FEA Homework 2

February 14, 2022

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[1]:
```
import sympy as sp
import matplotlib.pyplot as plt
from IPython.display import display, Latex

plt.style.use('maroon.mplstyle')

display_latex = lambda text: display(Latex(text))

def round_expr(expr, num_digits):
    return expr.xreplace({n : round(n, num_digits) for n in expr.atoms(sp.
↪Number)})
```

Contents

ME 6233 Homework 2 Gabe Morris
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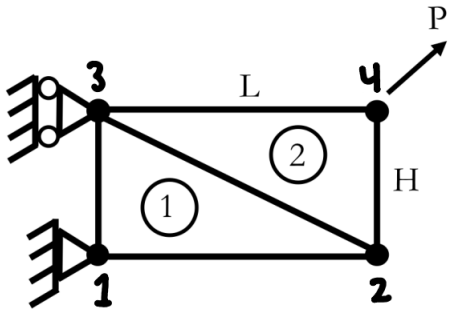
1 Problem 1

1.1 Given



$P = 150\,lb$, $L = 5\,in$, $H = 2\,in$, $t = 0.5\,in$, $E = 30 \cdot 10^6\,psi$, and $\nu = 0.30$

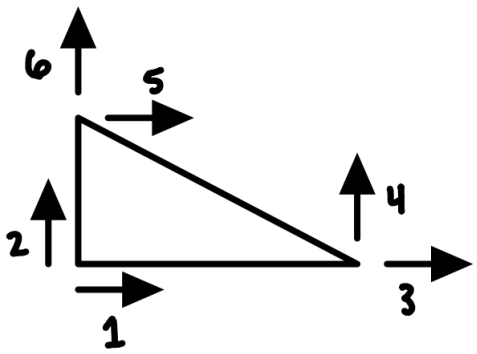
Notice that the global nodes have been rearranged. This was done to make the mapping easier.

1.2 Find

a. The global stiffness matrix
b. The displacements at each node
c. The stresses within each element
d. Plot the undeformed and deformed shape

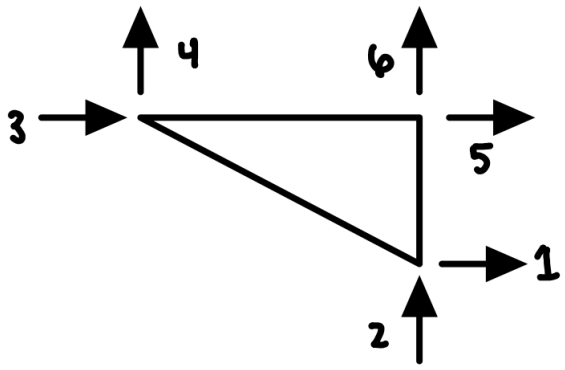
1.3 Solution

For the first element,



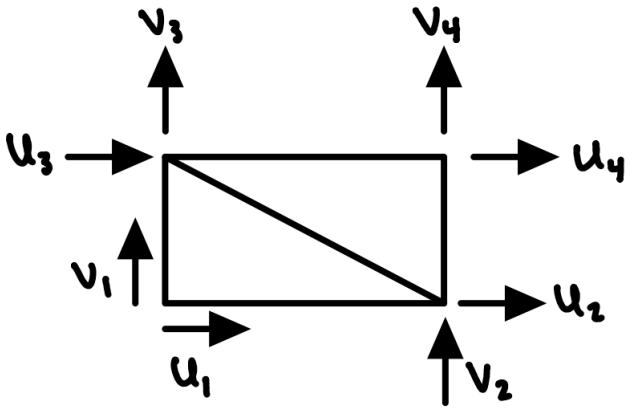
$\delta_1 = \delta_2 = \delta_5 = 0$
$\delta_3 = u_2$
$\delta_4 = v_2$
$\delta_6 = v_3$

For the second element,



$\delta_3 = 0$
$\delta_1 = u_2$
$\delta_2 = v_2$
$\delta_4 = v_3$
$\delta_5 = u_4$
$\delta_6 = v_4$

The global displacements are,



1.3.1 Part A

[2]:
```
# Define numerical inputs here
P_ = 150
L_, H_ = 5, 2
t_ = 0.5
E_, nu_ = 30e6, 0.3

# Define local coordinates for each element
x1 = [0, L_, 0]
```

```
y1 = [0, 0, H_]

x2 = [L_, 0, L_]
y2 = [0, H_, H_]

# The area
A_1 = ((x1[1] - x1[0])*(y1[2] - y1[0]) - (x1[2] - x1[0])*(y1[1] - y1[0]))/2
A_2 = -1*((x2[1] - x2[0])*(y2[2] - y2[0]) - (x2[2] - x2[0])*(y2[1] - y2[0]))/2
A_1, A_2
```

[2]:
```
(5.0, 5.0)
```

[3]:
```
# Define a symbolic B
A, y_23, y_31, y_12, x_32, x_13, x_21 = sp.symbols('A y_{23} y_{31} y_{12}␣
↪x_{32} x_{13} x_{21}')
B = 1/(2*A)*sp.Matrix([
    [y_23, 0, y_31, 0, y_12, 0],
    [0, x_32, 0, x_13, 0, x_21],
    [x_32, y_23, x_13, y_31, x_21, y_12]
])
B
```

[3]:
$$\begin{bmatrix} \frac{y_{23}}{2A} & 0 & \frac{y_{31}}{2A} & 0 & \frac{y_{12}}{2A} & 0 \\ 0 & \frac{x_{32}}{2A} & 0 & \frac{x_{13}}{2A} & 0 & \frac{x_{21}}{2A} \\ \frac{x_{32}}{2A} & \frac{y_{23}}{2A} & \frac{x_{13}}{2A} & \frac{y_{31}}{2A} & \frac{x_{21}}{2A} & \frac{y_{12}}{2A} \end{bmatrix}$$

[4]:
```
# Numeric B
# Remember that indices start at 0
B1 = B.subs([
    (A, A_1),
    (y_23, y1[1] - y1[2]),
    (y_31, y1[2] - y1[0]),
    (y_12, y1[0] - y1[1]),
    (x_32, x1[2] - x1[1]),
    (x_13, x1[0] - x1[2]),
    (x_21, x1[1] - x1[0])
])
B1
```

[4]:
$$\begin{bmatrix} -0.2 & 0 & 0.2 & 0 & 0 & 0 \\ 0 & -0.5 & 0 & 0 & 0 & 0.5 \\ -0.5 & -0.2 & 0 & 0.2 & 0.5 & 0 \end{bmatrix}$$

[5]:
```
B2 = B.subs([
    (A, A_2),
    (y_23, y2[1] - y2[2]),
    (y_31, y2[2] - y2[0]),
    (y_12, y2[0] - y2[1]),
```

```
    (x_32, x2[2] - x2[1]),
    (x_13, x2[0] - x2[2]),
    (x_21, x2[1] - x2[0])
])
B2
```

[5]:
$$\begin{bmatrix} 0 & 0 & 0.2 & 0 & -0.2 & 0 \\ 0 & 0.5 & 0 & 0 & 0 & -0.5 \\ 0.5 & 0 & 0 & 0.2 & -0.5 & -0.2 \end{bmatrix}$$

[6]:
```
E = E_/(1 - nu_**2)*sp.Matrix([
    [1, nu_, 0],
    [nu_, 1, 0],
    [0, 0, (1 - nu_)/2]
])
round_expr(E, 3)
```

[6]:
$$\begin{bmatrix} 32967032.967 & 9890109.89 & 0 \\ 9890109.89 & 32967032.967 & 0 \\ 0 & 0 & 11538461.538 \end{bmatrix}$$

[7]:
```
k1_local = t_*A_1*sp.transpose(B1)*E*B1
round_expr(k1_local, 3)
```

[7]:
$$\begin{bmatrix} 10508241.758 & 5357142.857 & -3296703.297 & -2884615.385 & -7211538.462 & -2472527.473 \\ 5357142.857 & 21758241.758 & -2472527.473 & -1153846.154 & -2884615.385 & -20604395.604 \\ -3296703.297 & -2472527.473 & 3296703.297 & 0 & 0 & 2472527.473 \\ -2884615.385 & -1153846.154 & 0 & 1153846.154 & 2884615.385 & 0 \\ -7211538.462 & -2884615.385 & 0 & 2884615.385 & 7211538.462 & 0 \\ -2472527.473 & -20604395.604 & 2472527.473 & 0 & 0 & 20604395.604 \end{bmatrix}$$

[8]:
```
k2_local = t_*A_2*sp.transpose(B2)*E*B2
round_expr(k2_local, 3)
```

[8]:
$$\begin{bmatrix} 7211538.462 & 0 & 0 & 2884615.385 & -7211538.462 & -2884615.385 \\ 0 & 20604395.604 & 2472527.473 & 0 & -2472527.473 & -20604395.604 \\ 0 & 2472527.473 & 3296703.297 & 0 & -3296703.297 & -2472527.473 \\ 2884615.385 & 0 & 0 & 1153846.154 & -2884615.385 & -1153846.154 \\ -7211538.462 & -2472527.473 & -3296703.297 & -2884615.385 & 10508241.758 & 5357142.857 \\ -2884615.385 & -20604395.604 & -2472527.473 & -1153846.154 & 5357142.857 & 21758241.758 \end{bmatrix}$$

[9]:
```
k1_global = k1_local.col_insert(6, sp.zeros(rows=6, cols=2)).row_insert(6, sp.
↪zeros(rows=2, cols=8))
round_expr(k1_global, 3)
```

[9]:

$$\begin{bmatrix} 10508241.758 & 5357142.857 & -3296703.297 & -2884615.385 & -7211538.462 & -2472527.473 & 0 & 0 \\ 5357142.857 & 21758241.758 & -2472527.473 & -1153846.154 & -2884615.385 & -20604395.604 & 0 & 0 \\ -3296703.297 & -2472527.473 & 3296703.297 & 0 & 0 & 2472527.473 & 0 & 0 \\ -2884615.385 & -1153846.154 & 0 & 1153846.154 & 2884615.385 & 0 & 0 & 0 \\ -7211538.462 & -2884615.385 & 0 & 2884615.385 & 7211538.462 & 0 & 0 & 0 \\ -2472527.473 & -20604395.604 & 2472527.473 & 0 & 0 & 20604395.604 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

[10]:
```
k2_global = k2_local.col_insert(0, sp.zeros(rows=6, cols=2)).row_insert(0, sp.
↪zeros(rows=2, cols=8))
round_expr(k2_global, 3)
```

[10]:
$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 7211538.462 & 0 & 0 & 2884615.385 & -7211538.462 & -2884615.385 \\ 0 & 0 & 0 & 20604395.604 & 2472527.473 & 0 & -2472527.473 & -20604395.604 \\ 0 & 0 & 0 & 2472527.473 & 3296703.297 & 0 & -3296703.297 & -2472527.473 \\ 0 & 0 & 2884615.385 & 0 & 0 & 1153846.154 & -2884615.385 & -1153846.154 \\ 0 & 0 & -7211538.462 & -2472527.473 & -3296703.297 & -2884615.385 & 10508241.758 & 5357142.857 \\ 0 & 0 & -2884615.385 & -20604395.604 & -2472527.473 & -1153846.154 & 5357142.857 & 21758241.758 \end{bmatrix}$$

[11]:
```
k_global = k1_global + k2_global
round_expr(k_global, 1)
```

[11]:
$$\begin{bmatrix} 10508241.8 & 5357142.9 & -3296703.3 & -2884615.4 & -7211538.5 & -2472527.5 & 0 & 0 \\ 5357142.9 & 21758241.8 & -2472527.5 & -1153846.2 & -2884615.4 & -20604395.6 & 0 & 0 \\ -3296703.3 & -2472527.5 & 10508241.8 & 0 & 0 & 5357142.9 & -7211538.5 & -2884615.4 \\ -2884615.4 & -1153846.2 & 0 & 21758241.8 & 5357142.9 & 0 & -2472527.5 & -20604395.6 \\ -7211538.5 & -2884615.4 & 0 & 5357142.9 & 10508241.8 & 0 & -3296703.3 & -2472527.5 \\ -2472527.5 & -20604395.6 & 5357142.9 & 0 & 0 & 21758241.8 & -2884615.4 & -1153846.2 \\ 0 & 0 & -7211538.5 & -2472527.5 & -3296703.3 & -2884615.4 & 10508241.8 & 5357142.9 \\ 0 & 0 & -2884615.4 & -20604395.6 & -2472527.5 & -1153846.2 & 5357142.9 & 21758241.8 \end{bmatrix}$$

1.3.2 Part B

[12]:
```
F_1, F_2, F_5 = sp.symbols('F_1 F_2 F_5')
F = sp.Matrix([
    [F_1],
    [F_2],
    [0],
    [0],
    [F_5],
    [0],
    [P_*sp.sqrt(2)/2],
    [F_*sp.sqrt(2)/2]
]).n()
F
```

[12]:

$$\begin{bmatrix} F_1 \\ F_2 \\ 0 \\ 0 \\ F_5 \\ 0 \\ 106.066017177982 \\ 106.066017177982 \end{bmatrix}$$

[13]:
```
u2, v2, v3, u4, v4 = sp.symbols('u_2 v_2 v_3 u_4 v_4')
d = sp.Matrix([
    [0],
    [0],
    [u2],
    [v2],
    [0],
    [v3],
    [u4],
    [v4]
])
d
```

[13]:
$$\begin{bmatrix} 0 \\ 0 \\ u_2 \\ v_2 \\ 0 \\ v_3 \\ u_4 \\ v_4 \end{bmatrix}$$

[14]:
```
system = sp.Eq(F, k_global*d)
round_expr(system, 3)
```

[14]:
$$\begin{bmatrix} F_1 \\ F_2 \\ 0 \\ 0 \\ F_5 \\ 0 \\ 106.066 \\ 106.066 \end{bmatrix} = \begin{bmatrix} -3296703.297u_2 - 2884615.385v_2 - 2472527.473v_3 \\ -2472527.473u_2 - 1153846.154v_2 - 20604395.604v_3 \\ 10508241.758u_2 - 7211538.462u_4 + 5357142.857v_3 - 2884615.385v_4 \\ -2472527.473u_4 + 21758241.758v_2 - 20604395.604v_4 \\ -3296703.297u_4 + 5357142.857v_2 - 2472527.473v_4 \\ 5357142.857u_2 - 2884615.385u_4 + 21758241.758v_3 - 1153846.154v_4 \\ -7211538.462u_2 + 10508241.758u_4 - 2472527.473v_2 - 2884615.385v_3 + 5357142.857v_4 \\ -2884615.385u_2 + 5357142.857u_4 - 20604395.604v_2 - 1153846.154v_3 + 21758241.758v_4 \end{bmatrix}$$

[15]:
```
solved = sp.solve(system)
for key, value in solved.items():
    display_latex(f'${sp.latex(key)}={sp.latex(value)}$')
```

$F_1 = -265.165042944938$

$F_2 = -106.066017177975$

$F_5 = 159.099025766958$

$u_2 = 2.42131523003677 \cdot 10^{-5}$

$u_4 = 6.89954607183987 \cdot 10^{-6}$

$v_2 = 6.54725387051039 \cdot 10^{-5}$

$v_3 = -1.4243030764922 \cdot 10^{-6}$

$v_4 = 6.83110553439691 \cdot 10^{-5}$

1.3.3 Part C

[16]:
```
strain1 = B1*sp.Matrix([
    [0],
    [0],
    [solved[u2]],
    [solved[v2]],
    [0],
    [solved[v3]]
])
strain1
```

[16]:
$$\begin{bmatrix} 4.84263046007354 \cdot 10^{-6} \\ -7.12151538246102 \cdot 10^{-7} \\ 1.30945077410208 \cdot 10^{-5} \end{bmatrix}$$

[17]:
```
stress1 = E*strain1
stress1
```

[17]:
$$\begin{bmatrix} 152.603901052738 \\ 24.4166241684383 \\ 151.090473934855 \end{bmatrix}$$

[18]:
```
strain2 = B2*sp.Matrix([
    [solved[u2]],
    [solved[v2]],
    [0],
    [solved[v3]],
    [solved[u4]],
    [solved[v4]]
])
strain2
```

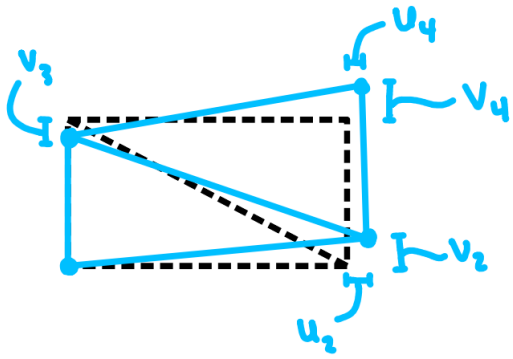
[18]:
$$\begin{bmatrix} -1.37990921436797 \cdot 10^{-6} \\ -1.41925831943259 \cdot 10^{-6} \\ -5.29026856982834 \cdot 10^{-6} \end{bmatrix}$$

[19]:
```
stress2 = E*strain2
stress2
```

[19]:

$$\begin{bmatrix} -59.5281333032225 \\ -60.4361895739445 \\ -61.0415604210962 \end{bmatrix}$$

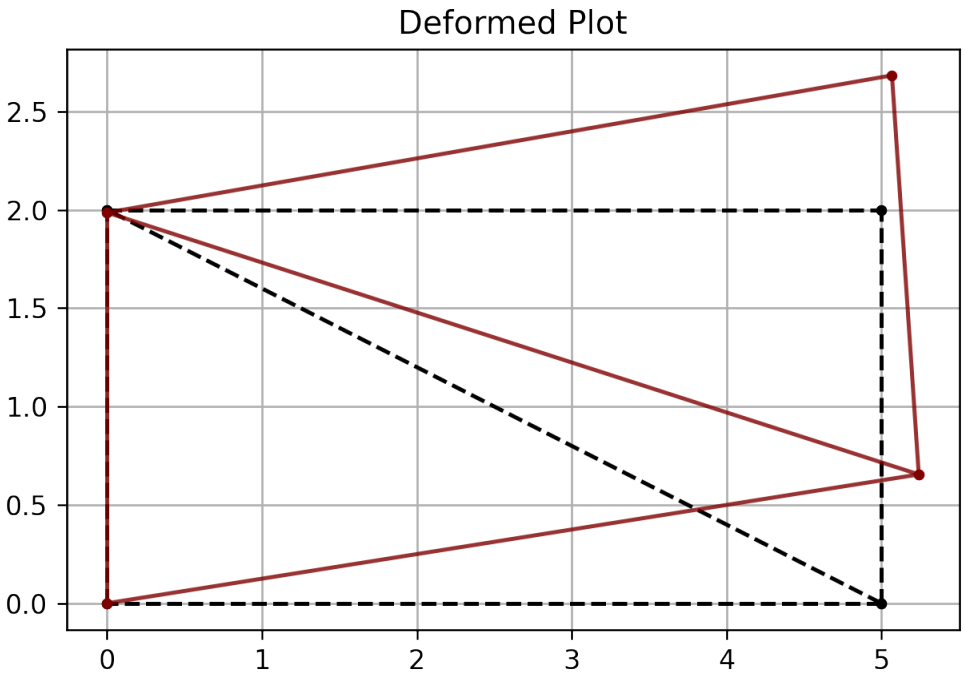
1.3.4 Part D



Remember, I changed the definition of the nodes to make things easier. Here is a plot to scale:

[20]:
```
undeformed = {'color': 'black', 'ls': '--', 'label': 'undeformed', 'marker': '.
 ↪'}
deformed = {'color': 'maroon', 'label': 'deformed', 'marker': '.', 'alpha': 0.8}
s_ = 10_000  # How exaggerated the results are going to be

plt.plot([0, 0], [0, H_], [L_, 0], [0, 0], [L_, 0], [0, H_], [L_, L_], [0, H_],␣
 ↪[0, L_], [H_, H_], **undeformed)
plt.plot([0, 0], [0, H_ + s_*solved[v3]], [0, L_ + s_*solved[u2]], [0,␣
 ↪s_*solved[v2]], [0, L_ + s_*solved[u2]], [H_ + s_*solved[v3],␣
 ↪s_*solved[v2]], [L_ + s_*solved[u4], 0], [H_ + s_*solved[v4], H_ +␣
 ↪s_*solved[v3]], [L_ + s_*solved[u2], L_ + s_*solved[u4]], [s_*solved[v2], H_␣
 ↪+ s_*solved[v4]], **deformed)
plt.title('Deformed Plot')
plt.show()
```


Deformed Plot
2.5
2.0
1.5
1.0
0.5
0.0
0
1
2
3
4
5