Lecture #14

02/10/2023

Important result from linear algebra:

Any induced matrix norm is sub-multiplicative.

$$\|AB\|_p \leq \|A\|_p \, \|B\|_p$$

for all $1 \leq p \leq \infty$

For example, when $p=2$, then

$$\|AB\|_2 \leq \|A\|_2 \, \|B\|_2 \quad \text{etc.}$$

Let us specialize the above result for nonsingular A, and $B = A^{-1}$

Then,
$$\| \underbrace{A A^{-1}}_{\substack{I \\ \text{(identity matrix)}}} \|_p \le \underbrace{\|A\|_p \; \|A^{-1}\|_p}_{\substack{\kappa_p(A) \\ \text{(p-norm condition number)}}}$$

$$\underbrace{}_{\|I\|_p = 1}$$

$$\iff \quad 1 \le \kappa_p(A) \quad \text{for all } 1 \le p \le \infty.$$

$$\boxed{\therefore \text{ In general: } \quad 1 \le \kappa_p(A) \le \infty}$$

The $\underbrace{\text{lower bound}}_{(1)}$ is achieved e.g., by $A = I$ or other orthogonal matrices

The upper bound (∞) is achieved by singular matrices.

For any nonsingular/invertible matrix A:

$$1 \le \kappa_p(A) < \infty .$$

If $\kappa_*(A) \approx 1$, then we say the matrix A is "<u>well-conditioned</u>".

Then, small changes in $\underline{b}$ vector will cause small changes in the solution $\underline{x}$ for $A\underline{x} = \underline{b}$

If $\kappa_*(A) \gg 1$, then we say the matrix A is "<u>ill-conditioned</u>"

much larger than

Example:

$$A = \begin{pmatrix} 5 & 7 & 6 & 5 \\ 7 & 10 & 8 & 7 \\ 6 & 8 & 10 & 9 \\ 5 & 7 & 9 & 10 \end{pmatrix}.$$

$\det(A) = 1 \neq 0$ (so there is unique solution for $A\underline{x} = \underline{b}$ for any $\underline{b} \neq \underline{0}$)

If we want to solve for $\underline{x}$ in the square linear system: $A\underline{x} = \underline{b}$ where

$$\underline{b} = \begin{pmatrix} 23 \\ 32 \\ 33 \\ 31 \end{pmatrix}.$$

$\rightsquigarrow$ You can do: A\b in MATLAB

Check that $\underline{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$

What if we try to compute $\underline{x}$ with

$$\underline{b} = \begin{pmatrix} 22.9 \\ 32.1 \\ 32.9 \\ 31.1 \end{pmatrix}$$

Check that $\underline{x} = \begin{pmatrix} -7.2 \\ 6 \\ 2.9 \\ -0.1 \end{pmatrix}$.

Try another:

$$\underline{b} = \begin{pmatrix} 22.99 \\ 32.01 \\ 32.99 \\ 31.01 \end{pmatrix} \Rightarrow \underline{x} = \begin{pmatrix} 0.18 \\ 1.5 \\ 1.19 \\ 0.89 \end{pmatrix}$$

Check for example: $\kappa_\infty(A) = \underbrace{\|A\|_\infty}\ \underbrace{\|A^{-1}\|_\infty}$

$= 33 \times 136$

$= 4488 \gg 1.$

i.e., matrix A is ill-conditioned.

Related MATLAB commands:

>> norm(x, p) ← for computing p-norm of a vector $\underline{x}$, for $0 < p \leq \infty$

>> norm(X, p) ← for computing induced p-norm of any rectangular matrix X, for p = 1, 2, Inf, "fro"

$\gg$ cond$(A, p) \leftarrow \kappa_p(A) := \|A\|_p \|A^{-1}\|_p$

Interpolation problem

Fit a curve passing through ALL given datapoints

x	y
x_1	y_1
x_2	y_2
$\vdots$	
x_n	y_n

Find $y(x)$ satisfying $y_i = y(x_i)$ for all $i = 1, 2, \ldots, n$

A common choice is:

$$y = p(x)$$
$$= c_1 + c_2 x + c_3 x^2 + c_4 x^3 + \dots + c_n x^{n-1}$$

i.e., $y = p(x)$ is a univariate polynomial in x of degree $(n-1)$.

where n = # of datapoints/samples

$\therefore$ $y_i = p(x_i)$ for all $i = 1, \dots, n$

$\therefore$ We get n equations in $\underbrace{n \text{ unknowns}}_{\underbrace{c_1, c_2, \dots, c_n}_{\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}}}$

In fact, we get n linear equations in n unknowns: $c_1, c_2, \dots, c_n$

$$\Updownarrow$$

$$\underbrace{\begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & & & & \\ 1 & x_{n-1} & x_{n-1}^2 & \cdots & x_{n-1}^{n-1} \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{pmatrix}}_{n \times n} \underbrace{\begin{pmatrix} c_1 \\ \vdots \\ c_{n-1} \\ c_n \end{pmatrix}}_{n \times 1} = \underbrace{\begin{pmatrix} y_1 \\ \vdots \\ y_{n-1} \\ y_n \end{pmatrix}}_{n \times 1}$$

Known matrix — Unknown vector — Known vector

known

$$\Leftrightarrow \quad X \underline{c} = \underline{y}$$

known

} Square linear system in unknown $\underline{c}$

This matrix

$$X := \begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & & & & \\ 1 & x_{n-1} & x_{n-1}^2 & \cdots & x_{n-1}^{n-1} \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{pmatrix}$$

is called Vandermonde matrix.

named after the French mathematician Vandermonde

Theorem:

det of Vandermonde matrix

$$\det(\underset{\text{Vandermonde matrix}}{X}) = \prod_{1 \leq i < j \leq n} (x_j - x_i)$$

Consequence:

Unique solution vector $\underline{c}$

if and only if $\underbrace{\det(X) \neq 0}$

$$\Updownarrow$$

$x_i \neq x_j$ for all $j \neq i$