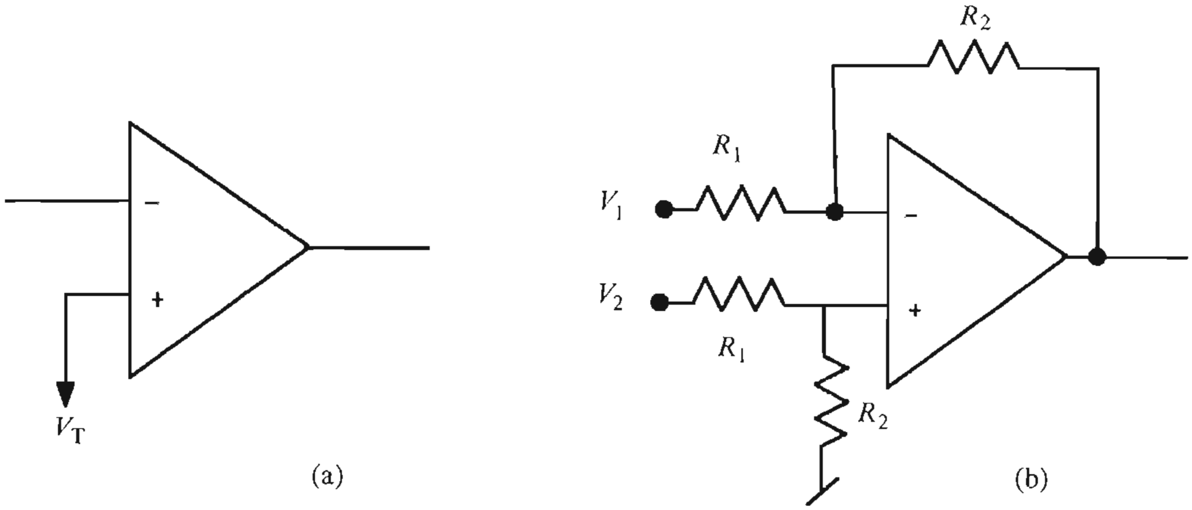


Fig. I.7. (a) A comparator or discriminator. (b) A fixed gain differential amplifier.

for reducing this 'pickup' noise is to use circuits which respond differentially, so that any common noise which is picked up by both inputs is rejected by the differential amplifier since it only, ideally, responds to the difference $V_+ - V_-$ ('common mode rejection'). A possible circuit is shown in Fig. I.7b.

$$V_2 - I(R_1 + R_2) = 0$$

$$V_+ = V_2 - IR_1 = V_2\left(\frac{R_2}{R_1 + R_2}\right) \cong V_- \tag{I.6a}$$

$$V_1 - I'(R_1 + R_2) = V_o$$

$$V_- = V_1 - I'R_1 \tag{I.6b}$$

$$V_o = (V_2 - V_1)(R_2/R_1)$$

This circuit can be thought of as a differential inverting amplifier. (See Eq. I.2 where with $V_2 \to 0$ in Eq. I.6b, the two expressions agree). Note that both inputs $V_1$ and $V_2$ 'see' the same input impedance, $R_1 + R_2$. This fact allows us to make a good cable termination which enhances common mode rejection.

## Reference

[I.1] *Linear Databook*, National Semiconductor Corporation (1982).

# Appendix J

## Statistics introduction

There has been no real explanation of statistics in the body of the text. However, we have freely quoted 'folding errors in quadrature', 'stochastic error' and other concepts. In order to give some minimal background, we expand a bit on the basic concepts of statistical error. References are provided at the end of the text (Chapter 13). The treatment here is completely without rigor.

Consider first a discrete series of $N$ measurements of a quantity $y$, $y_i$, $i = 1, \ldots, N$. The mean and mean square deviation from the mean of those measurements are $\bar{y}$ and $\sigma^2$.

$$\bar{y} = \sum_i^N y_i/N$$
$$\sigma^2 = \sum_i (y_i - \bar{y})^2/N \tag{J.1}$$

In the limit where $N \to \infty$ the distribution of results $y$ approaches a continuous distribution function $\frac{d\underline{P}}{dy}$ where the probability to observe $y$ between $y$ and $y + dy$ is defined to be $d\underline{P} = \frac{d\underline{P}}{dy}\, dy$. The expressions for $\bar{y}$ and $\sigma^2$ in the continuous case become

$$\bar{y} \to \int y(d\underline{P}/dy)dy = \int y\, d\underline{P}$$
$$\sigma^2 \to \int (y-\bar{y})^2(d\underline{P}/dy)dy = \int (y-\bar{y})^2 d\underline{P} \tag{J.2}$$
$$\int d\underline{P} = 1$$

The most commonly assumed distribution function is the Gaussian. It is often assumed to apply in experimental situations because, in the limit of large numbers of events, many distributions approach a Gaussian. The theoretical function is characterized by two parameters, a mean $<y>$ and a root mean square (RMS) deviation from the mean equal to $\Delta y$.

$$d\underline{P}(y, \langle y\rangle, \Delta y) = \frac{1}{\sqrt{2\pi\Delta y^2}} \exp\left[-\frac{1}{2}\left(\frac{y - \langle y\rangle}{\Delta y}\right)^2\right] dy \tag{J.3}$$

It is easy to show that the Gaussian given in Eq. J.3 is normalized to 1, and that the parameters $\langle y\rangle$ and $\Delta y$ are $\bar{y}$ and $\sigma$ respectively as defined in Eq. J.2. Note that $\bar{y}$ and $\sigma$ refer to a large experimental data set, while $\langle y\rangle$ and $\Delta y$ are parameters which define a theoretical distribution function, $d\underline{P}/dy$.

Since it is used so extensively in the text, we quickly derive the Poisson distribution. Examples used in the body of the text include mean free path, decay lifetime, and phototube photoelectron statistics. Consider a case where the probability of interacting in traversing dx is dx/$\langle L\rangle$; the mean free path is $\langle L\rangle$. The probability of getting no events in traversing $x$ is $\underline{P}(0,x)$.

$$d\underline{P}(0,x) = -\underline{P}(0,x)dx/\langle L\rangle$$

$$\underline{P}(0,x) = e^{-x/\langle L\rangle} \tag{J.4}$$

The exponential law for no interaction is already very familiar having been quoted in the discussion of cross sections. For getting $N$ events in $x$, the appropriate probability is $\underline{P}(N,x)$, where we find none in $dx$ and $N$ in $x$ (with the ordering of the events being assumed to be irrelevant). Assuming that joint probabilities multiply:

$$d\underline{P}(N,x) = \prod_{i}^{N} \frac{(dx_i/\langle L\rangle)}{N!} e^{-x/\langle L\rangle}$$

$$\underline{P}(N,x) = \frac{e^{-x/\langle L\rangle}}{N!}\left(\frac{x}{\langle L\rangle}\right)^N \tag{J.5}$$

$$= \frac{\langle N\rangle^N}{N!} e^{-\langle N\rangle}, \quad \langle N\rangle = x/\langle L\rangle$$

In terms of the mean number of events/traversal, $\langle N\rangle$, we recognize the Poisson photoelectron distribution, for example. The Poisson distribution approaches a Gaussian in the appropriate limit as $N$ becomes large.

Suppose the $N$ events yield measures $y_i$ of some variable with measurement error $\sigma_i$ and that we wish to make a hypothesis that they are described by a distribution function which predicts $y$ as $\bar{y}$ when characterized by a single parameter $\alpha$ (for simplicity). Note that $\bar{y}$ is the theoretical prediction in what follows and not the sample mean as given in Eq. J.1. For example, we measure $N$ decay times, $t_i$, and assume a distribution with a single lifetime, $\tau$. The joint probability of the $N$ independent measures defines a 'likelihood function' $L$. Recall that independent events have probabilities which multiply. For example, the chance to roll a five on a die is 1/6, while the change to roll a six and a five (in that order) is 1/36.

$$L(\alpha) = \prod_{i}^{N} d\underline{P}_i(y_i,\alpha) \tag{J.6}$$

The maximum of the likelihood would occur at a value of $\alpha$, $\langle\alpha\rangle$ for which we have the largest joint probability. In the special case of a Gaussian function $d\underline{P}$, we minimize the $\chi^2$ function i.e. the method of least squares. A maximum $L$, if $d\underline{P}$ is a Gaussian, means a minimum value of the argument of the exponential.

$$\begin{aligned} \chi^2 &\sim \ln L \\ &= \sum_{i=1}^{N} [y_i - \bar{y}(\alpha,x_i)]^2/\sigma_i^2 \\ L(\alpha)\big|_{\max} &\Rightarrow \chi^2(\alpha)\big|_{\min} \\ \partial\chi^2/\partial\alpha\big|_{\langle\alpha\rangle} &= 0 \end{aligned} \tag{J.7}$$

The minimum value of $\chi^2$, in the special case $\bar{y}=\alpha$ occurs for

$$\frac{\partial \chi^2}{\partial \bar{y}}=0$$

$$\bar{y}=\left(\sum_i y_i/\sigma_i^2\right)\bigg/\left(\sum_i \frac{1}{\sigma_i^2}\right) \tag{J.8}$$

In the special case that all errors are equal, $\sigma_i=\sigma$, $\bar{y}=\sum_i y_i/N$, and we recover Eq. J.1. The result Eq. J.8 is the best estimator of the mean of a set of measures assuming a Gaussian error distribution. We expect that it is related to the sample mean, as indeed we have shown.

The algebra for the error on the mean is straightforward. The value $\bar{y}$ occurs when $\chi^2$ is minimized. The estimate of $\sigma_{\bar{y}}$ comes from looking at how fast we deviate from the minimum $(\partial^2\chi^2/\partial^2\bar{y})$, $\partial^2\chi^2/\partial^2_{\bar{y}}=1\sigma/^2_{\bar{y}}$.

$$\sigma_{\bar{y}}^2=\frac{1}{\sum_i (1/\sigma_i^2)}$$

$$WT=\sum_i WT_i, \quad WT_i\equiv 1/\sigma_i^2 \tag{J.9}$$

$$\bar{y}=\left(\sum_i y_i WT_i\right)/WT$$

If the measurements have different errors, $\sigma_i$, then they are 'weighted' as the inverse square of that error in finding the best estimation of the mean. This is called the weighted mean technique. If all individual measurement errors are the same, $\sigma_i=\sigma_0$, then the best estimate of the error to be attached to the ensemble of $N$ measurements is, $NWT_0$ or $\sigma=\sigma_0/\sqrt{N}$. We remark on this $1/\sqrt{N}$ behavior in the body of the text several times. We see that error estimators improve only slowly with the number of measures. It is also intuitively clear that poor measurements with large errors (small weights) should not strongly influence the estimate for $\bar{y}$ relative to good measurements with small errors (large weights).

The idea of estimating $\alpha$ is that for a large data set the likelihood clusters about its most probable value, $L(\langle\alpha\rangle)=L_{\max}$, has a width $\sigma_\alpha$ (see Eq. J.2).

$$\sigma_\alpha^2=[\int(\alpha-\langle\alpha\rangle)^2 L\, d\alpha]/[\int L\, d\alpha] \tag{J.10}$$

For the special case of Gaussian errors, and $L(\alpha)\sim\exp\left[-\frac{1}{2}\left(\frac{\alpha-\langle\alpha\rangle}{\sigma_\alpha}\right)^2\right]$ and $\chi^2\sim\left(\frac{\alpha-\langle\alpha\rangle}{\sigma_\alpha}\right)^2$. Therefore, the best estimate of the error on $\alpha$ can be found using the expression for $\chi^2$ and differentiating, $\partial\chi^2/\partial\alpha=(\alpha-\langle\alpha\rangle)/\sigma_\alpha^2$ which is minimized when $(\alpha-\langle\alpha\rangle)=0$ by assumption. Differentiating again, $\partial\chi^2/\partial\alpha^2=1/\sigma_\alpha^2$, leading to an estimate for the error of $\alpha$ using the behavior of $\chi^2$ as a function of $\alpha$.

$$\sigma_\alpha^2 = \left(\frac{\partial^2\chi^2}{\partial\alpha^2}\right)^{-1}_{\langle\alpha\rangle} \tag{J.11}$$

Clearly, if the hypothesis is good, then $\chi^2 \sim 1$for a single variable, since in that case $\chi^2 \sim [(\alpha - \langle\alpha\rangle)/\sigma_\alpha]^2$. The test of how well a hypothesis fits the data depends on the size of $\chi^2$. The precise idea of 'goodness of fit, and significance' can be explored in the references.

For multiple parameters, $\alpha_i$, specifying the hypothesis, the error analysis becomes more difficult. Still, for $N$ measurements at location $x_i$ yielding values $y_i$ of a variable $y$, the $\chi^2$ is the same as in Eq. J.7, and we minimize $\chi^2$ with respect to the set $\alpha_j$, $j=1$, $M$ by simultaneously solving the set of $M$ least square equations.

$$\partial\chi^2/\partial\alpha_j = 0 \tag{J.12}$$

An example might be chamber measurements sampling a trajectory in a magnetic field. The orbit, $\bar{y}$, is a function of the parameters $\mathbf{x}_0\hat{\alpha}_0$, $q$, and $p$, as we have derived. A least squares fit to the helical path will yield the best fit value of the parameters defining the path of the track. In particular, we will determine the momentum of the track with some error. This outline of a procedure makes quite specific what we mean by using detectors to measure the momentum of a track.

With $M$ parameters, the errors on the parameters are estimated from a straightforward generalization of Eq. J.11. However, in this case there is an 'error matrix' $H^{-1}$, of dimension $M \times M$, with off diagonal elements indicating a correlation between the parameters.

$$\begin{gathered} H_{ij} = \partial\chi^2/\partial\alpha_i\partial\alpha_j \\ \sigma_{ai}^2 = (H^{-1})_{ii} \\ \langle\alpha_i - \langle\alpha_i\rangle\rangle\,\langle\alpha_j - \langle\alpha_j\rangle\rangle = (H^{-1})_{ij} \end{gathered} \tag{J.13}$$

The diagonal elements of the inverse $H$ matrix provide an estimator of the errors on the $M$ parameters $\alpha_j$.

For example, assume $N$ points at $x_i$ measuring $y_i$ with error $\sigma_i$. Assume no forces, so that the path $\bar{y}$ is a straight line. By hypothesis $M=2$, $\bar{y}=ax+b$. The remaining degrees of freedom are $N-M$, so generalizing our previous discussion, we expect $\chi^2_{\text{min}}/(N-M) \sim 1$ would indicate a reasonable fit to the hypothesis. The minimum value of $\chi^2$ with respect to $a$ and $b$ occurs when

$$\chi^2 = \sum_i^N (y_i - ax_i - b)^2/\sigma_i^2$$

$$\frac{\partial\chi^2}{\partial a} = 2\sum_i^N (y_i - ax_i - b)(-x_i)/\sigma_i^2 = 0 \tag{J.14}$$

$$\frac{\partial\chi^2}{\partial b} = 2\sum_i^N (y_i - ax_i - b)(-1)/\sigma_i^2 = 0$$

This is two equations in two unknowns. The solution is left as an exercise for the reader.

$$b = \begin{vmatrix} \sum_i y_i WT_i & \sum_i x_i WT_i \\ \sum_i x_i y_i WT_i & \sum_i x_i^2 WT_i \end{vmatrix} / \Delta$$

$$a = \begin{vmatrix} \sum_i WT_i & \sum_i y_i WT_i \\ \sum_i x_i WT_i & \sum_i x_i y_i WT_i \end{vmatrix} / \Delta \tag{J.15}$$

$$\Delta = \begin{vmatrix} \sum_i WT_i & \sum_i x_i WT_i \\ \sum_i x_i WT_i & \sum_i x_i^2 WT_i \end{vmatrix}$$

The error matrix on $a$ and $b$ follows from Eq. J.13. The algebraic details are again left to the reader.

$$\frac{\partial^2 \chi^2}{\partial a^2} = 2\sum_i^N x_i^2 WT_i$$

$$\frac{\partial^2 \chi^2}{\partial a\, \partial b} = 2\sum_i^N x_i WT_i$$

$$\frac{\partial^2 \chi^2}{\partial b^2} = 2\sum_i^N WT_i \tag{J.16}$$

$$\sigma_b^2 \sim \frac{1}{\Delta}\sum_i x_i^2 WT_i$$

$$\sigma_a^2 \sim \frac{1}{\Delta}\sum_i WT_i$$

These formulae serve us as a concrete example of the least squares method in a case of practical interest. They also illustrate the general method, Eq. J.12, Eq. J.13, in a particularly simple case. The reader is encouraged to fit some real data in order to gain experience.

Suppose we determine a variable $Y$ which is a function of $N$ variables $y_i$ each with a different error $\sigma_i$. The maximum likelihood for $Y, \bar{Y}$ occurs, we assume, when the $y_i$ attain their most likely values.

$$\begin{aligned} Y &= Y(y_i) \\ \bar{Y} &= Y(\bar{y}_i) \end{aligned} \tag{J.17}$$

Using the definition, Eq. J.1, for mean square deviation, we can find the error on $Y$ due to the errors on $y_i$. We assume that the fluctuations in the $y_i$ are uncorrelated so that all cross terms, proportional to $y_i\, y_j$, $i \neq j$, average to zero. The series Taylor expansion in $N$ dimensions is applied.

$$\sigma_Y^2 = (Y - \bar{Y})^2$$

$$\cong\left[\sum_i (y_i-\bar{y}_i)\frac{\partial Y}{\partial y_i}\right]^2$$

$$\cong\sum_i\left[(y_i-\bar{y}_i)\frac{\partial Y}{\partial y_i}\right]^2 \quad \text{uncorrelated} \tag{J.18}$$

$$\cong\sum_i [\sigma_y \partial Y/\partial y_i]^2$$

This result allows us to 'propagate' errors. If we know the error in $y$, then we can find the error on any function of $y=Y$. For a single variable the error in $Y$ becomes, $\sigma_Y=\sigma_y\ \partial Y/\partial y$. For a function of many variables, we can work out the result using Eq. J.18. For example, the product $Y=y_1y_2\ldots y_N$ has a fractional error which 'adds in quadrature'.

$$Y=y_1y_2\ldots y_N$$

$$(\sigma_Y/Y)^2=\sum_i(\sigma_{y_i}/y_i)^2 \tag{J.19}$$

Many volumes have been written on probability and statistics. The aim of this appendix has been only to introduce the concepts of a data set, the sample mean and standard deviation and its characterization by a distribution function $d\underline{P}(y)$ defined by parameters $\alpha$. The 'best' estimate of the mean is derived along with an estimate of the error on that best estimate assuming Gaussian errors and using the method of maximum likelihood/least sources. Propagation of errors is introduced as a final topic. Having provided this appendix, we have a self-contained explanation of the topics in statistics alluded to in the body of the text. More advanced references appear at the end of Chapter 13.

# Appendix K

## Monte Carlo models

Many of the problems encountered in making a realistic detailed model of a detector are so complex as to not be susceptible to analytic techniques. Nevertheless, we can break the problem of modeling a very complex system down into a series of choices for the relevant dynamical variables. The Monte Carlo method allows us to choose those variables and hence to construct such a model.

For example, suppose we want to model delta ray production in a medium by incident muons. The choices involved are first at what depth is the delta ray made. We choose $x$ out of a distribution $e^{-(N_0 \rho \sigma x/A)}$ where $\sigma$ is the cross section for delta production. If there is indeed delta production within the active volume, the dynamics depends on a single variable, as has been mentioned previously. We can, for example, pick the recoil electron kinetic energy from $T=0$ to $T=T_{max}$ from a distribution $d\sigma/dT \sim 1/T^2$. Having the energy, the recoil angle follows, as derived in Appendix A, from the two body kinematics. We now have the position and momentum vectors of the electron at the point of production. We can find the final state muon as well if we are to trace it further through the system.

If it is desired that the delta ray be followed in its path, other choices must be made. Suppose there are no fields, for simplicity, so that the path is, at least locally, a straight line. We pick a distance which is short with respect to both the energy loss, $dE_I/dx$, and for which the multiple scattering angle is small. Over this distance the energy and angles of the electron can be considered to be constant. We extrapolate the electron path as a straight line. We pick a new energy $T'$ by removing the ionization energy from the electron. We also pick new angles using a Gaussian distribution with RMS $=\theta_0$ about the initial direction due to multiple scattering in the medium. Clearly, we can make a rather complicated model out of individual, fairly simple, choices. In fact, almost all the formulae appearing in the body of this text would, at some point, be used in a complete Monte Carlo simulation of the complex detectors used today. Therefore, we need to be conversant with all the dynamics discussed in this text if we want to be able to make a realistic model of a detector.

How do we actually go about choosing the individual dynamical quantities which define the evolution of the system? First, we need a 'random number generator' which produces a uniformly distributed number $r$ in the interval (0, 1). These generators are widely available on almost all computer platforms.

$$0 < r < 1, \text{ uniformly distributed} \tag{K.1}$$

Suppose we have a probability distribution $d\underline{P}(x)/dx$ defined such that $d\underline{P}(x)$ is the probability for $x$ to occur between $x$ and $x+dx$. If $x$ is constrained to the interval

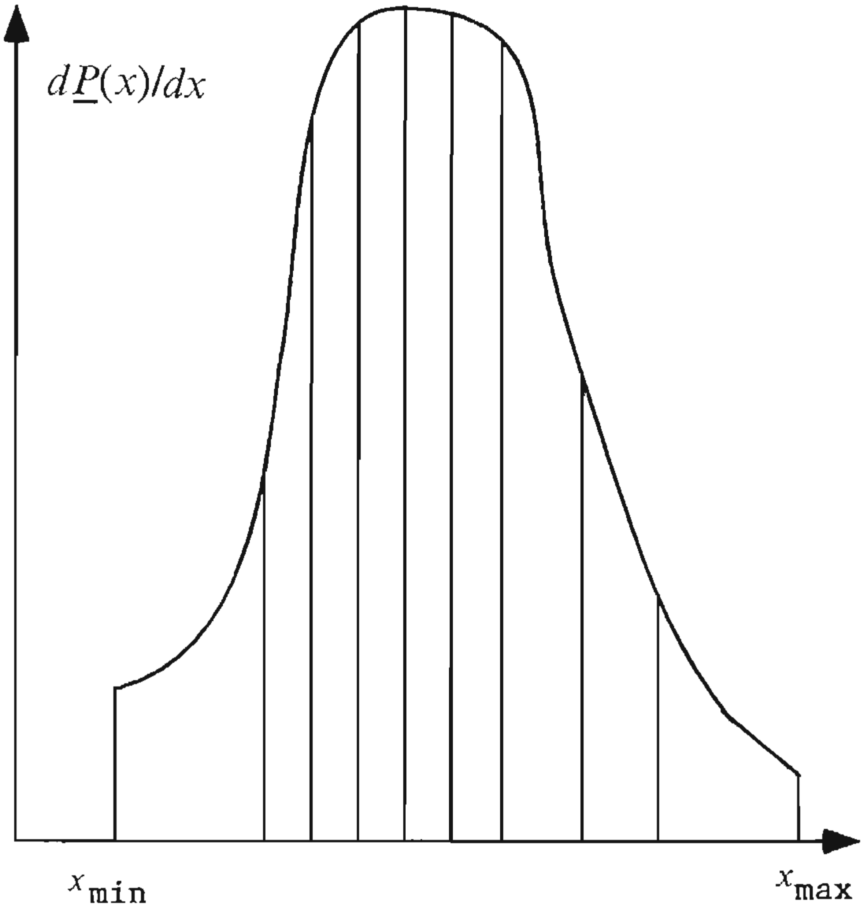


Fig. K.1. Visualization of intervals in $x$, $\Delta x$, over which the integral $\Delta\underline{P}$ is the same.

$(x_{min}, x_{max})$ then the integral probability is normalized to one, so that the cumulative probability from $x_{min}$ to $x$ should be uniformly distributed.

$$\int_{x_{min}}^{x} d\underline{P}(t) \Big/ \int_{x_{min}}^{x_{max}} d\underline{P}(t) = r = \int_{x_{min}}^{x} d\underline{P} \tag{K.2}$$

If $r=0$ then $x=x_{min}$, while if $r=1$, $x=x_{max}$.

A visual interpretation of $r=\int d\underline{P}$ is given in Fig. K.1. The discrete case is shown when we can perform the integral to find $N$ intervals of $x$ with equal probability, $\Delta\underline{P}=\int d\underline{P}\equiv g(x)$. If the inversion, $\Delta x=g^{-1}(\Delta\underline{P})$ is possible then $x$ can be chosen by choosing index $1<i<N$, with $x$ between $x_{min}$ and $x_{max}$.

If this expression can be analytically inverted, we can 'pick $x$ out of a distribution $d\underline{P}(x)/dx$'. Let us examine some possibilities. If $d\underline{P}(x)/dx$ is uniform, we can find a solution. An example arises in azimuthally uniform scattering, for example unpolarized Compton scattering. The aximuthal angle $\phi$ is chosen as,

$$\int_{x_{min}}^{x} dt \Big/ \int_{x_{min}}^{x_{max}} dt = r$$

$$x = x_{min} + r(x_{max} - x_{min}) \tag{K.3}$$

$$\phi = 2\pi r, \quad \phi_{min} = 0, \quad \phi_{max} = 2\pi$$

A more complex behavior is power law dynamics. For example, we can pick out of $1/\theta^4$ for Rutherford scattering, or from $1/T^2$ for recoil delta ray kinetic energies. By the way, the student is strongly urged to try a few of these examples – on the simplest computer that is accessible and sufficient. For example, $x_{min} = 0$, $x_{max} = 2\pi$, $\alpha = 2$, $x = r^{1/3}$, pick a random number, take the cube root and histogram the result.

$$\int_{x_{min}}^{x} t^{\alpha} dt \Big/ \int_{x_{min}}^{x_{max}} t^{\alpha}\, dt = r \tag{K.4}$$

$$x = [x_{min}^{\alpha+1} + r(x_{max}^{\alpha+1} - x_{min}^{\alpha+1}]^{1/(\alpha+1)}$$

The special case $\alpha = 0$ reproduces Eq. K.3.

We have mentioned particle lifetimes several times. A model for unstable particle decays would require picking a decay time out of an exponential distribution. Another example is picking a free path $L$ out of a mean free path, $\exp(-L/<L>)$.

$$\int_{x_{min}}^{x} e^{-t/\tau} dt \Big/ \int_{x_{min}}^{x_{max}} e^{-t/\tau} dt = r \tag{K.5}$$

$$x = -\tau \{\ln[e^{-x_{min}/\tau} + r(e^{-x_{min}/\tau} - e^{-x_{min}/\tau})]\}$$

In the energy rather than the time domain any unstable particle is described by a Lorentzian energy spectrum. The Fourier transform of a decaying exponential with lifetime $\tau$ is a Lorentzian with energy full width $\Gamma = \hbar/\tau$. The uncertainty relation $\Delta E \Delta t = \hbar$ becomes $\Gamma\tau = \hbar$ in this special case. This type of behavior also occurs in any damped resonant system.

$$\frac{\displaystyle\int_{x_{min}}^{x} \left[\frac{1}{(t - x_o)^2 + (\Gamma/2)^2}\right] dt}{\displaystyle\int_{x_{min}}^{x_{max}} \left[\frac{1}{(t - x_o)^2 + (\Gamma/2)^2}\right] dt} \tag{K.6}$$

$$\phi_{min} = 2(x_{min} - x_o)/\Gamma$$

$$\phi_{min} = 2(x_{max} - x_o)/\Gamma$$

$$x = x_o + \Gamma/2\{\tan[\tan^{-1}\phi_{min} + r(\tan^{-1}\phi_{max} - \tan^{-1}\phi_{min})]\}$$

The most common distribution that is used is, perhaps, the Gaussian distribution. The integral leads to a non-analytic error function, so it seems that the technique fails. However, a trick avails: using the joint probability and displacing the mean to zero, $x = x' - \langle x' \rangle$, $y = y' - \langle y' \rangle$.

$$d\underline{P}(x)d\underline{P}(y) = e^{-x^2/2\sigma^2}e^{-y^2/2\sigma^2}\,dx\,dy$$
$$= e^{-r^2/2\sigma^2}r dr\, d\phi \qquad (K.7)$$
$$= e^{-u/2\sigma^2}\frac{du\,d\phi}{2}, \quad u = r^2$$

Therefore, we can pick $\phi$ from a uniform distribution as in Eq. K.3 and $r^2$ from an exponential distribution as in Eq. K.5. The result is, $x = r\cos\,\phi$, $y = r\sin\,\phi$, two Gaussianly distributed variables $x$ and $y$ with zero mean and rms $= \sigma$. The mean can be restored by addition, $x' = x + \langle x'\rangle$, $y' = y + \langle y'\rangle$.

Let us now consider angular distributions. In the case of isotropic dynamics, e.g. decay of a spinless particle, we can use the previous results for uniform distributions for $\phi$ and $\cos\,\theta$. The solid angle is then a constant.

$$d\sigma/d\Omega = \text{const} = 1/4\pi$$
$$d\Omega = d(\cos\theta)d\phi$$
$$\phi = 2\pi r \qquad (K.8)$$
$$\cos\theta = \cos\theta_{min} + r(\cos\theta_{max} - \cos\theta_{min})$$

Suppose we have a more complicated angular distribution. If, for example, $d\sigma/d\Omega = \cos^2\theta$, then we can use the power law behavior, Eq. K.4.

$$d\sigma/d\Omega = \cos^2\theta$$
$$\cos\theta = [\cos^3\theta_{min} + r(\cos^3\theta_{max} - \cos^3\theta_{min})]^{1/3} \qquad (K.9)$$

What about the dipole behavior which is seen in non-relativistic radiation patterns? In that case the integral can be done, but it cannot be analytically inverted. In fact, for most distributions, an analytic inversion is not possible.

$$d\sigma/d\Omega = \sin^2\theta$$
$$\int \sin^2\,\theta\, d(\cos\,\theta) = \cos\theta - \frac{\cos^3\theta}{3} \qquad (K.10)$$

How do we proceed? Clearly, by performing the integral given in Eq. K.2 numerically. Assume that $x$ is contained in the interval $(x_{min}, x_{max})$. Assume also that within that interval there is a maximum value of $(d\underline{P}(x)/dx)$ called $(d\underline{P}(x)/dx)_{max}$ We use the 'rejection' method, see Fig. K.2.

$$\text{pick}r_1 : \text{distributing } x \text{ as } x = x_{min} + r_1(x_{max} - x_{min})$$
$$\text{pick}r_2 : \text{if } r_2 < (d\underline{P}(x)/d\underline{P}(x)dx)_{max} \text{ then accept } x \qquad (K.11)$$
$$\text{if } r_2 > (d\underline{P}(x)/d\underline{P}(x)dx)_{max} \text{ then reject } x$$

As can easily be seen in Fig. K.2, this procedure weights $x$ by $(d\underline{P}(x)/dx)$. Clearly, if $(d\underline{P}(x)/dx) = 0$, then $x$ is never accepted, while if $(d\underline{P}(x)/dx) \sim (d\underline{P}(x)/dx)_{max}$ then $x$ is almost always accepted. For example, in the case $d\sigma/d\Omega = \sin^2\,\theta = 1 - \cos^2\,\theta$ we pick $\cos\theta$ uniformly between $-1$ and 1. If $1 - \cos^2\,\theta < r$, then we accept that choice of $\cos\theta$ and continue. If not we repeat. The student should explicitly try this procedure and verify that it works.

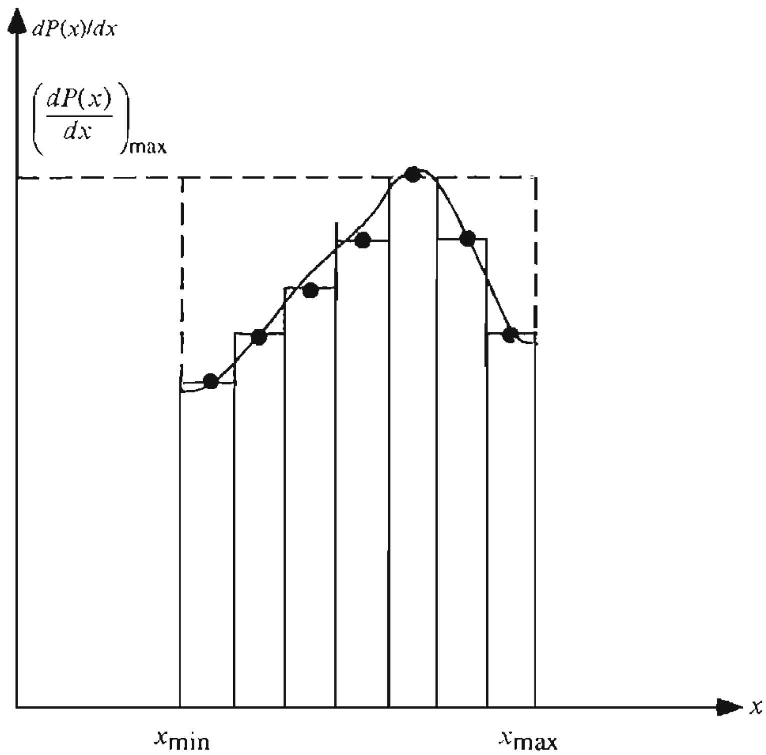


Fig. K.2. Plot of distribution function ($d\underline{P}(x)/dx$) indicating how one approximates it by choosing uniform width strips in $x$ and then accepting the strip $x$ if $(d\underline{P}(x)/dx)/(d\underline{P}(x)/dx)_{max} <$ a random number.

We can also combine methods by using the fact that joint probabilities are multiplicative. Suppose we factor $(d\underline{P}(x)/dx)$ into $g(x)$ and $h(x)$ where $0<g<1$ and $h>0$, $\int h(x)dx=1$.

$$d\underline{P}(x)/dx = g(x)h(x) \tag{K.12}$$

We might do this because $h(x)$ is analytically invertable, for example. We draw $x$ from $h(x)$ and then accept $x$ if $r<g(x)$. Clearly the probability to accept $x$ is $g(x)$ by the rejection technique.

Several 'processes' with probability $P_i$ can be used in a further generalization. For example, the total probability might be the total cross section while the individual probabilities would refer to the possible reactions making up the total reaction rate.

$$dP(x)/dx = \sum_i^N P_i g_i(x)h_i(x) \tag{K.13}$$

First choose $i$ by picking a process using a weight $P_i/\sum_i^N P_i$. Then draw $x$ from $h_i(x)$ either directly or by rejection. Then accept $x$ if $r<g_i(x)$. Clearly very complex distribution functions for $x$ can be built up this way.

# Glossary of symbols

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| $a$ | Radius of a system or acceleration or magnetic radius of curvature or stochastic term in calorimeter resolution. |
| $a_o$ | First Bohr radius = 0.53 Å. |
| $a_T$ | Transverse radius of curvature in a magnetic field. |
| $A$ | Atomic weight or area or electromagnetic vector potential, **A**. |
| Å | Angstrom = $10^{-10}$ m. |
| $A(\theta)$ | Scattering amplitude or transition amplitude. |
| $A^{\mu}$ | Four dimensional acceleration. |
| $\alpha$ | Fine-structure constant or first Townsend coefficient or $\hat{\alpha}$ direction cosines. |
| $\mathrm{\alpha}$ | Helium nucleus. |
| $\alpha_s$ | Strong interaction coupling constant. |
| $b$ | Impact parameter. Point of closest approach in a collision or constant term in calorimeter resolution. |
| b | barn = $10^{-24}$ cm$^2$. |
| **B** | Magnetic induction field, or binding energy per nucleon. |
| $\beta$ | Velocity with respect to $c = v/c$. |
| β | Electron from nuclear beta decay. |
| $\boldsymbol{\beta}_d$ | Drift velocity in **E** and **B** fields. |
| $c$ | Speed of light. |
| $C$ | Counting rate or capacitance. |
| $C_s$ | Source capacity. |
| $\underline{C}$ | Capacitance per unit length. |
| CB | Conduction band in a solid. |
| $\chi^2$ | The chi-squared function. |
| $d$ | Distance between detector electrodes or distance between wires in a PWC. |
| $d\bar{V}^2$ | Mean square noise voltage frequency distribution. |
| $d\Omega$ | Element of solid angle = $d(\cos\theta)d\phi$. |
| $D$ | Diffusion coefficient or displacement electric field. |
| $\delta$ | Phase shift or skin depth or difference of refractive index from one or Dirac delta function or gain/dynode in a PMT or optical phase difference. |
| $\Delta t$ | Sample thickness of an EM calorimeter in $X_o$ units. |
| $\Delta y$ | RMS of a distribution function. |

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| $\Delta E$ | Energy deposited in a calorimeter sample. |
| $\Delta \nu$ | Sampling thickness of a hadronic calorimeter in $\lambda_I$ units. |
| $e$ | Electronic charge. |
| $e/h$ | Ratio of the response of a calorimeter to EM and hadronic components of a cascade. |
| $e_n$ | Voltage noise in volts/$\sqrt{\text{Hz}}$ for a transistor. |
| eV | Electron volt. |
| $\mathbf{E}$ | Electric field strength or total system energy. |
| $E_g$ | Band gap in a solid between the top of the VB and the bottom of the CB. |
| $E_n$ | Bohr energy in a quantum state $n$. |
| $E_o$ | Rydberg ground state energy = 13.6 eV. |
| $E_c$ | Critical energy where ionization and radiative losses are equal. |
| $E_s$ | Characteristic multiple scattering energy, 21 MeV. |
| $E_{TH}$ | Threshold for pion production in a hadronic interaction. |
| ENC | Equivalent Noise Charge or noise referred to the input compared to source charge. |
| $\varepsilon$ | Particle energy or detection efficiency or electrical permitivity or polarization vector. |
| $\varepsilon(t)$, $\varepsilon(\nu)$ | Particle energy in a cascade as a function of depth $t$ or $\nu$. |
| $f$ | Frequency of a wave or focal length of a lens system. |
| $f_o$ | Fraction of electromagnetic energy in a single hadronic collision, $\sim 1\sqrt{3}$. |
| '$f_o$' | Fraction of electromagnetic energy in a hadronic cascade. |
| $f(\omega)$ | Transfer function or filter function in noise source studies. |
| fm | Fermi = $10^{-13}$ cm. |
| $F$ | Force on a particle. |
| $g_m$ | Transconductance of a front end transistor. |
| GeV | $10^9$ eV. |
| $\gamma$ | Ratio of particle energy/mass = $1/\sqrt{1-\beta^2}$. |
| γ | Photon |
| $\Gamma$ | Transition rate or inverse lifetime = decay width. |
| $H$ | System Hamiltonian or magnetic field. |
| $\hbar$ | Planck constant, $\hbar = h/2\pi$. |
| $i_s$ | Source current liberated in a detector by particle passage. |
| $I$ | Intensity = energy crossing unit with frequency $\omega$, $I(\omega)$ or ionization potential $\langle I \rangle$ or electric current. |
| $I(t)$ | Capacitively induced current flow between detector electrodes. |
| $\mathbf{j}$ | Current density or current/area. |
| $k$ | Wave number = $2\pi/\lambda$ or particle momentum in $\hbar$ units ($\mathbf{p}=\hbar\mathbf{k}$) or Boltzmann's constant or quadrupole magnet gradient constant. |
| K | Kaon or K meson. |
| $L$ | Mean free path, $\langle L \rangle$, or length of travel or total path length in a cascade or angular momentum or likelihood function. |
| $L_z$ | Angular momentum about the $z$ axis. |
| $\underline{L}$ | Inductance per unit length. |
| $\ell$ | Angular momentum quantum number or length. |

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| $\lambda$ | Wavelength or charge/length on an electrode or nuclear interaction length. |
| $\lambda\!\!\!^{-}$ | Compton wavelength of a particle $= \hbar/mc$. |
| $\lambda\!\!\!^{-}_{DB}$ | deBroglie wavelength $= h/p$. |
| $m$ | Mass of particle (light particles), or $L_z = m\hbar$ angular momentum projection quantum number. |
| $M$ | Mass of a particle (used for ions) or system of particles or matrix for magnetic beam transport or matrix of multiple scattering error. |
| MeV | $10^6$ eV. |
| MIP | Minimum Ionizing Particle. |
| $\mu$ | Magnetic permeability or mobility of charge carriers in an electric field or electron magnetic moment. |
| $n$ | Number density or principal Bohr quantum number or index of refraction or magnet current turns per unit length. |
| $N$ | Number of occurrences or noise. |
| $<N>$ | Mean number of particles produced in a hadronic collision. |
| $N_o$ | Avogadro's number. |
| $N^o(\nu)$, $N^{\pm}(\nu)$ | Number of neutral and charged particles at depth $\nu$ in a hadronic cascade. |
| $N(t)$, $N(\nu)$ | Number of particles in a cascade as a function of depth $t$ or $\nu$. |
| $\nu$ | Path length in nuclear interaction length units. |
| $\omega$ | Circular frequency, $\omega = 2\pi f$, or photon energy, $\varepsilon = \hbar\omega$. |
| $\omega_p$ | Plasma frequency. |
| $\Omega$ | Solid angle. |
| $p$ | Particle momentum or number density of positive charge carriers. |
| $p^*$ | Momentum in the CM frame. |
| $p^\mu$ | Four dimensional momentum. |
| $p_T$ | Transverse momentum. |
| $P$ | Probability of occurrence or gas pressure or 'pitch' (electrode spacing) of a silicon detector. |
| $\underline{P}$ | Power of a system or probability distribution function. |
| PMT | Photomultiplier tube. |
| $P_\ell$ | Legendre polynomial. |
| $\phi$ | Azimuthal angle or bend angle in a magnetic field. (Dipole $\phi_B$ or quadrupole $\phi_Q$.) |
| $\phi_L$ | Lorentz angle in combined electric and magnetic fields. |
| $\pi$ | Pion, or $\pi$ meson. |
| $\Psi$ | Wave function, where $\lvert\Psi\rvert^2\, d\mathbf{r}$ is the probability of finding a particle in a volume element $d\mathbf{r}$. |
| $q$ | Particle charge or momentum transfer in a collision. |
| $q_p$ | Charge induced on 'pads' of a wire chamber cathode. |
| $q_s$ | Source charge liberated in a detector by particle passage. |
| $q(t)$ | Charge in a detector appearing between the electrodes. |
| $Q$ | Ratio of recoil kinetic energy to mass in a collision. |
| $Q(t)$ | Charge induced on detector electrodes. |
| $r$ | Radius or random number. |
| $r_h$ | Characteristic transverse size of a hadronic shower. |
| $r_M$ | Moliere radius. Characteristic transverse EM cascade size. |

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| $R$ | Radial wave function or electrical resistance or reflection coefficient. |
| RMS | Root mean square of a distribution. |
| $\underline{R}$ | Range of a particle or resistance per unit length. |
| $\rho$ | Mass density or charge density of a system or electrical resistivity $= 1/\sigma$ or density of quantum states or magnetic radius of curvature. |
| $s$ | Four dimensional interval between two space–time events or path length for orbit in a magnetic field or spin quantum number. |
| $s_z$ | Spin angular momentum component about the $z$ axis. |
| $\sqrt{s}$ | Energy in the Center of Momentum frame. |
| STP | Standard Temperature and Pressure. |
| $\sigma$ | Cross section or electrical conductivity or charge per unit area (surface charge density) or root mean square deviation of a set of measurements. |
| $t$ | Time label of events or path length in radiation length units. |
| $T$ | Kinetic energy or wire tension or absolute temperature. |
| $\tau$ | Lifetime of process or mean time of occurrence or decay lifetime. |
| $\theta$ | Spherical angle. $\langle\theta^2\rangle$ = mean square scattering angle. |
| $\theta_{MS}$ | Multiple scattering angle. |
| $u$ | Energy density of a field or radial wave function (bound state). |
| $U$ | Energy of an electrical system or total potential energy of a system. |
| $U^\mu$ | Four dimensional velocity |
| $v$ | Particle velocity. |
| $v_d$ | Drift velocity. |
| $v_T$ | Thermal velocity. |
| $V$ | Electric potential, voltage ($U = qV$). |
| $\underline{V}$ | Volume of a system. |
| $\langle V^2\rangle$ | Mean square output noise voltage after frequency filtering. |
| $V_B$ | Magnetostatic potential. |
| $V_T$ | Threshold voltage. |
| $V(t)$ | Voltage as a function of time induced on detector electrodes. |
| VB | Valence band in a solid |
| $W$ | Ratio of sampled energy in a cascade to the total energy or sampling fraction. |
| $WT$ | 'Weight' of a measurement $= 1/\sigma^2$. |
| $x$ | Distance traveled along the path of a particle, or $\mathbf{x}$ vector labeling particle position. |
| $x'$ | Transverse 'velocity' $dx/ds$ (dimensionless). |
| $x^\mu$ | Four dimensional position. |
| $X_o$ | Radiation length. |
| $y$ | Energy in units of the critical energy or frequency in units of $\gamma\omega_p$ or transverse multiple scattering displacement or ratio of time to drift time. |
| $\bar{y}$ | Mean of a set of measurements of $y$. |
| $\langle y\rangle$ | Mean of a distribution function. |
| $Y_\ell^m$ | Spherical harmonic, angular solution of the Schroedinger equation for central forces. |
| $Z$ | Atomic number or electrical impedance of a circuit element. |
| $\xi$ | Formation zone length in transition radiation. |