Research Statement

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I am an algebraic topologist and broadly my work sheds light on the interplay between algebraic topology and number theory as well as algebraic topology and geometric topology.

One focal point of my research is the study of periodic phenomena in the homotopy groups of spheres. The homotopy groups of spheres encode maps between spheres up to continuous deformation. They are important because nice topological spaces (CW complexes) are built by out of spheres and the homotopy groups of spheres provide the gluing data for building these spaces. There is also a deep connection between the stable homotopy groups of spheres and cobordisms of framed manifolds. In the 1960's, J. F. Adams [2] first showed that Bott periodicity in the homotopy groups of the stable orthogonal group can be used to construct a periodic family of elements in the stable homotopy groups of spheres called the α-family. L. Smith [68] constructed a periodic family of longer wavelength called the β-family and H. Miller, D. Ravenel, and W. S. Wilson [56] generalized these families of elements to periodic families of chromatic complexity n called the n-th Greek letter families. These periodic families have deep connections to arithmetic, for example the α-family encodes special values of the Riemann zeta function by work of J. F. Adams [2] and the β-family encodes certain integral modular forms by work of M. Behrens [13]. In my work, I show that the β family, depending on a prime $p \geq 5$, is nontrivial in the Hurewicz image of iterated algebraic K-theory of integers. This indicates a deeper relationship between iterated algebraic K-theory and modular forms analogous to the relationship between algebraic K-theory of rings of integers in number fields and Dedekind zeta functions.

Another focal point of my research is the study of algebraic K-theory, which I approach computationally using a method known colloquially as trace methods. Classically, algebraic K-theory receives a ring as input and produces a graded abelian group. The algebraic K-theory of rings of integers in number fields encodes deep arithmetic information, the algebraic K-theory of the integral group ring of the fundamental group of a manifold encodes deep geometric information, and the algebraic K-theory of the coordinate ring of a variety encodes deep algebro-geometric information. Modern algebraic K-theory receives as input a category. In particular, from this perpective, it is possible to define algebraic K-theory of generalizations of rings and commutative rings known as E_1 rings and E_∞ rings. The algebraic K-theory of an E_1 ring is notoriously difficult to compute, but since the 1990's an approach known as trace methods, where one computes algebraic K-theory by successive approximations, has been quite successful, for example [35, 10]. In 2018, this approach was simplified even further by T. Nikolaus and P. Scholze [59] and this simpler description of approximations to algebraic K-theory was employed in recent groundbreaking work in the field of integral p-adic Hodge theory by B. Bhatt, M. Morrow, and P. Scholze [15].

These first two focal points are united through my work on a highly regarded family of conjectures known as the Ausoni-Rognes red-shift conjectures [11]. These conjectures generalize the Lichtenbaum-Quillen conjectures [48, 63] in number theory to higher chromatic heights and they suggest a deep connection between periodic phenomena in the homotopy groups of spheres and algebraic K-theory and related invariants. In particular, the philosophy of the red-shift conjecture is that iterating the algebraic K-theory construction increases the chromatic complexity of the periodic information that is detected. These conjectures are also closely tied to the computational approach to algebraic K-theory known as trace methods and often, in practice, the shifts in chromatic complexity are detected in a close enough approximation to algebraic K-theory such as topological cyclic homology (TC), topological periodic cyclic homology (TP), or topological negative cyclic homology (TC^-). Each of the invariants TC, TP, and TC^- are built out of extra structure on an invariant known as topological Hochschild homology (THH), which is a first-order approximation to algebraic K-theory, in a sense made precise using Goodwillie's calculus of functors [29], and it is more directly amenable to computations. In my research, I have developed new tools for computing topological Hochschild homology, and more generally factorization homology. I have also done computations that were not accessible before these tools were available, such as topological Hochschild homology of the image of J spectrum. And I am currently working on extending the state of knowledge of topological Hochschild homology of some of the most fundamental E_1-rings from the chromatic homotopy theory perspective. I have also given new evidence for three different red-shift type conjectures in algebraic K-theory, topological negative cyclic homology, and topological periodic cyclic homology.

The third focal point of my research is equivariant stable homotopy theory. Equivariant homotopy theory has a long rich history and it has recently been put in the spot light because it was a key tool in the resolution of a longstanding problem in manifold theory, called the Arf-Kervaire invariant one problem, by Hill-Hopkins-Ravenel [37]. One of the key tools used in this theory is the development of the Hill-Hopkins-Ravenel (HHR) norm functor N_H^G which is an explicit tensor induction construction. In [37], they defined these for all finite groups G with subgroup H, but for compact Lie groups the theory is more delicate. In Angeltveit et al [8], they give an explicit construction of $N_{\mu_n}^{\mathbb{T}}$ where $\mathbb{T}$ is the circle compact Lie group and μ_n is the subgroup of n-th roots of unity. In particular, when $n = 1$, they prove that this models topological Hochschild homology so that $N_{\mu_1}^{\mathbb{T}}(R) \simeq THH(R)$. In joint work with T. Gerhardt and M. Hill, I define the norm $N_{C_2}^{O(2)}$ where $O(2)$ is the compact Lie group of two by two orthogonal matrices and C_2 is the cyclic group of order 2. I also show that for a genuine commutative C_2 ring spectrum A, there are $\mathcal{R}$-equivalences

$$N_{D_2}^{O(2)}(A) \simeq A \otimes_{D_2} O(2) \simeq THR(A)$$

where $\mathcal{R}$ is the family of subgroups $\mathbb{T}$-free subgroups of $O(2)$; ie the subgroups that intersect with the subgroup $\mathbb{T}$ of $O(2)$ trivially and $THR(A)$ is the Bökstedt contruction of Real topological Hochschild homology, defined in [36].

We also define a theory of Witt vectors for discrete E_σ rings, which extends work of [27] to the non-commutative setting and gives new computational tools. In particular, this new theory of Witt vectors takes as input a discrete E_σ-ring $\underline{M}$ (an E_σ algebra in C_2 Mackey functors) and produces a C_2 Mackey functor $\underline{\mathbb{W}}(\underline{M}; p)$. For those unfamiliar with Mackey functors, a ring with anti-involution produces a discrete E_σ-ring, so our theory gives a new notion of p-typical Witt vectors for rings with anti-involution.

Also, in joint work with M. Merling and M. Péroux, we extend the definition of topological Hochschild homology to other equivariant contexts. For example, we generalize topological Hochschild homology and real topological Hochschild homology to a theory that we call quaternionic topological Hochschild homology (THQ). This new construction comes equipped with a Pin(2)-action and it may be thought of as a new construction of a HHR norm $N_{Q_4}^{\mathrm{Pin}(2)}$ where Q_4 is the cyclic group of order 4 and $Pin(2)$ the compact Lie group also called $Pin_-(2)$ in the literature. There is renewed interest in Pin(2)-equivariant homotopy theory as a careful analysis in Pin(2)-equivariant homotopy theory lead to C. Manolescu's resolution of the triangulation conjecture, stating that there exist non-triangulable n-dimensional topological manifolds for each $n \geq 5$ [52]. In the non-equivariant context, T. Lawson [46] showed that the Hodge-to-de-Rham cohomology spectral sequence of R. Lipshitz and D. Treumann [49] can be generalized using topological Hochschild homology. The longterm goal is to draw analogous connections between THQ and involutive Heegaard-Floer homology of K. Hendricks and C. Manolescu [33].

The first section contains tailored introductions to chromatic homotopy theory, algebraic K-theory, and equivariant homotopy theory. The remaining document is separated into three sections each containing results and future directions: Section 2 describes tools for computing factorization homology and topological Hochschild homology as well as computations, Section 3 describes results giving evidence for red-shift type conjectures for algebraic K-theory, topological negative cyclic homology, and topological periodic cyclic homology, and finally Section 4 describes results and future projects on equivariant topological Hochschild homology, Witt vectors, HHR norms, and applications to geometric topology.

1 Background

1.1 Chromatic homotopy theory

Based on groundbreaking new computations of stable homotopy groups [65] and foundational work of J.F. Adams, S. P. Novikov, D. Quillen, J. Morava, P. Bousfield [1, 60, 61, 58, 20], D. Ravenel presented an extremely clear picture of how to understand periodic phenomena stable homotopy theory in several well regarded conjectures, which circulated in the late 1970's [64]. In the late 1980's, E. Devinatz, M. Hopkins, and J. Smith [24, 41], resolved all but one of them and their results remain indispensible in modern chromatic homotopy theory. For example, they proved that the category of p-local finite spectra $\mathcal{F}_p$ has an unique filtration by thick subcategories

$$0 \subset \cdots \subset \mathcal{C}_2 \subset \mathcal{C}_1 \subset \mathcal{C}_0 = \mathcal{F}_p,$$

up to equivalence, where $\mathcal{C}_n = \{K(n-1)_*\text{-acyclics}\}$. Here $K(n)_*$ is the extraordinary homology theory known as Morava K-theory $K(n)_*$ with coeffients $\mathbb{F}_p[v_n^{\pm 1}]$ when $0 < n < \infty$ and $\mathbb{Q}$ when $n = 0$. This filtration is in fact an artifact of the filtration of the moduli of p-typical formal groups by height via a correspondence going back to work of Quillen [61].

To make this a bit more precise, we recall that by Brown representability, every extraordinary (co)homology theory (satisfying the Eilenberg-Steenrod axioms except for the dimension axiom) E^* is represented by a spectrum E. We therefore use the term (co)homology theory and spectrum interchangeably. Also, throughout, by an E_1 ring and an E_∞ ring we mean an an algebra over the E_1-operad or an algebra over the E_∞-operad in any of the modern models for the symmetric monoidal category for spectra. Therefore, E_1 rings and E_∞ rings correspond to multiplicative (co)homology theories. The Morava K-theory spectra $K(n)$ are E_1 rings and they are additionally complex oriented. We can associate a formal group to any complex oriented cohomology theory and the associated formal group to Morava K-theory $K(n)$ is the Honda height n formal group. We therefore think of spectra in $\mathcal{C}_n$ as having height $\geq n$ in a sense.

More precisely, we say a p-local finite spectrum X has type n if $X \in \mathcal{C}_n - \mathcal{C}_{n+1}$. By the periodicity theorem of [41], a type n spectrum X always has a periodic self map

$$v_n^{i_n} \colon \Sigma^{(2p^n-2)i_n} X \to X$$

for some positive integer i_n. One may form a height $n+1$ spectrum by coning off this self map, which we denote by $X/v_n^{i_n}$. Since the sphere spectrum has height 0, we may construct a type n spectrum as an iterated cofiber $V = S/(p^{i_0}, v_1^{i_1}, \ldots v_{n-1}^{i_{n-1}})$. We say an element in $\alpha \in \pi_* S$ is v_0-periodic if $\alpha \circ p^{k i_0}$ is not null homotopic for all k, other wise we say it is v_0 power torsion. If α is v_0-power torsion, then α factors through S/p^{i_0} and we write $\alpha_1 \colon S/p^{i_0} \to S$ for the associated map. We then say the map is v_1-periodic if the composite $\alpha_1 \circ v_1^{k i_1}$ is not null homotopic for all k and otherwise we say it is v_1-power torsion. Iterating this definition produces a filtration of the p-local homotopy groups of spheres

$$\cdots \subset F_2 \subset F_1 \subset \pi_* S_{(p)} \tag{1}$$

called the chromatic filtration, where $F_n = \{v_{n-1} \text{ power torsion elements}\} \subset \pi_* S$. For example, the divided α family α_{k/i_0} is known to exist for $p \geq 3$ and it is v_0-power torsion, but it is v_1-periodic so it is contained in $F_1 - F_2$. The β family $\beta_{k/i_0, i_1}$ exists for $p \geq 5$ and it is v_0-power torsion and v_1-power torsion, but it is v_2-periodic. The alpha family is deeply connected to the special values of the Riemann zeta function by work of J.F. Adams [2], and the beta family is deeply connected to certain integral Modular forms satisfying certain congruences by work of M. Behrens [13]. When $p \geq 7$, the γ family is also known to exist and it is hoped that this elements are connected to automorphic forms.

One key feature of chromatic homotopy theory is the study of the homotopy groups of a spectrum using its Bousfield localizations, which generalize the localization of a module over a ring. In fact, for any spectrum X and homology theory E, we may form the Bousfield localzation $L_E X$.

This allows us to approximate the homotopy groups of a spectrum. For example, by the chromatic convergence theorem [66, Theorem 7.5.7], the p-local sphere spectrum is equivalent to the limit in the diagram

$$S_{(p)} \simeq \lim_n L_n S \longrightarrow \dots \longrightarrow L_2 S \longrightarrow L_1 S \longrightarrow L_0 S \tag{2}$$

where $L_n S$ is the Bousfield localization of the sphere spectrum at the wedge $K(0) \vee K(1) \vee \cdots \vee K(n)$ of Morava K-theory spectra. Alternatively, one may consider the Bousfield localization at $T(0) \vee \cdots \vee T(n)$ denoted L_n^f. The only remaining conjecture from the Ravenel's conjectures [64], the telescope conjecture, states that these two localizations agree.

The key notion from chromatic homotopy theory that will be important for our work, is the notion of chromatic complexity of a spectrum. Complex oriented p-local spectra E have associated p-typical formal groups, which have an essentially unique filtration by height. For example, p-local complex K-theory $KU_{(p)}$ is associated to the multiplicative formal group of height 1. This gives one notion of chromatic complexity, however since many spectra we consider are not complex oriented, we use the following notions of chromatic complexity instead.

Definition 1.1. We say a spectrum X has *height* $\leq n$ if $L_{K(n)} E = 0$ for all $k > n$. We say a X has *type* $\geq n$ if $L_{K(k)} X = 0$ for $k < n$.

If X has height $\leq n$ and $L_{K(n)} X \neq 0$, then we say X has height n. If X has type $\geq n$, but $L_{K(n)} X \neq 0$, we say X has type n. For example Morava K-theory $K(n)$ has height n, it has type n, and its associated p-typical formal group has height n for all $0 \leq n < \infty$. Complex K-theory KU has height 1, its associated formal group has height 1, but its type is 0. The mod p Moore spectrum S/p has type 1, but height ∞ and it is not complex oriented. Therefore, in some cases, only one of these two notions captures the expected chromatic complexity. When R is an E_∞-ring, then R always has height n for some n and type 0 by [32] and when X is a nontrivial finite p-local spectrum then X always type n for some finite n by [41]. To end, we state three of conjectures that fit into the Ausoni-Rognes red-shift philosphy.

Conjecture 1.2. Let R be an E_1 ring.

1. If R has height n, then $K(R)$ has height $n + 1$.
2. If R has type n, then $L_{K(k)} TP(R) = 0$ for $1 \leq k \leq n$.
3. If the n-th Greek letter family is nontrivial in the Hurwicz image of R, then the $n+1$-st Greek letter family is nontrivial in the Hurewicz image of $TC^-(R)$, and consequently $K(R)$.

Conjecture 1. has also recently been conjectured by M. Land, L. Meier, and G. Tamme [44] and it is a weak form the one stated by J. Rognes in [67]. When R is a (suitably finite) $K(n)$ local spectrum, it is equivalent to the red-shift conjecture as stated in [11]. When $n = 0$, conjecture 1 is equivalent one of the Lichtenbaum-Quillen conjectures [48, 63] relating algebraic K-theory and zeta functions at $p \geq 3$ as first formulated by Waldhausen [69]. In the case $n = 0$, conjecture 3. gives a correspondence between algebraic K-theory and special values of the Riemann Zeta function and in the case $n = 1$ it gives a correspondence between integral modular forms and iterated algebraic K-theory of the the integers. In my work, I give evidence for each of these three red-shift type conjectures.

1.2 Trace methods

In the early 1990's, Bökstedt-Hsiang-Madsen [19] developed a new technique for computing algebraic K-theory, known as trace methods, which relies on tools from algebraic topology. In particular, algebraic K-theory of E_1 rings, which generalize rings, can be approximated by topological Hochschild homology (THH), which may be built in a similar fashion to Hochschild homology (HH) of rings by working over the deeper base of the sphere spectrum. It can also be described as the factorization homology

$$\mathrm{THH}(R) = \int_{\mathbb{T}} R$$

of the circle $\mathbb{T}$ compact Lie group with coefficients in the E_1 ring R and therefore has an action of the group of homeomorphisms of the circle by functoriality. There is a highly nontrivial map called the Bökstedt trace $\mathrm{K}_*(A) \to \mathrm{THH}_*(A)$, which refines the Dennis trace map to Hochschild homology. Moreover, topological Hochschild homology has the structure of a cyclotomic spectrum which includes an action of the circle group $\mathbb{T}$ along with $\mathbb{T}$ equivariant structure maps called the Tate valued Frobenius maps, following [59]. This extra structure allows one to build a further refinement of topological Hochschild homology called topological cyclic homology (TC). For a large class of E_1 rings A, the p-complete algebraic K-theory groups $\mathrm{K}_n(A;\mathbb{Z}_p)$ and the p-complete topological cyclic homology groups $\mathrm{TC}_n(A;\mathbb{Z}_p)$ are isomorphic for $n \geq 0$ by [34, 30].

To simplify exposition, we restrict to bounded below p-complete E_1 rings. Recent work of T. Nikolaus and P. Scholze [59], simplifies the definition of topological cyclic homology of connective ring spectra as the homotopy fiber in the homotopy fiber sequence

$$\mathrm{TC}(A;\mathbb{Z}_p) \longrightarrow \mathrm{TC}^-(A) \xrightarrow{\psi_p - can} \mathrm{TP}(A) \tag{3}$$

where the map *can* comes from the $\mathbb{T}$ equivariant structure and the map ψ_p comes from the cyclotomic structure. More precisely, $\mathrm{TC}^-(A)$, known as topological negative cyclic homology, is the $\mathbb{T}$-homotopy fixed points of $\mathrm{THH}(A)$ and $\mathrm{TP}(A)$, known as topological periodic cyclic homology, is the $\mathbb{T}$-Tate construction of $\mathrm{THH}(A)$. The map can is the canonical map from the homotopy fixed points to the Tate construction. The map ψ_p is then the homotopy $\mathbb{T}$-fixed points of the Tate valued Frobeneus map

$$\psi_p \colon THH(A) \to THH(A)^{tC_p}$$

where we implicitly identify $(THH(A)^{tC_p})^{h\mathbb{T}}$ and $TP(A)$ using our assumption on A via [59, Lemma II.4.2]. The invariants THH, TC^-, TP, and TC have proven to be interesting in their own right as well, by recent work of Bhatt-Morrow-Scholze [15] on integral p-adic Hodge theory.

1.3 Equivariant homotopy theory

Let G be a finite group. It is clear that for any subgroup $H < G$, there is an associated inclusion of fixed points, but one of the key features of stable equivariant homotopy theory is the existence of transfers maps, which are a kind of umkehr or "wrong-way" map. The data of inclusion of fixed points and transfers is encoded in the homotopy groups of a G spectrum by the algebraic structure of a G Mackey functor. When $G = C_2$ the cyclic group of order 2, a C_2 Mackey functor may be described by its Lewis diagram

$$M(C_2/e) \underset{res}{\overset{tr}{\rightleftarrows}} M(C_2/C_2)$$

as well as compatible actions of the Weyl group $W_G(H) = N_{C_2}H/H$ on $M(G/H)$.

The category of Mackey functors Mak_G is equipped with a symmetric monoidal product $\square$ called the box product and the unit is the Burnside Mackey functor $\underline{A}^G$ with $\underline{A}^G(G/H) = A(H)$. Here $A(H)$ is the Burnside ring which, as a group, is the free abelian group on conjugacy classes of subgroups of G. The associative monoids in this category are called associative Green functors and the commutative monoids in this category are called commutative Green functors.

We need additional structure to encode genuine commutativity, which is the data of a multiplicative transfer or norm map, which in the case of C_2 is of the form

$$N \colon M(C_2/e) \to M(C_2/C_2).$$

These genuine commutative monoids in Mackey functors are known as Tambara functors. There is also a genuine version of associative monoids in Mackey functors, known as E_σ algebras in C_2 Mackey functors, which we call discrete E_σ-rings. In particular, these recover the classical notion of associative rings with ant-involution and a recent definition of Hermitian Mackey functor due to Dotto-Ogle [28]. Since these these are foundational to our work, we recall (a less technical version of) the definition here.

Definition 1.3. A discrete E_σ ring is a C_2 Mackey functor $\underline{M}$ such that $\underline{M}(C_2/e)$ is an associative ring, $\underline{M}$ is a $N_e^{C_2}\iota_e^*\underline{M}$ module whose restriction to the trivial subgroup e is the canonical module structure of $M(C_2/e)\otimes M(C_2/e)^{\mathrm{op}}$ on $M(C_2/e)$, and a unit map $\underline{A}^{C_2}\to\underline{M}$.

One of the most important features of equivariant stable homotopy theory, for my work, is the Hill-Hopkins-Ravenel (HHR) norm functor N_H^G where H,G are finite groups and H is a subgroup of G and $[G:H]=n$, defined in [37]. We first define G/H weighted smash product as the composite

$$\wedge^{G/H}\colon \mathsf{Sp}^H \xrightarrow{\wedge^n} \mathsf{Sp}^{\Sigma_n\wr H} \xrightarrow{\lambda_n^*} \mathsf{Sp}^G$$

where λ_n^* is induced by the group homomorphism $G\to\Sigma_n\wr H$, which is equivalent to a choice of ordering on the set of cosets G/H, and Sp^G is the category of G orthogonal spectra indexed on a trivial universe. The HHR norm can then be defined as

$$N_H^G = I_{\mathbb{R}^\infty}^U \circ \wedge^{G/H} \circ I_{\tilde{U}}^{\mathbb{R}^\infty}$$

where the first and last functor in the composite are change of universe functors, which are equivalences of categories in for G orthogonal spectra, and the universe $\tilde{U}$ is the restriction of U to H, also denoted ι_H^*U. We may then define the norm in the category of Mackey functors as

$$N_H^G\underline{M} = \underline{\pi}_0 N_H^G H\underline{M}$$

where $H\underline{M}$ is the Eilenberg-MacLane spectrum associated to the Mackey functor $\underline{M}$.

The HHR norm functor was one of the key tools used by Hill-Hopkins-Ravenel [37] to resolve the Arf-Kervaire invariant one problem. They proved that manifolds with Arf-Kervaire invariant one only exist in dimensions $2^{i+1}-2$ for $i=1,2,3,4,5$ and possibly 6 improving on work of Browder [21] by resolving the cases $2^{i+1}-2$ for $i>6$. A framed manifold of dimension $4k-2$ can be surgically converted into a sphere whenever the Arf-Kervaire invariant is zero, so their work proves that all framed manifolds of dimension $4k-2$, except possibly those in dimension $2,6,14,30,62$ and 126, can be surgically converted into a sphere using framed cobordisms.

2 Results on THH and factorization homology

2.1 The topological Hochschild-May spectral sequence

Multiplicative filtrations of commutative rings are ubiquitous in algebra, but multiplicative filtrations of E_∞ rings have been less accessible. A simple reason is that ideals in ring theory are simple and well understood, but in the setting of E_∞ rings the notion of ideal is more complicated and there isn't an accepted definition. J. Smith suggested a notion of ideal of a ring spectrum, now called Smith ideals and the theory was further developed by Hovey in [42]. In [6], A. Salch and I develop a theory of multiplicatively filtered E_∞ ring that generalize Smith ideals and are a useful notion for generalizing the flavor of filtering by powers of an ideal. A multiplicatively filtered E_∞ ring may be concisely packaged as a commutative monoid in the category of functors from $\mathbb{N}^{\mathrm{op}}$ to spectra. In [6], we construct a large class of examples of such filtrations using the idea of the Whitehead tower from algebraic topology.

Theorem 2.1 (Angelini-Knoll, Salch [6]). There is an explicit model for a multiplicative Whitehead filtration

$$\ldots \to \tau_{\geq 3}R \to \tau_{\geq 2}R \to \tau_{\geq 1}R \to R$$

of a connective E_∞ ring R equipped with structure maps $\rho_{i,j} \colon \tau_{\geq i}R \wedge \tau_{\geq j}R \to \tau_{\geq i+j}R$ satisfying commutativity, associativity, unitality, and compatibility axioms as well as a cofibrancy condition.

The associated graded E_∞ ring associated to this filtration is

$$E_0 \tau_{\geq \bullet} R = H\pi_* R$$

or, in other words, the generalized Eilenberg-MacLane spectrum associated to the graded ring $\pi_* R$. In the special case of connective topological K-theory ku, whose homotopy groups are $\pi_* ku \cong \mathbb{Z}[\beta]$ where β is the Bott element in degree 2, the filtration exactly mimics filtering by powers of the ideal generated by β in algebra.

In [6, Thm. 3.4.8], we then prove that there is a spectral sequence in topological Hochschild homology associated to a general filtered E_∞ ring. Our spectral sequence was motivated by May's spectral sequence from [53] where he uses a filtration of a Hopf algebra by powers of the augmentation ideal to filter the bar construction and produce a spectral sequence computing the cohomology of that Hopf algebra. In particular, we prove that there is a May-type spectral sequence of the form

$$E^1_{*,*} = G_* \left(\mathrm{THH}(H\pi_* R)\right) \Rightarrow G_* \left(\mathrm{THH}(H\pi_* R)\right).$$

for any connective homology theory G and any connective E_∞ ring R, where the second grading on the input is the one coming the May filtration.

In fact, our theorem is much more general. By [54], when R is a E_∞ ring $\mathrm{THH}(R)$ may be constructed as the tensoring $S^1_\bullet \otimes R$ of R with a simplicial model for $S^1_\bullet$ in the category of E_∞ rings. Our spectral sequence also applies to tensoring with any simplicial finite set $X_\bullet$ producing a spectral sequence with signature

$$E^1_{*,*} = G_*(X_\bullet \otimes E_0^* I_\bullet) \Rightarrow G_*(X_\bullet \otimes I_0)$$

for any connective homology theory G and any multiplicatively filtered E_∞ ring $I_\bullet$, where the second grading on the input is the May filtration grading.

When M is a framed manifold and R is a E_∞ ring there is an equivalence

$$\mathrm{sing}_\bullet(U(M)) \otimes R \simeq \int_M R$$

by [12, Prop. 5.1] where $\mathrm{sing}_\bullet(U(M))$ is the singular simplicial set of the underlying topological space of M and $\int_M R$ is factorization homology. Our main theorem may then be described as follows.

Theorem 2.2 (Angelini-Knoll, Salch [6]). There is a May-type spectral sequence for factorization homology with signature

$$E^1_{*,*} = G_* \left(\int_M E_0^* I \right) \Rightarrow G_* \left(\int_M I_0 \right)$$

for any multiplicatively filtered E_∞ ring I, framed manifold M, and connective homology theory G.

2.2 Topological Hochschild homology of the $K(1)$-local sphere

Just as the the homotopy groups of spheres are fundamental to algebraic topology, the algebraic K-theory of the sphere spectrum, regarded as an E_∞ ring, is fundamental to algebraic K-theory. In particular, the algebraic K-theory space $\Omega^\infty K(S)$ contains $Q(S^0)$ as a retract as well as the the Whitehead space of a point $\mathrm{Wh}^{\mathrm{diff}}(*)$. In general, $\Omega \mathrm{Wh}^{\mathrm{diff}}(M)$ encodes information about stable h-cobordisms, so understanding the case $M = *$ is of fundamental importance and more generally $\mathrm{Wh}^{\mathrm{diff}}(M)$ splits off of $K(S[\Omega M])$ which is a $K(S)$-module.

In 1982, Waldhausen suggested a program for computing $K(S)$ by successively computing the tower

$$\begin{tikzcd}
K(S_{(p)}) \arrow[r] & \dots \arrow[r] & K(\tau_{\geq 0}L_n^fS) \arrow[r] & \dots \arrow[r] & K(\tau_{\geq 0}L_2^fS) \arrow[r] & K(\tau_{\geq 0}L_1^fS) \arrow[r] & K(\tau_{\geq 0}L_0^fS) \\
& & K(M_n^fS) \arrow[u] & & K(M_2^fS) \arrow[u] & K(M_1^fS) \arrow[u] & K(M_0^fS) \arrow[u, "="']
\end{tikzcd}$$

where $\tau_{\geq 0}$ is the connective cover functor, M_n^fS is the category of finite cell $\tau_{\geq 0}L_n^fS$-modules which are $\tau_{\geq 0}L_{n-1}^fS$-acyclic, and L_n^f is the Bousfield localization at the spectra $T(0) \vee \cdots \vee T(n)$ as defined in Section 1.1. One may also consider the analogous tower

$$K(S_{(p)}) \longrightarrow \dots \longrightarrow K(\tau_{\geq 0}L_nS) \dots \longrightarrow K(\tau_{\geq 0}L_2S) \longrightarrow K(\tau_{\geq 0}L_1S) \longrightarrow K(\tau_{\geq 0}L_0S)$$

and the two agree for $n = 0, 1$. Optimistically, at the time it was believed that these two towers agreed in general and McClure-Schwanzl [55] proved that for the second tower there is an equivalence

$$K(S_{(p)}) \simeq \lim K(\tau_{\geq 0}L_nS)$$

in the same spirit of the chromatic convergence theorem from Section 1.1 though it does not follow directly from chromatic convergence.

It is therefore highly disirable to compute $K(\tau_{\geq 0}L_nS)$ and when $n = 1$, this agrees with $K(\tau_{\geq 0}L_1^fS)$. The p-completion of $(\tau_{\geq 0}L_1S)_p$ is $\tau_{\geq 0}L_{K(1)}S$, or in other words the p-complete connective image of J spectrum j, and we need to p-complete for this computation to be amenable to the trace methods approach. We therefore proceed with computing successive approximations to algebraic K-theory of j.

For the red-shift program of Ausoni-Rognes, it is most useful to compute $S/(p, v_1) \wedge K(j)$ when $p \geq 5$. In particular, if it is a finitely generated $P(v_2)$-module, then $S/(p, v_1, v_2) \wedge K(j)$ is a finite spectrum and this implies $K(j)$ is $T(3)$ acyclic. Consequently, $L_{K(k)}K(j) = 0$ for all $k \geq 3$ by a theorem of Hahn [32]. The long term goal is therefore to answer the question.

Question 2.3. Is $S/(p, v_1) \wedge K(j)$ a finitely generated $P(v_2)$-module?

Rognes originally conjectured that $S/(p, v_1) \wedge K(j)$ is actually a finitely generated free $P(v_2)$-module in a talk from 2000 at Oberwolfach [67] predating the red-shift conjecture as it appears in print in 2008 [11]. I therefore computed $S/(p, v_1) \wedge THH(j)$ as the modulo (p, v_1) linear approximation to $S/(p, v_1) \wedge K(j)$ in the sense of Goodwillie's Calculus of functors [**?**]. My work uses the spectral sequence of Theorem 2.2 in a key way and it would not have been possible without developing this additional tool or some analogous one.

Theorem 2.4 (Angelini-Knoll [3])**.** There is an isomorphism of graded rings

$$V(1)_* \operatorname{THH}(j) \cong P(\mu) \otimes \Gamma(\sigma b) \otimes H_*(E(\alpha_1, \lambda_1', \lambda_2); d(\lambda_2) = \alpha_1\lambda_1').$$

for $p \geq 5$.

Here the notation $H_*(M; d)$ means the homology of the differential graded algebra M modulo the differential d. In parallel, Eva Höning has given a different computation of $V(1)_* \operatorname{THH}(j)$ using the Brun spectral sequence. Currently, in joint work in progress with Eva Höning we are developing tools for computing $V(1)_* \operatorname{TC}^-(j)$ and $V(1)_* \operatorname{TP}(j)$ in order to compute $V(1)_*TC(j)$ using the Nikolaus-Scholze equalizer and consequently $V(1)_*K(j)$ resolving Question 2.3.

2.3 THH of truncated Brown-Peterson spectra

The E_1-ring BP, known as the Brown-Peterson spectrum, is one of the most fundamental objects in chromatic homotopy theory as it exhibits connections between periodicity in the homotopy groups of spheres and p-typical formal groups. The cofficients of BP are a polynomial algebra over $\mathbb{Z}_{(p)}$ on generators v_n for $n \geq 1$ which correspond to p-typical formal groups of height n. By coning off generators v_m for $m > n$, we can form an E_1 ring $BP\langle n\rangle$ whose homotopy groups are the symmetric algebra on generators $v_1, \dots, v_n$ over $\mathbb{Z}_{(p)}$. Note that there is a prime p hidden in the notation of BP and $BP\langle n\rangle$.

For small n, the spectra $BP\langle n\rangle$ are well known. When $n = -1$, the spectrum $BP\langle -1\rangle$ is the Eilenberg-MacLane spectrum $H\mathbb{F}_p$, or in other words the commutative ring $\mathbb{F}_p$. When $n = 0$, $BP\langle 0\rangle$ is the Eilenberg-MacLane spectrum $H\mathbb{Z}_{(p)}$ and when $n = 1$, $BP\langle 1\rangle$ is the Adams summand ℓ, which is a retract of p-local connective complex topological K-theory ku. Until recently, this exhausted the list of examples $BP\langle n\rangle$ that were known to be E_∞-rings, but recently it was shown that $BP\langle 2\rangle$ has a model as an E_∞-ring by Lawson-Naumann [47] at $p = 2$ and Hill-Lawson [38] at $p = 3$ using the theory of topological modular forms and topological automorphic forms respectively.

Calculations of topological Hochschild homology $BP\langle n\rangle$ with coefficients in $BP\langle k\rangle$ for $-1 \leq k \leq n$ have been fundamental to the subject since its inception. When Bökstedt defined topological Hochschild homology [18, 17], the first calculations he did were the computations of $THH_*(BP\langle n\rangle)$ for $n = -1, 0$. In particular, Bökstedt's computation

$$THH_*(BP\langle -1\rangle) = P(\mu_0),$$

where $|\mu_0| = 2$ can be used to reprove one of the most fundamental results in algebraic topology, Bott periodicity [45]. More generally, it is known that topological Hochschild homology of $BP\langle n\rangle$ with coefficients in $H\mathbb{F}_p$ is isomorphic to

$$\mathrm{THH}_*(BP\langle n\rangle; H\mathbb{F}_p) \cong E(\lambda_1, \lambda_2, \lambda_3, \dots \lambda_n) \otimes P(\mu_{n+1})$$

where the generator μ_{n+1} conjecturally contributes higher chromatic height analogues of Bott periodicity in algebraic K-theory of $BP\langle n\rangle$.

A major achievement in the study of algebraic K-theory of ring spectra is Ausoni and Rognes's calculation of algebraic K-theory of $BP\langle 1\rangle$, which begins with the calculation of $THH_*(BP\langle 1\rangle; H\mathbb{F}_p)$. This computation gave significant evidence for the Ausoni-Rognes red-shift conjecture that algebraic K-theory increases chromatic complexity, that we will discuss in more detail in the next section.

In joint work in progress with D. Culver and Eva Höning, we compute $THH_*(BP\langle 2\rangle)$. As n increases calculations of $THH_*(BP\langle n\rangle)$ become significantly more complex. The way to make progress towards $THH_*(BP\langle n\rangle)$ is therefore to first consider the calculation $THH_*(BP\langle n\rangle; H\mathbb{F}_p)$ and then work towards more complicated cofficients. The standard technique for doing this is to use the Bockstein spectral sequences, such as

$$E^1_{*,*} = THH_*(BP\langle n\rangle; H\mathbb{F}_p)[v_1] \Longrightarrow THH_*(BP\langle n\rangle; k(1))$$

where $k(1)$ is the connective cover of the Morava K-theory spectrum $K(1)$. In joint work, with Dominic Culver and Eva Höning, we have computed the first three Bockstein spectral sequences.

Theorem 2.5 (A-K-Culver-Höning)**.** There is an isomorphism of gradad $\mathbb{Z}_{(p)}$-modules

$$THH_*(BP\langle 2\rangle; H\mathbb{Z}_{(p)}) \cong \otimes E_{\mathbb{Z}_{(p)}}(\lambda_1, \lambda_2) \oplus \mathbb{Z}_{(p)}\{\lambda_1^{\epsilon}\lambda_2^{\epsilon'}c_i : i \geq 1\}/\sim$$

where $p^{\nu_p(i)+1}\lambda_1^{\epsilon}\lambda_2^{\epsilon'}c_i \sim 0$ for $i \geq 1$ and $k = 1, 2$. There is an isomorphism of graded $\mathbb{F}_p$-vector spaces

$$THH_*(BP\langle 2\rangle; k(1)) \cong \left(P(v_1) \otimes \left(E(\lambda_1) \oplus \mathbb{F}_p\{\lambda_1^{\epsilon_1}z_{i,j}, \lambda_1^{\epsilon_2}z'_{i,j}, \lambda_1^{\epsilon_3}z''_{i,j}\}\right)\right) / \sim$$

where $v_1^{\nu_p(i)+1}\lambda_1^{\epsilon_1}z_{i,j} \sim v_1^{\nu_p(i)+1}\lambda_1^{\epsilon_2}z'_{i,j} \sim v_1^{\nu_p(i)+1}\lambda_1^{\epsilon_3}z''_{i,j} \sim 0$ and $\epsilon_1, \epsilon_2, \epsilon_3 \in \{0,1\}$. There is an isomorphism of graded $\mathbb{F}_p$-vector spaces

$$THH_*(BP\langle 2\rangle; k(2)) \cong E(\lambda_2) \oplus \mathbb{F}_p\{\lambda_2^{\epsilon_1}y_{i,j}, \lambda_2^{\epsilon_2}y'_{i,j}, \lambda_2^{\epsilon_3}y''_{i,j}\}/\sim$$

where $v_2^{\nu_p(i)+1}\lambda_2^{\epsilon_1}y_{i,j} \sim v_2^{\nu_p(i)+1}\lambda_2^{\epsilon_2}y'_{i,j} \sim v_2^{\nu_p(i)+1}\lambda_2^{\epsilon_3}y''_{i,j} \sim 0$ for $\epsilon_1, \epsilon_2, \epsilon_3 \in \{0,1\}$.

We have also employed the topological Hochschild-May spectral sequence as an additional tool for computing the diagonal in the square of spectral sequences

$$\begin{tikzcd}
THH_*(BP\langle 2\rangle; H\mathbb{F}_p)[v_0, v_1] \arrow[r, Rightarrow] \arrow[d, Rightarrow] \arrow[dr, Rightarrow, "HMSS"] & THH_*(BP\langle 2\rangle; k(1))[v_0] \arrow[d, Rightarrow] \\
THH_*(BP\langle 2\rangle; H\mathbb{Z}_{(p)})[v_1] \arrow[r, Rightarrow] & THH_*(BP\langle 2\langle; BP\langle 1\rangle).
\end{tikzcd}$$

The idea is not to replace the other Bockstein spectral sequences with the topological Hochschild May spectral sequence, but rather to compare all three ways of computing the bottom right corner. We have already made significant progress towards this goal.

For computing $THH_*(BP\langle 2\rangle; BP\langle 2\rangle/p)$ we have similar approach. In fact, we may also use the Brun spectral sequence [40]. We plan to use the Brun spectral sequence to compute the diagonal in the square and compare to the other Bockstein spectral sequences

$$\begin{tikzcd}
THH_*(BP\langle 2\rangle; H\mathbb{F}_p)[v_1, v_2] \arrow[r, Rightarrow] \arrow[d, Rightarrow] \arrow[dr, Rightarrow, "BSS"] & THH_*(BP\langle 2\rangle; k(1))[v_2] \arrow[d, Rightarrow] \\
THH_*(BP\langle 2\rangle; k(2))[v_1] \arrow[r, Rightarrow] & THH_*(BP\langle 2\langle; BP\langle 2\rangle/p)
\end{tikzcd}$$

Finally, to compute $THH_*(BP\langle 2\rangle)$ we plan to construct a coarser filtration of $BP\langle 2\rangle$ as a multiplicative filtered spectrum, which is additively given by the filtration

$$\cdots \to \tau_{\geq 4p^2-4}BP\langle 2\rangle \to \tau_{\geq 2p^2-2}BP\langle 2\rangle \to BP\langle 2\rangle.$$

The existence of such a multiplicatively filtered E_∞ ring spectrum is of independent interest and it would vastly simplify the computation of $THH_*(BP\langle 2\rangle)$.

3 Results on red-shift phenomena in algebraic K-theory

3.1 Iterated algebraic K-theory of the integers and integral modular forms

In the 1950's, J. F. Adams computed the image of the J homomorphism from the homotopy groups of the infinite orthogonal group to the stable homotopy groups of spheres [1]. In this celebrated work, he in particular demonstrated a direct connection between special values of the Riemann zeta function and the orders of the divided alpha family, a v_1-periodic family of elements in the image of the J homomorphism, whose periodicity is an artifact of Bott periodicity. In the 1970's, D. Quillen defined algebraic K-theory and presented the first complete calculation of algebraic K-theory, the algebraic K-theory of finite fields [62]. He also showed that the divided α-family is nontrivial in the Hurewicz image of algebraic K-theory of finite fields and consequently the algebraic K-theory of the integers for $p \geq 3$. This gave some of the first evidence for the conjectures of S. Lichtenbaum [48] and D. Quillen [63], now known as the Lichtenbaum-Quillen conjectures.

In my work, I give the first higher chromatic height analogue of this theorem. The β-family is a v_2-periodic family, which was first constructed and proven to be nontrivial by L. Smith [68]. The

β-family was proven to have a tight connection to certain integral modular forms satisfying certain congruence relations by M. Behrens [13]. I prove that the β-family, at $p \geq 5$, is detected in the iterated algebraic K-theory of finite fields of order q, where q is a topological generator of $\mathbb{Z}_p^\times$. Here we say a family of elements in the homotopy groups is detected in the homotopy groups of an E_1 ring R if this family of elements has nontrivial Hurewicz image

$$\pi_* S \to \pi_* R.$$

Recall that the β-family depends on a chosen prime p. I prove a higher chromatic height and an iterated algebraic K-theory analogue of the result J.F. Adams [1] and D. Quillen [62] on the relationship between algebraic K-theory of the integers and special values of the Riemann zeta function.

Theorem 3.1 (Angelini-Knoll [5])**.** The β-family is detected in iterated algebraic K-theory of of the integers $K(K(\mathbb{Z}))$ when $p \geq 5$.

This indicates, more generally, that there should be a dictionary between iterated algebraic K-theory of rings of integers O_F in number fields F and modular forms over rings of integers in number fields O_F satisfying certain congruence relations.

This also gives evidence for a version of the red-shift conjecture, which I call the Greek-letter family red-shift conjecture. Since there is speculation that the Greek letter family elements $\alpha_k^{(n)}$ are related to automorphic forms and special values of L functions of degree n, we view this as a higher chromatic height analogue of Lichtenbaums conjectures [48] on the relationship between algebraic K-theory and special values of Dedekind zeta functions.

Conjecture 3.2 (Greek letter family red-shift conjecture)**.** Let R be an E_1-ring. If the n-th Greek letter family is nontrivial in the Hurewicz image of R, then the $n+1$-st Greek letter family is nontrivial in the Hurewicz image of $\mathrm{K}(R)$.

In the case $n = 0$, we call p^k the 0-th Greek letter family. With this convention, the 0-th Greek letter family is nontrivial in $R = \mathbb{Z}$ for all k and the α-family is detected in $\mathrm{K}(\mathbb{Z})$, giving evidence of the conjecture. Theorem 3.1 gives evidence for the conjecture when $n = 1$.

3.2 Morava K-theory of algebraic K-theory

The study of Morava K-theory of algebraic K-theory and related invariants has been of longstanding interest. For example, S. Mitchell [57] proved that

$$L_{K(m)} K(\mathbb{Z}) = 0$$

for $m \geq 2$, which implies the same result for any $H\mathbb{Z}$-algebra and any scheme. This implies that $K(\mathbb{Z})$ has height ≤ 1 in the sense of Definition 1.1. Also, work of C. Ausoni and J. Rognes [10] implies that

$$L_{K(m)} K(\ell_p) = 0$$

for $m \geq 3$ where ℓ_p is a retract of p complete topological K-theory.

More recently, there has been renewed interest in Morava K-theory of algebraic K-theory and new advances in the subject by work of Clausen-Mathew-Naumann-Noell [23], Land-Meier-Tamme [44], and Bhatt-Clausen-Matthew [14]. The techniques we employ are completely independent of these other recent results and therefore they expand the toolset for approaching such questions.

Our main computational result is a proof that $K(BP\langle n\rangle)$ has height $\leq n+1$ for all n, p such that $BP\langle n\rangle$ is an E_∞-ring.

Theorem 3.3 (Angelini-Knoll, Salch [7])**.** Let $n \geq -1$ be an integer and $p > 2$ a prime such that the p-complete of $BP\langle 2\rangle$ has a model as an E_∞-ring, then

$$L_{K(m)} K\left(BP\langle n\rangle_p\right) = L_{T(m)} K\left(BP\langle n\rangle_p\right) = 0$$

for $m \geq n+2$ and consequently,

$$L_{T(m)}K\left(R\right) = 0$$

for $m \geq n+2$ for any $BP\langle n\rangle_p$-algebra R, for example $R = K(2)$ at $p = 3$ and $n = 2$.

When $n = -1, 0, 1$, this result was already known, but when $n = 2$ and $p = 3$ this result is completely new and it is one of the first red-shift type results at this chromatic height. We also remark, that it is left open whether $L_{K(n+1)}K(BP\langle n\rangle_p) \neq 0$. In the case $n = 2$ and $p = 3$, it is the aim of my joint work D. Culver and E. Höning to prove that $L_{K(3)}K(BP\langle 2\rangle_p) \neq 0$, see Section 2.3.

The proof of Theorem 3.3, is one of the main applications of technical result, which is of independent interest. We show that certain sequences of spectra, that are not necessarily uniformly bounded below, commute with smashing with Morava K-theory. In the following result, A_* denotes the dual Steenrod algebra and $H(M; Q_m)$ denotes the Margolis homology of a $E(Q_m)$-module M where Q_m is the m-th Milnor primitive in the Steenrod algebra A. We also write $H_*(X)$ for homology of X with coefficients in $\mathbb{F}_p$.

Theorem 3.4 (Angelini-Knoll, Salch [7])**.** Suppose that

$$\cdots \to T_i \to T_{i+1} \to \ldots$$

is sequence of spectra such that

$$\operatorname*{colim}_{i\to\infty} T_i = 0 \text{ and } \lim_{i\to-\infty} T_i = T.$$

Suppose that each T_i is bounded below, $H\mathbb{F}_p$-nilpotent complete, π_*T_i is a finite type $\mathbb{Z}_p$-module for each i, and the maximal degree of an A_*-comodule primitive in $H_*(T_i)$ is $M-1$, where M does not depend on i. Additionally, assume that

$$\lim_{i\to-\infty} H(H_*(T_i); Q_m) = 0.$$

Then, the $K(m)$-localization of T vanishes

$$L_{K(m)}T = 0.$$

In joint work with J.D. Quigley, I also apply this result to prove Morava K-theory vanishing results for topological periodic cyclic homology and algebraic K-theory, see Section 3.3.

3.3 Morava K-theory of topological periodic cyclic homology

For each prime p, there is a sequence of Thom spectra that can be constructed using the Thom construction associated to the filtration of spaces

$$* \to \Omega J_{p-1}(S^2) \to \Omega J_{p^2-1}(S^2) \to \ldots \to \Omega J(S^2)$$

lying over the classifying space of stable p-local spherical bundles $BGL_1S_{(p)}$, where $J(S^2)$ is the James construction. Mahowald [50], proved the existence of the associated the family of spectra

$$S \to y(1) \to y(2) \to \ldots \to y(\infty) = H\mathbb{F}_p$$

when $p = 2$ and this was extended to odd primes by Mahowald-Ravenel-Shick [51]. These spectra have type n in the sense that $L_{n-1}y(n) = 0$, but $L_{K(n)}y(n) \neq 0$. In joint work with J.D. Quigley [4], we show that the vanishing range of Morava K-theory of topological periodic cyclic homology of $y(n)$ increases strictly by 1.

Theorem 3.5 (Angelini-Knoll,Quigley [4])**.** There are isomorphisms

$$L_{K(m)}\,\mathrm{TP}(y(n)) = 0$$

for $1 \leq m \leq n+1$.

This demonstrates that shifts in chromatic complexity occur in topological periodic cyclic homology even before passing to topological cyclic homology or algebraic K-theory. I also emphasize that Tate red-shift result demonstrates a chromatic height shift at all chromatic heights and there are basically no other red-shift type results of this nature already in the literature.

We also show that relative algebraic K-theory at least preserves chromatic complexity.

Theorem 3.6 (Angelini-Knoll,Quigley [4])**.** There are isomorphisms

$$L_{K(n)}K(y(n), H\mathbb{F}_p) = 0$$

for $0 \leq m \leq n$ where $K(y(n), H\mathbb{F}_p) = \mathrm{fib}\left(K(y(n)) \to K(H\mathbb{F}_p)\right)$.

The result above has also been proven recently by Land-Meier-Tamme [44] in parallel, but they do not use trace methods, so they do not recover our TP result, and our methods are completely independent. In an early draft of our paper, we had a proof that $L_{K(n)}K(y(n)) = 0$, but this proof contained a flaw, which we quickly caught. In fact, we now believe this bug indicates that $L_{K(n)}K(y(n)) \neq 0$ and in fact TP is the correct invariant to study for this kind of red-shift type result.

4 Equivariant topological Hochschild homology

4.1 Real topological Hochschild homology, Witt vectors, and Norms

Real algebraic K-theory, due to L. Hesselholt and I. Madsen [36] and recently extended by B. Calmes et al [22], receives a ring with anti-involution and produces a graded C_2 Mackey functor, which is the C_2 equivariant analogue of an abelian group where C_2 is the cyclic group of order 2. More generally, the input can be an E_σ ring (an algebra in C_2 spectra over the equivariant little disc operad E_σ). Real algebraic K-theory has its roots in M. Karoubi and O. Villamayor's Hermitian K-theory [43] and M. Atiyah's Real topological K-theory [9]. There is also a tight connection between Real algebraic K-theory and L-theory, which has applications to surgery theory of manifolds, see [22] for a modern account of this.

Real algebraic K-theory of E_σ rings is even more difficult to compute than algebraic K-theory, so it is desireable to have new tools. One recent approach is the development of trace methods in this setting. There has been significant progress towards this goal by E. Dotto, A. Høgenhaven, K. Moi, I. Patchkoria, and S. Precht-Reeh [26, 39, 25], building on [36], but there are still unanswered questions. In particular, there are Real analogues of all of the classical invariants, such as Real topological Hochschild homology (THR).

Non-equivariantly the first instance of red-shift behavior appears in the shift from characteristic p in $\mathbb{F}_p$ to characteristic zero in D. Quillen's computation of the algebraic K-theory of a finite field $K(\mathbb{F}_p)_p \simeq HW(\mathbb{F}_p)$ where $W(\mathbb{F}_p)$ is the p-typical Witt vectors and H is the Eilenberg-MacLane spectrum functor [62]. One of my goals is to explore height shifting phenomena in this new equivariant setting. As a first step in this direction, in joint work with T. Gerhardt and M. Hill, we give a description of Witt vectors for discrete E_σ rings. This extends previous work of E. Dotto, K. Moi, and I. Patchkoria [27] to the non-commutative setting. Before describing our construction, we briefly recall the non-equivariant analogue of this story.

L. Hesselholt and I. Madsen [34] proved that there is a deep connection between the topological Hochschild homology of a commutative ring R and the Witt vectors of a commutative ring R. Recall that $\mathrm{THH}(R)$ is equipped with an action of the circle group $\mathbb{T}$ and a μ_{p^n} action by restriction, where μ_{p^n} are the p^n-th roots of unity inside of $\mathbb{T} \subset \mathbb{C}^\times$. In [34], Hesselholt-Madsen proved that there is an isomorphism

$$\mathrm{W}_{n+1}(A) \cong \underline{\pi}_0^{C_{p^n}} \mathrm{THH}(A)(C_{p^n}/C_{p^n})$$

where $\mathrm{W}_{n+1}(A)$ is the length $n+1$ p-typical Witt vectors of A. Recently, A. Blumberg, T. Gerhardt, M. Hill, and T. Lawson [16] generalize this to a theory of Witt vectors for associative Green functors. In particular, they construct a notion of Hochschild homology of Green functors $\underline{\mathrm{HH}}_*^G(M)$ where M is an H-Mackey functor, H and G are finite subgroups of $\mathbb{T}$, and H is a subgroup of G. In previous work [?], the authors along with Angeltveit and Mandell constructed a spectrum $\mathrm{THH}_H(A) = N_H^{\mathbb{T}}(A)$ for an H-ring spectrum using a twisted version of the cyclic bar construction. In [16], they then show that there is an isomorphism

$$\underline{\mathrm{HH}}_0^G(\underline{\pi}_0^H A) \cong \underline{\pi}_0^G \mathrm{THH}_H(A)$$

where G is a finite subgroup of $\mathbb{T}$ and H is a subgroup of G. This gives a description of topological Hochschild homology of an H-ring spectrum A in terms of an equivariant Hochschild homology construction, which they define to be the Witt vectors for Green functors. This is motivated by the fact that in the classical setting there is an isomorphism $\pi_0 \mathrm{THH}(A) \cong \mathrm{HH}_0(A)$.

In my work with T. Gerhardt and M. Hill, I define a new algebraic construction called Real Hochschild homology of E_σ algebras in Mackey functors. In the following definition, we use the definition of the Hill-Hopkins-Ravenel (HHR) norm in the Mackey functors from Section 1.3. By a discrete E_σ ring, we mean an E_σ algebra in C_2-Mackey functors. Throughout, we fix a splitting of the extension

$$1 \to \mathbb{T} \to O(2) \to C_2 \to 1$$

compatible with the splittings of the extensions

$$1 \to \mu_m \to O(2) \to C_2 \to 1$$

and we fix m odd to simplify exposition.

Definition 4.1. Let m be an odd integer. The Real Hochschild homology of a discrete E_σ ring $\underline{M}$, is defined as

$$\underline{HR}_*^{D_{2m}}(\underline{M}) = H_*(B_\bullet(N_{C_2}^{D_{2m}}\underline{M}, N_e^{D_{2m}}\iota_e^*\underline{M}, {}^{\gamma_m}N_{C_2}^{D_{2m}}\underline{M}))$$

where $B_\bullet(N_{C_2}^{D_{2m}}\underline{M}, N_e^{D_{2m}}\iota_e^*\underline{M}, {}^{\gamma_m}N_{C_2}^{D_{2m}}\underline{M})$ is the two sided bar construction and ${}^{\gamma_m}N_{C_2}^{D_{2m}}\underline{M}$ is the same as $N_{C_2}^{D_{2m}}\underline{M}$ as a Mackey functor, but it has twisted left $N_e^{D_{2m}}\iota_e^*\underline{M}$ action

$$N_e^{D_{2m}}\iota_e^*\underline{M} \wedge N_{C_2}^{D_{2m}}\underline{M} \xrightarrow{\gamma_m \wedge 1} N_e^{D_{2m}}\iota_e^*\underline{M} \wedge N_{C_2}^{D_{2m}}\underline{M} \to N_{C_2}^{D_{2m}}\underline{M}$$

where $\gamma_m = e^{(m-1)\pi i/m} \in \mu_m < D_{2m}$. Using this definition, we define a construction of p-typical Witt vectors for E_σ algebras in C_2 Mackey functors taking values in C_2-Mackey functors as the limit

$$\underline{W}(\underline{M}; p) = \lim_R \underline{HR}_0^{D_{2p^k}}(\underline{M}) \tag{4}$$

where R is an algebraic analogue of the Real restriction map, which comes from an algebraic Real cyclotomic structure on $\underline{HR}_*^{D_{2m}}(\underline{M})$, that we define.

. In addition to the connection between Witt vectors and Real Hochschild homology, our construction also has interest from the point of view of construction of HHR norms for compact Lie groups. We point the reader to Section 1.3 for a more detailed account. In particular, we give a new construction of the HHR norm $N_{C_2}^{O(2)}$ as

$$N_{C_2}^{O(2)}(-) = I_{\mathbb{R}^\infty}^{U} |B_\bullet^{\mathrm{di}}(I_{\tilde{U}}^{\mathbb{R}^\infty}(-))| \colon \mathsf{Sp}_{\tilde{U}}^{C_2} \to \mathsf{Sp}_{\tilde{U}}^{O(2)}$$

where U is a complete $O(2)$-universe, $\tilde{U} = \iota_{C_2}^* U$, and $\mathsf{Sp}_{\mathcal{V}}^G$ denotes the category of orthogonal G spectra indexed on a complete G-universe for any compact Lie group G. We prove that our new construction deserves to be called the HHR norm in the following sense.

Theorem 4.2 (Angelini-Knoll, Gerhardt, Hill)**.** The norm functor $N_{C_2}^{O(2)}$ is left Quillen adjoint

$$N_{C_2}^{O(2)} \colon \mathsf{Sp}_{\tilde{U}}^{C_2} \rightleftarrows \mathsf{Sp}_{U,\mathcal{R}}^{O(2)} : \iota_{C_2}^*$$

where the model structure on the source is the positive complete stable model structure, and the model structure of the target is the positive complete stable model structure with respect to the family $\mathcal{R}$ of subgroups that intersect the circle $\mathbb{T}$ trivially.

We also extend work of E. Dotto, K. Moi, I. Patchkoria, and S. Precht-Reeh [25] to the $O(2)$-equivariant setting. Showing that the norm $N_{C_2}^{O(2)}$ is a model for THR, defined using the Bökstedt construction in the sense that there is an $\mathcal{R}$-equivalence

$$N_{C_2}^{O(2)}(A) \simeq_{\mathcal{R}} THR(A).$$

So this gives a norm model for Real topological Hochschild homology. We hope to extend this to an equivalence as Real cyclotomic spectra in future work.

Additionally, we give a new description of the restriction to D_{2m} of $N_{C_2}^{D_{2m}}$ when m is odd.

Theorem 4.3 (Angelini-Knoll,Gerhardt, Hill)**.** Suppose A is a flat E_σ ring (a mild cofibrancy condition), then there are equivalences

$$\iota_{D_{2m}}^* N_{C_2}^{O(2)}(A) \simeq \iota_{D_{2m}}^* THR(A) \simeq N_{C_2}^{D_{2m}}(A) \wedge_{N_e^{D_{2m}}(\iota_e^* A)} {}^{\gamma_m} N_{C_2}^{D_{2m}}(A)$$

where $\gamma_m = e^{(m-1)\pi i/2}$ and ${}^{\gamma_m} N_{C_2}^{D_{2m}}(A)$ is the right action of $N_e^{D_{2m}}(\iota_e^* A)$ on ${}^{\gamma_m} N_{C_2}^{D_{2m}}(A)$ where ${}^{\gamma_m} N_{C_2}^{D_{2m}}(A)$ is the evident analogue in spectra of the construction ${}^{\gamma_m} N_{C_2}^{D_{2m}}(\underline{M})$ described in Definition (4.1).

As a consequence of our results, there is, in particular, a Real cyclotomic trace map

$$\underline{\pi}_0^{C_2} KR(A) \to \underline{\pi}_* THR(A) \to \underline{HR}_*^{C_2}(R).$$

in C_2-Mackey functors that extends the classical factorization of the Dennis trace through topological Hochschild homology.

4.2 Crossed simplicial homology and equivariant Floer homology

Connes' cyclic category is indispensible in the construction of topological Hochschild homology and the dihedral category is indispensible in the construction Real topological Hochschild homology. Since these categories extend the simplex category Δ any presheaf on one of these categories may be regarded as a simplicial set by restriction and it is known that the geometric realization has an $\mathbb{T}$-action in the cyclic case and a $O(2)$ action in the dihedral case.

These categories have a common generalization known as crossed simplicial groups, due to Fiedorwicz-Loday [31]. Generally, a crossed simplicial group $\Delta\mathfrak{G}$ has the same objects in Δ, but it has extra automorphisms with the property that any morphism can be factored as one extra automorphism composed with a morphism in Δ. The automorphisms themselves $\mathfrak{G}_\bullet$ form a simplicial set where $\mathfrak{G}_n$ is a group for each $n \geq 0$, but the structure maps are not group homomorphism, but rather crossed homomorphisms. In work in progress with M. Merling and M. Péroux, we consider generalizations of topological Hochschild homology associated to a crossed simplicial group $\Delta\mathfrak{G}$.

As a particular example, there is a crossed simplicial group called the quaternionic category, with associated simplicial set $Q_\bullet$ where Q_n is the quaternionic group of order $4n$. The geometric realization of any pre-sheaf on the quaternionic category comes equipped with a $Pin(2)$-cation, and in particular our construction of quaternionic topological Hochschild homology THQ comes equipped with a $Pin(2)$ action (this group $Pin(2)$ is also sometimes called $Pin_-(2)$ in the lterature). In our current work, we set up the necessary machinery and do some initial calculations. In particular, we define a HHR norm functor $N_{Q_1}^{Pin(2)}$ in an analogous way to the construction of the norm $N_{C_2}^{O(2)}$ using the quaternionic bar construction in place of the dihedral bar construction. We expect that there are also analogues of the results in the previous section in this setup, but we have not explored them. Instead, we aim to significantly generalize this construction to provide combinatorial models for the norms $N_{\mathfrak{G}_0}^{\mathfrak{G}}$ for any compact Lie group $\mathfrak{G}$ that arises as the geometric realization of the simplicial set $\mathfrak{G}_\bullet$ whose n-simplices $\mathfrak{G}_n$ are the additional automorphisms of the object $[n]$ in some crossed simplicial group $\Delta\mathfrak{G}$. In the present work, we plan to give different models for this new construction including a Norm model, a Bökstedt model, and a Hochschild-Mitchell construction as well as do some initial calculations.

As a long term goal, we aim to connect this theory to recent ground breaking results in manifold theory of C. Manolescu [52] and related work on involutive Heegaard-Floer homology of K. Hendricks and C. Manolescu [33]. For example, in the non-equivariant situation, T. Lawson [46] has given an explanation of the connection between descent properties of topological Hochschild homology and Heegaard-Floer homology by recovering the Hodge-to-de Rahm cohomology spectral sequence of R. Lipshitz and D. Treumann [49] from a spectral sequence in topological Hochschild homology.

Goal 4.4. Construct an equivariant non-commutative Hodge-to-de Rahm spectral sequence for topological crossed simplicial homology that can shed light on equivariant Heegaard-Floer homology.

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