A brief introduction to Algebraic K-theory

Gabriel Angelini-Knoll

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Preface

These are notes (in progress) for a 2 hour per week course in algebraic K-theory taught at the Freie Universität Berlin in Winter 2020/21. They will continue to be updated regularly. Please feel free to reach out at my email address gak@math.fu-berlin.de if you notice any typos or errors. These notes are hosted on my personal website.

These notes draw from the tome of Charles Weibel, known as the K-book [19] as well as the original papers in the subject by Quillen [11, 12], Segal [14], and Waldhausen [16, 17]. The goal is to give a brief survey of constructions of algebraic K-theory, fundamental theorems in algebraic K-theory, and applications to geometric topology, algebraic geometry, and number theory. The notes focus on the case of algebraic K-theory of rings, which are the common thread in the applications to each of these three subjects.

These notes briefly survey the field of algebraic K-theory from the time period 1950-1985. The advantage of focusing on this time period is that no previous knowledge of $(\infty, 1)$-categories is required and for those interested in the most modern constructions of algebraic K-theory, all of the essential ideas already existed in the work of Quillen, Segal, and Waldhausen.

It is assumed that students in this course have a firm background in the basics of algebra, linear algebra, category theory, and topology. Additional knowledge of geometry topology, algebraic geometry, and number theory is useful for understanding certain examples, but certainly not required for the bulk of the material. Previous knowledge of the theory of simplicial sets will certainly be helpful, but not required, so a short section on such material is included as an appendix for those unfamiliar.

Conventions

Throughout, by a ring we will mean an associative ring with unit. We will always specify that a ring is non-unital when we want to consider it without unit.

Let Mod_R be the category of left modules over a ring R. When $R = \mathbb{Z}$, we simply write Ab for this category and refer to it as the category of abelian groups and when R is a field k, we write Vec_k for this category and refer to it as the category of vector spaces over a field k. We let $M(R)$ denote the skeleton of the category of finitely generated left R modules and we let $P(R)$ denote the skeleton of the category of finitely generated projective left modules over a ring R. When k is a commutative ring, let $\mathsf{Rep}_k(G)$ be the skeleton of the category of finitely generated $k[G]$-modules. Usually, we we only consider this in the case k is a field.

Throughout, by a space we mean a compactly generated weak Hausforff space and we write Top for the category of compactly generated weak Hausdorff spaces. Write CW for the category of CW complexes and CW^f for the category of finite CW complexes. When X is a space, let $VB_{\mathbb{R}}(X)$ denote the skeleton of the category of real vector bundles over X and let $VB_{\mathbb{C}}(X)$ denote the skeleton of the category of complex vector bundles over X.

Let Set be the category of sets and let Fin be the skeleton of the category of finite sets. Let G be a finite group and let Fin_G the skeleton of the category of finite G-sets. When R is a ring, let $\mathsf{Rep}_R(G)$ be the skeleton of the category of finite dimensional $k[G]$-modules.

Chapter 1

Introduction

The 0-th algebraic K-theory group K_0 was first defined by Grothendieck in the late 1950's in order to generalize the Riemann-Roch Theorem to varieties [4]. The name K-theory comes from German word *Klassen* meaning classes and the reason for this name will be more clear after reading Section 2.1.[1] Even earlier, in the early 1950's, Whitehead studied the simply homotopy of of a finite CW complex and constructed an obstruction to two spaces which are homotopy equivalent being simple homotopy equivalent. It was later understood that this class lived in the first algebraic K-theory group K_1 of an integral group ring. It was then shown that these two algebraic K-theory groups could be related by a localization sequence and that there should in fact be a related group K_i for all integers i extending this localization sequence to the left and right.

Milnor constructed the group K_2 of a ring as the center of the Steinberg group of a ring, inspired in part by a theorem of Matsumoto [7], and used this to motivate his definition of the higher algebraic K-theory groups, now known as Milnor K-theory group, in 1970 [9]. However, as we will see, this theory is a not a rich invariant in the sense that for finite fields the Milnor K-theory groups K_n^M vanish for $n \geq 2$.

In 1972, Quillen defined higher algebraic K-theory groups using the $+$-construction [11]. One of his key insights was that the algebraic K-theory groups should be defined as the the homotopy groups of a space. In the same year [12], Quillen defined higher algebraic K-theory groups for a category equipped with notion of exact sequences called an *exact category*. This allowed for much broader input, in particular recovering examples of interest in algebraic geometry.

In 1974 [14], Segal defined the algebraic K-theory of a symmetric monoidal

[1] Though Grothendieck famously spent most of his life in France, he was in fact born in Berlin, Germany in 1939. Perhaps this is why Grothendieck chose the letter K from the German word Klassen rather then the French, but this is not well documented.

category. This notion is sensitive to the symmetric monoidal structure, so it not a special case of Quillen's Q-construction unless the symmetric monoidal structure is the direct sum in an additive category. Quilen's Q-construction is also not a special case of Segal's construction. One of Segal's motivations was to give new constructions of infinity loop spaces, which were known to represent cohomology theories by Brown representability [5].

In 1978, Waldhausen extended Quillen's Q-construction further so that the input could be a category with cofibrations and weak equivalences [16]. This allowed one to define the algebraic K-theory of spaces. This new definition extended the applications of algebraic K-theory to manifold theory [18].

Since 1985, there have been several new constructions of algebraic K-theory using the theory of $(\infty, 1)$-categories. These constructions have proven quite useful for demonstrating universal properties of algebraic K-theory. For example, in 2016, Barwick defined a version of Waldhausen's algebriac K-theory construction for small Waldhausen quasicategories in [1] and proved that algebraic K-theory may be considered as a homology theory, in an abstract sense, on the quasi-category of small Waldhausen quasi-categories. Blumberg–Gepner–Tabuada [3] prove that the connective algebraic K-theory of a small stable quasi-category is the universal additive invariant and non-connective algebraic K-theory of a small stable quasi-category is the universal localizing invariant. Additionally, Gepner-Groth-Nikolaus [6] prove universal properties of the algebraic K-theory of symmetric monoidal quasi-categories. However, will not discuss these more constructions further in the present notes.

Chapter 2

Classical Algebraic K-theory

We begin by studying the groups K_0 and K_1. These two groups arose independently in the 1950's from entirely different contexts. Later, it was proven they are related by a localization sequence.

The group K_2 of a ring was then defined by Milnor, inspired by work of Matsumoto [7], and Milnor used this to motivate his definition of higher algebraic K-theory groups K_*^M now know as Milnor K-theory. However, these groups are not as rich an invariant as the higher algebraic K-theory groups that we will discuss in the next chapters, due to Quillen [11, 12]. We will explicitly prove this in the case of finite fields.

At the start, I want to emphasize that there are really two flavors of algebraic K-theory: algebraic K-theory of symmetric monoidal categories and algebraic K-theory categories with a notion of exact sequences, such as exact categories. The two flavors of algebraic K-theory agree when we consider symmetric monoidal categories with respect to the coproduct and algebraic K-theory of categories with exact sequences in which these exact sequences split. For example, this is the case for the category of finitely generated projective R modules. Since the category of finitely generated projective R modules will be our central example throughout and the distinction may not always be clear.

I also want to emphasize that non of the results in this chapter are new and our treatment in this chapter is almost entirely contained in chapters I-III of Weibel's K-book [19]. In [19], Weibel goes into significantly more depth on this subject than we attempt to do here. We also point the reader towards books of Bass [2] and Milnor [10] from the time period before 1972, which give an even more thorough treatment of what was known at the time about algebraic K-theory groups. The first four chapters of Rosenberg's [13] are also an great reference for this material and he much more depth then I do in this first chapter.

2.1 The Grothendieck group

In order to give a very general definition of K_0, we will first briefly set up the the theory of monoidal categories and monoids in a monoidal category.

Monoidal categories are an abstraction of the properties enjoyed by the category of abelian groups Ab with respect to the tensor product $\otimes_{\mathbb{Z}}$ and the integers $\mathbb{Z}$. In particular, the category Ab is equipped with a functor

$$\otimes_{\mathbb{Z}} \colon \mathsf{Ab} \times \mathsf{Ab} \to \mathsf{Ab}$$

and a unit $\mathbb{Z}$ object in the sense that there are isomorphisms

$$M \otimes_{\mathbb{Z}} \mathbb{Z} \cong M \cong \mathbb{Z} \otimes_{\mathbb{Z}} M$$

for any abelian group M, which are natural in M. The tensor product is also associative

$$(M \otimes N) \otimes L \cong M \otimes (N \otimes L),$$

where this associativity is natural in M, N and L. There is also a factor swap map

$$B_{M,N} \colon M \otimes N \to N \otimes M$$

which is also natural in M and N. In addition, each of these pieces of data satisfy certain commutative diagrams. This data is abstracted to the definition of a symmetric monoidal category, which also applies in many other contexts.

Definition 2.1.1. A *symmetric monoidal category* $\mathcal{C}$ consists of a category $\mathcal{C}$ a functor

$$\otimes \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$$

and a unit $1_{\mathcal{C}}$ together with four natural isomorphisms:

1. an associator

$$a_{-,=,\equiv} \colon (- \otimes =) \otimes \equiv \xrightarrow{\cong} - \otimes (= \otimes \equiv)$$

2. a left unitor

$$\lambda_{-} \colon 1 \otimes (-) \xrightarrow{\cong} (-),$$

3. a right unitor

$$\rho_{-} \colon (-) \otimes 1 \xrightarrow{\cong} (-),$$

   and

4. a braiding

$$B_{-,=} \colon (-) \otimes (=) \xrightarrow{\cong} (=) \otimes (-).$$

These natural transformations must satisfy the triangle identity

$$\mathrm{id}_x \otimes \lambda_y \circ a_{x,1_{\mathcal{C}},y} = \rho_x \otimes \mathrm{id}_y \tag{2.1.2}$$

and pentagon identity

$$a_{w,x,y\otimes z} \circ a_{w\otimes x,y,z} = \mathrm{id}_w \otimes a_{x,y,z} \circ a_{w,x\otimes y,z} \circ a_{w,x,y} \otimes \mathrm{id}_z \tag{2.1.3}$$

the hexagon identities

$$a_{y,z,x} \circ B_{x,y\otimes z} \circ a_{x,y,z} = \mathrm{id}_y \otimes B_{x,z} \circ a_{y,x,z} \circ B_{x,y} \otimes \mathrm{id}_z \tag{2.1.4}$$

$$a^{-1}_{y,z,x} \circ B_{x,y\otimes z} \circ a_{x,y,z} = \mathrm{id}_y \otimes B_{x,z} \circ a_{y,x,z} \circ B_{x,y} \otimes \mathrm{id}_z \tag{2.1.5}$$

and the "squaring to identity" axiom

$$B_{y,x} \circ B_{x,y} = \mathrm{id}_{x\otimes y}\,. \tag{2.1.6}$$

We succinctly write $(\mathcal{C}, \otimes, 1_{\mathcal{C}})$ for all of this data.

If $\mathcal{C}$ has all of the other structure except that it is not equipped with a braiding natural transformation $B_{-,=}$ satisfying (2.1.4), (2.1.5), and (2.1.6), then we say $\mathcal{C}$ is a *monoidal category*.

Examples 2.1.7. The category $P(R)$ is a symmetric monoidal category with respect to $\oplus$ denoted $(P(R), \oplus, 0)$ and it is a monoidal category with respect to $\otimes_R$, denoted $(P(R), \otimes_R, R)$. Moreover, when R is a commutative ring then the monoidal category $(P(R), \otimes_R, R)$ is in fact a symmetric monoidal category.

The category $VB_k(X)$ for $k = \mathbb{R}$ or $k = \mathbb{C}$ is a symmetric monoidal category with Whitney sum $\oplus$ and it is a monoidal category with tensor product $\otimes$ denoted $(VB_k(X), \oplus, 0)$ and $(VB_k(X), \otimes, k)$ where k here denotes the trivial one dimensional k vector bundle.

The categories Set (respectively Fin) are symmetric monoidal categories with respect to the coproduct $(\mathsf{Set}, \amalg, \varnothing)$ (respectively $(\mathsf{Fin}, \amalg, \varnothing)$, and with respect to the product $(\mathsf{Set}, \times, *)$ (respectively $(\mathsf{Fin}, \times, *)$). Similarly, Fin_G is a symmetric monoidal category with respect to the coproduct $(\mathsf{Fin}_G, \amalg, \varnothing)$ and the product $(\mathsf{Fin}_G, \times, *)$. Let k be a general field. The category $\mathsf{Rep}_k(G)$ is a symmetric monoidal category with respect to the direct sum $(\mathsf{Rep}_k(G), \oplus, 0)$ and it is a monoidal category with respect to tensor product $(\mathsf{Rep}_k(G), \otimes_k, k)$.

We now discuss monoids in a general symmetric monoidal category.

Definition 2.1.8. A (unital) monoid M in a symmetric monoidal category $\mathcal{C}$ is an object M in $\mathcal{C}$ equipped with an operation

$$\mu\colon\ M \otimes M \to M$$

and a unit map

$$\eta\colon\ 1_{\mathcal{C}} \to M$$

from the unit object $1_{\mathcal{C}}$ in $\mathcal{C}$ to M satisfying:

1. the associativity axiom

$$\begin{tikzcd} M \otimes M \otimes M \arrow[r, "\mu\times 1"] \arrow[d, "1\times\mu"'] & M \otimes M \arrow[d, "\mu"] \\ M \otimes M \arrow[r, "\mu"] & M \end{tikzcd} \tag{2.1.9}$$

and

2. the unitality axiom

$$\begin{tikzcd} M \arrow[r, "\eta\times M"] \arrow[dr, equal] & M \otimes M \arrow[d, "\mu"] & M \arrow[l, "1\times\eta"'] \arrow[dl, equal] \\ & M. & \end{tikzcd} \tag{2.1.10}$$

If in addition, the commutativity axiom

$$\begin{tikzcd} M \otimes M \arrow[rr, "\tau"] \arrow[dr, "\mu"'] & & M \otimes M \arrow[dl, "\mu"] \\ & M & \end{tikzcd} \tag{2.1.11}$$

is satisfied, we say that M is a *commutative monoid in* $\mathcal{C}$.

When $(\mathcal{C}, \otimes_{\mathcal{C}}, 1_{\mathcal{C}}) = (\mathsf{Set}, \times, *)$ we will simply refer to (unital) monoids and commutative monoids in Set as monoids and commutative monoids.

Each of the examples $(P(R), \oplus, 0)$, $(M(R), \oplus, 0)$, $VB_k(X)$, $(\mathsf{Fin}, \amalg, \emptyset)$, $(\mathsf{Fin}, \times, *)$, $(\mathsf{Fin}_G, \amalg, \emptyset)$, $(\mathsf{Fin}_G, \times, *)$, $(\mathsf{Rep}_k(G), \oplus, 0)$, and $(\mathsf{Rep}_k(G), \otimes_k, k)$ may actually be regarded as commutative monoids by applying the forgetful functor from small categories to sets.

If $(M, +, 0)$ is a commutative monoid with operation $+$ and $(M, \times, 1)$ is a monoid with respect to a second operation $\times$ such that $(M, +, \times, 0, 1)$ forms a ring without additive inverses, then we say that M is a semi-ring. In fact, $(P(R), \oplus, \otimes_R, 0, R)$, $(\mathsf{VB}_k(X), \oplus, \otimes, 0, k)$, $(\mathsf{Fin}, \amalg, \times, \emptyset, *)$, $(\mathsf{Fin}_G, \amalg, \times, \emptyset, *)$, and $(\mathsf{Rep}_{\mathbb{C}}(G), \oplus, \otimes_{\mathbb{C}[G]}, 0, \mathbb{C}[G])$ are all examples of semi-rings.

We are now prepared to discuss our definition algebraic K-theory K_0.

Construction 2.1.12. Given a commutative monoid M we form the Grothendieck group completion of M, denoted M^{gp} as follows. We define an equivalence relation on elements $(m, n) \in M \times M$. We define an equivalence relation

$$(m, n) \sim (m + p, n' + p)$$

for any $p \in M$. We then define $M^{\mathrm{gp}} := M \times M / \sim$.

Exercise 2.1.13. Check that $(m,n) \sim (m+p, n'+p)$ is an equivalence relation.

Note that, by construction the abelian group M^{gp} has the universal property that given a map of commutative monoids $M \to A$, where A is an abelian group, then the map $M \to A$ factors as

$$\begin{tikzcd} M \arrow[r] \arrow[dr] & M^{\text{gp}} \arrow[d, dashed] \\ & A. \end{tikzcd} \tag{2.1.14}$$

In other words, there is an adjunction given by the isomorphism

$$\mathrm{Hom}_{\mathsf{CMon}}(M, A) \cong \mathrm{Hom}_{\mathsf{Ab}}(M^{\text{gp}}, A)$$

natural in M and A. In particular, the construction M^{gp} is functorial in M.

Alternatively, we could let $F(M)$ be the free abelian group on symbols $[m]$ where $m \in M$. We can then quotient by the subgroup $R(M)$ of $F(M)$ generated by the relations $[m+n]-[m]-[n]$. This construction also clearly satisfies the universal property 2.1.14. Consequently, we may give a different definition of M^{gp} that agrees with the previous construction up to natural isomorphism

Definition 2.1.15. Given a commutative monoid M, define

$$M^{\text{gp}} := F(M)/R(M)$$

where $F(M)$ and $R(M)$ are as defined above.

For $m \in M$ we will write $[m]$ for a general element in M^{gp}.

Definition 2.1.16. Let R be a ring. Then we define the 0-th algebraic K-theory group of R

$$K_0^{\oplus}(R) := (P(R), \oplus, 0)^+$$

where we regard the set of isomorphism classes of subgroups of $P(R)$ as commutative monoid via $\oplus$ and 0. In fact, since

$$(P(R), \oplus, \otimes_R, 0, R)$$

is a semi-ring, then $K_0(R)$ is a ring. When R is a commutative ring then $K_0(R)$ is also a commutative ring.

In fact, this is a special case of a more general construction.

Definition 2.1.17. Let $(\mathcal{C}, \otimes, 1)$ be a skeletally small symmetric monoidal concrete category with symmetric monoidal skeleton $\mathrm{sk}\,\mathcal{C}$. Then we may regard $\mathrm{sk}\mathcal{C}$ as a commutative monoid in Set with respect to $\otimes$ and 1 and define

$$K_0^{\otimes}(\mathcal{C}) := (\mathrm{sk}(\mathcal{C}), \otimes 1)^+.$$

Moreover, if

$$(\mathrm{sk}\mathcal{C}, \oplus, \otimes_{\mathcal{C}}, 0_{\mathcal{C}}, 1_{\mathcal{C}})$$

is a semi-ring. Then $K_0^{\oplus}(\mathcal{C})$ is a ring.

This general construction allows us to recover many examples of interest.

Examples 2.1.18. The 0-th complex topological K-theory of X is

$$KU^0(X) \cong K_0^{\oplus}(VB_{\mathbb{C}}(X))$$

and the 0-th real topological K-theory of X is

$$KO^0(X) \cong K_0^{\oplus}(VB_{\mathbb{R}}(X)).$$

In fact these are both rings because $(VB_k(X), \oplus, \otimes, 0, k)$ is a semi-ring when $k = \mathbb{C}$ or $k = \mathbb{R}$.

The Burnside ring of a finite group G is

$$A(G) = K_0^{\amalg}(\mathsf{Fin}_G)$$

where the ring structure comes from the fact that $(\mathsf{Fin}_G, \amalg, \times, \emptyset, *)$ is a semi-ring.

Let k be a field. The representation ring of G is

$$R_k(G) = K_0^{\oplus}(\mathsf{Rep}_k(G))$$

where the ring structure comes from the fact that $(\mathsf{Rep}_k(G), \oplus, \otimes_k, 0, \mathbb{C})$ is a semi-ring.

We finish with some basic computations. First, note that $(\mathbb{N}, +, 0)$ is a commutative monoid and its Grothendieck group completion is clearly

$$\mathbb{N}^+ = \mathbb{Z}.$$

Notice that there is always map of commutative monoids

$$\begin{aligned} \mathbb{N} &\to P(R). \\ n &\mapsto R^n \end{aligned}$$

and by functoriality of the Grothendieck construction, a group homomorphism

$$\mathbb{Z} \to K_0(R). \tag{2.1.19}$$

We say that R satisfies the left invariant basis property, denoted IBP, if R^n and R^m are not isomorphic whenever $n \neq m$. In this case, the rank of a free left R module does not depend on a choice of basis. All commutative rings satisfy this property and integral group rings $\mathbb{Z}[G]$ all satisfy the invariant basis property.

Example 2.1.20. The ring of k-linear endomorphisms $\mathrm{End}_k(k)$ does not satisfy the IBP property.

Exercise 2.1.21. Prove that there is an isomorphism of $\mathrm{End}_k(k)$ modules

$$\mathrm{End}_k(k) \cong \mathrm{End}_k(k) \oplus \mathrm{End}_k(k),$$

verifying the claim in Example 2.1.20.

Lemma 2.1.22. *When R satisfies the IBP, then the map*

$$\mathbb{Z} \to K_0(R)$$

induced by the map $n \mapsto R^n$ is injective.

Definition 2.1.23. We define the reduced K_0 group to be the cokernel of the map $\mathbb{Z} \to K_0(R)$ and denote it $\tilde{K}_0(R)$.

Proposition 2.1.24. *When k is a field, then*

$$\tilde{K}_0(k) = 0.$$

Proof. The rank of a vector space gives a map of commutative monoids

$$P(k) \to \mathbb{N}$$

sending $[k^n]$ to n, which is an isomorphism of commutative monoids. $\square$

Exercise 2.1.25. Prove that when R is a principle ideal domain, then

$$\tilde{K}_0(R) = 0.$$

Exercise 2.1.26. Prove that when R is a local ring, then

$$\tilde{K}_0(R) = 0.$$

The invariant $\tilde{K}_0(R)$ has interesting applations to geometry and number theory. For example, when G is a group and $R = \mathbb{Z}[G]$ is the associated integral group ring, then we define the 0-th Whitehead group

$$Wh_0(G) := \widetilde{K}_0(\mathbb{Z}[G]).$$

We will simply state a result of Wall's that shows that the 0-th Whitehead group is an interesting invariant in topology. We say that a topological space X is dominated by a CW complex K if it there is a map $K \to X$ with right homotopy inverse.

Theorem 2.1.27 (Wall's finiteness obstruction)**.** *Suppose that X is dominated by a finite CW complex K and let $G = \pi_1 X$. Then there is an associated obstruction class $w(X) \in Wh_0(G)$ such that $w(X) = 0$ if and only if X is homotopy equivalent to a finite CW complex.*

Remark 2.1.28. Note that we know by CW approximation that X is homotopy equivalent to a CW complex, so it suffices to consider the case when X is also a CW complex, however it is not at all clear that a homotopy retract of a finite CW complex is again finite CW complex.

Reduced K_0 also has applications to number theory. First, we recall a definition from commutative algebra.

Definition 2.1.29. A Dedekind domain R is an integral domain such that for all nontrivial ideals $J \subset I \subset R$, there exists an ideal K such that $IK = J$.

Remark 2.1.30. In a Dedekind domain R every ideal can be written as a product of prime ideals. However, it may not be able to be written as a product of prime ideals in a unique way. If every ideal can be written as a product of prime ideals in a unique way, then R is a PID. Every PID is also clearly a PID.

Definition 2.1.31. The *ideal class group* of a Dedekind domain R it the quotient

$$\mathrm{Cl}(R) = \{I : I \subset R\}/\sim$$

where I ranges over all ideals in R and the equivalence relation states that $I \sim J$ if there exist $x, y \in R$ such that $xI = yJ$ as subsets of R. The group structure is given by the product of ideals.

Exercise 2.1.32. Check that this is in fact an equivalence relation and that $Cl(R)$ is an abelian group.

The ideal class group measures the failure of a Dedekind domain to be a UFD, or in other words, the failure of a Dedekind domain to be a PID.

Again, we will not prove the following result, but we record it as another important application of K_0.

Theorem 2.1.33. *When R is a Dedekind domain, then there is an isomorphism*

$$\tilde{K}_0(R) \cong Cl(R).$$

The class group measures the failure of unique prime factorization. In other words, when R is also a UFD then $\tilde{K}_0(R) = 0$. To see that unique prime factorization can fail, simply consider the ring $\mathbb{Z}[\sqrt{-5}]$. In this ring,

$$(1 - \sqrt{-5})(1 + \sqrt{-5}) = 6 = 2 \cdot 3.$$

Since $\mathbb{Z}[\sqrt{-5}]$ is a Dedekind domain, we observe the following.

Lemma 2.1.34. *There is an isomorphism*

$$K_0(\mathbb{Z}[\sqrt{-5}]) = \mathbb{Z} \oplus \mathbb{Z}/2.$$

The result Theorem 2.1.33 is a special case of a more general result, which we again state without proof.

Theorem 2.1.35. *Let R be a commutative ring of Krull dimension ≤ 1, then there is an isomorphism*

$$\mathit{rank} \oplus \mathit{det} \colon K_0(R) \cong [\mathrm{Spec}(R), \mathbb{Z}] \oplus \mathrm{Pic}(R).$$

2.2 The Whitehead group $Wh_1(G)$

In the 1940's and 1950's, Whitehead developed the theory of simple homotopy types. We say that a finite CW complex Y has the same simple homotopy type as a finite CW complex X if they are homotopy equivalent and each homotopy can be described in terms of elementary expansions and collapses.

Whitehead defined a group which encoded the obstruction to two homotopy equivalent finite CW complexes having the same simple homotopy type. Suppose X and Y are CW complexes and there is a homotopy equivalence

$$X \xrightarrow{\simeq} Y.$$

Then clearly this homotopy equivalence induces an isomorphism $\pi_1 X \cong \pi_1 Y$. We would like to know whether X and Y are simple homotopy equivalent. There is an obstruction to this, which lies in a group $Wh(\pi_1 X)$, which is an ablian group that depends only on the group $\pi_1 X$. It was later noted that this group can be defined in terms of algebraic K-theory.

Let R be an associative ring and let $GL_n R$ be the group of invertible $n \times n$ matrices with coefficients in R. There is an inclusion

$$GL_n R \subset GL_{n+1} R$$

given by

$$A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$$

We can then form the union (the colimit) to define

$$GL(R) = \bigcup_{n \geq 1} GL_n(R).$$

In general, if we have a group G we can take the quotient by commutators to define

$$G^{\mathrm{ab}} := G/[G, G].$$

In fact this is a left adjoint to the forgetful functor from abelian groups groups so it satisfies a universal property, which is encoded in the natural isomorphism

$$\mathrm{Hom}_{\mathsf{Ab}}(G^{\mathrm{ab}}, A) \cong \mathrm{Hom}_{\mathsf{Gp}}(G, A).$$

Definition 2.2.1. Let R be a ring, then we define

$$K_1(R) := GL(R)^{\mathrm{ab}}.$$

In fact, there is a nice description of the commutator $[GL(R), GL(R)]$. Let $E_n(R)$ be the subgroup of $GL_n(R)$ consisting of the $n \times n$ matrices, which are *transvections*. A transvection is the sum of the identity matrix and a matrix with only one nonzero entry, where that nonzero entry does not occur on the diagonal. We write $e_{i,j}(r)$ for this matrix where r is the nonzero entry and it occurs in the i,j-th position where $i \neq j$. We may then define $E(R)$ in the same way that we defined $GL(R)$ as the union

$$E(R) = \bigcup_{n \geq 1} E_n(R).$$

Definition 2.2.2. A group G is *perfect* if

$$G = [G, G].$$

Note that for a perfect group $G^{\mathrm{ab}} = 0$. Such groups are quite interesting from the perspective of topology. For example, a path connected space such that $\pi_1 X$ is a nontrivial perfect group and $\pi_k X = 0$ for all $k > 0$ has the property that its homology is the same as the homology of a point and yet it X is not contractible.

Lemma 2.2.3. *When $n \geq 3$, then $E_n(R)$ is a perfect group.*

Proof. Whenever i, j, k are distinct, then

$$e_{i,j}(r) = [e_{i,k}(r), e_{k,i}(1)].$$

□

Exercise 2.2.4. Give an example where $E_2(R)$ is not perfect.

Exercise 2.2.5. Verify that, if $g \in GL_n(R)$, the identity

$$\begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix} = \begin{pmatrix} 1 & g \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -g^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & g \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

holds in $GL_{2n}(R)$.

The following example will be useful in the proof of Whitehead's lemma.

Example 2.2.6. A signed permutation matrix is a matrix that permutes the standard basis on R^n up to a sign. If we write $\{e_1, \ldots, e_n\}$ for the standard basis, then a signed permutation acts on the set $\{\pm e_1, \ldots, \pm e_n\}$. We observe that, for example, the signed permutation matrix

$$\overline{w}_{1,2} := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = e_{1,2}(1)e_{2,1}(1)e_{1,2}(1)$$

can be written as a product of transvections and therefore it is contained in $E_2(R)$ for any ring R. More generally,

$$\overline{w}_{i,j} \in E_n(R)$$

for $n \geq i, j$. We can then show that cyclic permutations of three basis elements are also contained in $E_n(R)$, since they can be written as $\overline{w}_{jk}\overline{w}_{i,j}$. Consequently, every matrix corresponding to an even permutation of basis elements is an element in $E_n(R)$ for some n. Thus, by Exercise 2.2.5 we know that $E_{2n}(R)$ contains the matrix

$$\begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix}.$$

The subgroup $E_n(R)$ is not necessarily a normal subgroup in $GL_n(R)$. It is often a normal subgroup in $GL_n(R)$ for sufficiently large n, but even this is too much to ask for in general. When R is a commutative ring, the situation is much easier and $E_n(R)$ is normal in $GL_n(R)$ for $n \geq 3$. Nevertheless, we have the following lemma due to Whitehead which, in particular, implies that $E(R)$ is normal in $GL(R)$.

Lemma 2.2.7 (Whitehead's Lemma)**.** *There is an isomorphism*

$$[GL(R), GL(R)] \cong E(R)$$

Proof. The fact that

$$E(R) \subset [GL(R), GL(R)]$$

follows from Lemma 2.2.3. Conversely, suppose $[A, B] \in [GL_n(R), GL_n(R)]$. Then we can write $[A, B]$ as the product

$$[A, B] = \begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & B^{-1} \end{pmatrix} \begin{pmatrix} (AB)^{-1} & 0 \\ 0 & AB \end{pmatrix}$$

By Example 2.2.6, we therefore know that $[A, B] \in E(R)$. □

This gives a new definition of $K_1(R)$.

Definition 2.2.8. Let R be a ring, then we define

$$K_1(R) := GL(R)/E(R).$$

In particular, $K_1(R)$ is a quotient of $GL(R)$ by a perfect normal subgroup. The definition as a quotient by a perfect normal subgroup will be important for the next chapter.

Definition 2.2.9. We define the Whitehead group of a group G as

$$Wh_1(G) := K_1(\mathbb{Z}[G]) / < \pm g : g \in G > .$$

where $g \in G$ is regarded as an element in $GL_1(\mathbb{Z}[G]) \subset GL(\mathbb{Z}[G])$ and

$$< \pm g : g \in G >$$

denotes the subgroup of $K_1(\mathbb{Z}[G])$ generated by the elements $\pm g \in K_1(\mathbb{Z}[G])$.

Again, we will simply cite a deep result that demonstrates that this group is useful for studying problems in topology.

Theorem 2.2.10 (Whitehead)**.** *Suppose K and L are finite CW complexes and there is a homotopy equivalence $f\colon K \to L$ inducing an isomorphism $\pi_1 K \cong \pi_1 L$. Let $G = \pi_1 K$. Then there is an associated class*

$$\tau(f) \in Wh_1(G),$$

called the Whitehead torsion of f, such that $\tau(f) = 0$ if and only if f is a simple homotopy equivalence.

In fact, there are other applications of Whitehead torsion to manifold theory. For those unfamiliar with these constructions in manifold theory, we do not plan to give full definitions as that would be too much of a diversion and these constructions will not be used later. We therefore just provide enough information to state the main results in order to indicate the depth of the subject of algebraic K-theory.

Let (W, M, M') be a triple of compact piecewise linear (PL) manifolds. We say this triple is an *h-cobordism* if W has boundary $M \amalg M'$ and both inclusions $M \subset W$ and $M' \subset W$ are homotopy equivalences. There is therefore a Whitehead torsion class $\tau \in Wh_1(\pi_1 M)$ associated to the inclusion $M \subset W$. We record the following deep result, proven by Mazur [8], without proof.

Theorem 2.2.11 (The s-cobordism theorem)**.** *Given an h-cobordism (W, M, M') of PL-manifolds, with M fixed and dim$(M) \geq 5$. Then there is a PL homeomorphism of triples*

$$(W, M, M') \cong (M \times [0, 1], M \times \{0\}, M \times \{1\})$$

if and only if $\tau = 0$. Moreover, every element $\tau \in Wh_1(\pi_1 M)$ arises as the Whitehead torsion of some h-cobordism (W, M, M').

This result can be used to prove a version of the generalized Poincare conjecture, which had originally proven by Smale [15] before the s-cobordism theorem was known..

Corollary 2.2.12. *Suppose N is a PL manifold with the same homotopy type as a sphere S^n and $n \geq 5$. Then N is PL-homeomorphic to a S^n.*

Proof. Form a PL manifold W by removing two disjoint n-discs D_1 and D_2 from N. Then we produce a PL cobordism $(W, S_1^{n-1}, S_2^{n-1})$ where S_i^{n-1} is the boundary of D_i in N. Since $\pi_1 S^{n-1} = 0$ when $n \geq 5$, we know that the Whitehead torsion $\tau \in Wh_1(0)$ vanishes. Thus, there is a PL homeomorphism $W \cong S^{n-1} \times [0, 1]$ by Theorem 2.2.11 and $N = W \cup D_1 \cup D_2$ is therefore PL homeomorphic to S^n. □

2.2.1 Relating K_0 and K_1

Finally, we prove that there is a localization sequence relating K_1 and K_0 in certain cases. Let I be an ideal in R and let $GL(I)$ be the kernel of the map $GL(R) \to GL(R/I)$. Let $E_n(R, I)$ be the normal subgroup of $E_n(R)$ generated by matrices $e_{i,j}(r)$ such that $r \in I$ and $1 \leq i \neq j \leq n$ and define $E(R, I)$ as the union

$$E(R, I) = \bigcup_{n \geq 1} E_n(R, I).$$

Lemma 2.2.13 (Relative Whitehead Lemma)**.** *The group $E(R, I)$ is normal in $GL(I)$ and*

$$[GL(I), GL(I)] \subset E(R, I).$$

Exercise 2.2.14. Prove Lemma 2.2.13.

Definition 2.2.15. Define the relative K_1 group as

$$K_1(R, I) := GL(I)/E(R, I).$$

We can also define a relative algebraic K-theory group K_0. Given a ring R and an ideal $I \in R$, we can for the trivial square-zero extension of R by I, denoted $R \oplus I$.

Definition 2.2.16. We define the relative K_0 group as

$$K_0(R, I) := \ker(K_0(R \oplus I) \to K_0(R)).$$

The definitions of relative K_1 and relative K_0 are a bit different. This is because $K_0(R, I)$ in fact does not depend on R. If $R \to S$ is a map of rings and I is mapped isomorphically onto an ideal of S, which we also call I, then $K_0(R, I) \cong K_0(S, I)$. We therefore sometimes simply write

$$K_0(I) := K_0(R, I).$$

The same is not true for $K_1(R, I)$ and in fact there are maps of rings $R \to S$ where I maps isomorphically onto an ideal of S, also denoted I, and yet

$$K_1(R, I) \ncong K_1(S, I).$$

It is known that $K_1(R, I)$ is independent of R if and only if $I^2 = I$, or in other words I is idempotent by [?qx, Vaserstein 14.2]

We will leave part of the proof of the localization sequence as an exercise.

Exercise 2.2.17. If $f\colon R \to S$ is a ring map sending I isomorphically onto an ideal of S, also denoted I, then prove that

$$K_0(R, I) \cong K_0(S, I).$$

Hint: Show that $GL(S)/GL(S \oplus I) = 1$. Then prove that if $I \cap J = 0$, then

$$K_0(I + J) = K_0(I) \oplus K_0(J).$$

Use this to prove that there is an exact sequence

$$1 \to GL(I) \to GL(R) \to GL(R/I) \xrightarrow{\delta} K_0(I) \to K_0(R) \to K_0(R/I). \quad (2.2.18)$$

Proposition 2.2.19. *There is an exact sequence*

$$K_1(R, I) \to K_1(R) \to K_0(R/I) \to K_0(I) \to K_0(R) \to K_0(R/I). \quad (2.2.20)$$

Proof. By Exercise 2.2.17, we know that there is an exact sequence (2.2.18). Passing to quotients by $E(R)$ and $E(R/I)$ gives exactness of the sequence (2.2.20) at $K_1(R/I)$. By Exercise 2.2.17, it therefore suffices to show that the sequence (2.2.20) is exact at $K_0(R)$. Let g be an element of the kernel of the composite

$$GL(R) \to K_1(R) \to K_1(R/I).$$

Then we know that the image of g in $GL(R/I)$, is in $E(R/I)$. Write $\overline{g}$ for this element of $E(R/I)$. Since the map $E(R) \to E(R/I)$ is surjective, there is an element $e \in E(R)$ mapping to $\overline{g}$. Consequently, ge^{-1} maps to $1 \in E(R/I) \subset GL(R)$. So ge^{-1} is in the kernel of $GL(R) \to GL(R/I)$, which we denoted $GL(I)$. Write $[ge^{-1}]$ for the equivalence class of ge^{-1} in $GL(I)/E(R, I)$.

To summarize, for any $g \in K_1(R)$ in the kernel $K_1(R) \to K_1(R/I)$, we have produced a well-defined element $[ge^{-1}]$ in $K_1(R, I)$ that maps to $K_1(R)$. Thus, the sequence (2.2.20) is exact at $K_1(R)$. □

The existence of this sequence was known already in the 1960's [2], but it was not known how to extend the sequence to the left. It was expected that there were groups K_n for all $n \in \mathbb{Z}$ that produce a long exact sequence, but it remained an open question until the 1970's. In Section 2.4, we discuss the first proposed definition of higher algebraic K-theory groups in the early 1970's due to Milnor. This was defined in order to extend the groups K_0, K_1, and K_2, where K_2 is defined in Section 2.3 and the choice of definition of the higher Milnor K-theory groups is clearly inspired by the definition of K_2.

2.3 The Steinberg group and its center

We now discuss K_2 of a ring and the higher Milnor K-theory groups of a field.

Definition 2.3.1. Let A be a ring. Let $n \geq 3$, then the Steinberg group $St_n(A)$ is the group with generators $x_{i,j}(r)$ for $a \in A$ and $i, j \in \{1, \dots, n\}$ with $i \neq j$

and relations

$$x_{i,j}(a)x_{i,j}(b) = x_{i,j}(a+b) \tag{2.3.2}$$

$$[x_{i,j}(a), x_{k.\ell}(s)] = \begin{cases} 1 & \text{if} j \neq k \text{ and } i \neq \ell \\ x_{i,\ell}(rs) & \text{if} j = k \text{ and } i \neq \ell \\ x_{k,j}(-sr) & \text{if} j \neq k \text{ and } i = \ell, \end{cases} \tag{2.3.3}$$

which are called the *Steinberg relations*.

Exercise 2.3.4. Show that the transvections $e_{i,j}(a)$ in $E_n(A)$ for $n \geq 3$ satisfy the Steinberg relations.

As a consequence of Exercise 2.3.4, there is a canonical surjective group homomorphism

$$St_n(A) \to E_n(A)$$

for $n \geq 3$ mapping $x_{i,j}(a)$ to $e_{i,j}(a)$. Since the Steinberg relations for n include the Steinberg relations for all $k < n$, there is a canonical inclusion

$$St_{n-1}(A) \to St_n(A)$$

and we define

$$St(A) = \bigcup St_n(A).$$

Exercise 2.3.5. Prove that a level map $\{A_i\} \to \{B_i\}$ of sequences of groups, which is a levelwise surjection induces a surjection

$$\operatorname*{colim}_i A_i \to \operatorname*{colim}_i B_i.$$

Definition 2.3.6. Let A be a ring. We define $K_2(A)$ to be the kernel of the canonical surjection

$$St(A) \to E(A).$$

As a consequence of the definition, there is an exact sequence

$$1 \to K_2(A) \to St(A) \to GL(A) \to K_1(A) \to 1.$$

As defined, it is not clear that $K_2(A)$ is an abelian group, because $St(A)$ is not necessarily abelian. However, it turns out that it is an abelian group. Moreover, we have the following result of Steinberg, but we omit the proof.

Theorem 2.3.7 (Steinberg)**.** *The group $K_2(A)$ is abelian and it is exactly the center of $St(A)$.*

We end by remarking that the group K_2 really deserves to be called K_2 in the following sense.

Theorem 2.3.8. *Let A be a Dedekind domain with field of fractions F, then there is a long exact sequence*

$$K_2(F) \to \bigoplus_{\mathfrak{p}\in\mathbb{P}} K_1(A/\mathfrak{p}) \to K_1(A) \to K_1(F) \to \bigoplus_{\mathfrak{p}\in\mathbb{P}} K_0(A/\mathfrak{p}) \to K_0(A) \to K_0(F) \to 0$$

where $\mathbb{P}$ is the set of prime ideals in A.

We will prove this as a consequence of the localization sequence and dévissage due to Quillen [12] in 1972 later in the course. However, it is important to know that the localization sequence as stated in Theorem 2.3.8 had already been proven by Milnor in 1971 and it appears in [10].

2.4 Milnor K-theory of fields

Milnor extended the definition of K_2 to higher algebraic K-theory groups of fields, now known as Milnor K-theory groups, in the 1970's.

Let k be a field. We define the tensor algebra of the group of units k to by

$$T(k^\times) = \bigoplus_{i\geq 0}(k^\times)^{\otimes i}$$

where $(k^\times)^{\otimes 0} = \mathbb{Z}$. This is also known as the free associative algebra on $k^\times$. Write $\ell(x)$ for an element in $k^\times$ in degree 1 corresponding to $x \in k^\times$. We can then define Milnor K-theory of a field as a graded ring all at once.

Definition 2.4.1. The Milnor K-theory groups of a field k are

$$K_*^M(k) := T(k^\times)/(\ell(x)\otimes\ell(1-x) : 1 \neq x \in k^\times)$$

Note that the ideal generated by the elements $\ell(x)\otimes\ell(1-x)$ is a homogeneous ideal and therefore it makes sense to form the quotient in graded rings.

It is clear that

$$K_0^M(k) = \mathbb{Z} = K_0(k) \text{ and}$$
$$K_1^M(k) = k^\times = K_1(k)$$

for any field k. By a theorem of Matsumoto, $K_2^M(k) \cong K_2(k)$.

Theorem 2.4.2 (Matsumoto)**.** *There is an isomorphism*

$$K_2^M(k) \cong K_2(k)$$

for any field k.

This motivated Milnor's definition of higher algebraic K-theory groups. Note that

$$K_0(\mathbb{F}_q) = K_0^M(\mathbb{F}_q) = \mathbb{Z} \text{ and}$$
$$K_1(\mathbb{F}_q) \cong \mathbb{F}_q^\times$$

which is a cyclic group of order $q-1$. In light of this, the following result gives a complete calculation of the Milnor K-theory of of finite fields.

Proposition 2.4.3. *The Milnor K-theory groups $K_n^M(\mathbb{F}_q)$ vanish for all $n \geq 2$. Consequently, there is an isomorphism of graded rings*

$$K_*^M(\mathbb{F}_q) \cong \mathbb{Z} \oplus \mathbb{F}_q^\times$$

where $\mathbb{Z} \oplus \mathbb{F}_q^\times$ is the trivial square zero extension of $\mathbb{Z}$ by the cyclic group $\mathbb{F}_q^\times$.

Proof. We first show that

$$\Big(\mathbb{F}_q^\times \otimes \mathbb{F}_q^\times / (x \otimes (1-x) : x \neq 1, 0)\Big) = 1.$$

We write $(x \otimes y) \cdot (z \otimes w)$ for the group operation and 1 for the unit. Note that $\mathbb{F}_q^\times$ is cyclic of order $q-1$ and consequently $\mathbb{F}_q^\times \otimes \mathbb{F}_q^\times$ is also cyclic of order $q-1$. This cyclic group is generated by $x \otimes x$ whenever x is a generator of $\mathbb{F}_q^\times$.

We split into two cases. If q is even, then we know $2x \otimes x = 0$ in $\mathbb{F}_q \otimes \mathbb{F}_q$. So $x \otimes x = x \otimes -x$. We also know that $x \otimes -x = x \otimes 1$ in $K_2^M(\mathbb{F}_q)$ by the relations and since 1 is the identity in $\mathbb{F}_q^\times$, the element $x \otimes 1$ is trivial in the group $K_2^M(\mathbb{F}_q)$. In other words, we conclude that

$$x \otimes x = 1$$

for all elements $x \otimes x \in K_2^M(\mathbb{F}_q)$ where x is a generator of $\mathbb{F}_q^\times$. This implies that the group $K_2^M(\mathbb{F}_q)$ is trivial. In fact, essentially the same argument implies that $K_n^M(\mathbb{F}_q) = 0$ for $n > 2$.

When q is odd, we still know that $x \otimes -x = x \otimes 1$ is trivial and consequently, we have skew-symmetry

$$(x \otimes y) \cdot (y \otimes x) = (x \otimes -xy) \cdot (y \otimes -xy) = xy \otimes -xy = 1$$

in $K_2^M(\mathbb{F}_q)$. This immediately implies that $(x \otimes x)^2 = 1$ and more generally one can show that

$$(x \otimes x)^{mn} = x^m \otimes x^n$$

when m, n are odd. The set of odd powers of elements in $K_2^M(\mathbb{F}_q)$ is exactly the same as the non-squares, by construction. If there exists a non-square u such that $1-u$ is also a non-square in $\mathbb{F}_q$, then all elements are divisible by the element $u \otimes (1-u) = 0$, or in other words all elements are trivial. To find such a u, we note that there is an involution $u \mapsto 1 - u$ on the set $\mathbb{F}_q = \{0, 1\}$. and the set $\mathbb{F}_q - \{0, 1\}$ consists of $(q-1)/2$ non squares, but only $(q-3)/2$ squares. In other words, there are strictly less squares than non-squares and there must be an orbit of the C_2-action that is completely contained in the non-squares. Again, essentially the same proof implies that $K_n^M(\mathbb{F}_q) = 0$ for $n > 2$. □

We include this result in order to indicate that Milnor K-theory, though very interesting in its own right, is this is not the richest invariant. We will see a different construction of higher algebraic K-theory groups where the algebraic K-theory of fields are nontrivial in arbitrarily high degrees.

Chapter 3

Group completion algebraic K-theory

As we hinted at in the beginning of the course, there are really two main flavors of algebraic K-theory: group completion algebraic K-theory and algebraic K-theory of of a category with a notion of exact sequences. Here we describe the constructions of group completion algebraic K-theory, due to Quillen and Segal, known as the $+$-construction and the $S^{-1}S$ construction. We then give a sketch of the computation of algebraic K-theory of finite fields due to Quillen.

3.1 The $+$-construction

Appendix A

Fundamentals

A.1 Categories

Category theory will be of fundamental importance in studying algebraic K-theory. We recall the basic notions here.

Definition A.1.1. A category $\mathcal{C}$ consists of

1. a class of objects denoted $\text{ob}(\mathcal{C})$
2. for each pair of objects c, c' a set $\mathcal{C}(c, c')$ of morphisms from c to c', and
3. for any triple c, c', c'' a map of sets

$$- \circ - \colon \mathcal{C}(c', c'') \times \mathcal{C}(c', c'') \to \mathcal{C}(c, c'')$$

4. For each object c an element

$$\text{id}_C \in \mathcal{C}(c, c).$$

satisfying

$$(f \circ g) \circ h = f \circ (g \circ h)$$

for each triple of maps

$$(f, g, h) \in \mathcal{C}(c''', c'') \times \mathcal{C}(c'', c') \times \mathcal{C}(c', c)$$

and

$$\text{id}_{c'} \circ f = f = f \circ \text{id}_c$$

for each map $f \colon c \to c'$ in $\mathcal{C}(c, c')$, where each of these identities are functorial in the appropriate sense.

Example A.1.2. Given a category $\mathcal{C}$, let $\mathcal{C}^{\mathrm{op}}$ be the category whose objects are the same as the objects in $\mathcal{C}$ and there is a unique morphism $f^{\mathrm{op}}\colon\ b \to a$ for every morphism $f\colon\ a \to b$ in $\mathcal{C}$.

Remark A.1.3. Note that we use the convention that all categories are locally small by requiring that $\mathcal{C}(c, c')$ is a set rather than a proper class. We say a category is a small category if, in addition, it has a set of objects $\mathrm{ob}(\mathcal{C})$.

Definition A.1.4. The skeleton $\mathrm{sk}\mathcal{C}$ of a cat egory $\mathcal{C}$ is the full subcategory of $\mathcal{C}$ consisting of one object for each isomorphism class of objects in $\mathcal{C}$. We say that $\mathcal{C}$ is *skeletally small* if $\mathrm{sk}\mathcal{C}$ is a small category.

Example A.1.5. The category of sets and set maps, denoted Set, is an example of a category that is not a small category. The category of finite sets and maps of finite sets is skeletally small and we write Fin for its skeleton.

Example A.1.6. Let $\mathcal{C}$ be a small category. Then there is a category $\mathsf{Arr}(\mathcal{C})$ whose objects are maps $f\colon a \to b$ in $\mathcal{C}$ and a morphism from $a \to b$ to $c \to d$ is a commuting square

$$\begin{tikzcd} a \arrow[r] \arrow[d] & b \arrow[d] \\ c \arrow[r] & d \end{tikzcd}$$

in $\mathcal{C}$. Composition is defined by vertical composition of squares. We call this category the *arrow category*.

Definition A.1.7. A functor $F\colon\ \mathcal{C} \to \mathcal{D}$ associates to each object $c \in \mathrm{ob}\,\mathcal{C}$ an object $F(c) \in \mathrm{ob}\,\mathcal{D}$, to each morphism $f\colon c \to c'$ a morphism $F(f)\colon\ F(C) \to F(c')$ in $\mathcal{D}$ such that $F(f \circ g) = F(f) \circ F(g)$.

Example A.1.8. Let Top denote the category of topological spaces and continuous maps. Then any topological space may be regarded as a set by forgetting the topology and any continuous map is in particular a map of sets, so this defines a functor

$$U\colon\ \mathsf{Top} \to \mathsf{Set}$$

called the forgetful functor.

Definition A.1.9. A subcategory $\mathcal{C}$ in $\mathcal{D}$ consists of a category C and a functor $\iota\colon\ C \to \mathcal{D}$ such that

$$\mathcal{C}(c, c') \to \mathcal{D}(\iota(c), \iota(c')) \tag{A.1.10}$$

is injective. In particular, this implies that the set of objects in c inject in the set of objects of $\mathcal{D}$.

We say $\mathcal{C}$ is a *full* subcategory if the map (A.1.10) is also surjective. More generally, we say a functor $F\colon\ \mathcal{C} \to \mathcal{D}$ is faithful if the associated map

$$\mathcal{C}(c, c') \to \mathcal{D}(F(c), F(c')) \tag{A.1.11}$$

Definition A.1.12. Given a pair functors $F, G\colon \mathcal{C} \to \mathcal{D}$, then a natural transformation $\gamma\colon F \Rightarrow G$ associates to an object x in $\mathcal{C}$ a map $\gamma_x\colon F(x) \to G(x)$ and to a map $f\colon x \to y$ a commutative diagram

$$\begin{tikzcd} F(x) \arrow[r, "F(f)"] \arrow[d, "\gamma_x"'] & F(y) \arrow[d, "\gamma_y"] \\ G(x) \arrow[r, "G(f)"] & G(y) \end{tikzcd} \tag{A.1.13}$$

and to a composable pair of morphisms $g \circ f$ in $\mathcal{C}$ where $f\colon x \to y$ and $g\colon y \to z$, then there is a commutative diagram

$$\begin{tikzcd} F(x) \arrow[r, "F(f)"] \arrow[d, "\gamma_x"'] & F(y) \arrow[r, "F(g)"] \arrow[d, "\gamma_y"] & F(z) \arrow[d, "\gamma_z"] \\ G(x) \arrow[r, "G(f)"] & G(y) \arrow[r, "G(g)"] & G(z). \end{tikzcd}$$

Definition A.1.14. The functor category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is the category whose objects are functors $F, G\colon \mathcal{C} \to \mathcal{D}$ and whose morphisms are natural transformations $\gamma\colon F \Rightarrow G$, where composition is given by composition of natural transformations where $\alpha \circ \beta$ is defined on objects by

$$(\alpha \circ \beta)_x := \alpha_x \circ \beta_x$$

and on morphisms $f\colon x \to y$ by vertical composition of squares of the form (A.1.13) and similarly for compositions of morphisms.

Definition A.1.15. A factorization system on a small category $\mathcal{C}$ consists of of a pair of subcategories (E, M) such that

1. E and M each contain all of the isomorphisms in $\mathcal{C}$ and consequently all objects in $\mathcal{C}$, and
2. every morphism $f\colon A \to C$ in $\mathcal{C}$ can be factored as $f = m \circ e$ where $e\colon A \to B$ is a map in E and $m\colon B \to C$ is a map in M.
3. this factorization defines a functor

$$\mathsf{Arr}(\mathcal{C}) \to \mathsf{Arr}(E) \times_{\mathcal{C}} \mathsf{Arr}(M)$$

sending f to (m, e) and a commutative square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & C \arrow[d, "g''"] \\ A' \arrow[r, "f'"] & C' \end{tikzcd}$$

to the composition of squares

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d, "g"'] & B \arrow[r, "m"] \arrow[d, "g'"] & C \arrow[d, "g''"] \\ A' \arrow[r, "e'"] & B' \arrow[r, "m'"] & C'. \end{tikzcd}$$

Here the category $\mathsf{Arr}(E) \times_{\mathcal{C}} \mathsf{Arr}(M)$ is the pullback in the category of small categories of the diagram

$$\mathsf{Arr}(M) \xrightarrow{t} \mathcal{C} \xleftarrow{s} \mathsf{Arr}(E)$$

where t sends an object $e\colon A \to B$ in $\mathsf{Arr}(E)$ to B and s sends an object $m\colon B \to C$ in $\mathsf{Arr}(M)$ to B. On morphisms these functors are defined in the evident way. In other words, the category $\mathsf{Arr}(E) \times_{\mathcal{C}} \mathsf{Arr}(M)$ has objects pairs of morphisms (e, m) such that, as maps in $\mathcal{C}$, they are composable.

A.2 Sets

Definition A.2.1. A partial order on a set P is a binary relation $\leq$ satisfying

1. $x \leq x$ (Reflexivity),
2. if $x \leq y$ and $y \leq x$, then $x = y$ (Anti-symmetry),
3. if $x \leq y$ and $y \leq z$, then $x \leq z$ (Transitivity).

We say x is related to y if either $x \leq y$ or $y \leq x$. A set with a partial order will be called a partially ordered set or a POSet for short.

Any POSet P can be considered as a small category with objects the elements of P and morphism sets

$$P(x, y) = \begin{cases} * & x \leq y \\ \emptyset & \text{otherwise} \end{cases}$$

and composition

$$P(y, z) \times P(x, y) \to P(x, z)$$

is defined in the obvious way using transitivity.

Definition A.2.2. A total order on a set X is a partial order satisfying the totality axiom: for each $x, y \in X$ either $x \leq y$ or $y \leq x$.

In other words, a totally ordered set is simply a partially ordered set in which any two elements are related.

Example A.2.3. The natural numbers

$$\mathbb{N}_0 = \{0, 1, 2, \dots\}$$

are a totally ordered set with the usual total order. The set $n = \{0, 1, \dots, n\}$ is a finite totally ordered set equipped with the restriction of the total order on $\mathbb{N}_0$. When we view the totally ordered set n as a category it can be depicted as

$$0 \leftarrow 1 \leftarrow 2 \leftarrow \cdots \leftarrow n.$$

Appendix B

Simplicial methods

B.1 The simplex category

Definition B.1.1. Let Ord denote the category of finite totally ordered sets and maps of finite sets that preserve the total order. Let Δ be the the skeleton of this category with exactly one object for each isomorphism class of objects in Ord. The objects in Δ are

$$[n] = \{0, 1, \ldots, n\}$$

for $n \geq 0$ and nondecreasing maps of finite sets.

Example B.1.2. There is a canonical factorization system (E, M) on the category Δ. The category E consists of all isomorphisms as well as the closure under composition of the maps of the form

$$\sigma_k \colon [n] \to [n-1]$$

for $0 \leq k \leq n$ such that

$$\sigma_k(i) = \begin{cases} i & \text{if } i < k \\ k & \text{if } i = k, k+1 \\ i-1 & \text{if } k+1 \leq i \leq n. \end{cases}$$

The category M consists of all isomorphisms as well as the closure under composition of the morphisms

$$\delta_k(i) = \begin{cases} i & \text{if } i < k \\ i+1 & \text{if } k \leq i \leq n. \end{cases}$$

We can therefore depict the category Δ as follows

$$\begin{tikzcd}
{[0]} \arrow[r, shift left=2] \arrow[r, shift right=2] & {[1]} \arrow[l] \arrow[r, shift left=4] \arrow[r] \arrow[r, shift right=4] \arrow[l, shift left=2, start anchor=east, end anchor=west, to=2-1, phantom] & {[2]} \arrow[l, shift left=2] \arrow[l, shift right=2] \arrow[r, shift left=6] \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[r, shift right=6] & \cdots \arrow[l, shift left=4] \arrow[l] \arrow[l, shift right=4]
\end{tikzcd} \tag{B.1.3}$$

Definition B.1.4. Given a category $\mathcal{C}$, a s*implicial object in* $\mathcal{C}$ is a functor

$$\Delta^{\mathrm{op}} \to \mathcal{C}.$$

This can be described as a collection of objects $\{X[i]\}_{i\geq 0}$ sitting in a diagram

$$X[0] \mathrel{\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow}} X[1] \mathrel{\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow}} X[2] \mathrel{\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow}} \ldots \tag{B.1.5}$$

where the maps in the diagram are called the face maps

$$\{\partial_i \colon X[n+1] \to X[n]\}_{0\leq i\leq n+1}$$

and degeneracy maps

$$\{s_i \colon X[n] \to X[n+1]\}_{0\leq i\leq n}$$

and these must satisfy the simplicial identities:

1. $\partial_i \circ \partial_j = \partial_{j-1} \circ \partial_i$ if $i < j$,

2. $s_i \circ s_j = s_j \circ s_{i-1}$ if $i > j$,

3. $\partial_i \circ s_j = \begin{cases} s_{j-1} \circ \delta_i & \text{if } i < j \\ \mathrm{id}_{X_j} & \text{if } i = j \text{ or } i = j+1 \\ s_j \circ \delta_{i-1} & \text{if } i > j+1 \end{cases}$

Morphisms of simplicial objects in $\mathcal{C}$ are simply natural transformations of functors $\Delta^{\mathrm{op}} \to \mathcal{C}$.

Example B.1.6. When $\mathcal{C} = \mathsf{Set}$, we call a simplicial object in Set simply a simplicial set and we write sSet for this category. We could also consider pointed sets Set_* and we will write sSet_* for the category of simplicial objects in pointed sets (or equivalently, pointed simplicial sets). When $\mathcal{C} = \mathsf{Top}$ we refer to the category of simplicial objects in Top simply as simplicial spaces and we denote the category of simplicial spaces by sTop.

Example B.1.7. Given an object $[n] \in \Delta$, we can form

$$\mathrm{Hom}_{\Delta}(-,[n]) \colon \Delta^{\mathrm{op}} \to \mathsf{Set}.$$

This is clearly a simplicial set. We denote this simplicial set

$$\Delta^n := \mathrm{Hom}_{\Delta}(-,[n]).$$

Exercise B.1.8. Use the Yoneda lemma to show that

$$\mathrm{Hom}_{\mathsf{sSet}}(\Delta^n, X) = X[n].$$

Hint: Write $\iota_n := \mathrm{id} \in \mathrm{Hom}_\Delta([n],[n])$. Given a simplicial map $\varphi\colon \Delta^n \to X$, associate to this map an n-simplex $\varphi(\iota_n) \in X[n]$. Prove that this gives a bijection.

Definition B.1.9. We define a topological space

$$|\Delta^n| = \{(t_0, \dots, t_n) \in [0,1]^{n+1} : \sum t_i = 1\}$$

equipped with the subspace topology. In fact, the spaces $|\Delta^n|$ form cosimplicial space

$$|\Delta^\bullet|(-)\colon \Delta \to \mathsf{Top}$$

such that $|\Delta^\bullet|([n]) = |\Delta^n|$. The map

$$\theta_*\colon |\Delta^n| \to |\Delta^m|$$

induced by $\theta\colon [n] \to [m]$ is defined by

$$\theta(t_0, t_1, \dots t_n) = (t'_0, \dots t'_m)$$

where

$$t'_i = \begin{cases} 0 & \text{if } \theta^{-1}(i) \neq \emptyset, \\ \sum_{j \in \theta^{-1} i} t_j & \text{if } \theta^{-1}(i) = \emptyset. \end{cases}$$

Remark B.1.10. There is also an analogous cosimplicial simplicial set

$$\Delta^n\colon \Delta \to \mathsf{sSet}$$

given on n-simplices by

$$\Delta^n = \mathrm{Hom}_\Delta(-, [n])$$

where the cosimplicial structure is given by regarding this as a functor in the second variable.

Definition B.1.11. Given a pair of simplicial sets X, Y, we define the product of simplicial sets $X \times Y$ by letting the n-simplices be simply

$$(X \times Y)[n] = X[n] \times Y[n]$$

and the face and degeneracy maps simply as

$$\partial_i^{X \times Y} = (\partial_i^X, \partial_i^Y) \text{ and } s_i^{X \times Y} = (s_i^X, s_i^Y).$$

Note that the non-degenerate n-simplices of $X \times Y$ are not simply the product of the non-degenerate n-simplices of X with the non-degenerate n-simplices of Y.

Definition B.1.12. We also define an internal hom $\underline{\mathrm{Hom}}(X, Y)$ to be the simplicial set with n-simplices

$$\underline{\mathrm{Hom}}(X, Y)[n] := \mathrm{Hom}_{\mathsf{sSet}}(X \times \Delta^n, Y)$$

and face and degeneracy maps induced by considering Δ^n as a cosimplicial object in sSet and applying the functor $\mathrm{Hom}_{\mathsf{sSet}}(X \times -, Y)$.

Proposition B.1.13. *The category* $(\mathsf{sSet}, \times, \Delta^0)$ *is a a symmetric monoidal category and there is a natural isomorphism*

$$Hom_{\mathsf{sSet}}(X \times Y, Z) \cong Hom_{\mathsf{sSet}}(X, \underline{Hom}(Y, Z)).$$

Definition B.1.14. We say $f\colon X \to Y$ is homotopy equivalent to $g\colon X \to Y$ if there is a map of simplicial sets

$$H\colon X \times \Delta^1 \to Y$$

such that $H|_{(\iota_0)_*(\Delta^0)} = f$ and $H|_{(\iota_1)_*(\Delta^0)} = g$ where $(\iota_j)_*\colon \Delta^0 \to \Delta^1$ is the map of simplicial sets induced by the unique map $\iota_j\colon [0] \to [1]$ map in Δ sending 0 to j; i.e.

$$(\iota_j)_* := \mathrm{Hom}(-, \iota_j)\colon \mathrm{Hom}(-, [0]) \to \mathrm{Hom}(-, [1]).$$

Let $\partial\Delta^n$ denote the smallest sub-simplicial set of Δ^n generated by the faces $d_j(\iota_n)$ for $0 \leq j \leq n$. where ι_n denotes the element

$$\iota_n = \mathrm{id} \in \mathrm{Hom}_\Delta([n], [n]) = \mathrm{Hom}_{\mathsf{sSet}}(\Delta^n, \Delta^n) = (\Delta^n)_n$$

Let Λ^n_k denote the smallest sub-simplicial set of Δ^n generated by the faces $d_i(\iota_n)$ for the face $d_k(\iota_n)$. We say a simplicial set is a Kan complex if there exists a unique lift in any diagram of the form

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d] & X \\ \Delta^n \arrow[ur, dashed, "\exists! h"'] & \end{tikzcd}$$

for $0 \leq k \leq n$. Where Λ^n_k is the sub-simplicial set of Δ^n generated by $d_j(\iota_n)$

We can consider the full subcategory of the category of simplicial sets whose objects are the category of Kan complexes.

Let X, Y be Kan complexes and define

$$[X, Y] = \mathrm{Hom}(X, Y)/\simeq$$

where $\simeq$ is the equivalence relation given by simplicial homotopy equivalence.

Definition B.1.15. Define the homotopy category of simplicial sets, denoted ho(sSet) to be the category whose objects are Kan complexes and morphisms from $X \to Y$ are $[X, Y]$. We define the homotopy category of Top, denoted ho(Top) to be the category whose objects are CW complexes and maps are homotopy classes of maps from X to Y, which we also denote $[X, Y]$.

The following is one of the fundamental theorems of simplicial sets.

Theorem B.1.16. *There is a natural isomorphism $\gamma_{-,-}$ of functors $sSet^{\mathrm{op}} \times \mathsf{Top} \to$ Set defined on objects by*

$$\gamma_{X,Y} \colon Hom_{\mathsf{Top}}(|X|, Y) \cong Hom_{\mathsf{sSet}}(X, sin(Y)).$$

Construction B.1.17. Let X be an object in Top. We define

$$\mathrm{sing}(X) = \mathrm{Hom}_{\mathsf{Top}}(|\Delta^{\bullet}|, X).$$

This is clearly a simplicial set. One can check that it is also a Kan complex when X is a CW complex.

We will now define geometric realization, more generally, of a simplicial space.

Construction B.1.18. Let $X_{\bullet}$ be an object in sTop. Then we define

$$|X_{\bullet}| := \mathrm{coeq}\left(\coprod_{f\colon i \to j \in \Delta^{op}} X_i \times |\Delta^j| \underset{\amalg X_f \times \mathrm{id}_{|\Delta^j|}}{\overset{\amalg\, \mathrm{id}_{X_i} \times |\Delta^f|}{\rightrightarrows}} \coprod_{[n] \in \Delta^{\mathrm{op}}} X_n \times |\Delta^n| \right)$$

where the coproducts are equipped with the coproduct topology and coequalizer is equipped with the quotient topology. Sometimes, we simply write this as

$$|X| = \left(\coprod_{n \geq 0} |\Delta^n| \times X_n| \right) / \sim$$

where $\sim$ is the equivalence relation generated by

$$(x, \partial_i y) \sim (\delta_i x, y) \text{ and } (x, s_i y) \simeq (\sigma_i x, y).$$

One can check that these two spaces are the same.

Example B.1.19. When X is a simplicial set, we define $|X|$ in the same way by equipping $X[n]$ with the discrete topology for all $n \geq 0$.

Construction B.1.20. There is an alternate way to construct the geometric realization of a simplicial set X that is less intuitive, but makes the proof of Theorem B.1.16 quite easy. Let $\Delta \downarrow X$ denote the category whose objects are maps of simplicial sets $x\colon \Delta^n \to X$ where $n \geq 0$ and whose maps $x \to y$ are commuting triangles

$$\begin{tikzcd} \Delta^n \arrow[r, "x"] \arrow[d, "\theta_*"'] & X \\ \Delta^m \arrow[ur, "y"'] & \end{tikzcd}$$

where $\theta\colon [n] \to [m]$ is a map in Δ. Composition is defined in the evident way. Then we can define $|X|$ to be the colimit

$$|X| = \operatorname*{colim}_{\Delta\downarrow X} |\Delta^n|$$

in the category of topological spaces.

Exercise B.1.21. Show that the definition of $|X|$ in Construction B.1.20 is the same as in Construction B.1.18 for simplicial sets up to homeomorphism.

Proof of Theorem B.1.16. Recall that $Hom_{\mathsf{Top}}(-, Y)$ sends limits in $\mathsf{Top}^{\mathrm{op}}$, or in other words colimits in Top, to limits in Set. Therefore,

$$\begin{aligned} \mathrm{Hom}_{\mathsf{sSet}}(|X|, Y) =& \mathrm{Hom}_{\mathsf{sSet}}(\operatorname*{colim}_{\Delta\downarrow X} |\Delta^n|, Y) && \text{(B.1.22)} \\ \cong& \lim_{\Delta\downarrow X} \mathrm{Hom}_{\mathsf{sSet}}(|\Delta^n|, Y) && \text{(B.1.23)} \\ =& \lim_{\Delta\downarrow X} \mathrm{sing}(Y)_n && \text{(B.1.24)} \\ =& \lim_{\Delta\downarrow X} \mathrm{Hom}_{\mathsf{sSet}}(\Delta^n, \mathrm{sing}(Y)) && \text{(B.1.25)} \\ \cong& \mathrm{Hom}_{\mathsf{sSet}}(\operatorname*{colim}_{\Delta\downarrow X} \Delta^n, \mathrm{sing}(Y)) && \text{(B.1.26)} \\ \cong& \mathrm{Hom}_{\mathsf{sSet}}(X, \mathrm{sing}(Y)) && \text{(B.1.27)} \end{aligned}$$

where the equality (B.1.22) is the definition of geometric realization, the isomorphism (B.1.23) follows because $\mathrm{Hom}_{\mathsf{Top}}(-, Y)$ send limits in $\mathsf{Top}^{\mathrm{op}}$ to limits in Set, as remarked above, the equality (B.1.24) holds by definition of $\mathrm{sing}(Y)$, the equality (B.1.25) holds by Exercise (B.1.8), the isomorphism (B.1.26) hold because $\mathrm{Hom}_{\mathsf{Top}}(-, Y)$ sends limits in $\mathsf{Top}^{\mathrm{op}}$ to limits in Set. The last isomorphism holds because X is a cocone for the functor from $\Delta \downarrow X \to \mathsf{sSet}$ sending $\Delta^n \to X$ to Δ^n. □

Recall that Cat denotes the category of small categories. Let $[n]$ be the small category

$$0 \to 1 \to 2 \to \cdots \to n$$

described in Example A.2.3. These form a cosimplicial object in Cat by letting the codegeneracy map σ_i be given by composing the morphisms with source i and $i + 1$. and the coface map δ_i given by inserting an identity map in the i-th position.

Definition B.1.28. Given a small category $\mathcal{C}$, we define a simplicial set by letting the n-simplices be

$$N\mathcal{C}[n] := \mathsf{Cat}([n], \mathcal{C}),$$

which should be viewed as a sequence of n composable morphisms, if $n > 0$ and, and simply the objects in $\mathcal{C}$ if $n = 0$. The face maps are given by composing two adjacent morphisms and the degeneracy maps are given by inclusion of the identity morphism. In other words, we simply use the fact that

$$[\bullet] \colon \Delta \to \mathsf{Cat}$$

is a cosimplicial object in small categories and therefore, by functoriality,

$$\mathsf{Cat}([n], \mathcal{C})$$

forms as simplicial set.

For example, any discrete group G can be regarded as a small category with one object $*$ and morphism set $G(*, *) = G$. By unpacking the definition, we see that

$$NG[n] = G^n$$

where $G^0 = *$. We can be even more explicit in this case. The face maps

$$\partial_i \colon G^{n+1} \to G^n$$

are defined by

$$\partial_i(g_1, \dots g_{n+1}) = \begin{cases} (g_2, g_3, \dots, g_{n+1}) & \text{if } i = 0 \\ (g_1, \dots, g_i g_{i+1}, \dots, g_{n+1}) & \text{if } 0 < i < n+1 \\ (g_1, g_3, \dots, g_n) & \text{if } i = n \end{cases}$$

and the degeneracy maps are defined by

$$s_i(g_1, \dots g_n) = (g_0, g_1, \dots g_{i-1}, 1, g_i, \dots, g_{n-1})$$

Note, that there was notion special about doing this construction for a discrete group. More generally, given a topological group G we define an object in sTop in the same way and we denote it

$$B(*, G, *) \colon \Delta^{\mathrm{op}} \to \mathsf{Top}.$$

Definition B.1.29. Given a category $\mathcal{C}$, we define

$$B\mathcal{C} := |N\mathcal{C}|.$$

Given a topological group G, we also define

$$BG = |B(*, G, *)|.$$

Remark B.1.30. Of course, when G is a discrete group then

$$NG = B(*, G, *)$$

so there isn't a conflict in notation.

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