# Lecture #22

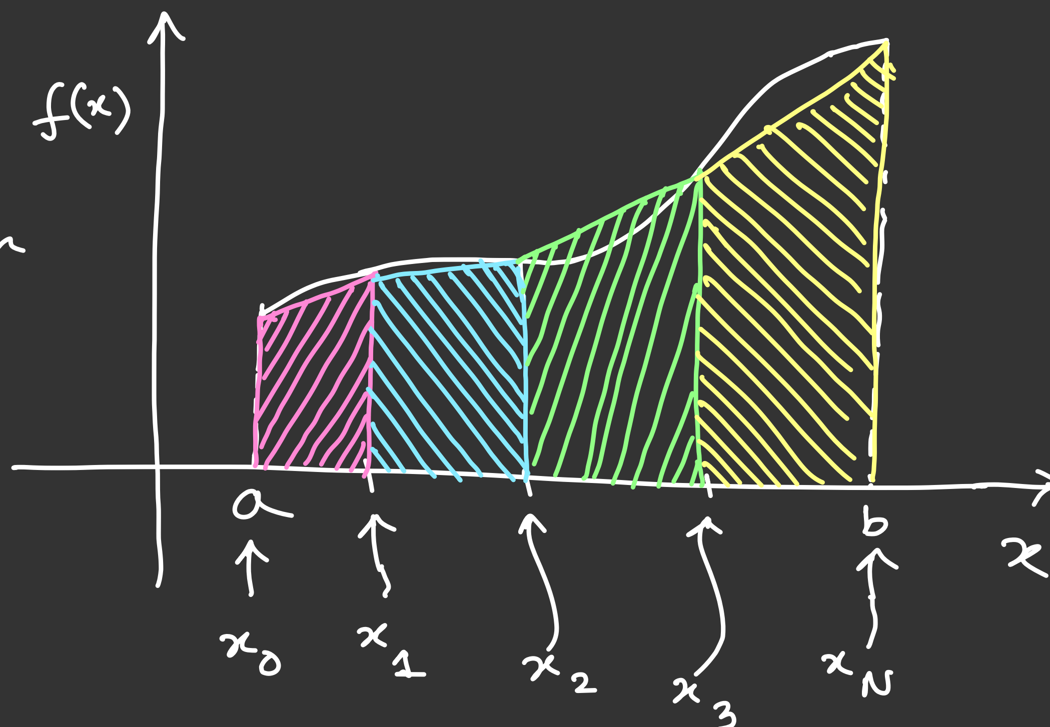
03/03/2023

## Trapezoid method

Area of trapezoid

$= \frac{1}{2} \times$ distance between parallel sides

$\times$ sum of the lengths of parallel sides




Here $N = 4$

For non-uniform spacings: $\underbrace{\Delta x_1}_{\substack{\text{length} \\ \text{of the first} \\ \text{subinterval}}}, \Delta x_2, \ldots, \Delta x_N$

length of the $N^{th}$ sub-interval

Then:

$$\int_a^b f(x)\,dx \approx \sum_{K=1}^{N} \underbrace{\frac{1}{2} \times (\Delta x_K) \times \left(f(x_{K-1}) + f(x_K)\right)}_{\text{area of the } K^{th} \text{ trapezoid}}$$

Special case:

Uniformly spaced points: $\{x_0 = a, x_1, \ldots, x_{N-1}, x_N = b\}$

where $\Delta x_1 = \Delta x_2 = \ldots = \Delta x_N = \dfrac{b-a}{N}$

Then :

$$\int_a^b f(x)\,dx \approx \underbrace{\frac{b-a}{N}\left\{\frac{f(x_0)+f(x_N)}{2} + \sum_{k=1}^{N-1} f(x_{k-1})\right\}}_{\text{special case of the more general formula in the prev. slide}}$$

In-built MATLAB command for trapezoid method:

>> trapz (·) ← please look up the documentation in MATLAB

>> trapz(x, y) ← 1 D integration

For multi-dimensional integration using trapz(.)

Suppose you want to do:

$$\int_{y=-5}^{y=+5} \int_{x=-3}^{x=+3} f(x,y)\,dx\,dy, \quad \text{where}$$

$$f(x,y) = x^2 \sin(y) + y\cos(x)$$

```
>> x = -3:0.1:3;   ← vector
>> y = -5:0.1:5;   ← vector
>> [X,Y] = meshgrid(x,y)
   matrices
>> F = (X.^2).*sin(Y) + Y.*cos(X)
```

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>> I = trapz(y, trapz(x, F, 2))
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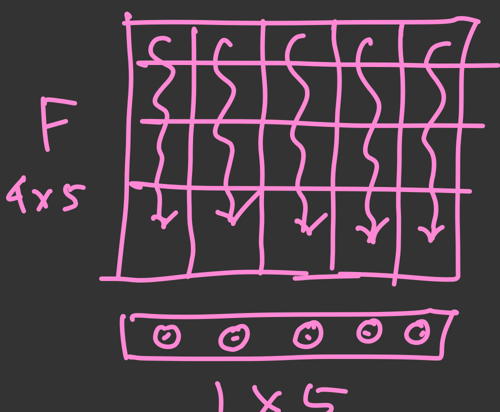
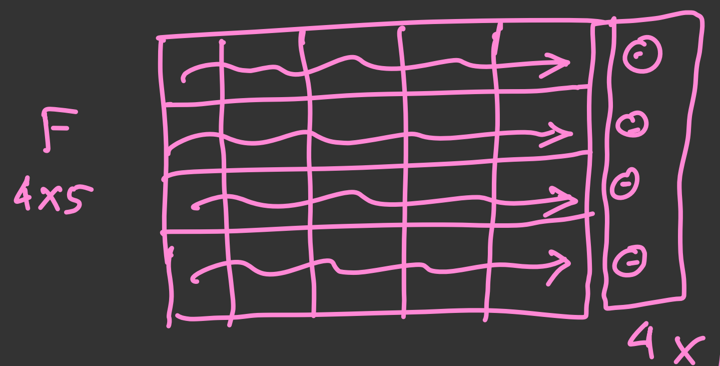
I: desired integral (scalar)

2: along row dimension

`trapz(F, 1)` ← along column dimension

`trapz(F, 2)` ← along row dimension



Simpson's three point method to approximate

$$I := \int_a^b f(x)\,dx$$

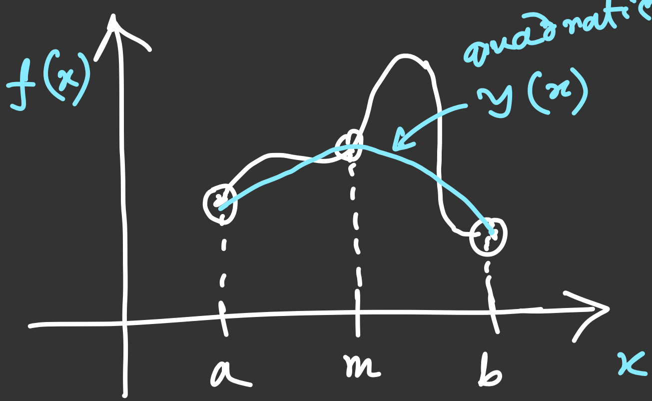
<u>IDEA:</u> Fit a quadratic $\boxed{y = c_0 + c_1 x + c_2 x^2}$

through 3 points: $(a, f(a))$

$(b, f(b))$

$(m, f(m))$

where $m := \dfrac{a+b}{2}$



We need to determine the coefficients $c_0, c_1, c_2$ by solving the square linear system:

$$\begin{pmatrix} 1 & a & a^2 \\ 1 & m & m^2 \\ 1 & b & b^2 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} f(a) \\ f(m) \\ f(b) \end{pmatrix}$$

solve for $c_0, c_1, c_2$

can be done by hand by elimination

Then:

$$I = \int_a^b f(x)\,dx \approx \int_a^b y(x)\,dx$$

$$= c_0(b-a) + \frac{c_1}{2}(b^2-a^2) + \frac{c_2}{3}(b^3-a^3)$$

$$= \frac{b-a}{6}\left[6c_0 + 3c_1(b+a) + 2c_2(b^2+ab+a^2)\right]$$

$$= \frac{b-a}{6}\left[ f(a) + 4f(m) + f(b) \right]$$

plugging in the solved $c_0, c_1, c_2$

Let $\Delta x := \dfrac{b-a}{2}$

Then: $I \approx \dfrac{\Delta x}{3}\left[ f(a) + 4f(m) + f(b) \right]$

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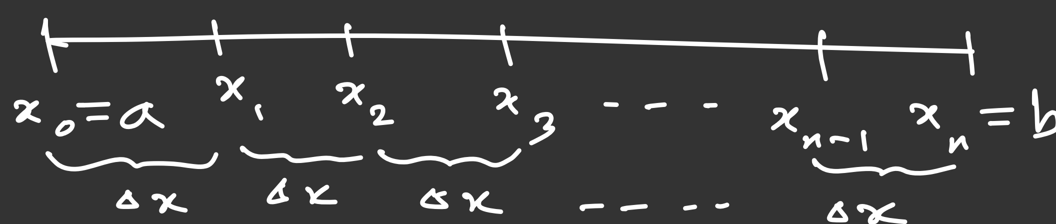
Extend this for $n$ subintervals with uniform spacings within the domain $[a, b]$ where $n$ is an even integer:




where $\Delta x = \dfrac{b-a}{n}$

Then:

$$I = \int_a^b f(x) \approx \frac{\Delta x}{3} \sum_{i=1}^{n/2} \left[ f(x_{2i-2}) + 4 f(x_{2i-1}) + f(x_{2i}) \right]$$

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Errors in approximating $\int_a^b f(x)\,dx$ for uniform spacing with $\Delta x = \frac{b-a}{n}$

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Midpoint rule: $O((\Delta x)^3)$

Trapezoid rule: $O((\Delta x)^3)$

Simpson's rule: $O((\Delta x)^4)$

Example: Compute $I = \int_0^1 \frac{\log(1+x)}{1+x^2}\,dx$

- True value: $I_{true} = \frac{\pi}{8}\log 2$

$\approx \boxed{0{\cdot}27219826 \text{ (upto 8 decimal places)}}$

- Midpoint method with x-axis partition: $[0, 1/2], [1/2, 1]$

$$I_{midpoint} = \frac{1}{2} \times \frac{\log(1+1/4)}{1+(1/4)^2} + \frac{1}{2} \times \frac{\log(1+3/4)}{1+(3/4)^2}$$

$$= \frac{8}{17}\log(5/4) + \frac{8}{25}\log(7/4) \approx \boxed{0{\cdot}28408578}$$

• Trapezoid method with the same partition: $[0, 1/2]$, $[1/2, 1]$

$$I_{trapz} = \left\{ \frac{1}{2} \times \frac{1}{2} \times \left( \cancelto{0}{\frac{\log(1)}{1}} + \frac{\log(1+1/2)}{1+(1/2)^2} \right) \right\}$$

$$+ \left\{ \frac{1}{2} \times \frac{1}{2} \times \left( \frac{\log(1+1/2)}{1+(1/2)^2} + \frac{\log(2)}{2} \right) \right\}$$

$$= \frac{1}{4} \left[ \frac{8}{5} \log(3/2) + \frac{1}{2} \log(2) \right]$$

$$\approx \boxed{0.24882944}$$

- Simpson's method with the same partition:

$$I_{\text{Simpson}} = \frac{1}{6}\left[ \cancelto{0}{\frac{\log(1)}{1}} + 4\cdot\frac{\log\left(1+\frac{1}{2}\right)}{1+\frac{1}{4}} + \frac{1}{2}\log(2) \right]$$

$$= \frac{1}{6}\left[ \frac{16}{5}\log\left(\frac{3}{2}\right) + \frac{1}{2}\log(2) \right]$$

$$\approx \boxed{0.27401032}$$

End of example.

Solving $\underbrace{\text{Ordinary Differential Equations}}_{\text{(ODEs)}}$ :

means determining curves / trajectories / signals :

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ODE initial value problems (IVPs) :

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These look like (for scalar ODEs) :

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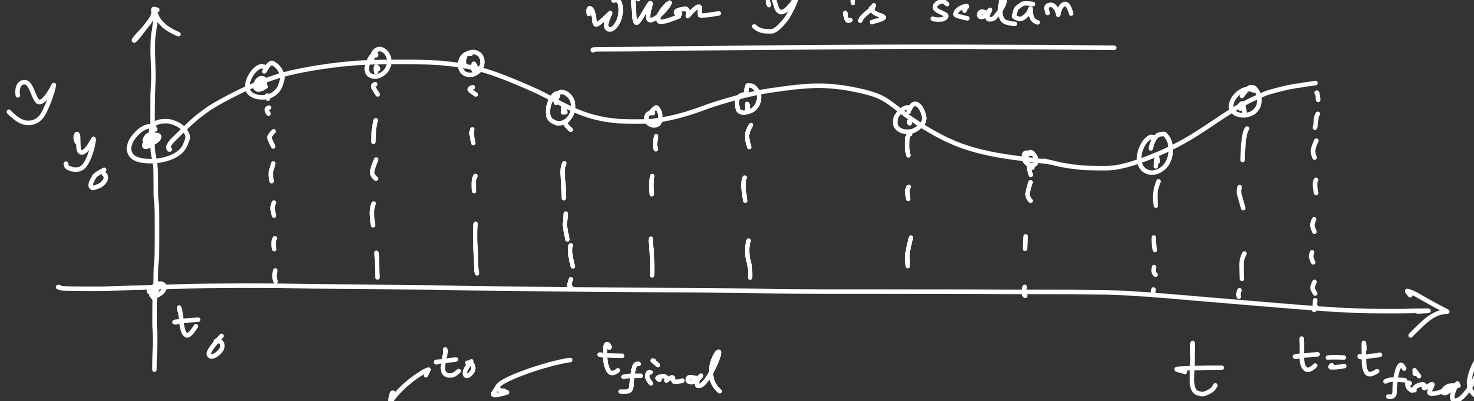
$$\underbrace{\frac{dy}{dt} = f(t, y(t))}_{\text{ODE}}, \quad \underbrace{y(t_0) = y_0}_{\text{initial value}}$$

known / given

Given $f, t_0, y_0$, solve for $y(t)$ where $t \in [t_0, t_{final}]$

given

when $y$ is scalar




$$t_{final}$$

If $t \in [\underset{t_0}{0}, T]$

then one option is to uniformly discretize $t$:

$$n \Delta t = T \iff \Delta t = \frac{T}{n}$$