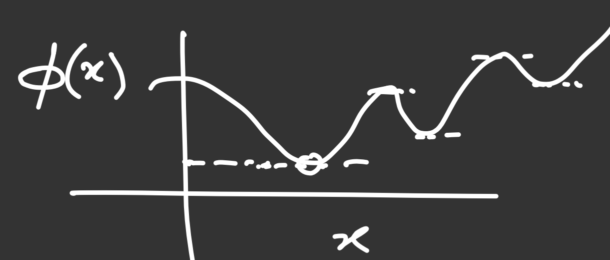
# Lecture # 9

## 01/30/2023

### Two remarks:

**Remark #1:** One application where solving nonlinear equation of the form $f(x) = 0$ arises is: $\underbrace{\text{optimization problem}}$




$$\underbrace{\min_{x} \phi(x)}$$

$$\frac{d\phi}{dx} \equiv \boxed{\begin{aligned} \underbrace{\phi'(x)} &= 0 \\ f(x) &= 0 \end{aligned}}$$

All critical points $x$ satisfy this e.g., local maxima/minima

<u>Remark #2</u>: Newton's method can be generalized to multiple dimensions:

<u>For example</u>: $\underbrace{\underline{f}(\underline{x}) = \underline{0}}$, $\underline{x} \in \mathbb{R}^2$

$$\boxed{\begin{aligned} f_1(x_1, x_2) &= 0 \\ f_2(x_1, x_2) &= 0 \end{aligned}} \qquad \begin{pmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{pmatrix} = 0_{2\times 1}$$

- For multivariate optimization problem: $\min\limits_{\underline{x} \in \mathbb{R}^n} \phi(\underline{x})$

$\underbrace{\text{critical points}}_{\text{local minima/maxima/saddles}}$ solve: $\underbrace{\nabla \phi(\underline{x})}_{\underline{f}(\underline{x})} = \begin{pmatrix} \frac{\partial \phi}{\partial x_1} \\ \vdots \\ \frac{\partial \phi}{\partial x_n} \end{pmatrix} = \underline{0}_{n\times 1}$.

Gradient of $\phi$

Newton's method for the multivariate case $\underline{f}(\underline{x}) = \underline{0}$ becomes:

$$\underbrace{\underline{x}_{k+1}}_{\substack{n \times 1 \\ \text{vector}}} = \underbrace{\underline{x}_k}_{\substack{n \times 1 \\ \text{vector}}} - \underbrace{\left[\nabla \underline{f}(\underline{x}_k)\right]^{-1}}_{\substack{n \times n \\ \text{matrix}}} \underbrace{\underline{f}(\underline{x}_k)}_{\substack{n \times 1 \\ \text{vector}}}$$

where $\underbrace{\nabla \underline{f}(\underline{x}_k)}_{}$ Jacobian matrix of $\underline{f}$ evaluated at $\underline{x}_k$

$$:= \left.\begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \cdots & \dfrac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \dfrac{\partial f_n}{\partial x_1} & \cdots & \dfrac{\partial f_n}{\partial x_n} \end{bmatrix}\right|_{\underline{x} = \underline{x}_k}$$

When $\underline{f} \equiv \nabla \phi$ then the recursion becomes:

$$\underline{x}_{k+1} = \underline{x}_k - \left[\underbrace{\nabla^2 \phi(\underline{x}_k)}_{\text{Hessian of } \phi}\right]^{-1} \nabla \phi(\underline{x}_k)$$

## Fixed point recursion/algorithm: (Back to our 1D problem)

Same math problem: solve for a real root of $f(x) = 0$

**IDEA:** Rewrite $f(x) = 0$ as $x = g(x)$

**Algorithm:**

- Make an initial guess $x_0$
- Then do: $x_{k+1} = g(x_k)$,

for $k = 0, 1, 2, \ldots$

<u>Theorem</u>: Suppose $g \in C^1$, and $S := |g'(r)| < 1$ where $r = g(r)$.

- Then the fixed point recursion converges to $x = r$ locally for $x_0$ close to $r$.

- Local rate of convergence is <u>linear</u>:

$$e_{k+1} \approx S e_k$$

for $k$ large enough

Multiple ways to write $x = g(x)$ for the same $f(x) = 0$ :

**Example :** $f(x) = x^3 + x - 1 = 0$

$$\left.\begin{array}{l} f(0) = -1 < 0 \\ f(1) = +1 > 0 \end{array}\right\} \Longrightarrow$$ there is a real root in $[0, 1]$

**One way to rewrite :** $x = \boxed{g_1(x) = 1 - x^3}$

**Another way :** $x = \boxed{g_2(x) = (1-x)^{1/3}}$

**Another way :** $x = \boxed{\dfrac{1}{x^2+1} = g_3(x)}$

Yet another way:

$$x = \boxed{g_4(x) = \frac{1+2x^3}{1+3x^2}}$$

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Example: $f(x) = \cos(x) - x = 0$

$$\Rightarrow \quad x = g(x) = \cos(x)$$

Simply do:

$$x_{k+1} = g(x_k) = \cos(x_k)$$

$$g'(x) = -\sin(x)$$

$$\Rightarrow |g'(x)| = |\sin(x)| < 1 \quad \forall \quad x \in [0, \pi/2)$$

$\therefore$ The recursion $x_{k+1} = \cos(x_k)$ will converge for any initial guess $x_0 \in [0, \pi/2)$.

Two results on fixed point recursion:

#1 (Existence) A continuous function $g : [a, b] \mapsto [a, b]$ must have at least one fixed point.

$\Longleftrightarrow$ There exists at least one solution $x \in [a, b]$ for the equation $x = g(x)$.

#2 : (uniqueness)

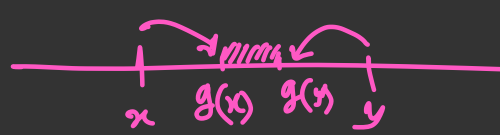
Contraction mapping theorem (1D) :

[ Sufficient condition for uniqueness ]

If there exists $\lambda \in [0, 1)$ such that

$$|g(x) - g(y)| \leq \lambda |x - y| \text{ for all } x, y \in [a, b]$$

then the equation $x = g(x)$ has unique fixed point in $[a, b]$.



**Example:** Kepler's equation in celestial mechanics

$$x = \underbrace{m + \varepsilon \sin(x)}_{g(x)}$$

Known parameters

$$0 < \varepsilon < 1$$

$$0 < m$$

Prove that Kepler's equation has a unique fixed point.

**Proof :**

$$|g(x) - g(y)|$$

$$= \varepsilon \,|\sin(x) - \sin(y)|$$

Because $\sin(\cdot)$ is continuous and differentiable in any interval $[x, y]$, hence by mean value theorem (earlier lectures), there exists $c \in [x, y]$ such that

$$\sin(x) - \sin(y) = \cos(c)\,(x-y)$$

$$\Rightarrow |\sin(x) - \sin(y)| = |\cos(c)|\,|x-y|$$

$$\leq |x-y|$$

since $|\cos(c)| \leq 1$

$$\therefore \ |g(x) - g(y)| = \varepsilon \, |\sin(x) - \sin(y)|$$
$$\leq \varepsilon \, |x - y|$$

But the problem tells that $0 < \varepsilon < 1$ (given)

$\therefore$ By contraction mapping theorem, we conclude that $g(x)$ has unique fixed point.

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