Background concept:

Taylor's theorem with remainder:

Consider real numbers x, x_0 such that $x \neq x_0$.

Let $\underbrace{f \in C^{k+1}([x, x_0])}_{\text{"}f\text{ is }(k+1)\text{ times continuously differentiable in }[x, x_0]\text{"}}$ ← This assumes $x < x_0$. You can use $[x_0, x]$ if $x > x_0$.

Then, there exists c between x and x_0, such that

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3 + \dots + \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + \frac{f^{(k+1)}(c)}{(k+1)!}(x - x_0)^{k+1}.$$

We next apply Taylor's theorem with remainder to derive the rate-of-convergence for Newton's method.

$\boxed{\text{Suppose } f \text{ is twice differentiable and } f' \neq 0.}$

Let x_k be the k^{th} iterate from Newton's method.

Let x_{true} be the true root, i.e., $f(x_{true}) = 0$.

By Taylor's theorem with remainder, we have:

$$\underbrace{f(x_{true})}_{0} = f(x_k) + (x_{true} - x_k) f'(x_k) + \frac{(x_{true} - x_k)^2}{2!} f''(c_k)$$

for some c_k between x_k and x_{true}.

$$\Rightarrow -\frac{f(x_k)}{f'(x_k)} = (x_{true} - x_k) + \frac{(x_{true} - x_k)^2}{2} \frac{f''(c_k)}{f'(x_k)}$$

$$\Rightarrow \underbrace{x_k - \frac{f(x_k)}{f'(x_k)}}_{x_{k+1}} - x_{true} = \frac{(x_{true} - x_k)^2}{2} \frac{f''(c_k)}{f'(x_k)}$$

$$\Rightarrow \underbrace{|x_{k+1} - x_{true}|} = \underbrace{(x_{true} - x_k)^2} \cdot \frac{1}{2} \left| \frac{f''(c_k)}{f'(x_k)} \right|$$

$$\Rightarrow \underbrace{e_{k+1}}_{\text{absolute error @ } (k+1)^{th} \text{ step}} = e_k^2 \cdot \frac{1}{2} \left| \frac{f''(c_k)}{f'(x_k)} \right|$$

$$\Rightarrow \frac{e_{k+1}}{e_k^2} = \frac{1}{2} \left| \frac{f''(c_k)}{f'(x_k)} \right|$$

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Since c_k is between x_{true} and x_k, if Newton's method locally converges, then we must have:

$$\lim_{k\to\infty} c_k = x_{true}, \qquad \lim_{k\to\infty} x_k = x_{true}$$

∴ Taking limit $k\to\infty$ to both sides of the formula @ the end of the prev. page, we get:

$$\lim_{k\to\infty} \frac{e_{k+1}}{e_k^2} = \underbrace{\frac{1}{2}\left|\frac{f''(x_{true})}{f'(x_{true})}\right|}_{\lambda\ \text{(asymptotic error constant)}}$$

$\alpha = 2$
(order of convergence)

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