Lecture # 5

01/20/2023

Summary of the fix we used for the quadratic equation to avoid the effect of round-off error:

If $4|ac| << b^2$ then $|b| \approx \sqrt{b^2 - 4ac}$ and we may loose numerical accuracy/precision

To fix this:

If $b > 0$, then use $x_{\pm} = \left\{ \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}, \; - \dfrac{2c}{b + \sqrt{b^2 - 4ac}} \right\}$

If $b < 0$, then use:

$$x_{\pm} = \left\{ \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad \frac{+2c}{-b + \sqrt{b^2 - 4ac}} \right\}.$$

Another example:

Compute $\exp(-5) = e^{-5}$

$$\exp(-x) = \sum_{k=0}^{\infty} \frac{(-x)^k}{k!}$$

Suppose we keep only 10 terms: then this way of evaluation gives:

$$\exp(-5) \approx \sum_{k=0}^{9} \frac{(-5)^k}{k!} = \frac{(-5)^0}{0!} + \frac{(-5)^1}{1!} + \frac{(-5)^2}{2!} + \dots$$

$$= 1 - 5 + \frac{25}{2} - \frac{125}{6} + \cdots$$

But another mathematically equivalent way to evaluate:

$$\exp(-5) = \frac{1}{\exp(5)}$$

$$\approx \frac{1}{\sum_{k=0}^{9} \frac{5^k}{k!}} = \frac{1}{\frac{5^0}{0!} + \frac{5^1}{1!} + \frac{5^2}{2!} + \cdots + \frac{5^9}{9!}}$$

But not equivalent in computer evaluation.

Try them out yourself in MATLAB

Calculus prelim. reminder:

Continuous functions: $f \in \underbrace{C}_{\text{continuous}}\big(\underbrace{[a,b]}_{\text{closed interval}}\big)$

$$\Updownarrow$$

"$f(x)$ is a continuous function for $a \le x \le b$".

$$a \le x \le b \iff x \in [a,b]$$

$$a < x < b \iff x \in (a,b)$$

$$a \le x < b \iff x \in [a,b)$$

Intermediate Value Theorem:

Suppose $f \in C([a, b])$

Take any y such that

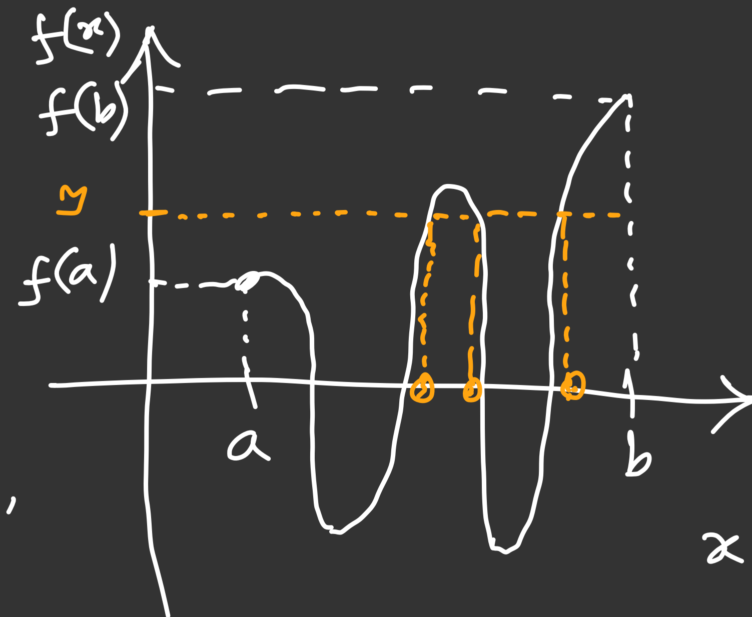
$$f(a) \leq y \leq f(b).$$

Then, there exists c

satisfying $a \leq c \leq b$,

such that

$$y = f(c)$$



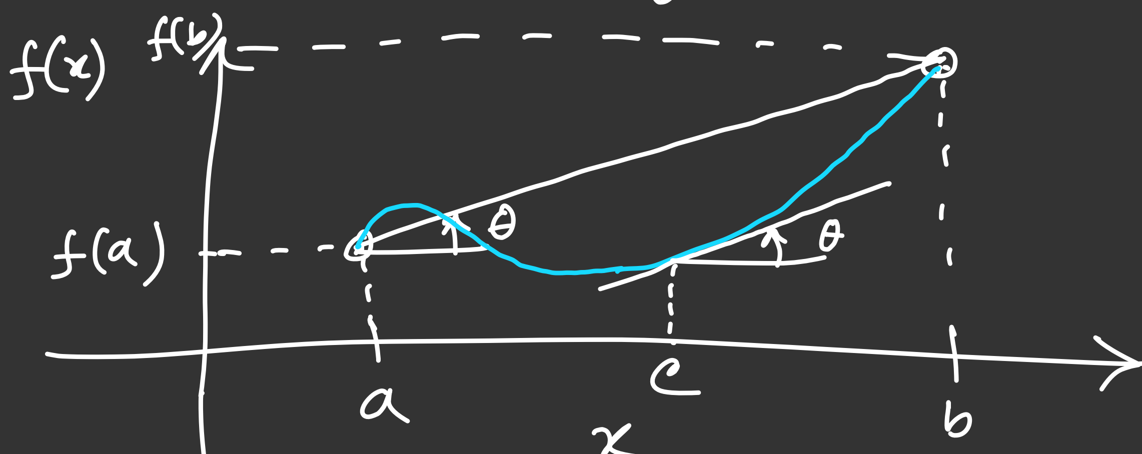
Example: Prove that $f(x) = x^2 - 3$, where $1 \leq x \leq 3$, must take values 0 AND 1.

This is because $f(1) = -2$, $f(3) = +6$.

Mean Value Theorem:

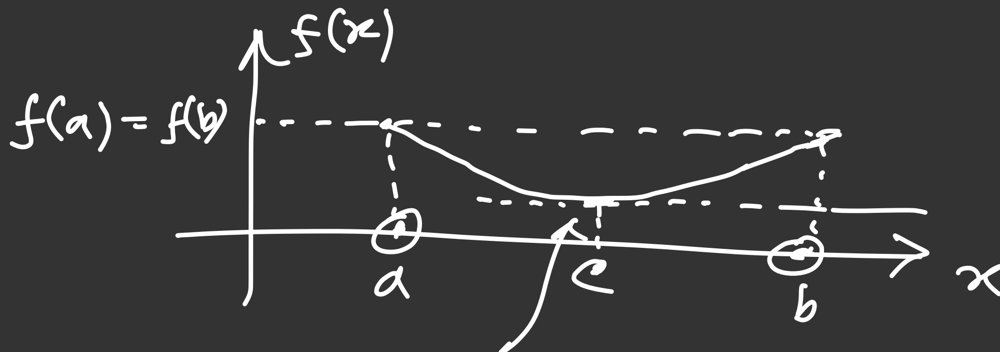
Suppose $f \in C([a,b])$ AND f is differentiable in (a,b). Then, there exists c satisfying $a \le c \le b$, such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$



Rolle's Theorem (Special case of Mean Value Theorem)

Specialize the Mean Value Theorem for $f(a) = f(b)$



$$f'(c) = 0$$

Taylor series: Let the function f be smooth

$\Updownarrow$

$$\underbrace{f \in C^{\infty}([x, x_0])}$$

$\Updownarrow$

continuously

"Function f is infinitely many times differentiable in $[x, x_0]$".

Then, the Taylor series approximation of f around the point x_0 is:

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \frac{f'''(x_0)}{3!}(x-x_0)^3 + \cdots + \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + \cdots$$

Our interest:

How to algorithmically solve scalar nonlinear equation: $\underbrace{f(x) = 0}_{\text{we want to find the real root(s)}}$

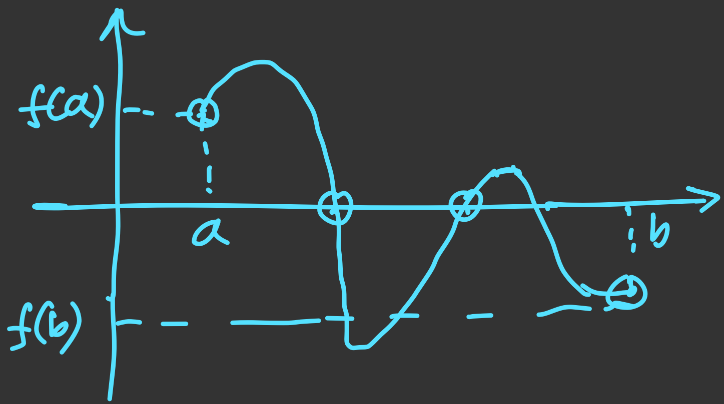
Examples:

- $\underbrace{\cos(x) - x}_{f(x)} = 0$, $x \approx 0.7391\ldots$
- $\overbrace{2x^5 - 5x^4 + 20x^3 - 10x^2 + 10x - 1}^{f(x)} = 0$
  $\underbrace{}_{\text{unique real root in } [0,1]}$

Algorithm # 1 for solving $f(x) = 0$:

Bisection method / algorithm :

Idea: Specialize intermediate value theorem :



Algorithm for bisection method:

Given $[a, b]$ such that $f(a)f(b) < 0$

Check if $\underbrace{f(a)f(b) \geq 0 \text{ (in MATLAB, >=)}}_{\text{Invalid input}}$

$\boxed{\text{while}}$ $\frac{b-a}{2} >$ tol~~erance~~ $\to \varepsilon$

$c = \frac{a+b}{2}$

if $f(c) == 0$

break;

end

if $f(a)\,f(c) < 0$

$b = c$

else

$a = c$

end

$\boxed{\text{end}}$ ending the while loop