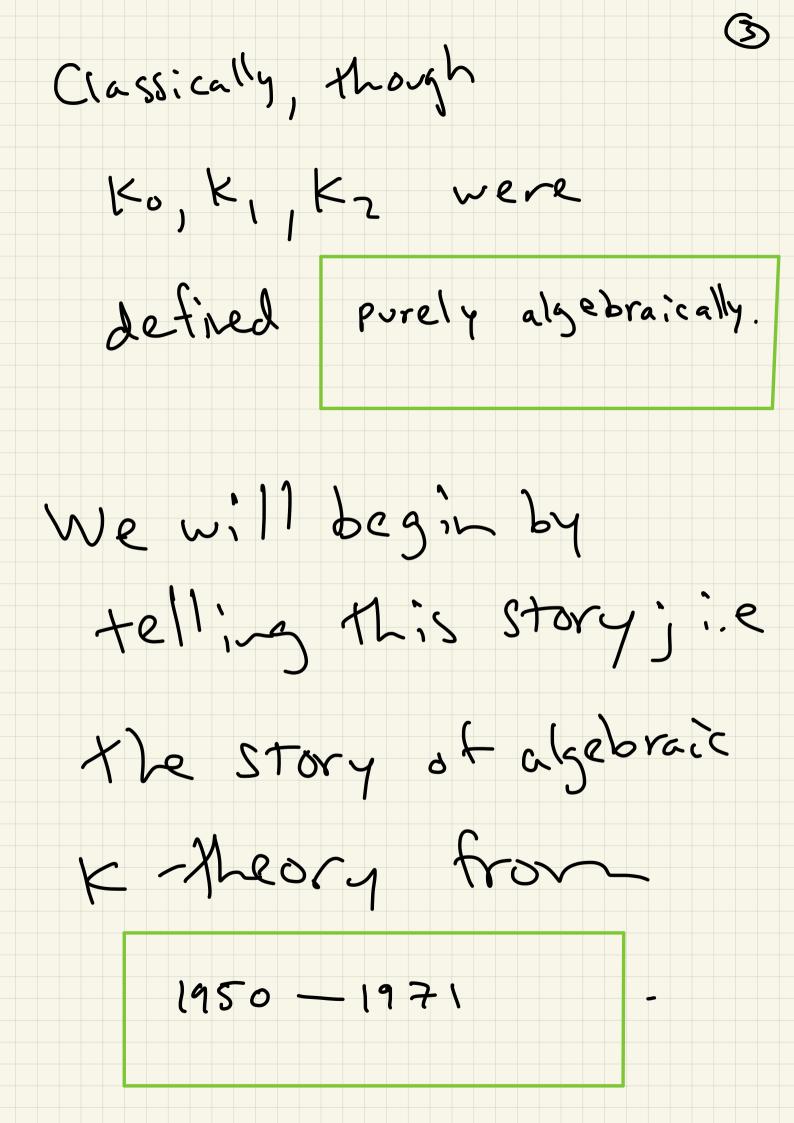
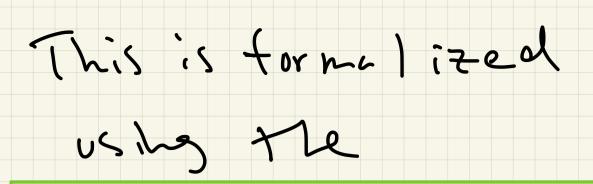
Lecture 1: The Grothendieck group

 $R = k \left(\frac{x}{4} \right)$ $k = k \left(\frac{x}{4} \right)$

Algebraic geometry It is useful to replace R by a category of modules over R P(R) Where ob P(R) = 25in. gen. projectivel mor P(R) = Eisomorphing 3 and replace KL(R) with a space (CR) Such that $\pi_n K(R) = K_n(R)$



IT The Grothen dieck group In the late 1950's, Grothen dieck defined Ko to generalize the Riemann - Roch theorem to varieties. To do This one reed; to not just consider vector spaces, but virtual vector spaces for example.

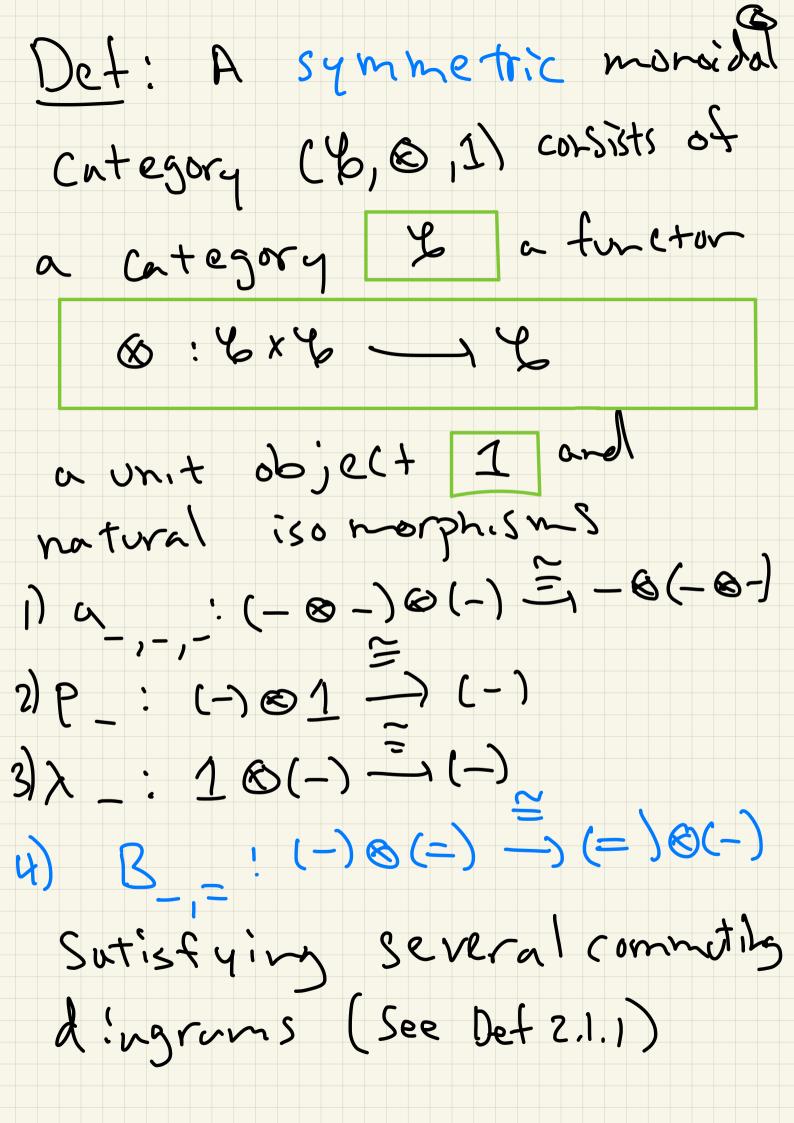


Grothendieck group

To define this at the right level of generality, we need the notion

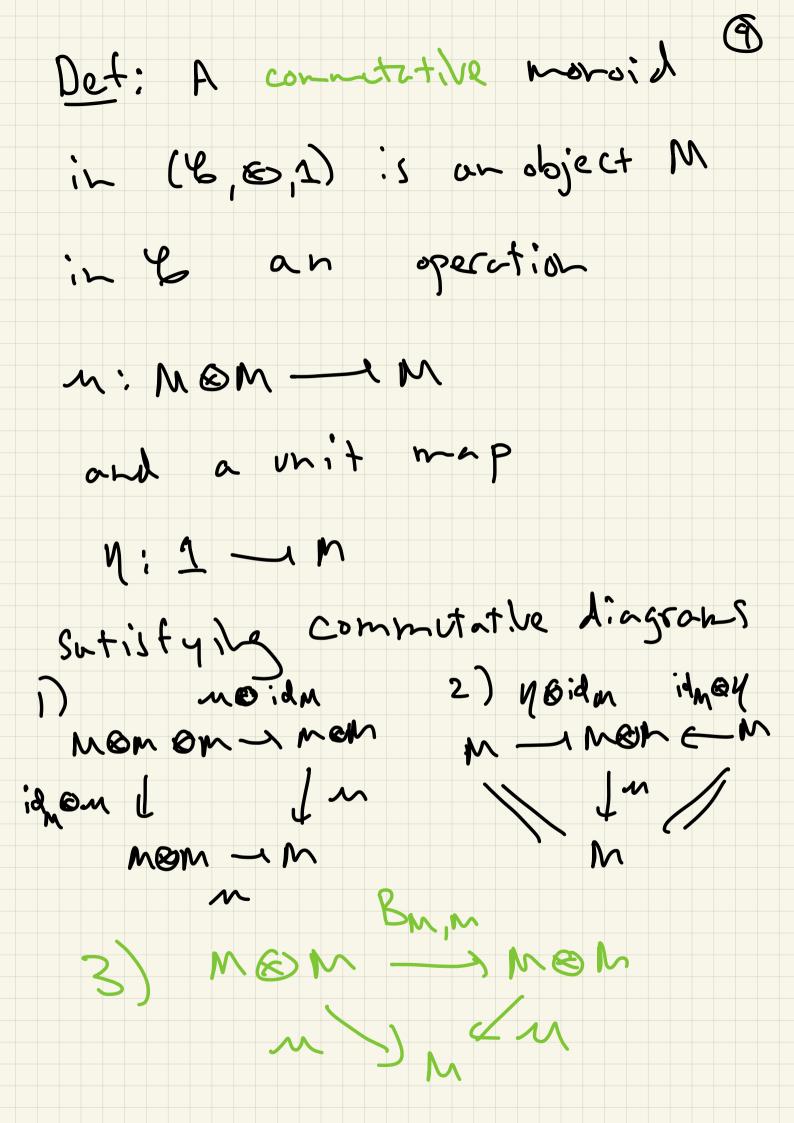
Symmetric monoilal Category

This abstructs the structure present in (Ab, 82, 2).



isomorphist Shitely ictive 126, 2000 Example: (P(R), (B), 0) monoidal category (Symmetric) (PCP1, &, R) monoidal Category CR commetative Def: X cu complex K=R, C VB_K(X) obVB_K(X) budles
over k mor VBK(X) isos Example: Synnetric monoidal (VBk(x), B, 0). (VBk(x), 8, k) terson buelle product Whithey sum of vector bundles

iso chases of Sets Def: ob FL Fin 130 morphisms mor F.L symmetric monoidal Examples: (Fin, 11, 0) (Fin, x, *) Det: Rep_k(G) = P(k(G)) Examples: Symnetric monoidal categories (RepkG), (RepkG) (RepkG) (K)



when (b, 8, 1) = (Set, x, *) (1) we simply (all a (Lommetative) monoid in Cset, x, x) a (commetative) monoid. Ex. Ccommutative) monoids

- · (P(R), B, O) (P(R), &), R)
 Commetative when
 R is commetative
 - · (VBk(x), B, O) (VBk(x), &, k)
- · (Fing, 4,0) (Fing, x, x)
- · (Repkie), @ ,0) (Repkin , &, K)

Construction! Let (M, +, 0) be a commutative monoid. Then

 $M^{9P} = M \times M / \sim$

where

(m,, n,)~(m2, n2)

where $m_2 = m_1 + p$

for some pEM.

// m, _ m, P

Then mg? is an abelian gror.

There is another construction (3) that clearly has the some universal property. Let F(M) be the free abelian group on cm? were mem and quotient by the free abelian group on The relations [m+n]-cm]-[n] dusted R(n) Det: Mar = F(m)/R(m).

We can how define algebraic 19 K-Ne bry In degree zero.

Def: Let Rbe an assoc. unital ring (B(R) = (P(R), B, 0)

More generally, let (4,8,1)

be a small symmetric monorder

cottegory. We may regard it

as a commetative monord in let

Def:

 $K_{o}(\mathcal{L}) = (\mathcal{L}, \otimes, 1)^{9P}$

Examples:

 $K_{o}(VB_{e}(x)) \stackrel{\sim}{=} KU^{o}(x)$

Ko(nBB(x)) = Koo(x)

Ko(Fing) = A(G) Burnsida.

Ko (Rep (G)) = R(G)

= R(G) representation

Ko (ReprCG))=ROCG)

Exercise: Prove that

K°(S) = 1

(More generally, when RisaPID oralocal ring show Ko(R)=ZZ)

0 III Applications 1) Geometric topology Let X bea cw complex adlet K be a finite cw conglex. We say X is domirated by K if there is a map S.T. ior ~ idx In other words, X is a retract in hotop of K.

Example: Ma conpect topological manifold then

M = x cw complex

f(M) = Xo = X
Thinte cu complex
So M is dom he ted

by a filite ou conglex

and we can ask whether

Mis the htply type of a finite our complex.

This will be true if m has a

triangulation

R we always (8) Given a ring have a map Ko(21) -1 Ko(R) and when R=2(G) for G grap or R connitative tren this is injective. Def: $\mathcal{K}_{o}(R) = cdeer(k_{o}(R) - 1k_{o}(R))$ Q: If X is dominated by a finite Cw complex k, the htpy type of a filite cu complex?

Thu [wall's filiteress obstruction] Suppose X is don. Lated by a filite au complex K and 6=17,(x). Ten there is an obstration class w(x) e Ko(266) Such that wcx)=o;f and only if X is honotopy equivalent to a filite Cw complex

Ex: M compact manifold w(m) = 0.

20 2) Number theory Def: A Dedekind donain R is an integral donnain 8.t. for all hontrivial ideals there exists an ideal K in R such that IK=J. Ex: Of rung of integers in a number field. Det: The ideal class grap

of a dedekid donain R

is the quotient CL(R) = ST: TCR3/n

T ~ 7; CD where there exist x, y & R Such that there is an equality XI = YJ of subsets of R. The group structure is the product of ideals.

Thm: Let R be a Dedekind domain, then there is an isomorphism $K_0(R) = CL(R)$. The class group neasurs The failure of unique Prive factorization. To see that this can OQ(5-5) fail, consider EV-5] Inthis my, (6) can Le written as a product of prile ideals in wo (1-(-5)(1+(-5)=(6)=(2)(3)

Example: $K_{o}(2155-57) \approx 72 \times 21/2$ 21/2 = 2(1), (2, 1-(5)) Thm: R commetative ring with Krull dimension <1.

$$K_{o}(R) \cong [Spec(R), 2] \otimes Pic(R)$$

(Co(R) -1 [Spec(R), 2]

P -> 2 -> din P® Re/elRa)
Weibel, Corollary 2.6.2
"K-book"