Lecture #16

02/15/2023

Example: (linear spline)

$$S(x) = \begin{cases} S_1(x) = x & \text{for } -1 \le x \le 0.5 \\ S_2(x) = 0.5 + 2(x - 0.5) & \text{for } 0.5 \le x \le 2 \\ S_3(x) = x + 1.5 & \text{for } 2 \le x \le 4 \end{cases}$$

Question: Is $S(x)$ a linear spline?

Check:

$$S_1(0.5) = 0.5 = S_2(0.5) \checkmark$$

$$S_2(2) = 3.5 = S_3(2) \checkmark$$

So, we have verified continuity at the two breakpoints: $x = 0.5$ and $x = 2$.

Example: (Cubic spline)

x_i	y_i
1	2
2	1
4	4
5	3

$$\underbrace{i = 0,\ 1,\ 2,\ 3}_{\text{4 datapoints}} \Rightarrow \begin{cases} \text{three intervals} \\ \text{and} \\ \text{1 cubic polynomial} \\ \text{per interval} \end{cases}$$

Claim:

$$S(x) = \begin{cases} S_1(x) = 2 - \frac{13}{8}(x-1) + \frac{5}{8}(x-1)^3 & \text{for } 1 \le x \le 2 \\ S_2(x) = 1 + \frac{x-2}{4} + \frac{15}{8}(x-2)^2 - 5/8\,(x-2)^3 & \text{for } 2 \le x \le 4 \\ S_3(x) = 4 + \frac{x-4}{4} - \frac{15}{8}(x-4)^2 + 5/8\,(x-4)^3 & \text{for } 4 \le x \le 5 \end{cases}$$

is a cubic spline

How to check/verify?

$$S_1(2) = S_2(2) \qquad\qquad S_2(4) = S_3(4)$$

$$S_1'(2) = S_2'(2) \qquad \text{AND} \qquad S_2'(4) = S_3'(4)$$

$$S_1''(2) = S_2''(2) \qquad\qquad S_2''(4) = S_3''(4)$$

Linear spline in general:

$$\underbrace{x_0 < x_1 < x_2 < \dots < x_n}_{(n+1) \text{ datapoints}} \leftarrow \text{ordered } x\text{-coordinates}$$

(n+1) datapoints

with $(n+1) - 2 = (n-1)$ breakpoints

There are n line segments to be determined

$\Rightarrow$ $2n$ parameters/coefficients (a_i, b_i) to be computed

Each piece has equation: $a_i x + b_i = 0$

for all $i = 1, \dots, n$

next pg.

How many equations we have got to determine these $2n$ unknowns?

Each breakpoint: 2 equations : total $2(n-1)$ equations

" boundary point: 1 equation : $1 \times 2 = 2$ equations

$\therefore$ Total # of equations : $2(n-1) + 2 = 2n$

Each equation is linear in unknowns (a_i, b_i)

$\therefore$ We have got a square linear system of size $2n$

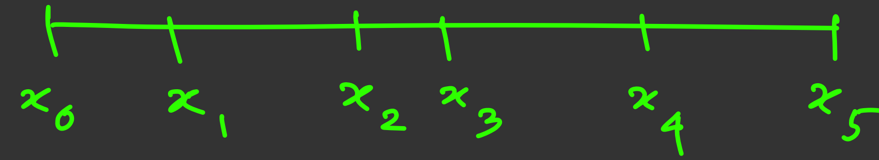
next pg.

$$\underbrace{A}_{\substack{2n \times 2n \\ \text{Known}}} \; \underset{\substack{2n \times 1 \\ \text{Unknown}}}{\underline{c}} = \underset{\substack{2n \times 1 \\ \text{known}}}{\underline{y}}$$

$$\underline{c} := \left(\begin{array}{c} a_1 \\ b_1 \\ \hline a_2 \\ b_2 \\ \hline a_3 \\ b_3 \\ \hline \vdots \\ \hline a_n \\ b_n \end{array} \right)$$

Example: (linear spline again) Suppose $n = 5$

We have total $2n = 2 \times 5 = 10$ unknowns



$n + 1 = 5 + 1 = 6$ datapoints

$y_0 = a_1 x_0 + b_1$

$y_1 = a_1 x_1 + b_1$

$y_1 = a_2 x_1 + b_2$

$y_2 = a_2 x_2 + b_2$

$\vdots$

$y_5 = a_5 x_5 + b_5$

Boundary point evaluations

Interior/breakpoint evaluations

In matrix - vector form :

$$
\underbrace{\underbrace{\begin{pmatrix}
x_0 & 1 & 0 & 0 & \cdots & 0 \\
x_1 & 1 & 0 & 0 & \cdots & 0 \\
0 & 0 & x_1 & 1 & 0 \cdots & 0 \\
0 & 0 & x_2 & 1 & 0 \cdots & 0 \\
\vdots & \vdots & & & & \vdots \\
0 & 0 & \cdots & 0 & x_5 & 1
\end{pmatrix}}_{A}}_{10 \times 10}
\underbrace{\underbrace{\begin{pmatrix}
a_1 \\ b_1 \\ a_2 \\ b_2 \\ \vdots \\ a_5 \\ b_5
\end{pmatrix}}_{\underline{c}}}_{10 \times 1}
=
\underbrace{\underbrace{\begin{pmatrix}
y_0 \\ y_1 \\ y_1 \\ y_2 \\ y_2 \\ y_3 \\ y_3 \\ y_4 \\ y_4 \\ y_5
\end{pmatrix}}_{\underline{y}}}_{10 \times 1}
$$

$$\Leftrightarrow \quad A\,\underline{c} = \underline{y}$$

solve by MATLAB A\y

Exercise : (Quadratic spline)

$x_0 \quad x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5$

Unknown coefficient vector to be computed

$$\underbrace{\begin{pmatrix} a_1 \\ b_1 \\ c_1 \\ a_2 \\ b_2 \\ c_2 \\ \vdots \\ a_5 \\ b_5 \\ c_5 \end{pmatrix}}_{15 \times 1}$$

$3 \times 5 = 15$ unknowns

leads to solving a square linear system with "natural boundary conditions"

See details in CANVAS File Section: folder "Supplementary Notes" $\longrightarrow$ degree d Splines.

Idea # 2 : Function approximation / regression :

Interpolation problems: Square linear system (solve)

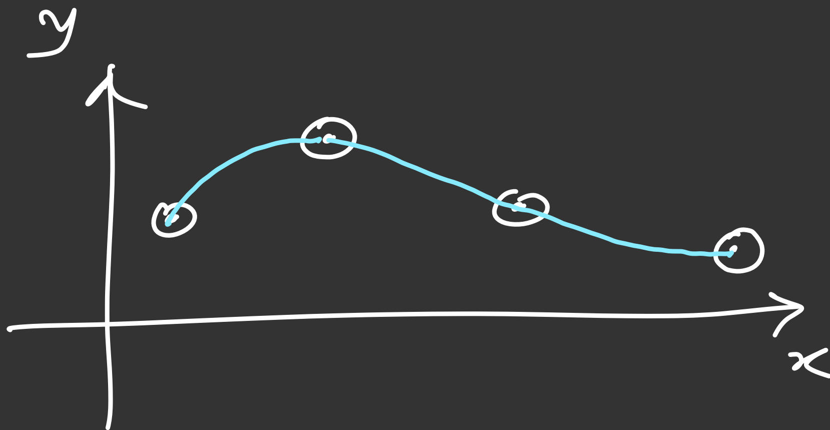
$$\underbrace{A}_{\substack{n\times n \\ (\text{known})}} \underbrace{\underline{x}}_{\substack{(\text{unknown}) \\ n\times 1}} = \underbrace{\underline{b}}_{\substack{n\times 1 \\ (\text{known})}}$$

Regression / Function approximation problems:

$$\underbrace{A}_{\substack{m\times n \\ (\text{known})}} \underbrace{\underline{x}}_{\substack{(\text{unknown}) \\ n\times 1}} = \underbrace{\underline{b}}_{\substack{m\times 1 \\ (\text{known})}}, \quad m > n$$

$\underbrace{\text{More}}_{(m)}$ equations, $\underbrace{\text{less}}_{(n)}$ unknowns $\Longleftrightarrow$ $\underbrace{A}_{m\times n}$ is a "tall" matrix.

Interpolation



Function approximation / regression