Lecture # 13

02/08/2023

Norms as a measure of magnitude :

Vector norms :

In general, vector p-norm :

$$\|\underline{x}\|_p := \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}, \quad \text{where } \underline{x} \in \mathbb{R}^n \text{ and } 0 < p < \infty$$

Special cases :

$\boxed{p=1}$ $\quad \|\underline{x}\|_1 = |x_1| + \dots + |x_n|$

$\boxed{p=2}$ $\quad \|\underline{x}\|_2 = \left(x_1^2 + \dots + x_n^2\right)^{1/2}$

$$\|\underline{x}\|_0 := \underbrace{\text{nnz}}_{\text{number of nonzero entries (cardinality)}}(\underline{x})$$

nnz(·) is also MATLAB command

$$\|\underline{x}\|_\infty := \max_{i=1,\dots,n} |x_i|$$

Matrix norms: (for any matrix $\underbrace{A \in \mathbb{R}^{m\times n}}_{\text{rectangular matrices}}$)

Matrix 1 norm:

$$\|A\|_1 := \max_{j=1,\dots,n} \sum_{i=1}^{n} |a_{ij}|$$

$$\|A\|_\infty := \max_{i=1,\dots,m} \sum_{j=1}^{n} |a_{ij}|$$ ← Matrix ∞ norm

$$\|A\|_2 := \sqrt{\underbrace{\lambda_{max}}_{\text{maximum eigen value}}(AA^T)}$$ ← Matrix 2 norm (also known as the spectral norm)

Frobenius norm (also known as the Hilbert-Schmidt norm)

$$\|A\|_{Frobenius} := \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}^2} = \sqrt{\operatorname{trace}(AA^T)}$$

$$\operatorname{trace}(M) := \sum_{i=1}^{n} M_{ii} \quad \text{for } M \in \mathbb{R}^{n\times n}$$

Very important quantity in numerical algorithms:

Condition number of a matrix M:

$$\kappa_*(M) := \|M\|_* \|M^{-1}\|_*$$ for a square matrix M

where $*$ can be any matrix norm.

For example, when $* \equiv 2$ then:

$$\underbrace{\kappa_2(M)}_{\text{2-norm condition number of the square matrix } M} = \|M\|_2 \|M^{-1}\|_2$$

Another example: when $* = \infty$ then

$$\kappa_\infty(M) = \|M\|_\infty \|M^{-1}\|_\infty$$

Suppose we want to solve a square linear system:

$$A \underline{x} = \underline{b}, \quad \underline{b} \neq \underline{0}, \quad A \text{ is nonsingular/invertible/} \det(A) \neq 0$$

$$\Rightarrow \underline{x} = A^{-1} \underline{b}$$

Does small change in $\underline{b}$ produce small change in $\underline{x}$?

$$\overset{\text{unchanged}}{A} \underbrace{(\underline{x} + \Delta \underline{x})}_{\text{perturbed solution}} = \underline{b} + \underbrace{\Delta \underline{b}}_{\text{perturbation in problem data } \underline{b}}$$

$$\Rightarrow \cancel{A \underline{x}} + A \Delta \underline{x} = \cancel{\underline{b}} + \Delta \underline{b}$$

$$\Rightarrow \quad \boxed{A \Delta \underline{x} = \Delta \underline{b}}$$

$$\Rightarrow \quad \Delta \underline{x} = A^{-1} (\Delta \underline{b})$$

$$\Rightarrow \quad \underbrace{\| \Delta \underline{x} \|_2}_{} = \| A^{-1} \Delta \underline{b} \|_2$$

$$\leq \| A^{-1} \|_2 \, \| \Delta \underline{b} \|_2 \quad \cdots\cdots (*)$$

absolute error

On the other hand, original solution: $\underline{b} = A \underline{x}$

$$\Rightarrow \| \underline{b} \|_2 = \| A \underline{x} \|_2$$

$$\leq \| A \|_2 \| \underline{x} \|_2$$

$$\Rightarrow \quad \frac{1}{\| \underline{x} \|_2} \leq \frac{\| A \|_2}{\| \underline{b} \|_2} \quad \cdots\cdots (**)$$

Combining (*) and (**):

$$\underbrace{\frac{\|\Delta \underline{x}\|_2}{\|\underline{x}\|_2}}_{\substack{\text{relative error} \\ \text{in solution}}} \leq \underbrace{\|A\|_2 \|A^{-1}\|_2}_{\kappa_2(A)} \underbrace{\frac{\|\Delta \underline{b}\|_2}{\|\underline{b}\|_2}}_{\substack{\text{relative} \\ \text{perturbation in} \\ \text{problem data}}}$$

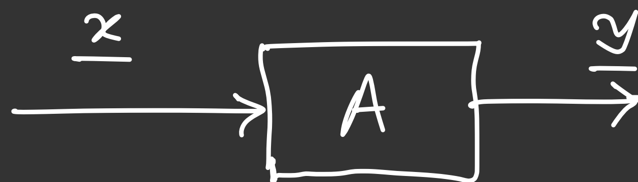
2-norm condition number

$\kappa_2(A)$ is small $\Rightarrow \dfrac{\|\Delta \underline{x}\|_2}{\|\underline{x}\|_2}$ is small when $\dfrac{\|\Delta \underline{b}\|_2}{\|\underline{b}\|_2}$ is small

If $\kappa_2(A)$ is large $\Longrightarrow$ $\dfrac{\|\Delta \underline{x}\|_2}{\|\underline{x}\|_2}$ could be large

even if $\dfrac{\|\Delta \underline{b}\|_2}{\|\underline{b}\|_2}$ is small.

Induced Matrix Norm



$$\underline{y} = A\,\underline{x}$$

Fix $1 \leq p \leq \infty$. Define induced matrix norm of A: next pg.

$$\|A\|_p := \max_{\underline{x} \neq \underline{0}} \frac{\|A\underline{x}\|_p}{\|\underline{x}\|_p}$$

So earlier examples $\|A\|_1$, $\|A\|_2$, $\|A\|_\infty$ are special cases of the above formula.

Example:

$$A = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

what does this matrix do?

Compute:

$$\|A\|_2 = \max_{\underline{x} \neq \underline{0}} \frac{\|A\underline{x}\|_2}{\|\underline{x}\|_2} = \max_{\underline{x} \neq \underline{0}} \frac{\cancel{\|\underline{x}\|_2}}{\cancel{\|\underline{x}\|_2}} = 1.$$