Lecture # 24
03/08/2023

Recap of basics on eig. values & eig. vectors

① Computing eig. values $\{\lambda_i\}_{i=1}^{n}$ of $A \in \mathbb{R}^{n \times n}$ reduces to solving for the roots of a monic polynomial equation in degree n:

$$\det(\lambda I - A) = 0 \Rightarrow \underbrace{\lambda^n + c_1 \lambda^{n-1} + c_2 \lambda^{n-2} + \ldots + c_{n-1}\lambda + c_n}_{\text{Characteristic polynomial}} = 0$$

$$\underbrace{\text{charpoly}(\lambda) = 0}$$

→ $\boxed{\text{has } n \text{ roots}}$

→ Some roots maybe real, some maybe complex conjugates, some maybe repeated

(2) Roots of charpoly(λ) have sensitive dependence on its coefficients $c_1, c_2, \ldots, c_n$, which in turn depend on a complicated manner on the entries of A:

For example, $A \in \mathbb{R}^{2\times 2}$:

$$\text{charpoly}(\lambda) \equiv \lambda^2 - \text{trace}(A)\lambda + \det(A) = 0$$

Here, $c_1 = -\text{trace}(A)$

$c_2 = \det(A)$

But for general $A \in \mathbb{R}^{n\times n}$:

$$\text{charpoly}(\lambda) \equiv \sum_{k=0}^{n} (-1)^k \underbrace{\text{trace}(\Lambda^k A)}_{\text{trace of the } k^{\text{th}} \text{ exterior power of } A} \lambda^{n-k}$$

③ Structural info. about A $\rightsquigarrow$ Structural info on eig. values

Example:

- If $A = A^T$ (symmetric) then all eig. values λ_i of A are real.

- If A is symmetric and positive (semi)definite, then $\lambda_i > 0 \ (\geq 0)$ for all $i = 1, \dots, n$

④ Similarity transformation: $A \mapsto MAM^{-1}$

where M is any nonsingular matrix, preserves the eig. values of A:

$$\lambda_i(A) = \lambda_i(MAM^{-1}) \text{ for all } i = 1, \dots, n$$

(5) Let $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{n\times m}$

Then: $\underbrace{AB}_{m\times m}$ and $\underbrace{BA}_{n\times n}$ have the same <u>nonzero</u> eig. values.

(6) The following sentences are equivalent:

A is (unitarily) diagonalizable

$\Updownarrow$

Matrix A is <u>normal</u>/ <u>non-defective</u>

$\Updownarrow$

$AA^T = A^TA$ (Matrix A commutes with its transpose)

$\Updownarrow$

$A = V \Lambda V^{-1}$ } Λ is the diagonal matrix with eig. values along main diagonal

and Matrix V is invertible
stacking of eig. vector columns

$$\Updownarrow$$

A has linearly independent eig. vectors $\{\underline{v}^{(j)}\}_{j=1}^{n}$

where $\underline{v}^{(1)}$, $\underline{v}^{(2)}$, $\ldots$, $\underline{v}^{(n)}$

eig. vector corresponding to λ_1

eig. vector corresponding to λ_2

eig. vector corresponding to λ_n

Last topic: Algorithms to solve special eig. value problems

Problem statement:

Compute the "dominant"/largest magnitude eig. value and its corresponding "dominant" eig. vector for a given $A \in \mathbb{R}^{n\times n}$ diagonalizable.

We order the eigen values as:

$$|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \ldots \geq |\lambda_n|$$

Strictly greater than ← needed for our algorithm for solving this problem

IDEA: perform vector fixed point recursion ::

- Make a random guess for initial vector $\underline{v}_0$
- for $k = 0, 1, 2, \ldots$

$$\underline{v}_{k+1} = \frac{A\underline{v}_k}{\|A\underline{v}_k\|_2} ;$$

end

This algorithm is called the Power Iteration Algorithm

We claim: $\{\underline{v}_k\} \xrightarrow{k \to \infty} \underline{v}$

dominant eig. vector of A

Once we get converged $\underline{v}$, then extract the corresponding dominant eig. value as :

$$\lambda = \frac{\underline{v}^T A \underline{v}}{\underline{v}^T \underline{v}}$$

because the definition of eig. value : $A\underline{v} = \lambda \underline{v}$

Power Iteration :

$\underline{v}_0$ = random initial guess vector

for $k = 1, 2, \ldots$

$\quad \underline{v}_k = \dfrac{A \underline{v}_{k-1}}{\| A \underline{v}_{k-1} \|_2}$;

end

$\lambda = (v(:,end))^T A \, \underline{v}(:,end)$;

Theorem: Given $A \in \mathbb{R}^{n \times n}$ diagonalizable with eig. values orderend as

$$|\lambda_1| > |\lambda_2| \geq \cdots \geq |\lambda_n| .$$

For "almost every" initial guess ($\iff$ with probability 1), the power iteration algorithm converges linearly to the eig. vector $\underline{v}^{(1)}$ corresponding to λ_1, with worst-case convergence rate

$$\left| \lambda_2 / \lambda_1 \right| .$$

Derivation / Proof:

Choose an initial vector $\underline{x}_0$ and decompose it into eig. vectors $\underline{v}^{(j)}$ of A:

$$\underline{x}_0 = \sum_{j=1}^{n} c_j \, \underline{v}^{(j)}$$

This is always possible for diagonalizable A

$\{c_j\}$ are constant coefficients

"almost every" $\Leftrightarrow c_1 \neq 0$

Then,
$$A\underline{x}_0 = \sum_{j=1}^{n} c_j \boxed{A\underline{v}^{(j)}}$$
$$= \sum_{j=1}^{n} c_j \boxed{\lambda_j \underline{v}^{(j)}}$$

$$\Rightarrow \quad A^2 \underline{x}_0 = A(A\underline{x}_0)$$
$$= \sum_{j=1}^{n} c_j \lambda_j \left(A\underline{v}^{(j)}\right)$$
$$= \sum_{j=1}^{n} c_j \lambda_j^2 \underline{v}^{(j)}$$
$$\vdots \qquad\qquad \vdots$$
$$A^k \underline{x}_0 = \sum_{j=1}^{n} c_j \lambda_j^k \underline{v}^{(j)} \quad \text{for any } k = 0, 1, 2, \ldots$$

$$\therefore \quad \underline{x}_{k+1} = \frac{A\,\underline{x}_k}{\|A\,\underline{x}_k\|_2}$$

$$= \frac{A^k\,\underline{x}_0}{\|A^k\,\underline{x}_0\|_2}$$

$$= \frac{\sum_{j=1}^{n} c_j \lambda_j^k \underline{v}^{(j)}}{\left\{ \sum_{j=1}^{n} (c_j \lambda_j)^2 \right\}^{1/2}}$$

because

$$\left(\underline{v}^{(j)}\right)^T \underline{v}^{(j)} = 1 \Rightarrow \|\underline{v}^{(j)}\|_2 = 1 \text{ and } \left(\underline{v}^{(j)}\right)^T \underline{v}^{(i)} = 0$$

for all $i \neq j$

$$\Rightarrow \underline{x}_{k+1} = \frac{\cancel{c_1} \cancel{\lambda_1^k} \left\{ \underline{v}^{(1)} + \sum_{j=2}^{n} \frac{c_j}{c_1} \left(\frac{\lambda_j}{\lambda_1}\right)^k \underline{v}^{(j)} \right\}}{\cancel{c_1} \cancel{\lambda_1^k} \left[1 + \sum_{j=2}^{n} \left(\frac{c_j}{c_1}\right)^2 \left(\frac{\lambda_j}{\lambda_1}\right)^{2k} \right]^{1/2}}$$

($\because c_1 \neq 0$ with probability 1) ($\because \lambda_1 \neq 0$)

$$\Rightarrow \lim_{k\to\infty} \underline{x}_{k+1} = \underline{v}^{(1)} \quad \text{since } |\lambda_1| > |\lambda_j| \text{ for all } j = 2, \dots, n$$

with convergence rate $\leq |\lambda_2 / \lambda_1|$

" " " $= |\lambda_2/\lambda_1|$ if $c_2 \neq 0$

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