Lecture 7: Covering spaces and classifying spaces of categories.

I. Quasi-Fibrations and Quillen's theorem B

To prove the key lemma finishing the proof of Quillen's theorem B, we first need three lemmas about quasi-fibrations.

<u>Lemma 1</u>: Let $p: E \to B$ be a continuous map and let $U, V \subseteq B$ be subspaces s.t. $U \cap V \neq \emptyset$ and $U \cup V = B$. If

$$p|_{p^{-1}(U)},\ p|_{p^{-1}(V)},\ \text{and}\ p|_{p^{-1}(U \cap V)}$$

are quasi-fibrations, then p is a quasi-fibration.

Lemma 2 Let $p: E \to B$ be a continuous map onto B, let $B' \subset B$ be a subspace and let $E' = p^{-1}(B')$. Suppose there is a fiber preserving deformation

$$\begin{array}{ccc} E & \xrightarrow{D_t} & E \\ \downarrow & & \downarrow \\ B & \xrightarrow{d_t} & B \end{array} \qquad t \in [0,1]$$

s.t.

$D_0 = id_E$, $d_0 = id_B$, $D_t(E') = E'$, $d_t(B') = B'$ and $D_1(E) \subset E'$ and $d_1(B) \subset B'$. Additionally, assume that $p^{-1}(x) \xrightarrow{\sim_{we}} p^{-1}(d_1(x))$ is a weak equivalence for all $x \in B$, then p is a quasi-fib.

Lemma 3: Let $p: E \to B$ be a continuous map. Assume B is a CW complex with n-skeleton B_i and assume $p|_{p^{-1}(B_i)}$ is a quasi-fib. for all $i \geq 0$, then p is a quasi-fibration.

Proof: Any compact subset of B lies in some B_i, so any compact subset of E lies in some $E_i = p^{-1}(B_i)$. Consequently, for any $x \in B_i$, $y \in p^{-1}(x)$

$$\begin{aligned}\pi_n(E, p^{-1}(x); y) &\cong \operatorname{colim}_j \pi_n(E_j, p^{-1}(x); y)\\ &\cong \operatorname{colim}_j \pi_n(B_j; x)\\ &\cong \pi_n(B, x). \quad \square\end{aligned}$$

Lemma (Technical lemma for Thm B)

Suppose I is a small category and $X : I \to \mathrm{Top}$ is a functor

Define a simplicial space X_I with n-simplices

$$[n] \longmapsto \coprod_{i_0 \to \dots \to i_n} X(i_0).$$

Assume $X(i) \to X(j)$ is a weak equivalence for all $i \to j$ in I.

Then the canonical map of simplicial sets

$$\coprod_{i_0 \to \dots \to i_n} X(i_0) \to \coprod_{i_0 \to \dots \to i_n} *$$

induces a quasi-fibration.

<u>Proof</u>: By Lemma 3, it suffices to check that

$$|X_I|_p \xrightarrow{\pi} (BI)_p \text{ is a quasi-fibration}$$

for all $p \geq 0$. On 0-cells

This is just

$$\coprod_{i \in ND_0(N_\bullet I)} X_i \longrightarrow ND_0(N_\bullet I)$$

$$\qquad\qquad\qquad\qquad\qquad\qquad \|$$

is

$$\coprod_{i \in \pi_0(BI)} X_i \longrightarrow \coprod_{\pi_0(BI)} *$$

So this is clearly a quasi-fibration.

We then induct up the cells. Assume

that

$$|X_I|_{p-1} \longrightarrow BI_{p-1}$$

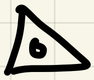
is a quasi-fibration.

We then consider the map of pushouts

$$\begin{tikzcd}
\coprod_{i_0 \prec \dots \prec i_p \in ND_p(N_\bullet I)} \partial|\Delta^p| \times X_{i_0} \arrow[r] \arrow[d] & |X_I|_{p-1} \arrow[d] \\
\coprod_{i_0 \prec \dots \prec i_p \in ND_p(N_\bullet I)} |\Delta^p| \times X_{i_0} \arrow[r] \arrow[ur, phantom, "\ulcorner"] & |X_I|_p
\end{tikzcd}$$

$$\downarrow$$

$$\begin{tikzcd}
\coprod_{i_0 \prec \dots \prec i_p} \partial|\Delta^p| \arrow[r] \arrow[d] & BI_{p-1} \arrow[d] \\
\coprod_{i_0 \prec \dots \prec i_p} |\Delta^p| \arrow[r] \arrow[ur, phantom, "\ulcorner"] & BI_p
\end{tikzcd}$$

Then let $U = BI_p$ with the barycenter (Ex: ) of each p-cell removed and

let $V = BI_p - BI_{p-1}$. Then

$U \cup V = BI$.

Since $BI_p - BI_{p-1} \cong \coprod_{i_0 \to \cdots \to i_p} |\Delta^p| - \partial|\Delta^p|$

and the map

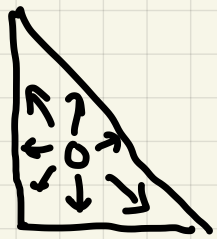
$$\begin{array}{ccc}
\coprod_{i_0 \to \cdots \to i_p} \left((|\Delta^p| \times X_{i_0}) - (\partial|\Delta^p| \times X_{i_0}) \right) & = & \coprod_{i_0 \to \cdots \to i_p} (|\Delta^p| - \partial|\Delta^p|) \times X_{i_0} \\
\big\downarrow & & \\
\coprod_{i_0 \to \cdots \to i_p} |\Delta^p| - \partial|\Delta^p| & &
\end{array}$$

is a trivial fibration, we know $\pi|_V$ and $\pi|_{U \cap V}$ are quasi-fibrations.

It therefore suffices to check that $p|_{p^{-1}(U)}$ is a quasifibration by Lemma 1.

We construct a fiber preserving deformation

$$\begin{tikzcd} p^{-1}(U) \arrow[r] \arrow[d, "D_t"'] & U \arrow[d, "d_t"] \\ p^{-1}(U) \arrow[r] & U \end{tikzcd}$$



w/ $d_1(U) \subseteq (BI)_{p-1}$

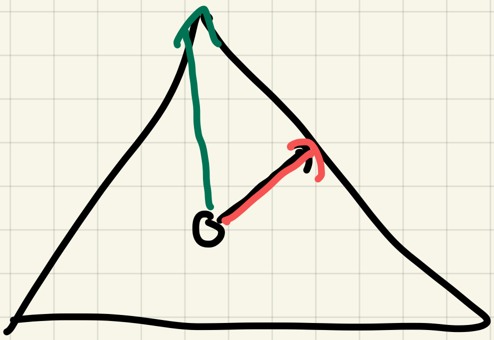
by considering a radial deformation of each p-cell onto the boundary.

We need to show

$$p^{-1}(x) \to p^{-1}(d_1(x))$$

is a weak equivalence for all $x \in U$ by Lemma 2.

Our deformation takes x



to some elt. in a lower cell

$$\coprod_{i_{j_0} \to i_{j_1} \to \cdots \to i_{j_q}} \Delta^q \ni x' = d_1(x)$$

for $\{j_0, \ldots, j_q\} \in \{q, \ldots, p\}$

Thus, $p^{-1}(x) = X_{i_0}$ and

$$p^{-1}(d_1(x)) = X_{i_{j_0}}$$

and the map

$$X_{i_0} = p\big|_V^{-1}(x) \to F{:}b(p|_V, x) = X_{i_{j_0}}$$

is induced by the map $i_0 \to i_{j_0}$

in I which is a weak equivalence

by assumption. $\square$

Corollary: Given a functor $\mathcal{C} \xrightarrow{f} \mathcal{C}'$,

then $B\,Tw(f) \to B\mathcal{C}'$

is a quasi-fibration whenever

$v\backslash f : y\backslash f \to y'\backslash f$ induces

a weak equivalence

$By\backslash f \to By'\backslash f$

for all $v : y \to y'$ in $\mathcal{C}'$.

Proof:

Let $X : (\mathcal{C}')^{op} \to Tw(f)$ be

$X_y = y\backslash f$. Then apply the

previous lemma.

II The fundamental groupoid of a category and covering spaces.

Let $\mathscr{C}$ be a small category. By a morphism inverting functor $F: \mathscr{C} \to \mathrm{Set}$ we mean a functor that sends all maps $v: y \to y'$ in $\mathscr{C}$ to isomorphisms.

<u>Def</u>: $\mathscr{C}[\mathrm{Arr}(\mathscr{C})^{-1}]$ is the category satisfying the universal property

$$\mathrm{Fun}(\mathscr{C}[\mathrm{Arr}(\mathscr{C})^{-1}], \mathrm{Set}) \cong \mathrm{Fun}'(\mathscr{C}, \mathrm{Set})$$

where

$$\mathrm{Fun}'(\mathscr{C}, \mathrm{Set}) \subseteq \mathrm{Fun}(\mathscr{C}, \mathrm{Set})$$

is the full subcategory of morphism inverting functors.

When $\mathcal{C}$ is a small category,

$\mathcal{C}[\mathrm{Arr}(\mathcal{C})^{-1}]$ is a groupoid

(a small category where all morphisms are invertible).

Ex: $\mathcal{C}$ has a single object, then

$\mathcal{C}[\mathrm{Arr}(\mathcal{C}^{-1})] = G$ is a group regarded as a category with one object.

In this case $G = \mathcal{C}^{gp}$ where $\mathcal{C}$ is regarded as a monoid.

A functor $G = \mathcal{C}[\mathrm{Arr}(\mathcal{C}^{-1})] \to \mathrm{Set}$

is then a G-set.

Thm: There is an equivalence of categories

$$\mathrm{Cov}(B\mathscr{C}) \rightleftarrows \mathrm{Fun}'(\mathscr{C}, \mathrm{Set}).$$

Proof. We first specify the functors in each direction.

Given a covering $E \xrightarrow{p} B\mathscr{C}$, we define a functor $E : \mathscr{C} \to \mathrm{Set}$ by $E_x = p^{-1}(x)$

$$E_{(x \to y)} = p^{-1}(x) \to p^{-1}(y).$$

By hypothesis, $E_{x \to y}$ is an iso of sets for all $x \to y$ in $\mathscr{C}$.

$$\begin{array}{c} E \\ \downarrow \\ B\mathscr{C} \end{array} \longmapsto E_{(-)} : \mathscr{C} \to Set$$

Now given a morphism inverting functor $f : \mathscr{C} \to Set$

we can just compose $f : \mathscr{C} \to Set \hookrightarrow Cat$ with the inclusion of Set in the category of small categories, sending a set S to the category $\underline{S}$ w/ $ob\,\underline{S} = S$ $Arr\,\underline{S} = S = \{id_s\}$.

We form the category $[0]\backslash f$ which is the comma category of $[0] \xrightarrow{[0]} Cat \hookleftarrow Set \xleftarrow{f} \mathscr{C}$.

$$ob([0]\backslash f) = (c \in ob\,\mathscr{C},\ * \to f(c))$$

$$(c, \alpha : * \to f(c)) \to (c', \alpha' : * \to f(c')) \qquad \begin{array}{c} c \to c' \in \mathscr{C} \\ * \\ \swarrow \quad \searrow \\ f(c) \to f(c') \end{array}$$

Lemma. The map

$$B_{[0]}\!\int\! f \to B\mathcal{C}$$

induced by the functor

$$(c,\ \alpha : * \to f(c)) \longmapsto c$$

$$\left(\begin{array}{c} c \\ \downarrow \\ c' \end{array},\ \begin{tikzcd}[row sep=small] & f(c) \arrow[d] \\ * \arrow[ur] \arrow[r] & f(c') \end{tikzcd} \right) \longmapsto \begin{array}{c} c \\ \downarrow \\ c' \end{array}$$

is a covering.

Proof: Exercise

Note: Both of these constructions are easily seen to be functorial.

We check that there is a natural iso

$$\begin{tikzcd} E \arrow[r] \arrow[d, "p"'] & B_{[0]}\backslash E_- \arrow[d] \\ B\mathcal{C} \arrow[r, "="'] & B\mathcal{C} \end{tikzcd}$$

and a natural iso

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d, "(B_{[0]}\backslash F)_-"'] & \mathcal{C} \arrow[d, "F"] \\ \mathrm{Set} \arrow[r, "="'] & \mathrm{Set} \end{tikzcd}$$

First, define

$$\begin{tikzcd} E \arrow[r] \arrow[dr] & B_{[0]}\backslash E_- \arrow[d] \\ & B\mathcal{C} \end{tikzcd}$$

$$\{e \in E_c\} \cong \{* \to E_c\}.$$

$$e^{\in E_c} \longmapsto (c, * \xrightarrow{e} E_c)$$

Similarly, define

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\mathrm{id}"] \arrow[dr] & \mathcal{C} \arrow[d, "F"] \\ & \mathrm{Set} \end{tikzcd}$$

$$c \mapsto (B_{[0]} \backslash F)_c = \{* \to F(c)\} = F(c)$$

then the diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\mathrm{id}"] \arrow[d, "(B_{[0]} \backslash f)_-"'] & \mathcal{C} \arrow[d, "f"] \\ \mathrm{Set} \arrow[r] & \mathrm{Set} \end{tikzcd}$$

so

$$(B_{[0]} f)_- = f \ .$$

Cor.

$$\pi_1(B\mathcal{C}, c) = \mathrm{Aut}_{\mathcal{C}[\mathrm{Arr}\,\mathcal{C}^{-1}]}(c)$$

We will use this to identify

$$\pi_0(\Omega BQP(R)) = \pi_1(BQP(R))$$

with $K_0(R)$

$P(R)$ = finitely generated proj & all maps of finitely generated proj.

III. K_0 of an exact category

<u>Def</u>: An Ab-enriched category is a category $\mathcal{C}$ w/

$$\mathrm{Hom}_{\mathcal{C}}(c, c') \in \mathrm{Ab}$$

for all $c, c' \in \mathrm{ob}\,\mathcal{C}$.

We say an Ab-enriched category is an additive category if it is closed under finite coproducts. (Consequently, it has a zero object 0 and all finite biproducts denoted $\oplus$.)

Def: We say an additive category is an abelian category if it is closed under finite limits and every map $f: A \to B$ factors as

$$A \overset{p}{\twoheadrightarrow} \operatorname{coker}(\ker f) \overset{\sim}{\to} \ker(\operatorname{coker} f) \overset{i}{\hookrightarrow} B$$

epi $\qquad$ mono

So we write

$$A \twoheadrightarrow \operatorname{im} f \hookrightarrow B$$

<u>Def:</u> An exact category A is an additive sub category of an abelian category $\mathcal{C}$. such that whenever

$$0 \to X \to Y \to Z \to 0$$

is an exact sequence in $\mathcal{C}$ and $X, Z \in \operatorname{ob} A$ then $Y \in \operatorname{ob} A$.

Note: A is not closed under ker & coker.

Let E be be the class of sequences

$$0 \to X \rightarrowtail Y \twoheadrightarrow Z \to 0 \quad \circledast$$

in A where $\circledast$ is exact in $\mathcal{C}$.

In this, case we say $X \rightarrowtail Y$ is an admissible monomorphism and $Y \twoheadrightarrow Z$ is an admissible epimorphism.

Examples:

- Any abelian category is an exact category
- $P(R)$ finitely generated proj. R-modules
- $M(R)$ finitely generated R-modules
- $VB(X) \subseteq$ families of vector spaces over X

  vector bundles over X

  X a space
- $VB(X) \subseteq \mathcal{O}_X$-mod

  $VB(X) =$ algebraic vector bundles

  X a scheme

Def: Given a small exact category $\mathcal{A}$ define

$$K_0(\mathcal{A}) = \frac{\mathbb{Z}[\mathcal{A}]}{\left(\begin{array}{c} [C]=[A]+[B] \\ \text{whenever} \\ 0\to A\to B\to C\to 0 \in \mathcal{E} \end{array} \right.}$$

Ex: A split exact category

is an exact category $\mathcal{A}$

in which every exact

sequence splits

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] \arrow[l, bend right] & 0 \end{tikzcd} .$$

$$B \cong A \oplus C$$

In this case,

$$K_0(\mathcal{A}) = K_0^{\oplus}(\mathcal{A})$$

Ex: $K_0(P(R)) = K_0^{\oplus}(P(R))$

$$\begin{array}{ccc} \| & & \| \\ \dfrac{\mathbb{Z}[P(R)]}{[A]+[C]=[B] = [A \oplus B]} & & \dfrac{\mathbb{Z}[P(R)]}{[A]+[C]=[A \oplus C]} \end{array}$$