Lecture # 10

02/01/2023

Math problem: solving a system of linear equations

Example:

$$
\underbrace{\begin{aligned}
5x - 2.8y + z &= 1 \\
x + 7y - 9.2z &= 7 \\
-3x + z &= 11.8
\end{aligned}}
$$

$$
\underbrace{\begin{bmatrix} 5 & -2.8 & 1 \\ 1 & 7 & -9.2 \\ -3 & 0 & 1 \end{bmatrix}}_{3\times 3 \text{ matrix}}
\underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{3\times 1 \text{ vector}}
=
\underbrace{\begin{pmatrix} 1 \\ 7 \\ 11.8 \end{pmatrix}}_{3\times 1 \text{ vector}}
$$

$\Longleftrightarrow$ $\underbrace{A}_{\substack{\text{square} \\ \text{matrix} \\ \text{of real entries}}} \underline{x} = \underline{b}$

column vector of real entries

Given A, $\underline{b}$, compute unknown $\underline{x}$.

In general, we want to solve: $A\underline{x} = \underline{b}$

$A \in \mathbb{R}^{n\times n}$, $\underline{x} \in \mathbb{R}^{n\times 1}$, $\underline{b} \in \mathbb{R}^{n\times 1}$

$\boxed{\begin{array}{ll} A, \underline{b} & \text{known} \\ \underline{x} & \text{unknown} \end{array}}$

In practice, n is large.

Recall facts from linear algebra:

Solving $A\underline{x} = \underline{b}$

mostly this class

$\underline{b} = \underline{0}$ (homogeneous system)

$\det(A) \neq 0$

$\underline{x} = \underline{0}$ is the unique solution

$\det(A) = 0$

There are infinitely many solutions (If $\underline{u}$ and $\underline{v}$ are solutions, then so is $\alpha\underline{u} + \beta\underline{v}$)

$\underline{b} \neq \underline{0}$ (non-homogeneous system)

$\det(A) \neq 0$

Unique solution

$\underline{x} = ?$

$\det(A) = 0$

depends on both $A, \underline{b}$

No solution (# of solution $= 0$)

Infinitely many solutions (# of solution $= \infty$)

Geometric meaning of solving a square linear system:

finding common point(s) of intersection of the hyperplanes

For $n = 3$: we have 3 planes in 3 dimensions.

If all 3 planes intersect at a common point, then unique solution.

→ All 3 planes are parallel to each other

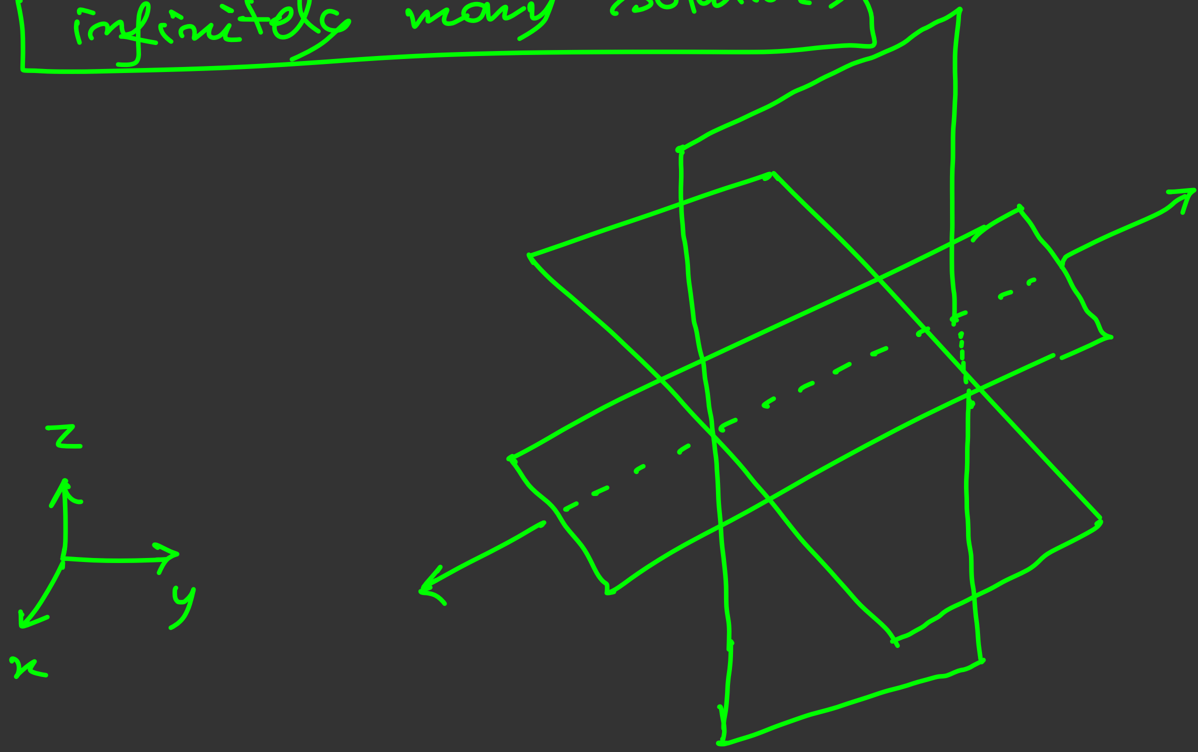
OR 2 planes " " " " " and 1 intersects them

OR No planes are parallel but the lines of intersection of each pair are parallel

No solutions

- If none of the 3 planes is parallel to any other two, but one passes through the line of intersection of the other two, then

$\boxed{\text{infinitely many solutions}}$



Suppose, we have a square linear system $A\underline{x} = \underline{b}$ with $\underline{b} \neq \underline{0}$, $\det(A) \neq 0$

<u>Question:</u> How to compute the unique solution $\underline{x}$?

<u>Example:</u> <u>Näive idea: Gauss elimination:</u>

$$\left.\begin{aligned} x_1 + 2x_2 - x_3 &= 3 \\ 2x_1 + x_2 - 2x_3 &= 3 \\ -3x_1 + x_2 + x_3 &= -6 \end{aligned}\right\} \quad \underbrace{\underbrace{\begin{bmatrix} & A & \end{bmatrix}}_{3\times 3}}_{A} \underbrace{\underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{3\times 1}}_{\underline{x}} = \underbrace{\underbrace{\begin{pmatrix} 3 \\ 3 \\ -6 \end{pmatrix}}_{3\times 1}}_{\underline{b}}$$

Elimination step (try to get an upper triangular matrix structure)

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 2 & 1 & -2 & 3 \\ -3 & 1 & 1 & -6 \end{array}\right] \quad \text{with } A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -2 \\ -3 & 1 & 1 \end{bmatrix},\ b = \begin{bmatrix} 3 \\ 3 \\ -6 \end{bmatrix}$$

$$\xrightarrow{\overbrace{R_2}^{\text{row \# 2}} - 2\overbrace{R_1}^{\text{row \# 1}}} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -3 & 0 & -3 \\ -3 & 1 & 1 & 6 \end{array}\right]$$

$$\xrightarrow{R_3 - (-3R_1)} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -3 & 0 & -3 \\ 0 & 7 & -2 & 3 \end{array}\right]$$

$$\xrightarrow{R_3 - (-7/3)R_2} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -3 & 0 & -3 \\ 0 & 0 & -2 & -4 \end{array}\right]$$

We have got:

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 3 \\ -3x_2 &= -3 \\ -2x_3 &= -4 \end{aligned}$$

Back substitution:

$$x_3 = \frac{-4}{-2} = +2$$

$$-3x_2 = -3 \Rightarrow x_2 = +1$$

$$x_1 + 2x_2 - x_3 = 3$$

$$\Rightarrow x_1 + \cancel{2} - \cancel{2} = 3$$

$$\Rightarrow x_1 = 3$$

$$\therefore \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$$

Done.

Operational count:

$$\underset{n\times n}{\underset{\sim}{A}}\ \underset{n\times 1}{\underset{\sim}{\underline{x}}} = \underset{n\times 1}{\underset{\sim}{\underline{b}}}$$

elimination step:

$$\underbrace{\frac{2}{3}n^3 + \frac{1}{2}n^2 - \frac{7}{6}n}_{\text{Operations}}$$

back substitution step: n^2 operations

Total: $O(n^3)$

"Big O-notation"

"Of the order of".

Algorithm: LU decomposition

L ← Lower triangular

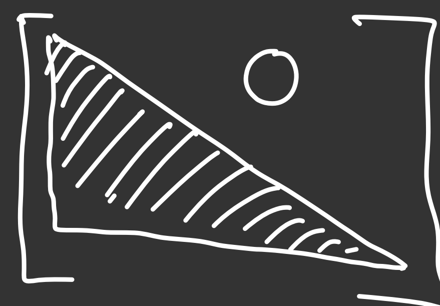
U ← upper triangular

Systematic version of Gauss elimination

Consider a square matrix $A = [a_{ij}]$

We call this matrix lower triangular if

$a_{ij} = 0$ for all $i < j$



Similarly, we say A is upper triangular matrix

if $a_{ij} = 0$ for all $i > j$

$A =$