Lecture #7

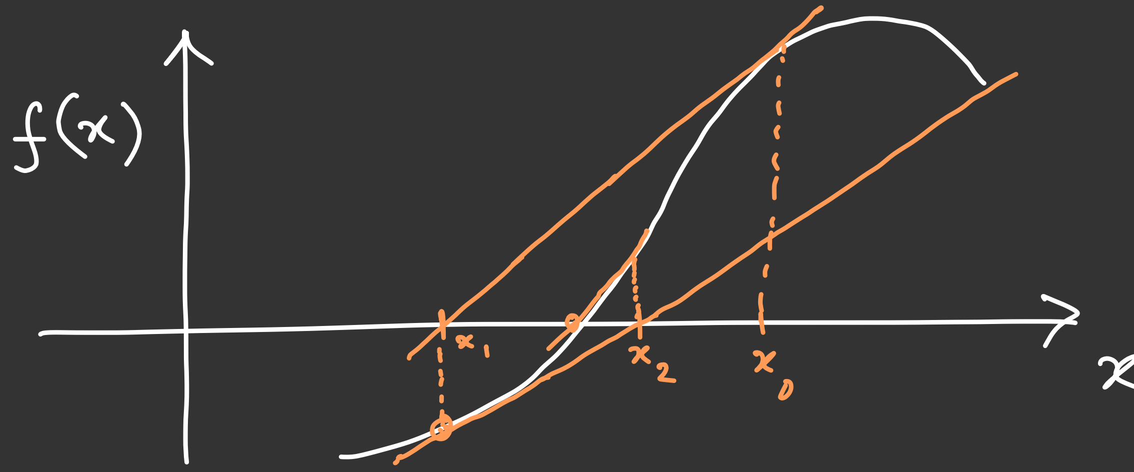
01/25/2023

Newton's method/Newton-Raphson method

To solve the same math problem as before: compute the real root(s) of $\boxed{f(x) = 0}$

Idea:
- Make an initial guess: x_0
- Then do: $$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

  where $k = 0, 1, 2, \ldots$
- Iterate for $\underbrace{\text{Num\_Iter}}_{\text{(for example, 1000)}}$ times



Observe that:

we need:

1. $f'(x_k)$ must exist
2. $f'(x_k) \neq 0$

$$
\begin{cases}
x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)} \\[2ex]
x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)} \\[2ex]
x_3 = x_2 - \dfrac{f(x_2)}{f'(x_2)} \\
\vdots
\end{cases}
$$

Example: Solve $\underbrace{x^3 + x - 1}_{f(x)} = 0$

It has 3 roots, only one of them real

want to compute this via

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

$$= x_k - \frac{x_k^3 + x_k - 1}{3x_k^2 + 1}$$

$$= \frac{2x_k^3 + 1}{3x_k^2 + 1}$$

Try this recursion in MATLAB:

$$x_{k+1} = \frac{2x_k^3 + 1}{3x_k^2 + 1}, \quad k = 0, 1, 2, \ldots$$

$$x_{true} = 0.68232780\ldots$$

Advantages	Disadvantages
If it converges (is may NOT), then it does so with better OR same rate as the bisection algorithm	• $f' = 0 \rightsquigarrow$ trouble! • f' does not exist $\rightsquigarrow$ trouble! • May NOT even converge (may diverge, oscillate forever)

Theorem: $\boxed{\text{If}}$ $f(x)=0$ has a "simple" real root, and if Newton's method converges to that root, $\boxed{\text{then}}$ it converges quadratically $(\alpha = 2)$

$$\therefore \lim_{n\to\infty} \frac{e_{n+1}}{e_n^2} = \lambda$$

Simple $\equiv$ "non-repeated"

Hint for proof: Use "Taylor theorem with remainder" to show that.

$$\lim_{n\to\infty} \frac{e_{n+1}}{e_n^{2}} = \frac{1}{2}\left|\frac{f''(x_{true})}{f'(x_{true})}\right| \rightarrow \lambda$$

$\alpha = 2$

Example: Compute $\sqrt{a}$. $a > 0$.

$$\left.\begin{aligned} & x = \sqrt{a} \\ \Leftrightarrow\ & x^2 = a \\ \Leftrightarrow\ & \underbrace{x^2 - a}_{f(x)} = 0 \end{aligned}\right\}$$

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

$$= x_k - \frac{x_k^2 - a}{2x_k}$$

$$\Rightarrow \boxed{x_{k+1} = \frac{x_k^2 + a}{2x_k}}$$

Guess x_0, then iterate in a loop

Example of $\underbrace{\text{repeated}}_{\text{non-simple}}$ root :

$$f(x) = \underbrace{x^2}_{f(x)} = 0$$

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

$$= x_k - \frac{x_k^2}{2x_k} = \frac{x_k}{2}$$

$$\Rightarrow \boxed{x_{k+1} = \frac{1}{2} x_k}$$

$$\Rightarrow x_1 = \frac{1}{2} x_0, \quad x_2 = \frac{1}{2} x_1 = \left(\frac{1}{2}\right)^2 x_0, \quad x_3 = \left(\frac{1}{2}\right)^3 x_0$$

$\therefore \quad x_k = \left(\frac{1}{2}\right)^k x_0 \quad \forall\ k = 0, 1, 2, \ldots$

$\Rightarrow \quad e_{k+1} = e_k/2$

$\therefore$ linear convergence.

General result:

Theorem. Let $f \in C^{m+1}([a,b])$. Suppose that the nonlinear equation $f(x) = 0$ has a root at $x = r$ with multiplicity $\tilde{m} := m+1 > 1$.

Then, Newton's method is locally convergent to r, and

$$\lim_{k \to \infty} \frac{e_{k+1}}{e_k} = \frac{\tilde{m} - 1}{\tilde{m}}$$

$$\therefore \quad \underbrace{\alpha = 1,}_{\text{linear convergence}} \qquad \lambda = \frac{\tilde{m} - 1}{\tilde{m}}.$$