Loday construction in functor categories

Gabe Angelini-Knoll

Wayne State University

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Motivation

Goal:

Describe a computational tool (using some categorical machinery) for computing higher order THH, which is an approximation to iterated algebraic K-theory.

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Algebraic K-theory $K(R)$	iterated algebraic K-theory $K^{(n)}(R)$
vector bundles	<i>n</i> -vector bundles
	$K(2$ -vector bundles $/k) \simeq K(K(k))$
	(Baas-Dundas-Richter-Rognes, Osorno)
E_1 -ring spectra	E_n -ring spectra
	Deligne conjecture for K-theory
	(Blumberg-Gepner-Tabuada, Barwick)
Quillen-Lichtenbaum	Ausoni-Rognes red-shift conjecture
	$K^{(n)}$: ht. m spectra $ ightarrow$ ht. $m+n$ spectra

Approximating iterated algebraic K-theory

For a commutative ring spectrum R, we can approximate iterated K-theory using the iterated trace map

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where $THH(R) = S^1 \otimes R$. Now we observe that

$$S^1 \otimes (S^1 \otimes \dots (S^1 \otimes R)) = T^n \otimes R$$

since $\mathsf{Comm}\mathcal{S}$ has all colimits weighted in sSets (McClure-Staffeldt). We'd therefore like to have tools for computing $T^n \otimes R$ and related invariants.



The Loday construction

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Definition

Define $X_{\bullet} \tilde{\otimes} I$ to be the simplicial object in \mathcal{C} with n-simplices $(X_{\bullet} \tilde{\otimes} I)_n = \bigotimes_{x \in X_n} I\{x\}$ and with face maps

$$d_i: \bigotimes_{x\in X_n} I\{x\} \to \bigotimes_{y\in X_{n-1}} I\{y\}$$

defined by

$$\bigotimes_{x \in X_n} I\{x\} \xrightarrow{\simeq} \bigotimes_{y \in X_{n-1}} \bigotimes_{x \in d_i^{-1}(y)} I\{x\} \longrightarrow \bigotimes_{y \in X_{n-1}} I\{y\}$$

and similarly for degeneracy maps.



Functor categories: symmetric monoidal product

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$$(I \otimes_{\mathit{Day}} J)(c) = \int^{(a,b) \in \mathcal{D} \times \mathcal{D}} \mathcal{D}(a \otimes b,c) \wedge I(a) \wedge I(b)$$

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$$(I \otimes_{Day} J)(c) = \int_{-\infty}^{(a,b) \in \mathcal{D} \times \mathcal{D}} \mathcal{D}(a \otimes b,c) \wedge I(a) \wedge I(b)$$

and on morphisms by induced maps of coends.

Theorem (Day)

The category $(\operatorname{Fun}(\mathcal{D},\mathcal{C}),\otimes_{\operatorname{Day}})$ is a closed symmetric monoidal category and the category of commutative monoids in $\operatorname{Fun}(\mathcal{D},\mathcal{C})$ is equivalent to the subcategory category of lax symmetric monoidal functors $\operatorname{Fun}^{LS}(\mathcal{D},\mathcal{C})$.



Functor categories: model structure

Let $\mathcal C$ be a combinatorial cofibrantly generated symmetric monoidal model category satisfying the SS monoid axiom and let $\mathcal D$ have virtually cofibrant function spaces.

Theorem (Isaacson)

The category $\operatorname{Fun}(\mathcal{D},\mathcal{C})$ with the projective model structure and Day convolution is a symmetric monoidal model category that satisfies the SS monoid axiom.

Following White, we say \mathcal{M} satisfies the strong commutative monoid axiom (SCMA) if for any (acyclic) cofibration h the map $h^{\square n}/\Sigma_n$ is also an (acyclic) cofibration.



Functor categories: model structure

Lemma (A-K)

In addition, let $\mathcal C$ satisfy the SCMA, and let $\mathcal D$ be a POSet enriched in $\mathcal C$ ($\mathcal D(a,b)=1_{\mathcal C}$ or 0). Then the functor category $\operatorname{Fun}(\mathcal D,\mathcal C)$ with the projective model structure satisfies the strong commutative monoid axiom.

White proved that if a model category $\mathcal M$ satisfies (SCMA) then $\mathsf{Comm}\mathcal M$ has the model structure inherited from $\mathcal M$ and cofibrations in $\mathsf{Comm}\mathcal M$ with cofibrant source forget to cofibrations in $\mathcal M$. In particular, cofibrant objects in $\mathsf{Fun}^{LS}(\mathcal D,\mathcal C)$ forget to cofibrant objects in $\mathsf{Fun}(\mathcal D,\mathcal C)$.

Filtered commutative ring spectra

Definition

A filtered commutative ring spectrum is a cofibrant object in $\operatorname{Fun}^{LS}(\mathbb{N}^{\operatorname{op}},\mathcal{S})$ with the model structure inherited from the projective model structure on $\operatorname{Fun}(\mathbb{N}^{\operatorname{op}},\mathcal{S})$

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This is the same data as a sequence of cofibrations

$$\dots I(2) \rightarrow I(1) \rightarrow I(0)$$

between cofibrant objects with structure maps

$$\rho_{i,j}: I(i) \wedge I(j) \to I(i+j)$$

satisfying commutativity, associativity, unitality, and compatibility.



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satisfying commutativity, associativity, unitality, and compatibility. Now, for a simplicial (finite) set X_{\bullet} we can form $X_{\bullet} \otimes I$ where $I \in \operatorname{Fun}^{LS}(\mathbb{N}^{\operatorname{op}},\mathcal{S})$ where \mathcal{S} is the category of symmetric spectra of simplicial sets (with the positive flat stable model structure).

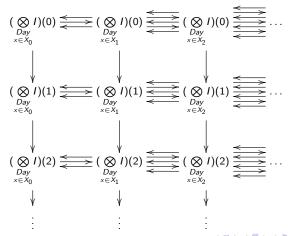
May filtration of the generalized bar construction

Let $I \in \operatorname{Fun}^{LS}(\mathbb{N}^{\operatorname{op}}, \mathcal{S})$, then the May filtration of $X_{\bullet} \otimes I(0)$ is the Loday construction in $\operatorname{Fun}^{LS}(\mathbb{N}^{\operatorname{op}}, \mathcal{S})$;



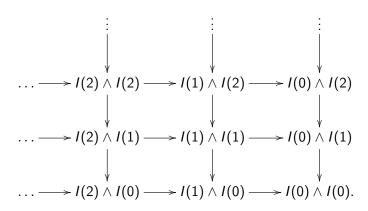
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Example:

The object $(I \otimes_{\text{Day}} I)(n)$ is the colimit of the *n*-th truncation of the diagram



Theorem (A-K, Salch)

If I is a cofibrant object in $\operatorname{Fun}^{LS}(\mathbb{N}^{\operatorname{op}},\mathcal{S})$ then $|X_{\bullet}\otimes I|$ is again a cofibrant object in $\operatorname{Fun}^{LS}(\mathbb{N}^{\operatorname{op}},\mathcal{S})$.

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Given a cofibrant object I in $\operatorname{Fun}^{LS}(\mathbb{N}^{\operatorname{op}},\mathcal{S})$ we can form a commutative ring spectrum E_0I , which is additively $E_0I = \bigvee_{i>0} I(i)/I(i+1)$.

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Theorem (A-K, Salch)

There is a spectral sequence

$$G_{*,*}(X_{\bullet}\otimes E_0I)\Rightarrow G_*(X_{\bullet}\otimes I(0))$$

for any generalized homology theory G.



The main step in constructing this spectral sequence is the proof that the construction of the associated graded E_0 commutes with the functor $X_{\bullet} \otimes -$.

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There is an equivalence in Comm S

$$E_0(X_{\bullet}\otimes I)\simeq X_{\bullet}\otimes E_0I.$$

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Theorem (A-K, Salch)

There is an equivalence in Comm S

$$E_0(X_{\bullet}\otimes I)\simeq X_{\bullet}\otimes E_0I.$$

(The object on the left is the associated graded object in Comm \mathcal{S} of the filtered commutative ring spectrum $X_{\bullet} \otimes I$. A priori, this is the E_1 -page of the THH-May spectral sequence.)



Computation

Let $j = K(\mathbb{F}_q)_p$ where $p \ge 5$ and q is a prime power that topologically generates \mathbb{Z}_p^{\times} .



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Theorem (A-K)

There is an isomorphism of graded rings

$$V(1)_* THH(j) \cong P(\mu_2) \otimes \Gamma(\sigma b) \otimes \mathbb{F}_p \{1, \alpha_1, \lambda_1', \lambda_2 \alpha_1, \lambda_2 \lambda_1' \alpha_1 \lambda_1' \lambda_2 \}.$$

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Proof sketch.

Construct the Whitehead tower $j^{\geq \bullet}$ of j as an object in $\operatorname{Fun}^{LS}(\mathbb{N}^{\operatorname{op}},\mathcal{S})$, then compute

$$V(1)_{*,*}THH(E_0j^{\geq \bullet}) \Rightarrow V(1)_*THH(j).$$



Thank You

Gabe Angelini-Knoll Wayne State University gak@math.wayne.edu