Research Statement

Gabriel J. Angelini-Knoll

I am an algebraic topologist and broadly my work sheds light on the interplay between algebraic topology, number theory, and geometric topology. My work uses techniques from stable homotopy theory to study algebraic K-theory and factorization homology. Both algebraic K-theory and factorization homology have applications to number theory and geometric topology and my work sheds new light on these connections.

Classically, algebraic K-theory takes a ring as input and its output is a graded abelian group. Algebraic K-theory has applications to number theory and geometric topology; for example, the algebraic K-theory of rings of integers in number fields generalizes the class group and the algebraic K-theory of the integral group ring of the fundamental group of a manifold has applications to h-cobordisms of manifolds. In the 1970's, Lichtenbaum and Quillen conjectured that there should be a relationship between special values of Dedekind zeta functions and quotients of orders of algebraic K-theory groups of rings of integers in number fields. Some of the first evidence for this conjecture came from Adams' work on the image of the J homomorphism in the stable homotopy groups of spheres. Adams proved that the periodic family of elements in the homotopy groups of spheres, known as the α-family, encodes special values of the Riemann zeta function and Quillen proved that the α-family has nontrivial Hurewicz image in algebraic K-theory of the integers.

In my work, I give evidence for several versions of the Lichtenbaum–Quillen conjectures for periodicities of longer wavelength, known as the Ausoni–Rognes red-shift conjectures. I shed new light on these these conjectures and I also give new formulations of these conjectures that also generalize the Lichtenbaum–Quillen conjectures to periodicities of longer wavelength. Specifically, I prove an analogue of the result of Adams and Quillen at periodicities of longer wavelength; I prove that the β-family is detected in the Hurewicz image of iterated algebraic K-theory of the integers in [AK18], which demonstrates a connection between iterated algebraic K-theory and integral modular forms. I also prove results about vanishing of Morava K-theory of algebraic K-theory [AKS20] and topological periodic cyclic homology [AKQ19], which sheds light on the Ausoni–Rognes red-shift conjectures. Additionally, I prove a result that is a first step towards calculating the algebraic K-theory of the Ravenel spectra $X(n)$ that filter between complex cobordism and the sphere spectrum [AKQ21].

My approach to algebraic K-theory uses tools from factorization homology. In particular, there is a highly nontrivial trace map from algebraic K-theory to factorization homology of the circle, which is a vast generalization of the trace of a matrix. Using extra structure on factorization homology, one can construct a refinement of the trace map that is a much closer approximation to algebraic K-theory, known as topological cyclic homology. In my work, I developed a new tool for computing factorization homology in [AKS18]. I have used this tool to make progress on calculations of factorization homology that were not possible before [AK21, AKCH20]. I am also working on the frontier of a new area of research in equivariant algebraic K-theory and equivariant factorization homology. My work gives new interpretations of multiplicative induction formulas for compact Lie groups and constructs equivariant non-commutative Witt vectors [AKGH20, AKMP20].

This document contains a background section (Section 1) and three sections on results and future directions: Factorization homology (Section 2), Red-shift phenomena in algebraic K-theory (Section 3), and equivariant factorization homology (Section 4).

1 Background

This section contains self contained background subsections on my three primary research areas: Chromatic homotopy theory (Section 1.1), Trace methods and factorization homology (Section 1.2), and Equivariant homotopy theory (Section 1.3).

1.1 Chromatic homotopy theory

One focal point of my research is the study of periodic phenomena in the homotopy groups of spheres. The homotopy groups of spheres encode maps between spheres up to continuous deformation. They are important because nice topological spaces (CW complexes) are built by out of spheres and the homotopy groups of spheres provide the gluing data for building these spaces. There is also a deep connection between the stable homotopy groups of spheres and cobordisms of framed manifolds.

Based on groundbreaking new computations of stable homotopy groups [Rav86], D. Ravenel presented a clear picture of periodic phenomena stable homotopy theory in a collection of well-regarded conjectures in [Rav84]. In [DHS88, HS98], E. Devinatz, M. Hopkins, and J. Smith, resolved all but one of these conjectures and their results remain indispensible in modern chromatic homotopy theory. For example, they proved that the category of p-local finite spectra $\mathcal{F}_p$ has an unique filtration by thick subcategories

$$0 \subset \cdots \subset \mathcal{C}_2 \subset \mathcal{C}_1 \subset \mathcal{C}_0 = \mathcal{F}_p, \tag{1}$$

up to equivalence, where $\mathcal{C}_n = \{K(n-1)_*\text{-acyclics}\}$. Here $K(n)_*$ is the generalized homology theory known as Morava K-theory $K(n)_*$ with coeffients $\mathbb{F}_p[v_n^{\pm 1}]$ when $0 < n < \infty$ and $\mathbb{Q}$ when $n = 0$. This filtration is in fact an artifact of the filtration of the moduli of p-typical formal groups by height via a correspondence going back to work of Quillen [Qui69].

To make this a bit more precise, we recall that by Brown representability, every generalized (co)homology theory (satisfying the Eilenberg-Steenrod axioms except for the dimension axiom) E^* is represented by a spectrum E. We therefore use the term (co)homology theory and spectrum interchangeably. Also, throughout, by an E_1 ring and an E_∞ ring we mean an an algebra over the E_1-operad or an algebra over the E_∞-operad in any of the modern models for the symmetric monoidal category for spectra. Therefore, E_1 rings and E_∞ rings correspond to multiplicative (co)homology theories. The Morava K-theory spectra $K(n)$ are E_1 rings and they are additionally complex oriented. We can associate a formal group to any complex oriented cohomology theory and the associated formal group to Morava K-theory $K(n)$ is the Honda height n formal group. We therefore think of spectra in $\mathcal{C}_n$ as having height $\geq n$ in a sense.

More precisely, we say a p-local finite spectrum X has type n if $X \in \mathcal{C}_n - \mathcal{C}_{n+1}$. By the periodicity theorem of [HS98], a type n spectrum X always has a periodic self map

$$v_n^{i_n} \colon \Sigma^{(2p^n-2)i_n} X \to X \tag{2}$$

for some positive integer i_n. One may form a height $n+1$ spectrum by coning off this self map, which we denote by $X/v_n^{i_n}$. Since the sphere spectrum has height 0, we may construct a type n spectrum as an iterated cofiber $V = S/(p^{i_0}, v_1^{i_1}, \ldots v_{n-1}^{i_{n-1}})$. We say an element in $\alpha \in \pi_*S$ is v_0-periodic if $\alpha \circ p^{ki_0}$ is not null homotopic for all k, other wise we say it is v_0-power torsion. If α is v_0-power torsion, then α factors through S/p^{i_0} and we write $\alpha_1 \colon S/p^{i_0} \to S$ for the associated map. We then say the map is v_1-periodic if the composite $\alpha_1 \circ v_1^{ki_1}$ is not null homotopic for all k and otherwise we say it is v_1-power torsion. Iterating this definition produces a filtration of the p-local homotopy groups of spheres

$$\cdots \subset F_2 \subset F_1 \subset \pi_* S_{(p)} \tag{3}$$

called the chromatic filtration, where $F_n = \{v_{n-1} \text{ power torsion elements}\} \subset \pi_* S$.

For example, the α family α_k for $p \geq 3$ is v_0-torsion, but it is v_1-periodic. When $p \geq 5$, the β family β_k is v_0-torsion and v_1-torsion, but it s v_2-periodic. The α-family is deeply connected to the special values of the Riemann zeta function by work of J.F. Adams [Ada66], and the β-family is deeply connected to certain integral Modular forms satisfying certain congruences by work of M. Behrens [Beh06]. When $p \geq 7$, the γ family is v_0-power torsion, v_1-power torsion and v_2-power torsion, but v_3-periodic. It has been speculated that this elements are connected to automorphic forms and special values of L-functions.

One key feature of chromatic homotopy theory is the study of the homotopy groups of a spectrum using its Bousfield localizations, which generalize the localization of a module over a ring. In fact, for any spectrum X and homology theory E, we may form the Bousfield localzation $L_E X$. This allows us to approximate the homotopy groups of a spectrum. For example, by the chromatic convergence theorem [Rav92, Theorem 7.5.7], the p-local sphere spectrum is equivalent to the limit in the diagram

$$S_{(p)} \simeq \lim_n L_n S \longrightarrow \ldots \longrightarrow L_2 S \longrightarrow L_1 S \longrightarrow L_0 S \tag{4}$$

where $L_n S$ is the Bousfield localization of the sphere spectrum at the wedge

$$K(0) \vee K(1) \vee \cdots \vee K(n)$$

of Morava K-theory spectra.

The key notion from chromatic homotopy theory that will be important for our work, is the notion of chromatic complexity of a spectrum.

Definition 1.1. We say a spectrum X has *height* $\leq n$ if $L_{K(n)} E = 0$ for all $k > n$. We say a X has *type* $\geq n$ if $L_{K(k)} X = 0$ for $k < n$.

As we hinted at earlier, the filtration of the finite p-local spectra from 1 stems from the essentially unique filtration by height of the moduli stack of formal groups $\mathcal{M}_{\mathrm{fg}}$ over the p-local integers by closed subschemes

$$\cdots \subset M(n+1) \subset M(n) \subset \cdots \subset M(1) \subset \mathcal{M}_{\mathrm{fg}}$$

and whether a spectrum is height n or type n is the question of whether our spectrum is supported on $M(n)$ or the open complement $\mathcal{M}_{\mathrm{fg}}^{\leq n}$ of $M(n+1)$. If a spectrum is both height n and type n, then it is supported on the intersection $\mathcal{H}(n) = M(n) \cap \mathcal{M}_{\mathrm{fg}}^{\leq n}$, for example Morava K-theory $K(n)$ itself has this property.

1.2 Trace methods and factorization homology

In the early 1990's, Bökstedt-Hsiang-Madsen [BHM93] developed a new technique for computing algebraic K-theory, known as trace methods, which relies on tools from algebraic topology. In particular, algebraic K-theory of E_1 rings, which generalize rings, can be approximated by topological Hochschild homology (THH), which may be built in a similar fashion to Hochschild homology (HH) of rings by working over the deeper base of the sphere spectrum. It can also be described as the factorization homology

$$\mathrm{THH}(R) = \int_{\mathbb{T}} R$$

of the circle $\mathbb{T}$ compact Lie group with coefficients in the E_1 ring R and therefore has an action of the group of homeomorphisms of the circle by functoriality. There is a highly nontrivial map called the Bökstedt trace $\mathrm{K}_*(A) \to \mathrm{THH}_*(A)$, which refines the Dennis trace map to Hochschild homology. More general versions of factorization homology also have applications to iterated algebraic K-theory. In particular, there is a trace map

$$K^{(n)}(A) \to THH^{(n)}(A)$$

from n-th iterated algebraic K-theory of an E_n-ring A to factorization homology of the n-torus.

Moreover, topological Hochschild homology has the structure of a cyclotomic spectrum which includes an action of the circle group $\mathbb{T}$ along with $\mathbb{T}$ equivariant structure maps called the Tate valued Frobenius maps, following [NS17]. This extra structure allows one to build a further refinement of topological Hochschild homology called topological cyclic homology (TC). For a large class of E_1 rings A, the p-complete algebraic K-theory groups $\mathrm{K}_n(A;\mathbb{Z}_p)$ and the p-complete topological cyclic homology groups $\mathrm{TC}_n(A;\mathbb{Z}_p)$ are isomorphic for $n \geq 0$ by [HM97, DGM13].

To simplify exposition, we restrict to bounded below p-complete E_1 rings. Recent work of T. Nikolaus and P. Scholze [NS17], simplifies the definition of topological cyclic homology of connective ring spectra as the homotopy fiber in the homotopy fiber sequence

$$\mathrm{TC}(A;\mathbb{Z}_p) \longrightarrow \mathrm{TC}^{-}(A) \xrightarrow{\psi_p - can} \mathrm{TP}(A) \tag{5}$$

where the map *can* comes from the $\mathbb{T}$ equivariant structure and the map ψ_p comes from the cyclotomic structure. More precisely, $\mathrm{TC}^{-}(A)$, known as topological negative cyclic homology, is the $\mathbb{T}$-homotopy fixed points of $\mathrm{THH}(A)$ and $\mathrm{TP}(A)$, known as topological periodic cyclic homology, is the $\mathbb{T}$-Tate construction of $\mathrm{THH}(A)$. The map can is the canonical map from the homotopy fixed points to the Tate construction. The map ψ_p is then the homotopy $\mathbb{T}$-fixed points of the Tate valued Frobeneus map

$$\psi_p\colon\ THH(A) \to THH(A)^{tC_p}$$

where we implicitly identify $(THH(A)^{tC_p})^{h\mathbb{T}}$ and $TP(A)$ using our assumption on A via [NS17, Lemma II.4.2]. The invariants THH, TC^{-}, TP, and TC have proven to be interesting in their own right as well, by recent work of Bhatt-Morrow-Scholze [BMS19] on integral p-adic Hodge theory.

1.3 Equivariant homotopy theory

Let G be a finite group. It is clear that for any subgroup $H < G$, there is an associated inclusion of fixed points, but one of the key features of stable equivariant homotopy theory is the existence of transfers maps, which are a kind of umkehr or "wrong-way" map. The data of inclusion of fixed points and transfers is encoded in the homotopy groups of a G spectrum by the algebraic structure of a G Mackey functor. When $G = C_2$ the cyclic group of order 2, a C_2 Mackey functor may be described by its Lewis diagram

$$M(C_2/e) \underset{res}{\overset{tr}{\rightleftarrows}} M(C_2/C_2)$$

as well as compatible actions of the Weyl group $W_{C_2}(H) = N_{C_2}H/H$ on $M(G/H)$.

The category of Mackey functors Mak_G is equipped with a symmetric monoidal product $\square$ called the box product and the unit is the Burnside Mackey functor $\underline{A}^G$ with $\underline{A}^G(G/H) = A(H)$. Here $A(H)$ is the Burnside ring which, as a group, is the free abelian group on conjugacy classes of subgroups of G. The associative monoids in this category are called associative Green functors and the commutative monoids in this category are called commutative Green functors.

We need additional structure to encode genuine commutativity, which is the data of a multiplicative transfer or norm map, which in the case of C_2 is of the form

$$N\colon\ M(C_2/e) \to M(C_2/C_2).$$

These genuine commutative monoids in Mackey functors are known as Tambara functors. There is also a genuine version of associative monoids in Mackey functors, known as E_σ algebras in C_2 Mackey functors, which we call discrete E_σ-rings. In particular, these recover the classical notion of associative rings with ant-involution and a recent definition of Hermitian Mackey functor due to Dotto–Ogle [DO19]. Since these these are foundational to our work, we recall (a less technical version of) the definition here.

Definition 1.2. A discrete E_σ ring is a C_2 Mackey functor $\underline{M}$ such that $\underline{M}(C_2/e)$ is an associative ring, $\underline{M}$ is a $N_e^{C_2}\iota_e^*\underline{M}$ module whose restriction to the trivial subgroup e is the canonical module structure of $M(C_2/e)\otimes M(C_2/e)^{\mathrm{op}}$ on $M(C_2/e)$, and a unit map $\underline{A}^{C_2} \to \underline{M}$.

One of the most important features of equivariant stable homotopy theory, for my work, is the Hill–Hopkins–Ravenel (HHR) norm functor N_H^G where H, G are finite groups and H is a subgroup of G and $[G : H] = n$, defined in [HHR16]. We first define G/H weighted smash product as the composite

$$\wedge^{G/H}\colon\ \mathsf{Sp}^H \xrightarrow{\wedge^n} \mathsf{Sp}^{\Sigma_n \wr H} \xrightarrow{\lambda_n^*} \mathsf{Sp}^G$$

where λ_n^* is induced by the group homomorphism $G \to \Sigma_n \wr H$, which is equivalent to a choice of ordering on the set of cosets G/H, and Sp^G is the category of G orthogonal spectra indexed on a trivial universe. The HHR norm can then be defined as

$$N_H^G = I_{\mathbb{R}^\infty}^U \circ \wedge^{G/H} \circ I_{\widetilde{U}}^{\mathbb{R}^\infty} \tag{6}$$

where the first and last functor in the composite are change of universe functors, which are equivalences of categories in for G orthogonal spectra, and the universe $\widetilde{U}$ is the restriction of U to H, also denoted $\iota_H^* U$. We may then define the norm in the category of Mackey functors as

$$N_H^G \underline{M} = \underline{\pi}_0 N_H^G H\underline{M} \tag{7}$$

where $H\underline{M}$ is the Eilenberg-MacLane spectrum associated to the Mackey functor $\underline{M}$.

The HHR norm functor was one of the key tools used by Hill–Hopkins–Ravenel [HHR16] to resolve the Arf–Kervaire invariant one problem. They proved that manifolds with Arf–Kervaire invariant one only exist in dimensions $2^{i+1} - 2$ for $i = 1, 2, 3, 4, 5$ and possibly 6 improving on work of Browder [Bro69] by resolving the cases $2^{i+1} - 2$ for $i > 6$. A framed manifold of dimension $4k - 2$ can be surgically converted into a sphere whenever the Arf–Kervaire invariant is zero, so their work proves that all framed manifolds of dimension $4k-2$, except possibly those in dimension $2, 6, 14, 30, 62$ and 126, can be surgically converted into a sphere using framed cobordisms.

2 Factorization homology

2.1 The topological Hochschild-May spectral sequence

Multiplicative filtrations of commutative rings are ubiquitous in algebra, but multiplicative filtrations of E_∞ rings have been less accessible. A simple reason is that ideals in ring theory are simple and well understood, but in the setting of E_∞ rings the notion of ideal is more complicated and there isn't an accepted definition. J. Smith suggested a notion of ideal of a ring spectrum, now called Smith ideals and the theory was further developed by Hovey in [Hov14].

In [AKS18], A. Salch and I develop a theory of *multiplicatively filtered E_∞ rings* that generalize Smith ideals and are a useful notion for generalizing the flavor of filtering by powers of an ideal. A multiplicatively filtered E_∞ ring may be concisely packaged as a commutative monoid in the category decreasingly filtered spectra. In [AKS18], we construct a large class of examples of such filtrations using the idea of the Whitehead tower from algebraic topology.

Theorem 2.1 (Angelini-Knoll–Salch [AKS18])**.** There is an explicit model for a multiplicative Whitehead filtration

$$\ldots \to \tau_{\geq 3} R \to \tau_{\geq 2} R \to \tau_{\geq 1} R \to R$$

of a connective E_∞ ring R equipped with structure maps $\rho_{i,j}\colon \tau_{\geq i} R \wedge \tau_{\geq j} R \to \tau_{\geq i+j} R$ satisfying commutativity, associativity, unitality, and compatibility axioms as well as a cofibrancy condition. In other words, $\tau_{\geq \bullet} R$ is a cofibrant multiplicatively filtered E_∞ ring.

The associated graded E_∞ ring associated to this filtration is

$$E_0 \tau_{\geq \bullet} R = H\pi_* R$$

or, in other words, the generalized Eilenberg-MacLane spectrum associated to the graded ring $\pi_* R$. In the special case of connective topological K-theory ku, whose homotopy groups are

$\pi_* ku \cong \mathbb{Z}[\beta]$ where β is the Bott element in degree 2, the filtration exactly mimics filtering by powers of the ideal generated by β in algebra.

In [AKS18, Thm. 3.4.8], we then prove that there is a spectral sequence in topological Hochschild homology associated to a general multiplicatively filtered E_∞ ring. Our spectral sequence was motivated by May's spectral sequence from [May65] where he uses a filtration of a Hopf algebra by powers of the augmentation ideal to filter the bar construction and produce a spectral sequence computing the cohomology of that Hopf algebra. In particular, we prove that there is a May-type spectral sequence of the form

$$E^1_{*,*} = G_* \left(\mathrm{THH}(H\pi_* R)\right) \Rightarrow G_* \left(\mathrm{THH}(H\pi_* R)\right).$$

for any connective homology theory G and any connective E_∞ ring R, where the second grading on the input is the one coming the May filtration.

In fact, our theorem is much more general. By [MSV97], when R is a E_∞ ring $\mathrm{THH}(R)$ may be constructed as the tensoring $S^1_\bullet \otimes R$ of R with a simplicial model for $S^1_\bullet$ in the category of E_∞ rings. Our spectral sequence also applies to tensoring with any simplicial finite set $X_\bullet$ producing a spectral sequence with signature

$$E^1_{*,*} = G_*(X_\bullet \otimes E_0^* I_\bullet) \Rightarrow G_*(X_\bullet \otimes I_0)$$

for any connective homology theory G and any multiplicatively filtered E_∞ ring $I_\bullet$, where the second grading on the input is the May filtration grading. When M is a framed manifold and R is a E_∞ ring there is an equivalence

$$\mathrm{sing}_\bullet(U(M)) \otimes R \simeq \int_M R$$

by [AF15, Prop. 5.1] where $\mathrm{sing}_\bullet(U(M))$ is the singular simplicial set of the underlying topological space of M and $\int_M R$ is factorization homology. Our main theorem may then be described as follows.

Theorem 2.2 (Angelini-Knoll–Salch [AKS18])**.** There is a May-type spectral sequence for factorization homology with signature

$$E^1_{*,*} = G_*\left(\int_M E_0^* I\right) \Rightarrow G_*\left(\int_M I_0\right)$$

for any multiplicatively filtered E_∞ ring I, framed manifold M, and connective homology theory G.

2.2 Factorization homology of the image of J

Just as the homotopy groups of spheres are fundamental to algebraic topology, the algebraic K-theory of the sphere spectrum, regarded as an E_∞ ring, is fundamental to algebraic K-theory. In particular, the algebraic K-theory space $\Omega^\infty K(S)$ contains $Q(S^0)$ as a retract as well as the the Whitehead space of a point $\mathrm{Wh}^{\mathrm{diff}}(*)$. In general, $\Omega\mathrm{Wh}^{\mathrm{diff}}(M)$ encodes information about stable h-cobordisms, so understanding the case $M = *$ is of fundamental importance and more generally $\mathrm{Wh}^{\mathrm{diff}}(M)$ splits off of $K(S[\Omega M])$ which is a $K(S)$-module.

In 1982, Waldhausen suggested a program for computing $K(S)$ by successively approximating algebraic K-theory of $S_{(p)}$ by the limit of the sequence

$$\dots \longrightarrow K(\tau_{\geq 0}L_3S) \longrightarrow K(\tau_{\geq 0}L_2S) \longrightarrow K(\tau_{\geq 0}L_1S) \longrightarrow K(\tau_{\geq 0}L_0S)$$

McClure–Schwanzl [MS93], then proved that in fact there is an equivalence

$$K(S_{(p)}) \simeq \lim K(\tau_{\geq 0}L_nS)$$

in the same spirit of the chromatic convergence theorem from (4).

It is therefore highly disirable to compute $K(\tau_{\geq 0}L_nS)$. The p-completion of $(\tau_{\geq 0}L_1S)_p$ is the p-complete connective image of J spectrum j. We therefore proceed with computing successive approximations to algebraic K-theory of j.

For the red-shift program of Ausoni–Rognes, it is most useful to compute $K(j)$ modulo (p, v_1) when $p \geq 5$. In particular, if it is a finitely generated $P(v_2)$-module, then $K(j)$ mod (p, v_1, v_2) is a finite spectrum and this implies $K(j)$ is $T(3)$ acyclic. Consequently, $L_{K(k)}K(j) = 0$ for all $k \geq 3$ by a theorem of Hahn [Hah16] and yet $L_{K(2)}K(j)$ is nontrivial. The long term goal is therefore to answer the question.

Question 2.3. Are the homotopy groups of $K(j)$ modulo (p, v_1) a finitely generated $P(v_2)$-module?

Rognes originally conjectured that mod (p, v_1) $K(j)$ is actually a finitely generated free $P(v_2)$-module in a talk from 2000 at Oberwolfach [Rog00] predating the red-shift conjecture as it appears in print in 2008 [AR08]. In my PhD thesis [AK17], I therefore computed $THH(j)$ modulo (p, v_1) as a first approximation to $K(j)$ modulo (p, v_1). My work uses the spectral sequence of Theorem 2.2 in a key way and it would not have been possible without developing a new tool.

Theorem 2.4 (Angelini-Knoll [AK21])**.** There is an isomorphism of graded rings

$$V(1)_*THH(j) \cong P(\mu) \otimes \Gamma(\sigma b) \otimes H_*(E(\alpha_1, \lambda_1', \lambda_2); d(\lambda_2) = \alpha_1\lambda_1').$$

for $p \geq 5$.

Here the notation $H_*(M; d)$ means the homology of the differential graded algebra M modulo the differential d. Concurrently, Eva Höning computed $V(1)_* \mathrm{THH}(j)$ using the Brun spectral sequence in her PhD thesis [Hön17]. Eva Höning and I are currently developing tools for computing $V(1)_* \mathrm{TC}^-(j)$ and $V(1)_* \mathrm{TP}(j)$ in order to compute $V(1)_*TC(j)$ using the Nikolaus-Scholze equalizer and consequently $V(1)_*K(j)$ and resolve Question 2.3. In particular, joint with Cary Malkiewich, I have sketched the existence of a version of the topological Hochschild-May spectral sequence for TC^- and TP and we also understand how the canonical map and Tate-valued Frobenius behave with respect to the May filtration. This gives a new tool for computing $TC(j)$ modulo (p, v_1) directly, but the input is already quite complicated if one uses the Whitehead filtration. The topological Hochschild homology of j mod (p, v_1) is also infinitely generated as a graded $\mathbb{F}_p$-algebra, so this leads to computational complexity. We therefore plan to construct operations on topological Hochschild homology that would simplify this computational complexity.

2.3 Factorization homology of truncated Brown-Peterson spectra

The E_1 ring BP, known as the Brown-Peterson spectrum, is one of the most fundamental objects in chromatic homotopy theory as it exhibits connections between periodicity in the homotopy groups of spheres and p-typical formal groups. The cofficients of BP are a polynomial algebra over $\mathbb{Z}_{(p)}$ on generators v_n for $n \geq 1$ which correspond to p-typical formal groups of height n. By coning off generators v_m for $m > n$, we can form an E_1 ring $BP\langle n\rangle$ whose homotopy groups are the symmetric algebra on generators $v_1, \ldots, v_n$ over $\mathbb{Z}_{(p)}$. Note that there is a prime p hidden in the notation of BP and $BP\langle n\rangle$.

For small n, the spectra $BP\langle n\rangle$ are well known. When $n = -1$, the spectrum $BP\langle -1\rangle$ is the Eilenberg-MacLane spectrum $H\mathbb{F}_p$ or in other words the commutative ring $\mathbb{F}_p$. When $n = 0$, $BP\langle 0\rangle$ is the Eilenberg-MacLane spectrum $H\mathbb{Z}_{(p)}$ and when $n = 1$, $BP\langle 1\rangle$ is the Adams summand ℓ, which is a retract of p-local connective complex topological K-theory ku. Until recently, this exhausted the list of examples $BP\langle n\rangle$ that were known to be E_∞-rings, but recently it was shown that $BP\langle 2\rangle$ has a model as an E_∞-ring by Lawson-Naumann [LN12] at $p = 2$ and Hill-Lawson [HL10] at $p = 3$ using the theory of topological modular forms and topological automorphic forms respectively.

Calculations of topological Hochschild homology $BP\langle n\rangle$ with coefficients in $BP\langle k\rangle$ for $-1 \leq k \leq n$ have been fundamental to the subject since its inception. When Bökstedt defined topological Hochschild homology [Bök87b, Bök87a], the first calculations he did were the computations of $\mathrm{THH}_*(BP\langle n\rangle)$ for $n = -1, 0$. In particular, Bökstedt's computation

$$THH_*(BP\langle -1\rangle) = P(\mu_0),$$

where $|\mu_0| = 2$ can be used to reprove one of the most fundamental results in algebraic topology, Bott periodicity [LH19]. More generally, it is known that topological Hochschild homology of $BP\langle n\rangle$ with coefficients in $H\mathbb{F}_p$ is isomorphic to

$$\mathrm{THH}_*(BP\langle n\rangle; H\mathbb{F}_p) \cong E(\lambda_1, \lambda_2, \lambda_3, \ldots \lambda_n) \otimes P(\mu_{n+1})$$

where the generator μ_{n+1} conjecturally contributes higher chromatic height analogues of Bott periodicity in algebraic K-theory of $BP\langle n\rangle$.

In joint work with D. Culver and Eva Höning, I compute $\mathrm{THH}_*(BP\langle 2\rangle; M)$ where the cofficients M for various $BP\langle 2\rangle$-modules M as approximations to $\mathrm{THH}_*(BP\langle 2\rangle)$. As n increases calculations of $\mathrm{THH}_*(BP\langle n\rangle)$ become significantly more complex. The way to make progress towards $\mathrm{THH}_*(BP\langle n\rangle)$ is therefore to first consider the calculation of $\mathrm{THH}_*(BP\langle n\rangle; H\mathbb{F}_p)$ and then work towards more complicated cofficients. The standard technique for doing this is to use the Bockstein spectral sequences, such as

$$E^1_{*,*} = THH_*(BP\langle n\rangle; H\mathbb{F}_p)[v_1] \Longrightarrow THH_*(BP\langle n\rangle; k(1))$$

where $k(1)$ is the connective cover of the Morava K-theory spectrum $K(1)$. In joint work, with Dominic Culver and Eva Höning, we computed the first three Bockstein spectral sequences.

Theorem 2.5 (Angelini-Knoll–Culver–Höning [AKCH20]). There is an isomorphism of gradad $\mathbb{Z}_{(p)}$-modules

$$THH_*(BP\langle 2\rangle; H\mathbb{Z}_{(p)}) \cong \otimes E_{\mathbb{Z}_{(p)}}(\lambda_1, \lambda_2) \oplus \mathbb{Z}_{(p)}\{\lambda_1^{\epsilon}\lambda_2^{\epsilon'}c_i : i \geq 1\}/\sim$$

where $p^{\nu_p(i)+1}\lambda_1^{\epsilon}\lambda_2^{\epsilon'}c_i \sim 0$ for $i \geq 1$ and $k = 1, 2$. There is an isomorphism of graded $\mathbb{F}_p$-vector spaces

$$THH_*(BP\langle 2\rangle; k(1)) \cong \big(P(v_1) \otimes \big(E(\lambda_1) \oplus \mathbb{F}_p\{\lambda_1^{\epsilon_1}z_{i,j}, \lambda_1^{\epsilon_2}z'_{i,j}, \lambda_1^{\epsilon_3}z''_{i,j}\}\big)\big) / \sim$$

where $v_1^{\nu_p(i)+1}\lambda_1^{\epsilon_1}z_{i,j} \sim v_1^{\nu_p(i)+1}\lambda_1^{\epsilon_2}z'_{i,j} \sim v_1^{\nu_p(i)+1}\lambda_1^{\epsilon_3}z''_{i,j} \sim 0$ and $\epsilon_1, \epsilon_2, \epsilon_3 \in \{0, 1\}$. There is an isomorphism of graded $\mathbb{F}_p$-vector spaces

$$THH_*(BP\langle 2\rangle; k(2)) \cong E(\lambda_2) \oplus \mathbb{F}_p\{\lambda_2^{\epsilon_1}y_{i,j}, \lambda_2^{\epsilon_2}y'_{i,j}, \lambda_2^{\epsilon_3}y''_{i,j}\}/\sim$$

where $v_2^{\nu_p(i)+1}\lambda_2^{\epsilon_1}y_{i,j} \sim v_2^{\nu_p(i)+1}\lambda_2^{\epsilon_2}y'_{i,j} \sim v_2^{\nu_p(i)+1}\lambda_2^{\epsilon_3}y''_{i,j} \sim 0$ for $\epsilon_1, \epsilon_2, \epsilon_3 \in \{0, 1\}$.

We have also employed the topological Hochschild-May spectral sequence as an additional tool for computing the diagonal in the square of spectral sequences

$$\begin{tikzcd} THH_*(BP\langle 2\rangle; H\mathbb{F}_p)[v_0, v_1] \arrow[r, Rightarrow] \arrow[d, Rightarrow] \arrow[dr, Rightarrow, "HMSS"] & THH_*(BP\langle 2\rangle; k(1))[v_0] \arrow[d, Rightarrow] \\ THH_*(BP\langle 2\rangle; H\mathbb{Z}_{(p)})[v_1] \arrow[r, Rightarrow] & THH_*(BP\langle 2\langle; BP\langle 1\rangle). \end{tikzcd}$$

The idea is not to replace the other Bockstein spectral sequences with the topological Hochschild May spectral sequence, but rather to compare all three ways of computing the bottom right corner. We have already made significant progress towards this goal.

For computing $\mathrm{THH}_*(BP\langle 2\rangle; BP\langle 2\rangle/p)$ we have similar approach. In fact, we may also use the Brun spectral sequence [H$\ddot{2}$0]. We plan to use the Brun spectral sequence to compute the diagonal in the square and compare to the other Bockstein spectral sequences

$$\begin{tikzcd} THH_*(BP\langle 2\rangle; H\mathbb{F}_p)[v_1, v_2] \arrow[r, Rightarrow] \arrow[d, Rightarrow] \arrow[dr, Rightarrow, "BSS"] & THH_*(BP\langle 2\rangle; k(1))[v_2] \arrow[d, Rightarrow] \\ THH_*(BP\langle 2\rangle; k(2))[v_1] \arrow[r, Rightarrow] & THH_*(BP\langle 2\langle; BP\langle 2\rangle/p) \end{tikzcd}$$

Finally, to compute $\mathrm{THH}_*(BP\langle 2\rangle)$ we plan to construct a coarser filtration of $BP\langle 2\rangle$ as a multiplicative filtered spectrum, which is additively given by the filtration

$$\cdots \to \tau_{\geq 4p^2-4}BP\langle 2\rangle \to \tau_{\geq 2p^2-2}BP\langle 2\rangle \to BP\langle 2\rangle.$$

One approach to this involves developing obstruction theory for multiplicatively filtered E_∞-rings. The existence of such a multiplicatively filtered E_∞ ring as well as obstruction theory for multiplicatively filtered E_∞ rings is of independent interest as and it would vastly simplify the computation of $\mathrm{THH}_*(BP\langle 2\rangle)$, as we have already observed computationally.

3 Results on red-shift phenomena in algebraic K-theory

3.1 Iterated algebraic K-theory of the integers and integral modular forms

In the 1950's, J. F. Adams computed the image of the J homomorphism from the homotopy groups of the infinite orthogonal group to the stable homotopy groups of spheres [Ada58]. In this celebrated work, he in particular demonstrated a direct connection between special values of the Riemann zeta function and the orders of the divided alpha family, a v_1-periodic family of elements in the image of the J homomorphism, whose periodicity is an artifact of Bott periodicity. In the 1970's, D. Quillen defined algebraic K-theory and presented the first complete calculation of algebraic K-theory, the algebraic K-theory of finite fields [Qui72]. He also showed that the divided α-family is nontrivial in the Hurewicz image of algebraic K-theory of finite fields and consequently the algebraic K-theory of the integers for $p \geq 3$. This gave some of the first evidence for the conjectures of S. Lichtenbaum [Lic73] and D. Quillen [Qui75], now known as the Lichtenbaum–Quillen conjectures.

In my work, I give the first higher chromatic height analogue of this theorem. The β-family is a v_2-periodic family, which was first constructed and proven to be nontrivial by L. Smith [Smi70]. The β-family was proven to have a tight connection to certain integral modular forms satisfying certain congruence relations by M. Behrens [Beh06]. I prove that the β-family, at $p \geq 5$, is detected in the iterated algebraic K-theory of finite fields of order q, where q is a topological generator of $\mathbb{Z}_p^\times$. Here we say a family of elements in the homotopy groups is detected in the homotopy groups of an E_1 ring R if this family of elements has nontrivial image

$$\pi_* S \to \pi_* R$$

under the Hurewicz map, which is induced by the unit map of R. Recall that the β-family depends on a chosen prime p. I prove a higher chromatic height and an iterated algebraic K-theory analogue of the result J.F. Adams [Ada58] and D. Quillen [Qui72] on the relationship between algebraic K-theory of the integers and special values of the Riemann zeta function.

Theorem 3.1 (Angelini-Knoll [AK18])**.** The β-family is detected in iterated algebraic K-theory of of the integers $K(K(\mathbb{Z}))$ when $p \geq 5$.

This indicates, more generally, that there should be a dictionary between iterated algebraic K-theory of rings of integers O_F in number fields F and modular forms over rings of integers in number fields O_F satisfying certain congruence relations. This also gives evidence for a version of the red-shift conjecture, which I call the Greek-letter family red-shift conjecture. Since it has been speculated that the Greek letter family elements $\alpha_k^{(n)}$ are related to automorphic forms and special values of L functions of degree n, we view this as a higher chromatic height analogue of Lichtenbaum's conjecture [Lic73] on the relationship between algebraic K-theory and special values of Dedekind zeta functions.

Conjecture 3.2 (Greek letter family red-shift conjecture)**.** Let R be an E_1 ring. If the n-th Greek letter family is nontrivial in the Hurewicz image of R, then the $n+1$-st Greek letter family is nontrivial in the Hurewicz image of $\mathrm{K}(R)$.

In the case $n = 0$, we call p^k the 0-th Greek letter family. With this convention, the 0-th Greek letter family is nontrivial in $R = \mathbb{Z}$ for all k and the α-family is detected in $\mathrm{K}(\mathbb{Z})$, giving evidence of the conjecture. Theorem 3.1 gives evidence for the conjecture when $n = 1$.

3.2 Morava K-theory of algebraic K-theory

The study of Morava K-theory of algebraic K-theory and related invariants has been of long-standing interest. For example, S. Mitchell [Mit90] proved that

$$L_{K(m)}K(\mathbb{Z}) = 0$$

for $m \geq 2$, which implies the same result for any $H\mathbb{Z}$-algebra and any variety. This implies that $K(\mathbb{Z})$ has height ≤ 1 in the sense of Definition 1.1. Also, work of C. Ausoni and J. Rognes [AR02] implies that

$$L_{K(m)}K(\ell_p) = 0$$

for $m \geq 3$ where ℓ_p is a retract of p complete topological K-theory.

More recently, there has been renewed interest in Morava K-theory of algebraic K-theory and new advances in the subject by work of Clausen-Mathew-Naumann-Noell [CMNN20], Land-Meier-Tamme [LMT20], and Bhatt-Clausen-Matthew [BCM20]. The techniques we employ are completely independent of these other recent results and therefore they expand the toolset for approaching such questions.

Our main computational result is a proof that $K(BP\langle n\rangle)$ has height $\leq n+1$ for all n, p such that $BP\langle n\rangle$ is an E_∞-ring.

Theorem 3.3 (Angelini-Knoll–Salch [AKS20])**.** Let $n \geq -1$ be an integer and $p > 2$ a prime such that the p-complete of $BP\langle 2\rangle$ has a model as an E_∞-ring, then

$$L_{K(m)}K\left(BP\langle n\rangle_p\right) = L_{T(m)}K\left(BP\langle n\rangle_p\right) = 0$$

for $m \geq n+2$ and consequently,

$$L_{T(m)}K\left(R\right) = 0$$

for $m \geq n+2$ for any $BP\langle n\rangle_p$-algebra R, for example $R = K(2)$ at $p = 3$ and $n = 2$.

When $n = -1, 0, 1$, this result was already known, but when $n = 2$ and $p = 3$ this result is completely new and it is one of the first red-shift type results at this chromatic height. We also remark, that it is left open whether $L_{K(n+1)}K(BP\langle n\rangle_p) \neq 0$. In the case $n = 2$ and $p = 3$, it is the aim of my joint work D. Culver and E. Höning to prove that $L_{K(3)}K(BP\langle 2\rangle_p) \neq 0$, see Section 2.3.

The proof of Theorem 3.3, is one of the main applications of a technical result, which is of independent interest. We show that certain sequences of spectra, that are not necessarily uniformly bounded below, commute with smashing with Morava K-theory. In the following result, A_* denotes the dual Steenrod algebra and $H(M; Q_m)$ denotes the Margolis homology of a $E(Q_m)$-module M where Q_m is the m-th Milnor primitive in the Steenrod algebra A. We also write $H_*(X)$ for homology of X with coefficients in $\mathbb{F}_p$.

Theorem 3.4 (Angelini-Knoll–Salch [AKS20])**.** Suppose that

$$\cdots \to T_i \to T_{i+1} \to \dots$$

is sequence of spectra such that

$$\operatorname*{colim}_{i\to\infty} T_i = 0 \text{ and } \lim_{i\to-\infty} T_i = T.$$

Suppose that each T_i is bounded below, $H\mathbb{F}_p$-nilpotent complete, π_*T_i is a finite type $\mathbb{Z}_p$-module for each i, and the maximal degree of an A_*-comodule primitive in $H_*(T_i)$ is $M-1$, where M does not depend on i. Additionally, assume that

$$\lim_{i\to-\infty} H(H_*(T_i);Q_m)=0.$$

Then, the $K(m)$-localization of T vanishes; i.e. $L_{K(m)}T=0$.

In joint work with J.D. Quigley, I also apply this result to prove Morava K-theory vanishing results for topological periodic cyclic homology and algebraic K-theory (Section 3.3).

3.3 Morava K-theory of topological periodic cyclic homology

For each prime p, there is a sequence of Thom spectra that can be constructed using the Thom construction associated to the filtration of spaces

$$*\to\Omega J_{p-1}(S^2)\to\Omega J_{p^2-1}(S^2)\to\ldots\to\Omega J(S^2)$$

lying over the classifying space of stable p-local spherical bundles $BGL_1S_{(p)}$, where $J(S^2)$ is the James construction. Mahowald [Mah79], proved the existence of the associated the family of spectra

$$S\to y(1)\to y(2)\to\ldots\to y(\infty)=H\mathbb{F}_p$$

when $p=2$ and this was extended to odd primes by Mahowald-Ravenel-Shick [MRS01]. These spectra have type n in the sense that $L_{n-1}y(n)=0$, but $L_{K(n)}y(n)\neq 0$. In joint work with J.D. Quigley [AKQ19], we show that the vanishing range of Morava K-theory of topological periodic cyclic homology of $y(n)$ increases strictly by 1.

Theorem 3.5 (Angelini-Knoll,Quigley [AKQ19])**.** There are equivalences

$$L_{K(m)}\,\mathrm{TP}(y(n))\simeq 0$$

for $1\leq m\leq n+1$.

This demonstrates that shifts in chromatic complexity occur in topological periodic cyclic homology even before passing to topological cyclic homology or algebraic K-theory. I also emphasize that this Tate red-shift result demonstrates a chromatic height shift at all chromatic heights and there are no other red-shift type results of this nature in the literature.

We also show that relative algebraic K-theory at preserves height (Definition 1.1).

Theorem 3.6 (Angelini-Knoll,Quigley [AKQ19])**.** There are isomorphisms

$$L_{K(n)}K(y(n),H\mathbb{F}_p)=0$$

for $0\leq m\leq n$ where $K(y(n),H\mathbb{F}_p)=\mathrm{fib}\left(K(y(n))\to K(H\mathbb{F}_p)\right)$.

The result above has also been proven recently by Land-Meier-Tamme [LMT20] in parallel. However, they do not use trace methods and therefore they cannot recover our TP result by their methods. Our methods are also completely independent.

4 Equivariant factorization homology

4.1 Real topological Hochschild homology, Witt vectors, and Norms

Real algebraic K-theory, due to L. Hesselholt and I. Madsen [HM18] and recently extended by B. Calmes et al [CDH+20], receives a ring with anti-involution and produces a graded C_2 Mackey functor, which is the C_2 equivariant analogue of an abelian group where C_2 is the cyclic group of order 2. More generally, the input can be an E_σ ring (an algebra in C_2 spectra over the equivariant little disc operad E_σ). Real algebraic K-theory has its roots in M. Karoubi and O. Villamayor's Hermitian K-theory [KV71] and M. Atiyah's Real topological K-theory [Ati66]. There is also a tight connection between Real algebraic K-theory and L-theory, which has applications to surgery theory of manifolds, see [CDH+20] for a modern account of this.

Real algebraic K-theory of E_σ rings is even more difficult to compute than algebraic K-theory, so it is desireable to have new tools. One recent approach is the development of trace methods in this setting. There has been significant progress towards this goal by E. Dotto, A. Høgenhaven, K. Moi, I. Patchkoria, and S. Precht-Reeh [Dot12, Hø16, DMPP17], building on [HM18], but there are still unanswered questions. In particular, there are Real analogues of all of the classical invariants, such as Real topological Hochschild homology (THR), which may be regarded as equivariant factorization homology.

Non-equivariantly the first instance of red-shift behavior appears in the shift from characteristic p in $\mathbb{F}_p$ to characteristic zero in D. Quillen's computation of the algebraic K-theory of a finite field $K(\mathbb{F}_p)_p \simeq HW(\mathbb{F}_p)$ where $W(\mathbb{F}_p)$ is the p-typical Witt vectors and H is the Eilenberg-MacLane spectrum functor [Qui72]. One of my goals is to explore height shifting phenomena in this new equivariant setting. As a first step in this direction, in joint work with T. Gerhardt and M. Hill, we give a description of Witt vectors for discrete E_σ rings. This extends previous work of E. Dotto, K. Moi, and I. Patchkoria [DMP19] to the non-commutative setting. Before describing our construction, we briefly recall the non-equivariant analogue of this story.

L. Hesselholt and I. Madsen [HM97] proved that there is a deep connection between the topological Hochschild homology of a commutative ring R and the Witt vectors of a commutative ring R. Recall that $\mathrm{THH}(R)$ is equipped with an action of the circle group $\mathbb{T}$ and a μ_{p^n} action by restriction, where μ_{p^n} are the p^n-th roots of unity inside of $\mathbb{T} \subset \mathbb{C}^\times$. In [HM97], Hesselholt-Madsen proved that there is an isomorphism

$$\mathrm{W}_{n+1}(A) \cong \underline{\pi}_0^{C_{p^n}} \mathrm{THH}(A)(C_{p^n}/C_{p^n})$$

where $\mathrm{W}_{n+1}(A)$ is the length $n+1$ p-typical Witt vectors of A. Recently, Blumberg–Gerhardt–Hill–Lawson [BGHL18] generalize this to a theory of Witt vectors for associative Green functors. In particular, they construct a notion of Hochschild homology of Green functors $\underline{\mathrm{HH}}_*^G(M)$ where M is an H-Mackey functor, H and G are finite subgroups of $\mathbb{T}$, and H is a subgroup of G. In [ABG+18], the authors along with Angeltveit and Mandell constructed a spectrum $\mathrm{THH}_H(A) = N_H^{\mathbb{T}}(A)$ for an H-ring spectrum using a twisted version of the cyclic bar construction. In [BGHL18], they then show that there is an isomorphism

$$\underline{\mathrm{HH}}_0^G(\underline{\pi}_0^H A) \cong \underline{\pi}_0^G \mathrm{THH}_H(A)$$

where G is a finite subgroup of $\mathbb{T}$ and H is a subgroup of G. This gives a description of topological Hochschild homology of an H-ring spectrum A in terms of an equivariant Hochschild homology construction, which they define to be the Witt vectors for Green functors. This is motivated by the fact that in the classical setting there is an isomorphism $\pi_0 \operatorname{THH}(A) \cong \operatorname{HH}_0(A)$.

In my work with T. Gerhardt and M. Hill, I define a new algebraic construction, called Real Hochschild homology, of dicrete E_σ rings (Definition 1.2). Throughout, we fix a splitting of the extension

$$1 \to \mathbb{T} \to O(2) \to C_2 \to 1$$

compatible with the splittings of the extensions

$$1 \to \mu_m \to D_{2m} \to C_2 \to 1$$

and we fix m odd to simplify exposition. Recall the norm in the category of C_2-Mackey functors defined by [**?**].

Definition 4.1. Let m be an odd integer. The Real Hochschild homology of a discrete E_σ ring $\underline{M}$, is defined as

$$\underline{HR}_*^{D_{2m}}(\underline{M}) = H_*(B_\bullet(N_{C_2}^{D_{2m}}\underline{M}, N_e^{D_{2m}}\iota_e^*\underline{M}, {}^{\gamma_m}N_{C_2}^{D_{2m}}\underline{M}))$$

where $B_\bullet(N_{C_2}^{D_{2m}}\underline{M}, N_e^{D_{2m}}\iota_e^*\underline{M}, {}^{\gamma_m}N_{C_2}^{D_{2m}}\underline{M})$ is the two sided bar construction and ${}^{\gamma_m}N_{C_2}^{D_{2m}}\underline{M}$ is the same as $N_{C_2}^{D_{2m}}\underline{M}$ as a Mackey functor, but it has twisted left $N_e^{D_{2m}}\iota_e^*\underline{M}$ action

$$N_e^{D_{2m}}\iota_e^*\underline{M} \wedge N_{C_2}^{D_{2m}}\underline{M} \xrightarrow{\gamma_m \wedge 1} N_e^{D_{2m}}\iota_e^*\underline{M} \wedge N_{C_2}^{D_{2m}}\underline{M} \to N_{C_2}^{D_{2m}}\underline{M}$$

where $\gamma_m = e^{(m-1)\pi i/m} \in \mu_m < D_{2m}$.

Using this definition, we define a construction of p-typical Witt vectors for E_σ algebras in C_2 Mackey functors taking values in C_2-Mackey functors as the limit

$$\underline{W}(\underline{M}; p) = \lim_R \underline{HR}_0^{D_{2p^k}}(\underline{M}) \tag{8}$$

where R is an algebraic analogue of the Real restriction map, which comes from an algebraic Real cyclotomic structure on $\underline{HR}_*^{D_{2m}}(\underline{M})$, that we define.

. In addition to the connection between Witt vectors and Real Hochschild homology, our construction also has interest from the point of view of construction of HHR norms for compact Lie groups. We point the reader to Section 1.3 for a more detailed account. In particular, we define norm $N_{C_2}^{O(2)}$ as

$$N_{C_2}^{O(2)}(-) = |B_\bullet^{\mathrm{di}}(-)| \colon \mathsf{Sp}^{C_2} \to \mathsf{Sp}^{O(2)}$$

We prove that this definition deserves to be called the HHR norm in the following sense.

Theorem 4.2 (Angelini-Knoll–Gerhardt–Hill)**.** The norm functor $N_{C_2}^{O(2)}$ is left Quillen adjoint

$$N_{C_2}^{O(2)} \colon \mathsf{Sp}_{\widetilde{U}}^{C_2} \rightleftarrows \mathsf{Sp}_{U,\mathcal{R}}^{O(2)} : \iota_{C_2}^*$$

where the model structure on the source is the positive complete stable model structure, and the model structure of the target is the positive complete stable model structure with respect to the family $\mathcal{R}$ of subgroups that intersect the circle $\mathbb{T}$ trivially. Here U is a complete $O(2)$ universe and $\widetilde{U} = \iota_{C_2}^* U$.

We also extend work of E. Dotto, K. Moi, I. Patchkoria, and S. Precht-Reeh [DMPP17] to the $O(2)$-equivariant setting. Showing that the norm $N_{C_2}^{O(2)}$ is a model for THR, defined using the Bökstedt construction in the sense that there is an $\mathcal{R}$-equivalence

$$N_{C_2}^{O(2)}(A) \simeq_{\mathcal{R}} THR(A).$$

This gives a norm model for Real topological Hochschild homology. We plan to extend this to an equivalence as Real cyclotomic spectra in future work. Additionally, we give a new description of the restriction to D_{2m} of $N_{C_2}^{D_{2m}}$ when m is odd.

Theorem 4.3 (Angelini-Knoll-Gerhardt-Hill [AKGH20])**.** Suppose A is a flat E_σ ring (a mild cofibrancy condition), then there are equivalences $\mathcal{R}_m$-equivalences

$$\iota_{D_{2m}}^* N_{C_2}^{O(2)}(A) \simeq \iota_{D_{2m}}^* THR(A) \simeq N_{C_2}^{D_{2m}}(A) \wedge_{N_e^{D_{2m}}(\iota_e^* A)} {}^{\gamma_m} N_{C_2}^{D_{2m}}(A)$$

where $\mathcal{R}_m$ is the family of subgroups that intersect $\mu_n \subset D_{2m}$ trivially, $\gamma_m = e^{(m-1)\pi i/2}$ and the $N_e^{D_{2m}}(\iota_e^* A)$-module ${}^{\gamma_m} N_{C_2}^{D_{2m}}(A)$ has a right action of $N_e^{D_{2m}}(\iota_e^* A)$ on ${}^{\gamma_m} N_{C_2}^{D_{2m}}(A)$ where ${}^{\gamma_m} N_{C_2}^{D_{2m}}(A)$ is the evident analogue in spectra of the construction ${}^{\gamma_m} N_{C_2}^{D_{2m}}(\underline{M})$ described in Definition 4.1.

As a consequence of our results, there is, in particular, a Real cyclotomic trace map

$$\underline{\pi}_0^{C_2} KR(A) \to \underline{\pi}_* THR(A) \to \underline{HR}_*^{C_2}(R).$$

in C_2-Mackey functors that extends the classical factorization of the Dennis trace through topological Hochschild homology.

4.2 Crossed simplicial homology and equivariant Floer homology

Connes' cyclic category is indispensible in the construction of topological Hochschild homology and the dihedral category is indispensible in the construction Real topological Hochschild homology. Since these categories extend the simplex category Δ any presheaf on one of these categories may be regarded as a simplicial set by restriction and it is known that the geometric realization has an $\mathbb{T}$-action in the cyclic case and a $O(2)$ action in the dihedral case.

These categories have a common generalization known as crossed simplicial groups, due to Fiedorwicz-Loday [FL91]. Generally, a crossed simplicial group $\Delta\mathfrak{G}$ has the same objects in Δ, but it has extra automorphisms with the property that any morphism can be factored as one extra automorphism composed with a morphism in Δ. The automorphisms themselves $\mathfrak{G}_\bullet$ form a simplicial set where $\mathfrak{G}_n$ is a group for each $n \geq 0$, but the structure maps are not group homomorphisms, but rather crossed homomorphisms. In work in progress with M. Merling and M. Péroux, we consider generalizations of topological Hochschild homology associated to a crossed simplicial group $\Delta\mathfrak{G}$.

As a particular example, there is a crossed simplicial group called the quaternionic category, with associated simplicial set $Q_\bullet$ where Q_n is the quaternionic group of order $4n$. The geometric realization of any pre-sheaf on the quaternionic category comes equipped with a $\mathrm{Pin}_{-}(2)$-action, and in particular our construction of quaternionic topological Hochschild homology THQ comes equipped with a $\mathrm{Pin}_{-}(2)$ action.

We aim to connect this theory to recent ground breaking results in manifold theory of C. Manolescu [Man16] and related work on equivariant Floer homology. For example, in the non-equivariant situation, T. Lawson [Law20] has given an explanation of the connection between descent properties of topological Hochschild homology and Heegaard-Floer homology by recovering the Hodge-to-de Rahm cohomology spectral sequence of R. Lipshitz and D. Treumann [LT16] from a spectral sequence in topological Hochschild homology.

Goal 4.4. Construct an equivariant non-commutative Hodge-to-de Rahm spectral sequence for topological crossed simplicial homology to shed light on equivariant Floer homology.

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