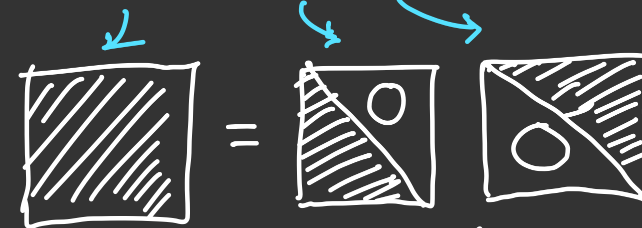
Lecture #11

02/03/2023

IDEA behind LU decomposition algorithm:

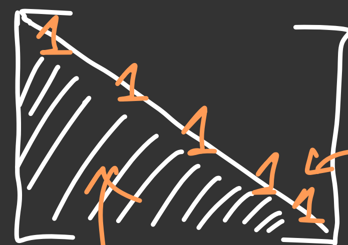
Given A

Decompose the coefficient matrix $A = L\,U$



The U matrix will be the exact same matrix that we obtained after the "elimination step" in Gauss elimination.

The L matrix will look like:

$L =$ 

ones along the main diagonal

This is a convention

These entries are the factors we used in subtracting the rows in Gauss elimination

Example: (LU decomposition of a 2×2 real matrix)

$$A = \begin{bmatrix} 1 & 1 \\ 3 & -4 \end{bmatrix} \xrightarrow{R_2 - 3R_1} \begin{bmatrix} 1 & 1 \\ 3 & -7 \end{bmatrix}$$

$$\therefore U = \begin{bmatrix} 1 & 1 \\ 0 & -7 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$$

$$\therefore A = LU$$

Check:

$LU \stackrel{?}{=} A$

$$LU = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -7 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 3 & -4 \end{bmatrix} = A.$$

checked!

verified

Example from last lecture (3x3 A in Gauss elimination)

$$\underbrace{\begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -2 \\ -3 & 1 & 1 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & -7/3 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & 2 & -1 \\ 0 & -3 & 0 \\ 0 & 0 & -2 \end{bmatrix}}_{U}$$

Advantages of LU decomposition:

- Computing determinant:

  Fact from linear algebra:

  det(triangular matrix) = product of the diag entries.

$$\det(A) = \det(LU)$$

$$= \underbrace{\det(L)}\ \underbrace{\det(U)}$$

$$= 1 \cdot \det(U)$$

because $\det(\cdot)$ of product equals product of det

For previous 3×3 example: $\det(A) = \det(U)$

$$= 1 \times (-2) \times (-3)$$

$$= +6$$

So, LU decomposition makes computing $\det(A)$ easy.

- Solving a square linear system $A\underline{x} = \underline{b}$

$$A = LU \iff A\underline{x} = \underline{b}$$

$$\Updownarrow$$

$$(LU)\underline{x} = \underline{b}$$

$$\Updownarrow$$

$$L(U\underline{x}) = \underline{b}$$

$$\Updownarrow$$

$$L\underline{y} = \underline{b} \quad \text{where } \underline{y} := U\underline{x}$$

Algorithm:

First do: $L\underline{y} = \underline{b}$ by forward substitution

$\downarrow$ get $\underline{y}$

Then do: $U\underline{x} = \underline{y}$ by backward substitution

$\downarrow$ get $\underline{x}$

Done!

Example: (Same 3×3 example as before)

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & -7/3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & -1 \\ 0 & -3 & 0 \\ 0 & 0 & -2 \end{bmatrix}}_{A} \underline{x} = \underbrace{\begin{pmatrix} 3 \\ 3 \\ -6 \end{pmatrix}}_{\underline{b}}$$

Step 1: Solve $\underset{\text{known}}{L}\ \underset{\text{unknown}}{\underline{y}} = \underset{\text{known}}{\underline{b}}$

$$\Leftrightarrow \left.\begin{aligned} y_1 &= 3 \\ 2y_1 + y_2 &= 3 \\ -3y_1 - \tfrac{7}{3}y_2 + y_3 &= -6 \end{aligned}\right\} \quad \underline{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ -4 \end{pmatrix}$$

Step 2: $U\underline{x} = \underline{y}$

$$\Rightarrow \begin{bmatrix} 1 & 2 & -1 \\ 0 & -3 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ -4 \end{pmatrix} \Rightarrow \begin{aligned} x_1 + 2x_2 - x_3 &= 3 \\ -3x_2 &= -3 \\ -2x_3 &= -4 \end{aligned}$$

$$\Rightarrow \quad \left.\begin{aligned} x_1 &= 3 \\ x_2 &= +1 \\ x_3 &= +2 \end{aligned}\right\} \quad \underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}.$$

<u>Complexity of LU decomposition:</u>

is still $O(n^3)$ just like Gauss elimination.

However, there is an advantage compared to Gauss elimination:

If we want to solve for k different $\underline{b}$ vectors for the same A:

$$\left.\begin{aligned} A\underline{x}_1 &= \underline{b}_1 \\ A\underline{x}_2 &= \underline{b}_2 \\ &\vdots \\ A\underline{x}_k &= \underline{b}_k \end{aligned}\right\}$$

Gauss elimination complexity:

$$k\left[\left\{\frac{2}{3}n^3 + \frac{n^2}{2} - \frac{7}{6}n\right\} + n^2\right]$$

$$= O\left(\frac{2}{3}kn^3\right)$$

But the complexity for LU becomes : $\frac{2}{3}n^3 + 2kn^2$

$\therefore$ LU decomposition allows significant numerical benefit compared to Gauss elimination when k is large .

- LU decomposition also makes computing A^{-1} easy :

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3 x 3 example :

$$\underbrace{\underbrace{A}_{3\times 3}}_{\text{known}} \underbrace{X}_{3\times 3} = \underbrace{\underbrace{I}_{3\times 3}}_{\text{known}}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$X = \left[\underbrace{[\;]}_{3\times 1} \underbrace{[\;]}_{3\times 1} \underbrace{[\;]}_{3\times 1} \right]$$

$$\begin{cases} A\underline{x}_1 = \underline{b}_1 \\ A\underline{x}_2 = \underline{b}_2 \\ A\underline{x}_3 = \underline{b}_3 \end{cases}$$ where $\underline{b}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\underline{b}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\underline{b}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$$X = \begin{bmatrix} \underline{x}_1 & \underline{x}_2 & \underline{x}_3 \end{bmatrix}.$$

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solve 3 square linear systems with the same A but 3 different $\underline{b}$'s

Existence of LU decomposition:

Fact: Any square real matrix A admits a factorization/decomposition of the form:

$$\underbrace{PA = LU}_{\text{PLU decomposition}}$$

P: Permutation matrix (square)

If $P \equiv I$, then we say simply LU decomposition

What's a permutation matrix?

It is a binary matrix such that:

- every row has exactly single entry 1
- " column " " " " 1
- rest of the entries are all zeros.

Example: 2×2 permutation matrices:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

No other permutation matrix of size 2×2 is possible, as per the prev. page's definition.