A brief introduction to Algebraic K-theory

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Preface

These are notes (in progress) for a 2 hour per week course in algebraic K-theory taught at the Freie Universität Berlin in Winter 2020/21. They will continue to be updated regularly. Please feel free to reach out at my email address gak@math.fu-berlin.de if you notice any typos or errors. These notes are hosted on my personal website.

These notes draw from the tome of Charles Weibel, known as the K-book [25], Algebraic K-theory and its applications by Jonathon Rosenberg [16], Algebraic K-theory by Srinivas [19], as well as the original papers in the subject by Quillen [14, 15], Segal [17], and Waldhausen [22, 23]. In other words, I do not claim originality of any results or proofs. The goal is to give a brief survey of constructions of algebraic K-theory, fundamental theorems in algebraic K-theory, and applications to geometric topology, algebraic geometry, and number theory. The notes focus on the case of algebraic K-theory of rings, which are the common thread in the applications to each of these three subjects.

These notes briefly survey the field of algebraic K-theory from the time period 1950-1985. The advantage of focusing on this time period is that no previous knowledge of $(\infty, 1)$-categories is required and for those interested in modern constructions of algebraic K-theory, most of the essential ideas already existed in the work of Quillen, Segal, and Waldhausen.

It is assumed that students in this course have a firm background in the basics of algebra, linear algebra, category theory, and topology. Additional knowledge of geometry topology, algebraic geometry, and number theory is useful for understanding certain examples, but certainly not required for the bulk of the material. Previous knowledge of the theory of simplicial sets will certainly be helpful, but a short section on such material is included as an appendix for those unfamiliar.

Conventions

Throughout, by a ring we will mean an associative ring with unit. We will always specify that a ring is non-unital when we want to consider it without unit.

Let Mod_R be the category of left modules over a ring R. When $R = \mathbb{Z}$, we simply write Ab for this category and refer to it as the category of abelian groups and when R is a field k, we write Vec_k for this category and refer to it as the category of vector spaces over a field k. We let $M(R)$ denote the skeleton of the category of finitely generated left R modules and we let $P(R)$ denote the skeleton of the category of finitely generated projective left modules over a ring R. When k is a commutative ring, let $\mathsf{Rep}_k(G)$ be the skeleton of the category of finitely generated $k[G]$-modules. Usually, we we only consider this in the case k is a field.

Throughout, by a space we mean a compactly generated weak Hausforff space and we write Top for the category of compactly generated weak Hausdorff spaces. Write CW for the category of CW complexes and CW^f for the category of finite CW complexes. When X is a space, let $VB_{\mathbb{R}}(X)$ denote the skeleton of the category of real vector bundles over X and let $VB_{\mathbb{C}}(X)$ denote the skeleton of the category of complex vector bundles over X.

Let Set be the category of sets and let Fin be the skeleton of the category of finite sets. Let G be a finite group and let Fin_G the skeleton of the category of finite G-sets. When R is a ring, let $\mathsf{Rep}_R(G)$ be the skeleton of the category of finite dimensional $k[G]$-modules.

Chapter 1

Introduction

The 0-th algebraic K-theory group K_0 was first defined by Grothendieck in the late 1950's in order to generalize the Riemann-Roch Theorem to varieties [4]. The name K-theory comes from German word *Klassen* meaning classes and the reason for this name will be more clear after reading Section 2.1.[1] Even earlier, in the early 1950's, Whitehead studied the simply homotopy of of a finite CW complex and constructed an obstruction to two spaces which are homotopy equivalent being simple homotopy equivalent. It was later understood that this class lived in the first algebraic K-theory group K_1 of an integral group ring. It was then shown that these two algebraic K-theory groups could be related by a localization sequence and that there should in fact be a related group K_i for all integers i extending this localization sequence to the left and right.

Milnor constructed the group K_2 of a ring as the center of the Steinberg group of a ring, inspired in part by a theorem of Matsumoto [10], and used this to motivate his definition of the higher algebraic K-theory groups, now known as Milnor K-theory group, in 1970 [12]. However, as we will see, this theory is a not a rich invariant in the sense that for finite fields the Milnor K-theory groups K_n^M vanish for $n \geq 2$.

In 1972, Quillen defined higher algebraic K-theory groups using the $+$-construction [14]. One of his key insights was that the algebraic K-theory groups should be defined as the the homotopy groups of a space. In the same year [15], Quillen defined higher algebraic K-theory groups for a category equipped with notion of exact sequences called an *exact category*. This allowed for much broader input, in particular recovering examples of interest in algebraic geometry.

In 1974 [17], Segal defined the algebraic K-theory of a symmetric monoidal

[1] Though Grothendieck famously spent most of his life in France, he was in fact born in Berlin, Germany in 1939. Perhaps this is why Grothendieck chose the letter K from the German word Klassen rather then the French, but this is not well documented.

category. This notion is sensitive to the symmetric monoidal structure, so it not a special case of Quillen's Q-construction unless the symmetric monoidal structure is the direct sum in an additive category. Quilen's Q-construction is also not a special case of Segal's construction. One of Segal's motivations was to give new constructions of infinity loop spaces, which were known to represent cohomology theories by Brown representability [5].

In 1978, Waldhausen extended Quillen's Q-construction further so that the input could be a category with cofibrations and weak equivalences [22]. This allowed one to define the algebraic K-theory of spaces. This new definition extended the applications of algebraic K-theory to manifold theory [24].

Since 1985, there have been several new constructions of algebraic K-theory using the theory of $(\infty, 1)$-categories. These constructions have proven quite useful for demonstrating universal properties of algebraic K-theory. For example, in 2016, Barwick defined a version of Waldhausen's algebriac K-theory construction for small Waldhausen quasicategories in [1] and proved that algebraic K-theory may be considered as a homology theory, in an abstract sense, on the quasi-category of small Waldhausen quasi-categories. Blumberg–Gepner–Tabuada [3] prove that the connective algebraic K-theory of a small stable quasi-category is the universal additive invariant and non-connective algebraic K-theory of a small stable quasi-category is the universal localizing invariant. Additionally, Gepner-Groth-Nikolaus [7] prove universal properties of the algebraic K-theory of symmetric monoidal quasi-categories. However, will not discuss these more constructions further in the present notes.

Chapter 2

Classical Algebraic K-theory

We begin by studying the groups K_0 and K_1. These two groups arose independently in the 1950's from entirely different contexts. Later, it was proven they are related by a localization sequence.

The group K_2 of a ring was then defined by Milnor, inspired by work of Matsumoto [10], and Milnor used this to motivate his definition of higher algebraic K-theory groups K_*^M now know as Milnor K-theory. However, these groups are not as rich an invariant as the higher algebraic K-theory groups that we will discuss in the next chapters, due to Quillen [14, 15]. We will explicitly prove this in the case of finite fields.

At the start, I want to emphasize that there are really two flavors of algebraic K-theory: algebraic K-theory of symmetric monoidal categories and algebraic K-theory categories with a notion of exact sequences, such as exact categories. The two flavors of algebraic K-theory agree when we consider symmetric monoidal categories with respect to the coproduct and algebraic K-theory of categories with exact sequences in which these exact sequences split. For example, this is the case for the category of finitely generated projective R modules. Since the category of finitely generated projective R modules will be our central example throughout and the distinction may not always be clear.

I also want to emphasize that non of the results in this chapter are new and our treatment in this chapter is almost entirely contained in chapters I-III of Weibel's K-book [25]. In [25], Weibel goes into significantly more depth on this subject than we attempt to do here. We also point the reader towards books of Bass [2] and Milnor [13] from the time period before 1972, which give an even more thorough treatment of what was known at the time about algebraic K-theory groups. The first four chapters of Rosenberg's [16] are also an great reference for this material and he much more depth then I do in this first chapter.

2.1 The Grothendieck group

In order to give a very general definition of K_0, we will first briefly set up the the theory of monoidal categories and monoids in a monoidal category.

Monoidal categories are an abstraction of the properties enjoyed by the category of abelian groups Ab with respect to the tensor product $\otimes_{\mathbb{Z}}$ and the integers $\mathbb{Z}$. In particular, the category Ab is equipped with a functor

$$\otimes_{\mathbb{Z}} \colon \mathsf{Ab} \times \mathsf{Ab} \to \mathsf{Ab}$$

and a unit $\mathbb{Z}$ object in the sense that there are isomorphisms

$$M \otimes_{\mathbb{Z}} \mathbb{Z} \cong M \cong \mathbb{Z} \otimes_{\mathbb{Z}} M$$

for any abelian group M, which are natural in M. The tensor product is also associative

$$(M \otimes N) \otimes L \cong M \otimes (N \otimes L),$$

where this associativity is natural in M, N and L. There is also a factor swap map

$$B_{M,N} \colon M \otimes N \to N \otimes M$$

which is also natural in M and N. In addition, each of these pieces of data satisfy certain commutative diagrams. This data is abstracted to the definition of a symmetric monoidal category, which also applies in many other contexts.

Definition 2.1.1. A *symmetric monoidal category* $\mathcal{C}$ consists of a category $\mathcal{C}$ a functor

$$\otimes \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$$

and a unit $1_{\mathcal{C}}$ together with four natural isomorphisms:

1. an associator

$$a_{-,=,\equiv} \colon (- \otimes =) \otimes \equiv \xrightarrow{\cong} - \otimes (= \otimes \equiv)$$

2. a left unitor

$$\lambda_{-} \colon 1 \otimes (-) \xrightarrow{\cong} (-),$$

3. a right unitor

$$\rho_{-} \colon (-) \otimes 1 \xrightarrow{\cong} (-),$$

and

4. a braiding

$$B_{-,=} \colon (-) \otimes (=) \xrightarrow{\cong} (=) \otimes (-).$$

These natural transformations must satisfy the triangle identity

$$\mathrm{id}_x \otimes \lambda_y \circ a_{x,1_{\mathcal{C}},y} = \rho_x \otimes \mathrm{id}_y \tag{2.1.2}$$

and pentagon identity

$$a_{w,x,y\otimes z} \circ a_{w\otimes x,y,z} = \mathrm{id}_w \otimes a_{x,y,z} \circ a_{w,x\otimes y,z} \circ a_{w,x,y} \otimes \mathrm{id}_z \tag{2.1.3}$$

the hexagon identities

$$a_{y,z,x} \circ B_{x,y\otimes z} \circ a_{x,y,z} = \mathrm{id}_y \otimes B_{x,z} \circ a_{y,x,z} \circ B_{x,y} \otimes \mathrm{id}_z \tag{2.1.4}$$

$$a^{-1}_{y,z,x} \circ B_{x,y\otimes z} \circ a_{x,y,z} = \mathrm{id}_y \otimes B_{x,z} \circ a_{y,x,z} \circ B_{x,y} \otimes \mathrm{id}_z \tag{2.1.5}$$

and the "squaring to identity" axiom

$$B_{y,x} \circ B_{x,y} = \mathrm{id}_{x\otimes y}\,. \tag{2.1.6}$$

We succinctly write $(\mathcal{C}, \otimes, 1_{\mathcal{C}})$ for all of this data.

If $\mathcal{C}$ has all of the other structure except that it is not equipped with a braiding natural transformation $B_{-,=}$ satisfying (2.1.4), (2.1.5), and (2.1.6), then we say $\mathcal{C}$ is a *monoidal category*.

Examples 2.1.7. The category $P(R)$ is a symmetric monoidal category with respect to $\oplus$ denoted $(P(R), \oplus, 0)$ and it is a monoidal category with respect to $\otimes_R$, denoted $(P(R), \otimes_R, R)$. Moreover, when R is a commutative ring then the monoidal category $(P(R), \otimes_R, R)$ is in fact a symmetric monoidal category.

The category $VB_k(X)$ for $k = \mathbb{R}$ or $k = \mathbb{C}$ is a symmetric monoidal category with Whitney sum $\oplus$ and it is a monoidal category with tensor product $\otimes$ denoted $(VB_k(X), \oplus, 0)$ and $(VB_k(X), \otimes, k)$ where k here denotes the trivial one dimensional k vector bundle.

The categories Set (respectively Fin) are symmetric monoidal categories with respect to the coproduct $(\mathsf{Set}, \amalg, \emptyset)$ (respectively $(\mathsf{Fin}, \amalg, \emptyset)$, and with respect to the product $(\mathsf{Set}, \times, *)$ (respectively $(\mathsf{Fin}, \times, *)$). Similarly, Fin_G is a symmetric monoidal category with respect to the coproduct $(\mathsf{Fin}_G, \amalg, \emptyset)$ and the product $(\mathsf{Fin}_G, \times, *)$. Let k be a general field. The category $\mathsf{Rep}_k(G)$ is a symmetric monoidal category with respect to the direct sum $(\mathsf{Rep}_k(G), \oplus, 0)$ and it is a monoidal category with respect to tensor product $(\mathsf{Rep}_k(G), \otimes_k, k)$.

We now discuss monoids in a general symmetric monoidal category.

Definition 2.1.8. A (unital) monoid M in a symmetric monoidal category $\mathcal{C}$ is an object M in $\mathcal{C}$ equipped with an operation

$$\mu\colon\ M \otimes M \to M$$

and a unit map

$$\eta\colon\ 1_{\mathcal{C}} \to M$$

from the unit object $1_{\mathcal{C}}$ in $\mathcal{C}$ to M satisfying:

1. the associativity axiom

$$\begin{tikzcd} M \otimes M \otimes M \arrow[r, "\mu\times 1"] \arrow[d, "1\times\mu"'] & M \otimes M \arrow[d, "\mu"] \\ M \otimes M \arrow[r, "\mu"] & M \end{tikzcd} \tag{2.1.9}$$

and

2. the unitality axiom

$$\begin{tikzcd} M \arrow[r, "\eta\times M"] \arrow[dr, equal] & M \otimes M \arrow[d, "\mu"] & M \arrow[l, "1\times\eta"'] \arrow[dl, equal] \\ & M. & \end{tikzcd} \tag{2.1.10}$$

If in addition, the commutativity axiom

$$\begin{tikzcd} M \otimes M \arrow[rr, "\tau"] \arrow[dr, "\mu"'] & & M \otimes M \arrow[dl, "\mu"] \\ & M & \end{tikzcd} \tag{2.1.11}$$

is satisfied, we say that M is a *commutative monoid in* $\mathcal{C}$.

When $(\mathcal{C}, \otimes_{\mathcal{C}}, 1_{\mathcal{C}}) = (\mathsf{Set}, \times, *)$ we will simply refer to (unital) monoids and commutative monoids in Set as monoids and commutative monoids.

Each of the examples $(P(R), \oplus, 0)$, $(M(R), \oplus, 0)$, $VB_k(X)$, $(\mathsf{Fin}, \amalg, \emptyset)$, $(\mathsf{Fin}, \times, *)$, $(\mathsf{Fin}_G, \amalg, \emptyset)$, $(\mathsf{Fin}_G, \times, *)$, $(\mathsf{Rep}_k(G), \oplus, 0)$, and $(\mathsf{Rep}_k(G), \otimes_k, k)$ may actually be regarded as commutative monoids by applying the forgetful functor from small categories to sets.

If $(M, +, 0)$ is a commutative monoid with operation $+$ and $(M, \times, 1)$ is a monoid with respect to a second operation $\times$ such that $(M, +, \times, 0, 1)$ forms a ring without additive inverses, then we say that M is a semi-ring. In fact, $(P(R), \oplus, \otimes_R, 0, R)$, $(\mathsf{VB}_k(X), \oplus, \otimes, 0, k)$, $(\mathsf{Fin}, \amalg, \times, \emptyset, *)$, $(\mathsf{Fin}_G, \amalg, \times, \emptyset, *)$, and $(\mathsf{Rep}_{\mathbb{C}}(G), \oplus, \otimes_{\mathbb{C}[G]}, 0, \mathbb{C}[G])$ are all examples of semi-rings.

We are now prepared to discuss our definition algebraic K-theory K_0.

Construction 2.1.12. Given a commutative monoid M we form the Grothendieck group completion of M, denoted M^{gp} as follows. We define an equivalence relation on elements $(m, n) \in M \times M$. We define an equivalence relation

$$(m, n) \sim (m + p, n' + p)$$

for any $p \in M$. We then define $M^{\mathrm{gp}} := M \times M / \sim$.

Exercise 2.1.13. Check that $(m, n) \sim (m + p, n' + p)$ is an equivalence relation.

Note that, by construction the abelian group M^{gp} has the universal property that given a map of commutative monoids $M \to A$, where A is an abelian group, then the map $M \to A$ factors as

$$\begin{tikzcd} M \arrow[r] \arrow[dr] & M^{\mathrm{gp}} \arrow[d, dashed] \\ & A. \end{tikzcd} \tag{2.1.14}$$

In other words, there is an adjunction given by the isomorphism

$$\mathrm{Hom}_{\mathsf{CMon}}(M, A) \cong \mathrm{Hom}_{\mathsf{Ab}}(M^{\mathrm{gp}}, A)$$

natural in M and A. In particular, the construction M^{gp} is functorial in M.

Alternatively, we could let $F(M)$ be the free abelian group on symbols $[m]$ where $m \in M$. We can then quotient by the subgroup $R(M)$ of $F(M)$ generated by the relations $[m + n] - [m] - [n]$. This construction also clearly satisfies the universal property 2.1.14. Consequently, we may give a different definition of M^{gp} that agrees with the previous construction up to natural isomorphism

Definition 2.1.15. Given a commutative monoid M, define

$$M^{\mathrm{gp}} := F(M)/R(M)$$

where $F(M)$ and $R(M)$ are as defined above.

For $m \in M$ we will write $[m]$ for a general element in M^{gp}.

Definition 2.1.16. Let R be a ring. Then we define the 0-th algebraic K-theory group of R

$$K_0^{\oplus}(R) := (P(R), \oplus, 0)^+$$

where we regard the set of isomorphism classes of subgroups of $P(R)$ as commutative monoid via $\oplus$ and 0. In fact, since

$$(P(R), \oplus, \otimes_R, 0, R)$$

is a semi-ring, then $K_0(R)$ is a ring. When R is a commutative ring then $K_0(R)$ is also a commutative ring.

In fact, this is a special case of a more general construction.

Definition 2.1.17. Let $(\mathcal{C}, \otimes, 1)$ be a skeletally small symmetric monoidal concrete category with symmetric monoidal skeleton $\mathrm{sk}\,\mathcal{C}$. Then we may regard $\mathrm{sk}\mathcal{C}$ as a commutative monoid in Set with respect to $\otimes$ and 1 and define

$$K_0^{\otimes}(\mathcal{C}) := (\mathrm{sk}(\mathcal{C}), \otimes 1)^+.$$

Moreover, if

$$(\mathrm{sk}\mathcal{C}, \oplus, \otimes_{\mathcal{C}}, 0_{\mathcal{C}}, 1_{\mathcal{C}})$$

is a semi-ring. Then $K_0^{\oplus}(\mathcal{C})$ is a ring.

This general construction allows us to recover many examples of interest.

Examples 2.1.18. The 0-th complex topological K-theory of X is

$$KU^0(X) \cong K_0^{\oplus}(VB_{\mathbb{C}}(X))$$

and the 0-th real topological K-theory of X is

$$KO^0(X) \cong K_0^{\oplus}(VB_{\mathbb{R}}(X)).$$

In fact these are both rings because $(VB_k(X), \oplus, \otimes, 0, k)$ is a semi-ring when $k = \mathbb{C}$ or $k = \mathbb{R}$.

The Burnside ring of a finite group G is

$$A(G) = K_0^{\amalg}(\mathsf{Fin}_G)$$

where the ring structure comes from the fact that $(\mathsf{Fin}_G, \amalg, \times, \varnothing, *)$ is a semi-ring.

Let k be a field. The representation ring of G is

$$R_k(G) = K_0^{\oplus}(\mathsf{Rep}_k(G))$$

where the ring structure comes from the fact that $(\mathsf{Rep}_k(G), \oplus, \otimes_k, 0, \mathbb{C})$ is a semi-ring.

We finish with some basic computations. First, note that $(\mathbb{N}, +, 0)$ is a commutative monoid and its Grothendieck group completion is clearly

$$\mathbb{N}^+ = \mathbb{Z}.$$

Notice that there is always map of commutative monoids

$$\begin{aligned} \mathbb{N} \to& P(R). \\ n \mapsto& R^n \end{aligned}$$

and by functoriality of the Grothendieck construction, a group homomorphism

$$\mathbb{Z} \to K_0(R). \tag{2.1.19}$$

We say that R satisfies the left invariant basis property, denoted IBP, if R^n and R^m are not isomorphic whenever $n \neq m$. In this case, the rank of a free left R module does not depend on a choice of basis. All commutative rings satisfy this property and integral group rings $\mathbb{Z}[G]$ all satisfy the invariant basis property.

Example 2.1.20. The ring of k-linear endomorphisms $\mathrm{End}_k(k)$ does not satisfy the IBP property.

Exercise 2.1.21. Prove that there is an isomorphism of $\mathrm{End}_k(k)$ modules

$$\mathrm{End}_k(k) \cong \mathrm{End}_k(k) \oplus \mathrm{End}_k(k),$$

verifying the claim in Example 2.1.20.

Lemma 2.1.22. *When R satisfies the IBP, then the map*

$$\mathbb{Z} \to K_0(R)$$

induced by the map $n \mapsto R^n$ is injective.

Definition 2.1.23. We define the reduced K_0 group to be the cokernel of the map $\mathbb{Z} \to K_0(R)$ and denote it $\tilde{K}_0(R)$.

Proposition 2.1.24. *When k is a field, then*

$$\tilde{K}_0(k) = 0.$$

Proof. The rank of a vector space gives a map of commutative monoids

$$P(k) \to \mathbb{N}$$

sending $[k^n]$ to n, which is an isomorphism of commutative monoids. $\square$

Exercise 2.1.25. Prove that when R is a principle ideal domain, then

$$\tilde{K}_0(R) = 0.$$

Exercise 2.1.26. Prove that when R is a local ring, then

$$\tilde{K}_0(R) = 0.$$

The invariant $\tilde{K}_0(R)$ has interesting applations to geometry and number theory. For example, when G is a group and $R = \mathbb{Z}[G]$ is the associated integral group ring, then we define the 0-th Whitehead group

$$Wh_0(G) := \widetilde{K}_0(\mathbb{Z}[G]).$$

We will simply state a result of Wall's that shows that the 0-th Whitehead group is an interesting invariant in topology. We say that a topological space X is dominated by a CW complex K if it there is a map $K \to X$ with right homotopy inverse.

Theorem 2.1.27 (Wall's finiteness obstruction)**.** *Suppose that X is dominated by a finite CW complex K and let $G = \pi_1 X$. Then there is an associated obstruction class $w(X) \in Wh_0(G)$ such that $w(X) = 0$ if and only if X is homotopy equivalent to a finite CW complex.*

Remark 2.1.28. Note that we know by CW approximation that X is homotopy equivalent to a CW complex, so it suffices to consider the case when X is also a CW complex, however it is not at all clear that a homotopy retract of a finite CW complex is again finite CW complex.

Reduced K_0 also has applications to number theory. First, we recall a definition from commutative algebra.

Definition 2.1.29. A Dedekind domain R is an integral domain such that for all nontrivial ideals $J \subset I \subset R$, there exists an ideal K such that $IK = J$.

Remark 2.1.30. In a Dedekind domain R every ideal can be written as a product of prime ideals. However, it may not be able to be written as a product of prime ideals in a unique way. If every ideal can be written as a product of prime ideals in a unique way, then R is a PID. Every PID is also clearly a PID.

Definition 2.1.31. The *ideal class group* of a Dedekind domain R it the quotient

$$\mathrm{Cl}(R) = \{I : I \subset R\}/\sim$$

where I ranges over all ideals in R and the equivalence relation states that $I \sim J$ if there exist $x, y \in R$ such that $xI = yJ$ as subsets of R. The group structure is given by the product of ideals.

Exercise 2.1.32. Check that this is in fact an equivalence relation and that $Cl(R)$ is an abelian group.

The ideal class group measures the failure of a Dedekind domain to be a UFD, or in other words, the failure of a Dedekind domain to be a PID.

Again, we will not prove the following result, but we record it as another important application of K_0.

Theorem 2.1.33. *When R is a Dedekind domain, then there is an isomorphism*

$$\tilde{K}_0(R) \cong Cl(R).$$

The class group measures the failure of unique prime factorization. In other words, when R is also a UFD then $\tilde{K}_0(R) = 0$. To see that unique prime factorization can fail, simply consider the ring $\mathbb{Z}[\sqrt{-5}]$. In this ring,

$$(1 - \sqrt{-5})(1 + \sqrt{-5}) = 6 = 2 \cdot 3.$$

Since $\mathbb{Z}[\sqrt{-5}]$ is a Dedekind domain, we observe the following.

Lemma 2.1.34. *There is an isomorphism*

$$K_0(\mathbb{Z}[\sqrt{-5}]) = \mathbb{Z} \oplus \mathbb{Z}/2.$$

The result Theorem 2.1.33 is a special case of a more general result, which we again state without proof.

Theorem 2.1.35. *Let R be a commutative ring of Krull dimension ≤ 1, then there is an isomorphism*

$$\mathit{rank} \oplus \mathit{det} \colon\ K_0(R) \cong [\mathrm{Spec}(R), \mathbb{Z}] \oplus \mathrm{Pic}(R).$$

2.2 The Whitehead group $Wh_1(G)$

In the 1940's and 1950's, Whitehead developed the theory of simple homotopy types. We say that a finite CW complex Y has the same simple homotopy type as a finite CW complex X if they are homotopy equivalent and each homotopy can be described in terms of elementary expansions and collapses.

More precisely, let (K, L) be finite CW pair. Then we write $K \searrow^e L$ and say that *K collapses to L via an elementary collapse* if the following hold:

1. K is build from L by attaching two cells; i.e. $K = L \cup e_{n-1} \cup e_n$ where $e_1, e_2 \not\subset K$.

2. there exists a pair (D^n, D^{n-1}) and a map

$$\psi\colon\ D^n \to K$$

such that

$$\begin{tikzcd} S^{n-1} \arrow[r] \arrow[d] & L \arrow[d] \\ D^{n-1} \arrow[r, "\psi|_{D^{n-1}}"'] & L \cup e_{n-1} \end{tikzcd}$$

and

$$\begin{tikzcd} S^{n} \arrow[r] \arrow[d] & L \arrow[d] \\ D^{n-1} \arrow[r, "\psi"'] & L \cup e_{n-1} \end{tikzcd}$$

such that the closure $\mathrm{cl}(\delta D^n - D^{n-1}) \subset L^{n-1}$, where L^{n-1} is the n-skeleton of L.

In this situation, we also write $L \nearrow^e K$ and say that *L expands to K via an elementary expansions*.

Whitehead defined a group which encoded the obstruction to two homotopy equivalent finite CW complexes having the same simple homotopy type. Suppose X and Y are CW complexes and there is a homotopy equivalence

$$X \overset{\simeq}{\to} Y.$$

Then clearly this homotopy equivalence induces an isomorphism $\pi_1 X \cong \pi_1 Y$. We would like to know whether X and Y are simple homotopy equivalent. There is an obstruction to this, which lies in a group $Wh(\pi_1 X)$, which is an ablian group that depends only on the group $\pi_1 X$. It was later noted that this group can be defined in terms of algebraic K-theory.

Let R be an associative ring and let $GL_n R$ be the group of invertible $n \times n$ matrices with coefficients in R. There is an inclusion

$$GL_n R \subset GL_{n+1} R$$

given by

$$A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$$

We can then form the union (the colimit) to define

$$GL(R) = \bigcup_{n \geq 1} GL_n(R).$$

In general, if we have a group G we can take the quotient by commutators to define

$$G^{\mathrm{ab}} := G/[G,G].$$

In fact this is a left adjoint to the forgetful functor from abelian groups groups so it satisfies a universal property, which is encoded in the natural isomorphism

$$\mathrm{Hom}_{\mathsf{Ab}}(G^{\mathrm{ab}}, A) \cong \mathrm{Hom}_{\mathsf{Gp}}(G, A).$$

Definition 2.2.1. Let R be a ring, then we define

$$K_1(R) := GL(R)^{\mathrm{ab}}.$$

In fact, there is a nice description of the commutator $[GL(R), GL(R)]$. Let $E_n(R)$ be the subgroup of $GL_n(R)$ consisting of the $n \times n$ matrices, which are *transvections*. A transvection is the sum of the identity matrix and a matrix with only one nonzero entry, where that nonzero entry does not occur on the diagonal. We write $e_{i,j}(r)$ for this matrix where r is the nonzero entry and it occurs in the i,j-th position where $i \neq j$. We may then define $E(R)$ in the same way that we defined $GL(R)$ as the union

$$E(R) = \bigcup_{n \geq 1} E_n(R).$$

Definition 2.2.2. A group G is *perfect* if

$$G = [G,G].$$

Note that for a perfect group $G^{\mathrm{ab}} = 0$. Such groups are quite interesting from the perspective of topology. For example, a path connected space such that $\pi_1 X$ is a nontrivial perfect group and $\pi_k X = 0$ for all $k > 0$ has the property that its homology is the same as the homology of a point and yet it X is not contractible.

Lemma 2.2.3. *When $n \geq 3$, then $E_n(R)$ is a perfect group.*

Proof. Whenever i, j, k are distinct, then

$$e_{i,j}(r) = [e_{i,k}(r), e_{k,i}(1)].$$

□

Exercise 2.2.4. Give an example where $E_2(R)$ is not perfect.

Exercise 2.2.5. Verify that, if $g \in GL_n(R)$, the identity

$$\begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix} = \begin{pmatrix} 1 & g \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -g^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & g \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

holds in $GL_{2n}(R)$.

The following example will be useful in the proof of Whitehead's lemma.

Example 2.2.6. A signed permutation matrix is a matrix that permutes the standard basis on R^n up to a sign. If we write $\{e_1, \ldots, e_n\}$ for the standard basis, then a signed permutation acts on the set $\{\pm e_1, \ldots, \pm e_n\}$. We observe that, for example, the signed permutation matrix

$$\overline{w}_{1,2} := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = e_{1,2}(1)e_{2,1}(1)e_{1,2}(1)$$

can be written as a product of transvections and therefore it is contained in $E_2(R)$ for any ring R. More generally,

$$\overline{w}_{i,j} \in E_n(R)$$

for $n \geq i, j$. We can then show that cyclic permutations of three basis elements are also contained in $E_n(R)$, since they can be written as $\overline{w}_{jk}\overline{w}_{i,j}$. Consequently, every matrix corresponding to an even permutation of basis elements is an element in $E_n(R)$ for some n. Thus, by Exercise 2.2.5 we know that $E_{2n}(R)$ contains the matrix

$$\begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix}.$$

The subgroup $E_n(R)$ is not necessarily a normal subgroup in $GL_n(R)$. It is often a normal subgroup in $GL_n(R)$ for sufficiently large n, but even this is too much to ask for in general. When R is a commutative ring, the situation is much easier and $E_n(R)$ is normal in $GL_n(R)$ for $n \geq 3$. Nevertheless, we have the following lemma due to Whitehead which, in particular, implies that $E(R)$ is normal in $GL(R)$.

Lemma 2.2.7 (Whitehead's Lemma)**.** *There is an isomorphism*

$$[GL(R), GL(R)] \cong E(R)$$

Proof. The fact that
$$E(R) \subset [GL(R), GL(R)]$$
follows from Lemma 2.2.3. Conversely, suppose $[A, B] \in [GL_n(R), GL_n(R)]$. Then we can write $[A, B]$ as the product
$$[A, B] = \begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & B^{-1} \end{pmatrix} \begin{pmatrix} (AB)^{-1} & 0 \\ 0 & AB \end{pmatrix}$$
By Example 2.2.6, we therefore know that $[A, B] \in E(R)$. ☐

This gives a new definition of $K_1(R)$.

Definition 2.2.8. Let R be a ring, then we define
$$K_1(R) := GL(R)/E(R).$$

In particular, $K_1(R)$ is a quotient of $GL(R)$ by a perfect normal subgroup. The definition as a quotient by a perfect normal subgroup will be important for the next chapter.

Definition 2.2.9. We define the Whitehead group of a group G as
$$Wh_1(G) := K_1(\mathbb{Z}[G]) / < \pm g : g \in G > .$$
where $g \in G$ is regarded as an element in $GL_1(\mathbb{Z}[G]) \subset GL(\mathbb{Z}[G])$ and
$$< \pm g : g \in G >$$
denotes the subgroup of $K_1(\mathbb{Z}[G])$ generated by the elements $\pm g \in K_1(\mathbb{Z}[G])$.

Again, we will simply cite a deep result that demonstrates that this group is useful for studying problems in topology.

Theorem 2.2.10 (Whitehead)**.** *Suppose K and L are finite CW complexes and there is a homotopy equivalence $f\colon K \to L$ inducing an isomorphism $\pi_1 K \cong \pi_1 L$. Let $G = \pi_1 K$. Then there is an associated class*
$$\tau(f) \in Wh_1(G),$$
called the Whitehead torsion of f, such that $\tau(f) = 0$ if and only if f is a simple homotopy equivalence.

It happens quite often that a homotopy equivalence between finite CW complexes is in fact a simple homotopy equivalence. For an example of non-vanishing Whitehead torsion, consider the Lens spaces
$$L(p, q) = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\} / (z_1, z_2) \simeq (\zeta z_1, \zeta^q z_2)$$

where $p, q \geq 1$ are integers and ζ is a primitive p-th root of unity. Then there exists a homotopy equivalence

$$f\colon\ L(7,1) \xrightarrow{\simeq} L(7,2)$$

and one can prove that any such homotopy equivalence cannot be a *simple* homotopy equivalence [3, p.98], giving an example when

$$\tau(f) \neq 0 \in \mathrm{Wh}_1(\pi_1(L(7,1)) = \mathrm{Wh}_1(\mathbb{Z}/7).$$

In fact, there are other applications of Whitehead torsion to manifold theory. For those unfamiliar with these constructions in manifold theory, we do not plan to give full definitions as that would be too much of a diversion and these constructions will not be used later. We therefore just provide enough information to state the main results in order to indicate the depth of the subject of algebraic K-theory.

Let (W, M, M') be a triple of compact piecewise linear (PL) manifolds. We say this triple is an *h-cobordism* if W has boundary $M \amalg M'$ and both inclusions $M \subset W$ and $M' \subset W$ are homotopy equivalences. There is therefore a Whitehead torsion class $\tau \in Wh_1(\pi_1 M)$ associated to the inclusion $M \subset W$. We record the following deep result, proven by Mazur [11], without proof.

Theorem 2.2.11 (The s-cobordism theorem)**.** *Given an h-cobordism (W, M, M') of PL-manifolds, with M fixed and dim$(M) \geq 5$. Then there is a PL homeomorphism of triples*

$$(W, M, M') \cong (M \times [0,1], M \times \{0\}, M \times \{1\})$$

if and only if $\tau = 0$. Moreover, every element $\tau \in Wh_1(\pi_1 M)$ arises as the Whitehead torsion of some h-cobordism (W, M, M').

This result can be used to prove a version of the generalized Poincare conjecture, which had originally proven by Smale [18] before the s-cobordism theorem was known..

Corollary 2.2.12. *Suppose N is a PL manifold with the same homotopy type as a sphere S^n and $n \geq 5$. Then N is PL-homeomorphic to a S^n.*

Proof. Form a PL manifold W by removing two disjoint n-discs D_1 and D_2 from N. Then we produce a PL cobordism $(W, S_1^{n-1}, S_2^{n-1})$ where S_i^{n-1} is the boundary of D_i in N. Since $\pi_1 S^{n-1} = 0$ when $n \geq 5$, we know that the Whitehead torsion $\tau \in Wh_1(0)$ vanishes. Thus, there is a PL homeomorphism $W \cong S^{n-1} \times [0,1]$ by Theorem 2.2.11 and $N = W \cup D_1 \cup D_2$ is therefore PL homeomorphic to S^n. ☐

2.2.1 Relating K_0 and K_1

Finally, we prove that there is a localization sequence relating K_1 and K_0 in certain cases. Let I be an ideal in R and let $GL(I)$ be the kernel of the map

$GL(R) \to GL(R/I)$. Let $E_n(R,I)$ be the normal subgroup of $E_n(R)$ generated by matrices $e_{i,j}(r)$ such that $r \in I$ and $1 \leq i \neq j \leq n$ and define $E(R,I)$ as the union

$$E(R,I) = \bigcup_{n \geq 1} E_n(R,I).$$

Lemma 2.2.13 (Relative Whitehead Lemma)**.** *The group $E(R,I)$ is normal in $GL(I)$ and*

$$[GL(I), GL(I)] \subset E(R,I).$$

Exercise 2.2.14. Prove Lemma 2.2.13.

Definition 2.2.15. Define the relative K_1 group as

$$K_1(R,I) := GL(I)/E(R,I).$$

We can also define a relative algebraic K-theory group K_0. Given a ring R and an ideal $I \in R$, we can for the trivial square-zero extension of R by I, denoted $R \oplus I$. We will use an explicit model for this square zero extension as the pullback

$$\begin{tikzcd} R \times_{R/I} R \arrow[r] \arrow[d] & R \arrow[d] \\ R \arrow[r] & R/I \end{tikzcd}$$

where the pullback is

$$R \times_{R/I} R = \{(r_1, r_2) : r_1 - r_2 \in I\}.$$

This pullback has a splitting $R \to R \times_{R/I} R$ given by sending r to (r,r) and the kernel is I so we see that it is abstractly isomorphic to $R \oplus I$, but the description in terms of the pullback can be useful for relating the quotient $R \oplus I \to R$ to the quotient $R \to R/I$.

Definition 2.2.16. We define the relative K_0 group as

$$K_0(R,I) := \ker(K_0(R \oplus I) \to K_0(R)).$$

The definitions of relative K_1 and relative K_0 are a bit different. This is because $K_0(R,I)$ in fact does not depend on R. If $R \to S$ is a map of rings and I is mapped isomorphically onto an ideal of S, which we also call I, then $K_0(R,I) \cong K_0(S,I)$. We therefore sometimes simply write

$$K_0(I) := K_0(R,I).$$

The same is not true for $K_1(R,I)$ and in fact there are maps of rings $R \to S$ where I maps isomorphically onto an ideal of S, also denoted I, and yet

$$K_1(R,I) \ncong K_1(S,I).$$

It is known that $K_1(R, I)$ is independent of R if and only if $I^2 = I$, or in other words I is idempotent by [21, 14.2].

Given a map of commutative rings $R \to S$ such that and ideal I in R maps isomorphically onto an ideal of S, the the square

$$\begin{tikzcd} R \arrow[r] \arrow[d] & S \arrow[d] \\ R/I \arrow[r] & S/I \end{tikzcd} \tag{2.2.17}$$

is a pullback of associative rings. We say that the square above is a *Milnor square*.

Construction 2.2.18. Given a Milnor square, we can construct an R-module $M = (M_1, g, M_2)$ from the data of an S-module M_1 and R/I-module M_2 and an isomorphism

$$g\colon\ M_2 \otimes_{R/I} S/I \cong S/I \otimes_S M_1$$

of S/I-modules by defining M to be the pullback

$$\begin{tikzcd} M \arrow[r] \arrow[d] & M_1 \arrow[d] \\ M_2 \arrow[r] & M_1/IM_1 \end{tikzcd}$$

in the category of R-modules.

Lemma 2.2.19. *Every finitely generated projective R-module P can be written as*

$$P = (P_1, P_2, g)$$

for some $P_1 \in P(S)$, $P_2, \in P(R/I)$ and

$$g\colon\ P_2 \otimes_{R/I} S/I \cong S/I \otimes_S P_1$$

in $P(S/I)$.

In particular, given a matrix $A \in GL_n(S/I) \subset GL_n(S/I)$, we can define a finitely generated projective R-module

$$P = (P_1, P_2, A)$$

where

$$A\colon\ P_2 \otimes_{R/I} S/I \cong S/I \otimes_S P_1$$

is our isomorphism and so P_2 is a free R/I-module and P_1 is a free S-module. This defines a map

$$\delta\colon\ GL(S/I) \to K_0(R).$$

Moreover, given a Milnor square this map fits into the following Mayer-Vietoris sequence.

Theorem 2.2.20 (Mayer-Vietoris)**.** *Given a Milnor square (2.2.17), there is an exact sequence*

$$GL(S/I) \to K_0(R) \to K_0(R/I) \oplus K_0(S) \to K_0(S/I)$$

Exercise 2.2.21. If $f\colon R \to S$ is a ring map sending I isomorphically onto an ideal of S, also denoted I, then prove that

$$K_0(R, I) \cong K_0(S, I).$$

Hint: Show that $GL(S)/GL(S \oplus I) = 1$. Then prove that if $I \cap J = 0$, then

$$K_0(I + J) = K_0(I) \oplus K_0(J).$$

Use this to prove that there is an exact sequence

$$1 \to GL(I) \to GL(R) \to GL(R/I) \xrightarrow{\delta} K_0(I) \to K_0(R) \to K_0(R/I). \tag{2.2.22}$$

Proposition 2.2.23. *There is an exact sequence*

$$K_1(R, I) \to K_1(R) \to K_0(R/I) \to K_0(I) \to K_0(R) \to K_0(R/I). \tag{2.2.24}$$

Proof. By Exercise 2.2.21, we know that there is an exact sequence (2.2.22). Passing to quotients by $E(R)$ and $E(R/I)$ gives exactness of the sequence (2.2.24) at $K_1(R/I)$. By Exercise 2.2.21, it therefore suffices to show that the sequence (2.2.24) is exact at $K_0(R)$. Let g be an element of the kernel of the composite

$$GL(R) \to K_1(R) \to K_1(R/I).$$

Then we know that the image of g in $GL(R/I)$, is in $E(R/I)$. Write $\overline{g}$ for this element of $E(R/I)$. Since the map $E(R) \to E(R/I)$ is surjective, there is an element $e \in E(R)$ mapping to $\overline{g}$. Consequently, ge^{-1} maps to $1 \in E(R/I) \subset GL(R)$. So ge^{-1} is in the kernel of $GL(R) \to GL(R/I)$, which we denoted $GL(I)$. Write $[ge^{-1}]$ for the equivalence class of ge^{-1} in $GL(I)/E(R, I)$.

To summarize, for any $g \in K_1(R)$ in the kernel $K_1(R) \to K_1(R/I)$, we have produced a well-defined element $[ge^{-1}]$ in $K_1(R, I)$ that maps to $K_1(R)$. Thus, the sequence (2.2.24) is exact at $K_1(R)$. □

The existence of this sequence was known already in the 1960's [2], but it was not known how to extend the sequence to the left. It was expected that there were groups K_n for all $n \in \mathbb{Z}$ that produce a long exact sequence, but it remained an open question until the 1970's. In Section 2.4, we discuss the first proposed definition of higher algebraic K-theory groups in the early 1970's due to Milnor. This was defined in order to extend the groups K_0, K_1, and K_2, where K_2 is defined in Section 2.3 and the choice of definition of the higher Milnor K-theory groups is clearly inspired by the definition of K_2.

2.3 The Steinberg group and its center

We now discuss K_2 of a ring and the higher Milnor K-theory groups of a field.

Definition 2.3.1. Let A be a ring. Let $n \geq 3$, then the Steinberg group $St_n(A)$ is the group with generators $x_{i,j}(r)$ for $a \in A$ and $i, j \in \{1, \ldots, n\}$ with $i \neq j$ and relations

$$x_{i,j}(a)x_{i,j}(b) = x_{i,j}(a+b) \tag{2.3.2}$$

$$[x_{i,j}(a), x_{k.\ell}(s)] = \begin{cases} 1 & \text{if} j \neq k \text{ and } i \neq \ell \\ x_{i,\ell}(rs) & \text{if} j = k \text{ and } i \neq \ell \\ x_{k,j}(-sr) & \text{if} j \neq k \text{ and } i = \ell, \end{cases} \tag{2.3.3}$$

which are called the *Steinberg relations*.

Exercise 2.3.4. Show that the transvections $e_{i,j}(a)$ in $E_n(A)$ for $n \geq 3$ satisfy the Steinberg relations.

As a consequence of Exercise 2.3.4, there is a canonical surjective group homomorphism

$$St_n(A) \to E_n(A)$$

for $n \geq 3$ mapping $x_{i,j}(a)$ to $e_{i,j}(a)$. Since the Steinberg relations for n include the Steinberg relations for all $k < n$, there is a canonical inclusion

$$St_{n-1}(A) \to St_n(A)$$

and we define

$$St(A) = \bigcup St_n(A).$$

Exercise 2.3.5. Prove that a level map $\{A_i\} \to \{B_i\}$ of sequences of groups, which is a levelwise surjection induces a surjection

$$\operatorname*{colim}_i A_i \to \operatorname*{colim}_i B_i.$$

Definition 2.3.6. Let A be a ring. We define $K_2(A)$ to be the kernel of the canonical surjection

$$St(A) \to E(A).$$

As a consequence of the definition, there is an exact sequence

$$1 \to K_2(A) \to St(A) \to GL(A) \to K_1(A) \to 1.$$

As defined, it is not clear that $K_2(A)$ is an abelian group, because $St(A)$ is not necessarily abelian. However, it turns out that it is an abelian group. Moreover, we have the following result of Steinberg, but we omit the proof.

Theorem 2.3.7 (Steinberg)**.** *The group $K_2(A)$ is abelian and it is exactly the center of $St(A)$.*

We end by remarking that the group K_2 really deserves to be called K_2 in the following sense.

Theorem 2.3.8. *Let A be a Dedekind domain with field of fractions F, then there is a long exact sequence*

$$K_2(F) \to \bigoplus_{\mathfrak{p}\in\mathbb{P}} K_1(A/\mathfrak{p}) \to K_1(A) \to K_1(F) \to \bigoplus_{\mathfrak{p}\in\mathbb{P}} K_0(A/\mathfrak{p}) \to K_0(A) \to K_0(F) \to 0$$

where $\mathbb{P}$ is the set of prime ideals in A.

We will prove this as a consequence of the localization sequence and dévissage due to Quillen [15] in 1972 later in the course. However, it is important to know that the localization sequence as stated in Theorem 2.3.8 had already been proven by Milnor in 1971 and it appears in [13].

2.4 Milnor K-theory of fields

Milnor extended the definition of K_2 to higher algebraic K-theory groups of fields, now known as Milnor K-theory groups, in the 1970's.

Let k be a field. We define the tensor algebra of the group of units k to by

$$T(k^\times) = \bigoplus_{i\geq 0}(k^\times)^{\otimes i}$$

where $(k^\times)^{\otimes 0} = \mathbb{Z}$. This is also known as the free associative algebra on $k^\times$. Write $\ell(x)$ for an element in $k^\times$ in degree 1 corresponding to $x \in k^\times$. We can then define Milnor K-theory of a field as a graded ring all at once.

Definition 2.4.1. The Milnor K-theory groups of a field k are

$$K_*^M(k) := T(k^\times)/(\ell(x) \otimes \ell(1-x) : 1 \neq x \in k^\times)$$

Note that the ideal generated by the elements $\ell(x) \otimes \ell(1-x)$ is a homogeneous ideal and therefore it makes sense to form the quotient in graded rings.

It is clear that

$$K_0^M(k) = \mathbb{Z} = K_0(k) \text{ and}$$
$$K_1^M(k) = k^\times = K_1(k)$$

for any field k. By a theorem of Matsumoto, $K_2^M(k) \cong K_2(k)$.

Theorem 2.4.2 (Matsumoto)**.** *There is an isomorphism*

$$K_2^M(k) \cong K_2(k)$$

for any field k.

This motivated Milnor's definition of higher algebraic K-theory groups. Note that

$$K_0(\mathbb{F}_q) = K_0^M(\mathbb{F}_q) = \mathbb{Z} \text{ and}$$
$$K_1(\mathbb{F}_q) \cong \mathbb{F}_q^\times$$

which is a cyclic group of order $q-1$. In light of this, the following result gives a complete calculation of the Milnor K-theory of of finite fields.

Proposition 2.4.3. *The Milnor K-theory groups $K_n^M(\mathbb{F}_q)$ vanish for all $n \geq 2$. Consequently, there is an isomorphism of graded rings*

$$K_*^M(\mathbb{F}_q) \cong \mathbb{Z} \oplus \mathbb{F}_q^\times$$

where $\mathbb{Z} \oplus \mathbb{F}_q^\times$ is the trivial square zero extension of $\mathbb{Z}$ by the cyclic group $\mathbb{F}_q^\times$.

Proof. We first show that

$$\Big(\mathbb{F}_q^\times \otimes \mathbb{F}_q^\times / (x \otimes (1-x) : x \neq 1, 0)\Big) = 1.$$

We write $(x \otimes y) \cdot (z \otimes w)$ for the group operation and 1 for the unit. Note that $\mathbb{F}_q^\times$ is cyclic of order $q-1$ and consequently $\mathbb{F}_q^\times \otimes \mathbb{F}_q^\times$ is also cyclic of order $q-1$. This cyclic group is generated by $x \otimes x$ whenever x is a generator of $\mathbb{F}_q^\times$.

We split into two cases. If q is even, then we know $2x \otimes x = 0$ in $\mathbb{F}_q \otimes \mathbb{F}_q$. So $x \otimes x = x \otimes -x$. We also know that $x \otimes -x = x \otimes 1$ in $K_2^M(\mathbb{F}_q)$ by the relations and since 1 is the identity in $\mathbb{F}_q^\times$, the element $x \otimes 1$ is trivial in the group $K_2^M(\mathbb{F}_q)$. In other words, we conclude that

$$x \otimes x = 1$$

for all elements $x \otimes x \in K_2^M(\mathbb{F}_q)$ where x is a generator of $\mathbb{F}_q^\times$. This implies that the group $K_2^M(\mathbb{F}_q)$ is trivial. In fact, essentially the same argument implies that $K_n^M(\mathbb{F}_q) = 0$ for $n > 2$.

When q is odd, we still know that $x \otimes -x = x \otimes 1$ is trivial and consequently, we have skew-symmetry

$$(x \otimes y) \cdot (y \otimes x) = (x \otimes -xy) \cdot (y \otimes -xy) = xy \otimes -xy = 1$$

in $K_2^M(\mathbb{F}_q)$. This immediately implies that $(x \otimes x)^2 = 1$ and more generally one can show that

$$(x \otimes x)^{mn} = x^m \otimes x^n$$

when m, n are odd. The set of odd powers of elements in $K_2^M(\mathbb{F}_q)$ is exactly the same as the non-squares, by construction. If there exists a non-square u such that $1-u$ is also a non-square in $\mathbb{F}_q$, then all elements are divisible by the element $u \otimes (1-u) = 0$, or in other words all elements are trivial. To find such a u, we note that there is an involution $u \mapsto 1-u$ on the set $\mathbb{F}_q = \{0, 1\}$. and the set $\mathbb{F}_q - \{0, 1\}$ consists of $(q-1)/2$ non squares, but only $(q-3)/2$ squares. In other words, there are strictly less squares than non-squares and there must be an orbit of the C_2-action that is completely contained in the non-squares. Again, essentially the same proof implies that $K_n^M(\mathbb{F}_q) = 0$ for $n > 2$. $\square$

We include this result in order to indicate that Milnor K-theory, though very interesting in its own right, is this is not the richest invariant. We will see a different construction of higher algebraic K-theory groups where the algebraic K-theory of fields are nontrivial in arbitrarily high degrees.

Chapter 3

Group completion algebraic K-theory

As we hinted at in the beginning of the course, there are really two main flavors of algebraic K-theory: group completion algebraic K-theory and algebraic K-theory of of a category with a notion of exact sequences. Here we describe a construction of group completion algebraic K-theory, due to Quillen [14], known as the $+$-construction. We then give a sketch of the computation of algebraic K-theory of finite fields due to Quillen. In the following section, we define algebraic K-theory of a symmetric monoidal category via the $S^{-1}S$ construction of Quillen and Segal, see [8].

3.1 The $+$-construction

This section draws heavily from [19, Chp. 2]. We gave a construction of the classifying space of a (topological) group in Definition B.1.31. In particular, we can consider the infinite general linear group $GL(R)$ of a ring R and form its classifying space $BGL(R)$. This has the property that

$$\pi_k(BGL(R)) = \begin{cases} GL(R) & \text{if } k = 1 \\ 0 & \text{otherwise.} \end{cases}$$

and any other CW complex with this property will be homotopy equivalent to the model for $BGL(R)$ we gave in Definition B.1.31.

In particular, we note that if we want to form a space $K(R)$ with $\pi_1(K(R) = K_1(R)$ then we will have to attach cells to $BGL(R)$ to kill the perfect normal subgroup $E(R)$ in $GL(R)$. We would also like to not change the space $BGL(R)$ too much, so we attach cells so that the resulting CW complex $BGL(R)^+$ has the property that

$$H_*(BGLR(R)^+, BGL(R); L) \cong 0$$

for any $\mathbb{Z}[K_1(R)]$-module L. In fact, the space $BGL(R)^+$ can be characterized by this property. In particular, the induced map

$$H_*(BGL(R);\mathbb{Z}) \to H_*(BGL(R)^+;\mathbb{Z})$$

is an isomorphism. We will give a different characterization of the $+$-construction, which uses the following definition.

Definition 3.1.1. An H-space is topological space X equipped with continuous maps

$$\mu\colon\ X \times X \to X$$

and

$$e\colon\ * \to X$$

such that the diagram

$$\begin{tikzcd} X \arrow[r, "e\times \mathrm{id}_X"] \arrow[dr, "\mathrm{id}_X"'] & X \times X \arrow[d, "\mu"] & X \arrow[l, "\mathrm{id}_X \times e"'] \arrow[dl, "\mathrm{id}_X"] \\ & X & \end{tikzcd}$$

commutes up to homotopy, in the sense that

$$\mu \circ (e \times \mathrm{id}_X) \simeq \mathrm{id}_X \simeq \mu \circ (\mathrm{id}_X \times e).$$

If in addition X is a commutative group object in the homotopy category of topological spaces ho(Top), then we say that X is a commutative H-group.

Example 3.1.2. Let X be a based topological space. Recall that $\Omega X = \mathrm{Map}_*(S^1, X)$, the space of pointed maps from $S^1 \to X$, has an operation

$$\Omega X \times \Omega X \to \Omega X$$

given by concatonation of loops and a unit

$$e\colon\ * \to \Omega X$$

which sends the basepoint to the constant loop at the basepoint of X. This does not satisfy unitality on the nose, but it does satisfy unitality up to homotopy so ΩX is an H-space.

Definition 3.1.3. The space $BGL(R)^+$ is the initial H-space receiving a map from $BGL(R)$, in other words, whenever there is a map $BGL(R) \to Y$ where Y is an H-space, then the map factors as

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & BGL(R)^+ \arrow[d, dashed] \\ & Y. \end{tikzcd}$$

Remark 3.1.4. Note that $\pi_1(Y)$ is abelian whenever Y is an H-space by the Eckman-Hilton argument. Therefore, $\pi_1(BGL(R)^+)$ must be the abelianization of $\pi_1(BGL(R)) = GL(R)$ as desired.

We may therefore think of the $BGL(R)^+$ as the *H-spacification* of $BGL(R)$. This specifies the space $BGL(R)^+$ up to homeomorphism, but we need to know that such a space exists. We therefore give an explicit construction. First, we give our definition of the algebraic K-theory space of a ring R

Definition 3.1.5. Given a ring R, the algebraic K-theory space of R is

$$K(R) := K_0(R) \times BGLR^+$$

where $K_0(R)$ is regarded as a discrete space. Moreover,

$$K_n(R) := \pi_n(K(R)).$$

We now give an explicit construction of X^+ for a general path connected based space X of the homotopy type of a CW complex such that $[\pi_1 X, \pi_1 X]$ is a perfect normal subgroup. In fact, there is a plus construction associated to any perfect normal subgroup N of $\pi_1(X)$, but we leave that generalization to the reader. In fact, this is the context in which the +-construction was originally defined by Kervaire in [9], before work of Quillen [14]. Kervaire studied this construction in the context of n-manifolds that are homology spheres, so manifolds M such that $\pi_1(M)$ is a perfect subgroup and M^+ is homotopy n-sphere for $n \geq 3$.

Construction 3.1.6. Let X be a path connected based space of the homotopy type of a CW complex such that $[\pi_1 X, \pi_1 X]$ is a perfect normal subgroup. We will construct a relative CW complex (X^+, X) by attaching cells to X so that $\pi_1(X^+) = H_1(X; \mathbb{Z})$ and the relative homology $H_*(\tilde{X}^+, \hat{X}; \mathbb{Z})$ is trivial where $\tilde{X}$ is the universal cover of X^+ and $\hat{X}$ is the pullback of the universal cover along the map $X \to X^+$. We can actually accomplish this by only attaching 2-cells and 3-cells to X to form X^+.

Write $N = [\pi_1 X, \pi_1 X]$. Choose a minimal set of generators e_α of the subgroup N of $\pi_1 X$. Choose a basepoint preserving loop $\gamma_\alpha \colon S^1 \to X$ for each homotopy class $e_\alpha \in \pi_1 X$. We can then form X_1 as the the pushout

$$\begin{tikzcd} \coprod_\alpha S^1 \arrow[r, "\amalg \gamma_\alpha"] \arrow[d] & X \arrow[d] \\ \coprod_\alpha S^1 \arrow[r] & X_1 \end{tikzcd}$$

and by construction $\pi_1(X_1) = \pi_1(X)/N$ as desired. However, we note that by the exact sequence in homology

$$0 \to H_2(X; \mathbb{Z}) \to H_2(X_1; \mathbb{Z}) \to H_1(\coprod_\alpha S^1; \mathbb{Z}) \xrightarrow{0} H_1(X; \mathbb{Z}) \xrightarrow{\cong} H_1(X_1; \mathbb{Z})$$

the map $H_2(X) \to H_2(X_1)$ is not an isomorphism as desired.

Let $\widetilde{X}_1 \to X_1$ be the universal cover of X_1 and let $\widehat{X}$ be the pullback

$$\begin{tikzcd} \widehat{X} \arrow[r] \arrow[d] & \widehat{X}_1 \arrow[d, "\pi"] \\ X \arrow[r] & X_1. \end{tikzcd}$$

Note that $\widetilde{X}_1$ is formed from $\widehat{X}$ by attaching 2-cells as well and $\tilde{X}_1$ is connected so $\widehat{X}$ is also connected. We therefore observe that $\widehat{X}$ is a Galois covering space of X corresponding to the normal subgroup N in $\pi_1(X)$ and associated Galois group $\pi_1(X)/N$. For each 2-cell a_α in the relative CW complex (X_1, X), we can pullback to a collection of 2-cells $\pi^{-1}(a_\alpha)$ of the relative CW complex $(\tilde{X}_1, \hat{X})$ and $\pi_1(X)$ acts transitively on these 2-cells with stabilizer group $\pi_1 X/N$. Therefore, $H_*(\widetilde{X}_1, \widehat{X}; \mathbb{Z})$ is a free $\mathbb{Z}[\pi_1 X/N]$-module on generators $[\tilde{a}_\alpha]$, where $\tilde{a}_\alpha$ is a lift of a 2-cell a_α of (X_1, X) to $(\widetilde{X}_1, \widehat{X})$. We then apply the Hurewicz homomorphism vertically to the long exact sequence in homotopy associated to the pair $(\widetilde{X}_1, \widehat{X})$ to form the commutative diagram

$$\begin{tikzcd} \pi_2(\widehat{X}) \arrow[r] \arrow[d] & \pi_2(\widetilde{X}_1) \arrow[r] \arrow[d] & \pi_2(\widetilde{X}_1, \widehat{X}) \arrow[r] \arrow[d] & \pi_1(\widehat{X}_1) \arrow[d] \\ H_2(\widehat{X}; \mathbb{Z}) \arrow[r] & H_2(\widetilde{X}; \mathbb{Z}) \arrow[r] & H_2(\widetilde{X}_1, \widehat{X}; \mathbb{Z}) \arrow[r] & H_1(\widehat{X}; \mathbb{Z}). \end{tikzcd}$$

Since N is perfect, $H_1(\hat{X}; \mathbb{Z}) = N/[N, N] = 0$. Also, since $\pi_1(\widetilde{X}_1) = \pi_0(\widetilde{X}_1) = 0$, the Hurewicz theorem implies that $\pi_2(\widetilde{X}) \cong H_2(\widetilde{X}; \mathbb{Z})$. We know therefore know the composite map

$$\pi_2(\widetilde{X}_1) \to H_2(\widetilde{X}_1; \mathbb{Z}) \to H_2(\widetilde{X}_1, \widehat{X}; \mathbb{Z})$$

is surjective. For each $[\tilde{a}_\alpha]$ we choose a lift $[\tilde{f}_\alpha]$ to $\pi_2(\widetilde{X}_1)$ represented by a basepoint preserving map $\tilde{f}_\alpha \colon S^2 \to \widetilde{X}_1$. Define

$$f_\alpha = \pi \circ \tilde{f}_\alpha$$

and let X^+ be the pushout

$$\begin{tikzcd} \coprod_\alpha S^2 \arrow[r, "f_\alpha"] \arrow[d] & X_1 \arrow[d] \\ \coprod_\alpha D^3 \arrow[r] & X^+. \end{tikzcd}$$

Remark 3.1.7. At various points in the construction, we made choices. We leave it to the reader to check that the choices we made do not matter in the sense that a different choice would lead to a space that is homeomorphic to the space X^+.

We now have to check that this construction has the desired properties. First, note that we only attached 3-cells to X_1 to form X^+, so

$$\pi_1(X^+) \cong \pi_1(X)/N$$

as desired.

Write $\widetilde{X}^+$ for the universal cover of X^+. Then we have a diagram

$$\begin{tikzcd} \widehat{X} \arrow[r] \arrow[d] & \widetilde{X}_1 \arrow[r] \arrow[d] & \widetilde{X}^+ \arrow[d] \\ X \arrow[r] & X_1 \arrow[r] & X^+ \end{tikzcd}$$

where each square in the diagram is a pullback. Again, since X^+ was formed from X by only attaching only 2-cells and 3-cells, we know that $\widetilde{X}^+$ is formed from $\widehat{X}$ by attaching only 2-cells and 3-cells. Consequently, the relative cellular chains $C_*(\widetilde{X}^+, \widehat{X}; \mathbb{Z})$ are concentrated in two degrees

$$0 \to C_3(\widetilde{X}^+, \widehat{X}) \xrightarrow{d} C_2(\widetilde{X}^+, \widehat{X}) \to 0. \tag{3.1.8}$$

We then observe that there are isomophisms

$$C_3(\widetilde{X}^+, \widehat{X}) \cong C_3(\widetilde{X}^+, \widetilde{X}_1) \cong H_3(\widetilde{X}^+, \widetilde{X}_1; \mathbb{Z})$$
$$C_2(\widetilde{X}^+, \widehat{X}) \cong C_2(\widetilde{X}_1, \widehat{X}) \cong H_2(\widetilde{X}_1, \widehat{X}; \mathbb{Z})$$

and the boundary map d in (3.1.8) is exactly the boundary map

$$d\colon\ H_3(\widetilde{X}^+, \widetilde{X}_1; \mathbb{Z}) \to H_2(\widetilde{X}_1, \widehat{X}; \mathbb{Z})$$

associated to the long exact sequence of the triple $(\widetilde{X}^+, \widetilde{X}_1, \widehat{X})$. Therefore, the map d from (3.1.8) factors as

$$d\colon\ H_3(\widetilde{X}^+, \widetilde{X}_1; \mathbb{Z}) \xrightarrow{\partial} H_2(\widetilde{X}_1; \mathbb{Z}) \xrightarrow{j} H_2(\widetilde{X}_1, \widehat{X}; \mathbb{Z})$$

where ∂ is the boundary map in the long exact sequence for the pair $(\widetilde{X}^+, \widetilde{X}_1)$ and j is induced by the canonical map of pairs $(\widetilde{X}_1, \emptyset) \to (\widetilde{X}_1, \hat{X})$.

We then examine the diagram

$$\begin{tikzcd} \dots \arrow[r] & \pi_3(\widetilde{X}^+) \arrow[r] \arrow[d] & \pi_3(\widetilde{X}^+, \widetilde{X}_1) \arrow[r] \arrow[d] & \pi_2(\widetilde{X}_1) \arrow[d] \\ \dots \arrow[r] & H_3(\widetilde{X}^+; \mathbb{Z}) \arrow[r] & H_3(\widetilde{X}^+, \widetilde{X}_1; \mathbb{Z}) \arrow[r, "\partial"] & H_2(\widetilde{X}_1; \mathbb{Z}) \end{tikzcd}$$

where the vertical maps are the Hurewicz maps and the horizontal maps are the long exact sequence of a pair. Given an element

$$[\tilde{b}_\alpha] \in C_3(\tilde{X}^+, \tilde{X}_1; \mathbb{Z}) \cong H_3(\tilde{X}, \tilde{X}_1; \mathbb{Z})$$

corresponding to a 3-cell $\tilde{b}_\alpha$ of the relative CW complex $(\widetilde{X}^+, \tilde{X})$. The attaching map $\tilde{f}_\alpha \colon S^2 \to \widetilde{X}_1$ of the 3-cell of $(\widetilde{X}^+, \widetilde{X}_1)$ corresponds to a class $[\tilde{f}_\alpha]$ of $\pi_2(\widetilde{X}_1)$ and maps to a class $[\tilde{f}_\alpha]$ in $H_2(\tilde{X}_1; \mathbb{Z})$. So by inspection $[\tilde{b}_\alpha]$ maps to $[\tilde{f}_\alpha]$ under the map

$$\delta\colon\ H_3(\tilde{X}^+, \widetilde{X}_1; \mathbb{Z}) \to H_2(\tilde{X}_1; \mathbb{Z}).$$

In addition, the map

$$j\colon\ H_2(\tilde{X}_1; \mathbb{Z}) \to H_2(\widetilde{X}_1, \widehat{X}; \mathbb{Z})$$

sends $[\tilde{f}_\alpha]$ to $[\tilde{a}_\alpha]$.

By the same argument as before, $H_3(\widetilde{X}^+, \widetilde{X}_1; \mathbb{Z})$ is a free $\mathbb{Z}[\pi_1(X)/N]$-module on generators $[\tilde{b}_\alpha]$ and similarly $H_2(\widetilde{X}^+, \widehat{X})$ is a free $\mathbb{Z}[\pi_1(X)/N]$-module on generators $[\tilde{a}_\alpha]$ where α ranges over the same indexing set I. Thus, the map d is an isomorphism of $\mathbb{Z}[\pi_1(X)/N]$-modules and consequently $C_*(\tilde{X}^+, \widehat{X})$ is acyclic.

Moreover, for any $\mathbb{Z}[\pi_1(X)/N]$-module L, the chains $C_*(\widetilde{X}^+, \widehat{X}; L)$ whose homology is $H_*(X^+, X; L)$ are defined to be

$$C_*(\widetilde{X}^+, \widehat{X}; L) \cong L \otimes_{\mathbb{Z}[\pi_1(X)/N]} C_*(\widetilde{X}^+, \widehat{X}; \mathbb{Z})$$

and consequently $C_*(\widetilde{X}^+, \widehat{X}; L)$ is also an acyclic chain complex. In particular,

$$H_*(X; \mathbb{Z}) \to H_*(X^+; \mathbb{Z})$$

is an isomorphism. Moreover, we have shown the following.

Lemma 3.1.9. *For any $\mathbb{Z}[\pi_1 X]$-module L, the groups*

$$H^*(X^+, X; L) \cong 0.$$

We still need to show that X^+ satisfies the universal property of Definition (3.1.3). The proof will be by obstruction theory. First, we give a general setup of obstruction theory.

Let (X, A) be a based relative CW complex with base point $x_0 \in A$. Assume that X and A are path connected. Given a continuous pointed map $f\colon\ A \to Y$ where Y is path connected,

1. when does there exist an extension F

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & Y \\ X \arrow[ur, dashed, "F"'] & \end{tikzcd}$$

such that the diagram commutes, where $i\colon A \to X$ is the canonical inclusion?

2. Given two such extensions $F, F'\colon X \to Y$ such that $F|_A = F'|_A = f$, when is there a homotopy $H\colon X \times I \to Y$ rel A from F to F' ?

Write X_n for the n-skeleton of X. Then we can always extend

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"] & Y \\ X_1 \arrow[ur, dashed, "F_1"'] & \end{tikzcd}$$

because if $g\colon \coprod S^0 \to A$ is the attaching map for attaching the 1-cells to A to form X_1, when composing this map with f

$$\coprod S^0 \to A \to Y$$

necessarily produces a null homotopy map, because Y is path connected. therefore, we can let F_1 be any null homotopic basepoint preserving map $F_1\colon X_1 \to Y$. Note that $\pi_1(X_2) \cong \pi_1(X)$, so if it is possible to extend further

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & Y \\ X_1 \arrow[d] & \\ X_2 \arrow[uur, dashed, "F_2"'] & \end{tikzcd} \tag{3.1.10}$$

then there would be a homomorphism θ in a commuting diagram

$$\begin{tikzcd} \pi_1 A \arrow[r, "f"] \arrow[d] & \pi_1 Y \\ \pi_1 X \arrow[ur, dashed, "\theta"'] & \end{tikzcd} \tag{3.1.11}$$

In fact, this is an if and only if, so if there is a group homomorphism $\theta\colon \pi_1 X \to \pi_1 Y$ making the diagram commute, then there exists an extension F_2 making the diagram (3.1.10) commute.

Now suppose A extends to X_2 and fix a group homomorphism $\theta\colon \pi_1 X \to \pi_1 Y$ such that the diagram (3.1.11) commutes. Now suppose we want to extend A to X_3

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & Y \\ X_3 \arrow[ur, dashed, "F_3"'] & \end{tikzcd} \tag{3.1.12}$$

compatible with the extensions F_1 and F_2, which amounts to asking that on π_1 the diagram (3.1.12) is the diagram (3.1.11) up to isomorphism. Recall that $\pi_1(Y)$ acts on $\pi_i(Y)$ for $i \geq 1$, so $\pi_i(Y)$ is a $\mathbb{Z}[\pi_1 Y]$-module for all $i \geq 2$ (and all $i \geq 1$ if Y is an H-space, as is the case in our main example of interest). Since we have a group homomorphism $\theta\colon \pi_1(X) \to \pi_1(Y)$ by assumption, this induces a map $\mathbb{Z}[\pi_1 X] \to \mathbb{Z}[\pi_1 Y]$ of group rings and therefore we may regard $\pi_i(Y)$ as a $\mathbb{Z}[\pi_1 X]$-module $\theta^*\pi_i(Y)$ via restriction.

We therefore proceed by induction up the skeleta of X. Suppose we have an extension $F_n\colon X_n \to Y$ of f to X_n for some $n \geq 2$ such that $\pi_1(F_n) = \theta\colon \pi_1(X_n) = \pi_1 X \to \pi_1 Y$. Then we can define an obstruction cocycle giving an obstruction class

$$\mathrm{Ob}^{n+1}(\theta, F_n) \in H^{n+1}(X, A, \theta^*\pi_n(Y))$$

in the cohomology of the pair (X, A) with coefficients in the $\mathbb{Z}[\pi_1 X]$-module $\theta^*\pi_n(Y)$. The vanishing of this obstruction class is a necessary and sufficient condition for F_n to extend further to $F_{n+1}\colon X_{n+1} \to Y$ such that $F_{n+1}|_{X_n} = F_n$. When $H^n(X, A, \theta^*\pi_n(Y)) \cong 0$ for all $n \geq 2$, then f extends to X.

The second question may actually be regarded as a special case of the first for the pair $(X \times I, X \times \{0,1\} \cup A \times I)$. Recall that by Van Kampen

$$\pi_1(X \times \{0,1\} \cup A \times I) \cong \pi_1(X) *_{\pi_1(A)} \pi_1(X)$$

and the map

$$\pi_1(X \times \{0,1\} \cup A \times I) \to \pi_1(X)$$

is the canonical quotient. Defining $\tilde{f}\colon X \times \{0,1\} \cup A \times I \to Y$ by

$$\begin{aligned} \tilde{f}|_{X\times 0} &= F \\ \tilde{f}|_{X\times 1} &= F' \\ \tilde{f}|_{A\times I} &= f \circ \pi_1(x,t) \end{aligned}$$

where $\pi_1\colon A \times I \to A$ is projection onto the first factor. Then we can say that F is homotopic to F' rel A if and only if the map $\tilde{f}$ extends to a map $\tilde{F}\colon X \times I \to Y$. Again, $\tilde{f}$ will extend to the 2-skeleton of $X \times I$ if $\tilde{\theta} = \pi_1(F) = \pi_1(F')\colon \pi_1 X \to \pi_1 Y$ by the calculation of fundamental groups above. For $n \geq 1$, the obstructions lie in

$$H^{n+2}(X \times I, X \times \{0,1\} \cup A \times I; \tilde{\theta}^*\pi_{n+1}(Y)))$$

but by inspection of the suspension homomorphism these groups are isomorphic to
$$H^{n+1}(X, A; \tilde{\theta}^* \pi_{n+1}(Y))).$$
We therefore have an obstruction class
$$\mathrm{Ob}^{n+1}(\theta, F, F') \in H^{n+1}(X, A; \tilde{\theta}^* \pi_{n+1}(Y)))$$
which is trivial if and only if there is exists a homotopy $H\colon F \simeq F'$ rel A. If $H^n(X, A; \tilde{\theta}^* \pi_n(Y))) \cong 0$ for $n \geq 2$, then there always exists a homotopy $H\colon F \simeq F'$ rel A.

We now summarize in our special case.

Theorem 3.1.13. *Let X be a based path connected space. The construction X^+ of (3.1.6) has the universal property that for any based path connected commutative H-group Y with a map $X \to Y$, then there exists a lift h, that is unique up to homotopy*

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & X^+ \arrow[d, dashed, "h"] \\ & Y \end{tikzcd}$$

Proof. Since Y is path connected, there is no obstruction to extending to the 1-skeleton of X^+ (the one skeleton of X^+ is X anyways, so there is nothing to show). The obstruction to extending X to the 2-skeleton of X^+ is the existence of a group homomorphism $\theta\colon \pi_1(X^+) \to \pi_1(Y)$ such that the diagram

$$\begin{tikzcd} \pi_1(X) \arrow[r] \arrow[dr] & \pi_1(X^+) \arrow[d, "\theta"] \\ & \pi_1(Y) \end{tikzcd}$$

commutes. This group homomorphism exists by the universal property of $\pi_1(X^+) = \pi_1(X)/[\pi_1(X), \pi_1(X)] =: \pi_1(X)^{\mathrm{ab}}$ since $\pi_1(Y)$ is necessarily an abelian group. Finally, the obstructions to existence of an extension $h\colon X^+ \to Y$ and uniqueness of such a map live in $H^*(X^+, X; \theta^* \pi_* Y)$, but these groups vanish by Lemma 3.1.9. Therefore, there exists an extension $X^+ \to Y$ and this extension is unique up to homotopy rel X as desired. □

Remark 3.1.14. Note that we still need to show that X^+ is in fact an H-space. We will do this shortly. First, we give an example.

Example 3.1.15. Consider the colimit
$$\Sigma := \cup_{n \geq 0} \Sigma_n$$
of Σ_n along the inclusions $\Sigma_n \to \Sigma_{n+1}$ sending an automorphism of the finite set $\{1, \ldots, n\}$ to the corresponding automorphism of the finite set $\{1, \ldots, n +$

$1\}$ fixing $n+1$. Then the inclusion of the alternating $A_n \to \Sigma_n$ is compatible and produces an inclusion of $A = \cup_{n\geq 0} A_n$ in Σ with quotient $\mathbb{Z}/2$. The group A is in fact a perfect normal subgroup of Σ and we can form the $+$-construction with respect to this perfect normal subgroup. This gives a space $B\Sigma^+$ such that $\pi_1(B\Sigma^+) \cong \mathbb{Z}/2$. In fact, we have the following theorem of Barratt-Priddy-Quillen.

Theorem 3.1.16 (Barratt-Priddy-Quillen)**.** *There is an isomorphism*

$$\pi_k^s(S^0) \to \pi_k(\mathbb{Z} \times B\Sigma^+)$$

for $k \geq 0$, where $\pi_k^s S^0$ is the k-th stable homotopy groups of spheres.

Remark 3.1.17. We should think of $\mathbb{Z} \times B\Sigma^+$ as an explicit model for the algebraic K-theory space $K(\mathsf{Fin})$ of the category of finite sets. We will prove this later.

Theorem 3.1.18. *The space $BGL(R)^+$ is an commutative H-group and the map $BGL(R)^+ \to Y$ is a map of H-spaces whenever Y is an H-space.*

We first define a multiplication

$$\mu\colon\ BGL(R)^+ \times BGL(R)^+ \to BGL(R)^+.$$

As we noted in the appendix, geometric realization commutes with finite products, so we first define an operation

$$\mu_0\colon\ GL(R) \times GL(R) \to GLR(R)$$

and then define a map

$$B\mu_0 \circ \mathrm{h}\colon\ BGL(R) \times BGL(R) \to BGL(R)$$

where h is the homotopy equivalence

$$\mathrm{h}\colon\ BGL(R) \times BGL(R) \xrightarrow{\simeq} B(GL(R) \times GL(R)).$$

We define $\mu_0(A, B)$ for matrices $A, B \in GL(R)$ by the formula

$$\mu_0(A, B) = (c_{i,j})$$

for $0 \leq i, j \leq n+m$ where

$$c_{i,j} = \begin{cases} a_{k,\ell} & \text{if } i = 2k-1, j = 2\ell - 1 \\ b_{k,\ell} & \text{if } i = 2k, j = 2\ell \\ 0 & \text{otherwise} \end{cases}$$

where $A = (a_{k,\ell})$ for $0 \leq k, \ell \leq n$ and $B = (b_{k,\ell})$ for $0 \leq k, \ell \leq m$. We leave the following results as simple exercises.

Exercise 3.1.19. The map μ_0 is a group homomorphism.

Exercise 3.1.20. There is a homotopy equivalence

$$h' \colon BGL(R)^+ \times BGL(R)^+ \to (BGL(R) \times BGL(R))^+ .$$

We therefore have an induced map

$$\mu \colon BGL(R)^+ \times BGL(R)^+ \to BGL(R)^+$$

We now construct an inverse. First, let $u \colon \mathbb{N} \to \mathbb{N}$ be an injective self-map of the set of positive integers. Define

$$u_\bullet \colon GL(R) \to GL(R)$$

where $A = (a_{i,j})$ by

$$u_\bullet(A) = \begin{cases} a_{i,j} & \text{if } (i,j) = (u(i), u(j)) \\ \delta_{i,j} & \text{otherwise} \end{cases}$$

where $\delta_{i,j} = 1$ if $i = j$ and 0 otherwise.

Exercise 3.1.21. The map $u_\bullet$ is a group homomorphism.

Lemma 3.1.22. *For each $u_\bullet$, there is an induced homotopy equivalence*

$$u^+ \colon BGL(R)^+ \to BGL(R)^+.$$

Proof. By Exercise 3.1.21, the map $u_\bullet$ induces a map $u^+ \colon BGL(R)^+ \to BGL(R)^+$. This induces a $K_1(R)$-equivariant self-map

$$\overline{u} \colon BE(R)^+ \to BE(R)^+$$

of the universal cover $\widetilde{B}GL(R)^+ = BE(R)^+$. Since $BE(R)^+$ is simply connected, it suffices to check that

$$H_*(\overline{u}; \mathbb{Z}) \colon H_*(BE(R)^+; \mathbb{Z}) \to H_*(BE(R)^+; \mathbb{Z})$$

is an isomorphism by the Whitehead theorem. To see this, we note that $\underline{u}$ is induced by the map $u_\bullet|_{E(R)} \colon E(R) \to E(R)$, which is equal to conjugation by some matrix C so the induced map $H_*(\underline{u})$ is also given by conjugation by C, but inner conjugation induces the identity map on group homology with integer coefficients. □

Exercise 3.1.23. Let M be the monoid of injective self maps of $\mathbb{N}$ under composition, then $K_0(M) \cong 0$.

Consequently, any u^+ is homotopic to the identity via basepoint preserving maps.

We say group homomorphisms $f, g\colon G \to GL(R)$ are pseudo-conjugate if there exists $u \in M$, from Exercise 3.1.23, such that either $u_\bullet \circ f$ and g or f and $u_\bullet \circ g$ are conjugate by an element in $GL(R)$.

Proposition 3.1.24. *If $f, g\colon G \to GL(R)$ are pseudo-conjugate, then the induced maps $f^+, g^+\colon BG \to BGL(R)^+$ are homotopy as maps preserving the basepoints*

Proof. For any map $f\colon G \to GL(R)$ write f' for

$$f'(-) = \mu(f(-), 1).$$

Then f' is homotopic to f by a base point preserving homotopy because $f' = v_\bullet \circ f$. where $v \in M$ is defined by $v(s) = 2s - 1$.

Suppose $f, g\colon G \to GL(R)$ satisfy $g(-) = \alpha(u_\bullet \circ f(-))\alpha^{-1} := (u_\bullet \circ f(-))^\alpha$ for $\alpha \in GL(R)$. Letting $\beta = \mu(\alpha, \alpha^{-1}) \in E(R)$, then $\mu(g, 1) = \mu(u_\bullet \circ f, 1)^\beta$. So the induced maps

$$BG \to BGL(R)$$

are homotopic such that under a homotopy, the image of the base point of BG is is a loop homotopic to $[\beta] \in \pi_1 BGL(R) = GL(R)$. The induced maps $BG \to BGL(R)^+$ are homotopic preserving the base points since $[\beta] = 0 \in \pi_1(BGL(R)^+)$. □

We can now prove that $BGL(R)^+$ is a commutative H-space with operation

$$\mu\colon BGL(R)^+ \times BGL(R)^+ \to BGL(R)^+$$

and identity given by inclusion of the basepoint $e\colon * \to BGL(R) \to BGL(R)^+$.

Proof of Theorem 3.1.18. Let $v, w \in M$ of Exercise 3.1.23 be given by $v(s) = 2s - 1$ and $w(s) = 2s$ for all $s \in \mathbb{N}$. Then for any $A \in GL(R)$, we have that $v_\bullet \circ A = \mu(A, 1)$ and $w_\bullet \circ A = \mu(1, A)$. By construction, the composite

$$BGL(R)^+ \times * \to BGL(R)^+ \times BGL(R)^+ \xrightarrow{\mu^+} BGL(R)^+$$

is homotopic to v^+ and the the composite

$$* \times BGL(R)^+ \to BGL(R)^+ \times BGL(R)^+ \xrightarrow{\mu^+} BGL(R)^+$$

is homotopic to w^+. So the inclusion $e\colon * \to BGL(R)^+$ is a two-sided identity up to pointed homotopy. Consequently, $BGL(R)^+$ is an H-space.

There exists $v', w' \in M$ such that for any $x, y, z \in GL(R)$

$$\mu(x, y) = v' \circ \mu(y, x)$$
$$\mu(\mu(x, y), z) = w' \circ \mu(x, \mu(y, z))$$

so $BGL(R)^+$ is homotopy commutative and homotopy associative. Since it is path-connected and it is the homotopy type of a CW complex it is in fact a commutative H-group by [19, Cor. A.47]. □

3.1.1 Algebraic K-theory of finite fields

Chapter 4

The Q-construction and fundamental theorems

In this chapter, we first setup and prove Quillen's famous Theorem A and Theorem B. These will be used in a key way for proving theorems about the Q-construction of algebraic K-theory. We then give Quillen's Q-construction of algebraic K-theory and prove some basic properties. We then prove some of the fundamental theorems of algebraic K-theory using this construction. The primary reference for this material is the original paper of Quillen [15].

4.1 Quillen's Theorem A and Theorem B

Recall from Definition B.1.30, that given a small category $\mathcal{C}$, we can form an associated simplicial set $N_\bullet\mathcal{C}$.

Definition 4.1.1. Let $\mathcal{C}$ be a small category and let $N_\bullet\mathcal{C}$ be the simplicial set with n-simplices

$$N_n\mathcal{C} = \mathrm{Fun}([n], \mathcal{C})$$

then we define the *classifying space of* $\mathcal{C}$ to be

$$B\mathcal{C} := |N_\bullet\mathcal{C}|.$$

In this section, we will study the homotopy theory of small categories, by which we mean the homotopy theory of spaces that arise as the classifying space of a small category. Throughout this section, we assume all categories are *small* categories unless otherwise specified. The following definition will be used in several places.

Definition 4.1.2. The comma category $S \downarrow T$ associated to a pair of functors

$$S\colon \mathcal{A} \to \mathcal{C} \leftarrow \mathcal{B} : T$$

has objects $(a \in \mathcal{A}, b \in \mathcal{B}, \alpha\colon S(A) \to T(B))$ and morphisms $(a, b, \alpha) \to (a', b', \alpha')$ given by triples

$$f\colon a \to a'$$

$$g\colon b \to b'$$

and a commuting diagram

$$\begin{tikzcd} S(a) \arrow[r, "S(f)"] \arrow[d, "\alpha"'] & S(a') \arrow[d, "\alpha'"] \\ T(b) \arrow[r, "T(g)"'] & T(b') \end{tikzcd}$$

where composition is defined in the evident way.

Examples 4.1.3. Given a category $\mathcal{C}$ and an object $X \in \mathcal{C}$, we define the slice category $\mathcal{C}/X$ to be the comma category corresponding to the span of categories

$$\mathcal{C} \xrightarrow{\mathrm{id}_{\mathcal{C}}} \mathcal{C} \xleftarrow{X} [0]$$

and we define the coslice category $X\backslash\mathcal{C}$ to be the comma category associated to the span of categories

$$[0] \xrightarrow{X} \mathcal{C} \xleftarrow{\mathrm{id}_{\mathcal{C}}} \mathcal{C}.$$

We also define f/Y, where $f\colon \mathcal{C} \to \mathcal{C}'$ is a functor and Y is an object in $\mathcal{C}'$ to be the comma category associated to the span of categories

$$\mathcal{C} \xrightarrow{f} \mathcal{C}' \xleftarrow{X} [0]$$

and similarly we define $X\backslash f$ to be the comma category associated to the span of categories

$$[0] \xrightarrow{X} \mathcal{C}' \xleftarrow{f} \mathcal{C}$$

The following lemmas will be of fundamental importance.

Lemma 4.1.4. *A natural transformation $\eta\colon f \Rightarrow g$ between functors*

$$f, g\colon \mathcal{C} \to \mathcal{D}$$

induces a homotopy

$$H\colon B\mathcal{C} \times I \to \mathcal{D}$$

between Bf and Bg.

Proof. The triple (f, g, η) defines a functor

$$(f, g, \eta)\colon \mathcal{C} \times [1] \to \mathcal{D}$$

sending $(c, 0)$ to $f(c)$, $(c, 1)$ to $g(c)$ and $(c, 0 \to 1)$ to $\eta\colon f(c) \to g(c)$. It is defined on morphisms in the evident way. We then apply the nerve and note that

$$N_\bullet(\mathcal{C} \times [1]) = N_\bullet(\mathcal{C}) \times \Delta^1$$

and by a result of Milnor geometric realization commutes with finite products so the functor (f, g, η) induces a continuous map

$$H\colon B\mathcal{C} \times I \to B\mathcal{D}$$

such that $H|_{B\mathcal{C}\times\{0\}} = Bf$ and $H|_{B\mathcal{C}\times\{1\}} = Bg$ as desired. □

As a consequence of this, we can also prove the following lemma of fundamental importance.

Lemma 4.1.5. *If a functor*

$$f\colon \mathcal{C} \to \mathcal{D}$$

has a right or a left adjoint, then f induces a homotopy equivalence

$$Bf\colon\ : B\mathcal{C} \simeq B\mathcal{D}.$$

Proof. Suppose f has a right adjoint g. The proof when f has a left adjoint is essentially the same so we omit it. Then there are natural transformations

$$\eta\colon\ \mathrm{id}_{\mathcal{C}} \to g \circ f \text{ and } \epsilon\colon\ f \circ g \to \mathrm{id}_{\mathcal{D}}$$

which induce homotopies

$$H_1\colon\ B\mathcal{C} \times I \to B\mathcal{C} \text{ and } H_2\colon\ B\mathcal{D} \times I \to B\mathcal{D}$$

from $\mathrm{id}_{B\mathcal{C}}$ to $Bg \circ Bf$ and from $Bf \circ Bg$ to $\mathrm{id}_{B\mathcal{D}}$ respectively. □

Definition 4.1.6. We say a category $\mathcal{C}$ has an initial object 0 if for all objects $c \in \mathcal{C}$ there exists a unique morphism $0 \to c$. Similarly, we say that $\mathcal{C}$ has a terminal object 1 if for all objects $c \in \mathcal{C}$ there exists a unique morphism $c \to 1$.

Lemma 4.1.7. *If $\mathcal{C}$ has an initial or terminal object then $B\mathcal{C}$ is contractible.*

Proof. The existence of an initial object in $\mathcal{C}$ is equivalent to the existence of a left adjoint to the functor $[0] \to \mathcal{C}$ sending 0 to the initial object 0 of $\mathcal{C}$. Similarly, the existence of a terminal object is equivalent to the existence of a right adjoint to the functor $[0] \to \mathcal{C}$ sending 0 to the terminal object 1 of $\mathcal{C}$. □

We first give a characterization of $\pi_1(B\mathcal{C})$, but we will not give a complete proof. See [19, Lemma 6.1.] for a careful proof of most of the statements. First, we give a definition.

Definition 4.1.8. We say that a functor $F\colon \mathcal{C} \to \mathcal{D}$ is *morphism inverting* if for all morphisms f in $\mathcal{C}$, then $F(f)$ is an isomorphism.

Theorem 4.1.9. *There is an equivalence of categories between the category of coverings of $B\mathcal{C}$, denote $Cov(B\mathcal{C})$ and the category of morphism inverting functors $\mathcal{C} \to \mathsf{Set}$, which we denote $Fun^{inv}(\mathcal{C}, \mathsf{Set})$. Moreover,*

$$\pi_1(\mathcal{C}, X) \cong Hom_{\mathcal{C}}(X, X)^{gp}$$

where $Hom_{\mathcal{C}}(X, X)^{gp}$ is group completion of the monoid $Hom_{\mathcal{C}}(X, X)$.

We will just sketch the proof. Given a covering space $E \to B\mathcal{C}$, we consider the fiber $E(X)$ over an object $X \in \mathcal{C}$ regarded as a 0-cell of the CW complex $B\mathcal{C}$. A map $f\colon X \to X'$ induces an bijection $E(f)\colon E(X) \cong E(X')$ of sets satisfying $E(f \circ g) = E(f) \circ E(g)$. Thus, we may regard E as a functor

$$E\colon \mathcal{C} \to \mathsf{Set}$$

that sends all maps f to bijections $E(f)$ as desired.

In the other direction, given a functor $F\colon C \to \mathsf{Set}$ we post-compose with the inclusion $\mathsf{Set} \to \mathsf{Cat}$ sending a set S to the category with objects S and only identity morphisms, and denote this functor F by abuse of notation. Then we define the $[0]\backslash F$ to be the comma category associated to the functors

$$[0] \xrightarrow{[0]} \mathsf{Cat} \xleftarrow{F} \mathcal{C}$$

where the left functor sends 0 to the category $[0]$. In other words, unpacking the definition, $[0]\backslash F$ has objects specified by a pair $(X, * \to F(X))$ and morphisms $f\colon X \to X'$ along with a basepoint preserving map $F(X) \to F(X')$.

Lemma 4.1.10. *The forgetful functor $[0]\backslash F \to \mathcal{C}$ induces a covering*

$$B([0]\backslash F) \to B\mathcal{C}$$

when $F\colon \mathcal{C} \to \mathsf{Set}$ is morphism inverting.

Proof. This follows from [6, Appendix I 3.2J]. □

There is a way to formally add inverses to all morphisms in a small category $\mathcal{C}$ to form a groupoid $G = \mathcal{C}[\mathrm{Arr}(\mathcal{C})^{-1}]$ [6, Chapter I, 1.1]. The group $\pi_1(B\mathcal{C}, X)$ is then the automorphisms of X in this groupoid G. Moreover, the category of morphism inverting functors

$$\mathcal{C} \to \mathsf{Set}$$

is equivalent to the category of functors

$$G \to \mathsf{Set}$$

which we refer to as the category G-Set of G -sets. When $B\mathcal{C}$ is path connected with a 0-simplex X, then the category of coverings of $B\mathcal{C}$ is equivalent to the category G_X-Set of G_X-sets in the traditional sense where G_X is the automorphism group of X in the groupoid $G = \mathcal{C}[\mathrm{Arr}(\mathcal{C})^{-1}]$.

Definition 4.1.11. We say a category $\mathcal{I}$ is a *filtered category* if it is nonempty and for any two objects x, y there exists an object z and morphisms $x \to z$ and $y \to z$ and for any two morphisms $f, g\colon x \to y$ there exists a morphism $h\colon y \to z$ such that $h \circ f = h \circ g$.

Proposition 4.1.12. *Given a functor*

$$I \to \mathsf{Cat}$$

from a filtered category I to the category of small categories Cat *with limit $\mathcal{C}$, where we denote $\mathcal{C}_i$ the functor evaluated at i, then*

$$\operatorname*{colim}_i \pi_n B\mathcal{C}_i \simeq \pi_n B\mathcal{C}$$

for all $n \geq 0$.

If in addition, $B\mathcal{C}_i \to B\mathcal{C}_j$ is a homotopy equivalence for all maps $i \to j$ in I, then there is a homotopy equivalence

$$B\mathcal{C}_i \simeq B\mathcal{C}$$

for all i.

Corollary 4.1.13. *Any filtered category I is contractible.*

Theorem 4.1.14 (Theorem A)**.** *If $f\colon \mathcal{C} \to \mathcal{C}'$ is a functor and the category $A\backslash f$ is contractible for all objects $A \in \mathcal{C}'$, then*

$$Bf\colon B\mathcal{C} \to B\mathcal{C}'$$

is a homotopy equivalence. Similarly, when f/Y is contractible for all objects $A \in \mathcal{C}'$, then the map Bf is a homotopy equivalence.

We will just prove the case $A\backslash f$ since the other version is formally dual in a precise sense. First, we will need the following definition.

Definition 4.1.15. We also need the following variant of the twisted arrow category, denoted $\mathsf{Tw}(f)$ where $f\colon \mathcal{C} \to \mathcal{C}'$ is a functor. The objects in $\mathsf{Tw}(f)$ are the same as the objects of the comma category associated to

$$\mathcal{C}' \xrightarrow{\mathrm{id}} \mathcal{C}' \xrightarrow{F} \mathcal{C}$$

and the morphisms from $(x, y, \alpha\colon x \to F(y))$ to $(x', y', \alpha'\colon x' \to F(y'))$ consist of maps $u\colon x \to x'$ and $w\colon y' \to y$ such that the diagram

$$\begin{tikzcd}
x \arrow[r, "\alpha"] \arrow[d, "u"'] & F(y) \\
x' \arrow[r, "\alpha'"'] & F(y') \arrow[u, "F(w)"']
\end{tikzcd}$$

commutes. Composition is defined by evident vertical composition of squares. When $f\colon \mathcal{C}' \to \mathcal{C}'$ is the identity $\mathrm{id}_{\mathcal{C}'}$, we simply write $\mathsf{Tw}(\mathcal{C}') := \mathsf{Tw}(\mathrm{id}_{\mathcal{C}'})$ and we observe that this is the usual twisted arrow category.

The following construction will also be integral to the proof.

Construction 4.1.16. Given a bisimplicial space, or in other words a functor

$$X_{\bullet,\bullet}\colon \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \to \mathsf{Top}$$

we can form the geometry realizations

$$|X_{\bullet,n}|\colon \Delta^{\mathrm{op}} \to \mathsf{Top}$$

and

$$|X_{n,\bullet}|\colon \Delta^{\mathrm{op}} \to \mathsf{Top}$$

for each $[n]$ in Δ^{op} and these are functorial in the variable $[n]$. One can also form a simplicial object $\triangle^* X_{\bullet,\bullet}$ with n-simplices $X_{n,n}$ by precomposition with the diagonal

$$\triangle^* X_{\bullet,\bullet}\colon \Delta^{\mathrm{op}} \xrightarrow{\triangle} \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \to \mathcal{C}.$$

Lemma 4.1.17. *There are homeomorphisms*

$$\begin{aligned} |[n] \mapsto |X_{n,\bullet}|| \cong& |[n] \mapsto |X_{\bullet,n}|| \\ \cong& |\triangle^* X_{\bullet,\bullet}| \end{aligned}$$

Proof of Theorem A.. Let $f\colon \mathcal{C} \to \mathcal{C}'$ be a functor. Consider the diagram of categories

$$(\mathcal{C}')^{\mathrm{op}} \xleftarrow{\pi_1} \mathsf{Tw}(f) \xrightarrow{\pi_2} \mathcal{C}$$

where $\pi_1(x, y, \alpha\colon x \to f(y)) = x$ and $\pi_2(x, y, \alpha\colon x \to f(y)) = y$ and these functors are defined on morphisms and composition in the evident way. We form a bisimplicial set $T_{p,q}$ with p, q simplices given by pairs

$$(y_p \to \cdots \to y_0 \to f(x_0), x_0 \to \ldots x_q)$$

where $y_p \to \cdots \to y_0 \to f(x_0)$ is an object in $\mathsf{Fun}([p], (\mathcal{C}'/f(x_0))^{\mathrm{op}})$ and $x_0 \to \ldots x_q$ is an object in $\mathsf{Fun}([q], \mathcal{C})$. where face maps in p direction (resp. q direction) composes $x_{i+1} \to x_i \to x_{i-1}$ to form a map $x_{i+1} \to x_{i-1}$ and degeneracies are given by inserting an identity, in the same way that we defined the nerve of a category. By forgetting the first component, there is therefore an evident map

$$T_{p,q} \to N_q\mathcal{C}$$

of bisimplicial sets, where we regard $N_q\mathcal{C}$ is constant in p. Upon geometric realizations, using any of the three equivalent constructions, we produce a map

$$B\pi_2\colon B\mathsf{Tw}(f) \to B\mathcal{C}.$$

If we first, geometrically realize with respect to p, then we produce a map of simplicial spaces defined on q-simplices by

$$\coprod_{x_0 \to \dots x_q} B(\mathcal{C}'/f(x_0)^{\mathrm{op}}) \to \coprod_{x_0 \to \dots x_q} *$$

Since $\mathcal{C}'/f(x_0)$ has a final object and $B\mathcal{D} \cong B\mathcal{D}^{\mathrm{op}}$, we know that $B(\mathcal{C}'/f(x_0)^{\mathrm{op}})$ is contractible this map is a weak equivalence. Since both sides are Reedy cofibrant (we will not go into detail on this point) this map induces a weak equivalence

$$B\mathsf{Tw}(f) \simeq B\mathcal{C}$$

We similarly have a functor

$$T_{p,q} \to N_\bullet((\mathcal{C}')^{\mathrm{op}}).$$

Taking geometric realization with respect to q, we produce a map

$$\coprod_{y_p \to \dots \to y_0} B(y_o \backslash f) \to \coprod_{y_p \to \dots y_0} * = N_p(\mathcal{C}')^{\mathrm{op}}.$$

By assumption, $B(y_o \backslash f)$ is contracible so the map is a weak equivalence.

$$B\mathsf{Tw}(f) \to B(\mathcal{C}')^{\mathrm{op}} \cong B(\mathcal{C}').$$

This shows that $B\mathcal{C}'$ and $B\mathcal{C}$ are homotopy equivalent, but we still want to know that the functor $f\colon \mathcal{C} \to \mathcal{C}'$ induces a homotopy equivalence. Note that the construction $\mathsf{Tw}(f)$ is functorial in functors to $\mathcal{C}'$, so we have a functor

$$\mathsf{Tw}(f) \to \mathsf{Tw}(\mathrm{id}_{\mathcal{C}'}) = \mathsf{Tw}(\mathcal{C}')$$

and the projection maps are also natural so there is a commutative diagram

$$\begin{tikzcd}
(\mathcal{C}')^{\mathrm{op}} \arrow[d, "\mathrm{id}_{\mathcal{C}'}"] & \mathsf{Tw}(f) \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[d] & \mathcal{C} \arrow[d, "f"] \\
(\mathcal{C}')^{\mathrm{op}} & \mathsf{Tw}(\mathcal{C}') \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & \mathcal{C}'
\end{tikzcd}$$

The same argument as before proves that the functor $B\pi_1\colon B\mathsf{Tw}(\mathcal{C}') \to B\mathcal{C}'$ is a homotopy equivalence by simply observing that

$$B(y_0 \backslash \mathrm{id}_{\mathcal{C}'}) = B(y_0 \backslash \mathcal{C}')$$

which clearly has an initial object. Therefore, we observe that the functor $f\colon \mathcal{C} \to \mathcal{C}'$ induces a homotopy equivalence $BF\colon B\mathcal{C} \to B\mathcal{C}'$ as desired. $\square$

The following special case is also of interest. For this special case, we need a definition.

Definition 4.1.18. We say $f\colon \mathcal{C} \to \mathcal{C}'$ is *pre-fibered* if and only if the subcategory $f^{-1}(y)$, which is the full subcategory of $\mathcal{C}$ whose objects are x such that $f(x) = y$, equipped with a canoncial functor

$$f^{-1}(y) \to y\backslash f$$

sending x to $(x, \mathrm{id}\colon f(x) \to y)$ has a right adjoint. We say f is *pre-cofibered* if the same canonical functor has a left adjoint. Denote the right adjoint (resp. left adjoint) by (x, v) maps to v^*x (resp. (x, v) maps to v_*X) when f is pre-fibered (resp. cofibered). Then for any map $v\colon y \to y'$ in $\mathcal{C}'$, there is a functor

$$v^*\colon f^{-1}(y) \to f^{-1}(y')$$

(respectively there is a functor $v_*\colon f^{-1}(y) \to f^{-1}(y')$). We say f is *fibered* if for every pair of composable morphism v, w in $\mathcal{C}'$ there is a natural isomorphism

$$v^*w^* \to (v \circ w)^*$$

of functors (resp. we say f is *cofibered* if there is a natural isomorphism $(v \circ w)_* \to v_* \circ w_* \to$). We call v^* base change and v_* cobase change.

Corollary 4.1.19. *Suppose $f\colon \mathcal{C} \to \mathcal{C}'$ is either pre-fibered or pre-cofibered and suppose that $f^{-1}(y)$ is contractible for all $y \in \mathcal{C}'$, then Bf is a homotopy equivalance.*

We first introduce some terminology.

Definition 4.1.20. Given a map of based topological spaces $f\colon (E, e) \to (B, b)$, we can form the pullback

$$\begin{tikzcd} F(f) \arrow[r] \arrow[d] & B^I \arrow[d] \\ E \times * \arrow[r, "f \times b"] & B \times B \end{tikzcd}$$

which we call the *homotopy fiber* of the map f. We say $f\colon E \to B$ is a *quasi-fibration* if the map $f^{-1}(b) \to F(f)$ is a weak equivalence, meaning it induces an isomorphism on all homotopy groups for all basepoints.

Construction 4.1.21. Since the homotopy fiber sequence $F(f) \to E \to B$ induces a long exact sequence in homotopy groups

$$\cdots \to \pi_{i+1}(B, b) \to \pi_i(F(f), \bar{e}) \to \pi_i(E, e) \to \pi_i(B, b) \to \dots$$

where $\bar{e} = (e, \mathrm{const}_b)$, we have a long exact sequence in homotopy groups

$$\cdots \to \pi_{i+1}(B, b) \to \pi_i(f^{-1}(b), e) \to \pi_i(E, e) \to \pi_i(B, b) \to \dots$$

as well whenever $E \to B$ is a quasi-fibration.

The following is a generalization of this construction that will be useful for our setup.

Definition 4.1.22. Given a diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ Z \arrow[r] & W \end{tikzcd} \tag{4.1.23}$$

in Top, define the homotopy pullback to be the pullback of the diagram

$$\begin{tikzcd} hPB \arrow[r] \arrow[d] & W^I \arrow[d] \\ Z \times Y \arrow[r] & W \times W \end{tikzcd}$$

and we say (4.1.23) is a homotopy pullback diagram if $X \to hPB$ is a homotopy equivalence equivalence.

As a special case note that, when $Z = *$, then (4.1.23) is a homotopy pullback diagram when $Y \to W$ is a quasi-fibration, in fact the diagram (4.1.23) is a homotopy pullback whenever Z is contractible and $Y \to W$ is homotopy equivalent to quasi-fibration.

Theorem 4.1.24 (Theorem B). *Let $f\colon \mathcal{C} \to \mathcal{C}'$ be a functor such that for every map $v\colon y \to y'$ the induced functor $v\backslash f\colon y\backslash f \to y'\backslash f$ is a homotopy equivalence. Then for every $y \in \mathcal{C}'$, there is a homotopy pullback*

$$\begin{tikzcd} By\backslash f \arrow[r] \arrow[d] & B\mathcal{C} \arrow[d] \\ By\backslash \mathcal{C}' \arrow[r] & B\mathcal{C}' \end{tikzcd}$$

where $By\backslash\mathcal{C}'$ is contractible because the category $y\backslash\mathcal{C}'$ has an initial object. Moreover, there is a long exact sequence in homotopy groups

$$\cdots \to \pi_{i+1}(B\mathcal{C}', y) \to \pi_i(y\backslash f, \overline{x}) \to \pi_i(B\mathcal{C}, x) \to \pi_i(B\mathcal{C}', y) \to \dots$$

where $\overline{x} = (x, \mathrm{id}_y\colon y \to f(x))$.

Remark 4.1.25. Again there is a dual formulation in terms of the categories f/y, but we leave the formulation and the proof to the reader.

First, we need three lemmas about quasi-fibrations.

Lemma 4.1.26. *Let $p\colon E \to B$ be a continuous map, and let $U, V \subset B$ be open subsets such that $U \cup V = B$. Suppose $p|_{p^{-1}(U)}$, $p|_{p^{-1}(V)}$, and $p|_{p^{-1}(U\cap V)}$ are quasi-fibrations, then p is a quasi-fibration.*

Proof. We know $U \cap V$ is the pullback of

$$U \hookrightarrow B \hookleftarrow V$$

so $\mathrm{Fib}(p^{-1}(U \cap V) \to U \cap V)$ is the pullback of $\mathrm{Fib}(p^{-1}(U) \to U)$ and $\mathrm{Fib}(p^{-1}(V) \to V)$ along $\mathrm{Fib}(p)$. If $x \in B$ then there is a weak equivalence $p^{-1}(V) \cap p^{-1}(x) \simeq_{we} \mathrm{Fib}(p^{-1}(V) \to V)$ if $x \in V$ and there is a weak equivalence $p^{-1}(U) \cap p^{-1}(x) \simeq_{we} \mathrm{Fib}(p^{-1}(U) \to U)$ if $x \in U$ by assumption. Also, if $x \in U \cap V$, then there is a weak equivalence $p^{-1}(U \cap V) \cap p^{-1}(x) \simeq_{we} \mathrm{Fib}(p^{-1}(U \cap V) \to U \cap V)$ by assumption. Therefore,

$$\begin{aligned}
p^{-1}(x) =& p^{-1}(B) \cap p^{-1}(x) \\
=& (p^{-1}(U) \cup p^{-1}(V)) \cap p^{-1}(x) \\
=& (p^{-1}(U) \cap p^{-1}(x)) \cup_{p^{-1}(U \cap V) \cap p^{-1}(x)} (p^{-1}(V) \cap p^{-1}(x)) \\
\simeq_{we}& \mathrm{Fib}(p^{-1}(U)) \cup_{\mathrm{Fib}(p^{-1}(U \cap V))} \mathrm{Fib}(p^{-1}(V)) \\
\simeq_{we}& \mathrm{Fib}(p).
\end{aligned}$$

as desired. □

Lemma 4.1.27. *Let $p\colon E \to B$ be a continuous map onto B, let $B' \subset B$ be a subspace and let $E' = p^{-1}(B')$. Suppose there is a* fiber preserving deformation*; i.e. suppose there are homotopies*

$$D\colon E \times I \to E \text{ and } d\colon B \times I \to B$$

such that $D(-,0) = id_E$, $d(-,0) = \mathrm{id}_B$, $D_t(E') = E'$, $d_t(B') = B'$ and $D_1(E) \subset E'$ and $d_1(B) \subset B'$.

Additionally, assume that $p^{-1}(x) \to p^{-1}(d_1(x))$ is a weak equivalence for all $x \in B$ then p is a quasi-fibration.

Lemma 4.1.28. *Let $p\colon E \to B$ be continuous map. Assume that B is CW complex with n-skeleton B_i and assume that $p|_{p^{-1}(B_i)}$ is a quasi-fibration for each i, then p is a quasi-fibration.*

Proof. Any compact subset of B lies inside of some B_i so any compact subset of E lies inside of some $p^{-1}B_i = E_i$. Consequently, given $x \in B_i$ and $y \in p^{-1}(x)$, then the homotopy groups of the pair $E, p^{01}(x)$ satisfy

$$\begin{aligned}
\pi_n(E, p^{-1}(x); y) \cong& \operatorname*{colim}_j \pi_n(p^{-1}(B_j), p^{-1}(x); y) \\
\cong& \operatorname*{colim}_j \pi_n(B_j, x) \cong \qquad \pi_n(B, x).
\end{aligned}$$

since each map $E_i \to B_i$ is a quasi-fibration. □

Proposition 4.1.29. *Suppose I is a small category and $X\colon I \to \mathsf{Top}$ is a functor. We form a simplicial space by*

$$\coprod_{i_0 \to i_1 \to \dots i_n} X(i_0)$$

where the face and degeneracy maps are induced by functoriality of X and the nerve construction of I, which we index over. There is an obvious map of simplicial spaces from this simplicial space to the nerve of I given on n-simplices by

$$\coprod_{i_0 \to i_1 \to \dots i_n} X(i_0) \to \coprod_{i_0 \to i_1 \to \dots i_n} *.$$

Suppose that $X_i \to X_j$ is a weak equivalence for all maps $i \to j$ in I, then the map induced on geometric realizations

$$\pi\colon X_I \to BI$$

is a quasi-fibration where

$$X_I := |[n] \mapsto \coprod_{i_0 \to i_1 \to \dots i_n} X(i_0)|$$

Proof. We need to check that the canonical map $\pi^{-1}(i) \to \mathrm{Fib}(\pi)$ is a homotopy equivalence. It suffices to check that when restricting to the p-skeleton of BI; i.e. we need to check that the map $\pi^{-1}((BI)_p) \to (BI)_p$ is a quasi-fibration for each $p \geq 0$. We proceed by induction up the skeleta. The inductive hypothesis holds because on zero 0-cells the map is a trivial fibration. We consider the map from the diagram

$$\begin{tikzcd}
\coprod_{i_0 \to \dots \to i_p \in ND_p(N_\bullet I)} X_{i_0} \times \delta\Delta^p \arrow[r] \arrow[d] & \pi^{-1}((BI)_{p-1}) \arrow[d] \\
\coprod_{i_0 \to \dots \to i_p \in ND_p(N_\bullet I)} X_{i_0} \times \Delta^p \arrow[r] & \pi^{-1}((BI)_p)
\end{tikzcd}$$

to the diagram

$$\begin{tikzcd}
\coprod_{i_0 \to \dots \to i_p \in ND_p(N_\bullet I)} \delta\Delta^p \arrow[r] \arrow[d] & \pi^{-1}((BI)_{p-1}) \arrow[d] \\
\coprod_{i_0 \to \dots \to i_p \in ND_p(N_\bullet I)} \Delta^p \arrow[r] & \pi^{-1}((BI)_p),
\end{tikzcd}$$

where $ND_p(N_\bullet I)$ denotes the non-degenerate p-simplices of $N_\bullet I$, in order to do an induction up the skeleta. We assume that we know that $p|_{(BI)_p}$ is a quasi-fibration.

Let U be the space formed from $(BI)_p$ by removing the barycenter of each p-cell. Let $V = (BI)_p - (BI)_{p-1}$. It suffices, by Lemma 4.1.26 to show that $\pi|_U$, $\pi|_V$ and $\pi|_{U\cap V}$ are all quasi-fibrations. This is clear for $\pi|_V$ and $\pi_{U\cap V}$ because each map is homeomorphic to a trivial fibration.

By our inductive hypothesis, we can assume that $\pi|_{BI_{p-1}}$ is a quasi-fibration. To apply Lemma 4.1.27, we construct a fiber preserving deformation D of $\pi|_U$ to $\pi_{BI_{p-1}}$ by considering the radial deformation of Δ^p without its barycenter to $\delta\Delta^p$. Thus, we just need to check that, if our deformation D takes $x \in U$ to $x' \in BI_{p-1}$, then the map $g^{-1}(x) \to g^{-1}(x')$ induced by D is a weak equivalence. It suffices to consider $x \in BI_p - BI_{p-1}$. Consequently, we can let x come from the interior point z of Δ^p, where the interior of Δ^p is $\Delta^p - \delta\Delta^p$. Suppose this Δ^p corresponds to the non-degenerate p-simplex $s = (i_0 \to \dots i_p)$. Our radial deformation takes z to some open face of Δ^p, say the open face with vertices $j_0 < j_1 < \dots j_q$, where $\{j_0, \dots, j_q\} \subset \{0, \dots, p\}$. Then $g^{-1}(x) = X_{i_0}$ and $g^{-1}(x') = X_k$ where $k = i_{j_0}$ and the map $X_{i_0} = g^{-1}(x) \to g^{-1}(x') = X_k$ is the map induced by $i_0 \to i_k$ coming from the face of s (note that this is some composite of the face maps of $N_\bullet I$, which are all given by composing adjacent composable morphisms). Since we assumed that these maps are weak equivalences, the proof is complete. ☐

Corollary 4.1.30. *Under the hypotheses of Theorem 4.1.24, the induced map*

$$B\pi_2\colon B\mathsf{Tw}(f) \to B(\mathcal{C}')^{\mathrm{op}}$$

is a quasi-fibration for any functor $f\colon \mathcal{C} \to \mathcal{C}'$.

Proof. Consider the functor $X\colon (\mathcal{C}')^{\mathrm{op}} \to \mathsf{Top}$ sending y to $X(y) = By\backslash f$. Then since for $v\colon y \to y'$ the induced map $Bv\backslash f$ is a homotopy equivalence we can apply Proposition 4.1.29. In this case $X_{(\mathcal{C}')^{\mathrm{op}}} = B\mathsf{Tw}(f)$ as we noted earlier so the proposition implies that

$$B\mathsf{Tw}(f) \to B(\mathcal{C}')^{\mathrm{op}}$$

is a quasi-fibration. ☐

Proof of Theorem B. We will prove Theorem 4.1.24 assuming Lemma 4.1.30. Consider the following diagram as in the proof of Theorem 4.1.14

$$\begin{tikzcd}
y\backslash f \arrow[r] \arrow[d] & \mathsf{Tw}(f) \arrow[r, "\simeq"] \arrow[d, "f'"] & \mathcal{C} \arrow[d, "f"] \\
y\backslash \mathcal{C} \arrow[r] \arrow[d, "\simeq"] & \mathsf{Tw}(\mathcal{C}') \arrow[r, "\simeq"] \arrow[d, "\simeq"] & \mathcal{C}' \\
{[0]} \arrow[r] & \mathcal{C}' &
\end{tikzcd}$$

where the maps indicated with $\simeq$ are maps inducing homotopy equivalences on classifying spaces, as we have already discussed. The vertical map

$$B\mathsf{Tw}(f) \to B\mathsf{Tw}(\mathcal{C}') \to B\mathcal{C}'$$

is $B\pi_2$, which is homotopy equivalent to the map $Bf\colon B\mathcal{C} \to B\mathcal{C}'$ in the top right square as we have already shown in the proof of Theorem 4.1.14. Since $B\mathsf{Tw}(f) \to B(\mathcal{C}')^{\mathrm{op}}$ is a quasi-fibration by Lemma 4.1.30, we know that the two squares on the left are homotopy pullbacks. This implies the theorem. $\square$

4.2 The Q-construction

We first need to define the input of our construction.

Definition 4.2.1. A Ab-*enriched category* is a category $\mathcal{C}$ such that

$$\mathrm{Hom}_{\mathcal{C}}(c, c')$$

is an abelian group for all $c, c' \in \mathrm{ob}\,\mathcal{C}$ and the composition

$$\mathrm{Hom}_{\mathcal{C}}(c, c') \times \mathrm{Hom}_{\mathcal{C}}(c', c'') \to \mathrm{Hom}_{\mathcal{C}}(c, c'')$$

is bilinear. An Ab-enriched category is called an *additive category* if it admits all finite coproducts. An Ab-enriched category is called *abelian* if it has all finite limits and colimits, and every morphism $f\colon A \to B$ decomposes as

$$A \xrightarrow{p} \mathrm{coker}(\ker f) \xrightarrow{\overline{f}} \ker(\mathrm{coker} f) \xrightarrow{i} B$$

where p is an epimorphism, i is a monomorphism, and $\overline{f}$ is an isomorphism.

An *exact category* $\mathcal{C}'$ is an additive subcategory $\mathcal{C}' \subset \mathcal{C}$ of an abelian category $\mathcal{C}$ such that $\mathcal{C}'$ is close under extensions in the sense that if there is an exact sequence

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

where X, Y are in $\mathcal{C}'$ then Y is also in $\mathcal{C}'$. Given an exact category $\mathcal{C}'$, let E be the collection of exact sequences in $\mathcal{C}'$ that are also exact in $\mathcal{C}$. Given an exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

in E, we say that the map $A \to B$ is an *admissible monomorphism* and we say $B \to C$ is an admissible epimorphism.

Remark 4.2.2. Note that an exact category is not necessarily closed under all kernels and cokernels.

Remark 4.2.3. An exact category $\mathcal{C}'$ with class of exact sequences has the following properties:

1. E contains all split exact sequences

$$0 \longrightarrow A \longrightarrow A \oplus C \longrightarrow C \longrightarrow 0$$

2. The admissible epimorphisms and admissible monomorphisms in $\mathcal{C}'$ are closed under composition.

3. Given an admissible epimorphism $B \twoheadrightarrow C$ and any morphism $C' \to C$ then the pullback of $B' \to C'$ is an admissible epi-morphism.

Examples 4.2.4. Let R be a ring, let X be a space, and let Y be a scheme.

1. The category $P(R)$ of finitely generated projective R modules is an exact category regarded as a subcategory of the abelian category of all left R-modules Mod_R. The exact sequences are all split exact.

2. The category $M(R)$ of finitely generated R-modules is an exact category regarded as a subcategory of the abelian category of all left R-modules Mod_R.

3. The category $\mathsf{VB}(X)$ of vector bundles over a topological space X is an exact category. It is a subcategory of the category of families of vector spaces parametrized by X.

4. The category $\mathsf{VB}(Y)$ is the category of algebraic vector bundles over Y is an exact category, regarded as a subcategory of the abelian category of $\mathcal{O}_X$-modules, denoted $\mathcal{O}_X$-Mod.

As a warm up, we will define K_0 of an exact category.

Definition 4.2.5. Given a small exact category $\mathcal{C}'$ with class of exact sequences E, define

$$K_0(\mathcal{C}') = F(\mathrm{iso}\mathcal{C}')/([A] + [C] = [B] : 0 \to A \to B \to C \to 0 \in E)$$

where $F(\mathrm{iso}\mathcal{C}')$ is the free abelian group on the set of isomorphism classes of of objects in $\mathcal{C}'$.

Example 4.2.6. When $\mathcal{C}'$ is an exact category with class of exact sequences E such that every exact sequence in E splits, then

$$K_0(\mathcal{C}') \cong K_0^{\oplus}(\mathcal{C}').$$

4.2.1 The Q-construction

We now define the algebraic K-theory of an exact category (in the sense of Quillen).

Definition 4.2.7. Given an exact category $\mathcal{C}$, we define a category $Q\mathcal{C}$ whose objects are the same as the objects in $\mathcal{C}$ and whose morphisms from A to B are isomorphism classes of spans

$$\begin{tikzcd} A & W \arrow[l, two heads, "j"] \arrow[r, tail, "i"] & B \end{tikzcd}$$

where j is an admissible epimorphism and i is an admissible monomorphism. Here we say that two spans are in the same isomorphism class if there is a commuting diagram

$$\begin{tikzcd} A \arrow[d, "\mathrm{id}_A"'] & W \arrow[l, two heads, "j"] \arrow[r, tail, "i"] \arrow[d, "\cong"] & B \arrow[d, "\mathrm{id}_B"] \\ A & W' \arrow[l, two heads, "j"] \arrow[r, tail, "i"] & B \end{tikzcd}$$

where the outer vertical maps are the identity and the middle vertical morphism is an isomorphism. We then define the composition of two (isomorphism classes) of spans from A to B and from B to C by the digram

$$\begin{tikzcd} & & W_3 \arrow[dl, two heads] \arrow[dr, tail] & & \\ & W_1 \arrow[dl, two heads] \arrow[dr, tail] & & W_2 \arrow[dl, two heads] \arrow[dr, tail] & \\ A & & B & & C \end{tikzcd}$$

where W_3 is the pullback $W_1 \times_B W_2$. By a slight abuse of terminology we refer to the morphism $A \rightarrowtail B$ in $Q\mathcal{C}$ where $W = A$ as as an admissible monomorphism and we refer to the morphism $A \twoheadleftarrow B$ where $W = B$ as an admissible epimorphism.

Remark 4.2.8. Suppose $A \twoheadleftarrow W_1 \rightarrowtail B$ is an isomorphism in QQ with inverse $B \twoheadleftarrow W_2 \rightarrowtail A$ then the composite

$$\begin{tikzcd} & & W_3 \arrow[dl, two heads] \arrow[dr, tail] & & \\ & W_1 \arrow[dl, two heads] \arrow[dr, tail] & & W_2 \arrow[dl, two heads] \arrow[dr, tail] & \\ A & & B & & A \end{tikzcd}$$

is isomorphic to the map $\mathrm{id}_A \colon\ A \twoheadleftarrow A \rightarrowtail A$ and this implies that all the maps in the composite are isomorphisms. Therefore the map $A \to B$ is an isomorphism in *mathcalC* and there is a one to one correspondence between isomorphisms in $\mathcal{C}$ and isomorphisms in $Q\mathcal{C}$.

Proposition 4.2.9. *The CW complex $BQ\mathcal{C}$ is connected and there is a bijection*

$$\pi_1(BQ\mathcal{C}) \cong K_0(\mathcal{C})$$

where $[A]$ in $K_0(\mathcal{C})$ corresponds to the based loop composed of the two edges from 0 *to A*

$$0 \rightarrowtail A \twoheadrightarrow 0$$

(regarded as maps in $Q\mathcal{C}$ and consequently paths in $BQ\mathcal{C}$).

Proof. By qxqx, it suffices to show that there is an equivalence of categories between the category of morphism inverting functors $F\colon Q\mathcal{C} \to \mathsf{Set}$ and the category of $K_0(\mathcal{C})$-sets where $K_0(\mathcal{C})$ is defined as in qxqx. The correspondence sends F to $F(0)$ so it suffices to show that $K_0(\mathcal{C})$ acts naturally on $F(0)$ and that this gives an equivalence of categories. We write $i_M\colon\ 0 \rightarrowtail M$ for the unique admissible monomorphism in $Q\mathcal{C}$ from 0 to M and $j_M\colon\ M \twoheadrightarrow 0$ for the unique admissible epimorphism in $Q\mathcal{C}$ from M to 0. Write $\mathcal{F}$ for the full subcategory of the category of morphism inverting functors $Q\mathcal{C} \to \mathsf{Set}$ such that $F(M) = F(0)$ and $F(i_M) = \mathrm{id}_M$ for all M in $Q\mathcal{C}$. Given any morphism functor F' there exists a functor F in $\mathcal{F}$ with $F(0) = F'(0)$ and we observe form the commutative diagram

$$\begin{tikzcd}
F'(M) \arrow[r] \arrow[d] & F'(0) \arrow[d, "\mathrm{id}"] \\
F(M) \arrow[r, "\mathrm{id}"] & F(0)
\end{tikzcd}$$

which implies that $F(M) \cong F'(M)$ and consequently there is an equivalence of categories between the category of morphism inverting functors $Q\mathcal{C} \to \mathsf{Set}$ and the full subcategory $\mathcal{F}$. We therefore want to to describe an equivalence of categories between $\mathcal{F}$ and the category of $K_0(\mathcal{C})$-sets. Given a $K_0(\mathcal{C})$-set S, define $F_S\colon Q\mathcal{C} \to \mathsf{Set}$ by $F_S(M) = S$, F_S sends admissible monomorphisms in $Q\mathcal{C}$ to id_S and admissible epimorphisms to $j\colon M \twoheadrightarrow M'$ in $Q\mathcal{C}$ to the action of the class $[\ker j] \in K_0(\mathcal{C})$ on S. It's clear that this takes values in $\mathcal{F}$. Now given an F in $\mathcal{F}$ and and admissible monomorphism $i\colon\ M \rightarrowtail M'$ in $Q\mathcal{C}$ then its clear that $i \circ i'_M = i_M$ so $F(i) = \mathrm{id}_M$. Given a sequence

$$M' \overset{i}{\rightarrowtail} M \overset{j}{\twoheadrightarrow} M''$$

where i is an admissible monomorphisms in $Q\mathcal{C}$ and j is an admissible epimorphism in M such that as a sequence in $\mathcal{C}$ it is an admissible exact sequence, then we have $j \circ i_{M''} = i \circ j_{M'}$ so $F(j) = F(j_{M'}) \in \mathrm{Aut}(F(0))$ and

$$F(j_M) = F(j \circ j_{M''}) = F(j_{M'}) \circ F(j_{M''})$$

so by the universal property of $K_0(\mathcal{C})$ there is a group homomorphism $K_0(\mathcal{C}) \to \mathrm{Aut}(F(0))$ sending M to $F(j_M)$. Thus, there is a natural action of $K_0(\mathcal{C})$ on $F(0)$ for any F in $\mathcal{F}$. It is not hard to check that these two functors actually give an equivalence of categories, so we leave the rest to the reader. $\square$

This motivates the following definition.

Definition 4.2.10. Let $\mathcal{C}$ be a small exact category. Then we define the Quillen algebraic K-theory space of $\mathcal{C}$ as

$$K(\mathcal{C}) := \Omega BQ\mathcal{C}.$$

Definition 4.2.11. Recall that for X a scheme then an $\mathcal{O}_X$ -module $\mathcal{F}$ is called quasi-coherent if X can be covered by affine opens $U_i = \mathrm{spec}(A_i)$ such that for each i there is an A_i-module M with $\mathcal{F}|_{U_i} = \widetilde{M}$ and we say $\mathcal{F}$ is coherent if this module is finitely generated.

.

Examples 4.2.12. We can now define algebraic K-theory of some familiar objects. When R is a ring, we define

$$K(R) := K(P(R))$$

more generally we can consider the category of finitely generated R-modules denoted $M(R)$, and we define the G-theory of R to be

$$G(R) := K(M(R)).$$

When X is a scheme we define

$$K(X) := K(VB(X))$$

where $VB(X)$ is the category of locally free $\mathcal{O}_X$ modules of finite rank at each stalk. When X is a Noetherian ring, we define $M(X)$ as the category of coherent $\mathcal{O}_X$-modules and we define

$$G(X) := K(M(X)).$$

4.2.2 Reduction by resolution

4.2.3 Dévissage

4.2.4 Localization sequences

Chapter 5

The $S_\bullet$-construction and fundamental theorems

In this section, we discuss results of Waldhausen [22]. We take one departure from the classical approach of Waldhausen and describe a universal property of algebraic K-theory following [20].

5.1 The $S_\bullet$-construction

For simplicity, we begin by defining the $S_\bullet$-construction for an exact category $\mathcal{C}$. We will then remark that the same construction holds for a more general input: a *category with cofibrations and weak equivalences* in the sends of Waldhausen.

Recall that there is a fully faithful embedding $\Delta \subset \mathsf{Cat}$ of the simplex category in the category of small categories. Recall that this also defines a cosimplicial small category with coface maps δ_i and codegeneracies σ_i. We abuse notation and write $[n]$ for the image of $[n]$ in Cat. We write $\mathsf{Cat}(\mathcal{C}, \mathcal{D})$ for the category of functors from $\mathcal{C}$ to $\mathcal{D}$. For example, the arrow category may be defined as

$$\mathsf{Arr}(\mathcal{C}) := \mathsf{Cat}([1], \mathcal{C}).$$

Construction 5.1.1. Let $\mathcal{C}$ be an exact category and consider the full subcategory

$$S_n(\mathcal{C}) \subset \mathsf{Cat}(\mathsf{Arr}([n]), \mathcal{C})$$

of those functors $A\colon \mathsf{Arr}([n]) \to \mathcal{C}$ such that

1. for every functor $\mu\colon [0] \to [n]$, we have $A(\mu \circ \sigma_0) = 0$
2. for every functor $\gamma\colon [2] \to [m]$, the sequence

$$0 \longrightarrow A(\gamma \circ \delta_2) \longrightarrow A(\gamma \circ \delta_1) \longrightarrow A(\gamma \circ \delta_0) \longrightarrow 0$$

is exact in $\mathcal{C}$.

This forms a simplicial object in small categories by functoriality of the composite

$$\mathsf{Cat}(\mathsf{Cat}([1], -), \mathcal{C})\colon\ \Delta^{\mathrm{op}} \to \mathcal{C}.$$

We write

$$d_i\colon\ S_n\mathcal{C} \to S_{n-1}\mathcal{C}$$

for the face maps and

$$s_i\colon\ S_n\mathcal{C} \to S_{n+1}\mathcal{C}$$

for the degeneracy maps.

A sequence

$$0 \to A' \to A \to A'' \to 0$$

is an *exact sequence* if for every functor $\theta\colon\ [1] \to [n]$ the induced sequence

$$0 \to A'(\theta) \to A(\theta) \to A''(\theta) \to 0$$

is an object-wise exact sequence in $\mathcal{C}$. A morphism $A \to A'$ is an isomorphism if for every functor $\theta\colon\ [1] \to [n]$ the map

$$A(\theta) \to A'(\theta)$$

is an object-wise isomorphism. With this class of exact sequences and these isomorphisms, the category $S_n\mathcal{C}$ is in fact an exact category. Writing Exact for the category of small exact categories, we observe that

$$S_\bullet\mathcal{C}\colon\ \Delta^{\mathrm{op}} \to \mathsf{Exact}$$

is a simplicial object in the category of small exact categories. We then write

$$wS_\bullet\mathcal{C}$$

for the groupoid with the same objects, but only isomorphisms.

Definition 5.1.2. Given an exact category $\mathcal{C}$, we define

$$K^{\mathrm{wald}}(\mathcal{C}) := \Omega|N_\bullet(wS_\bullet\mathcal{C})|$$

Here we note that $N_\bullet(wS_\bullet\mathcal{C})$ is in fact a bisimplicial set so by the geometric realization we mean any of the three constructions of geometric realization that produce the same space up to homeomorphism.

Moreover, since $S_\bullet\mathcal{C}$ takes values in the category of small exact categories, we may iterate the construction.

Definition 5.1.3. Given an exact category $\mathcal{C}$, we define

$$S_\bullet^{(n)} := S_\bullet(\dots(S_\bullet\mathcal{C}))$$

where we have iterated $S_\bullet$ n times. We define a sequence of spaces

$$K^{\mathrm{wald}}(\mathcal{C})_n := \Omega|N_\bullet(wS_\bullet^{(n+1)}\mathcal{C})|$$

for all $n \geq 0$.

5.2 The Additivity Theorem

Recall that an Ω-spectrum is a sequence of spaces X_i for $i \geq 0$ with structure maps

$$X_0 \to \Omega X_1 \to \Omega^2 X_2 \to \dots$$

such that all horizontal maps are homeomorphisms. The main result we want to prove is the following.

Theorem 5.2.1. *The sequence of spaces*

$$K^{wald}(\mathcal{C})_0 \to \Omega K^{wald}(\mathcal{C})_1 \to \Omega^2 K^{wald}(\mathcal{C})_2 \to \dots$$

forms an Ω-spectrum.

In fact, one of the most fundamental properties of algebraic K-theory follows from this.

Definition 5.2.2. Given exact categories $\mathcal{A}$, $\mathcal{C}$, $\mathcal{B}$ and functors $f\colon \mathcal{A} \to \mathcal{C}$ and $g\colon \mathcal{B} \to \mathcal{C}$, define $\mathcal{E}(\mathcal{A},\mathcal{C},\mathcal{B})$ to be the pullback

$$\begin{tikzcd} \mathcal{E}(\mathcal{A},\mathcal{C},\mathcal{B}) \arrow[r] \arrow[d] & cof(\mathcal{C})^+ \arrow[d] \\ \mathcal{A} \times \mathcal{B} \arrow[r] & \mathcal{C} \times \mathcal{C}. \end{tikzcd}$$

where $cof(\mathcal{C}) \subset \mathsf{Arr}(\mathcal{C})$ is the full subcategory of the admissible monomorphisms in $\mathcal{C}$ and $cof(\mathcal{C})^+$ is the category equivalent to $cof(\mathcal{C})$ whose objects are admissible monomorphisms $A \to B$ with a choice of quotient B/A. When $\mathcal{A} = \mathcal{B} = \mathcal{C}$, we simply write $\mathcal{E}(\mathcal{C})$ for this construction. Note that $\mathcal{E}(\mathcal{A},\mathcal{C},\mathcal{B})$ is evidently an exact category. An object in $\mathcal{E}(\mathcal{A},\mathcal{C},\mathcal{B})$ may be regarded as an exact sequence in $\mathcal{C}$ of the form

$$0 \to f(A) \to C \to g(B) \to 0$$

where A is an object in $\mathcal{A}$ and B is an object in $\mathcal{B}$.

Theorem 5.2.3 (Additivity)**.** *There is an isomoprhism*

$$((d_0)_*, (d_2)_*)\colon K^{wald}(\mathcal{E}(\mathcal{C})) \cong K(\mathcal{C}) \times K(\mathcal{C})$$

where $(d_0)_*$ is induced by the exact functor

$$d_0\colon \mathcal{E}(\mathcal{C}) \to \mathcal{C}$$

sending a sequence $A \to B \to B/A$ to A and $(d_2)_*$ is induced by the exact functor

$$d_2\colon \mathcal{E}(\mathcal{C}) \to \mathcal{C}$$

sending an exact sequence $A \to B \to B/A$ to B/A.

Theorem 5.2.4 (Additivity version 2)**.** *Given an sequence of functors*

$$F' \to F \to F'' \colon \mathcal{C}' \to \mathcal{C}$$

such that for all objects c in $\mathcal{C}'$ the sequence

$$0 \to F'(c) \to F(c) \to F''(c) \to 0$$

is exact in $\mathcal{C}$, then there is a weak equivalence

$$F_* \simeq F'_* \vee F''_* \colon K(\mathcal{C}') \to K(\mathcal{C}).$$

All of these constructions and results hold in a more general setting.

Definition 5.2.5. A *category with cofibrations* consists of a pointed category $\mathcal{C}$ with zero object 0 and a subcategory $\mathrm{cof}(\mathcal{C})$ such that all isomorphisms are in $\mathcal{C}$, the unique map $0 \to c$ is in $\mathrm{cof}(\mathcal{C})$ for all object c in $\mathcal{C}$, and pushouts

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d] & B \arrow[d] \\ C \arrow[r] & C \coprod_A B \end{tikzcd}$$

in $\mathcal{C}$ along a cofibration $A \to B$ exist in $\mathcal{C}$ and the canonical arrow $C \to C \coprod_A B$ is again in $\mathrm{cof}(\mathcal{C})$.

In particular, we write $C/B = 0 \coprod_B C$ for the pushout

$$\begin{tikzcd} B \arrow[r, tail] \arrow[d] & C \arrow[d] \\ 0 \arrow[r] & 0 \coprod_B C \end{tikzcd}$$

along a cofibration $B \to C$.

Definition 5.2.6. A *Waldhausen category* is a category with cofibrations $(\mathcal{C}, \mathrm{cof}(\mathcal{C})$ with a subcategory $w\mathcal{C}$ of $\mathcal{C}$ containing all isomorphisms and satisfying the gluing lemma: if the commutative diagram

$$\begin{tikzcd} B \arrow[d, "\simeq"] & A \arrow[l, tail] \arrow[r] \arrow[d, "\simeq"] & C \arrow[d, "\simeq"] \\ B' & A' \arrow[l, tail] \arrow[r] & C' \end{tikzcd}$$

where $A \to B$ and $A' \to B'$ are maps in $\mathrm{cof}(\mathcal{C})$ and the vertical maps are maps in $w\mathcal{C}$, then the induced map

$$B \coprod_A C \to B' \coprod_{A'} C'$$

is in $w\mathcal{C}$.

Example 5.2.7. Consider the full subcategory of topological spaces over and under a fixed space X

$$R(X) \subset X \backslash Top/X$$

and let $R(X)$ be the full subcategory of objects $X \to Y \to X$ where (Y, X) is a relative CW complex with finitely many cells. The cofibrations are inclusions of sub CW complexes and the weak equivalences are weak equivalences after forgetting to Top. We call this the Waldhausen category of *retractive spaces*.

5.3 Equivalent formulations of Additivity

We will first prove that Theorem 5.2.3 and Theorem 5.2.4 are equivalent formulations of the additivity theorem. We will then prove that both of them follow from Theorem 5.2.1. In fact Theorem 5.2.1 is also equivalent to Theorem 5.2.3 and Theorem 5.2.4, but we defer the proof of that until later. Here we will write

$$\mathcal{E}(\mathcal{C}) := S_2\mathcal{C}.$$

Proof that Theorem 5.2.3 implies Theorem 5.2.4. We first give an intermediate statement: there is a weak equivalence

$$(d_0)_* \vee (d_2)_* \simeq (d_1)_*\colon\ |wS_\bullet\mathcal{E}(\mathcal{C})| \to |wS_\bullet\mathcal{C}|. \tag{5.3.1}$$

We first prove that this implies Theorem 5.2.4. To see this, note that an exact sequence of functors

$$F' \rightarrowtail F \twoheadrightarrow F''\colon\ \mathcal{C}' \to \mathcal{C}$$

is equivalent data to a functor

$$G\colon\ \mathcal{C}' \to \mathcal{E}(\mathcal{C})$$

such that $d_0 \circ G = F'$, $d_1 \circ G$, and $d_2 \circ G = F''$. So if $(d_0)_* \vee (d_2)_* \simeq (d_1)_*$, then

$$F'_* \vee F''_* = ((d_0)_* \vee (d_2)_*) \circ G_* \simeq (d_1)_* \circ G_* = F_*.$$

It therefore suffices to prove that Theorem 5.2.3 implies that $(d_0)_* \vee (d_2)_* \simeq (d_1)_*$. We note that after precomposing with the exact functor

$$\vee\colon\ \mathcal{C} \times \mathcal{C} \to \mathcal{E}(\mathcal{C})$$

sending (A, B) to $A \to A \vee B \to B$, it is clear that $((d_0)_* \vee d_1)_*) \circ (\vee)_* \simeq (d_1)_* \circ (\vee)_*$. We therefore just need to argue that the map

$$(\vee)_*\colon\ wS_\bullet\mathcal{C} \times wS_\bullet\mathcal{C} \to wS_\bullet\mathcal{E}(\mathcal{C})$$

induces a homotopy equivalence on geometric realizations. However, it is clear that $(d_0)_* \vee (d_2)_* \circ (\vee)_* = \mathrm{id}$ and we assumed that $(d_0)_* \vee (d_2)_*$ induces a homotopy equivalence, so this implies that $(\vee)_*$ does as well. $\square$

Proof that Theorem 5.2.4 implies Theorem 5.2.3. First, we note that Theorem 5.2.3 is a special case of the statement that the functor

$$(d_0, d_2)\colon\ \mathcal{E}(\mathcal{A}, \mathcal{C}, \mathcal{B}) \to \mathcal{A} \times \mathcal{B}$$

sending a cofiber sequence $A \rightarrowtail C \twoheadrightarrow B$, where A is in the essential image of $\mathcal{A}$ and B is in the essential image of $\mathcal{B}$ to (A, B), induces a homotopy equivalence

$$((d_0)_*, (d_2)_*)\colon\ |wS_\bullet\mathcal{E}(\mathcal{A}, \mathcal{C}, \mathcal{B})| \simeq |wS_\bullet\mathcal{A}| \times |wS_\bullet\mathcal{B}|.$$

To see this, we first observe that the functor

$$(\vee)_*\colon\ |wS_\bullet\mathcal{A}| \times |wS_\bullet\mathcal{B}| \to |wS_\bullet\mathcal{E}(\mathcal{A}, \mathcal{C}, \mathcal{B})|$$

satisfies $((d_0)_*, (d_2)_*) \circ (\vee)_* = \mathrm{id}$. The result will follow if we can show that $(\vee)_* \circ ((d_0)_*, (d_2)_*) \simeq \mathrm{id}$. To see this, we rewrite $(\vee)_* \circ ((d_0)_*, (d_2)_*)$ as $F'_* \vee F''_*$ where

$$F'(\ A \rightarrowtail B \twoheadrightarrow C\ ) =\ A \xrightarrow{\mathrm{id}} A \xrightarrow{\mathrm{id}} 0\,, \tag{5.3.2}$$

$$F(\ A \rightarrowtail B \twoheadrightarrow C\ ) =\ A \rightarrowtail B \twoheadrightarrow C\,,\ \text{and} \tag{5.3.3}$$

$$F''(\ A \rightarrowtail B \twoheadrightarrow C\ ) =\ 0 \longrightarrow C \xrightarrow{\mathrm{id}} C\,. \tag{5.3.4}$$

Then clearly $F' \rightarrowtail F \twoheadrightarrow F''$ is a cofiber sequence of exact functors and, by Theorem 5.2.4, we know that

$$(\vee)_* \circ ((d_0)_*, (d_2)_*) = F'_* \vee F''_* \simeq F = \mathrm{id}\,.$$

□

Finally, we prove that Theorem 5.2.1 implies Theorem 5.2.4 and consequently Theorem 5.2.3.

Proof that Theorem 5.2.1 implies Theorem 5.2.4. Recall that Theorem 5.2.4 is equivalent to the statement that the there is a weak equivalence

$$(d_0)_* \vee (d_2)_* \simeq (d_1)_*\colon\ |wS_\bullet\mathcal{E}(\mathcal{C})| \to |wS_\bullet\mathcal{C}|.$$

We claim that this is true after post-composing with the map

$$j\colon\ |wS_\bullet\mathcal{C}| \to \Omega|wS_\bullet^{(2)}\mathcal{C}|;$$

that is, $j \circ ((d_0)_* \vee (d_2)_*) \simeq j \circ (d_1)_*$. Therefore, if j is a homotopy equivalence, then this implies Theorem 5.2.4 as desired. Since j being a homotopy equivalence is implied by Theorem 5.2.1, we are done.

It therefore suffices to prove the claim. Note that we form the 2-skeleton of $|sS_\bullet\mathcal{C}|$, denoted $|wS_\bullet\mathcal{C}|_{(2)}$ by

$$(\coprod_{k=0}^{2} BwS_k\mathcal{C} \times |\Delta^k|)/\simeq .$$

As we noted earlier, the 1-skeleton is $S^1 \wedge BwS_1\mathcal{C} \simeq S^1 \wedge Bw\mathcal{C}$ so we form the 2-skeleton as a homotopy coequalizer of the diagram

$$BwS_2\mathcal{C} \times |\Delta^2| \rightrightarrows BwS_1\mathcal{C} \wedge S^1.$$

The map j is the composite of the adjoint of the map $BwS_1\mathcal{C} \wedge S^1 \to |wS_\bullet\mathcal{C}|_{(2)}$ with the inclusion of the two skeleton. Restricting to the three inclusions $|\Delta^0| \to |\Delta^1|$ gives the maps $(d_0)_*$, $(d_1)_*$, and $(d_2)_*$, so the coequalizer gives a simplicial homotopy from $(d_0)_* \vee (d_2)_*$ to $(d_1)_*$ after post-composing with the map j. □

5.4 Proof of the Additivity theorem

Recall that the minimal choice of weak equivalences is the the class of isomorphisms. We will prove that the Theorem 5.2.3 follows from the special case where the class of weak equivalences is the class of isomorphisms. First, we reduce the case when the weak equivalences are isomorphisms to an even simpler case. This requires the following definition and lemma.

Definition 5.4.1. Let $\mathcal{C}$ be a category with cofibrations, then we define

$$s_n\mathcal{C} := \mathrm{ob}S_n\mathcal{C}.$$

Lemma 5.4.2. *An exact functor $f\colon \mathcal{C} \to \mathcal{C}'$ between categories with cofibrations induces a map*

$$s_\bullet\mathcal{C} \to s_\bullet\mathcal{C}'$$

and a natural isomorphism $\eta\colon f \overset{\cong}{\Rightarrow} g$ between two such functors induces a homotopy between f and g. In particular, an exact equivalence of categories $h\colon \mathcal{C} \to \mathcal{C}'$ induces a homotopy equivalence

$$|s_\bullet h|\colon |s_\bullet\mathcal{C}| \overset{\simeq}{\to} |s_\bullet\mathcal{C}'|.$$

In particular, when $\mathcal{C}$ is a category with cofibrations and weak equivalences then there is a homotopy equivalence

$$|s_\bullet\mathcal{C}| \simeq |isoS_\bullet\mathcal{C}|$$

Proof. We leave the proof as an exercise. However, it helps to observe that a simplicial homotopy from $X \to Y$ is the same data as a natural transformation of functors from

$$\eta\colon X \circ i \to Y \circ i$$

where $i\colon (\Delta/[1])^{\mathrm{op}} \to \Delta$ is the forgetful functor which takes an object $[n] \to [1]$ to $[n]$. ☐

Therefore, the special case of the additivity theorem where the class of weak equivalences is the class of isomorphisms is equivalent to the following proposition

Proposition 5.4.3. *The exact functor*

$$d_0 \vee d_1\colon\ S_2\mathcal{C} \to \mathcal{C} \times \mathcal{C}$$

induces a homotopy equivalence

$$|s_\bullet S_2\mathcal{C}| \xrightarrow{\simeq} |s_\bullet\mathcal{C}| \times |s_\bullet\mathcal{C}|.$$

We defer the proof of this proposition until later, but we will now show that it implies the additivity theorem.

Proof of Theorem 5.2.3 assuming Proposition 5.4.3. We define

$$\mathcal{C}(m,w) \subset \mathsf{Cat}([m],\mathcal{C})$$

to be the full sub category of functors $\mathsf{Cat}([m],\mathcal{C})$ taking values in $w\mathcal{C}$, regarded as a subcategory with cofibrations of $\mathsf{Cat}([m],\mathcal{C})$. We observe that $\mathcal{C}(\bullet,w)$ forms a simplicial category with cofibrations. Proposition 5.4.3 implies that

$$|s_\bullet S_2(\mathcal{C}(\bullet,w))| \to |s_\bullet(\mathcal{C}(\bullet,w)| \times |s_\bullet(\mathcal{C}(\bullet,w))|$$

is a homotopy equivalence (since it is induced by a level-wise weak equivalence between Reedy cofibrant simplicial spaces). We then observe that there is a natural isomorphism of bisimplicial sets

$$N_\bullet wS_\bullet\mathcal{C} \cong s_\bullet(\mathcal{C}(\bullet,w))$$

and therefore this proves Theorem 5.2.3. ☐

It therefore suffices to prove Proposition 5.4.3. First, we note some lemmas that will be used to prove the proposition. The first two lemmas follow from Quillen's Theorem A and Theorem B.

Lemma 5.4.4 (Lemma A)**.** *Let $y \in Y_n$, which by the Yoneda lemma corresponds to a map of simplicial sets*

$$\Delta^n \to Y$$

and let $f\colon X \to Y$ be a map of simplicial sets. Then we let $f/(n,y)$ denote the pullback

$$\begin{tikzcd} f/(n,y) \arrow[r] \arrow[d] & X \arrow[d] \\ \Delta^n \arrow[r] & Y. \end{tikzcd} \tag{5.4.5}$$

If $f/(n,y)$ is contractible for every (n,y), then $X \to Y$ is a homotopy equivalence.

Lemma 5.4.6 (Lemma B)**.** *If for every $a\colon [m] \to [n]$ and every $y \in Y_n$ the induced map*

$$f/(m, a^*y) \to f/(n, y)$$

is a homotopy equivalence, then for every (n, y) the pullback 5.4.5 is a homotopy pullback.

Proof sketch. Let Δ/Y be the category whose objects are pairs $([n], y)$ and morphisms $([n], y) \to ([n'], y')$ are morphisms $a\colon [n] \to [n']$ in Δ such that $a^*y = y'$. We regard this as a functor

$$\Delta/-\colon\ sSet \to \mathsf{Cat}$$

and apply $\Delta/-$ to the pullback diagram 5.4.5. Then one observes that the category $\Delta/(f/(n, y))$ is naturally isomorphic to $(\Delta/f)/(n, y)$. One then uses that there is a homotopy equivalence $N_\bullet(\Delta/Y) \simeq Y$. $\square$

Lemma 5.4.7. *The exact functor d_0 induces a map*

$$(d_0)_*\colon\ s_\bullet S_2\mathcal{C} \to s_\bullet\mathcal{C}$$

of simplicial sets satisfying the hypotheses of Lemma 5.4.6.

Proof. For every $y \in S_n\mathcal{C}$ and every map $w\colon [n] \to [m]$ in Δ, we need to show that the map

$$w_*\colon\ f/(m, w^*y) \to f/(n, y)$$

induces a homotopy equivalence. Since every map $[m] \to [n]$ can be embedded in a commuting triangle

$$\begin{tikzcd}
{[m]} \arrow[rr, "w"] & & {[n]} \\
& {[0]} \arrow[ul, "u"] \arrow[ur, "v"'] &
\end{tikzcd}$$

and if u_* and v_* induce homotopy equivalences then w_* does as well.

Let $*$ denote the unique zero simplex of $s_\bullet\mathcal{C}$. It suffices to show that the map

$$v_i\colon\ [0] \to [n]$$

sending 0 to i induces an equivalence

$$(v_i)_*\colon\ f/(0, *) \to f/(n, y')$$

for each $y' \in s_n\mathcal{C}$.

We can identify an m-simplex of $s_\bullet S_2\mathcal{C}$ with an object of $S_2(S_m\mathcal{C})$, or in other words an cofiber seqeunce $A' \to A \to A''$ of objects in $S_m\mathcal{C}$. An m-simplex of $f/(n, y')$ then consists of an m-simplex $A' \to A \to A''$ of $s_\bullet S_2\mathcal{C}$ and a map $u\colon [m] \to [n]$ such that A' is equal to

$$\mathsf{Arr}([m]) \xrightarrow{u_*} \mathsf{Arr}([n]) \xrightarrow{y'} \mathcal{C}.$$

the map $(d_2)_*$ sends $A' \to A \to A''$ to A'' and it induces a map

$$p\colon\ f/(n, y') \to s_\bullet\mathcal{C}.$$

This map is a left inverse to the composite map

$$s_\bullet\mathcal{C} \xrightarrow{j} f/(0, *) \xrightarrow{(v_i)_*} f/(n, y')$$

for each i, so if we can show p induces a homotopy equivalence, then this implies $(v_i)_*$ induces a homotopy equivalence since we have already observed that j induces a homotopy equivalence.

To show p induces a homotopy equivalence, it suffices to show that the composite $(v_n)_* j_* p_*$ is homotopic to the identity on $f/(n, y')$ so that $(v_n)_* j_*$ is a homotopy inverse to p_* We will construct an explicit homotopy from $(v_n)_* j_* p_*$ to the identity.

We will proceed by lifting the simplicial homotopy that contracts Δ^n to its last vertex. This simplicial homotopy is a natural transformation of functors from

$$\Delta/[1]^{\mathrm{op}} \to \Delta^{\mathrm{op}} \xrightarrow{\Delta^n} \mathsf{Set}$$

to itself by

$$(v\colon\ [m] \to [1]) \mapsto ((u\colon\ [m] \to [n]) \mapsto (\overline{u}\colon\ [m] \to [n]))$$

where $\overline{u}$ is defined as the composite

$$\overline{u}\colon\ [m] \xrightarrow{(u,v)} [n] \times [1] \xrightarrow{w} [n]$$

and where $w(j, 0) = j$ and $w(j, 1) = n$.

A lifting of this homotopy to one on $f/(n, y')$ is a map sending

$$(v\colon\ [m] \to [1])$$

to

$$(A' \to A \to A'', u\colon\ [n] \to [m]) \mapsto (\overline{A}' \to \overline{A} \to \overline{A}'', \overline{u}\colon\ [n] \to [m])$$

where $\overline{u}$ is defined in the same way as before and $\overline{A}'$ is the composite

$$\mathsf{Arr}([m]) \xrightarrow{\overline{u}} \mathsf{Arr}([n]) \xrightarrow{y'} \mathcal{C}.$$

To see that this actually does lift the simplicial contracting homotopy on Δ^n as desired, we note that for each $j \in [m]$, $u(j) \leq \overline{u}(j)$, so there is a natural transformation of functors

$$(u\colon [m] \to [n]) \mapsto (\underline{u}\colon [m] \to [n]).$$

Consequently, there is a natural transformation of functors

$$(u_*\colon \mathsf{Arr}([m]) \to \mathsf{Arr}([n])) \to (\overline{u}_*\colon \mathsf{Arr}([m]) \to \mathsf{Arr}([n])$$

and $\overline{u}_*$ induces a natural transformation of the composite functors

$$\mathsf{Arr}([m]) \to \mathsf{Arr}([n]) \xrightarrow{y} \mathcal{C}$$

or in other words a map from y' to $\overline{A}'$ in $S_m\mathcal{C}$. Such a map is necessarily unique because it is induced by a map $\mathsf{Arr}([m]) \to \mathsf{Arr}([n])$ of partially ordered sets. We define $\overline{A}' \to \overline{A} \to \overline{A}''$ by letting $A'' = \overline{A}''$ and letting $\overline{A}$ be a particular choice of pushout

$$\begin{tikzcd} A' \arrow[r, tail] \arrow[d] & A \arrow[d] \\ \overline{A}' \arrow[r, tail] & \overline{A} \end{tikzcd}$$

so that by construction $\overline{A}' \to \overline{A} \to \overline{A}''$ is a cofiber sequence. We therefore ask that $\overline{A}$ satisfies the following:

1. we let $\overline{A}$ be the object-wise pushout so that for each functor $\theta\colon [1] \to [n]$, there are pushouts

$$\begin{tikzcd} A'(\theta) \arrow[r, tail] \arrow[d] & A(\theta) \arrow[d] \\ \overline{A}'(\theta) \arrow[r, tail] & \overline{A}(\theta) \end{tikzcd}$$

2. if $A' \to \overline{A}'$ is the identity, then we choose $A \to \overline{A}$ to be the identity,

3. if $\overline{A}' = 0$, then we ask that $A \to \overline{A}$ is exactly the map $A \to A''$.

Since making these choices satisfies the universal property of the pushout, we can insist that this is our explicit model for the pushout.

It suffices to check that the construction of $\overline{A}' \to \overline{A} \to \overline{A}''$ is compatible with the maps in $\Delta/[1]$. Given a map $[m'] \to [m]$ over $[1]$, we need to check that the constructions of $\overline{A}' \to \overline{A} \to \overline{A}''$ are compatible. Using our explicit choice of pushout, this compatibility follows. $\square$

The only part of the proof that is outstanding is therefore the proof of Proposition 5.4.3.

Proof of Proposition 5.4.3. We apply Lemma 5.4.6 to the map

$$(d_0)_*\colon\ s_\bullet S_2\mathcal{C} \to s_\bullet\mathcal{C}.$$

In particular, since $s_0\mathcal{C} = *$, we have a homotopy fiber sequence

$$f/(0,*) \to s_\bullet S_2\mathcal{C} \to s_\bullet\mathcal{C} \tag{5.4.8}$$

where $f/(0,*) = s_\bullet S_2'\mathcal{C}$ where $S_2'\mathcal{C} \subset S_2\mathcal{C}$ is the subcategory of cofibrations of the form $0 \to B \cong B$ and therefore the category $S_2'\mathcal{C}$ is equivalent to $\mathcal{C}$. The fiber sequence is equivalent to

$$s_\bullet\mathcal{C} \to s_\bullet S_2\mathcal{C} \to s_\bullet\mathcal{C}. \tag{5.4.9}$$

We consider the map

$$\begin{tikzcd}
s_\bullet\mathcal{C} \arrow[r] \arrow[d, equal] & s_\bullet\mathcal{C} \times s_\bullet\mathcal{C} \arrow[r] \arrow[d, "\vee"] & s_\bullet\mathcal{C} \arrow[d, equal] \\
s_\bullet\mathcal{C} \arrow[r] & s_\bullet S_2\mathcal{C} \arrow[r] & s_\bullet\mathcal{C}.
\end{tikzcd} \tag{5.4.10}$$

of fiber sequence from the trivial fibration where the middle vertical map is induced by the exact functor sending (A,B) to $A \to A \vee B \to B$. Therefore, the middle vertical map is a homotopy equivalence on geometric realizations. The map

$$(d_0)_* \vee (d_1)_*\colon\ s_\bullet S_2\mathcal{C} \to s_\bullet\mathcal{C} \times s_\bullet\mathcal{C} \tag{5.4.11}$$

is a retract of this other map and therefore the map (5.4.11) induces a homotopy equivalence on geometric realization as well. □

5.5 Consequences of the Additivity theorem

Let $\mathcal{C}$ be a Waldhausen category. We have already shown that if the sequential spectrum $\mathbb{K}(\mathcal{C})$ is an ω-spectrum then this implies that $\mathbb{K}(\mathcal{C})$ satisfies the additivity theorem. Our goal is to prove the converse: that the additivity theorem implies that $\mathbb{K}(\mathcal{C})$ is an Ω-spectrum. We begin with a construction.

Definition 5.5.1 (Decalage)**.** Given a simplicial object

$$X_\bullet\colon\ \Delta^{op} \to \mathcal{C}$$

in a category $\mathcal{C}$, then the *Decalage* of $X_\bullet$, or *path object* of $X_\bullet$, is the simplicial object $PX_\bullet$ with

$$P(X_\bullet)_n = X_{n+1}, \tag{5.5.2}$$

$$d_k^{n,P(X_\bullet)} = d_k^{n+1,X_\bullet} \qquad \text{for } 0 \le k \le n, \tag{5.5.3}$$

$$s_k^{n,P(X_\bullet)} = s_k^{n+1,X_\bullet} \qquad \text{for } 0 \le k \le n. \tag{5.5.4}$$

Lemma 5.5.5. *The map*

$$d_0^{1,X_\bullet} \colon PX_\bullet \to X_0$$

of simplicial sets is a simplicial homotopy equivalence.

Proof. The composite

$$X_0 \xrightarrow{s_0^{0,X_\bullet}} P(X_\bullet) \xrightarrow{d_0^{1,X_\bullet}} X_0$$

is the identity map by the simplicial identities, so it suffices to show that there is a simplicial homotopy $s_0^{0,X_\bullet} \circ d_0^{1,X_\bullet} \simeq \mathrm{id}_{P(X_\bullet)}$. We specify a simpicial homotopy using an explicit natural transformation from the functor

$$\Delta/[1]^{\mathrm{op}} \to \Delta^{\mathrm{op}} \xrightarrow{P(X_\bullet)} \mathcal{C}$$

to itself. This homotopy sends $[n] \to [1]$ to $\phi_a^* \colon X_{n+1} \to X_{n+1}$ where ϕ_a^* is induced by a map $\phi_a \colon [n+1] \to [n+1]$ defined by

$$\phi_a(s) = \begin{cases} 0 & \text{if } s = 0 \\ s+1 & \text{if } a(j) = 1 \\ 0 & \text{if } a(j) = 0 \end{cases}$$

□

We observe that there is a map of simplicial sets

$$d_0^{k+1,X_\bullet} \colon P(X_\bullet)_k \to X_k$$

and a map from the constant simplicial set X_1 into $P(X_\bullet)$ since X_1 is the 0-simplices of $P(X_\bullet)$.

Example 5.5.6. Let $\mathcal{C}$ be a Waldhausen category. Then we have a sequence of bisimplicial sets

$$N_\bullet wS_1\mathcal{C} \to P(N_\bullet wS_\bullet\mathcal{C}) \to N_\bullet wS_\bullet\mathcal{C}$$

and the composite factors through $N_\bullet wS_0\mathcal{C} = *$ and by Lemma 5.5.5, the bisimplicial set $P(N_\bullet wS_\bullet\mathcal{C})$ is contractible. This produces a map

$$|w\mathcal{C}| \to \Omega|wS_\bullet\mathcal{C}|$$

and in fact this is the same map that we constructed earlier by inspection of the explicit homotopy equivalence $PN_\bullet wS_\bullet\mathcal{C} \simeq N_\bullet wS_0\mathcal{C} = *$. By substituting $\mathcal{C}$ with $S_\bullet\mathcal{C}$, we produce a sequence

$$|N_\bullet wS_\bullet\mathcal{C}| \to |PN_\bullet wS_\bullet^{(2)}\mathcal{C}| \to |N_\bullet wS_\bullet^{(2)}\mathcal{C}|.$$

Proposition 5.5.7. *The sequence*

$$|N_\bullet wS_\bullet\mathcal{C}| \to |PN_\bullet wS_\bullet^{(2)}\mathcal{C}| \to |N_\bullet wS_\bullet^{(2)}\mathcal{C}|$$

is a homotopy fiber sequence and consequently

$$|N_\bullet wS_\bullet^{(n)}\mathcal{C}| \simeq \Omega|N_\bullet wS_\bullet^{(n+1)}\mathcal{C}|$$

for all $n \geq 1$.

In fact, we will prove a more general theorem and this proposition will be a special case. For the more general theorem, we need some setup.

Definition 5.5.8. Let $f\colon \mathcal{A} \to \mathcal{B}$ be an exact functor between Waldhausen categories. Then define $S_\bullet(A \to B)$ to be the pullback

$$\begin{tikzcd}
S_\bullet(\mathcal{A} \to \mathcal{B}) \arrow[r] \arrow[d] & PS_\bullet\mathcal{B} \arrow[d, "d_0^{\bullet,S_\bullet\mathcal{B}}"] \\
S_\bullet(\mathcal{A}) \arrow[r, "S_\bullet f"] & S_\bullet(\mathcal{B})
\end{tikzcd}$$

in simplicial Waldhausen categories. In particular, there are pullbacks

$$\begin{tikzcd}
S_n(\mathcal{A} \to \mathcal{B}) \arrow[r] \arrow[d] & S_{n+1}\mathcal{B} \arrow[d, "d_0^{n+1,S_\bullet\mathcal{B}}"] \\
S_n\mathcal{A} \arrow[r, "S_n f"] & S_n\mathcal{B}
\end{tikzcd}$$

and the vertical map on the right has a section (which is not compatible with the face maps) so $S_n(\mathcal{A} \to \mathcal{B})$ may be identified with the fiber product

$$S_n(\mathcal{A}) \times_{S_n\mathcal{B}} S_{n+1}\mathcal{B}$$

so it has objects pairs (A, B) where A is a functor $A\colon \mathsf{Arr}([n]) \to \mathcal{A}$ and B is a functor $B\colon \mathsf{Arr}([n+1]) \to \mathcal{B}$ such that

$$d_0^{n+1,S_\bullet\mathcal{B}}(B) \cong B'$$

where B' is the functor $B'\colon \mathsf{Arr}([n]) \xrightarrow{A} \mathcal{A} \xrightarrow{f} \mathcal{B}$.

We note that there is a sequence

$$\mathcal{B} \to PS_\bullet\mathcal{B} \to S_\bullet\mathcal{B}$$

of simplicial Waldhausen categories, whose composite factors through a point $*$ where $\mathcal{B}$ denotes the constant simplicial Waldhausen category. By letting the map $\mathcal{B} \to S_\bullet\mathcal{A}$ be the map that factors through a point, we get a sequence

$$\mathcal{B} \to S_\bullet(\mathcal{A} \to \mathcal{B}) \to S_\bullet\mathcal{A}$$

such that the composite factors through a point.

Proposition 5.5.9. *The sequence*

$$|N_\bullet wS_\bullet \mathcal{B}| \to |N_\bullet S_\bullet^{(2)}(\mathcal{A} \to \mathcal{B})| \to |N_\bullet S_\bullet^{(2)} \mathcal{A}|$$

is a homotopy fiber sequence. In particular, the sequence

$$|N_\bullet S_\bullet \mathcal{A}| \to |PN_\bullet wS_\bullet^{(2)} \mathcal{A}| \to |N_\bullet wS_\bullet^{(2)} \mathcal{A}|$$

is a homotopy fiber sequence, which implies that $\mathbb{K}(\mathcal{C})$ is an Ω-spectrum.

Proof. By [?qx, Lem. 5.2], it suffices to show that the sequence

$$|N_\bullet wS_\bullet \mathcal{B}| \to |N_\bullet wS_\bullet S_n(\mathcal{A} \to \mathcal{B})| \to |N_\bullet wS_\bullet S_n \mathcal{A}|$$

is a fiber sequence for each n, since $|N_\bullet wS_\bullet S_n \mathcal{A}|$ is connected for each n. The idea is to use the Additivity theorem to show that this sequence is homotopy equivalent to the trivial fiber sequence

$$|N_\bullet wS_\bullet \mathcal{B}| \to |N_\bullet wS_\bullet \mathcal{B}| \times |N_\bullet wS_\bullet S_n(\mathcal{A})| \to |N_\bullet wS_\bullet S_n \mathcal{A}|.$$

Unpacking an object in $S_n(\mathcal{A} \to \mathcal{B})$, we see that it amounts to a sequence

$$A_{0,1} \rightarrowtail A_{0,2} \rightarrowtail \cdots \rightarrowtail A_{0,n}$$

of cofibrations in $\mathcal{A}$ and a sequence

$$B_{0,1} \rightarrowtail \cdots \rightarrowtail B_{0,n+1} \tag{5.5.10}$$

of cofibrations in $\mathcal{B}$ such that

$$\begin{tikzcd}
f(A_{0,1}) \arrow[r, tail] \arrow[d, "\cong"] & f(A_{0,2}) \arrow[r, tail] \arrow[d, "\cong"] & \cdots \arrow[r, tail] & f(A_{0,n}) \arrow[d, "\cong"] \\
B_{0,2}/B_{0,1} \arrow[r, tail] & B_{0,2}/B_{0,1} \arrow[r, tail] & \cdots \arrow[r, tail] & B_{0,n+1}/B_{0,1}.
\end{tikzcd}$$

Consider the full subcategory of $S'_n(\mathcal{A} \to \mathcal{B}) \subset S_n(\mathcal{A} \to \mathcal{B})$ consisting of objects such that all maps in the sequence (5.5.10) are identity maps and $A_{0,k} = 0$ for all $1 \le k \le n$. It's clear that there is an equivalence of categories between $\mathcal{B}$ and $S'_n(\mathcal{A} \to \mathcal{B})$. Let $S''_n(\mathcal{A} \to \mathcal{B})$ be the full subcategory of $S_n(\mathcal{A} \to \mathcal{B})$ where $B_0 = 0$. Then there is an equivalence of categories between $S''_n(\mathcal{A} \to \mathcal{B})$ and $S_n\mathcal{A}$. Consider the cofiber sequence of endofunctors

$$j' \rightarrowtail \mathrm{id} \twoheadrightarrow j'' : S_n(\mathcal{A} \to \mathcal{B}) \to S_n(\mathcal{A} \to \mathcal{B})$$

where

$$\begin{aligned}
j'(A_{\bullet,\bullet}, B_{\bullet,\bullet}) &= (\, 0 \rightarrowtail \cdots \rightarrowtail 0\,,\ B_0 \xrightarrow{\mathrm{id}} B_{0,1} \xrightarrow{\mathrm{id}} \cdots \xrightarrow{\mathrm{id}} B_{0,1}\,) \\
\mathrm{id}(A_{\bullet,\bullet}, B_{\bullet,\bullet}) &= (A_{\bullet,\bullet}, B_{\bullet,\bullet}) \\
j''(A_\bullet, B_\bullet) &= (A_{\bullet,\bullet},\ 0 \rightarrowtail f(A_{0,1}) \rightarrowtail \cdots \rightarrowtail f(A_{0,n})
\end{aligned}$$

so j' takes values in $S'_n(\mathcal{A} \to \mathcal{B})$ and j'' takes values in $S''_n(\mathcal{A} \to \mathcal{B})$. By the Additivity Theorem, the identity is homotopy equivalent to $j'_* + j''_*$. Thus the map

$$|N_\bullet wS_\bullet S_n\mathcal{A}| \times |N_\bullet wS_\bullet B| \to |N_\bullet wS_\bullet S_n(\mathcal{A} \to \mathcal{B})| \tag{5.5.11}$$

induced by the exact functor

$$S_n\mathcal{A} \times \mathcal{B} \to S_n(\mathcal{A} \to \mathcal{B})$$

defined by

$$s\colon \left(A_{0,1} \rightarrowtail A_{0,2} \rightarrowtail \cdots \rightarrowtail A_{0,n}, B\right) \mapsto$$

$$\left(A_{0,1} \rightarrowtail A_{0,2} \rightarrowtail \cdots \rightarrowtail A_{0,n}, B \xrightarrow{\mathrm{id}} B \xrightarrow{\mathrm{id}} \cdots \xrightarrow{\mathrm{id}} B \right).$$

Then letting $r = j'_* + j'_{**}$ (where we do not include the equivalence of categories in the notation), then by inspection $r \circ s = \mathrm{id}$. Also, the additivity theorem implies that $s \circ j'_* + j'_{**} \simeq \mathrm{id}$, so this implies the map (5.5.11) is a homotopy equivalence. This map also agrees with the map from the product fibration to our sequence, so up to homotopy the sequence is the trivial homotopy fiber sequence. □

Corollary 5.5.12. *The spectrum $\mathbb{K}(\mathcal{C})$ is an Ω-spectrum.*

5.6 A universal property of algebraic K-theory

The slogan is that algebraic K-theory is the universal functor that splits exact sequences. We call such functors additive functors. Such functors are part of the data of what we will call a *global Euler characteristic* after [20]. For this definition, let Wald be the category of small Waldhausen categories and exact functors between them (those preserving 0, cofibrations and weak equivalences and commuting with pushouts along cofibrations). Also, write $w_1\mathcal{C}$ for the full subcategory of $\mathsf{Arr}(\mathcal{C})$ with objects given by morphisms in $w\mathcal{C}$.

Definition 5.6.1. A *global Euler characteristic* is a pair (E, χ) where E is a functor

$$E\colon \mathit{Wald} \to \mathit{sSet}_*$$

and χ is a natural transformation

$$\chi\colon \mathrm{ob}(-) \to E(-),$$

such that for any $\mathcal{C}, \mathcal{D}$ in Wald the functor E satisfies:

1. the canonical map $E(\mathcal{C} \times \mathcal{D}) \to E(\mathcal{C}) \times E(\mathcal{D})$ is a weak equivalence,

2. the canonical functor
$$s_0\colon \mathcal{C} \to w_1\mathcal{C}$$
sending c to id_c induces a weak equivalence
$$E(\mathcal{C}) \simeq E(w_1\mathcal{C}),$$

3. the additivity theorem (Theorem 5.2.3) holds for E,

4. the functor E is group-like the simplicial set $E(\mathcal{C})$ is a group-like H-space with multiplication
$$E(\mathcal{C}) \times E(\mathcal{C}) \overset{(s_*,q_*)}{\underset{\simeq}{\leftarrow}} E(\mathcal{E}(\mathcal{C})) \xrightarrow{t_*} E(\mathcal{C})$$

Example 5.6.2. The functor $K(\mathcal{C})$ is an additive functor and it comes equipped with a natural transformation
$$\chi_{\mathrm{univ}}\colon\ \mathrm{ob}\,\mathcal{C} \to K(\mathcal{C})$$
via inclusion of 0-simplices.

We let Eul be the category whose objects are global Euler characteristics and whose morphisms $(E,\chi_E) \to (F,\chi_F)$ are natural transformations $a\colon\ E \Rightarrow F$ such that $a \circ \chi_E = \chi_F$. We say a morphisms $a\colon\ (E,\chi_E) \to (F,\chi_F)$ is a *weak equivalence* if $a_{\mathcal{C}}\colon\ E(\mathcal{C}) \to F(\mathcal{C})$ is a weak equivalence for all small Waldhausen categories $\mathcal{C}$.

Definition 5.6.3. Let $\mathrm{Ho}(\mathsf{Eul})$ be the category with the same objects as Eul and with morphisms
$$\mathrm{Hom}_{\mathrm{Ho}(\mathsf{Eul})}(E,F) := \mathrm{Hom}_{\mathsf{Eul}}(E,F)/\sim$$
where $f \sim g$ if there is a zigzag of weak equivalences $f \simeq g$.

Implicitly, we claim that this notion of weak equivalence is an equivalence relation. (To define the homotopy category more carefully, we would use the technique of formally inverting the class of weak equivalences $\mathcal{W}$ so that
$$\mathrm{Ho}(\mathsf{Eul}) = \mathsf{Eul}[\mathcal{W}^{-1}],$$
but we will not spell out this construction at the moment for brevity.)

We now formulate the universal property of algebraic K-theory.

Theorem 5.6.4 (Universal property)**.** *The Euler characteristic (K,χ_{univ}) is the initial object in $Ho(\mathsf{Eul})$.*

Construction 5.6.5 (Additive approximation)**.** We define a symmetric spectrum $\mathbf{P}F\mathcal{C}$ with n-th space

$$\mathbf{P}'F_n\mathcal{C} := \operatorname*{hocolim}_{k\in\mathcal{I}} \Omega^k\Sigma^n|F(wS^{(k)}_\bullet\mathcal{C})|$$

where $\mathcal{I}$ is the category of finite sets and injective maps. (For a functor

$$X\colon J \to \mathsf{Top}$$

where J is a small category, we define

$$\operatorname*{hocolim}_{k\in J} X_k := |[m] \to \coprod_{j\in N_m J} X_{j_0}|$$

where $j = (j_0 \to j_1 \to \cdots \to j_m)$.)

We then define the additive approximation to be

$$F^{\mathrm{add}}(\mathcal{C}) := \operatorname*{hocolim}_{n\in\mathbb{N}} \Omega^n \mathbf{P}'F_n\mathcal{C} = \Omega^\infty\,\mathbf{P}F\mathcal{C}$$

and it is equipped with a natural transformation

$$\eta\colon F \to F^{\mathrm{add}}.$$

Examples 5.6.6. By Theorem 5.2.1, we have a natural transformation

$$\chi_{univ}\colon \mathrm{ob}(\mathcal{C}) \to \mathrm{ob}^{\mathrm{add}}(\mathcal{C}) \simeq K(\mathcal{C})$$

and $K(\mathcal{C})$ is the additive approximation to $\mathrm{ob}(-)$.

Theorem 5.6.7. *Given a functor $F\colon Wald \to sSet_*$, the associated functor F^{add} is the universal additive functor equipped with a natural transformation*

$$F \to F^{add}.$$

Proof. See [20]. (Fill in later. Didn't get to this in class) □

The universal property of algebraic K-theory clearly follows from this theorem.

5.6.1 Waldhausen's fibration theorem

For fibration sequences in Waldhausen algebraic K-theory, we need some extra assumptions on our Waldhausen category: existence of a *cylinder functor* satisfying the *cylinder axiom* and the *saturation axiom* and *extension axiom*. We begin with these notions.

Definition 5.6.8. We say a Waldhausen category $\mathcal{C}$ has a *cylinder functor* if it is equipped with a functor

$$T\colon \mathsf{Arr}(\mathcal{C}) \to \mathcal{C}$$

equipped with natural transformations of functors $\mathsf{Arr}(\mathcal{C}) \to \mathcal{C}$

$$j_1\colon s(-) \to T(-) \tag{5.6.9}$$
$$j_2\colon t(-) \to T(-) \tag{5.6.10}$$
$$p\colon T(-) \to t(-) \tag{5.6.11}$$
$$\tag{5.6.12}$$

such that the diagram

$$\begin{tikzcd} s(-) \arrow[r, "j_1"] \arrow[dr] & T(-) \arrow[d, "p"] & t(-) \arrow[l, "j_2"'] \arrow[dl, equal] \\ & t(-) & \end{tikzcd}$$

commutes where $s(f) = A$ and $t(f) = B$ for a map $f\colon A \to B$ and the natural transformation $s(-) \to T(-)$ send f to the map $f\colon A \to B$.

Let $c_1\mathcal{C}$ be the full subcategory of the arrow category whose objects are cofibrations regarded as a Waldhausen category in the the evident way. We ask that $T(-)$ satisfies:

1. The functor
$$\mathsf{Arr}(\mathcal{C}) \to c_1\mathcal{C}$$
mapping f to $j_1(f) \vee j_2(f)$ is exact.

2. We have $T(0 \to A) = A$, for every A in $\mathcal{A}$ and $j_1(0 \to A) = p(0 \to A) = \mathrm{id}_A$

Example 5.6.13. The cylinder functor should remind us of the mapping cylinder. For example, the Waldhausen category $R^f(X)$ has a cylinder functor defined by

$$T(Y \to Y') = X \cup_{X \times [0,1]} Y \times [0,1] \cup_{Y \times 1} Y'$$

Definition 5.6.14. Suppose $\mathcal{C}$ is a Waldhausen category with a cylinder functor. We say $\mathcal{C}$ satisfies the *cylinder axiom* if the the natural transformation $p\colon T(f) \to t(f)$ is in $w\mathcal{C}$ for all f in $\mathsf{Arr}(\mathcal{C})$.

Definition 5.6.15. Given a Waldhausen category $\mathcal{C}$, we say that $\mathcal{C}$ satisfies the *saturation axiom* if $w\mathcal{C}$ satisfies the 2 out of 3 property; i.e. for any composable pair of morphisms $g \circ f$ in $\mathcal{C}$ and any two of the set $\{f, g, g \circ f\}$ are morphisms in $w\mathcal{C}$ then the third is also.

Remark 5.6.16. As a consequence, when $\mathcal{C}$ is a Waldhausen category with a cylinder functor satisfying the cylinder functor and it satisfies the saturation axiom, it follows that $j_2(f)$ is always in $w\mathcal{C}$ and that $j_1(f)$ is in $w\mathcal{C}$ if and only if f is in $w\mathcal{C}$.

From a cylinder functor, we can give a few other common constructions in algebraic topology.

Definition 5.6.17. The *cone functor* $C(-)$ of a Waldhausen category $\mathcal{C}$ with a cylinder functor T is the composite

$$C\colon\ \mathcal{C} \to \mathsf{Arr}(\mathcal{C}) \xrightarrow{T} \mathcal{C}$$

where the first functor $\mathcal{C} \to \mathsf{Arr}(\mathcal{C})$ sends A to $A \to 0$; in other words $C(A) = T(A \to 0)$.

Definition 5.6.18. The *suspension functor* is the functor

$$\Sigma(-)\colon\ \mathcal{C} \to \mathcal{C}$$

is the cofiber of the natural transformation

$$\mathrm{id} \rightarrowtail C(-)\ .$$

Example 5.6.19. The Waldhausen category $R^f(X)$ has a cylinder functor satisfying the cylinder axiom and it satisfies the the saturation axiom, when equipped with the homotopy equivalences as weak equivalences. If one equips $R^f(X)$ with a Waldhausen category structure where the weak equivalences are homeomorphisms, however, then this does not satisfy the cylinder axiom since the mapping space $X \cup_{X\times[0,1]} Y \times [0,1] \cup_{Y\times 1} Y'$ is homotopy equivalent to Y' in $R^f(X)$, but it is not homeomorphic to Y'.

Example 5.6.20. Exact categories do not usually have cylinder functors, however let $\mathbf{Ch}^b(\mathcal{C})$ be the category of bounded chain complexes in an exact category $\mathcal{C}$ such as the category of finitely generated R module $M(R)$ where R is a commutative ring. Then $\mathbb{C}^b(\mathcal{C})$ is a Waldhausen category with cofibrations the level-wise admissible monomorphisms and weak equivalences the quasi-isomorphisms (maps that are quasi-isomorphisms in the ambient abelian category $\mathbf{Ch}^b(\mathcal{A})$). This Waldhausen category has a cylinder functor given by the usual mapping cylinder sending $f\colon\ A_\bullet \to B_\bullet$ to the chain complex $T(f)$ with

$$T(f)_n = A_n \oplus A_{n-1} \oplus B_n$$

and the suspension functor ΣA is the shift operator $A_\bullet[-1]$ so that

$$(\Sigma A)_n = A_{n-1}.$$

Lemma 5.6.21. *If $\mathcal{C}$ is a Waldhausen category with a cylinder functor then $S_n\mathcal{C}$ is a Waldhausen category with a cylinder functor*

$$S_nT\colon \mathsf{Arr}(S_n\mathcal{C}) = S_n(\mathsf{Arr}((C))) \to S_n(\mathcal{C}))$$

with natural transformations S_nj_1, S_nj_2, and S_np. When the cylinder functor of $\mathcal{C}$ satisfies the cylinder axiom, then so does the cylinder functor of $S_n\mathcal{C}$. The $\mathcal{C}$ satisfies the saturation axiom, then $S_n\mathcal{C}$ also satisfies the saturation axiom.

Proof. Left as an exercise. □

Note that as a consequence of the additivity theorem, algebraic K-theory is an H-space with operation induced by the composite exact functor

$$\mathcal{C} \times \mathcal{C} \overset{\vee}{\to} S_2\mathcal{C} \overset{d_1}{\to} \mathcal{C}$$

sending (A, B) to $A \vee B$. The suspension functor allows us to define a homotopy inverse for this operation when our Waldhausen category $\mathcal{C}$ has a cylinder functor that also satisfies the cylinder axiom.

Proposition 5.6.22. *If $\mathcal{C}$ is a Waldhausen category with a cylinder functor satisfying the cylinder axiom, the the suspension functor induces a map*

$$\Sigma\colon K(\mathcal{C}) \to K(\mathcal{C}$$

which represents a homotopy inverse in the H-space structure on $K(\mathcal{C})$ give by sum. Consequently, $K(\mathcal{C})$ is a group-like *H-space.*

Proof. Consider the cofiber sequence

$$\mathrm{id} \rightarrowtail C \twoheadrightarrow \Sigma : \mathcal{C} \to \mathcal{C}$$

of exact functors. Then by the additivity theorem, we know that $(\Sigma) \vee (\mathrm{id})_* \simeq (C)_*$. Since $CA \to 0$ is a weak equivalence by assumption for all A in $\mathcal{C}$, $(C)_*$ is nullhomotopic and consequently $(\Sigma)_* \vee (\mathrm{id})_*$ is null homotopic. Thus, $\Sigma A + A = 0$ in the H-space structure on $K((C)$. □

Definition 5.6.23. We say that a Waldhausen category $\mathcal{C}$ satisfies the *extension axiom* if for each map of cofiber sequences

$$\begin{tikzcd}
A \arrow[r, tail] \arrow[d] & B \arrow[r, two heads] \arrow[d] & C \arrow[d] \\
A' \arrow[r, tail] & B' \arrow[r, two heads] & C'
\end{tikzcd}$$

such that $A \to A'$ and $C \to C'$ are weak equivalences then $B \to B'$ is also a weak equivalence.

Theorem 5.6.24 (Waldhausen's fibration theorem)**.** *Let $(\mathcal{C}, c\mathcal{C})$ be a category with cofibrations equipped with two wide subcategories $v\mathcal{C} \subset w\mathcal{C}$ of weak equivalences such that $(\mathcal{C}, c\mathcal{C}, w\mathcal{C})$ and $(\mathcal{C}, c\mathcal{C}, v\mathcal{C})$ are Waldhausen categories. Suppose that $(\mathcal{C}, c\mathcal{C}, w\mathcal{C})$ is satisfies the saturation axiom, the extension axiom, and it is equipped with a cylinder functor satisfying the cylinder axiom. Let $\mathcal{C}^w$ be the full sub Waldhausen category of $(\mathcal{C}, c\mathcal{C}, v\mathcal{C})$ consisting of A in $\mathcal{C}$ such that $0 \to A$ is a map in $w\mathcal{C}$. Then there is a homotopy fiber sequence*

$$K(\mathcal{C}^w) \to K((\mathcal{C}, c\mathcal{C}, v\mathcal{C}) \to K((\mathcal{C}, c\mathcal{C}, w\mathcal{C}).$$

[Gabe: Add remarks about bicategories.]

Definition 5.6.25. We define a bicategory $vw\mathcal{C}$ with bimorphisms given by commutative squares

$$\begin{tikzcd} a \arrow[r, "w"] \arrow[d, "v"] & b \arrow[d, "v'"] \\ a' \arrow[r, "w'"] & b' \end{tikzcd}$$

where horizontal morphisms w and w' are maps in $w\mathcal{C}$ and the vertical morphisms v and v' are maps in $v\mathcal{C}$.

Lemma 5.6.26. *If $(\mathcal{C}, c\mathcal{C}, w\mathcal{C})$ is a Waldhausen category with a cylinder functor satisfying the cylinder axiom and the saturation axiom, then the inclusion of the subcategory of acyclic cofibrations $\underline{w}\mathcal{C} \subset w\mathcal{C}$ (where $\underline{w}\mathcal{C} = c\mathcal{C} \cap w\mathcal{C}$) induces a homotopy equivalence*

$$|N_\bullet \underline{w}\mathcal{C}| \simeq |N_\bullet w\mathcal{C}|.$$

Proof. Let $i\colon \underline{w}\mathcal{C} \hookrightarrow w\mathcal{C}$ be the inclusion. By Quillen's theorem A, it suffices to show that $i\backslash B$ is contractible for all B in $w\mathcal{C}$. Recall that an object in $i\backslash B$ is a pair $(A, f\colon A \to B)$ where f is in $w\mathcal{C}$. Since the cylinder functor satisfies the cylinder axiom, the map $T(f) \to B$ is in $w\mathcal{C}$ so we can define a functor $t\colon i\backslash B \to i\backslash B$ by $t(A, f) = (T(f), p)$. Then $j_1(f)$ and $j_2(f)$ are also in $w\mathcal{C}$ by the saturation axiom and the cylinder axiom. They are also cofibrations by the axioms of a cylinder functor so they are maps in $\overline{w}\mathcal{C}$. Then there is a natural isomorphism $p \circ j_1(f) \simeq f$ and $p \circ j_2(f\colon A \to B) = \mathrm{id}_B$. Thus there are homotopies

$$\mathrm{id}_{i/B} \simeq t \simeq \mathrm{const}_B$$

so i/B must be contractible. Since this argument did not depend on a particular choice of B, we have prove then claim. ☐

Lemma 5.6.27 (Swallowing lemma)**.** *Let $\mathcal{A}$ be a subcategory of $\mathcal{B}$ and let $\mathcal{AB}$ be the bicategory of the commutative squares with vertical arrows in $\mathcal{A}$ and horizontal arrows in $\mathcal{B}$, then the map*

$$\mathcal{B} \to \mathcal{AB}$$

induces a homotopy equivalence

$$|N_\bullet \mathcal{B}| \simeq |N_\bullet^{hor} N_\bullet^{ver} \mathcal{AB}|.$$

Proof. If suffices to prove that for each n the map

$$N_n \mathcal{B} \to N_n^{\mathrm{ver}} \mathcal{AB}$$

induces an equivalence on geometric realizations. For fixed n, we define a map

$$N_n^{\mathrm{vert}} \mathcal{AB} \to \mathcal{B}$$

taking a sequence $A_0 \to \dots A_n$ to A_0. This is clearly a left inverse to the inclusion, so it suffices to shoe that the other composite is homotopic to the identity. This takes a sequence $A_0 \to \dots A_n$ to the sequence $A_0 \to \dots A_0$ of identity maps. We produce a natural transformation from this functor to the identity by way of the commuting square

$$\begin{tikzcd}
A_0 \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"] & A_0 \arrow[r, "\mathrm{id}"] \arrow[d, "a_1"'] & \dots \arrow[r, "\mathrm{id}"] & A_0 \arrow[d, "a_n \circ \cdots \circ a_1"] \\
A_0 \arrow[r, "a_1"'] & A_1 \arrow[r, "a_2"'] & \dots \arrow[r, "a_n"'] & A_n
\end{tikzcd}$$

and therefore $\mathcal{B}$ is a deformation retract of $N_n^{\mathrm{vert}} \mathcal{AB}$. □

Proof of Theorem 5.6.24. Consider $w\mathcal{C}$ a a bicategory which is vertically constant so that there is an evident map of bicategories

$$w\mathcal{C} \to vw\mathcal{C}$$

and applying the nerve in the vertical direction we have a map

$$w\mathcal{C} \to N_\bullet^{\mathrm{vert}} vw\mathcal{C}$$

of simplicial categories that induces a homotopy equivalence after applying the nerve and geometric realization by Lemma 5.6.35. After passing to nerves and diagonalizing there is a map

$$N_\bullet^{\mathrm{vert}} N_\bullet^{\mathrm{hor}} vw\mathcal{C} \to N_\bullet w\mathcal{C}$$

which is a left inverse to the inclusion and again by Lemma 5.6.35 this is a homotopy equivalence.

By first applying the $S_\bullet$-construction we produce a simplicial bicategory $vwS_\bullet\mathcal{C}$ and again this is homotopy equivalence to $wS_\bullet\mathcal{C}$ after passing to nerves. Let $v\overline{w}\mathcal{C}$ denote the sub-bicategory of $vw\mathcal{C}$ such that the horizontal morphisms are also cofibrations. Then the inclusion

$$v\overline{w}\mathcal{C} \to vw\mathcal{C}$$

induces a homotopy equivalence by Lemma 5.6.26, which uses the cylinder functor satisfying the cylinder axiom and the saturation axiom. Again, we can form a simplicial bicategory $v\overline{w}S_\bullet\mathcal{C}$ in the same way. The square of interest for our theorem is the outer square in the diagram

$$\begin{tikzcd} vS_\bullet\mathcal{C}^w \arrow[r] \arrow[d] & v\overline{w}S_\bullet\mathcal{C}^w \arrow[r] \arrow[d] & vwS_\bullet\mathcal{C}^w \arrow[r] \arrow[d] & wS_\bullet\mathcal{C}^w \arrow[d] \\ vS_\bullet\mathcal{C} \arrow[r] & v\overline{w}S_\bullet\mathcal{C} \arrow[r] & vwS_\bullet\mathcal{C} \arrow[r] & wS_\bullet\mathcal{C}. \end{tikzcd} \tag{5.6.28}$$

We have already shown that the horizontal maps in the middle and the right induce homotopy equivalences upon geometric realization, so it suffices to show the square on the left is induces a homotopy pullback after geometric realization and that $wS_\bullet\mathcal{C}^w$ is contractible after geometric realization. However, since $0 \to A$ is in $w\mathcal{C}$ for each A in $\mathcal{C}^w$, the category $w\mathcal{C}^w$ is equivalent to the terminal category and similarly for $wS_\bullet\mathcal{C}^w$. We therefore just need to show that the left square in the diagram (5.6.28) induces a pullback after applying geometric realization.

We will do this by identifying $v\overline{w}\mathcal{C}$ with $vS_\bullet^{(2)}f$ after applying geometric realizations and then producing the desired homotopy fiber sequence by applying Proposition 5.5.9 and the additivity theorem.

To make this identification, we note that by the extension axiom of $(\mathcal{C}, c\mathcal{C}, w\mathcal{C})$ a cofibration in $\mathcal{C}$ that is also a weak equivalence is the same data as a cofibration whose quotient lies in $\mathcal{C}^w$. In particular, we have an equivalence of categories between $S_1\mathcal{C}$ and $\overline{w}\mathcal{C}$. More generally, we produce an equivalence of categories between $S_n\mathcal{C}$ and $N_n\overline{w}\mathcal{C}$ and therefore an equivalence of categories between $vS_n\mathcal{C}$ and $N_n^{texthor}v\overline{w}\mathcal{C}$. By the same argument, we get an equivalence of categories between vS_mS_nf and $N_n^{\mathrm{vert}}v\overline{w}S_m\mathcal{C}$ and therefore after applying nerves and geometric realization we get a homotopy equivalence

$$|N_\bullet vS_\bullet^{(2)}| \simeq |N_\bullet^{\mathrm{hor}}N_\bullet^{\mathrm{vert}}v\overline{w}S_\bullet\mathcal{C}|$$

and by inspection this homotopy equivalence is compaible with the maps from $vS_\bullet\mathcal{C}$ as required. $\square$

As an application, we will prove the Gillet-Waldhausen theorem which reduces the algebraic K-theory of bounded chain complex in an exact category $\mathcal{C}$ to the algebraic K-theory of $\mathcal{C}$.

Theorem 5.6.29 (Gillet-Waldhausen Theorem)**.** *Let $\mathcal{C}$ be an exact category embedded in an abelian category $\mathcal{A}$, which is closed under kernels of surjections in $\mathcal{A}$. Then the exact inclusion*

$$\mathcal{C} \to \mathbf{Ch}^b(\mathcal{C})$$

of Waldhausen categories induces a homotopy equivalence

$$K(\mathcal{C}) \simeq K(\mathbf{Ch}^b(\mathcal{C})).$$

Definition 5.6.30. We say that a sequence $0 \to A_n \to \dots A_0 \to 0$ in an exact category $\mathcal{C}$ is *admissibly exact* if each map decomposes as

$$A_{i+1} \twoheadrightarrow B_i \rightarrowtail A_i$$

such that

$$B_i \rightarrowtail A_i \twoheadrightarrow B_{i+1}$$

is an exact sequence in $\mathcal{C}$. Let $\mathcal{C}^{[a,b]}_{\text{exact}}$ denote the Waldhausen category of admissibly exact sequences of length $b-a$ with weak equivalences defined to be level-wise weak equivalences and cofibrations $A \to A'$ defined to be level-wise cofibrations $A_i \to A_i'$ such that the pushout $A_i \coprod_{B_i} B_i' \to A_i'$ is a cofibration.

Remark 5.6.31. By the Additivity theorem, one can prove by induction that there is a homotopy equivalence

$$K(\mathcal{C}^{[0,n]}_{\text{exact}}) \simeq \prod_{i=1}^{n} K(\mathcal{C}).$$

Proof of Theorem 5.6.29. We will apply the localization sequence in the case where $(\mathcal{C}, c\mathcal{C})$ is the category of bounded chain complexes bf$Ch^b(\mathcal{A})$ in an exact category $\mathcal{A}$ and the cofibrations are level-wise admissible monomorphisms. The weak equivalences $w\mathcal{C}$ are the quasi-isomorphisms so the Waldhausen category $(\mathbf{Ch}^b(\mathcal{A}), c\mathbf{Ch}^b(\mathcal{A}), w\mathbf{Ch}^b(\mathcal{A})$ has a cylinder functor satisfying the cylinder axiom and it satisfies the saturation axiom and the extension axiom. We let $v\mathbf{Ch}^b(\mathcal{A})$ be the wide subcategory whose morphisms are isomorphisms. We therefore have a fiber sequence

$$K(\mathbf{Ch}^b(\mathcal{A})^w) \to K(\mathbf{Ch}^b(\mathcal{A}), v) \to K(\mathbf{Ch}^b(\mathcal{A}), w) \tag{5.6.32}$$

where $\mathbf{Ch}^b(\mathcal{A})^w$ is the full sub Waldhausen category of $(\mathbf{Ch}^b(\mathcal{A}, c\mathbf{Ch}^b(\mathcal{A}, v\mathbf{Ch}^b(\mathcal{A})$ whose objects are quasi-isomorphic to 0. We first consider the full sub Waldhausen category $\mathbf{Ch}^{[a,b]}(\mathcal{A})c\mathbf{Ch}^{[a,b]}(\mathcal{A}), v\mathbf{Ch}^{[a,b]}(\mathcal{A})) \subset (\mathbf{Ch}^b(\mathcal{A}), c\mathbf{Ch}^b(\mathcal{A}), v\mathbf{Ch}^b(\mathcal{A}))$ consisting of chain complexes $C_\bullet$ such that $C_i = 0$ whenever i doesn't satisfy $a \le i \le b$. Then there is also a corresepong sub Waldhausen category

$$\mathbf{Ch}^{[a,b]}(\mathcal{A})^w$$

of those chain complexes $C_\bullet$ bounded between a and b such that $C_\bullet$ is quasi-isomorphic to 0. Such chain complexes can be identified with objects in $\mathcal{A}^{[a,b]}_{\text{exact}}$. Moreover, we can identify

$$K(\mathbf{Ch}^{[a,b]}(\mathcal{A})c\mathbf{Ch}^{[a,b]}(\mathcal{A}), v\mathbf{Ch}^{[a,b]}(\mathcal{A})) \simeq \prod_{a}^{b} K(\mathcal{A})$$

by the additivity theorem. We therefore produce a fiber sequence

$$\prod_{a+1}^{b} K(\mathcal{A}) \to \prod_{a}^{b} K(\mathcal{A}) \xrightarrow{\chi} K(\mathcal{A})$$

where χ is induced by the Euler characteristic. Passing to colimits over a and b we produce a homotopy fiber sequence

$$K(\mathbf{Ch}^b(\mathcal{A})^w) \to K(\mathbf{Ch}^b(\mathcal{A}), v) \to K(\mathcal{A})$$

which is compatible with maps from the localization sequence (5.6.32). □

Exercise 5.6.33. Use the additivity theorem to prove that

$$K(\mathbf{Ch}^{[a,b]}(\mathcal{C})) \simeq \prod_{i=a}^{b} K(\mathcal{C}).$$

5.6.2 Agreement of the $S_\bullet$-construction and the Q-construction

We've seen that every exact category is a Waldhausen category. We would therefore like to know that there is a homotopy equivalence between the two notions of algebraic K-theory in this case. Even though the the Q-construction is recovered from Waldhausen's $S_\bullet$-construction, the Q-construction has some advantages for proving fundamental theorems. The Dévissage theorem, for example, isn't always known to have a counterpart in the setting of Waldhausen categories.

To show that the Q-construction and the $S_\bullet$-construction produce the same K-theory space, we need to introdue the edgewise subdivision of a simplicial object.

Definition 5.6.34. We define a functor

$$\mathrm{sd}^e\colon \Delta \to \Delta$$

so that $\mathrm{sd}^e([n]) = [2n+1]$ and on morphisms $\theta\colon [n] \to [m]$ by

$$\mathrm{sd}^e(\theta)\colon [2n+1] \to [2m+1] \tag{5.6.35}$$

$$\mathrm{sd}^e(\theta)(s) = \begin{cases} \theta(s) & \text{if } 0 \leq i \leq n \\ \theta(s-n-1)+(m+1) & \text{if } n \leq i \leq 2n+1 \end{cases} \tag{5.6.36}$$

Writing $[2n+1] = [n] \coprod [n]$ then this is simply defining morphisms by sending θ to

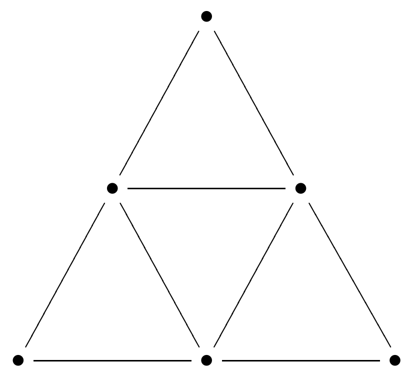
$$\mathrm{sd}^e(\theta) = \theta \amalg \theta\colon [n] \amalg [n] \to [m] \amalg [m].$$

We then define the *edgewise subdivision* of a simplicial object in $\mathcal{C}$ denoted $X_\bullet$ as

$$X_\bullet \circ (\mathrm{sd}^e)^{\mathrm{op}}\colon \Delta^{\mathrm{op}} \xrightarrow{(\mathrm{sd}^e)^{\mathrm{op}}} \Delta^{\mathrm{op}} \xrightarrow{X_\bullet} \mathcal{C}.$$

For short we write $X^e_\bullet := X_\bullet \circ (\mathrm{sd}^e)^{\mathrm{op}}$ and we write d^e_i and s^e_i for its face and degeneracy maps.

Example 5.6.37. In particular, let Δ^2 be the standard simplicial 2-simplex. Then $(\Delta^2)^e$ can be drawn as



(There is a difference between the edgewise subdivision and the Quillen-Segal subdivision. Sometimes the edgewise subdivision we use here is denoted sd_2 to distinguish it from the Quillen-Segal subdivision which is sometimes denoted sd_e.)

Remark 5.6.38. The face and degeneracy maps in $X^e_\bullet$ with structure maps s^e_i and d^e_i satisfy the following compatibility with the face and degeneracy maps s_i and d_i of $X_\bullet$:

$$\begin{tikzcd} X^e[n] \arrow[r, "d^e_i"] \arrow[d, equal] & X^e[n-1] \arrow[d, equal] \\ X[2n+1] \arrow[r, "d_{n-i}\circ d_{n+i+1}"] & X[2n-1] \end{tikzcd}$$

$$\begin{tikzcd} X^e[n] \arrow[r, "s^e_i"] \arrow[d, equal] & X^e[n+1] \arrow[d, equal] \\ X[2n+1] \arrow[r, "s_{n-i}\circ s_{n+i+1}"] & X[2n+1] \end{tikzcd}$$

Theorem 5.6.39. *There is a canonical map*

$$X^e_\bullet \to X_\bullet$$

of simplicial sets inducing a homeomorphism

$$|X^e_\bullet| \to |X_\bullet|$$

on geometric realizations.

We leave the proof for later, but it will be a key ingredient in the following comparison between Q-construction and the $S_\bullet$ construction. Let $\mathcal{C}$ be an exact category. We will write iso$\mathcal{B} \subset \mathcal{B}$ for the subcategory whose arrows consist of all the isomorphisms of $\mathcal{B}$ so $(\mathcal{C}, c\mathcal{C}, \text{iso}\mathcal{C})$ forms a Waldhausen category where cofibrations are the admissible monomorphisms. We can also regard the nerve $N_\bullet Q\mathcal{C}$ as a simplicial category by considering the *category* of functors $[n] \to Q\mathcal{C}$ and natural isomorphisms. We write iso$N_\bullet Q\mathcal{C}$ for this simplicial category. We first remark that the nerve of this simplicial category is homotopy equivalent to the nerve of $Q\mathcal{C}$ on geometric realizations:

Lemma 5.6.40. *There is a homotopy equivalence*

$$|N_\bullet Q\mathcal{C}| \simeq |N_\bullet isoN_\bullet Q\mathcal{C}|.$$

Proof. The proof is very similar to that of Proposition 5.6.35, so we omit it here. □

We can now prove the comparison theorem

Theorem 5.6.41. *There is a homotopy equivalence*

$$BisoS\bullet\mathcal{C} \xrightarrow{\cong} |(N_\bullet(isoS\bullet\mathcal{C}))^e| \xrightarrow{W} BQ\mathcal{C}$$

where the first map is the inverse of the canonical homeomorphism of Theorem 5.6.39. Consequently, there is a homotopy equivalence

$$K^W(\mathcal{C}) := \Omega BisoS\bullet\mathcal{C} \simeq \Omega BQ\mathcal{C} =: K^Q(\mathcal{C})$$

when $\mathcal{C}$ is an exact category regarded as a Waldhausen category with cofibration the admissible monomorphisms and weak equivalences the isomorphisms.

The first goal will be to define a simplicial map

$$W\colon (s_\bullet\mathcal{C})^e \to N_\bullet Q\mathcal{C}.$$

To do this we first need to define a map

$$W_k\colon s_{2k+1}\mathcal{C} \to N_k Q\mathcal{C}$$

on k-simplices. The map is the identity when $k = 0$. More generally, note that an object in $s_{2k+1}\mathcal{C}$ is a digram of the form

$$\begin{tikzcd}
A_{0,1} \arrow[r, tail] & A_{0,2} \arrow[r, tail] \arrow[d, two heads] & \cdots \arrow[r, tail] & A_{0,2k} \arrow[d, two heads] \\
& A_{1,2} \arrow[r, tail] & \cdots \arrow[r, tail] & A_{1,2k} \arrow[d, two heads] \\
& & & \vdots \arrow[d, two heads] \\
& & & A_{2k-1,2k}
\end{tikzcd}$$

which corresponds to composite of k-spans sitting inside this diagram of the form

$$\begin{tikzcd}
& A_{k-1,k+1} \arrow[dl, two heads] \arrow[dr, tail] & & \cdots \arrow[dl, two heads] \arrow[dr, tail] & & A_{0,2k-1} \arrow[dl, two heads] \arrow[dr, tail] & \\
A_{k,k+1} & & A_{k-1,k+2} & & \cdots & & A_{0,2k}
\end{tikzcd}$$

Lemma 5.6.42. *This map is actually a simplicial map.*

Proof. This is left as an exercise for now. $\square$

Lemma 5.6.43. *The map*

$$W_k \colon s_{2k+1}\mathcal{C} \to N_k Q\mathcal{C}$$

is a surjection.

Proof. Let $C\colon [k] \to Q\mathcal{C}$ be a k-simplex of $N_\bullet Q\mathcal{C}$. Define $C_i = A_{k-i,k+i+1}$ and for $\alpha_{i,i-1}\colon i \to i-1$ a map in $[n]$ choose representatives for the morphisms $C(\alpha_{i,i-1})\colon C_i \to C_{i-1}$

$$A_{k-i,k+i+1} \twoheadleftarrow A_{k-i-1,k+i+1} \rightarrowtail A_{k-i-1,k+i+2}.$$

We have therefore defined $A_{i,j}$ for $i < j$ and $i + j = 2n + 1$ as well as $i + j = 2n$. To define $A_{i,j}$ for $i + j = 2n - 1$ and $2n - 2$, we take pushouts and pullbacks to form a commuting diagram where each square is a pullback

$$\begin{tikzcd}
A_{n-i-1,n+i} \arrow[r, tail] \arrow[d, two heads] & A_{n-i-1,n+i+1} \arrow[r, tail] \arrow[d, two heads] & A_{n-i-1,n+i+2} \arrow[d, two heads] \\
A_{n-i,n+i} \arrow[r, tail] \arrow[d, two heads] & A_{n-i,n+i+1} \arrow[r, tail] \arrow[d, two heads] & A_{n-i,n+i+2} \arrow[d, two heads] \\
A_{n-i+1,n+i} \arrow[r, tail] & A_{n-i+1,n+i+1} \arrow[r, tail] & A_{n-i+1,n+i+2}
\end{tikzcd}$$

We then proceed inductively to define $A_{i,j}$ for the remaining i, j. This defines an object $A \in s_{2k+1}\mathcal{C}$ that maps to $N_k Q\mathcal{C}$. $\square$

Note that the map W_k extends to a functor so that we can regard the map

$$W_\bullet\colon (\mathrm{iso} S_\bullet \mathcal{C})^e \to \mathrm{iso} N_\bullet Q\mathcal{C}$$

is a map of simplicial categories. The result we just proved is a statement about the map on objects.

Lemma 5.6.44. *The functor*

$$W_k\colon (isoS_\bullet\mathcal{C})^e \to isoN_\bullet Q\mathcal{C}$$

is an equivalence of categories for each k.

Proof. We showed that the map is surjective on objects, so we just need to show that the functor is fully faithful. We already observed in Remark qxqx that there is a bijection

$$\mathrm{iso}\mathcal{C}(A, B) \cong \mathrm{iso}Q\mathcal{C}(A, B)$$

which shows that the functor

$$W_0\colon \mathrm{iso} S_1 \mathcal{C} \to \mathrm{iso} Q\mathcal{C}$$

is fully faithful. More generally, we want to show that the map

$$\mathrm{iso} S_{2k+1}\mathcal{C}(A, A') \to \mathrm{Fun}([n], \mathrm{iso}Q\mathcal{C})(W_k(A), W_k(A'))$$

is a bijection. For surjectivity, let

$$t\colon W_k(A) \to W_k(A')$$

be a natural isomorphism of functors $[k] \to Q\mathcal{C}$. For each $0 \le i \le k$, there is a map $t(i)$ represented by a commuting diagram

$$\begin{tikzcd}
A_{k-i,k+i+1} \arrow[d, "\cong"', "f_0"] & A_{k-i-1,k+i+1} \arrow[l, two heads, "j_{0,1}"'] \arrow[r, tail, "i_{0,1}"] \arrow[d, "\cong"', "f_{0,1}"] & A_{k-i-1,k+i+2} \arrow[d, "\cong"', "f_1"] \\
A'_{k-i,k+i+1} & A'_{k-i-1,k+i+1} \arrow[l, two heads, "j'_{0,1}"'] \arrow[r, tail, "i'_{0,1}"] & A'_{k-i-1,k+i+2}
\end{tikzcd} \tag{5.6.45}$$

in $Q\mathcal{C}$ where the vertical morphisms are isomorphisms in $Q\mathcal{C}$. We claim that this implies that the maps $A_{k-i-1,k+i+1} \to A'_{k-i-1,k+i+1}$ must also be isomorphisms. To see this, we note that commutativity of the diagram (5.6.45) implies that the map

$$
\begin{tikzcd}
A_{k-i-1,k+i+1} \arrow[r, "f_{0,1}", "\cong"'] \arrow[d, two heads, "j_{0,1}"'] & A'_{k-i-1,k+i+1} \arrow[r, tail, "i'_{0,1}"] \arrow[d, two heads, "j'_{0,1}"] & A'_{k-i-1,k+i+2} \\
A_{k-i,k+i+1} \arrow[r, "f_0", "\cong"'] \arrow[d, "\cong"', "f_0"] & A'_{k-i,k+i+1} & \\
A'_{k-i,k+i+1} & &
\end{tikzcd}
$$

and the map

$$
\begin{tikzcd}
A'_{k-i-1,k+i+1} \arrow[r, tail, "i_{0,1}"] \arrow[d, two heads, "j_{0,1}"] & A_{k-i-1,k+i+2} \arrow[r, tail, "f_1", "\cong"'] & A'_{k-i-1,k+i+2} \\
A'_{k-i,k+i+1} & &
\end{tikzcd}
$$

in $Q\mathcal{C}$ are in the same equivalence class and consequently the map

$$A_{k-i-1,k+i+1} \to A'_{k-i-1,k+i+1}$$

is an isomorphism. Since both pullbacks and pushouts preserve isomorphisms, we can inductively show that $A \to A'$ is also an isomorphism, or in other words a map in $\mathrm{iso}S_k\mathcal{C}$. To prove injectivity, consider two morphisms

$$t_0, t_1 \colon A \to A'$$

in $\mathrm{iso}S_k\mathcal{C}$ such that $W_k(t_0) = W_k(t_1)$, then we know

$$t_0 = t_1 \colon A_{i,j} \to A'_{i,j}$$

for $i + j = 2n$ and $i + j = 2n + 1$. By functoriality of pullback and pushouts, the same inductive argument as before proves that

$$t_0 = t_1 \colon A_{i,j} \to A'_{i,j}$$

for the remaining i, j as desired. $\square$

Now we can prove that there is an equivalence between Quillen's Q-construction and the Waldhausen $S_\bullet$-construction as desired.

Proof of Theorem 5.6.41. By Lemma 5.6.44, we have an equivalence of simplicial categories

$$W_k \colon (\mathrm{iso}S_\bullet\mathcal{C})^e \to \mathrm{iso}N_\bullet Q\mathcal{C}.$$

Applying the nerve to both sides, we know that on geometric realizations we have a homotopy equivalence

$$|N_\bullet(\mathrm{iso}\mathcal{S}_\bullet\mathcal{C})^e| \to |N_\bullet\mathrm{iso}N_\bullet Q\mathcal{C}|.$$

precomposing with the homeomorphism

$$|N_\bullet(\mathrm{iso}\mathcal{S}_\bullet\mathcal{C}| \cong |N_\bullet(\mathrm{iso}\mathcal{S}_\bullet\mathcal{C})^e|$$

of Theorem 5.6.39 and post-composing with the homotopy equivalence

$$|N_\bullet\mathrm{iso}N_\bullet Q\mathcal{C}| \simeq |N_\bullet Q\mathcal{C}|.$$

gives our desired homotopy equivalence. □

Appendix A

Fundamentals

A.1 Categories

Category theory will be of fundamental importance in studying algebraic K-theory. We recall the basic notions here.

Definition A.1.1. A category $\mathcal{C}$ consists of

1. a class of objects denoted $\mathrm{ob}(\mathcal{C})$
2. for each pair of objects c, c' a set $\mathcal{C}(c, c')$ of morphisms from c to c', and
3. for any triple c, c', c'' a map of sets
$$- \circ -\colon\ \mathcal{C}(c', c'') \times \mathcal{C}(c', c'') \to \mathcal{C}(c, c'')$$
4. For each object c an element
$$\mathrm{id}_C \in \mathcal{C}(c, c).$$

satisfying
$$(f \circ g) \circ h = f \circ (g \circ h)$$
for each triple of maps
$$(f, g, h) \in \mathcal{C}(c''', c'') \times \mathcal{C}(c'', c') \times \mathcal{C}(c', c)$$
and
$$\mathrm{id}_{c'} \circ f = f = f \circ \mathrm{id}_c$$
for each map $f\colon\ c \to c'$ in $\mathcal{C}(c, c')$, where each of these identities are functorial in the appropriate sense.

Example A.1.2. Given a category $\mathcal{C}$, let $\mathcal{C}^{\mathrm{op}}$ be the category whose objects are the same as the objects in $\mathcal{C}$ and there is a unique morphism $f^{\mathrm{op}}\colon\ b \to a$ for every morphism $f\colon\ a \to b$ in $\mathcal{C}$.

Remark A.1.3. Note that we use the convention that all categories are locally small by requiring that $\mathcal{C}(c, c')$ is a set rather than a proper class. We say a category is a small category if, in addition, it has a set of objects $\mathrm{ob}(\mathcal{C})$.

Definition A.1.4. The skeleton sk$\mathcal{C}$ of a cat egory $\mathcal{C}$ is the full subcategory of $\mathcal{C}$ consisting of one object for each isomorphism class of objects in $\mathcal{C}$. We say that $\mathcal{C}$ is *skeletally small* if sk$\mathcal{C}$ is a small category.

Example A.1.5. The category of sets and set maps, denoted Set, is an example of a category that is not a small category. The category of finite sets and maps of finite sets is skeletally small and we write Fin for its skeleton.

Example A.1.6. Let $\mathcal{C}$ be a small category. Then there is a category $\mathsf{Arr}(\mathcal{C})$ whose objects are maps $f\colon a \to b$ in $\mathcal{C}$ and a morphism from $a \to b$ to $c \to d$ is a commuting square

$$\begin{tikzcd} a \arrow[r] \arrow[d] & b \arrow[d] \\ c \arrow[r] & d \end{tikzcd}$$

in $\mathcal{C}$. Composition is defined by vertical composition of squares. We call this category the *arrow category*.

Definition A.1.7. A functor $F\colon\ \mathcal{C} \to \mathcal{D}$ associates to each object $c \in \mathrm{ob}\,\mathcal{C}$ an object $F(c) \in \mathrm{ob}\,\mathcal{D}$, to each morphism $f\colon c \to c'$ a morphism $F(f)\colon\ F(C) \to F(c')$ in $\mathcal{D}$ such that $F(f \circ g) = F(f) \circ F(g)$.

Example A.1.8. Let Top denote the category of topological spaces and continuous maps. Then any topological space may be regarded as a set by forgetting the topology and any continuous map is in particular a map of sets, so this defines a functor

$$U\colon\ \mathsf{Top} \to \mathsf{Set}$$

called the forgetful functor.

Definition A.1.9. A subcategory $\mathcal{C}$ in $\mathcal{D}$ consists of a category $\mathcal{C}$ and a functor $\iota\colon\ \mathcal{C} \to \mathcal{D}$ such that

$$\mathcal{C}(c, c') \to \mathcal{D}(\iota(c), \iota(c')) \tag{A.1.10}$$

is injective. In particular, this implies that the set of objects in c inject in the set of objects of $\mathcal{D}$.

We say $\mathcal{C}$ is a *full* subcategory if the map (A.1.10) is also surjective. More generally, we say a functor $F\colon\ \mathcal{C} \to \mathcal{D}$ is faithful if the associated map

$$\mathcal{C}(c, c') \to \mathcal{D}(F(c), F(c')) \tag{A.1.11}$$

is is injective and we say it is fully faithful if this map is also surjective. In this case, we say that there is a fully faithful embedding of $\mathcal{C}$ in $\mathcal{D}$ since the essential image of $\mathcal{C}$ in $\mathcal{D}$ is necessarily a full subcategory of $\mathcal{D}$.

Definition A.1.12. Given a pair functors $F, G\colon \mathcal{C} \to \mathcal{D}$, then a natural transformation $\gamma\colon F \Rightarrow G$ associates to an object x in $\mathcal{C}$ a map $\gamma_x\colon F(x) \to G(x)$ and to a map $f\colon x \to y$ a commutative diagram

$$\begin{tikzcd} F(x) \arrow[r, "F(f)"] \arrow[d, "\gamma_x"'] & F(y) \arrow[d, "\gamma_y"] \\ G(x) \arrow[r, "G(f)"] & G(y) \end{tikzcd} \tag{A.1.13}$$

and to a composable pair of morphisms $g \circ f$ in $\mathcal{C}$ where $f\colon x \to y$ and $g\colon y \to z$, then there is a commutative diagram

$$\begin{tikzcd} F(x) \arrow[r, "F(f)"] \arrow[d, "\gamma_x"'] & F(y) \arrow[r, "F(g)"] \arrow[d, "\gamma_y"] & F(z) \arrow[d, "\gamma_z"] \\ G(x) \arrow[r, "G(f)"] & G(y) \arrow[r, "G(g)"] & G(z). \end{tikzcd}$$

Definition A.1.14. The functor category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is the category whose objects are functors $F, G\colon \mathcal{C} \to \mathcal{D}$ and whose morphisms are natural transformations $\gamma\colon F \Rightarrow G$, where composition is given by composition of natural transformations where $\alpha \circ \beta$ is defined on objects by

$$(\alpha \circ \beta)_x := \alpha_x \circ \beta_x$$

and on morphisms $f\colon x \to y$ by vertical composition of squares of the form (A.1.13) and similarly for compositions of morphisms.

Definition A.1.15. A factorization system on a small category $\mathcal{C}$ consists of of a pair of subcategories (E, M) such that

1. E and M each contain all of the isomorphisms in $\mathcal{C}$ and consequently all objects in $\mathcal{C}$, and

2. every morphism $f\colon A \to C$ in $\mathcal{C}$ can be factored as $f = m \circ e$ where $e\colon A \to B$ is a map in E and $m\colon B \to C$ is a map in M.

3. this factorization defines a functor

$$\mathsf{Arr}(\mathcal{C}) \to \mathsf{Arr}(E) \times_{\mathcal{C}} \mathsf{Arr}(M)$$

sending f to (m, e) and a commutative square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & C \arrow[d, "g''"] \\ A' \arrow[r, "f'"] & C' \end{tikzcd}$$

to the composition of squares

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d, "g"'] & B \arrow[r, "m"] \arrow[d, "g'"] & C \arrow[d, "g''"] \\ A' \arrow[r, "e'"] & B' \arrow[r, "m'"] & C'. \end{tikzcd}$$

Here the category $\mathsf{Arr}(E) \times_{\mathcal{C}} \mathsf{Arr}(M)$ is the pullback in the category of small categories of the diagram

$$\mathsf{Arr}(M) \xrightarrow{t} \mathcal{C} \xleftarrow{s} \mathsf{Arr}(E)$$

where t sends an object $e\colon A \to B$ in $\mathsf{Arr}(E)$ to B and s sends an object $m\colon B \to C$ in $\mathsf{Arr}(M)$ to B. On morphisms these functors are defined in the evident way. In other words, the category $\mathsf{Arr}(E) \times_{\mathcal{C}} \mathsf{Arr}(M)$ has objects pairs of morphisms (e, m) such that, as maps in $\mathcal{C}$, they are composable.

A.2 Sets

Definition A.2.1. A partial order on a set P is a binary relation $\leq$ satisfying

1. $x \leq x$ (Reflexivity),
2. if $x \leq y$ and $y \leq x$, then $x = y$ (Anti-symmetry),
3. if $x \leq y$ and $y \leq z$, then $x \leq z$ (Transitivity).

We say x is related to y if either $x \leq y$ or $y \leq x$. A set with a partial order will be called a partially ordered set or a POSet for short.

Any POSet P can be considered as a small category with objects the elements of P and morphism sets

$$P(x, y) = \begin{cases} * & x \leq y \\ \emptyset & \text{otherwise} \end{cases}$$

and composition

$$P(y, z) \times P(x, y) \to P(x, z)$$

is defined in the obvious way using transitivity.

Definition A.2.2. A total order on a set X is a partial order satisfying the totality axiom: for each $x, y \in X$ either $x \leq y$ or $y \leq x$.

In other words, a totally ordered set is simply a partially ordered set in which any two elements are related.

Example A.2.3. The natural numbers

$$\mathbb{N}_0 = \{0, 1, 2, \dots\}$$

are a totally ordered set with the usual total order. The set $n = \{0, 1, \dots, n\}$ is a finite totally ordered set equipped with the restriction of the total order on $\mathbb{N}_0$. When we view the totally ordered set n as a category it can be depicted as

$$0 \leftarrow 1 \leftarrow 2 \leftarrow \cdots \leftarrow n.$$

Appendix B

Simplicial methods

B.1 The simplex category

Definition B.1.1. Let Ord denote the category of finite totally ordered sets and maps of finite sets that preserve the total order. Let Δ be the the skeleton of this category with exactly one object for each isomorphism class of objects in Ord. The objects in Δ are

$$[n] = \{0, 1, \dots, n\}$$

for $n \geq 0$ and the maps are nondecreasing maps of finite sets.

Example B.1.2. There is a canonical factorization system (E, M) on the category Δ. The category E consists of all isomorphisms as well as the closure under composition of the maps of the form

$$\sigma_k \colon [n] \to [n-1]$$

for $0 \leq k \leq n$ such that

$$\sigma_k(i) = \begin{cases} i & \text{if } i < k \\ k & \text{if } i = k, k+1 \\ i-1 & \text{if } k+1 \leq i \leq n. \end{cases}$$

The category M consists of all isomorphisms as well as the closure under composition of the morphisms

$$\delta_k(i) = \begin{cases} i & \text{if } i < k \\ i+1 & \text{if } k \leq i \leq n. \end{cases}$$

We can therefore depict the category Δ as follows

$$\begin{tikzcd}
{[0]} \arrow[r, shift left=2] \arrow[r, leftarrow] \arrow[r, shift right=2] & {[1]} \arrow[r, shift left=4] \arrow[r, leftarrow, shift left=2] \arrow[r] \arrow[r, leftarrow, shift right=2] \arrow[r, shift right=4] & {[2]} \arrow[r, shift left=6] \arrow[r, leftarrow, shift left=4] \arrow[r, shift left=2] \arrow[r, leftarrow] \arrow[r, shift right=2] \arrow[r, leftarrow, shift right=4] \arrow[r, shift right=6] & \cdots
\end{tikzcd} \tag{B.1.3}$$

Definition B.1.4. Given a category $\mathcal{C}$, a *simplicial object in* $\mathcal{C}$ is a functor

$$\Delta^{\mathrm{op}} \to \mathcal{C}.$$

This can be described as a collection of objects $\{X[i]\}_{i\geq 0}$ sitting in a diagram

$$X[0] \;\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow}\; X[1] \;\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow}\; X[2] \;\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow}\; \dots \tag{B.1.5}$$

where the maps in the diagram are called the face maps

$$\{\partial_i\colon\ X[n+1] \to X[n]\}_{0\leq i\leq n+1}$$

and degeneracy maps

$$\{s_i\colon\ X[n] \to X[n+1]\}_{0\leq i\leq n}$$

and these must satisfy the simplicial identities:

1. $\partial_i \circ \partial_j = \partial_{j-1} \circ \partial_i$ if $i < j$,
2. $s_i \circ s_j = s_j \circ s_{i-1}$ if $i > j$,
3. $\partial_i \circ s_j = \begin{cases} s_{j-1} \circ \delta_i & \text{if } i < j \\ \mathrm{id}_{X_j} & \text{if } i = j \text{ or } i = j+1 \\ s_j \circ \delta_{i-1} & \text{if } i > j+1 \end{cases}$

Morphisms of simplicial objects in $\mathcal{C}$ are simply natural transformations of functors $\Delta^{\mathrm{op}} \to \mathcal{C}$.

Example B.1.6. When $\mathcal{C} = \mathsf{Set}$, we call a simplicial object in Set simply a simplicial set and we write sSet for this category. We could also consider pointed sets Set_* and we will write sSet_* for the category of simplicial objects in pointed sets (or equivalently, pointed simplicial sets). When $\mathcal{C} = \mathsf{Top}$ we refer to the category of simplicial objects in Top simply as simplicial spaces and we denote the category of simplicial spaces by sTop.

Example B.1.7. Given an object $[n] \in \Delta$, we can form

$$\mathrm{Hom}_{\Delta}(-,[n])\colon\ \Delta^{\mathrm{op}} \to \mathsf{Set}.$$

This is clearly a simplicial set. We denote this simplicial set

$$\Delta^n := \mathrm{Hom}_{\Delta}(-,[n]).$$

Exercise B.1.8. Use the Yoneda lemma to show that

$$\mathrm{Hom}_{\mathsf{sSet}}(\Delta^n, X) = X[n].$$

Hint: Write $\iota_n := \mathrm{id} \in \mathrm{Hom}_{\Delta}([n],[n])$. Given a simplicial map $\varphi\colon\ \Delta^n \to X$, associate to this map an n-simplex $\varphi(\iota_n) \in X[n]$. Prove that this gives a bijection.

Definition B.1.9. We define a topological space

$$|\Delta^n| = \{(t_0, \dots, t_n) \in [0,1]^{n+1} : \sum t_i = 1\}$$

equipped with the subspace topology. In fact, the spaces $|\Delta^n|$ form cosimplicial space

$$|\Delta^\bullet|(-) \colon \Delta \to \mathsf{Top}$$

such that $|\Delta^\bullet|([n]) = |\Delta^n|$. The map

$$\theta_* \colon |\Delta^n| \to |\Delta^m|$$

induced by $\theta \colon [n] \to [m]$ is defined by

$$\theta(t_0, t_1, \dots t_n) = (t'_0, \dots t'_m)$$

where

$$t'_i = \begin{cases} 0 & \text{if } \theta^{-1}(i) \neq \emptyset, \\ \sum_{j \in \theta^{-1} i} t_j & \text{if } \theta^{-1}(i) = \emptyset. \end{cases}$$

Remark B.1.10. There is also an analogous cosimplicial simplicial set

$$\Delta^n \colon \Delta \to \mathsf{sSet}$$

given on n-simplices by

$$\Delta^n = \mathrm{Hom}_\Delta(-, [n])$$

where the cosimplicial structure is given by regarding this as a functor in the second variable.

Definition B.1.11. Given a pair of simplicial sets X, Y, we define the product of simplicial sets $X \times Y$ by letting the n-simplices be simply

$$(X \times Y)[n] = X[n] \times Y[n]$$

and the face and degeneracy maps simply as

$$\partial_i^{X \times Y} = (\partial_i^X, \partial_i^Y) \text{ and } s_i^{X \times Y} = (s_i^X, s_i^Y).$$

Note that the non-degenerate n-simplices of $X \times Y$ are not simply the product of the non-degenerate n-simplices of X with the non-degenerate n-simplices of Y.

Definition B.1.12. We also define an internal hom $\underline{\mathrm{Hom}}(X, Y)$ to be the simplicial set with n-simplices

$$\underline{\mathrm{Hom}}(X, Y)[n] := \mathrm{Hom}_{\mathsf{sSet}}(X \times \Delta^n, Y)$$

and face and degeneracy maps induced by considering Δ^n as a cosimplicial object in sSet and applying the functor $\mathrm{Hom}_{\mathsf{sSet}}(X \times -, Y)$.

Proposition B.1.13. *The category* $(\mathsf{sSet}, \times, \Delta^0)$ *is a a symmetric monoidal category and there is a natural isomorphism*

$$Hom_{\mathsf{sSet}}(X \times Y, Z) \cong Hom_{\mathsf{sSet}}(X, \underline{Hom}(Y, Z)).$$

Definition B.1.14. We say $f\colon X \to Y$ is homotopy equivalent to $g\colon X \to Y$ if there is a map of simplicial sets

$$H\colon X \times \Delta^1 \to Y$$

such that $H|_{(\iota_0)_*(\Delta^0)} = f$ and $H|_{(\iota_1)_*(\Delta^0)} = g$ where $(\iota_j)_*\colon \Delta^0 \to \Delta^1$ is the map of simplicial sets induced by the unique map $\iota_j\colon [0] \to [1]$ map in Δ sending 0 to j; i.e.

$$(\iota_j)_* := \mathrm{Hom}(-, \iota_j)\colon \mathrm{Hom}(-, [0]) \to \mathrm{Hom}(-, [1]).$$

Let $\partial\Delta^n$ denote the smallest sub-simplicial set of Δ^n generated by the faces $d_j(\iota_n)$ for $0 \le j \le n$. where ι_n denotes the element

$$\iota_n = \mathrm{id} \in \mathrm{Hom}_\Delta([n], [n]) = \mathrm{Hom}_{\mathsf{sSet}}(\Delta^n, \Delta^n) = (\Delta^n)_n$$

Let Λ^n_k denote the smallest sub-simplicial set of Δ^n generated by the faces $d_i(\iota_n)$ for the face $d_k(\iota_n)$. We say a simplicial set is a Kan complex if there exists a unique lift in any diagram of the form

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d] & X \\ \Delta^n \arrow[ur, dashed, "\exists! h"'] & \end{tikzcd}$$

for $0 \le k \le n$. Where Λ^n_k is the sub-simplicial set of Δ^n generated by $d_j(\iota_n)$

We can consider the full subcategory of the category of simplicial sets whose objects are the category of Kan complexes.

Let X, Y be Kan complexes and define

$$[X, Y] = \mathrm{Hom}(X, Y)/\simeq$$

where $\simeq$ is the equivalence relation given by simplicial homotopy equivalence.

Definition B.1.15. Define the homotopy category of simplicial sets, denoted $\mathrm{ho}(\mathsf{sSet})$ to be the category whose objects are Kan complexes and morphisms from $X \to Y$ are $[X, Y]$. We define the homotopy category of Top, denoted $\mathrm{ho}(\mathsf{Top})$ to be the category whose objects are CW complexes and maps are homotopy classes of maps from X to Y, which we also denote $[X, Y]$.

Construction B.1.16. Let X be an object in Top. We define

$$\mathrm{sing}(X) = \mathrm{Hom}_{\mathsf{Top}}(|\Delta^\bullet|, X).$$

This is clearly a simplicial set. One can check that it is also a Kan complex when X is a CW complex.

We will now define geometric realization, more generally, of a simplicial space.

Construction B.1.17. Let $X_\bullet$ be an object in sTop. Then we define

$$|X_\bullet| := \mathrm{coeq}\left(\coprod_{f\colon i \to j \in \Delta^{op}} X_i \times |\Delta^j| \underset{\amalg X_f \times \mathrm{id}_{|\Delta^j|}}{\overset{\amalg\, \mathrm{id}_{X_i} \times |\Delta^f|}{\rightrightarrows}} \coprod_{[n] \in \Delta^{\mathrm{op}}} X_n \times |\Delta^n| \right)$$

where the coproducts are equipped with the coproduct topology and coequalizer is equipped with the quotient topology. Sometimes, we simply write this as

$$|X| = \left(\coprod_{n \geq 0} |\Delta^n| \times X_n| \right) / \sim$$

where $\sim$ is the equivalence relation generated by

$$(x, \partial_i y) \sim (\delta_i x, y) \text{ and } (x, s_i y) \simeq (\sigma_i x, y).$$

One can check that these two spaces are the same.

Example B.1.18. When X is a simplicial set, we define $|X|$ in the same way by equipping $X[n]$ with the discrete topology for all $n \geq 0$.

Exercise B.1.19. The geometric realization of a simplicial set is always a CW complex.

We can therefore consider the geometric realization as taking values in compactly generated weak Hausdorff spaces. For the next result of Milnor, it is important to note that all constructions take place in this category.

Theorem B.1.20 (Milnor)**.** *Geometric realization commutes with finite limits. In particular, there is a homeomorphism*

$$|X_\bullet \times Y_\bullet| \cong |X_\bullet| \times |Y_\bullet|.$$

The following is one of the fundamental theorems of simplicial sets.

Theorem B.1.21. *There is a natural isomorphism $\gamma_{-,-}$ of functors $sSet^{\mathrm{op}} \times \mathsf{Top} \to$ Set defined on objects by*

$$\gamma_{X,Y} \colon \mathit{Hom}_{\mathsf{Top}}(|X|, Y) \cong \mathit{Hom}_{\mathsf{sSet}}(X, \mathit{sing}(Y)).$$

Construction B.1.22. There is an alternate way to construct the geometric realization of a simplicial set X that is less intuitive, but makes the proof of Theorem B.1.21 quite easy. Let $\Delta \downarrow X$ denote the category whose objects are maps of simplicial sets $x\colon \Delta^n \to X$ where $n \geq 0$ and whose maps $x \to y$ are commuting triangles

$$\begin{tikzcd} \Delta^n \arrow[r, "x"] \arrow[d, "\theta_*"'] & X \\ \Delta^m \arrow[ur, "y"'] & \end{tikzcd}$$

where $\theta\colon [n] \to [m]$ is a map in Δ. Composition is defined in the evident way. Then we can define $|X|$ to be the colimit

$$|X| = \operatorname*{colim}_{\Delta\downarrow X} |\Delta^n|$$

in the category of topological spaces.

Exercise B.1.23. Show that the definition of $|X|$ in Construction B.1.22 is the same as in Construction B.1.17 for simplicial sets up to homeomorphism.

Proof of Theorem B.1.21. Recall that $\mathrm{Hom}_{\mathsf{Top}}(-, Y)$ sends limits in $\mathsf{Top}^{\mathrm{op}}$, or in other words colimits in Top, to limits in Set. Therefore,

$$\begin{aligned} \mathrm{Hom}_{\mathsf{sSet}}(|X|, Y) &= \mathrm{Hom}_{\mathsf{sSet}}(\operatorname*{colim}_{\Delta\downarrow X} |\Delta^n|, Y) && \text{(B.1.24)} \\ &\cong \lim_{\Delta\downarrow X} \mathrm{Hom}_{\mathsf{sSet}}(|\Delta^n|, Y) && \text{(B.1.25)} \\ &= \lim_{\Delta\downarrow X} \mathrm{sing}(Y)_n && \text{(B.1.26)} \\ &= \lim_{\Delta\downarrow X} \mathrm{Hom}_{\mathsf{sSet}}(\Delta^n, \mathrm{sing}(Y)) && \text{(B.1.27)} \\ &\cong \mathrm{Hom}_{\mathsf{sSet}}(\operatorname*{colim}_{\Delta\downarrow X} \Delta^n, \mathrm{sing}(Y)) && \text{(B.1.28)} \\ &\cong \mathrm{Hom}_{\mathsf{sSet}}(X, \mathrm{sing}(Y)) && \text{(B.1.29)} \end{aligned}$$

where the equality (B.1.24) is the definition of geometric realization, the isomorphism (B.1.25) follows because $\mathrm{Hom}_{\mathsf{Top}}(-, Y)$ send limits in $\mathsf{Top}^{\mathrm{op}}$ to limits in Set, as remarked above, the equality (B.1.26) holds by definition of $\mathrm{sing}(Y)$, the equality (B.1.27) holds by Exercise (B.1.8), the isomorphism (B.1.28) hold because $\mathrm{Hom}_{\mathsf{Top}}(-, Y)$ sends limits in $\mathsf{Top}^{\mathrm{op}}$ to limits in Set. The last isomorphism holds because X is a cocone for the functor from $\Delta \downarrow X \to \mathsf{sSet}$ sending $\Delta^n \to X$ to Δ^n. □

Recall that Cat denotes the category of small categories. Let $[n]$ be the small category

$$0 \to 1 \to 2 \to \cdots \to n$$

described in Example A.2.3. These form a cosimplicial object in Cat by letting the codegeneracy map σ_i be given by composing the morphisms with source i and $i+1$. and the coface map δ_i given by inserting an identity map in the i-th position.

Definition B.1.30. Given a small category $\mathcal{C}$, we define a simplicial set by letting the n-simplices be

$$N\mathcal{C}[n] := \mathsf{Cat}([n], \mathcal{C}),$$

which should be viewed as a sequence of n composable morphisms, if $n > 0$ and, and simply the objects in $\mathcal{C}$ if $n = 0$. The face maps are given by composing two adjacent morphisms and the degeneracy maps are given by inclusion of the identity morphism. In other words, we simply use the fact that

$$[\bullet] \colon \Delta \to \mathsf{Cat}$$

is a cosimplicial object in small categories and therefore, by functoriality,

$$\mathsf{Cat}([n], \mathcal{C})$$

forms as simplicial set.

For example, any discrete group G can be regarded as a small category with one object $*$ and morphism set $G(*,*) = G$. By unpacking the definition, we see that

$$NG[n] = G^n$$

where $G^0 = *$. We can be even more explicit in this case. The face maps

$$\partial_i \colon G^{n+1} \to G^n$$

are defined by

$$\partial_i(g_1, \dots g_{n+1}) = \begin{cases} (g_2, g_3, \dots, g_{n+1}) & \text{if } i = 0 \\ (g_1, \dots, g_i g_{i+1}, \dots, g_{n+1}) & \text{if } 0 < i < n+1 \\ (g_1, g_3, \dots, g_n) & \text{if } i = n \end{cases}$$

and the degeneracy maps are defined by

$$s_i(g_1, \dots g_n) = (g_0, g_1, \dots g_{i-1}, 1, g_i, \dots, g_{n-1})$$

Note, that there was notion special about doing this construction for a discrete group. More generally, given a topological group G we define an object in sTop in the same way and we denote it

$$B(*, G, *) \colon \Delta^{\mathrm{op}} \to \mathsf{Top}.$$

Definition B.1.31. Given a category $\mathcal{C}$, we define

$$B\mathcal{C} := |N\mathcal{C}|.$$

Given a topological group G, we also define

$$BG = |B(*, G, *)|.$$

Remark B.1.32. Of course, when G is a discrete group then

$$NG = B(*, G, *)$$

so there isn't a conflict in notation.

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