Lecture # 8

01/27/2023

How to find out the multiplicity $\tilde{m}$ for the repeated roots ?

Suppose $f \in C^{m+1}$. If $f(x) = 0$ has root $x = r$ with multiplicity $\tilde{m} = m+1$, then

$$f(x) = (x-r)^{m+1} g(x), \qquad g(r) \neq 0$$

$$\boxed{\therefore \; f(r) = 0, \; f'(r) = 0, \; f''(r) = 0, \; \ldots, \; f^{(m)}(r) = 0.}$$

$$f^{(m+1)}(r) \neq 0$$

Example:

Solve:

$$f(x) = \sin(x) + x^2\cos(x) - x^2 - x = 0$$

Notice: $x = 0$ is a root since $f(0) = 0$.

$$f'(0) = \underbrace{\cos(0)}_{1} + \underbrace{2 \cdot 0 \cdot \cos(0)}_{0} + \underbrace{(0)^2(-\sin(0))}_{0} - \underbrace{2 \cdot 0}_{0} - 1$$

$$= 1 - 1 = 0$$

$$f''(0) = 0$$

$$f'''(0) = -1 \neq 0$$

$\therefore$ Multiplicity of the root $x = 0$ is $\tilde{m} = 3$

$$\therefore \lambda = \frac{\tilde{m} - 1}{\tilde{m}} = \frac{3-1}{3} = \frac{2}{3}$$

Having known $\tilde{m}$ for our specific $f(x)$, we can modify the Newton's method to recover quadratic convergence !!

Theorem: $f \in C^{m+1}([a,b])$

$f(x) = 0$ has a root $x = r$ has multiplicity $\tilde{m} > 1$

Then do:
$$\boxed{x_{k+1} = x_k - \tilde{m} \frac{f(x_k)}{f'(x_k)}}$$

This will locally converge to $x = r$ quadratically

Example: Newton's method may locally settle into an oscillation

Solve for real root of:

$$f(x) := 4x^4 - 6x^2 - \frac{11}{4} = 0.$$

$$f(-\infty) = +\infty > 0$$

$$f(0) = 0 - 0 - \frac{11}{4} < 0$$

$$f(+\infty) = +\infty > 0$$

Make an initial guess x_0

Newton's method/recursion for this problem becomes:

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

$$= x_k - \frac{4x_k^4 - 6x_k^2 - 1/4}{16x_k^3 - 12x_k}$$

$$= \frac{12x_k^4 - 6x_k^2 + 1/4}{16x_k^3 - 12x_k}$$