Lecture #12

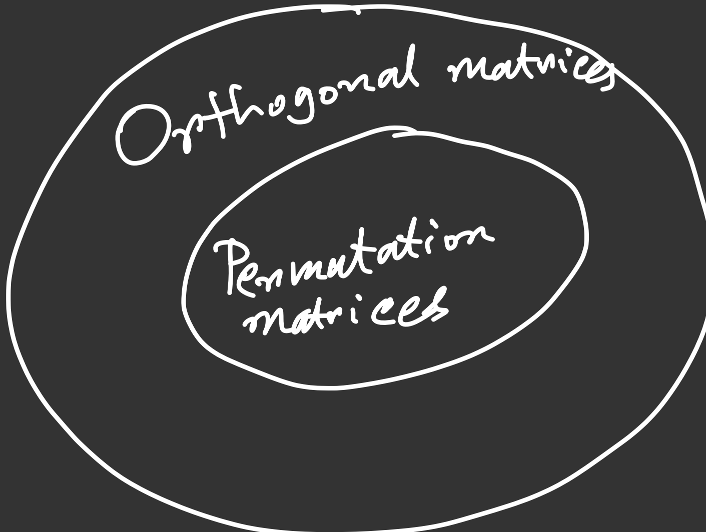
02/06/2023

Example of 3×3 permutation matrices:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

In general $n!$ permutation matrices of size $n \times n$.



Orthogonal matrices:

$n \times n$ real matrices Q such that:

$$Q^{-1} = Q^T$$

$$\Leftrightarrow \quad QQ^T = I_n = Q^TQ$$

$$\Leftrightarrow \quad \underbrace{\det(Q)}\,\underbrace{\det(Q^T)} = \underbrace{\det(I_n)}$$

$$\Rightarrow \quad \det(Q) \cdot \det(Q) = 1$$

$$\Rightarrow \quad (\det(Q))^2 = 1$$

$$\Rightarrow \quad \det(Q) = \pm 1.$$

Uniqueness of LU decomposition:

Consider $A = LU$ exists.

Such LU decomposition/factorization may NOT be unique unless we impose extra constraint on the diagonal entries.

Why?

$$
\begin{aligned}
A &= L\,U \\
&= L\,(I)\,U \\
&= L\,(D D^{-1})\,U, \\
&= (L D)\,(D^{-1} U) \\
&= L_1 \quad U_1
\end{aligned}
$$

where D is any diagonal matrix with nonzero diag. entries

This is why we artificially imposed the constraint earlier that $L_{ii} = 1$ for all $i = 1, \ldots, n$.

This constraint makes LU decomposition unique provided it exists.

In many practical science & engineering problems, the matrix $A \in \mathbb{R}^{n \times n}$ has additional structure: for example, often A happens to be "positive (semi) definite" matrix.

<u>Definition</u>: (Positive (semi) definite matrix)

Matrix $A \in \mathbb{R}^{n \times n}$ is called positive (semi) definite if

- A is symmetric ($A = A^T$)
- $\underline{x}^T A \underline{x} \geq 0$ for all $\underline{x} \neq \underline{0}$

semi-definite

> 0 for all $\underline{x} \neq \underline{0}$

definite

A is positive semi-definite $\iff \lambda_i(A) \geq 0$

for all $i = 1, \dots, n$

" " " definite $\iff \lambda_i > 0$

Suppose we want to solve : $A\underline{x} = \underline{b}$

where $\underline{b} \neq \underline{0}$, and we know that

the matrix A is positive definite.

Then, $A = LU$ exists, is unique, $U = L^T$.

$$\iff A = LU = \underbrace{LL^T}_{\text{Cholesky decomposition.}}$$

Example: (positive definite matrix and Cholesky decomposition)

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

$$\underline{x}^T A \underline{x} = (x_1 \quad x_2 \quad x_3) A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$= (x_1 \quad x_2 \quad x_3) \begin{pmatrix} 2x_1 - x_2 \\ -x_1 + 2x_2 - x_3 \\ -x_2 + 2x_3 \end{pmatrix}$$

$$= 2x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_2 - 2x_2x_3$$

$$= (x_1 - x_2)^2 + (x_2 - x_3)^2 + x_3^2 + x_1^2$$

$\therefore \quad \underline{x}^T A \underline{x} > 0 \quad$ for all $\underline{x} \neq \underline{0}$

$\therefore$ A is positive definite matrix.

$$(\lambda_1, \lambda_2, \lambda_3) = (0.5858, 2, 3.4142)$$

$$A = L L^T$$

$$L = \begin{bmatrix} 1.4142 & 0 & 0 \\ -0.7071 & 1.2247 & 0 \\ 0 & -0.8165 & 1.1547 \end{bmatrix}$$

MATLAB commands:

>> [L,U,P] = lu(·)

>> [L,U] = lu(·)

$\Big\}$ LU decomposition

>> chol(·) ← Cholesky decomposition

>> A\b ← will return the unique solution vector $\underline{x}$ satisfying

$$A\underline{x} = \underline{b}$$