Machine Design Homework 2

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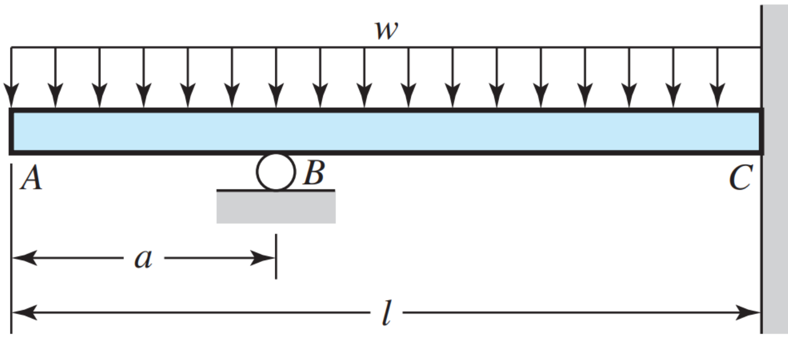
[1]:
```
import matplotlib.pyplot as plt
import sympy as sp
from IPython.display import display

plt.style.use('maroon_ipynb.mplstyle')
```

Contents

1 Problem 4-118

1.1 Given

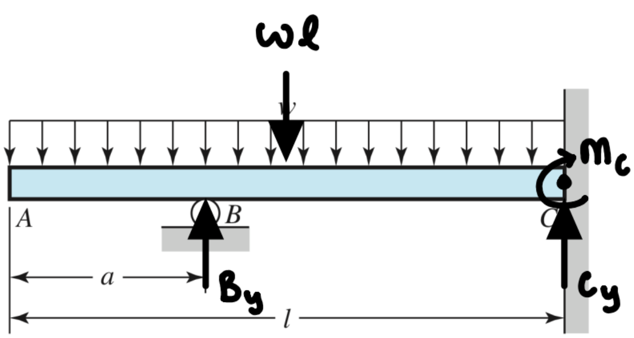


1.2 Find

Determine the support reactions using Castigliano's theory.

1.3 Solution

The free body diagram yields two equations with three unknowns,

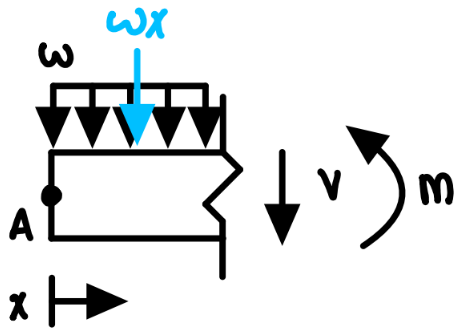


[2]:
```
By, w, l, Cy, Mc, a = sp.symbols('B_y w l C_y M_c a')
eq1 = sp.Eq(By + Cy, w*l)  # Forces in y direction
eq2 = sp.Eq(Mc + By*(l - a), w*l*(l/2))
display(eq1, eq2)
```

$B_y + C_y = lw$

$B_y(-a + l) + M_c = \frac{l^2 w}{2}$

The bending and shear diagram equations as a function of x may be extracted like so,



The above figure is for $0 \leq x \leq a$.

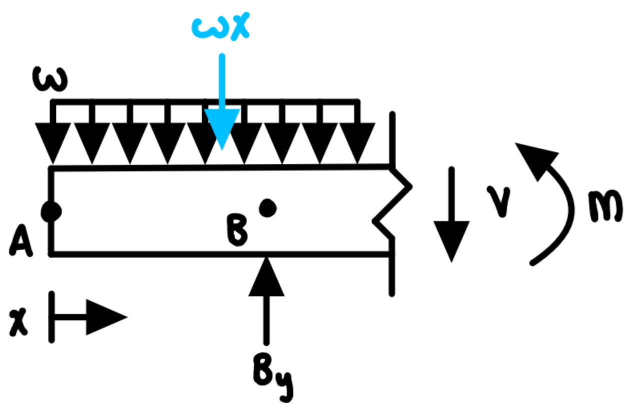
[3]:
```
# Shear equation
x = sp.Symbol('x')
V1 = -w*x
V1
```

[3]: $-wx$

[4]:
```
# Moment equation
M1 = -sp.S('0.5')*w*x**2
M1
```

[4]: $-0.5wx^2$

For $a \leq x \leq l$,



[5]:
```
V2 = By - w*x
V2
```

[5]: $B_y - wx$

[6]:
```
M2 = By*(x - a) - sp.S('0.5')*w*x**2
M2
```

[6]: $B_y(-a + x) - 0.5wx^2$

All together, the moment and shear equation may be represented as the piecewise functions below.

[7]:
```
V = sp.Piecewise((V1, (x >= 0) & (x <= a)), (V2, (x >= a) & (x <= l)))
M = sp.Piecewise((M1, (x >= 0) & (x <= a)), (M2, (x >= a) & (x <= l)))
display(V, M)
```

$$\begin{cases} -wx & \text{for } a \geq x \wedge x \geq 0 \\ B_y - wx & \text{for } l \geq x \wedge a \leq x \end{cases}$$

$$\begin{cases} -0.5wx^2 & \text{for } a \geq x \wedge x \geq 0 \\ B_y\left(-a+x\right) - 0.5wx^2 & \text{for } l \geq x \wedge a \leq x \end{cases}$$

Castigliano's theory involves computing the total energy, which is

$$U = \int \frac{M^2}{2EI}dx + \int \frac{CV^2}{2AG}dx$$

We can integrate across each section. Watch as `sympy` impressively solves this huge integral for us, symbolically.

[8]:
```
C, E, I, A, G, U_sym = sp.symbols('C E I A G U')
U1 = sp.Integral(M1**2/(2*E*I) + C*V1**2/(2*A*G), (x, 0, a))
U2 = sp.Integral(M2**2/(2*E*I) + C*V2**2/(2*A*G), (x, a, l))
U = U1 + U2
sp.Eq(U_sym, U)
```

[8]:

$$U = \int\limits_a^l \left(\frac{\left(B_y\left(-a+x\right) - 0.5wx^2\right)^2}{2EI} + \frac{C\left(B_y - wx\right)^2}{2AG}\right) dx + \int\limits_0^a \left(\frac{0.125w^2x^4}{EI} + \frac{Cw^2x^2}{2AG}\right) dx$$

[9]:
```
U_doit = U.doit().expand().n(6)
sp.Eq(U_sym, U_doit)
```

[9]:

$$U = -\frac{0.166667B_y^2a^3}{EI} + \frac{0.5B_y^2a^2l}{EI} - \frac{0.5B_y^2al^2}{EI} + \frac{0.166667B_y^2l^3}{EI} - \frac{0.0416667B_ya^4w}{EI} + \frac{0.166667B_yal^3w}{EI} - \frac{0.125B_yl^4w}{EI} + \frac{0.025l^5w^2}{EI} - \frac{0.5B_y^2Ca}{AG} + \frac{0.5B_y^2Cl}{AG} + \frac{0.5B_yCa^2w}{AG} - \frac{0.5B_yCl^2w}{AG} + \frac{0.166667Cl^3w^2}{AG}$$

Castigliano's Theory is,

$$\delta_i = \frac{\partial U}{\partial F_i}$$

We know the deflection at point B is 0.

[10]:
```
# Take the derivative of the expression above and set equal to 0
eq3 = sp.Eq(U_doit.diff(By).expand().n(6), 0)
eq3
```

[10]:

$$-\frac{0.333333B_ya^3}{EI} + \frac{1.0B_ya^2l}{EI} - \frac{1.0B_yal^2}{EI} + \frac{0.333333B_yl^3}{EI} - \frac{0.0416667a^4w}{EI} + \frac{0.166667al^3w}{EI} - \frac{0.125l^4w}{EI} - \frac{1.0B_yCa}{AG} + \frac{1.0B_yCl}{AG} + \frac{0.5Ca^2w}{AG} - \frac{0.5Cl^2w}{AG} = 0$$

[11]:

```
sol = sp.solve([eq1, eq2, eq3], (By, Cy, Mc), dict=True)[0]
for key, value in sol.items():
    display(sp.Eq(key, value.simplify()))
```

$$B_y = \frac{w\left(-AGa^3 - AGa^2 l - AGal^2 + 3.0AGl^3 + 12.0CEIa + 12.0CEIl\right)}{8.0AGa^2 - 16.0AGal + 8.0AGl^2 + 24.0CEI}$$

$$C_y = \frac{w\left(AGa^3 + 9.0AGa^2 l - 15.0AGal^2 + 5.0AGl^3 - 12.0CEIa + 12.0CEIl\right)}{8.0AGa^2 - 16.0AGal + 8.0AGl^2 + 24.0CEI}$$

$$M_c = \frac{w\left(-AGa^4 + 4.0AGa^2 l^2 - 4.0AGal^3 + AGl^4 + 12.0CEIa^2\right)}{8.0AGa^2 - 16.0AGal + 8.0AGl^2 + 24.0CEI}$$