FEA Homework 2

February 14, 2022

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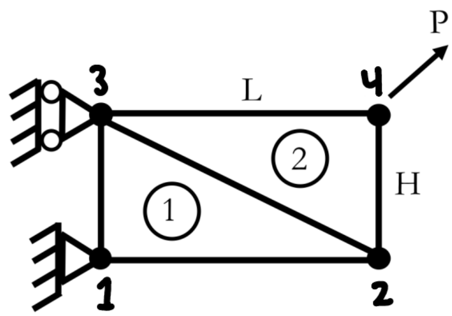
[1]:
```python
import sympy as sp
import matplotlib.pyplot as plt
from IPython.display import display, Latex

plt.style.use('maroon.mplstyle')

display_latex = lambda text: display(Latex(text))

def round_expr(expr, num_digits):
    return expr.xreplace({n : round(n, num_digits) for n in expr.atoms(sp.
 ↪Number)})
```

Contents

1 Problem 1

1.1 Given



$P = 150\,lb$, $L = 5\,in$, $H = 2\,in$, $t = 0.5\,in$, $E = 30 \cdot 10^6\,psi$, and $\nu = 0.30$. The angle of P is assumed to be 45°.

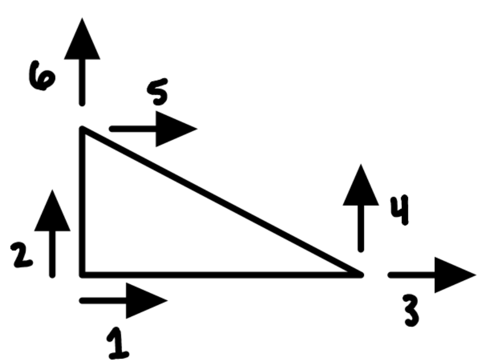
Notice that the global nodes have been rearranged. This was done to make the mapping easier.

1.2 Find

a. The global stiffness matrix
b. The displacements at each node
c. The stresses within each element
d. Plot the undeformed and deformed shape

1.3 Solution

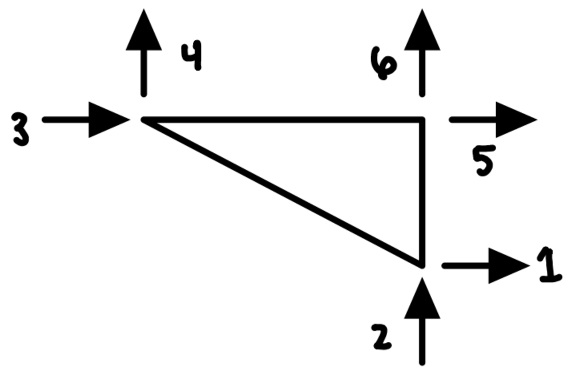
For the first element,



$\delta_1 = \delta_2 = \delta_5 = 0$
$\delta_3 = u_2$

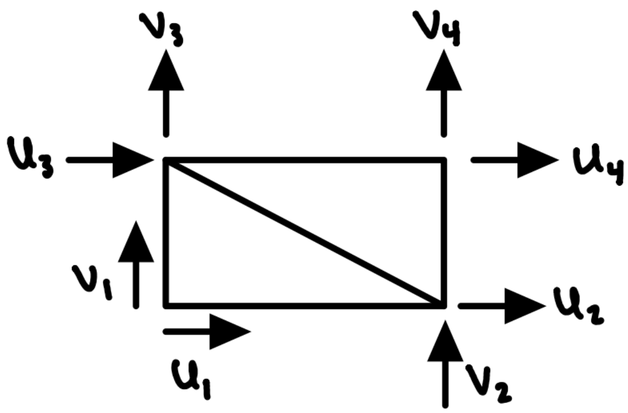
$\delta_4 = v_2$
$\delta_6 = v_3$

For the second element,



$\delta_3 = 0$
$\delta_1 = u_2$
$\delta_2 = v_2$
$\delta_4 = v_3$
$\delta_5 = u_4$
$\delta_6 = v_4$

The global displacements are,



1.3.1 Part A

[2]:
```
# Define numerical inputs here
P_ = 150
L_, H_ = 5, 2
t_ = 0.5
E_, nu_ = 30e6, 0.3
```

```python
# Define local coordinates for each element
x1 = [0, L_, 0]
y1 = [0, 0, H_]

x2 = [L_, 0, L_]
y2 = [0, H_, H_]

# The area
A_1 = ((x1[1] - x1[0])*(y1[2] - y1[0]) - (x1[2] - x1[0])*(y1[1] - y1[0]))/2
A_2 = -1*((x2[1] - x2[0])*(y2[2] - y2[0]) - (x2[2] - x2[0])*(y2[1] - y2[0]))/2
A_1, A_2
```

[2]: (5.0, 5.0)

[3]:
```python
# Define a symbolic B
A, y_23, y_31, y_12, x_32, x_13, x_21 = sp.symbols('A y_{23} y_{31} y_{12}␣
↪x_{32} x_{13} x_{21}')
B = 1/(2*A)*sp.Matrix([
    [y_23, 0, y_31, 0, y_12, 0],
    [0, x_32, 0, x_13, 0, x_21],
    [x_32, y_23, x_13, y_31, x_21, y_12]
])
B
```

[3]:
$$\begin{bmatrix} \frac{y_{23}}{2A} & 0 & \frac{y_{31}}{2A} & 0 & \frac{y_{12}}{2A} & 0 \\ 0 & \frac{x_{32}}{2A} & 0 & \frac{x_{13}}{2A} & 0 & \frac{x_{21}}{2A} \\ \frac{x_{32}}{2A} & \frac{y_{23}}{2A} & \frac{x_{13}}{2A} & \frac{y_{31}}{2A} & \frac{x_{21}}{2A} & \frac{y_{12}}{2A} \end{bmatrix}$$

[4]:
```python
# Numeric B
# Remember that indices start at 0
B1 = B.subs([
    (A, A_1),
    (y_23, y1[1] - y1[2]),
    (y_31, y1[2] - y1[0]),
    (y_12, y1[0] - y1[1]),
    (x_32, x1[2] - x1[1]),
    (x_13, x1[0] - x1[2]),
    (x_21, x1[1] - x1[0])
])
B1
```

[4]:
$$\begin{bmatrix} -0.2 & 0 & 0.2 & 0 & 0 & 0 \\ 0 & -0.5 & 0 & 0 & 0 & 0.5 \\ -0.5 & -0.2 & 0 & 0.2 & 0.5 & 0 \end{bmatrix}$$

[5]:
```python
B2 = B.subs([
    (A, A_2),
    (y_23, y2[1] - y2[2]),
```

```
    (y_31, y2[2] - y2[0]),
    (y_12, y2[0] - y2[1]),
    (x_32, x2[2] - x2[1]),
    (x_13, x2[0] - x2[2]),
    (x_21, x2[1] - x2[0])
])
B2
```

[5]:

$$\begin{bmatrix}0 & 0 & 0.2 & 0 & -0.2 & 0\\0 & 0.5 & 0 & 0 & 0 & -0.5\\0.5 & 0 & 0 & 0.2 & -0.5 & -0.2\end{bmatrix}$$

[6]:

```
E = E_/(1 - nu_**2)*sp.Matrix([
    [1, nu_, 0],
    [nu_, 1, 0],
    [0, 0, (1 - nu_)/2]
])
round_expr(E, 3)
```

[6]:

$$\begin{bmatrix}32967032.967 & 9890109.89 & 0\\9890109.89 & 32967032.967 & 0\\0 & 0 & 11538461.538\end{bmatrix}$$

[7]:

```
k1_local = t_*A_1*sp.transpose(B1)*E*B1
round_expr(k1_local, 3)
```

[7]:

$$\begin{bmatrix}10508241.758 & 5357142.857 & -3296703.297 & -2884615.385 & -7211538.462 & -2472527.473\\5357142.857 & 21758241.758 & -2472527.473 & -1153846.154 & -2884615.385 & -20604395.604\\-3296703.297 & -2472527.473 & 3296703.297 & 0 & 0 & 2472527.473\\-2884615.385 & -1153846.154 & 0 & 1153846.154 & 2884615.385 & 0\\-7211538.462 & -2884615.385 & 0 & 2884615.385 & 7211538.462 & 0\\-2472527.473 & -20604395.604 & 2472527.473 & 0 & 0 & 20604395.604\end{bmatrix}$$

[8]:

```
k2_local = t_*A_2*sp.transpose(B2)*E*B2
round_expr(k2_local, 3)
```

[8]:

$$\begin{bmatrix}7211538.462 & 0 & 0 & 2884615.385 & -7211538.462 & -2884615.385\\0 & 20604395.604 & 2472527.473 & 0 & -2472527.473 & -20604395.604\\0 & 2472527.473 & 3296703.297 & 0 & -3296703.297 & -2472527.473\\2884615.385 & 0 & 0 & 1153846.154 & -2884615.385 & -1153846.154\\-7211538.462 & -2472527.473 & -3296703.297 & -2884615.385 & 10508241.758 & 5357142.857\\-2884615.385 & -20604395.604 & -2472527.473 & -1153846.154 & 5357142.857 & 21758241.758\end{bmatrix}$$

[9]:

```
k1_global = k1_local.col_insert(6, sp.zeros(rows=6, cols=2)).row_insert(6, sp.
↪zeros(rows=2, cols=8))
round_expr(k1_global, 3)
```

[9]:

$$\begin{bmatrix} 10508241.758 & 5357142.857 & -3296703.297 & -2884615.385 & -7211538.462 & -2472527.473 & 0 & 0 \\ 5357142.857 & 21758241.758 & -2472527.473 & -1153846.154 & -2884615.385 & -20604395.604 & 0 & 0 \\ -3296703.297 & -2472527.473 & 3296703.297 & 0 & 0 & 2472527.473 & 0 & 0 \\ -2884615.385 & -1153846.154 & 0 & 1153846.154 & 2884615.385 & 0 & 0 & 0 \\ -7211538.462 & -2884615.385 & 0 & 2884615.385 & 7211538.462 & 0 & 0 & 0 \\ -2472527.473 & -20604395.604 & 2472527.473 & 0 & 0 & 20604395.604 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

[10]:
```
k2_global = k2_local.col_insert(0, sp.zeros(rows=6, cols=2)).row_insert(0, sp.zeros(rows=2, cols=8))
round_expr(k2_global, 3)
```

[10]:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 7211538.462 & 0 & 0 & 2884615.385 & -7211538.462 & -2884615.385 \\ 0 & 0 & 0 & 20604395.604 & 2472527.473 & 0 & -2472527.473 & -20604395.604 \\ 0 & 0 & 0 & 2472527.473 & 3296703.297 & 0 & -3296703.297 & -2472527.473 \\ 0 & 0 & 2884615.385 & 0 & 0 & 1153846.154 & -2884615.385 & -1153846.154 \\ 0 & 0 & -7211538.462 & -2472527.473 & -3296703.297 & -2884615.385 & 10508241.758 & 5357142.857 \\ 0 & 0 & -2884615.385 & -20604395.604 & -2472527.473 & -1153846.154 & 5357142.857 & 21758241.758 \end{bmatrix}$$

[11]:
```
k_global = k1_global + k2_global
round_expr(k_global, 1)
```

[11]:

$$\begin{bmatrix} 10508241.8 & 5357142.9 & -3296703.3 & -2884615.4 & -7211538.5 & -2472527.5 & 0 & 0 \\ 5357142.9 & 21758241.8 & -2472527.5 & -1153846.2 & -2884615.4 & -20604395.6 & 0 & 0 \\ -3296703.3 & -2472527.5 & 10508241.8 & 0 & 0 & 5357142.9 & -7211538.5 & -2884615.4 \\ -2884615.4 & -1153846.2 & 0 & 21758241.8 & 5357142.9 & 0 & -2472527.5 & -20604395.6 \\ -7211538.5 & -2884615.4 & 0 & 5357142.9 & 10508241.8 & 0 & -3296703.3 & -2472527.5 \\ -2472527.5 & -20604395.6 & 5357142.9 & 0 & 0 & 21758241.8 & -2884615.4 & -1153846.2 \\ 0 & 0 & -7211538.5 & -2472527.5 & -3296703.3 & -2884615.4 & 10508241.8 & 5357142.9 \\ 0 & 0 & -2884615.4 & -20604395.6 & -2472527.5 & -1153846.2 & 5357142.9 & 21758241.8 \end{bmatrix}$$

1.3.2 Part B

[12]:
```
F_1, F_2, F_5 = sp.symbols('F_1 F_2 F_5')
F = sp.Matrix([
    [F_1],
    [F_2],
    [0],
    [0],
    [F_5],
    [0],
    [P_*sp.sqrt(2)/2],
    [P_*sp.sqrt(2)/2]
]).n()
F
```

[12]:

$$\begin{bmatrix} F_1 \\ F_2 \\ 0 \\ 0 \\ F_5 \\ 0 \\ 106.066017177982 \\ 106.066017177982 \end{bmatrix}$$

[13]:

```
u2, v2, v3, u4, v4 = sp.symbols('u_2 v_2 v_3 u_4 v_4')
d = sp.Matrix([
    [0],
    [0],
    [u2],
    [v2],
    [0],
    [v3],
    [u4],
    [v4]
])
d
```

[13]:

$$\begin{bmatrix} 0 \\ 0 \\ u_2 \\ v_2 \\ 0 \\ v_3 \\ u_4 \\ v_4 \end{bmatrix}$$

[14]:

```
system = sp.Eq(F, k_global*d)
round_expr(system, 3)
```

[14]:

$$\begin{bmatrix} F_1 \\ F_2 \\ 0 \\ 0 \\ F_5 \\ 0 \\ 106.066 \\ 106.066 \end{bmatrix} = \begin{bmatrix} -3296703.297u_2 - 2884615.385v_2 - 2472527.473v_3 \\ -2472527.473u_2 - 1153846.154v_2 - 20604395.604v_3 \\ 10508241.758u_2 - 7211538.462u_4 + 5357142.857v_3 - 2884615.385v_4 \\ -2472527.473u_4 + 21758241.758v_2 - 20604395.604v_4 \\ -3296703.297u_4 + 5357142.857v_2 - 2472527.473v_4 \\ 5357142.857u_2 - 2884615.385u_4 + 21758241.758v_3 - 1153846.154v_4 \\ -7211538.462u_2 + 10508241.758u_4 - 2472527.473v_2 - 2884615.385v_3 + 5357142.857v_4 \\ -2884615.385u_2 + 5357142.857u_4 - 20604395.604v_2 - 1153846.154v_3 + 21758241.758v_4 \end{bmatrix}$$

[15]:

```
solved = sp.solve(system)
for key, value in solved.items():
    display_latex(f'${sp.latex(key)}={sp.latex(value)}$')
```

$F_1 = -265.165042944938$

$F_2 = -106.066017177975$

$F_5 = 159.099025766958$

$u_2 = 2.42131523003677 \cdot 10^{-5}$

$u_4 = 6.89954607183987 \cdot 10^{-6}$

$v_2 = 6.54725387051039 \cdot 10^{-5}$

$v_3 = -1.4243030764922 \cdot 10^{-6}$

$v_4 = 6.83110553439691 \cdot 10^{-5}$

1.3.3 Part C

[16]:
```python
strain1 = B1*sp.Matrix([
    [0],
    [0],
    [solved[u2]],
    [solved[v2]],
    [0],
    [solved[v3]]
])
strain1
```

[16]:

$$\begin{bmatrix} 4.84263046007354 \cdot 10^{-6} \\ -7.12151538246102 \cdot 10^{-7} \\ 1.30945077410208 \cdot 10^{-5} \end{bmatrix}$$

[17]:
```python
stress1 = E*strain1
stress1
```

[17]:

$$\begin{bmatrix} 152.603901052738 \\ 24.4166241684383 \\ 151.090473934855 \end{bmatrix}$$

[18]:
```python
strain2 = B2*sp.Matrix([
    [solved[u2]],
    [solved[v2]],
    [0],
    [solved[v3]],
    [solved[u4]],
    [solved[v4]]
])
strain2
```

[18]:

$$\begin{bmatrix} -1.37990921436797 \cdot 10^{-6} \\ -1.41925831943259 \cdot 10^{-6} \\ -5.29026856982834 \cdot 10^{-6} \end{bmatrix}$$

[19]:
```python
stress2 = E*strain2
stress2
```

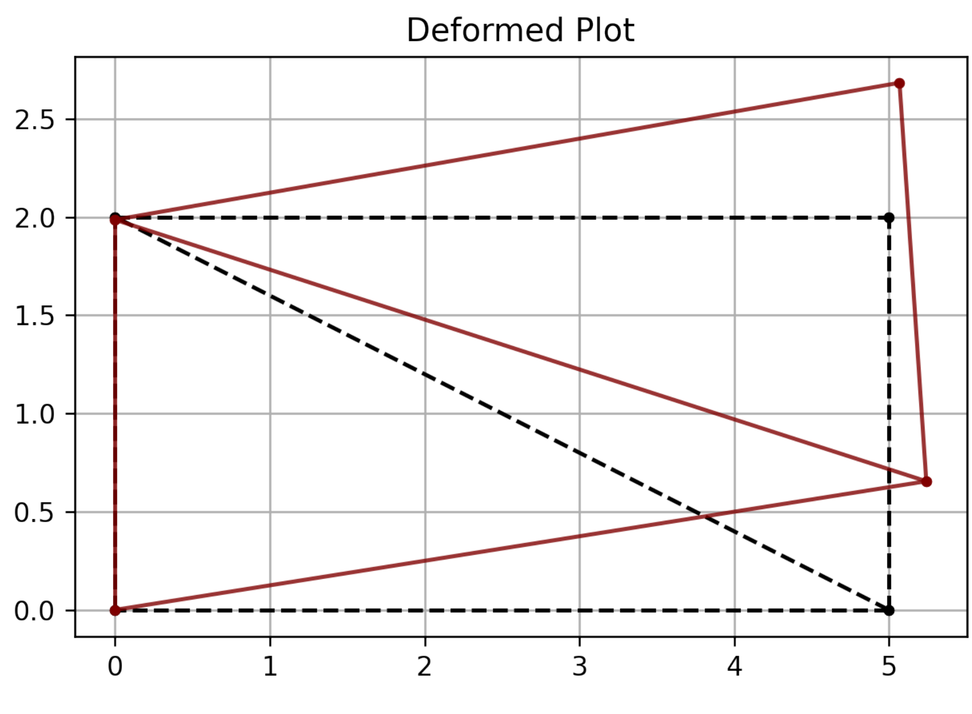
[19]:

$$\begin{bmatrix} -59.5281333032225 \\ -60.4361895739445 \\ -61.0415604210962 \end{bmatrix}$$

1.3.4 Part D

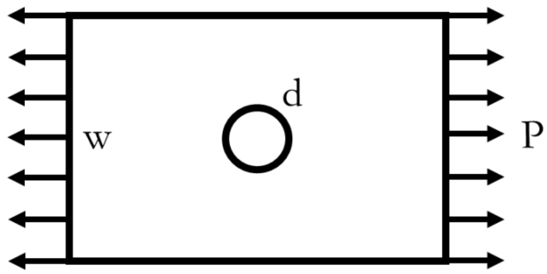
Here is a plot of the deformed shape.

[20]:
```
undeformed = {'color': 'black', 'ls': '--', 'label': 'undeformed', 'marker': '.
↪'}
deformed = {'color': 'maroon', 'label': 'deformed', 'marker': '.', 'alpha': 0.8}
s_ = 10_000  # How exaggerated the results are going to be

plt.plot([0, 0], [0, H_], [L_, 0], [0, 0], [L_, 0], [0, H_], [L_, L_], [0, H_],␣
↪[0, L_], [H_, H_], **undeformed)
plt.plot([0, 0], [0, H_ + s_*solved[v3]], [0, L_ + s_*solved[u2]], [0,␣
↪s_*solved[v2]], [0, L_ + s_*solved[u2]], [H_ + s_*solved[v3],␣
↪s_*solved[v2]], [L_ + s_*solved[u4], 0], [H_ + s_*solved[v4], H_ +␣
↪s_*solved[v3]], [L_ + s_*solved[u2], L_ + s_*solved[u4]], [s_*solved[v2], H_␣
↪+ s_*solved[v4]], **deformed)
plt.title('Deformed Plot')
plt.show()
```



2 Problem 1

2.1 Given



2.2 Find

a. The stress concentration factor if $\frac{d}{w} = 0.2$.
b. Does the stress concentration factor change with increasing number of elements? Confirm or deny with three different mesh densities.
c. Compare the FE solution with a theoretical stress concentration factor (provide references and supporting material).
d. Discuss the mesh used in these simulations including the element type(s) used, why they were chosen, and how they affected the result.

2.3 Solution

The part that I created has the parameters,

$d = 8\,mm$, $w = 40\,mm$, $L = 40\,mm$, and $\frac{d}{w} = 0.2$

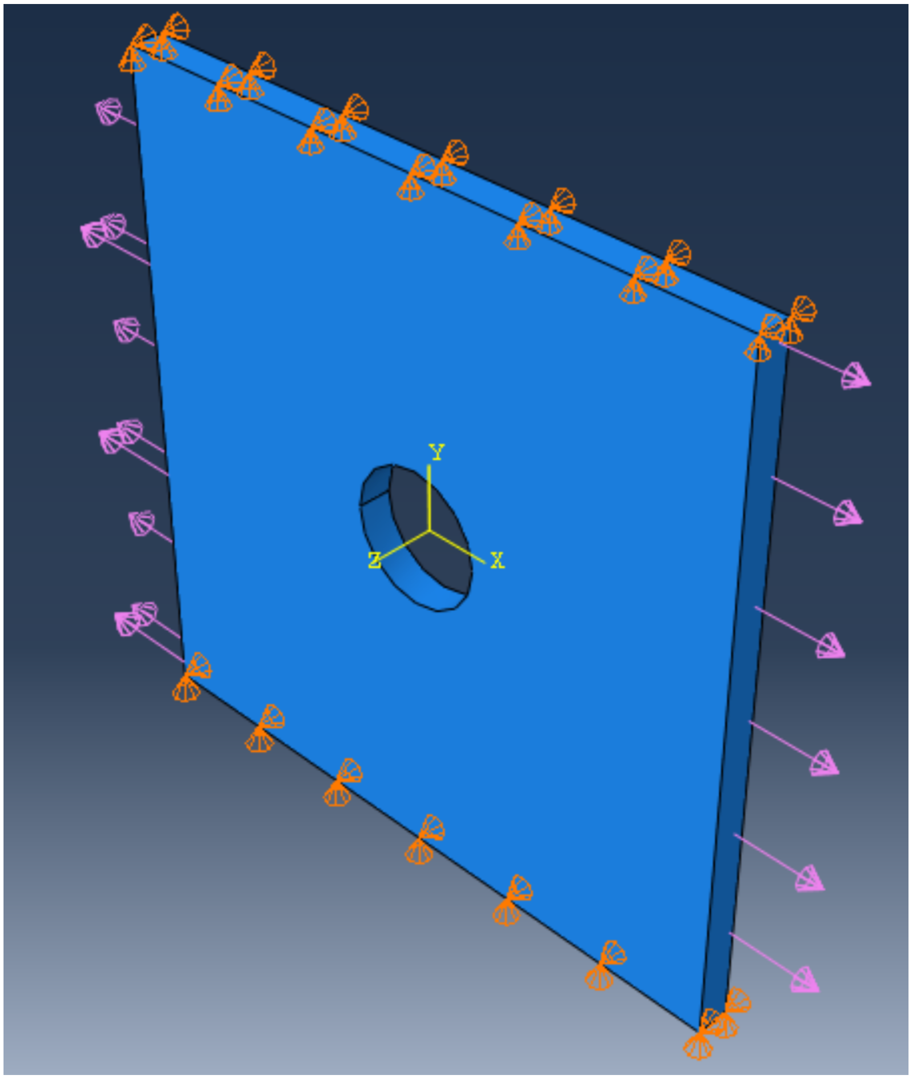
The material properties are,

$E = 69,000\,MPa$ and $\nu = 0.35$ (Aluminum)

The load is,

$P = 5\,MPa$

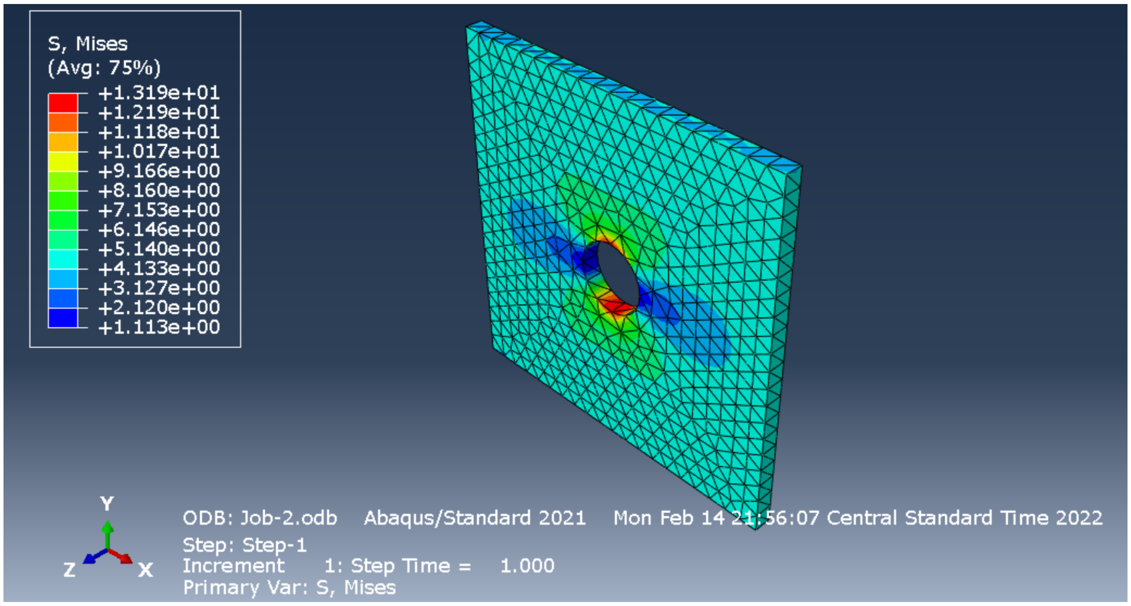
The part has a small thickness of 2 mm. Here are the loads,



The boundary conditions allow for motion in the U1 direction.

2.3.1 Part A

Using a 2 mm mesh with a C3D10 element type,

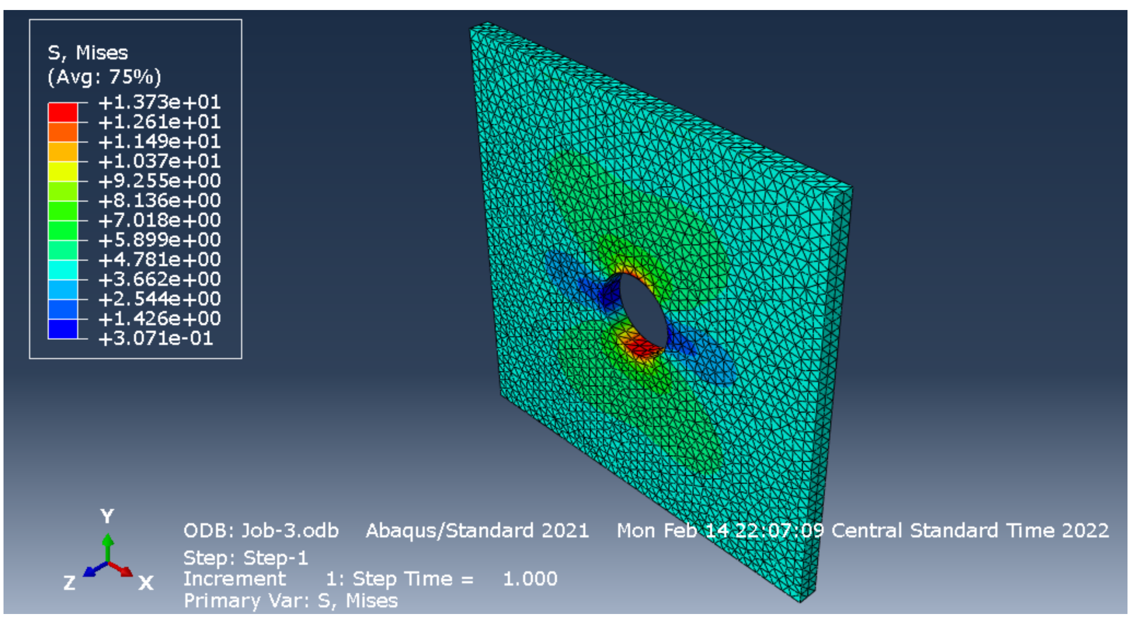


The stress concentration may have several definitions, but for our case, let's say that the stress concentration is the maximum stress divided by the nominal stress. For this 2 mm mesh,

$\frac{\sigma_{max}}{\sigma_{nom}} = \frac{13.19\,MPa}{5\,MPa} = 2.638$

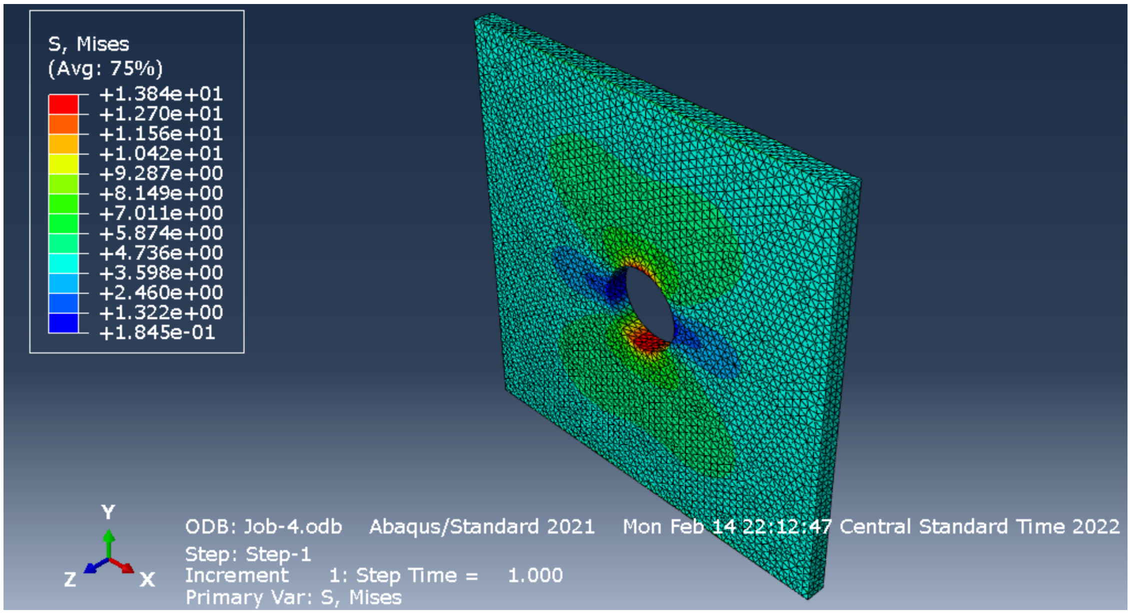
2.3.2 Part B

Using a mesh size of 1 mm,



The stress concentration factor increased to a value of $\frac{13.73\,MPa}{5\,MPa} = 2.746$.

Using a mesh size of 0.75 mm,



The stress concentration factor increased further to a value of $\frac{13.84\,MPa}{5\,MPa} = 2.768$.

2.3.3 Part C

According to this figure found in *Mechanics of Materials* by R.C. Hibbeler,

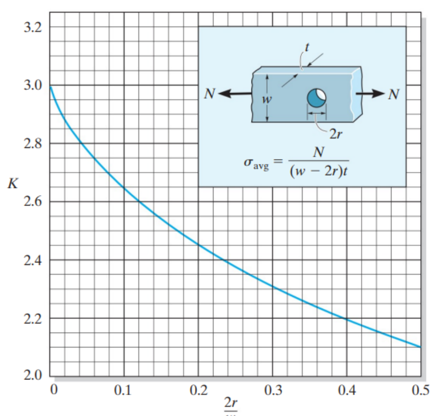


Fig. 4–24

The theoretical value should be around 2.45. This value is less than the value from the FEA results.

2.3.4 Part D

Mesh Size (mm)	Stress Concentration
2	2.638
1	2.746
0.75	2.768

The element used was C3D10 (tetrahedral). The element was used because it is a worthy representative for general purpose application. As seen in the table above, as the mesh size decreases (more elements), the stress concentration factor increases. The value has some significant deviation from the theoretical value. This could be due to the way in which the stress concentration factor is defined. That graph defines the stress concentration factor as the ratio of the maximum stress to the average stress across the central cross-section. The average stress for our case, would be greater than $5\,MPa$, meaning that the FEA results could agree more with the theoretical value had I used the average instead.