Lecture # 26

03/13/2023

The additional assumption needed on S is "primitivity" of a matrix.

Definition: A matrix $M \in \mathbb{R}^{n \times n}$ is called primitive if there exists some natural number/positive integer K such that

$$M^K > 0$$

(elementwise)

<u>Theorem:</u> (Very important) (Perron-Frobenius theorem)

(one version important for us)

If S (already column-stochastic) is also primitive, then

(i) $\lambda = 1$ is a simple eig.value

(ii) $\underline{p} > \underline{0}$

(elementwise)

However, in web search, the column stochastic matrix S is NOT necessarily primitive.

Larry Page & Sergey Brin (1996):

Their idea: From S, create a perturbed version of this matrix while preserving column stochasticity but guaranteeing primitivity

The perturbed S they proposed

$$\underbrace{G}_{\text{Google matrix}} := \alpha S + \frac{1-\alpha}{n} J,$$

S: column stochastic but may NOT be primitive

J: all ones matrix (ones(n) in MATLAB), equal to $\mathbb{1}\mathbb{1}^T$

where $0 < \alpha < 1$

Page & Brin's original choice: $\alpha = 0.85$

Notice that each column sum of G equals:

$$= \sum_i \left(\alpha S + \frac{1-\alpha}{n} J \right)$$

$$= \alpha \left(\sum_i S \right) + \frac{1-\alpha}{n} \left(\sum_i J \right)$$

$$= \alpha \cdot 1 + \frac{1-\alpha}{\cancel{n}} \cdot \cancel{n}$$

$$= \cancel{\alpha} + 1 - \cancel{\alpha}$$

$$= 1$$

All entries of G are nonnegative too.

$\therefore$ G is a column-stochastic matrix.

But on top of that, now we have:

$$G_{ij} > 0 \quad \text{for all } i, j = 1, \dots, n$$

$$\boxed{\therefore \; G \text{ is also a primitive matrix}}$$

$$\therefore \; \lambda_{max}(G) = 1$$

$\Leftrightarrow$ The dominant magnitude eig. value is unmoved & is simple

$\therefore$ By Perron-Frobenius Theorem, $\underline{p} > \underline{0}$

elementwise

But the PageRank vector/dominant eig. vector $\underline{p}$ now becomes a function of α.

$\therefore$ By construction, we now have :

$$|\lambda_1(G) = 1| > |\lambda_2(G)| \geq |\lambda_3(G)| \geq \cdots \geq |\lambda_n(G)|$$

because $\lambda_1(G) = 1$ is simple

$\therefore$ We can now compute $\underline{p}(\alpha)$ satisfying the fixed point equation :

$$\underline{p}(\alpha) = G\,\underline{p}(\alpha)$$

using the $\boxed{\text{Power Iteration Algorithm}}$

"damped PageRank"

where α satisfying $0 < \alpha < 1$ is called the damping fator

$\therefore$ We can simply implement the following for loop for computing the damped PageRank $\underline{p}(\alpha)$:

for $k = 0, 1, 2, \ldots$

$$\underline{p}_{k+1} = G\,\underline{p}_k$$

end

no need to normalize since G is column-stochastic by construction and $\|\underline{p}_k\|_1 = 1$ as $\underline{p}_k$ is a probability vector

US patent 6285999 filed on 1998, granted by 2001

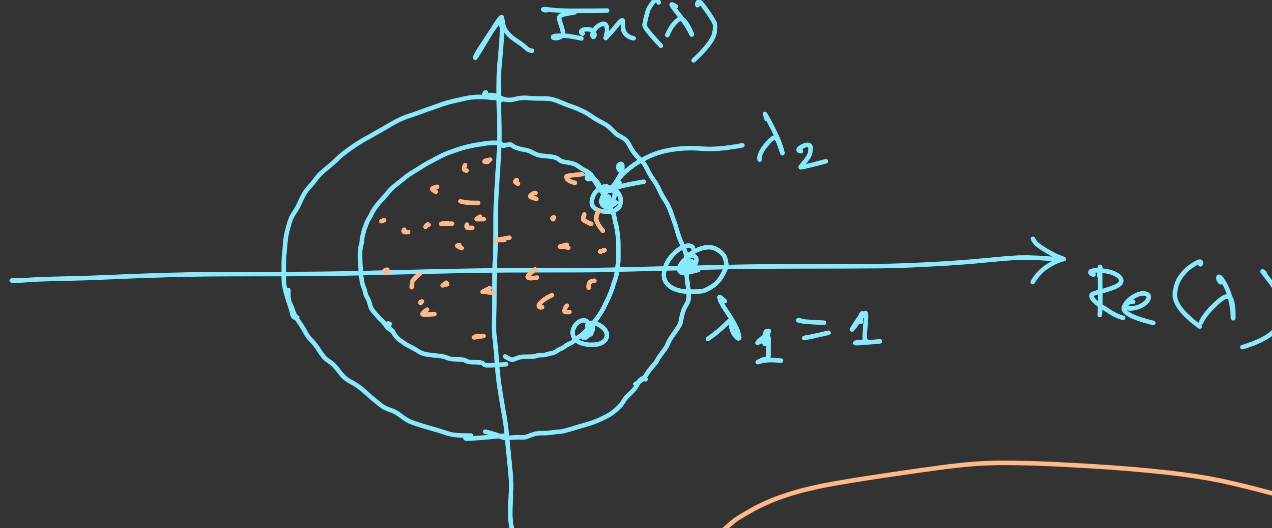
We know that power iteration converges <u>linearly</u> with worst-case rate $\left|\frac{\lambda_2}{\lambda_1}\right|$

In our case: $\lambda_1(G) = 1$

$\therefore$ Worst-case rate of the damped PageRank/power iteration is: $|\lambda_2(G)|$

<u>Theorem:</u> $|\lambda_2(G)| \leq \alpha$, where α is the damping factor in the definition of G

$\Longleftrightarrow$ The worst-case convergence rate is α.



$\therefore$ Smaller $\alpha \iff$ faster convergence but lesser accuracy

One variant of the basic PageRank Algorithm:

Original PageRank:

$$G = \alpha S + \frac{1-\alpha}{n} \underbrace{\mathbb{1}\mathbb{1}^T}_{J}$$

(all ones matrix)

$\alpha \approx 0$ then uniform random surfer model

Modified version:

(Directed surfer)

$$G = \alpha S + (1-\alpha)\, \underline{q}\, \mathbb{1}^T$$

where $\underline{q}$ is a personalization/preference vector

$q_1 + \dots + q_n = 1$

$\underline{q} \geq \underline{0}$ (elementwise)

Then solving

$$G\,\underline{p}(\alpha) = \underline{p}(\alpha)$$

$$\Rightarrow \left(\alpha S + (1-\alpha)\,\underline{q}\,\mathbb{1}^T\right)\underline{p}(\alpha) = \underline{p}(\alpha)$$

$$\Rightarrow \alpha S\,\underline{p}(\alpha) + \underbrace{(1-\alpha)\,\underline{q}\,\mathbb{1}^T\,\underline{p}(\alpha)}_{=\underline{q}} = \underline{p}(\alpha)$$

because $\boxed{\mathbb{1}^T\underline{p}(\alpha) = 1}$

$$\Rightarrow \alpha S\,\underline{p}(\alpha) + (1-\alpha)\,\underline{q} = \underline{p}(\alpha)$$

$$\Rightarrow \boxed{(I - \alpha S)\,\underline{p}(\alpha) = (1-\alpha)\,\underline{q}}$$

$\Rightarrow \; (I - \alpha S)\,\underline{p}(\alpha) = (1-\alpha)\,\underline{q}$

To be solved for unknown vector $\underline{p}(\alpha)$

Given S, α, $\underline{q}$

Square linear system: $A \backslash b$

It can be shown that the coefficient matrix $(I - \alpha S)$ is nonsingular for arbitrary $0 < \alpha < 1$ and any column stochastic S.

See Practice Prob. 11(a)(b) in CANVAS