Vibrations and Controls Homework 9

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[1]:
```
# Notebook Preamble
%config ZMQInteractiveShell.ast_node_interactivity = 'all'
import sympy as sp
import numpy as np
import matplotlib.pyplot as plt
from scipy.integrate import odeint

t, s = sp.symbols('t s')

plt.style.use('maroon_ipynb.mplstyle')
```

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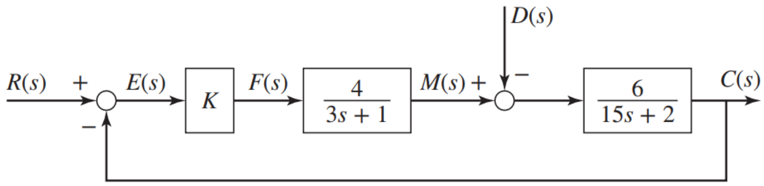
1 Problem 10.4

1.1 Given



1.2 Find

Derive the output $C(s)$, error $E(s)$, and actuator $M(s)$ equations for the diagram above and obtain the characteristic polynomial.

1.3 Solution

The output of the functions mentioned above may be acquired using algebra of the block diagram.

[2]:
```
# Use algebra to determine the functions
K = sp.Symbol('K')
R, E, F, M, D, C, T = [sp.Function(i)(s) for i in ['R', 'E', 'F', 'M', 'D',␣
↪'C', 'T']]
eq1 = sp.Eq(E, R - C)
eq1
eq2 = sp.Eq(F, E*K)
eq2
eq3 = sp.Eq(M, F*4/(3*s + 1))
eq3
eq4 = sp.Eq(C, (M - D)*6/(15*s + 2))
eq4
```

[2]: $E(s) = -C(s) + R(s)$

[2]: $F(s) = KE(s)$

[2]: $$M(s) = \frac{4F(s)}{3s + 1}$$

[2]: $$C(s) = \frac{-6D(s) + 6M(s)}{15s + 2}$$

The characteristic polynomial may be determined by solving for $\frac{C(s)}{R(s)}$.

[3]:
```
subs = eq4.subs([
    (M, eq3.rhs),
    (F, eq2.rhs),
    (E, eq1.rhs),
```

```
    (C, T*R)
])
sol = sp.solve(subs, T)[0]
sp.Eq(T, sol)
```

[3]:

$$T(s) = \frac{6 \cdot (4KR(s) - 3sD(s) - D(s))}{(24K + 45s^2 + 21s + 2)\, R(s)}$$

$\frac{C(s)}{R(s)}$ may be obtained by setting $D(s)$ equal to 0.

[4]:

```
sp.Eq(C/R, sol.subs(D, 0))
```

[4]:

$$\frac{C(s)}{R(s)} = \frac{24K}{24K + 45s^2 + 21s + 2}$$

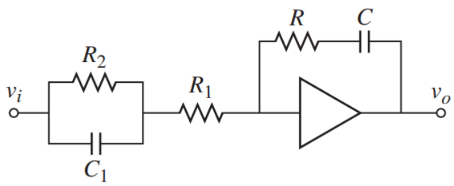
It is not necessary to try and rewrite $E(s)$, $M(s)$, and $F(s)$ in any other forms. The denominator of the $\frac{C(s)}{R(s)}$ function above is the characteristic polynomial,

$$45s^2 + 21s + 2 + 24K$$

2 Problem 10.11

2.1 Given

Figure 10.4.7 Op-amp implementation of PID action.



$$\frac{V_o(s)}{V_i(s)} = -\left(K_P + \frac{K_I}{s} + K_D s\right)\frac{1}{\beta R_1 C_1 s + 1}$$

$$K_P = \beta\frac{RC + R_2 C_1}{R_2 C} \qquad K_I = \frac{\beta}{R_2 C}$$

$$K_D = \beta R C_1 \qquad \beta = \frac{R_2}{R_1 + R_2}$$

2.2 Find

a. Determine the resistance values to obtain an op-amp PID controller with $K_P = 10$, $K_I = 1.4$, and $K_D = 4$. The circuit should limit frequencies above 100 rad/s. Take one capacitance to be $1\,\mu F$.
b. Plot the frequency response of the circuit.

2.3 Solution

Using the relationships on page 646, we can obtain 4 equations and 4 unknowns.

[5]:
```
C, R1, R2, R, C1, beta = sp.symbols(r'C R_1 R_2 R C_1 \beta')
Vo, Vi = sp.Function('V_o')(s), sp.Function('V_i')(s)
KP = beta*(R*C + R2*C1)/(R2*C)
eq1 = sp.Eq(KP, 10)
KI = beta/(R2*C)
eq2 = sp.Eq(KI, 1.4)
KD = beta*R*C1
eq3 = sp.Eq(KD, 4)
eq4 = sp.Eq(beta, R2/(R1 + R2))
eq1
eq2
eq3
eq4
```

[5]: $\frac{\beta\left(CR + C_1R_2\right)}{CR_2} = 10$

[5]:

[5]:

$$\frac{\beta}{CR_2} = 1.4$$

[5]:

$$C_1 R\beta = 4$$

[5]:

$$\beta = \frac{R_2}{R_1 + R_2}$$

Solving the system in terms of C and C_1,

[6]:
```python
sol = sp.solve([eq1, eq2, eq3, eq4], (R1, R2, R, beta), dict=True)[-1]
for key, value in sol.items():
    sp.Eq(key, value)
```

[6]:

$$R_1 = \frac{0.142857142857143\,(-2.97728445445476C + 5.0C_1)}{CC_1}$$

[6]:

$$R_2 = \frac{0.425326350636394}{C_1}$$

[6]:

$$R = \frac{6.71753079222075}{C}$$

[6]:

$$\beta = \frac{0.595456890890952C}{C_1}$$

Choosing $C = 1 \cdot 10^{-6}\,\mu F$,

[7]:
```python
sol2 = {k: v.subs(C, 1e-6) for k, v in sol.items()}
for key, value in sol2.items():
    sp.Eq(key, value)
```

[7]:

$$R_1 = \frac{142857.142857143 \cdot (5.0C_1 - 2.97728445445476 \cdot 10^{-6})}{C_1}$$

[7]:

$$R_2 = \frac{0.425326350636394}{C_1}$$

[7]:

$$R = 6717530.79222075$$

[7]:

$$\beta = \frac{5.95456890890952 \cdot 10^{-7}}{C_1}$$

Since R_1 has to be greater than 0, we can use the first equation above to solve for C_1 where R_1 is 0.

[8]:
```python
C1_lower = sp.solve(sol2[R1], C1)[0]
C1_lower
```

[8]:

$$5.95456890890952 \cdot 10^{-7}$$

Notice from the figure above, the natural frequency of the system is $\frac{1}{\beta R_1 C_1}$. This means that we need to choose a value for C_1 such that $\frac{1}{\beta R_1 C_1} \geq 100$.

[9]:
```python
C1_upper = sp.solve(1/(sol2[beta]*sol2[R1]*C1) - 100, C1)[0]
C1_upper
```

[9]:

$$6.09793975209791 \cdot 10^{-7}$$

Therefore, $C_1 = 6 \cdot 10^{-7}\,F$ is a worthy option. The selections for the resistors and capacitors are,

[10]:
```python
C1_ = 6e-7
for key, value in sol2.items():
    sp.Eq(key, value.subs(C1, C1_))
sp.Eq(C1, C1_)
sp.Eq(C, 1e-6)
```

[10]: $R_1 = 5408.46322505693$

[10]: $R_2 = 708877.251060657$

[10]: $R = 6717530.79222075$

[10]: $\beta = 0.99242815148492$

[10]: $C_1 = 6.0 \cdot 10^{-7}$

[10]: $C = 1.0 \cdot 10^{-6}$

2.3.1 Frequency Response

Going back to the original transfer function,

[11]:
```python
%config ZMQInteractiveShell.ast_node_interactivity = 'last_expr'
T = (-(KP + KI/s + KD*s)*1/(beta*R1*C1*s + 1)).subs(beta, R2/(R1 + R2))
sp.Eq(Vo/Vi, T.simplify())  # K values are substituted
T_ = T.subs([
    (R1, sol2[R1].subs(C1, C1_)),
    (R2, sol2[R2].subs(C1, C1_)),
    (R, sol2[R].subs(C1, C1_)),
    (C, 1e-6),
    (C1, C1_)
])
sp.Eq(Vo/Vi, T_)
w = sp.Symbol(r'\omega')
T_jw = T_.subs(s, sp.I*w)
sp.Eq(sp.Function('T')(sp.I*w), T_jw)
```

[11]: $$\frac{\mathrm{V_o}(s)}{\mathrm{V_i}(s)} = \frac{-CC_1RR_2s^2 - s\,(CR + C_1R_2) - 1}{Cs\,(C_1R_1R_2s + R_1 + R_2)}$$

[11]: $$\frac{\mathrm{V_o}(s)}{\mathrm{V_i}(s)} = \frac{-4.0s - 10.0 - \frac{1.4}{s}}{0.00322050669649045s + 1}$$

[11]: $$T(i\omega) = \frac{-4.0i\omega - 10.0 + \frac{1.4i}{\omega}}{0.00322050669649045i\omega + 1}$$

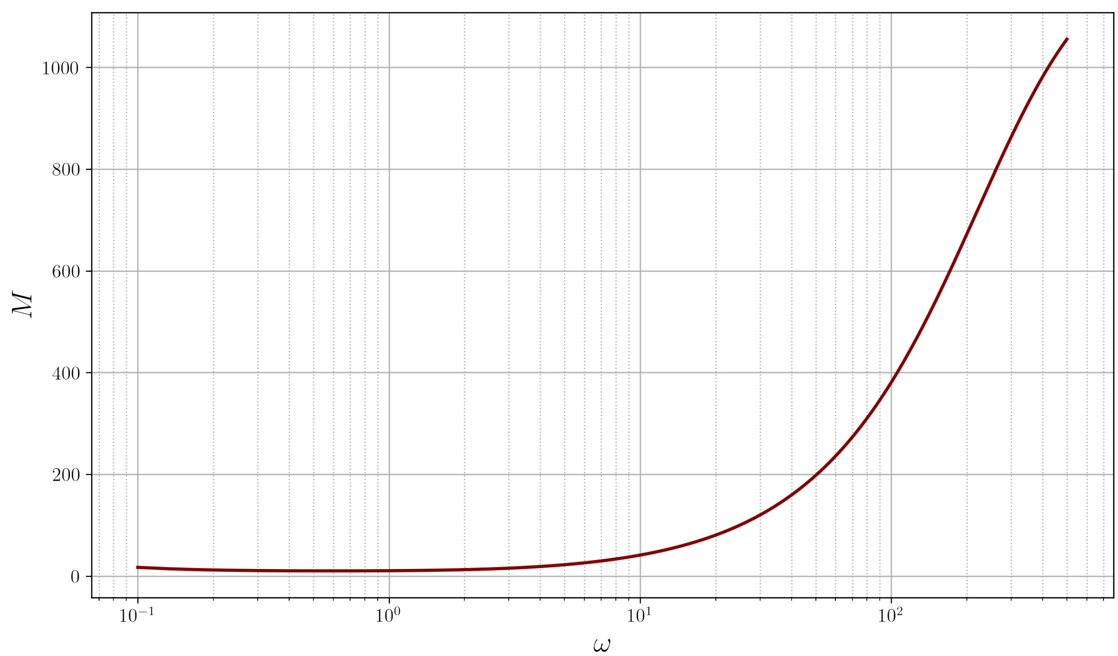
[12]:
```python
%config ZMQInteractiveShell.ast_node_interactivity = 'all'
mag = sp.Abs(T_jw)
phi = sp.arg(T_jw)
mag_lamb = sp.lambdify(w, mag, modules='numpy')
phi_lamb = sp.lambdify(w, phi, modules='numpy')
```

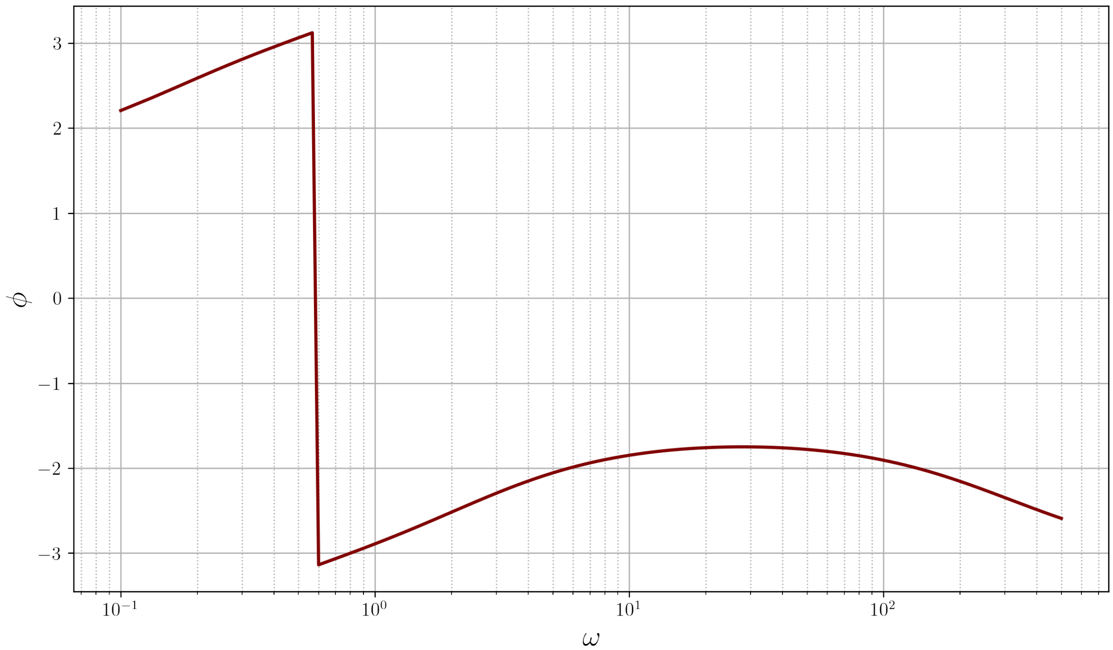
```python
w_range = np.linspace(0.1, 500, 15_000)

fig1, ax1 = plt.subplots()
ax1.plot(w_range, mag_lamb(w_range))
ax1.set_xlabel(r'$\omega$')
ax1.set_ylabel(r'$M$')
ax1.set_xscale('log')
ax1.grid(which='minor', ls=':')

fig2, ax2 = plt.subplots()
ax2.plot(w_range, phi_lamb(w_range))
ax2.set_xlabel(r'$\omega$')
ax2.set_ylabel(r'$\phi$')
ax2.set_xscale('log')
ax2.grid(which='minor', ls=':')

plt.show()
```





3
2
1
0
−1
−2
−3
ϕ
10−1
100
101
102
ω

3 Problem 10.12 Part B and Part D

3.1 Given

b. $\frac{Y(s)}{F(s)} = \frac{8s-5}{10s^2+4s+7}$ and $f(t) = 8u_s(t)$
c. $\frac{Y(s)}{F(s)} = \frac{5s+8}{s^2+2s-9}$ and $f(t) = 8u_s(t)$

3.2 Find

The steady state response if possible.

3.3 Solution

The steady state is possible if the system is stable (all roots of the characteristic equation have negative real parts).

3.3.1 Part B

[13]:
```
T_b = (8*s - 5)/(10*s**2 + 4*s + 7)
F = 8/s
T_b
poly = sp.fraction(T_b)[1]
poly
for root in sp.roots(poly):
    root
```

[13]: $$\frac{8s-5}{10s^2+4s+7}$$

[13]: $$10s^2+4s+7$$

[13]: $$-\frac{1}{5}-\frac{\sqrt{66}i}{10}$$

[13]: $$-\frac{1}{5}+\frac{\sqrt{66}i}{10}$$

The steady state exists.

[14]:
```
lim_b = sp.Limit(s*T_b*F, s, 0)
sp.Eq(lim_b, lim_b.doit())
```

[14]: $$\lim_{s\to 0^+}\left(\frac{8\cdot(8s-5)}{10s^2+4s+7}\right) = -\frac{40}{7}$$

3.3.2 Part D

[15]:
```
T_d = (5*s + 8)/(s**2 + 2*s - 9)
T_d
poly = sp.fraction(T_d)[1]
poly
for root in sp.roots(poly):
    root
```

[15]: $\frac{5s+8}{s^2+2s-9}$

[15]: s^2+2s-9

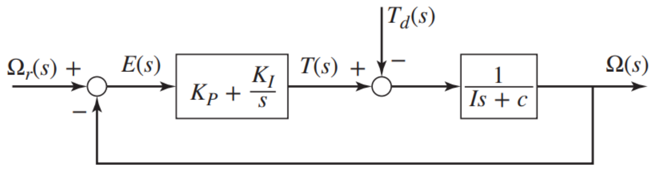
[15]: $-\sqrt{10}-1$

[15]: $-1+\sqrt{10}$

The final value theorem does not work because not all roots are negative.

4 Problem 10.22

4.1 Given



Suppose that $I = c = 4$ for the PI controller shown above. The performance specifications require that $\tau = 0.2$.

4.2 Find

a. Compute the required gain values for each of the following cases: $\zeta = 0.707$, $\zeta = 1$, and a root separation factor of 10.
b. Use a computer method to plot the unit step command responses for each of the cases in part (a). Compare the performance of each case.

4.3 Solution

The transfer function is,

[16]:
```
ohm, ohm_r = sp.Function(r'\Omega')(s), sp.Function(r'\Omega_r')(s)
KP, KI = sp.symbols('K_P K_I')
eq = sp.Eq((ohm_r - ohm)*(KP + KI/s)*1/(4*s + 4), ohm)
eq
T_s = sp.Function('T')(s)
sol = sp.solve(eq.subs(ohm, T_s*ohm_r), T_s)[0]
sp.Eq(ohm/ohm_r, sol)
```

[16]:
$$\frac{\left(\frac{K_I}{s} + K_P\right)\left(-\Omega(s) + \Omega_r(s)\right)}{4s + 4} = \Omega(s)$$

[16]:
$$\frac{\Omega(s)}{\Omega_r(s)} = \frac{K_I + K_P s}{K_I + K_P s + 4s^2 + 4s}$$

4.3.1 Part A

$$\tau = \frac{8}{4 + K_P} = 0.2$$

[17]:
```
eq1 = sp.Eq(8/(4 + KP), 0.2)
eq1
KP_ = sp.solve(eq1, KP)[0]
sp.Eq(KP, KP_)
```

[17]:

$$\frac{8}{K_P + 4} = 0.2$$

[17]:

$$K_P = 36.0$$

$$\zeta = \frac{c}{2\sqrt{mk}}$$

[18]:

```
# For 0.707
eq2 = sp.Eq((4 + KP_)/(2*sp.sqrt(4*KI)), 0.707)
eq2
KI_707 = sp.solve(eq2, KI)[0]
sp.Eq(KI, KI_707)

# For 1
eq3 = sp.Eq((4 + KP_)/(2*sp.sqrt(4*KI)), 1)
eq3
KI_1 = sp.solve(eq3, KI)[0]
sp.Eq(KI, KI_1)
```

[18]:

$$\frac{10.0}{\sqrt{K_I}} = 0.707$$

[18]:

$$K_I = 200.06041824631$$

[18]:

$$\frac{10.0}{\sqrt{K_I}} = 1$$

[18]:

$$K_I = 100.0$$

Thus, the gain values are $K_P = 36$ for $\zeta = 0.707$ and $\zeta = 1$ and $K_I = 200$ for $\zeta = 0.707$ and $K_I = 100$ for $\zeta = 1$. For the separation factor of 10 (this just means that the other root of the polynomial is some multiple $10 \cdot$ *other root*), we know that the other root is $-\frac{1}{\tau} = -5$. This means that the other root is $-5 \cdot 10 = -50$.

[19]:

```
poly = (s + 5)*(s + 50)
eq = sp.Eq(poly, poly.expand())
eq
```

[19]:

$$(s + 5)(s + 50) = s^2 + 55s + 250$$

But in order to compare this characteristic equation with the denominator of the transfer function, the value of m must be equivalent.

$$4s^2 + 220s + 1000$$

[20]:

```
%config ZMQInteractiveShell.ast_node_interactivity = 'last_expr'
eq = sp.Eq(4 + KP, 220)
eq
KP_fact = sp.solve(eq, KP)[0]
sp.Eq(KP, KP_fact)
```

[20]:
$K_P + 4 = 220$

[20]:
$K_P = 216$

Thus, $K_P = 216$ and $K_I = 1000$ for the factor of 10 case.

4.3.2 Part B

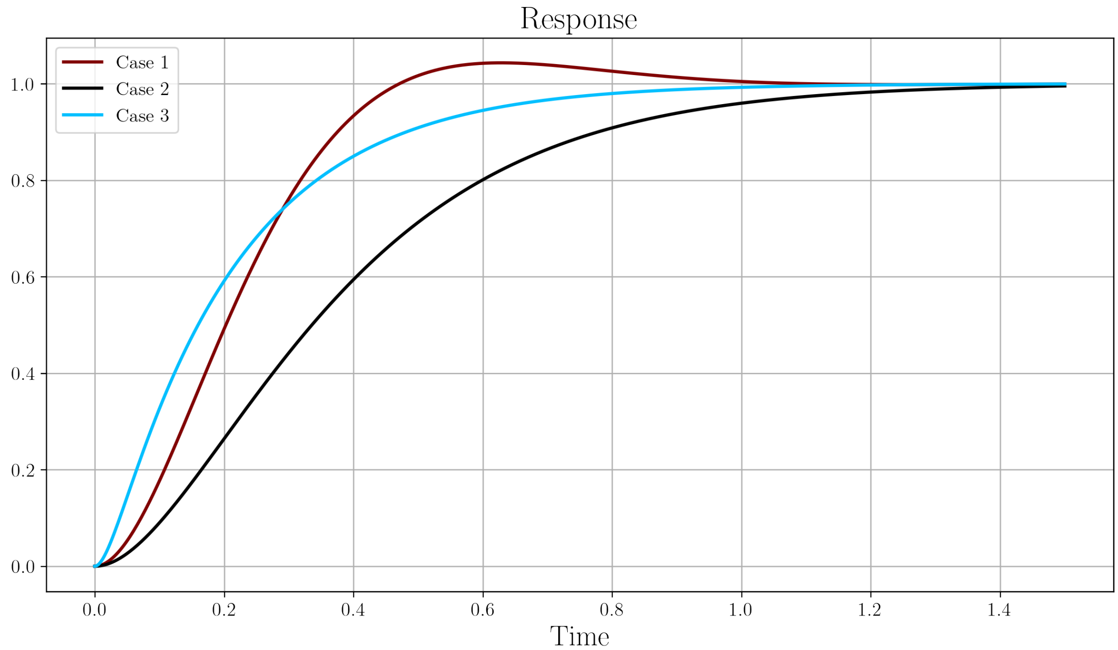
The responses can be compared by looking at the plot. In the differential equation form with the unit step as the input (just the constant 1 for values greater than 0), the system is,

$$(4s^2 + (4 + K_P)s + K_I)\Omega(s) = (K_I + K_P s)\frac{1}{s}$$
$$4\ddot{\omega} + (4 + K_P)\dot{\omega} + K_I\omega = K_I$$

The K_P value on the right-hand side goes away because the inverse laplace transform is technically $K_P\delta(t)$, but we are only considering the values greater than 0 ($K_P\delta(t)$ for t > 0 is 0).

[21]:
```
# The best way is to solve numerically
def diffs(x, _, KP__, KI__):
    return [
        x[1],
        1/4*(KI__ - (4 + KP__)*x[1] - KI__*x[0])
    ]

cases = [(KP_, KI_707), (KP_, KI_1), (KP_fact, 1000)]
omega = sp.Function(r'\omega')(t)
time = np.linspace(0, 1.5, 500)
fig, ax = plt.subplots()
for i, case in enumerate(cases):
    KP_value, KI_value = case
    sol = odeint(diffs, (0, 0), time, args=(KP_value, KI_value))
    ax.plot(time, sol[:, 0], label=f'Case {i+1}')
ax.set_xlabel('Time')
ax.set_title('Response')
ax.legend()
plt.show()
```


Response
Case 1
Case 2
Case 3
0.0
0.2
0.4
0.6
0.8
1.0
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
Time