Sharper lower bound for $\kappa_p(A)$:

Claim: For any nonsingular $A = [\underline{a}_1 \ \underline{a}_2 \cdots \underline{a}_n] \in \mathbb{R}^{n\times n}$ and $1 \leq p \leq \infty$, we have :

$$\kappa_p(A) \geq \frac{\max_{i=1,\ldots,n} \|\underline{a}_i\|_p}{\min_{i=1,\ldots,n} \|\underline{a}_i\|_p}$$

Proof: Recall the def$^{\underline{n}}$. of induced p norm :

$$\|A\|_p := \max_{\underline{x} \neq \underline{0}} \frac{\|A\underline{x}\|_p}{\|\underline{x}\|_p} = \max_{\|\underline{x}\|_p = 1} \|A\underline{x}\|_p$$

next pg.

$\therefore \quad \|A\|_p \geq \|A\underline{x}\|_p$ for all $\underline{x} \in \mathbb{R}^n$ with $\|\underline{x}\|_p = 1$

In particular, for all standard basis vectors $\underline{e}_1, \ldots, \underline{e}_n$

we get:

$$\|A\|_p \geq \|A\underline{e}_1\|_p = \|\underline{a}_1\|_p$$

$$\vdots \qquad\qquad \vdots$$

$$\|A\|_p \geq \|A\underline{e}_n\|_p = \|\underline{a}_n\|_p$$

$$\Rightarrow \boxed{\|A\|_p \geq \max_{i=1,\ldots,n} \|\underline{a}_i\|_p} \quad \text{-----} \ (*)$$

next pg.

On the other hand:

$$\|A^{-1}\|_p = \max_{\underline{x} \neq \underline{0}} \frac{\|A^{-1}\underline{x}\|_p}{\|\underline{x}\|_p}$$

$$= \max_{\underline{y} \neq \underline{0}} \frac{\|\underline{y}\|_p}{\|A\underline{y}\|_p} \qquad \text{let } \underline{y} := A\underline{x}$$

$$= \max_{\|\underline{y}\|_p = 1} \frac{1}{\|A\underline{y}\|_p}$$

$$= \frac{1}{\min\limits_{\|\underline{y}\|_p = 1} \|A\underline{y}\|_p}$$

next pg.

But for all standard basis vectors $\underline{e}_1, \ldots, \underline{e}_n$, we have:

$$\min_{\|\underline{y}\|_p = 1} \|A\underline{y}\|_p \leq \|A\underline{e}_1\|_p = \|\underline{a}_1\|_p$$

$$\vdots \qquad\qquad \vdots$$

$$\min_{\|\underline{y}\|_p = 1} \|A\underline{y}\|_p \leq \|A\underline{e}_n\|_p = \|\underline{a}_n\|_p$$

$$\Rightarrow \min_{\|\underline{y}\|_p = 1} \|A\underline{y}\|_p \leq \min_{i=1,\ldots,n} \|\underline{a}_i\|_p$$

$$\Rightarrow \boxed{\|A^{-1}\|_p = \frac{1}{\min_{\|\underline{y}\|_p = 1} \|A\underline{y}\|_p} \geq \frac{1}{\min_{i=1,\ldots,n} \|\underline{a}_i\|_p}} \quad \ldots (**)$$

Combining (*) and (**), we conclude:

$$\kappa_p(A) := \|A\|_p \|A^{-1}\|_p \geq \frac{\max_{i=1,\dots,n} \|\underline{a}_i\|_p}{\min_{i=1,\dots,n} \|\underline{a}_i\|_p} . \blacksquare$$