Lecture 1:

The Grothendieck group

I. Motivation

R associative unital ring

$\boxed{K_n(R)}$ is an abelian group for all $n \in \mathbb{Z}$

Examples / Applications:

- $R = \mathbb{Z}[G] \xrightarrow{K_n(\mathbb{Z}[G])}$ $\boxed{\text{geometric topology}}$

  integral group ring

- $R = \mathcal{O}_F \xrightarrow{K_n(\mathcal{O}_F)}$ $\boxed{\text{Number theory}}$

  F number field

  $\mathcal{O}_F$ ring of integers

- $R = k[x,y]/(f(x,y)) \xrightarrow{K_n(k[x,y]/(f(x,y)))}$ $\boxed{\text{Algebraic geometry}}$

  k a field

It is useful

to replace $\boxed{R}$ by

a category of modules

over R $\boxed{P(R)}$ where

$\mathrm{ob}\,P(R) = \left\{ \begin{array}{l} \text{iso. classes of} \\ \text{fin. gen. projective} \\ \quad R\text{-modules} \end{array} \right\}$

$\mathrm{mor}\,P(R) = \{\text{isomorphisms}\}$

and replace $\boxed{K_n(R)}$

with a space $\boxed{K(R)}$

such that

$$\boxed{\pi_n K(R) = K_n(R)}.$$

Classically, though

K_0, K_1, K_2 were

defined $\boxed{\text{purely algebraically.}}$

We will begin by

telling this story; i.e

the story of algebraic

K-theory from

$\boxed{1950 - 1971}$.

II The Grothendieck group

In the late 1950's, Grothendieck defined K_0 to generalize the

Riemann-Roch theorem

to varieties. To do this one needs to not just consider vector spaces, but **virtual vector spaces** for example.

This is formalized using the

$$\boxed{\text{Grothendieck group}}.$$

To define this at the right level of generality, we need the notion of a

$$\boxed{\text{Symmetric monoidal category}}.$$

This abstracts the structure present in $(\mathrm{Ab}, \otimes_{\mathbb{Z}}, \mathbb{Z})$.

Def: A symmetric monoidal category $(\mathcal{C}, \otimes, \mathbb{1})$ consists of

a category $\boxed{\mathcal{C}}$ a functor

$$\boxed{\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}}$$

a unit object $\boxed{\mathbb{1}}$ and natural isomorphisms

1) $a_{-,-,-} : (- \otimes -) \otimes (-) \xrightarrow{\cong} - \otimes (- \otimes -)$

2) $\rho_{-} : (-) \otimes \mathbb{1} \xrightarrow{\cong} (-)$

3) $\lambda_{-} : \mathbb{1} \otimes (-) \xrightarrow{\cong} (-)$

4) $B_{-,=} : (-) \otimes (=) \xrightarrow{\cong} (=) \otimes (-)$

Satisfying several commuting diagrams (See Def 2.1.1)

Example: isomorphism classes of finitely generated projective R-modules & isomorphism

$(P(R), \oplus, 0)$ symmetric monoidal category

$(P(R), \otimes_R, R)$ (symmetric) monoidal category

(R commutative).

__Def__: X CW complex $k = \mathbb{R}, \mathbb{C}$

$VB_k(X)$ $\quad$ $\text{ob}\, VB_k(X)$ k-vector bundles over k

$\text{mor}\, VB_k(X)$ isos

Example: symmetric monoidal

$(VB_k(X), \oplus, 0)$. $(VB_k(X), \otimes, k)$

Whitney sum

tensor product of vector bundles

trivial bundle

Def: Fin

ob Fin: iso classes of finite sets

mor Fin: isomorphisms

Examples: symmetric monoidal

$(\mathrm{Fin}, \amalg, \emptyset)$ $(\mathrm{Fin}, \times, *)$

Def: k field

$\mathrm{Rep}_k(G) = P(k[G])$

Examples: Symmetric monoidal categories

$(\mathrm{Rep}_k(G), \oplus, 0)$ $(\mathrm{Rep}_k(G), \otimes_k, k)$

<u>Def:</u> A commutative monoid in $(\mathcal{C}, \otimes, 1)$ is an object M in $\mathcal{C}$, an operation

$\mu : M \otimes M \longrightarrow M$

and a unit map

$\eta : 1 \longrightarrow M$

satisfying commutative diagrams

1)
$$\begin{tikzcd}
M \otimes M \otimes M \arrow[r, "\mu \otimes id_M"] \arrow[d, "id_M \otimes \mu"'] & M \otimes M \arrow[d, "\mu"] \\
M \otimes M \arrow[r, "\mu"'] & M
\end{tikzcd}$$

2)
$$\begin{tikzcd}
M \arrow[r, "\eta \otimes id_M"] \arrow[dr, equal] & M \otimes M \arrow[d, "\mu"] & M \arrow[l, "id_M \otimes \eta"'] \arrow[dl, equal] \\
& M &
\end{tikzcd}$$

3)
$$\begin{tikzcd}
M \otimes M \arrow[rr, "B_{M,M}"] \arrow[dr, "\mu"'] & & M \otimes M \arrow[dl, "\mu"] \\
& M &
\end{tikzcd}$$

when $(\mathcal{C}, \otimes, 1) = (Set, \times, *)$ (10)

we simply call a (commutative) monoid in $(Set, \times, *)$ a (commutative) monoid.

Ex: (commutative) monoids

- $(P(R), \oplus, 0)$ $(P(R), \otimes, R)$

  Commutative when R is commutative

- $(VB_k(X), \oplus, 0)$ $(VB_k(X), \otimes, k)$

- $(Fin_G, \sqcup, \phi)$ $(Fin_G, \times, *)$

- $(Rep_k(G), \oplus, 0)$ $(Rep_k(G), \otimes, k)$

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Construction: Let $(M, +, 0)$ be a commutative monoid. Then

$$M^{gp} = M \times M / \sim$$

where

$(m_1, n_1) \sim (m_2, n_2)$

where $\quad m_2 = m_1 + p$

$n_2 = n_1 + p$

for some $p \in M$.

" $\dfrac{m_1}{n_1} = \dfrac{m_1 \cdot p}{n_1 \cdot p}$ "

Then M^{gp} is an abelian group.

The construction has a

$$\boxed{\text{Universal property}}$$

written as

$$\begin{tikzcd}
 & M^+ \arrow[d, dotted] \\
M \arrow[ur] \arrow[r, bend right] & A \in \mathrm{Ab}
\end{tikzcd}$$

$\mathrm{CMon} \ni$

or, in other words, there is an $\boxed{\text{adjunction}}$ exhibited by the natural isomorphism

$$\mathrm{Hom}_{\mathrm{CMon}}(M, A) \cong \mathrm{Hom}_{\mathrm{Ab}}(M^+, A).$$

There is another construction that clearly has the same universal property.

Let $F(M)$ be the free abelian group on $[m]$ where $m \in M$ and quotient by the free abelian group on the relations
$[m+n]-[m]-[n]$
denoted $R(M)$

Def: $$M^{gp} = F(M)/R(M).$$

We can now define algebraic K-theory in degree zero.

Def: Let R be an assoc. unital ring

$$K_0^{\oplus}(R) = (P(R), \oplus, 0)^{gp}$$

More generally, let $(\mathcal{C}, \otimes, 1)$ be a small symmetric monoidal category. We may regard it as a commutative monoid in Set

Def:

$$K_0^{\otimes}(\mathcal{C}) = (\mathcal{C}, \otimes, 1)^{gp}$$

Examples:

$$K_0(VB_{\mathbb{C}}(X)) \cong KU^0(X)$$

$$K_0(VB_{\mathbb{R}}(X)) \cong KO^0(X)$$

$$K_0(Fin_G) = A(G)$$ Burnside ring of G

$$K_0(Rep_{\mathbb{C}}(G)) = R(G)$$ representation ring

$$K_0(Rep_{\mathbb{R}}(G)) = RO(G)$$

Exercise: Prove that

$$K_0(\mathbb{Z}) \cong \mathbb{Z}$$

(More generally, when R is a PID or a local ring show $K_0(R) \cong \mathbb{Z}$)

III Applications

① Geometric topology

Let X be a CW complex and let K be a finite CW complex. We say X is **dominated** by K if there is a map

$$\begin{tikzcd} X \arrow[r, "i"'] & K \arrow[l, bend right, "r"'] \end{tikzcd}$$

s.t.

$$i \circ r \simeq \mathrm{id}_X.$$

In other words, X is a retract in hoTop of K.

Example: M a compact topological manifold then

$$M \xrightarrow{\simeq} X \qquad X \text{ cw complex}$$

and

$$f(M) \subseteq X_0 \subseteq X$$

finite cw complex

So M is dominated by a finite cw complex and we can ask whether M is the htpy type of a finite cw complex.

This will be true if M has a $\boxed{\text{triangulation}}$.

Given a ring R we always (18)

have a map

$$K_0(\mathbb{Z}) \to K_0(R)$$
$$\begin{matrix} \text{\rotatebox{90}{$\cong$}} \\ \mathbb{Z} \end{matrix}$$

and when $R = \mathbb{Z}[G]$ for G group or R commutative then this is injective.

<u>Def</u>:

$$\tilde{K}_0(R) = \operatorname{coker}(K_0(\mathbb{Z}) \to K_0(R))$$

Q: If X is dominated by a finite CW complex K, then is X the htpy type of a finite CW complex?

<u>Thm</u> [Wall's finiteness obstruction]

Suppose X is dominated by a finite CW complex K and $G = \pi_1(X)$. Then there is an obstruction class

$$\boxed{w(X) \in \tilde{K}_0(\mathbb{Z}[G])}$$

such that

$w(X) = 0$ if and only if X is homotopy equivalent to a finite CW complex.

Ex: M compact manifold $w(M) = 0$.

② Number theory

Def: A Dedekind domain R

is an integral domain s.t.

for all nontrivial ideals

$$J \subset I \subset R$$

there exists an ideal K

in R such that $IK = J$.

Ex: $\mathcal{O}_F$ ring of integers in a number field.

Def: The ideal class group

of a dedekind domain R

is the quotient

$$Cl(R) = \{I : I \subset R\}/\sim$$

where $I \sim J$ if there exist $x, y \in R$ such that there is an equality

$$xI = yJ$$ of subsets of R.

The group structure is the product of ideals.

<u>Thm</u>: Let R be a Dedekind domain, then there is an isomorphism

$$\widetilde{K}_0(R) = Cl(R).$$

The class group measures the failure of unique prime factorization.

To see that this can fail, consider $\mathbb{Z}[\sqrt{-5}]$ $= \mathcal{O}_{\mathbb{Q}(\sqrt{-5})}$

In this ring, (6) can be written as a product of prime ideals in two ways

$$(1-\sqrt{-5})(1+\sqrt{-5}) = (6) = (2)(3).$$

Example: $K_0(\mathbb{Z}[\sqrt{-5}]) \cong \mathbb{Z} \oplus \mathbb{Z}/2$

$\mathbb{Z}/2 = \{(1), (2, 1-\sqrt{-5})\}$

Thm: R commutative ring with Krull dimension ≤ 1.

$$K_0(R) \cong [\mathrm{Spec}(R), \mathbb{Z}] \oplus \mathrm{Pic}(R)$$

$$K_0(R) \to [\mathrm{Spec}(R), \mathbb{Z}]$$

$$P \longmapsto \mathfrak{q} \longmapsto \dim \frac{P \otimes R_{\mathfrak{q}}/\mathfrak{q}(R_{\mathfrak{q}})}{R_{\mathfrak{q}}/\mathfrak{q}(R_{\mathfrak{q}})}$$

Weibel "K-book" Corollary 2.6.2