=5, "\varepsilon\times 1_G"'] \arrow[l, "\mu"] \arrow[l, shift left=5, "1_G\times\varepsilon"] & \cdots \arrow[l, shift right=6] \arrow[l, shift right=3] \arrow[l, shift left=3] \arrow[l, shift left=6] \arrow[l, shift right=1.5, leftarrow] \arrow[l, shift left=1.5, leftarrow] \arrow[l, shift left=4.5, leftarrow] \end{tikzcd}$$

(write: $\varepsilon : G \to *$ for the canonical map to the terminal object in Set.)

$$\begin{array}{ccc} N_{n+1}G & \longrightarrow & N_n G \\ \parallel & & \parallel \\ d_i : G^{\times n+1} & \longrightarrow & G^{\times n} : s_i \end{array}$$

$$d_i(g_1, \dots, g_{n+1}) = \begin{cases} (g_2, \dots, g_{n+1}) & i = 0 \\ (g_1, \dots, g_j \cdot g_{j+1}, \dots, g_{n+1}) & i = j \\ (g_1, \dots, g_n) & i = n \end{cases}$$

$$s_i(g_1, \dots, g_n) = (g_1, \dots, g_{i-1}, 1, g_{i+1}, \dots, g_n)$$

Note: This second definition also makes sense when G is a topological group. In this case $N_\bullet G$ is a simplicial space.

<u>Construction</u>: Given a simplicial space $X_\bullet : \Delta^{op} \to Top$ we form the following topological space $|X_\bullet|$ called the geometric realization of $X_\bullet$.

$$|X_\bullet| = \left(\coprod_{n \geq 0} |\Delta^n| \times X_n \right) \Big/ \sim$$

where $\coprod_{n \geq 0} |\Delta^n| \times X_n$ has the coproduct topology and $\sim$ is an equivalence relation.

The equivalence relation is generated by

$$(\sigma_i x, y) \sim (x, s_i y)$$

$$(\delta_i x, y) \sim (x, d_i y).$$

Explicitly, $|X_\bullet|$ is the coequalizer

$$\coprod_{\substack{f:[n]\to[m] \\ \text{in } \Delta}} |\Delta^n| \times X_m \underset{|\Delta^n| \times X_f}{\overset{|\Delta^f| \times \mathrm{id}_{X_m}}{\rightrightarrows}} \coprod_{[n]\in\Delta} |\Delta^n| \times X_n .$$

Def: (Comma category)

Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be categories

w/ functors

$$\mathcal{A} \xrightarrow{S} \mathcal{C} \xleftarrow{T} \mathcal{B}$$

then $(S \downarrow T)$ is a category

with

$$\mathrm{ob}(S \downarrow T) = (A, B, h),$$

$A \in \mathrm{ob}\,\mathcal{A}$ $\quad B \in \mathrm{ob}\,\mathcal{B}$ $\quad h : S(A) \to T(B)$

$$S \downarrow T\big((A_1, B_1, h_1), (A_2, B_2, h_2)\big)$$

$$\rotatebox{90}{$\in$}$$

$$\left(f : A_1 \to A_2,\ g : B_1 \to B_2,\ \begin{tikzcd} S(A_1) \arrow[r, "h_1"] \arrow[d, "S(f)"'] \arrow[dr, phantom, "\circlearrowleft"] & S(B_1) \arrow[d, "S(g)"] \\ S(A_2) \arrow[r, "h_2"'] & S(B_2) \end{tikzcd} \right)$$

Example: $\Delta \xrightarrow{\text{Yoneda}} sSet \xleftarrow{X_\bullet} *$

$[n] \mapsto \Delta^n$

Write $\Delta \downarrow X_\bullet$ for the associated comma category.

An object in $\Delta \downarrow X_\bullet$ is a map $\Delta^n \to X_\bullet$ of simplicial sets.

Note: $\mathrm{Hom}_{sSet}(\Delta^n, X_\bullet) = Y_n$ by the Yoneda lemma.

A map in $\Delta \downarrow X_\bullet$ is a commuting triangle
$$\begin{tikzcd} \Delta^n \arrow[r, "\theta"] \arrow[dr] & \Delta^m \arrow[d] \\ & X \end{tikzcd}$$
where θ is induced by $[n] \to [m]$ in Δ.

Also, $X_\bullet$ determines a functor

$$\Delta \downarrow X_\bullet \longrightarrow Top$$
$$(\Delta^n \to X_\bullet) \longmapsto |\Delta^n|$$
$$\left(\begin{tikzcd} \Delta^n \arrow[r] \arrow[dr] & \Delta^m \arrow[d] \\ & X_\bullet \end{tikzcd}\right) \longmapsto |\Delta^n| \to |\Delta^m|.$$

Def. [Geometric Realization 2.0]

$$|X_\bullet| = \varinjlim_{\Delta \downarrow X_\bullet} |\Delta^n|$$

Thm: There is an adjunction

$$|-| : sSet \rightleftarrows Top : Sing_\bullet(-)$$

exhibited by the natural isomorphism

$$Hom_{Top}(|X_\bullet|, Y) \cong Hom_{sSet}(X_\bullet, Sing_\bullet(Y)).$$

Proof: There are natural isomorphisms

$$\mathrm{Hom}_{\mathrm{Top}}(|X_\bullet|, Y) = \mathrm{Hom}_{\mathrm{Top}}\Big(\varinjlim_{\Delta \downarrow X_\bullet} |\Delta^n|, Y\Big)$$

$$\cong \varprojlim_{\Delta \downarrow X_\bullet} \mathrm{Hom}(|\Delta^n|, Y)$$

$$\cong \varprojlim_{\Delta \downarrow X_\bullet} \mathrm{sing}_n(Y)$$

$$= \varprojlim_{\Delta \downarrow X_\bullet} \mathrm{Hom}_{\mathrm{sSet}}(\Delta^n, \mathrm{sing}_\bullet(Y))$$

Yoneda $\cong \mathrm{Hom}_{\mathrm{sSet}}\Big(\varinjlim_{\Delta \downarrow X_\bullet} \Delta^n, \mathrm{sing}_\bullet(Y)\Big)$

$(*)$ $\cong \mathrm{Hom}_{\mathrm{sSet}}(X_\bullet, \mathrm{sing}_\bullet(Y)).$

To see ✪ note that,

$\operatorname{hocolim}_{\Delta^n \downarrow X_\bullet} \Delta^n$ satisfies the universal property

$$\begin{tikzcd}
\Delta^k \arrow[rr, shift left=2] \arrow[rr, shift right] \arrow[dr] & & \Delta^m \arrow[ll, shift left=0.5] \arrow[dl] \arrow[rr] \arrow[rrd, dashed] \arrow[rrdd, bend right] & & \operatorname{hocolim}_{\Delta^n \downarrow X_\bullet} \Delta^n \arrow[d, "\cong"] \\
& X_\bullet & & & X_\bullet \arrow[d, dotted] \\
& & & & Y_\bullet
\end{tikzcd}$$

$X_\bullet$ also satisfies the universal property of the colimit so by abstract nonsense there is a natural isomorphism

$$\operatorname{hocolim}_{\Delta \downarrow X_\bullet} \Delta^n \cong X_\bullet$$

□

Products:

Def: Given $X_\bullet, Y_\bullet : \Delta^{op} \to \mathcal{C}$,

$$X_\bullet \times Y_\bullet = \Delta^{op} \xrightarrow{\Delta} \Delta^{op} \times \Delta^{op} \to \mathcal{C}.$$

In particular, if

$X_\bullet, Y_\bullet : \Delta^{op} \to \mathrm{Set}$ then

$$(X_\bullet \times Y_\bullet)_n = X_n \times Y_n$$

$$d_i^{X_\bullet \times Y_\bullet} := (d_i^{X_\bullet}, d_i^{Y_\bullet})$$

$$s_i^{X_\bullet \times Y_\bullet} := (s_i^{X_\bullet}, s_i^{Y_\bullet}).$$

Warning: There are more non-degenerate n-simplices than products elts (x,y) where both are non degenerate.

Internal Hom

Def: Let

$$\underline{\mathrm{Hom}}(X_\bullet, Y_\bullet) : \Delta^{op} \longrightarrow \mathrm{Set}$$

be the simplicial set defined on n-simplices by

$$\underline{\mathrm{Hom}}(X_\bullet, Y_\bullet)[n] = \mathrm{Hom}(X_\bullet \times \Delta^n, Y_\bullet).$$

Exercise: $|X_\bullet|$ is a CW complex.

We can therefore consider

$$|-| : s\mathrm{Set} \longrightarrow T$$

compactly generated weak Hausdorf spaces.

Prop: [Milnor]

$$|X_\bullet \times Y_\bullet| \cong |X_\bullet| \times |Y_\bullet| \text{ in } T.$$

<u>Prop</u>: There is an adjunction exhibited by a natural isomorphism

$$\mathrm{Hom}_{sSet}(X_\bullet \times Y_\bullet, Z_\bullet) \cong \mathrm{Hom}_{sSet}(X_\bullet, \underline{\mathrm{Hom}}(Y_\bullet, Z_\bullet))$$

<u>Proof</u>: When $X_\bullet = \Delta^m$.

$$\mathrm{Hom}(\Delta^m \times Y_\bullet, Z_\bullet) = \mathrm{Hom}(\Delta^m, \underline{\mathrm{Hom}}(Y_\bullet, Z_\bullet)),$$

by the Yoneda lemma.

More generally, $X_\bullet \times Y_\bullet = \left(\operatorname{colim}_{\Delta \downarrow X_\bullet} \Delta^n\right) \times Y_\bullet$

$$= \operatorname{colim}_{\Delta \downarrow X_\bullet} (\Delta^n \times Y_\bullet)$$

So there are natural isomorphisms

$$\mathrm{Hom}(X_\bullet \times Y_\bullet, Z_\bullet) \cong \varprojlim_{\Delta \downarrow X_\bullet} \mathrm{Hom}(\Delta^n \times Y_\bullet, Z_\bullet)$$

$$\cong \lim_{\Delta \downarrow X_\bullet} \mathrm{Hom}(\Delta^n, \underline{\mathrm{Hom}}(Y_\bullet, Z_\bullet))$$

$$\cong \mathrm{Hom}(X_\bullet, \underline{\mathrm{Hom}}(Y_\bullet, Z_\bullet)).$$

Recall: $\Delta^1 = \mathrm{Hom}_{sSet}(-,[1])$

This takes the place of I in homotopy theory.

<u>Def</u>: A simplicial homotopy between $f,g: X_\bullet \to Y_\bullet$ is a map

$$H: X_\bullet \times \Delta^1 \longrightarrow Y \quad (H: f \simeq g)$$

such that

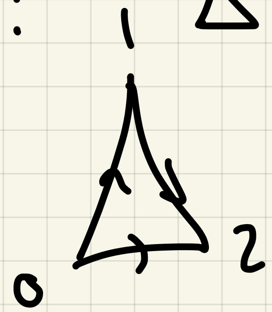
$$H \circ (id_{X_\bullet} \times d_0) = f \quad \text{and}$$

$$H \circ (id_{X_\bullet} \times d_1) = g \quad \text{where}$$

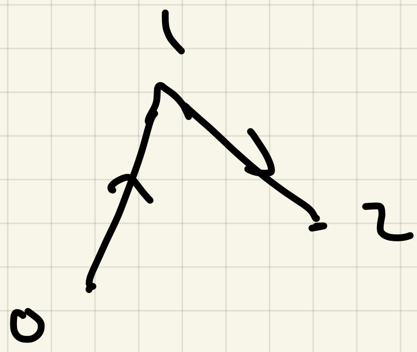
$$H \circ (id_{X_\bullet} \times d_i): X_\bullet \times \Delta^0 \xrightarrow{id_{X_\bullet} \times d_i} X_\bullet \times \Delta^1 \to Y_\bullet.$$

Def: Let Λ^n_k be the subsimplicial set of Δ^n generated by $d_i(\Delta^n)$ for $i \neq k$.

Ex: Δ^2



Λ^2_1



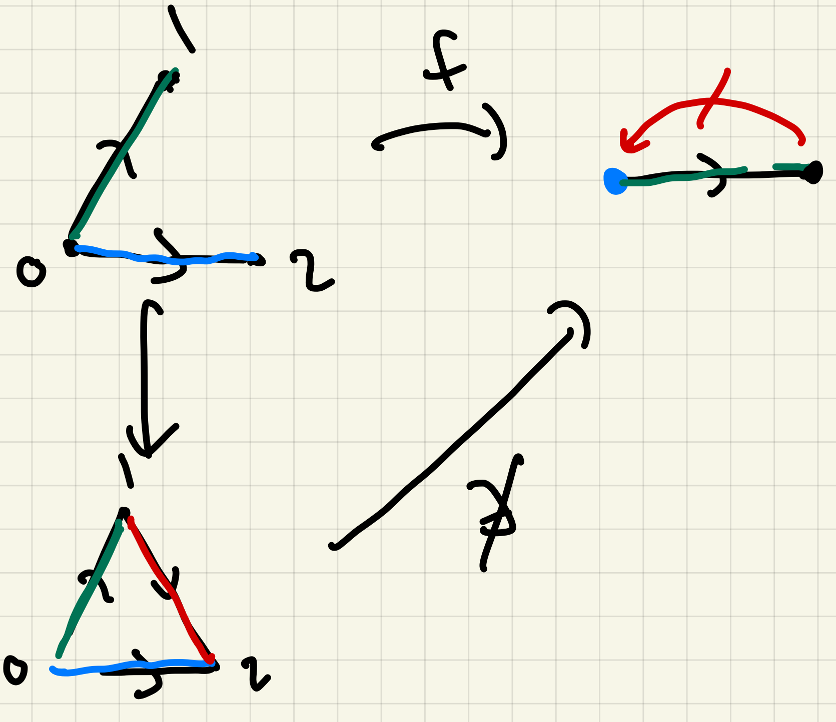
We say $X_\bullet$ is a Kan complex if for every diagram

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d] & X_\bullet \\ \Delta^n \arrow[ur, dashed, "\exists h"'] & \end{tikzcd} \qquad 0 \leq k \leq n.$$

Ex: $\mathrm{Sing}_\bullet(X)$ is always a Kan complex.

Ex: Δ^1 is not a Kan complex

$$\begin{tikzcd} \Lambda^2_0 \arrow[r, "f"] \arrow[d] & \Delta^1 \\ \Delta^2 \arrow[ur, dashed, "\nexists"'] & \end{tikzcd}$$



When $Y_\bullet$ is a Kan complex

simplicial homotopy between maps

$$f, g : X_\bullet \longrightarrow Y_\bullet$$

is an equivalence relation.

$$[X_\bullet, Y_\bullet] = \mathrm{Hom}_{sSet}(X_\bullet, Y_\bullet) \Big/ \sim$$

simplicial htpy equivalence.

<u>Def:</u>

$$ob(hosSet) = \{\text{Kan complexes}\}$$

$$\mathrm{Hom}_{hosSet}(X_\bullet, Y_\bullet) = [X_\bullet, Y_\bullet]$$

<u>Prop</u>: The adjunction $(|-|, \mathrm{sing}_\bullet(-))$ induces an equivalence of categories

$$|-| : ho(sSet) \rightleftarrows ho(T) : \mathrm{sing}_\bullet(\,)$$

exhibited by a natural isomorphism

$$[|X_\bullet|, Y]_{Top} \cong [X_\bullet, \mathrm{sing}_\bullet(Y)]_{sSet}$$

for $X_\bullet \in \mathrm{ob}\, sSet$, $Y \in \mathrm{ob}\, T$.

In particular, if $H : f \simeq g$ is a simplicial homotopy

$$H : X_\bullet \times \Delta^1 \longrightarrow Y_\bullet$$

between Kan complexes $X_\bullet, Y_\bullet$ then $|X_\bullet| \times |\Delta^1| \cong |X_\bullet| \times I \longrightarrow |Y_\bullet|$ is a homotopy between $|f|$ and $|g|$.