AM 147: Computational Methods and Applications

University of California, Santa Cruz

Winter 2023

Final Exam

Name:____ Student ID:______________________________

For this exam you only need a pen/pencil. Calculators or other electronic devices are NOT allowed. Write your answers in the spaces provided. If you need more space, work on the other side of the page.

1. Consider vectors $\boldsymbol{p}, \boldsymbol{q}$, each of size 2×1. [10 + (1 + 9) + (1 + 9) = 30 points]

   (a) For $\boldsymbol{p} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ and $\boldsymbol{q} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$, compute the matrix $\boldsymbol{A} = \boldsymbol{p}\,\boldsymbol{q}^{\top}$.

   Solution:
   $$\boldsymbol{A} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \begin{pmatrix} 2 & 5 \end{pmatrix} = \begin{pmatrix} 6 & 15 \\ 8 & 20 \end{pmatrix}.$$

   (b) Consider the matrix $\boldsymbol{A}$ from part (a). For a given 2×1 vector $\boldsymbol{b} \neq \boldsymbol{0}$, can the system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ have unique solution vector $\boldsymbol{x}$? Why/why not?

   Solution: No, we cannot have unique solution for vector $\boldsymbol{x}$. This is because $\det(\boldsymbol{A}) = 6 \times 20 - 8 \times 15 = 0$.

   (c) If we change the numerical values of the entries of $\boldsymbol{p}, \boldsymbol{q}$ in part(a), will your answer to part (b) change? Why/why not?

   Solution: No, the answer will not change. To see why, take any $\boldsymbol{p} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$ and $\boldsymbol{q} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$. Then $\boldsymbol{A} = \boldsymbol{p}\,\boldsymbol{q}^{\top} = \begin{pmatrix} p_1q_1 & p_1q_2 \\ p_2q_1 & p_2q_2 \end{pmatrix}$, and hence we always have $\det(\boldsymbol{A}) = p_1p_2q_1q_2 - p_1p_2q_1q_2 = 0$.

2. Consider the data in the following table. [(5 + 5) + 15 + (2 + 3) = 30 points]

x_i	y_i
1	2
2	3
4	5
5	6

(a) We want to compute a least squares approximation of the form $y = \alpha + \beta x$ for the above dataset. Clearly write down the matrix $\boldsymbol{A}$ and vector $\boldsymbol{y}$ for converting this least squares problem in the standard form:

$$\underset{\boldsymbol{\theta} := \begin{pmatrix}\alpha\\ \beta\end{pmatrix}}{\text{minimize}} \; \|\boldsymbol{A\theta} - \boldsymbol{y}\|_2^2.$$

Solution: For the given dataset, $\boldsymbol{A} = \begin{pmatrix}1 & 1\\ 1 & 2\\ 1 & 4\\ 1 & 5\end{pmatrix}$, and $\boldsymbol{y} = \begin{pmatrix}2\\ 3\\ 5\\ 6\end{pmatrix}$.

(b) Use your answer in part (a) and the least squares solution $\boldsymbol{\theta} = \left(\boldsymbol{A}^\top \boldsymbol{A}\right)^{-1} \boldsymbol{A}^\top \boldsymbol{y}$ to numerically compute the least squares estimates for α, β. Use fractions (that is, rational number format) in all your calculations. Show all the steps.

Solution: From part (a), we have

$$\boldsymbol{A}^\top \boldsymbol{A} = \begin{pmatrix}1 & 1 & 1 & 1\\ 1 & 2 & 4 & 5\end{pmatrix} \begin{pmatrix}1 & 1\\ 1 & 2\\ 1 & 4\\ 1 & 5\end{pmatrix} = \begin{pmatrix}4 & 12\\ 12 & 46\end{pmatrix},$$

which gives (using the formula supplied in last page)

$$\left(\boldsymbol{A}^\top \boldsymbol{A}\right)^{-1} = \frac{1}{(4 \times 46) - (12 \times 12)} \begin{pmatrix}46 & -12\\ -12 & 4\end{pmatrix} = \frac{1}{40} \begin{pmatrix}46 & -12\\ -12 & 4\end{pmatrix} = \begin{pmatrix}\frac{23}{20} & -\frac{3}{10}\\ -\frac{3}{10} & \frac{1}{10}\end{pmatrix}.$$

On the other hand, $\boldsymbol{A}^\top \boldsymbol{y} = \begin{pmatrix}1 & 1 & 1 & 1\\ 1 & 2 & 4 & 5\end{pmatrix} \begin{pmatrix}2\\ 3\\ 5\\ 6\end{pmatrix} = \begin{pmatrix}16\\ 58\end{pmatrix}$. Thus, we obtain

$$\boldsymbol{\theta} := \begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{pmatrix}\frac{23}{20} & -\frac{3}{10}\\ -\frac{3}{10} & \frac{1}{10}\end{pmatrix} \begin{pmatrix}16\\ 58\end{pmatrix} = \begin{pmatrix}\frac{368-348}{20}\\ \frac{116-96}{20}\end{pmatrix} = \begin{pmatrix}1\\ 1\end{pmatrix}.$$

(c) From your answer in part (b), the least squares linear approximation $y = \alpha + \beta x$ passes through how many points of the given dataset? Explain your answer.

Solution: From part (b), the least squares linear approximation is $y = 1 + x$, which (from the given table) passes through all the four datapoints.

The reason is that each y_i in the given dataset is one more than the corresponding x_i. Hence the least squares linear approximation passes through all of them.

3. We want to numerically approximate the first derivative of $f(x) = \exp(x)$ at $x = x_0$ with stepsize $h > 0$. In answering the following questions, you can use the Taylor expansion $\exp(\pm h) = 1 \pm h + \frac{h^2}{2} \pm \frac{h^3}{6} + \ldots$ [10 + 10 + 10 = 30 points]

(a) For the above $f(x)$, compute the forward difference approximation for $f'(x_0)$ of the form

$$\frac{f(x_0 + h) - f(x_0)}{h} = g_{\text{Forward}}(x_0) + O(h).$$

That is, find $g_{\text{Forward}}(x_0)$. Show all your steps.

Solution: We have

$$\frac{f(x_0 + h) - f(x_0)}{h} = \frac{\exp(x_0)\left(\exp(h) - 1\right)}{h} = \frac{\exp(x_0)\left((1 + h + \frac{h^2}{2} + \frac{h^3}{6} + \ldots) - 1\right)}{h}$$
$$= \exp(x_0) + O(h).$$

Thus, $g_{\text{Forward}}(x_0) = \exp(x_0)$.

(b) For the above $f(x)$, compute the backward difference approximation for $f'(x_0)$ of the form

$$\frac{f(x_0) - f(x_0 - h)}{h} = g_{\text{Backward}}(x_0) + O(h).$$

That is, find $g_{\text{Backward}}(x_0)$. Show all your steps.

Solution: We have

$$\frac{f(x_0) - f(x_0 - h)}{h} = \frac{\exp(x_0)\left(1 - \exp(-h)\right)}{h} = \frac{\exp(x_0)\left(1 - (1 - h + \frac{h^2}{2} - \frac{h^3}{6} + \ldots)\right)}{h}$$
$$= \exp(x_0) + O(h).$$

Thus, $g_{\text{Backward}}(x_0) = \exp(x_0)$.

(c) For the above $f(x)$, compute the central difference approximation for $f'(x_0)$ of the form

$$\frac{f(x_0+h)-f(x_0-h)}{2h} = g_{\text{Central}}(x_0) + O(h^2).$$

That is, find $g_{\text{Central}}(x_0)$. Show all your steps.

Solution: We have

$$\begin{aligned}
\frac{f(x_0+h)-f(x_0-h)}{2h} &= \frac{\exp(x_0)\left(\exp(h)-\exp(-h)\right)}{2h} \\
&= \frac{\exp(x_0)\left((1+h+\frac{h^2}{2}+\frac{h^3}{6}+...)-(1-h+\frac{h^2}{2}-\frac{h^3}{6}+...)\right)}{2h} \\
&= \frac{\exp(x_0)\left(2h+\frac{h^3}{3}+...\right)}{2h} \\
&= \exp(x_0) + O(h^2).
\end{aligned}$$

Thus, $g_{\text{Central}}(x_0) = \exp(x_0)$.

4. For each the following statements, ONLY ONE among the three options is correct. Choose the correct option for each. You DO NOT need to provide any explanation.

[5 × 2 = 10 points]

(a) Consider the ODE initial value problem: $\frac{\mathrm{d}y}{\mathrm{d}t} = f(t,y)$, $y(t_0) = y_0$ (given), and let $t_k := t_0 + k\Delta t$, $y_k := y(t_k)$ for all $k = 0, 1, \ldots$. Then

$$y_{k+1} - y_k = \int_{t_k}^{t_{k+1}} f(t,y)\mathrm{d}t.$$

Approximating the above integral by trapezoid method

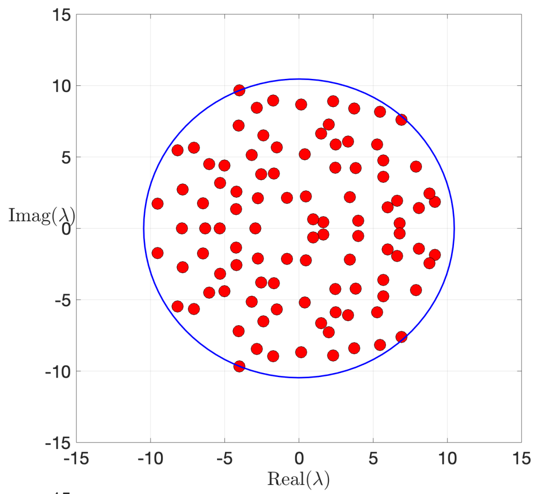
(i) gives an explicit method to solve the ODE initial value problem.
(ii) gives an implicit method to solve the ODE initial value problem.
(iii) makes it impossible to solve the ODE initial value problem.

(b) The function $S(x) = \begin{cases} x & \text{for} \quad -1 \le x \le 0.5, \\ 0.5 + 2(x-0.5) & \text{for} \quad 0.5 \le x \le 2, \\ x + 1.6 & \text{for} \quad 2 \le x \le 4, \end{cases}$ is

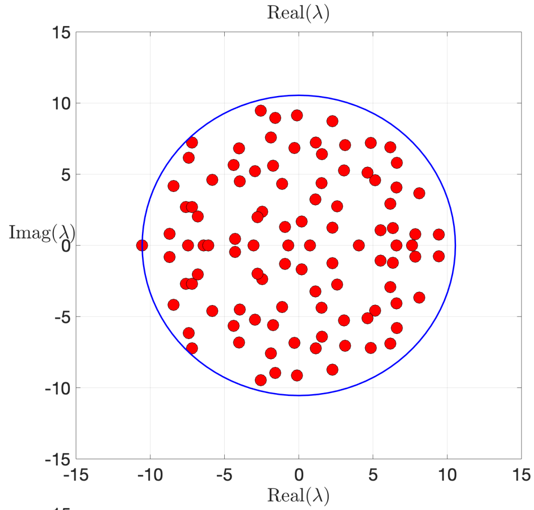
(i) not a spline.
(ii) a spline but not a linear spline.
(iii) a linear spline.

(c) Each figure below plots the eigenvalues of a real diagonalizable 100×100 matrix in the complex plane along with a circle centered at origin with radius equal to the spectral radius of that matrix. <u>For which of the following, the power iteration will converge</u>?

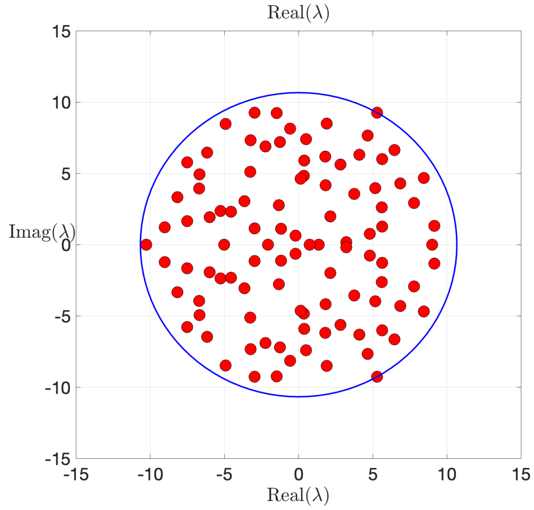
(i)



(ii)



(iii)



(d) For $N > 0$ and r non-zero real, the Newton's method to compute $N^{1/r}$ is

(i) $x_{k+1} = \dfrac{1}{r}\left((r+1)x_k + \dfrac{N}{x_k^{r-1}}\right)$.
(ii) $x_{k+1} = \dfrac{1}{r}\left((r-1)x_k + \dfrac{N}{x_k^{r-1}}\right)$.
(iii) $x_{k+1} = \dfrac{1}{r}\left((r+1)x_k + \dfrac{N}{x_k^{r+1}}\right)$.

(e) In PageRank algorithm, smaller numerical value of the damping factor α where $0 < \alpha < 1$, will lead to

(i) faster convergence but less accuracy.
(ii) slower convergence but more accuracy.
(iii) slower convergence and less accuracy.

Some useful information

- The inverse of a 2×2 nonsingular matrix can be computed as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

- Area of a trapezoid
  $= \frac{1}{2} \times$ distance between parallel sides $\times$ sum of the lengths of parallel sides.

- Newton's method to solve a nonlinear equation $f(x) = 0$, is the recursion
  $x_{k+1} = x_k - \dfrac{f(x_k)}{f'(x_k)}$, $k = 0, 1, \ldots$.