Details on deriving RK2 (explicit) method:

Assume that the solution $\underline{y}(t) \in C^2([0,T])$.

Expand $\underline{y}$ in Taylor series in the neighborhood of t_k, <u>correct upto 2^{nd} order</u>:

$$\underline{y}(t_{k+1}) = \underline{y}(t_k) + (\Delta t)\, \underline{y}'(t_k) + \frac{(\Delta t)^2}{2}\, \underline{y}''(t_k) + O\left((\Delta t)^3\right)$$

From the given ODE:

$$\underline{y}' := \frac{d\underline{y}}{dt} = \underline{f}(t, \underline{y})$$

$$\Rightarrow \underline{y}'' = \frac{d^2 \underline{y}}{dt^2} = \frac{d}{dt}\underline{f}(t, \underline{y}) = \frac{\partial \underline{f}}{\partial t} + \frac{\partial \underline{f}}{\partial \underline{y}} \cancelto{\underline{f}}{\frac{d\underline{y}}{dt}}$$

$$= \frac{\partial \underline{f}}{\partial t} + \frac{\partial \underline{f}}{\partial \underline{y}}\, \underline{f}$$

$\therefore$ Our Taylor expansion becomes:

$$\underline{y}_{k+1} = \underline{y}_k + (\Delta t)\,\underline{f}(t_k, \underline{y}_k) + \frac{(\Delta t)^2}{2}\left(\frac{\partial \underline{f}}{\partial t} + \frac{\partial \underline{f}}{\partial \underline{y}}\,\underline{f}\right)\Bigg|_{(t_k,\underline{y}_k)} + O\left((\Delta t)^3\right) \quad \text{------} \ (*)$$

Recall that we want:

$$\underline{y}_{k+1} = \underline{y}_k + \left(a\,\underline{k}_1 + b\,\underline{k}_2\right) \quad \text{....} \ (**)$$

where

$$\underline{k}_1 := (\Delta t)\,\underline{f}(t_k, \underline{y}_k)$$

$$\underline{k}_2 := (\Delta t)\,\underline{f}\left(t_k + \alpha\Delta t,\ \underline{y}_k + \beta\,\underline{k}_1\right)$$

Now expand the right-hand-side of $\underline{k}_2$ correct upto 2^{nd} order:

$$\underline{k}_2 := (\Delta t)\, \underline{f}\left(t_k + \alpha \Delta t,\ \underline{y}_k + \beta \underline{k}_1\right)$$

$$= (\Delta t)\left\{ \underline{f}(t_k, \underline{y}_k) + \alpha \Delta t \left.\frac{\partial \underline{f}}{\partial t}\right|_{(t_k, \underline{y}_k)} + \beta \left.\frac{\partial \underline{f}}{\partial \underline{y}}\right|_{(t_k, \underline{y}_k)} \underline{k}_1 + O\left((\Delta t)^2\right) \right\}$$

$$\underbrace{= (\Delta t)\left\{ \underline{f}(t_k, \underline{y}_k) + \alpha \Delta t \left.\frac{\partial \underline{f}}{\partial t}\right|_{(t_k, \underline{y}_k)} + \beta \left.\frac{\partial \underline{f}}{\partial \underline{y}}\right|_{(t_k, \underline{y}_k)} \underline{k}_1 \right\} + O\left((\Delta t)^3\right)}_{\text{substitute this expression in } (**)}$$

We obtain:

$$\underline{y}_{k+1} = \underline{y}_k + \left\{ a(\Delta t)\, \underline{f}(t_k, \underline{y}_k) + b(\Delta t) \left(\underline{f}(t_k, \underline{y}_k) + \alpha \Delta t \left.\frac{\partial \underline{f}}{\partial t}\right|_{(t_k, \underline{y}_k)} + \beta \left.\frac{\partial \underline{f}}{\partial \underline{y}}\right|_{(t_k, \underline{y}_k)} \underline{K}_1 \right) \right\}$$

$$+ O\left((\Delta t)^3\right) \quad \ldots\ldots (***)$$

Compare the right-hand-sides of (*) and (***):

Equating the coefficient of $\underline{f}(t_k, \underline{y}_k)$:

$$(a+b)(\Delta t) = \Delta t \Rightarrow \boxed{a+b = 1}$$

next pg.

Equating coefficient of $\left.\dfrac{\partial f}{\partial t}\right|_{(t_k, \underline{y}_k)}$:

$$\alpha b (\Delta t)^2 = \frac{(\Delta t)^2}{2}$$

$$\Rightarrow \boxed{\alpha b = 1/2}$$

Equating coefficient of $\left.\left(\dfrac{\partial f}{\partial \underline{y}} \underline{y}\right)\right|_{(t_k, \underline{y}_k)}$:

$$\beta b (\Delta t)^2 = \frac{(\Delta t)^2}{2}$$

$$\Rightarrow \boxed{\beta b = 1/2}$$

next pg.

$\therefore$ To determine the 4 parameters (a, b, α, β), we got only 3 equations relating them.

$\therefore$ Infinitely many solutions possible :

popular choice : $\underbrace{\alpha = \beta = 1, \quad a = b = 1/2}_{\text{substitute this back in } (**) \text{ to get :}}$

$$\left.\begin{aligned} \underline{y}_{k+1} &= \underline{y}_k + \frac{1}{2}\left(\underline{k}_1 + \underline{k}_2\right) \\ \text{where } \quad \underline{k}_1 &= (\Delta t)\, \underline{f}\left(t_k, \underline{y}_k\right) \\ \underline{k}_2 &= (\Delta t)\, \underline{f}\left(t_k + \Delta t, \underline{y}_k + \underline{k}_1\right) \end{aligned}\right\} \text{This is RK2}$$

$\blacksquare$