A brief introduction to Algebraic K-theory

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Preface

These are notes (in progress) for a 2 hour per week course in algebraic K-theory taught at the Freie Universität Berlin in Winter 2020/21. They will continue to be updated regularly. Please feel free to reach out at my email address gak@math.fu-berlin.de if you notice any typos or errors. These notes are hosted on my personal website.

These notes draw from the tome of Charles Weibel, known as the K-book [18] as well as the original papers in the subject by Quillen [11, 12], Segal [13], and Waldhausen [15, 16]. The goal is to give a brief survey of constructions of algebraic K-theory, fundamental theorems in algebraic K-theory, and applications to geometric topology, algebraic geometry, and number theory. The notes focus on the case of algebraic K-theory of rings, which are the common thread in the applications to each of these three subjects.

These notes briefly survey the field of algebraic K-theory from the time period 1950-1985. The advantage of focusing on this time period is that no previous knowledge of $(\infty, 1)$-categories is required and for those interested in the most modern constructions of algebraic K-theory, all of the essential ideas already existed in the work of Quillen, Segal, and Waldhausen.

It is assumed that students in this course have a firm background in the basics of algebra, linear algebra, category theory, and topology. Additional knowledge of geometry topology, algebraic geometry, and number theory is useful for understanding certain examples, but certainly not required for the bulk of the material. Previous knowledge of the theory of simplicial sets will certainly be helpful, but not required, so a short section on such material is included as an appendix for those unfamiliar. Throughout, by a ring we will mean an associative ring with unit. We will always specify that a ring is non-unital when we want to consider it without unit.

Let Mod_R be the category of left modules over a ring R. When $R = \mathbb{Z}$, we simply write Ab for this category and refer to it as the category of abelian groups and when R is a field k, we write Vec_k for this category and refer to it as the category of vector spaces over a field k. We let $M(R)$ denote the skeleton of the category of finitely generated left R modules and we let $P(R)$ denote the skeleton of the category of finitely generated projective left modules over a ring R. When k is a commutative ring, let $\mathsf{Rep}_k(G)$ be the skeleton of the category of finitely generated $k[G]$-modules. Usually, we we only consider this in the case k is a field.

Throughout, by a space we mean a compactly generated weak Hausforff space and we write Top for the category of compactly generated weak Hausdorff spaces. Write CW for the category of CW complexes and CW^f for the category of finite CW complexes. When X is a space, let $VB_{\mathbb{R}}(X)$ denote the skeleton of the category of real vector bundles over X and let $VB_{\mathbb{C}}(X)$ denote the skeleton of the category of complex vector bundles over X.

Let Set be the category of sets and let Fin be the skeleton of the category of finite sets. Let G be a finite group and let Fin_G the skeleton of the category of finite G-sets. When R is a ring, let $\mathsf{Rep}_R(G)$ be the skeleton of the category of finite dimensional $k[G]$-modules.

Chapter 1

Introduction

The 0-th algebraic K-theory group K_0 was first defined by Grothendieck in the late 1950's in order to generalize the Riemann-Roch Theorem to varieties [4]. The name K-theory comes from German word *Klassen* meaning classes and the reason for this name will be more clear after reading Section 2.1.[1] Even earlier, in the early 1950's, Whitehead studied the simply homotopy of of a finite CW complex and constructed an obstruction to two spaces which are homotopy equivalent being simple homotopy equivalent. It was later understood that this class lived in the first algebraic K-theory group K_1 of an integral group ring. It was then shown that these two algebraic K-theory groups could be related by a localization sequence and that there should in fact be a related group K_i for all integers i extending this localization sequence to the left and right.

Milnor constructed the group K_2 of a ring as the center of the Steinberg group of a ring, inspired in part by a theorem of Matsumoto [7], and used this to motivate his definition of the higher algebraic K-theory groups, now known as Milnor K-theory group, in 1970 [9]. However, as we will see, this theory is a not a rich invariant in the sense that for finite fields the Milnor K-theory groups K_n^M vanish for $n \geq 2$.

In 1972, Quillen defined higher algebraic K-theory groups using the $+$-construction [11]. One of his key insights was that the algebraic K-theory groups should be defined as the the homotopy groups of a space. In the same year [12], Quillen defined higher algebraic K-theory groups for a category equipped with notion of exact sequences called an *exact category*. This allowed for much broader input, in particular recovering examples of interest in algebraic geometry.

In 1974 [13], Segal defined the algebraic K-theory of a symmetric monoidal

[1] Though Grothendieck famously spent most of his life in France, he was in fact born in Berlin, Germany in 1939. Perhaps this is why Grothendieck chose the letter K from the German word Klassen rather then the French, but this is not well documented.

category. This notion is sensitive to the symmetric monoidal structure, so it not a special case of Quillen's Q-construction unless the symmetric monoidal structure is the direct sum in an additive category. Quilen's Q-construction is also not a special case of Segal's construction. One of Segal's motivations was to give new constructions of infinity loop spaces, which were known to represent cohomology theories by Brown representability [5].

In 1978, Waldhausen extended Quillen's Q-construction further so that the input could be a category with cofibrations and weak equivalences [15]. This allowed one to define the algebraic K-theory of spaces. This new definition extended the applications of algebraic K-theory to manifold theory [17].

Since 1985, there have been several new constructions of algebraic K-theory using the theory of $(\infty, 1)$-categories. These constructions have proven quite useful for demonstrating universal properties of algebraic K-theory. For example, in 2016, Barwick defined a version of Waldhausen's algebriac K-theory construction for small Waldhausen quasicategories in [1] and proved that algebraic K-theory may be considered as a homology theory, in an abstract sense, on the quasi-category of small Waldhausen quasi-categories. Blumberg–Gepner–Tabuada [3] prove that the connective algebraic K-theory of a small stable quasi-category is the universal additive invariant and non-connective algebraic K-theory of a small stable quasi-category is the universal localizing invariant. Additionally, Gepner-Groth-Nikolaus [6] prove universal properties of the algebraic K-theory of symmetric monoidal quasi-categories. However, will not discuss these more constructions further in the present notes.

Chapter 2

Classical Algebraic K-theory

We begin by studying the groups K_0 and K_1. These two groups arose independently in the 1950's from entirely different contexts. Later, it was proven they are related by a localization sequence.

The group K_2 of a ring was then defined by Milnor, inspired by work of Matsumoto [7], and Milnor used this to motivate his definition of higher algebraic K-theory groups K_*^M now know as Milnor K-theory. However, these groups are not as rich an invariant as the higher algebraic K-theory groups that we will discuss in the next chapters, due to Quillen [11, 12]. We will explicitly prove this in the case of finite fields.

At the start, I want to emphasize that there are really two flavors of algebraic K-theory: algebraic K-theory of symmetric monoidal categories and algebraic K-theory categories with a notion of exact sequences, such as exact categories. The two flavors of algebraic K-theory agree when we consider symmetric monoidal categories with respect to the coproduct and algebraic K-theory of categories with exact sequences in which these exact sequences split. For example, this is the case for the category of finitely generated projective R modules. Since the category of finitely generated projective R modules will be our central example throughout and the distinction may not always be clear.

I also want to emphasize that non of the results in this chapter are new and our treatment in this chapter is strictly contained in chapters I-III of Weibel's K-book [18]. In [18], goes into significantly more depth on this subject than we attempt to do here. We also point the reader towards books of Bass [2] and Milnor [10] from the time period before 1972, which give an even more thorough treatment of what was known at the time about algebraic K-theory groups. In the remaining chapters, our treatment more closely follow the original papers of [11–13, 15].

2.1 The Grothendieck group

In order to give a very general definition of K_0, we will first briefly set up the the theory of monoidal categories and monoids in a symmetric monoidal category.

Monoidal categories are an abstraction of the properties enjoyed by the category of abelian groups Ab with respect to the tensor product $\otimes_{\mathbb{Z}}$ and the integers $\mathbb{Z}$. In particular, the category Ab is equipped with a functor

$$\otimes_{\mathbb{Z}} \colon \mathsf{Ab} \times \mathsf{Ab} \to \mathsf{Ab}$$

and a unit $\mathbb{Z}$ object in the sense that there are isomorphisms

$$M \otimes_{\mathbb{Z}} \mathbb{Z} \cong M \cong \mathbb{Z} \otimes_{\mathbb{Z}} M$$

for any abelian group M, which are natural in M. The tensor product is also associative

$$(M \otimes N) \otimes L \cong M \otimes (N \otimes L),$$

where this associativity is natural in M, N and L. There is also a factor swap map

$$B_{M,N} \colon M \otimes N \to N \otimes M$$

which is also natural in M and N. In addition, each of these pieces of data satisfy certain commutative diagrams. This data is abstracted to the definition of a symmetric monoidal category, which also applies in many other contexts.

Definition 2.1.1. A *symmetric monoidal category* $\mathcal{C}$ consists of a category $\mathcal{C}$ a functor

$$\otimes \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$$

and a unit $1_{\mathcal{C}}$ together with four natural isomorphisms:

1. an associator

$$a_{-,=,\equiv} \colon (- \otimes =) \otimes \equiv \xrightarrow{\cong} - \otimes (= \otimes \equiv)$$

2. a left unitor

$$\lambda_{-} \colon 1 \otimes (-) \xrightarrow{\cong} (-),$$

3. a right unitor

$$\rho_{-} \colon (-) \otimes 1 \xrightarrow{\cong} (-),$$

and

4. a braiding

$$B_{-,=} \colon (-) \otimes (=) \xrightarrow{\cong} (=) \otimes (-).$$

These natural transformations must satisfy the triangle identity

$$\mathrm{id}_x \otimes \lambda_y \circ a_{x,1_{\mathcal{C}},y} = \rho_x \otimes \mathrm{id}_y \tag{2.1.2}$$

and pentagon identity

$$a_{w,x,y\otimes z} \circ a_{w\otimes x,y,z} = \mathrm{id}_w \otimes a_{x,y,z} \circ a_{w,x\otimes y,z} \circ a_{w,x,y} \otimes \mathrm{id}_z \tag{2.1.3}$$

the hexagon identities

$$a_{y,z,x} \circ B_{x,y\otimes z} \circ a_{x,y,z} = \mathrm{id}_y \otimes B_{x,z} \circ a_{y,x,z} \circ B_{x,y} \otimes \mathrm{id}_z \tag{2.1.4}$$

$$a^{-1}_{y,z,x} \circ B_{x,y\otimes z} \circ a_{x,y,z} = \mathrm{id}_y \otimes B_{x,z} \circ a_{y,x,z} \circ B_{x,y} \otimes \mathrm{id}_z \tag{2.1.5}$$

and the "squaring to identity" axiom

$$B_{y,x} \circ B_{x,y} = \mathrm{id}_{x\otimes y}\,. \tag{2.1.6}$$

We succinctly write $(\mathcal{C}, \otimes, 1_{\mathcal{C}})$ for all of this data.

If $\mathcal{C}$ has all of the other structure except that it is not equipped with a braiding natural transformation $B_{-,=}$ satisfying (2.1.4), (2.1.5), and (2.1.6), then we say $\mathcal{C}$ is a *monoidal category*.

Examples 2.1.7. The category $P(R)$ is a symmetric monoidal category with respect to $\oplus$ denoted $(P(R), \oplus, 0)$ and it is a monoidal category with respect to $\otimes_R$, denoted $(P(R), \otimes_R, R)$. Moreover, when R is a commutative ring then the monoidal category $(P(R), \otimes_R, R)$ is in fact a symmetric monoidal category.

The category $VB_k(X)$ for $k = \mathbb{R}$ or $k = \mathbb{C}$ is a symmetric monoidal category with Whitney sum $\oplus$ and it is a monoidal category with tensor product $\otimes$ denoted $(VB_k(X), \oplus, 0)$ and $(VB_k(X), \otimes, k)$ where k here denotes the trivial one dimensional k vector bundle.

The categories Set (respectively Fin) are symmetric monoidal categories with respect to the coproduct $(\mathsf{Set}, \amalg, \varnothing)$ (respectively $(\mathsf{Fin}, \amalg, \varnothing)$, and with respect to the product $(\mathsf{Set}, \times, *)$ (respectively $(\mathsf{Fin}, \times, *)$). Similarly, Fin_G is a symmetric monoidal category with respect to the coproduct $(\mathsf{Fin}_G, \amalg, \varnothing)$ and the product $(\mathsf{Fin}_G, \times, *)$. Let k be a general field. The category $\mathsf{Rep}_k(G)$ is a symmetric monoidal category with respect to the direct sum $(\mathsf{Rep}_k(G), \oplus, 0)$ and it is a monoidal category with respect to tensor product $(\mathsf{Rep}_k(G), \otimes_k, k)$.

We now discuss monoids in a general symmetric monoidal category.

Definition 2.1.8. A (unital) monoid M in a symmetric monoidal category $\mathcal{C}$ is an object M in $\mathcal{C}$ equipped with an operation

$$\mu\colon\ M \otimes M \to M$$

and a unit map

$$\eta\colon\ 1_{\mathcal{C}} \to M$$

from the unit object $1_{\mathcal{C}}$ in $\mathcal{C}$ to M satisfying:

1. the associativity axiom

$$\begin{tikzcd} M\otimes M\otimes M \arrow[r, "\mu\times 1"] \arrow[d, "1\times\mu"'] & M\otimes M \arrow[d, "\mu"] \\ M\otimes M \arrow[r, "\mu"] & M \end{tikzcd} \tag{2.1.9}$$

and

2. the unitality axiom

$$\begin{tikzcd} M \arrow[r, "\eta\times M"] \arrow[dr, equal] & M\otimes M \arrow[d, "\mu"] & M \arrow[l, "1\times\eta"'] \arrow[dl, equal] \\ & M. & \end{tikzcd} \tag{2.1.10}$$

If in addition, the commutativity axiom

$$\begin{tikzcd} M\otimes M \arrow[rr, "\tau"] \arrow[dr, "\mu"'] & & M\otimes M \arrow[dl, "\mu"] \\ & M & \end{tikzcd} \tag{2.1.11}$$

is satisfied, we say that M is a *commutative monoid in* $\mathcal{C}$.

When $(\mathcal{C}, \otimes_{\mathcal{C}}, 1_{\mathcal{C}}) = (\mathsf{Set}, \times, *)$ we will simply refer to (unital) monoids and commutative monoids in Set as monoids and commutative monoids.

Each of the examples $(P(R), \oplus, 0)$, $(M(R), \oplus, 0)$, $VB_k(X)$, $(\mathsf{Fin}, \amalg, \emptyset)$, $(\mathsf{Fin}, \times, *)$, $(\mathsf{Fin}_G, \amalg, \emptyset)$, $(\mathsf{Fin}_G, \times, *)$, $(\mathsf{Rep}_k(G), \oplus, 0)$, and $(\mathsf{Rep}_k(G), \otimes_k, k)$ may actually be regarded as commutative monoids by applying the forgetful functor from small categories to sets.

If $(M, +, 0)$ is a commutative monoid with operation $+$ and $(M, \times, 1)$ is a monoid with respect to a second operation $\times$ such that $(M, +, \times, 0, 1)$ forms a ring without additive inverses, then we say that M is a semi-ring. In fact, $(P(R), \oplus, \otimes_R, 0, R)$, $(\mathsf{VB}_k(X), \oplus, \otimes, 0, k)$, $(\mathsf{Fin}, \amalg, \times, \emptyset, *)$, $(\mathsf{Fin}_G, \amalg, \times, \emptyset, *)$, and $(\mathsf{Rep}_{\mathbb{C}}(G), \oplus, \otimes_{\mathbb{C}[G]}, 0, \mathbb{C}[G])$ are all examples of semi-rings.

We are now prepared to discuss our definition algebraic K-theory K_0.

Construction 2.1.12. Given a commutative monoid M we form the Grothendieck group completion of M, denoted M^+ as follows. We define an equivalence relation on elements $(m, n) \in M \times M$. We define an equivalence relation

$$(m, n) \sim (m + p, n' + p)$$

for any $p \in M$. We then define $M^+ := M \times M / \sim$.

Exercise 2.1.13. Check that $(m, n) \sim (m + p, n' + p)$ is an equivalence relation.

Note that, by construction the abelian group M^+ has the universal property that given a map of commutative monoids $M \to A$, where A is an abelian group, then the map $M \to A$ factors as

$$\begin{tikzcd} M \arrow[r] \arrow[dr] & M^+ \arrow[d, dashed] \\ & A. \end{tikzcd} \tag{2.1.14}$$

In other words, there is an adjunction given by the isomorphism

$$\mathrm{Hom}_{\mathsf{CMon}}(M, A) \cong \mathrm{Hom}_{\mathsf{Ab}}(M^+, A)$$

natural in M and A. In particular, the construction M^+ is functorial in M.

Alternatively, we could let $F(M)$ be the free abelian group on symbols $[m]$ where $m \in M$. We can then quotient by the subgroup $R(M)$ of $F(M)$ generated by the relations $[m + n] - [m] - [n]$. This construction also clearly satisfies the universal property 2.1.14. Consequently, we may give a different definition of M^+ that agrees with the previous construction up to natural isomorphism

Definition 2.1.15. Given a commutative monoid M, define

$$M^+ := F(M)/R(M)$$

where $F(M)$ and $R(M)$ are as defined above.

For $m \in M$ we will write $[m]$ for a general element in M^+.

Definition 2.1.16. Let R be a ring. Then we define the 0-th algebraic K-theory group of R

$$K_0(R) := P(R)^+$$

where we regard the set of isomorphism classes of subgroups of $P(R)$ as commutative monoid via $\oplus$ and 0. In fact, since

$$(P(R), \oplus, \otimes_R, 0, R)$$

is a semi-ring, then $K_0(R)$ is a ring. When R is a commutative ring then $K_0(R)$ is also a commutative ring.

In fact, this is a special case of a more general construction.

Definition 2.1.17. Let $(\mathcal{C}, \otimes, 1)$ be a skeletally small symmetric monoidal concrete category with skeleton $\mathrm{sk}\,\mathcal{C}$. Then we may regard $\mathrm{sk}\mathcal{C}$ as a commutative monoid in Set with respect to $\otimes$ and 1 and define

$$K_0^{\otimes}(\mathcal{C}) := \mathrm{sk}(\mathcal{C})^+.$$

Moreover, if

$$(\mathcal{C}, \oplus, \otimes_{\mathcal{C}}, 0_{\mathcal{C}}, 1_{\mathcal{C}})$$

is a semi-ring. Then $K_0^{\oplus}(\mathcal{C})$ is a ring.

This general construction allows us to recover many examples of interest.

Examples 2.1.18. The 0-th complex topological K-theory of X is

$$KU^0(X) = K_0^{\oplus}(VB_{\mathbb{C}}(X))$$

and the 0-th real topological K-theory of X is

$$KO^0(X) = K_0^{\oplus}(VB_{\mathbb{R}}(X)).$$

In fact these are both rings because $(VB_k(X), \oplus, \otimes, 0, k)$ is a semi-ring when $k = \mathbb{C}$ or $k = \mathbb{R}$.

The Burnside ring of a finite group G is

$$A(G) = K_0^{\amalg}(\mathsf{Fin}_G)$$

where the ring structure comes from the fact that $(\mathsf{Fin}_G, \amalg, \times, \varnothing, *)$ is a semi-ring.

Let k be a field. The representation ring of G is

$$R_k(G) = K_0^{\oplus}(\mathsf{Rep}_k(G))$$

where the ring structure comes from the fact that $(\mathsf{Rep}_k(G), \oplus, \otimes_k, 0, \mathbb{C})$ is a semi-ring.

We finish with some basic computations. First, note that $(\mathbb{N}, +, 0)$ is a commutative monoid and its Grothendieck group completion is clearly

$$\mathbb{N}^+ = \mathbb{Z}.$$

Notice that there is always map of commutative monoids

$$\begin{aligned} \mathbb{N} &\to P(R). \\ n &\mapsto R^{n+1} \end{aligned}$$

and by functoriality of the Grothendieck construction, a group homomorphism

$$\mathbb{Z} \to K_0(R). \tag{2.1.19}$$

We say that R satisfies the left invariant basis property, denoted IBP, if R^n and R^m are not isomorphic whenever $n \neq m$. In this case, the rank of a free left R module does not depend on a choice of basis. All commutative rings satisfy this property and integral group rings $\mathbb{Z}[G]$ all satisfy the invariant basis property.

Example 2.1.20. The ring of k-linear endomorphisms $\mathrm{End}_k(k)$ does not satisfy the IBP property.

Exercise 2.1.21. Prove that there is an isomorphism of $\mathrm{End}_k(k)$ modules

$$\mathrm{End}_k(k) \cong \mathrm{End}_k(k) \oplus \mathrm{End}_k(k),$$

verifying the claim in Example 2.1.20.

Lemma 2.1.22. *When R satisfies the IBP, then the map*

$$\mathbb{Z} \to K_0(R)$$

induced by the map $n \mapsto R^n$ is injective.

Definition 2.1.23. We define the reduced K_0 group to be the cokernel of the map $\mathbb{Z} \to K_0(R)$ and denote it $\tilde{K}_0(R)$.

Proposition 2.1.24. *When k is a field, then*

$$\tilde{K}_0(k) = 0.$$

Proof. The rank of a vector space gives a map of commutative monoids

$$P(k) \to \mathbb{N}$$

sending $[k^n]$ to n, which is an isomorphism of commutative monoids. $\square$

Exercise 2.1.25. Prove that when R is a principle ideal domain, then

$$\tilde{K}_0(R) = 0.$$

Exercise 2.1.26. Prove that when R is a local ring, then

$$\tilde{K}_0(R) = 0.$$

The invariant $\tilde{K}_0(R)$ has interesting applations to geometry and number theory. For example, when G is a group and $R = \mathbb{Z}[G]$ is the associated integral group ring, then we define the 0-th Whitehead group

$$Wh_0(G) := \widetilde{K}_0(\mathbb{Z}[G]).$$

We will simply state a result of Wall's that shows that the 0-th Whitehead group is an interesting invariant in topology. We say that a topological space X is dominated by a CW complex K if it there is a map $K \to X$ with right homotopy inverse.

Theorem 2.1.27 (Wall's finiteness obstruction)**.** *Suppose that X is dominated by a finite CW complex K and let $G = \pi_1 X$. Then there is an associated obstruction class $w(X) \in Wh_0(G)$ such that $w(X) = 0$ if and only if X is homotopy equivalent to a finite CW complex.*

Remark 2.1.28. Note that we know by CW approximation that X is homotopy equivalent to a CW complex, so it suffices to consider the case when X is also a CW complex, however it is not at all clear that a homotopy retract of a finite CW complex is again finite CW complex.

Reduced K_0 also has applications to number theory. First, we recall a definition from commutative algebra.

Definition 2.1.29. A Dedekind domain R is an integral domain such that for all nontrivial ideals $J \subset I \subset R$, there exists an ideal K such that $IK = J$.

Definition 2.1.30. The *ideal class group* of a Dedekind domain R it the quotient

$$\mathrm{Cl}(R) = \{I : I \subset R\}/\sim$$

where I ranges over all ideals in R and the equivalence relation states that $I \sim J$ if there exist $x, y \in R$ such that $xI = yJ$ as subsets of R. The group structure is given by the product of ideals.

Exercise 2.1.31. Check that this is in fact an equivalence relation and that $Cl(R)$ is an abelian group.

Again, we will not prove the following result, but we record it as another important application of K_0.

Theorem 2.1.32. *When R is a Dedekind domain, then there is an isomorphism*

$$\tilde{K}_0(R) \cong Cl(R).$$

The class group measures the failure of unique prime factorization. In other words, when R is also a UFD then $\tilde{K}_0(R) = 0$. To see that unique prime factorization can fail, simply consider the ring $\mathbb{Z}[\sqrt{-5}]$. In this ring,

$$(1 - \sqrt{-5})(1 + \sqrt{-5}) = 6 = 2 \cdot 3.$$

2.2 The Whitehead group $Wh_1(G)$

In the 1940's and 1950's, Whitehead developed the theory of simple homotopy types. We say that a finite CW complex Y has the same simple homotopy type as a finite CW complex X if they are homotopy equivalent and each homotopy can be described in terms of elementary expansions and collapses.

Whitehead defined a group which encoded the obstruction to two homotopy equivalent finite CW complexes having the same simple homotopy type. Suppose X and Y are CW complexes and there is a homotopy equivalence

$$X \xrightarrow{\simeq} Y.$$

Then clearly this homotopy equivalence induces an isomorphism $\pi_1 X \cong \pi_1 Y$. We would like to know whether X and Y are simple homotopy equivalent. There is an obstruction to this, which lies in a group $Wh(\pi_1 X)$, which is an ablian group that depends only on the group $\pi_1 X$. It was later noted that this group can be defined in terms of algebraic K-theory.

Let R be an associative ring and let $GL_n R$ be the group of invertible $n \times n$ matrices with coefficients in R. There is an inclusion

$$GL_n R \subset GL_{n+1} R$$

given by

$$A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$$

We can then form the union (the colimit) to define

$$GL(R) = \bigcup_{n \geq 1} GL_n(R).$$

In general, if we have a group G we can take the quotient by commutators to define

$$G^{\mathrm{ab}} := G/[G, G].$$

In fact this is a left adjoint to the forgetful functor from abelian groups groups so it satisfies a universal property, which is encoded in the natural isomorphism

$$\mathrm{Hom}_{\mathsf{Ab}}(G^{\mathrm{ab}}, A) \cong \mathrm{Hom}_{\mathsf{Gp}}(G, A).$$

Definition 2.2.1. Let R be a ring, then we define

$$K_1(R) := GL(R)^{\mathrm{ab}}.$$

In fact, there is a nice description of the commutator $[GL(R), GL(R)]$. Let $E_n(R)$ be the subgroup of $GL_n(R)$ consisting of the $n \times n$ matrices, which are *transvections*. A transvection is the sum of the identity matrix and a matrix with only one nonzero entry, where that nonzero entry does not occur on the diagonal. We write $e_{i,j}(r)$ for this matrix where r is the nonzero entry and it occurs in the i, j-th position where $i \neq j$. We may then define $E(R)$ in the same way that we defined $GL(R)$ as the union

$$E(R) = \bigcup_{n \geq 1} E_n(R).$$

Definition 2.2.2. A group G is *perfect* if

$$G = [G, G].$$

Note that for a perfect group $G^{\mathrm{ab}} = 0$. Such groups are quite interesting from the perspective of topology. For example, a path connected space such that $\pi_1 X$ is a nontrivial perfect group and $\pi_k X = 0$ for all $k > 0$ has the property that its homology is the same as the homology of a point and yet it X is not contractible.

Lemma 2.2.3. *When $n \geq 3$, then $E_n(R)$ is a perfect group.*

Proof. Whenever i, j, k are distinct, then

$$e_{i,j}(r) = [e_{i,k}(r), e_{k,i}(1)].$$

□

Exercise 2.2.4. Give an example where $E_2(R)$ is not perfect.

Exercise 2.2.5. Verify that, if $g \in GL_n(R)$, the identity

$$\begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix} = \begin{pmatrix} 1 & g \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -g^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & g \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

holds in $GL_{2n}(R)$.

The following example will be useful in the proof of Whitehead's lemma.

Example 2.2.6. A signed permutation matrix is a matrix that permutes the standard basis on R^n up to a sign. If we write $\{e_1, \ldots, e_n\}$ for the standard basis, then a signed permutation acts on the set $\{\pm e_1, \ldots, \pm e_n\}$. We observe that, for example, the signed permutation matrix

$$\overline{w}_{1,2} := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = e_{1,2}(1) e_{2,1}(1) e_{1,2}(1)$$

can be written as a product of transvections and therefore it is contained in $E_2(R)$ for any ring R. More generally,

$$\overline{w}_{i,j} \in E_n(R)$$

for $n \geq i, j$. We can then show that cyclic permutations of three basis elements are also contained in $E_n(R)$, since they can be written as $\overline{w}_{jk}\overline{w}_{i,j}$. Consequently, every matrix corresponding to an even permutation of basis elements is an element in $E_n(R)$ for some n. Thus, by Exercise 2.2.5 we know that $E_{2n}(R)$ contains the matrix

$$\begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix}.$$

The subgroup $E_n(R)$ is not necessarily a normal subgroup in $GL_n(R)$. It is often a normal subgroup in $GL_n(R)$ for sufficiently large n, but even this is too much to ask for in general. When R is a commutative ring, the situation is much easier and $E_n(R)$ is normal in $GL_n(R)$ for $n \geq 3$. Nevertheless, we have the following lemma due to Whitehead which, in particular, implies that $E(R)$ is normal in $GL(R)$.

Lemma 2.2.7 (Whitehead's Lemma)**.** *There is an isomorphism*

$$[GL(R), GL(R)] \cong E(R)$$

Proof. The fact that

$$E(R) \subset [GL(R), GL(R)]$$

follows from Lemma 2.2.3. Conversely, suppose $[A, B] \in [GL_n(R), GL_n(R)]$. Then we can write $[A, B]$ as the product

$$[A, B] = \begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & B^{-1} \end{pmatrix} \begin{pmatrix} (AB)^{-1} & 0 \\ 0 & AB \end{pmatrix}$$

By Example 2.2.6, we therefore know that $[A, B] \in E(R)$. □

This gives a new definition of $K_1(R)$.

Definition 2.2.8. Let R be a ring, then we define

$$K_1(R) := GL(R)/E(R).$$

In particular, $K_1(R)$ is a quotient of $GL(R)$ by a perfect normal subgroup. This second definition will be important for the next chapter.

We can now define the Whitehead group $Wh_1(G)$ of a group G. We write $\tilde{K}_1(R)$ for the cokernel of the map

$$K_1(\mathbb{Z}) \to K_1(R)$$

induced by the unit map $\mathbb{Z} \to R$. Note that this is consistent with our definition of $\tilde{K}_0(R)$ as the cokernel of the map

$$\mathbb{Z} \cong K_0(\mathbb{Z}) \to K_0(R)$$

by Exercise 2.1.25.

Definition 2.2.9. We define the Whitehead group of a group G as

$$Wh_1(G) := \tilde{K}_1(\mathbb{Z}[G]).$$

Again, we will simply cite a deep result that demonstrates that this group is useful for studying problems in topology.

Theorem 2.2.10 (Whitehead)**.** *Suppose K and L are finite CW complexes and there is a homotopy equivalence $f\colon K \to L$ inducing an isomorphism $\pi_1 K \cong \pi_1 L$. Let $G = \pi_1 K$. Then there is an associated class*

$$\tau(f) \in Wh_1(G),$$

called the Whitehead torsion of f, such that $\tau(f) = 0$ if and only if f is a simple homotopy equivalence.

In fact, there are other applications of Whitehead torsion to manifold theory. For those unfamiliar with these constructions in manifold theory, we do not plan to give full definitions as that would be too much of a diversion and these constructions will not be used later. We therefore just provide enough information to state the main results in order to indicate the depth of the subject of algebraic K-theory.

Let (W, M, M') be a triple of compact piecewise linear (PL) manifolds. We say this triple is an *h-cobordism* if W has boundary $M \amalg M'$ and both inclusions $M \subset W$ and $M' \subset W$ are homotopy equivalences. There is therefore a Whitehead torsion class $\tau \in Wh_1(\pi_1 M)$ associated to the inclusion $M \subset W$. We record the following deep result, proven by Mazur [8], without proof.

Theorem 2.2.11 (The s-cobordism theorem)**.** *Given an h-cobordism (W, M, M') of PL-manifolds, with M fixed and dim$(M) \geq 5$. Then there is a PL homeomorphism of triples*

$$(W, M, M') \cong (M \times [0,1], M \times \{0\}, M \times \{1\})$$

if and only if $\tau = 0$. Moreover, every element $\tau \in Wh_1(\pi_1 M)$ arises as the Whitehead torsion of some h-cobordism (W, M, M').

This result can be used to prove a version of the generalized Poincare conjecture, which had originally proven by Smale [14] before the s-cobordism theorem was known..

Corollary 2.2.12. *Suppose N is a PL manifold with the same homotopy type as a sphere S^n and $n \geq 5$. Then N is PL-homeomorphic to a S^n.*

Proof. Form a PL manifold W by removing two disjoint n-discs D_1 and D_2 from N. Then we produce a PL cobordism $(W, S_1^{n-1}, S_2^{n-1})$ where S_i^{n-1} is the boundary of D_i in N. Since $\pi_1 S^{n-1} = 0$ when $n \geq 5$, we know that the Whitehead torsion $\tau \in Wh_1(0)$ vanishes. Thus, there is a PL homeomorphism $W \cong S^{n-1} \times [0,1]$ by Theorem 2.2.11 and $N = W \cup D_1 \cup D_2$ is therefore PL homeomorphic to S^n. $\square$

2.2.1 Relating K_0 and K_1

Finally, we prove that there is a localization sequence relating K_1 and K_0 in certain cases. Let I be an ideal in R and let $GL(I)$ be the kernel of the map $GL(R) \to GL(R/I)$. Let $E_n(R, I)$ be the normal subgroup of $E_n(R)$ generated by matrices $e_{i,j}(r)$ such that $r \in I$ and $1 \leq i \neq j \leq n$ and define $E(R, I)$ as the union

$$E(R, I) = \bigcup_{n \geq 1} E_n(R, I).$$

Lemma 2.2.13 (Relative Whitehead Lemma)**.** *The group $E(R, I)$ is normal in $GL(I)$ and*

$$[GL(I).GL(I)] \subset E(R, I).$$

Exercise 2.2.14. Prove Lemma 2.2.13.

Definition 2.2.15. Define the relative K_1 group as

$$K_1(R, I) := GL(I)/E(R, I).$$

We can also define a relative algebraic K-theory group K_0. Given a ring R and an ideal $I \in R$, we can for the trivial square-zero extension of R by I, denoted $R \oplus I$.

Definition 2.2.16. We define the relative K_0 group as

$$K_0(R, I) := \ker(K_0(R \oplus I) \to K_0(R)).$$

The definitions of relative K_1 and relative K_2 are a bit different. This is because $K_0(R, I)$ in fact does not depend on R. If $R \to S$ is a map of rings and I is mapped isomorphically onto an ideal of S, which we also call I, then $K_0(R, I) \cong K_0(S, I)$. We therefore sometimes simply write

$$K_0(I) := K_0(R, I).$$

The same is not true for $K_1(R, I)$ and in fact there are maps of rings $R \to S$ where I maps isomorphically onto an ideal of S, also denoted I, and yet

$$K_1(R, I) \ncong K_1(S, I).$$

It is known that $K_1(R, I)$ is independent of R if and only if $I^2 = I$, or in other words I is idempotent by [?qx, Vaserstein 14.2]

We will leave part of the proof of the localization sequence as an exercise.

Exercise 2.2.17. If $f\colon R \to S$ is a ring map sending I isomorphically onto an ideal of S, also denoted I, then prove that

$$K_0(R, I) \cong K_0(S, I).$$

Hint: Show that $GL(S)/GL(S \oplus I) = 1$. Then prove that if $I \cap J = 0$, then

$$K_0(I + J) = K_0(I) \oplus K_0(J).$$

Use this to prove that there is an exact sequence

$$1 \to GL(I) \to GL(R) \to GL(R/I) \xrightarrow{\delta} K_0(I) \to K_0(R) \to K_0(R/I). \tag{2.2.18}$$

Proposition 2.2.19. *There is an exact sequence*

$$K_1(R, I) \to K_1(R) \to K_0(R/I) \to K_0(I) \to K_0(R) \to K_0(R/I). \tag{2.2.20}$$

Proof. By Exercise 2.2.17, we know that there is an exact sequence (2.2.18). Passing to quotients by $E(R)$ and $E(R/I)$ gives exactness of the sequence (2.2.20) at $K_1(R/I)$. By Exercise 2.2.17, it therefore suffices to show that the sequence (2.2.20) is exact at $K_0(R)$. Let g be an element of the kernel of the composite

$$GL(R) \to K_1(R) \to K_1(R/I).$$

Then we know that the image of g in $GL(R/I)$, is in $E(R/I)$. Write $\overline{g}$ for this element of $E(R/I)$. Since the map $E(R) \to E(R/I)$ is surjective, there is an element $e \in E(R)$ mapping to $\overline{g}$. Consequently, ge^{-1} maps to $1 \in E(R/I) \subset GL(R)$. So ge^{-1} is in the kernel of $GL(R) \to GL(R/I)$, which we denoted $GL(I)$. Write $[ge^{-1}]$ for the equivalence class of ge^{-1} in $GL(I)/E(R, I)$.

To summarize, for any $g \in K_1(R)$ in the kernel $K_1(R) \to K_1(R/I)$, we have produced a well-defined element $[ge^{-1}]$ in $K_1(R, I)$ that maps to $K_1(R)$. Thus, the sequence (2.2.20) is exact at $K_1(R)$. □

The existence of this sequence was known already in the 1960's [2], but it was not known how to extend the sequence to the left. It was expected that there were groups K_n for all $n \in \mathbb{Z}$ that produce a long exact sequence, but it remained an open question until the 1970's. In Section 2.4, we discuss the first proposed definition of higher algebraic K-theory groups in the early 1970's due to Milnor. This was defined in order to extend the groups K_0, K_1, and K_2, where K_2 is defined in Section 2.3 and the choice of definition of the higher Milnor K-theory groups is clearly inspired by the definition of K_2.

2.3 The Steinberg group and its center

We now discuss K_2 of a ring and the higher Milnor K-theory groups of a field.

Definition 2.3.1. Let A be a ring. Let $n \geq 3$, then the Steinberg group $St_n(A)$ is the group with generators $x_{i,j}(r)$ for $a \in A$ and $i, j \in \{1, \ldots, n\}$ with $i \neq j$ and relations

$$x_{i,j}(a)x_{i,j}(b) = x_{i,j}(a+b) \tag{2.3.2}$$

$$[x_{i,j}(a), x_{k.\ell}(s)] = \begin{cases} 1 & \text{if} j \neq k \text{ and } i \neq \ell \\ x_{i,\ell}(rs) & \text{if} j = k \text{ and } i \neq \ell \\ x_{k,j}(-sr) & \text{if} j \neq k \text{ and } i = \ell, \end{cases} \tag{2.3.3}$$

which are called the *Steinberg relations*.

Exercise 2.3.4. Show that the transvections $e_{i,j}(a)$ in $E_n(A)$ for $n \geq 3$ satisfy the Steinberg relations.

As a consequence of Exercise 2.3.4, there is a canonical surjective group homomorphism

$$St_n(A) \to E_n(A)$$

for $n \geq 3$ mapping $x_{i,j}(a)$ to $e_{i,j}(a)$. Since the Steinberg relations for n include the Steinberg relations for all $k < n$, there is a canonical inclusion

$$St_{n-1}(A) \to St_n(A)$$

and we define

$$St(A) = \bigcup St_n(A).$$

Exercise 2.3.5. Prove that a level map $\{A_i\} \to \{B_i\}$ of sequences of groups, which is a levelwise surjection induces a surjection

$$\operatorname*{colim}_i A_i \to \operatorname*{colim}_i B_i.$$

Definition 2.3.6. Let A be a ring. We define $K_2(A)$ to be the kernel of the canonical surjection

$$St(A) \to E(A).$$

As a consequence of the definition, there is an exact sequence

$$1 \to K_2(A) \to St(A) \to GL(A) \to K_1(A) \to 1.$$

As defined, it is not clear that $K_2(A)$ is an abelian group, because $St(A)$ is not necessarily abelian. However, it turns out that it is an abelian group. Moreover, we have the following result of Steinberg, but we omit the proof.

Theorem 2.3.7 (Steinberg)**.** *The group $K_2(A)$ is abelian and it is exactly the center of $St(A)$.*

We end by remarking that the group K_2 really deserves to be called K_2 in the following sense.

Theorem 2.3.8. *Let A be a Dedekind domain with field of fractions F, then there is a long exact sequence*

$$K_2(F) \to \bigoplus_{\mathfrak{p} \in \mathbb{P}} K_1(A/\mathfrak{p}) \to K_1(A) \to K_1(F) \to \bigoplus_{\mathfrak{p} \in \mathbb{P}} K_0(A/\mathfrak{p}) \to K_0(A) \to K_0(F) \to 0$$

where $\mathbb{P}$ is the set of prime ideals in A.

We will prove this as a consequence of the localization sequence and dvissage due to Quillen [12] in 1972 later in the course. However, it is important to know that the localization sequence as stated in Theorem 2.3.8 was had already been proven by Milnor an it appears in [10] from 1971.

2.4 Milnor K-theory of fields

Milnor extended the definition of K_2 to higher algebraic K-theory groups of fields, now known as Milnor K-theory groups, in the 1970's.

Let k be a field. We define the tensor algebra of the group of units k to by

$$T(k^\times) = \bigoplus_{i \geq 0} (k^\times)^{\otimes i}$$

where $(k^\times)^{\otimes 0} = \mathbb{Z}$. This is also known as the free associative algebra on $k^\times$. Write $\ell(x)$ for an element in $k^\times$ in degree 1 corresponding to $x \in k^\times$. We can then define Milnor K-theory of a field as a graded ring all at once.

Definition 2.4.1. The Milnor K-theory groups of a field k are

$$K_*^M(k) := T(k^\times)/(\ell(x) \otimes \ell(1-x) : 1 \neq x \in k^\times)$$

Note that the ideal generated by the elements $\ell(x) \otimes \ell(1-x)$ is a homogeneous ideal and therefore it makes sense to form the quotient in graded rings.

It is clear that

$$K_0^M(k) = \mathbb{Z} = K_0(k) \text{ and}$$
$$K_1^M(k) = k^\times = K_1(k)$$

for any field k. By a theorem of Matsumoto, $K_2^M(k) \cong K_2(k)$.

Theorem 2.4.2 (Matsumoto)**.** *There is an isomorphism*

$$K_2^M(k) \cong K_2(k)$$

for any field k.

This motivated Milnor's definition of higher algebraic K-theory groups.
Note that

$$K_0(\mathbb{F}_q) = K_0^M(\mathbb{F}_q) = \mathbb{Z} \text{ and}$$
$$K_1(\mathbb{F}_q) \cong \mathbb{F}_q^\times$$

which is a cyclic group of order $q-1$. In light of this, the following result gives a complete calculation of the Milnor K-theory of of finite fields.

Proposition 2.4.3. *The Milnor K-theory groups $K_n^M(\mathbb{F}_q)$ vanish for all $n \geq 2$. Consequently, there is an isomorphism of graded rings*

$$K_*^M(\mathbb{F}_q) \cong \mathbb{Z} \oplus \mathbb{F}_q^\times$$

where $\mathbb{Z} \oplus \mathbb{F}_q^\times$ is the trivial square zero extension of $\mathbb{Z}$ by the cyclic group $\mathbb{F}_q^\times$.

Proof. We first show that

$$\left(\mathbb{F}_q^\times \otimes \mathbb{F}_q^\times / (x \otimes (1-x) : x \neq 1, 0)\right) = 1.$$

We write $(x \otimes y) \cdot (z \otimes w)$ for the group operation and 1 for the unit. Note that $\mathbb{F}_q^\times$ is cyclic of order $q-1$ and consequently $\mathbb{F}_q^\times \otimes \mathbb{F}_q^\times$ is also cyclic of order $q-1$. This cyclic group is generated by $x \otimes x$ whenever x is a generator of $\mathbb{F}_q^\times$.

We split into two cases. If q is even, then we know $2x \otimes x = 0$ in $\mathbb{F}_q \otimes \mathbb{F}_q$. So $x \otimes x = x \otimes -x$. We also know that $x \otimes -x = x \otimes 1$ in $K_2^M(\mathbb{F}_q)$ by the relations and since 1 is the identity in $\mathbb{F}_q^\times$, the element $x \otimes 1$ is trivial in the group $K_2^M(\mathbb{F}_q)$. In other words, we conclude that

$$x \otimes x = 1$$

for all elements $x \otimes x \in K_2^M(\mathbb{F}_q)$ where x is a generator of $\mathbb{F}_q^\times$. This implies that the group $K_2^M(\mathbb{F}_q)$ is trivial. In fact, essentially the same argument implies that $K_n^M(\mathbb{F}_q) = 0$ for $n > 2$.

When q is odd, we still know that $x \otimes -x = x \otimes 1$ is trivial and consequently, we have skew-symmetry

$$(x \otimes y) \cdot (y \otimes x) = (x \otimes -xy) \cdot (y \otimes -xy) = xy \otimes -xy = 1$$

in $K_2^M(\mathbb{F}_q)$. This immediately implies that $(x \otimes x)^2 = 1$ and more generally one can show that

$$(x \otimes x)^{mn} = x^m \otimes x^n$$

when m, n are odd. The set of odd powers of elements in $K_2^M(\mathbb{F}_q)$ is exactly the same as the non-squares, by construction. If there exists a non-square u such that $1-u$ is also a non-square in $\mathbb{F}_q$, then all elements are divisible by the element $u \otimes (1-u) = 0$, or in other words all elements are trivial. To find such a u, we note that there is an involution $u \mapsto 1-u$ on the set $\mathbb{F}_q = \{0, 1\}$. and the set $\mathbb{F}_q - \{0, 1\}$ consists of $(q-1)/2$ non squares, but only $(q-3)/2$ squares. In other words, there are strictly less squares than non-squares and there must be an orbit of the C_2-action that is completely contained in the non-squares. Again, essentially the same proof implies that $K_n^M(\mathbb{F}_q) = 0$ for $n > 2$. □

We include this result in order to indicate that Milnor K-theory, though very interesting in its own right, is this is not the richest invariant. We will see a different construction of higher algebraic K-theory groups in the next section the fixes this defect.

Bibliography

[1] Clark Barwick, *On the algebraic K-theory of higher categories*, J. Topol. **9** (2016), no. 1, 245–347. MR3465850

[2] Hyman Bass, *Algebraic K-theory*, W. A. Benjamin, Inc., New York-Amsterdam, 1968. MR0249491

[3] Andrew J. Blumberg, David Gepner, and Gonçalo Tabuada, *A universal characterization of higher algebraic K-theory*, Geom. Topol. **17** (2013), no. 2, 733–838. MR3070515

[4] Armand Borel and Jean-Pierre Serre, *Le théorème de Riemann-Roch*, Bull. Soc. Math. France **86** (1958), 97–136. MR116022

[5] Edgar H. Brown Jr., *Cohomology theories*, Ann. of Math. (2) **75** (1962), 467–484. MR138104

[6] David Gepner, Moritz Groth, and Thomas Nikolaus, *Universality of multiplicative infinite loop space machines*, Algebr. Geom. Topol. **15** (2015), no. 6, 3107–3153. MR3450758

[7] Hideya Matsumoto, *Sur les sous-groupes arithmétiques des groupes semi-simples déployés*, Ann. Sci. École Norm. Sup. (4) **2** (1969), 1–62. MR240214

[8] Barry Mazur, *Relative neighborhoods and the theorems of Smale*, Ann. of Math. (2) **77** (1963), 232–249. MR150786

[9] John Milnor, *Algebraic K-theory and quadratic forms*, Invent. Math. **9** (1969/70), 318–344. MR260844

[10] ——, *Introduction to algebraic K-theory*, Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1971. Annals of Mathematics Studies, No. 72. MR0349811

[11] Daniel Quillen, *On the cohomology and K-theory of the general linear groups over a finite field*, Ann. of Math. (2) **96** (1972), 552–586. MR315016

[12] ——, *Higher algebraic K-theory. I*, Algebraic *K*-theory, I: Higher *K*-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), 1973, pp. 85–147. Lecture Notes in Math., Vol. 341. MR0338129

[13] Graeme Segal, *Categories and cohomology theories*, Topology **13** (1974), 293–312. MR353298

[14] Stephen Smale, *Generalized Poincaré's conjecture in dimensions greater than four*, Ann. of Math. (2) **74** (1961), 391–406. MR137124

[15] Friedhelm Waldhausen, *Algebraic K-theory of topological spaces. I*, Algebraic and geometric topology (Proc. Sympos. Pure Math., Stanford Univ., Stanford, Calif., 1976), Part 1, 1978, pp. 35–60. MR520492

[16] ——, *Algebraic K-theory of topological spaces. II*, Algebraic topology, Aarhus 1978 (Proc. Sympos., Univ. Aarhus, Aarhus, 1978), 1979, pp. 356–394. MR561230

[17] ——, *Algebraic K-theory of spaces, a manifold approach*, Current trends in algebraic topology, Part 1 (London, Ont., 1981), 1982, pp. 141–184. MR686115

[18] Charles A. Weibel, *The K-book*, Graduate Studies in Mathematics, vol. 145, American Mathematical Society, Providence, RI, 2013. An introduction to algebraic *K*-theory. MR3076731