Smart Material Chapter 4 Solutions

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[1]:
```
# toc
import sympy as sp
import numpy as np
import matplotlib.pyplot as plt

plt.style.use('maroon_ipynb.mplstyle')
```

Contents

Problem 4.1

Compute the stress required to produce 100 microstrain in APC 856 when the applied electric field is held constant at zero Compute the stress required to produce 100 microstrain when the electric displacement is held equal to zero.

Property	Unit	Symbol	APC 856	PZT-5H	PVDF
Relative dielectric constant	unitless	ε_r	4100	3800	12–13
Curie temperature	°C	T_c	150	250	
Coupling coefficient	unitless	k_{33}	0.73	0.75	
		k_{31}	0.36		0.12
		k_{15}	0.65		
Strain coefficient	10^{-12} C/N or m/V	d_{33}	620	650	−33
		$-d_{31}$	260	320	−23
		d_{15}	710		
Elastic compliance	10^{-12} m^2/N	s_{11}^E	15	16.1	250–500
		s_{33}^E	17	20	
Density	g/cm^3	ρ	7.5	7.8	1.78

Solution

We can use the constitutive equations.

$$S = sT + dE$$

$$D = dT + \epsilon E$$

[2]:
```
S = 100e-6
s = 17e-12  # m^2/N
T = S/s
T  # Pa
```

[2]: 5882352.941176471

For finding the open circuit stress, we need the open circuit compliance.

$$s^D = s^E(1 - k^2)$$

[3]:
```
SD = s*(1 - 0.73**2)
T = S/SD
T  # Pa
```

[3]: 12593348.193484202

Problem 4.2

A new composition of piezoelectric material is found to have a compliance at zero electric field of $18.2 \mu m^2/N$, a piezoelectric strain coefficient of $330 pm/V$, and a relative permittivity of 1500.

a. Write the one-dimensional constitutive relationship for the material with strain and electric displacement as the dependent variables.
b. Write the one-dimensional constitutive relationship with stress and electric field as the dependent variables.

Solution

Part A

[4]:
```
S, D, s, d, eps, T, E = sp.symbols(r'S D s d \epsilon T E')
A = sp.Matrix([[s, d], [d, eps]])
x = sp.Matrix([T, E])
b = sp.Matrix([S, D])
eq = sp.Eq(b, sp.MatMul(A, x))
eq
```

[4]:

$$\begin{bmatrix} S \\ D \end{bmatrix} = \begin{bmatrix} s & d \\ d & \epsilon \end{bmatrix} \begin{bmatrix} T \\ E \end{bmatrix}$$

$\epsilon = \epsilon_r \cdot \epsilon_o$

[5]:
```
subs = {s: sp.S('18.2e-6'), d: sp.S('330e-12'), eps: 1500*sp.S('8.854e-12')}
eq.subs(subs)
```

[5]:

$$\begin{bmatrix} S \\ D \end{bmatrix} = \begin{bmatrix} 1.82 \cdot 10^{-5} & 3.3 \cdot 10^{-10} \\ 3.3 \cdot 10^{-10} & 1.3281 \cdot 10^{-8} \end{bmatrix} \begin{bmatrix} T \\ E \end{bmatrix}$$

Part B

[6]:
```
eq = sp.Eq(x, sp.MatMul(A.inv(), b))
eq
```

[6]:

$$\begin{bmatrix} T \\ E \end{bmatrix} = \begin{bmatrix} \frac{\epsilon}{\epsilon s - d^2} & -\frac{d}{\epsilon s - d^2} \\ -\frac{d}{\epsilon s - d^2} & \frac{s}{\epsilon s - d^2} \end{bmatrix} \begin{bmatrix} S \\ D \end{bmatrix}$$

[7]:
```
eq.subs(subs)
```

[7]:

$$\begin{bmatrix} T \\ E \end{bmatrix} = \begin{bmatrix} 54945.0796995757 & -1365.24932616971 \\ -1365.24932616971 & 75295568.8978449 \end{bmatrix} \begin{bmatrix} S \\ D \end{bmatrix}$$

Problem 4.4

The short-circuit mechanical compliance of a piezoelectric material has been measured to be 20 $\mu m^2/N$ and the open-circuit mechanical compliance has been measured to be 16.2 $\mu m^2/N$. If the stress-free relative permittivity is equal to 2800, compute the relative permittivity of the material when the strain is constrained to be zero.

Solution

We can use the equation for the open circuit compliance to find the piezoelectric coupling coefficient, and apply that to the no strain permittivity relationship.

$$s^D = s^E(1 - k^2)$$
$$\epsilon^S = \epsilon^T(1 - k^2)$$

[8]:
```
sD, sE, epsT = 16.2e-6, 20e-6, 2800
k = np.sqrt(1 - sD/sE)
k
```

[8]: 0.43588989435406744

[9]:
```
epsS = epsT*(1 - k**2)
epsS
```

[9]: 2268.0

Problem 4.5

Compute the sensitivity between strain and electric displacement for a piezoelectric material whose material parameters are $s^E = 16 \mu m^2/N$ and $d = 220 pm/V$, with a relative permittivity of 1800. Assume that the signal conditioning circuit for the sensor maintains a zero electric field.

Solution

The sensitivity is going to be defined as the ratio between electric displacement and strain. With the zero electric field,

$$S = s^E T$$

$$D = dT$$

These equations can be re-arranged to find the sensitivity.

$$\frac{D}{S} = \frac{d}{s^E}$$

[10]:
```
sE, d, eps = 16e-12, 220e-12, 1800
d/sE  # C/m^2
```

[10]:
```
13.75
```

Problem 4.6

Beginning with equation 4.55, write the constitutive relationships for a piezoelectric material with an electric field applied in the direction of polarization (the 3 direction). Assume that the applied stress is zero except for the 1 and 2 material directions.

Solution

Equation 4.55 is

$$\begin{Bmatrix} \mathrm{S}_1 \\ \mathrm{S}_2 \\ \mathrm{S}_3 \\ \mathrm{S}_4 \\ \mathrm{S}_5 \\ \mathrm{S}_6 \end{Bmatrix} = \begin{bmatrix} \frac{1}{Y_1^{\mathrm{E}}} & -\frac{\nu_{12}}{Y_1^{\mathrm{E}}} & -\frac{\nu_{13}}{Y_1^{\mathrm{E}}} & 0 & 0 & 0 \\ -\frac{\nu_{12}}{Y_1^{\mathrm{E}}} & \frac{1}{Y_1^{\mathrm{E}}} & -\frac{\nu_{23}}{Y_1^{\mathrm{E}}} & 0 & 0 & 0 \\ -\frac{\nu_{31}}{Y_3^{\mathrm{E}}} & -\frac{\nu_{32}}{Y_3^{\mathrm{E}}} & \frac{1}{Y_3^{\mathrm{E}}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{23}^{\mathrm{E}}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{13}^{\mathrm{E}}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{12}^{\mathrm{E}}} \end{bmatrix} \begin{Bmatrix} \mathrm{T}_1 \\ \mathrm{T}_2 \\ \mathrm{T}_3 \\ \mathrm{T}_4 \\ \mathrm{T}_5 \\ \mathrm{T}_6 \end{Bmatrix} + \begin{bmatrix} 0 & 0 & d_{13} \\ 0 & 0 & d_{23} \\ 0 & 0 & d_{33} \\ 0 & d_{24} & 0 \\ d_{15} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \mathrm{E}_1 \\ \mathrm{E}_2 \\ \mathrm{E}_3 \end{Bmatrix}$$

$$\begin{Bmatrix} \mathrm{D}_1 \\ \mathrm{D}_2 \\ \mathrm{D}_3 \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & d_{15} & 0 \\ 0 & 0 & 0 & d_{24} & 0 & 0 \\ d_{13} & d_{23} & d_{33} & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \mathrm{T}_1 \\ \mathrm{T}_2 \\ \mathrm{T}_3 \\ \mathrm{T}_4 \\ \mathrm{T}_5 \\ \mathrm{T}_6 \end{Bmatrix} + \begin{bmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{22} & 0 \\ 0 & 0 & \varepsilon_{33} \end{bmatrix} \begin{Bmatrix} \mathrm{E}_1 \\ \mathrm{E}_2 \\ \mathrm{E}_3 \end{Bmatrix}$$

[11]:
```python
S1, S2, S3, S4, S5, S6 = sp.symbols(r'S1:7')
T1, T2, T3, T4, T5, T6 = sp.symbols(r'T1:7')
E1, E2, E3 = sp.symbols(r'E1:4')
D1, D2, D3 = sp.symbols(r'D1 D2 D3')
eps11, eps22, eps33 = sp.symbols(r'\varepsilon_{11} \varepsilon_{22} \varepsilon_{33}')
d13, d23, d33, d24, d15 = sp.symbols(r'd13 d23 d33 d24 d15')
Y1E, Y3E, G23E, G13E, G12E = sp.symbols(r'Y_1^E Y_3^E G_{23}^E G_{13}^E G_{12}^E')
nu12, nu13, nu23, nu31, nu32 = sp.symbols(r'\nu_{12} \nu_{13} \nu_{23} \nu_{31} \nu_{32}')

sE = sp.Matrix([
    [1/Y1E, -nu12/Y1E, -nu13/Y1E, 0, 0, 0],
    [-nu12/Y1E, 1/Y1E, -nu23/Y1E, 0, 0, 0],
    [-nu31/Y3E, -nu32/Y3E, 1/Y3E, 0, 0, 0],
    [0, 0, 0, 1/G23E, 0, 0],
    [0, 0, 0, 0, 1/G13E, 0],
    [0, 0, 0, 0, 0, 1/G12E]
])
S = sp.Matrix([S1, S2, S3, S4, S5, S6])
```

```
T = sp.Matrix([T1, T2, T3, T4, T5, T6])
d = sp.Matrix([
    [0, 0, d13],
    [0, 0, d23],
    [0, 0, d33],
    [0, d24, 0],
    [d15, 0, 0],
    [0, 0, 0]
])
E = sp.Matrix([E1, E2, E3])

D = sp.Matrix([D1, D2, D3])
eps = sp.Matrix([
    [eps11, 0, 0],
    [0, eps22, 0],
    [0, 0, eps33]
])

eq1 = sp.Eq(S, sp.Add(sp.MatMul(sE, T), sp.MatMul(d, E)))
eq2 = sp.Eq(D, sp.Add(sp.MatMul(d.transpose(), T), sp.MatMul(eps, E)))
display(eq1, eq2)
```

$$\begin{bmatrix}S_1\\S_2\\S_3\\S_4\\S_5\\S_6\end{bmatrix} = \begin{bmatrix}0 & 0 & d_{13}\\0 & 0 & d_{23}\\0 & 0 & d_{33}\\0 & d_{24} & 0\\d_{15} & 0 & 0\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}E_1\\E_2\\E_3\end{bmatrix} + \begin{bmatrix}\frac{1}{Y_1^E} & -\frac{\nu_{12}}{Y_1^E} & -\frac{\nu_{13}}{Y_1^E} & 0 & 0 & 0\\-\frac{\nu_{12}}{Y_1^E} & \frac{1}{Y_1^E} & -\frac{\nu_{23}}{Y_1^E} & 0 & 0 & 0\\-\frac{\nu_{31}}{Y_3^E} & -\frac{\nu_{32}}{Y_3^E} & \frac{1}{Y_3^E} & 0 & 0 & 0\\0 & 0 & 0 & \frac{1}{G_{23}^E} & 0 & 0\\0 & 0 & 0 & 0 & \frac{1}{G_{13}^E} & 0\\0 & 0 & 0 & 0 & 0 & \frac{1}{G_{12}^E}\end{bmatrix}\begin{bmatrix}T_1\\T_2\\T_3\\T_4\\T_5\\T_6\end{bmatrix}$$

$$\begin{bmatrix}D_1\\D_2\\D_3\end{bmatrix} = \begin{bmatrix}\varepsilon_{11} & 0 & 0\\0 & \varepsilon_{22} & 0\\0 & 0 & \varepsilon_{33}\end{bmatrix}\begin{bmatrix}E_1\\E_2\\E_3\end{bmatrix} + \begin{bmatrix}0 & 0 & 0 & 0 & d_{15} & 0\\0 & 0 & 0 & d_{24} & 0 & 0\\d_{13} & d_{23} & d_{33} & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}T_1\\T_2\\T_3\\T_4\\T_5\\T_6\end{bmatrix}$$

[12]:
```
subs = {T3: 0, T4: 0, T5: 0, T6: 0, E1: 0, E2: 0}
eq1 = eq1.subs(subs)
eq2 = eq2.subs(subs)
display(eq1, eq2)
```

$$\begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & d_{13} \\ 0 & 0 & d_{23} \\ 0 & 0 & d_{33} \\ 0 & d_{24} & 0 \\ d_{15} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ E_3 \end{bmatrix} + \begin{bmatrix} \frac{1}{Y_1^E} & -\frac{\nu_{12}}{Y_1^E} & -\frac{\nu_{13}}{Y_1^E} & 0 & 0 & 0 \\ -\frac{\nu_{12}}{Y_1^E} & \frac{1}{Y_1^E} & -\frac{\nu_{23}}{Y_1^E} & 0 & 0 & 0 \\ -\frac{\nu_{31}}{Y_3^E} & -\frac{\nu_{32}}{Y_3^E} & \frac{1}{Y_3^E} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{23}^E} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{13}^E} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{12}^E} \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix} = \begin{bmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{22} & 0 \\ 0 & 0 & \varepsilon_{33} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ E_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & d_{15} & 0 \\ 0 & 0 & 0 & d_{24} & 0 & 0 \\ d_{13} & d_{23} & d_{33} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

[13]:

```
display(eq1.doit(), eq2.doit())
```

$$\begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \end{bmatrix} = \begin{bmatrix} E_3 d_{13} + \frac{T_1}{Y_1^E} - \frac{T_2 \nu_{12}}{Y_1^E} \\ E_3 d_{23} - \frac{T_1 \nu_{12}}{Y_1^E} + \frac{T_2}{Y_1^E} \\ E_3 d_{33} - \frac{T_1 \nu_{31}}{Y_3^E} - \frac{T_2 \nu_{32}}{Y_3^E} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ E_3 \varepsilon_{33} + T_1 d_{13} + T_2 d_{23} \end{bmatrix}$$

So, the final equations would be,

[14]:

```
for eq in [eq1, eq2]:
    for i, rhs in enumerate(eq.rhs.doit()):
        if rhs != 0:
            display(sp.Eq(list(eq.lhs)[i], rhs))
```

$$S_1 = E_3 d_{13} + \frac{T_1}{Y_1^E} - \frac{T_2 \nu_{12}}{Y_1^E}$$

$$S_2 = E_3 d_{23} - \frac{T_1 \nu_{12}}{Y_1^E} + \frac{T_2}{Y_1^E}$$

$$S_3 = E_3 d_{33} - \frac{T_1 \nu_{31}}{Y_3^E} - \frac{T_2 \nu_{32}}{Y_3^E}$$

$$D_3 = E_3 \varepsilon_{33} + T_1 d_{13} + T_2 d_{23}$$

Problem 4.9

A piezoelectric material operating in the 33 mode has the material properties $d_{33} = 450pm/V$ and $Y_3^E = 63GPa$.

a. Compute the blocked stress and free strain under the application of an electric field of $0.75MV/m$.
b. Compute the voltage required to achieve this blocked stress or free displacement for a wafer that is $250\mu m$ thick.

Solution

Part A

The blocking stress as a result of zeroing displacement is

$$T_{bl} = d_{33}Y_3^E E_3$$

The free strain as a result of zeroing stress is

$$S_{fr} = d_{33}E_3$$

[15]:
```python
d33, Y3E, E3 = 450e-12, 63e9, 0.75e6
T_bl = d33*Y3E*E3
T_bl  # Pa
```

[15]: 21262500.0

[16]:
```python
S_fr = d33*E3
S_fr
```

[16]: 0.0003375

Part B

[17]:
```python
# Voltage = Electric field * thickness
V = E3*250e-6
V  # V
```

[17]: 187.5

Problem 4.10

		PZT-A	PZT-B	PZT-C	PZT-D	PZT-E
ε_{33}		1400	1400	1100	5440	1800
ε_{11}		1350	1300	1400	5000	2000
s_{11}^{E}	(μm^2/N)	12.7	13.1	11.2	14.8	16.5
s_{33}^{E}	(μm^2/N)	15.4	15.6	15.2	18.1	19.9
d_{13}	(pm/V)	-133	-132	-99	-287	-198
d_{33}	(pm/V)	302	296	226	635	417

For each material above,

a. Plot the blocked stress and free strain on a single plot.
b. Compute the volumetric energy density of each type of material for operation in the 33 mode for an electric field of $1MV/m$.

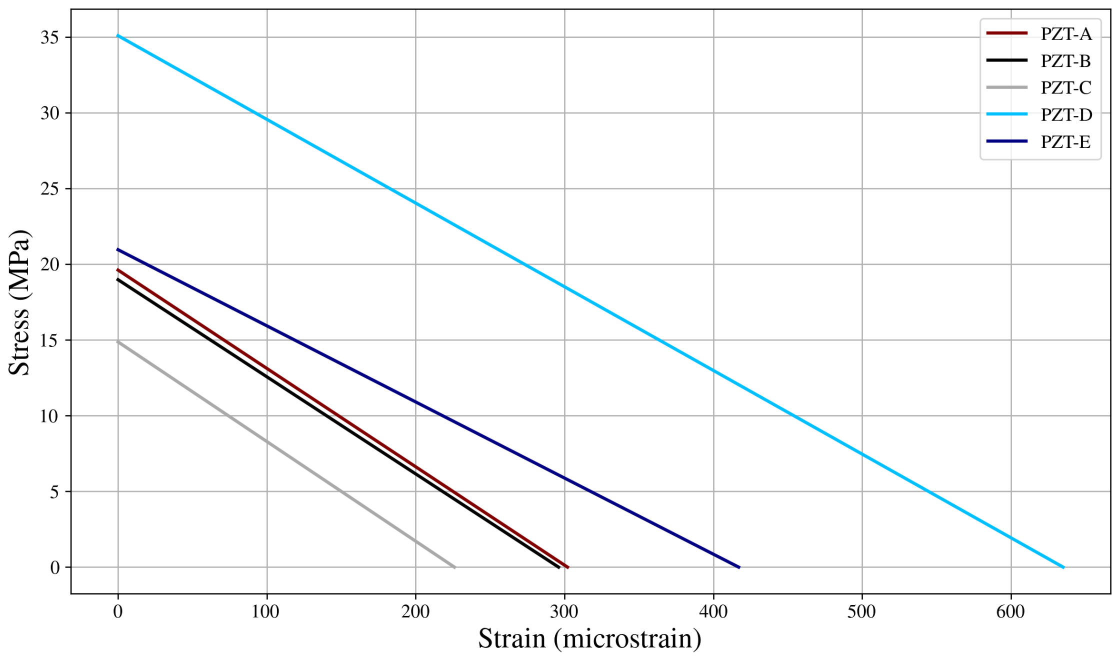
Solution

Part A

[18]:
```
d33 = np.array([302, 296, 226, 635, 417])*1e-12
s33 = np.array([15.4, 15.6, 15.2, 18.1, 19.9])*1e-12
materials = ['PZT-A', 'PZT-B', 'PZT-C', 'PZT-D', 'PZT-E']
Y33 = 1/s33  # Pa

fig, ax = plt.subplots()
E = 1e6  # V/m
for d, Y, mat in zip(d33, Y33, materials):
    T = d*Y*E*1e-6  # MPa
    S = d*E*1e6  # microstrain
    ax.plot([0, S], [T, 0], label=mat)
    print(mat)
    print('Blocking Stress:', T, 'MPa')
    print('Free Strain:', S, 'microstrain')
    print()
ax.legend()
ax.set_xlabel('Strain (microstrain)')
ax.set_ylabel('Stress (MPa)')
plt.show()
```

```
PZT-A
Blocking Stress: 19.610389610389607 MPa
Free Strain: 301.99999999999994 microstrain

PZT-B
Blocking Stress: 18.974358974358974 MPa
Free Strain: 296.0 microstrain

PZT-C
```

```
Blocking Stress: 14.868421052631579 MPa
Free Strain: 226.0 microstrain

PZT-D
Blocking Stress: 35.0828729281768 MPa
Free Strain: 634.9999999999999 microstrain

PZT-E
Blocking Stress: 20.954773869346738 MPa
Free Strain: 417.0 microstrain
```



Part B

The volumetric energy density is given by

$$E_v = \frac{1}{2} Y_3^E d_{33}^2 E_3^2$$

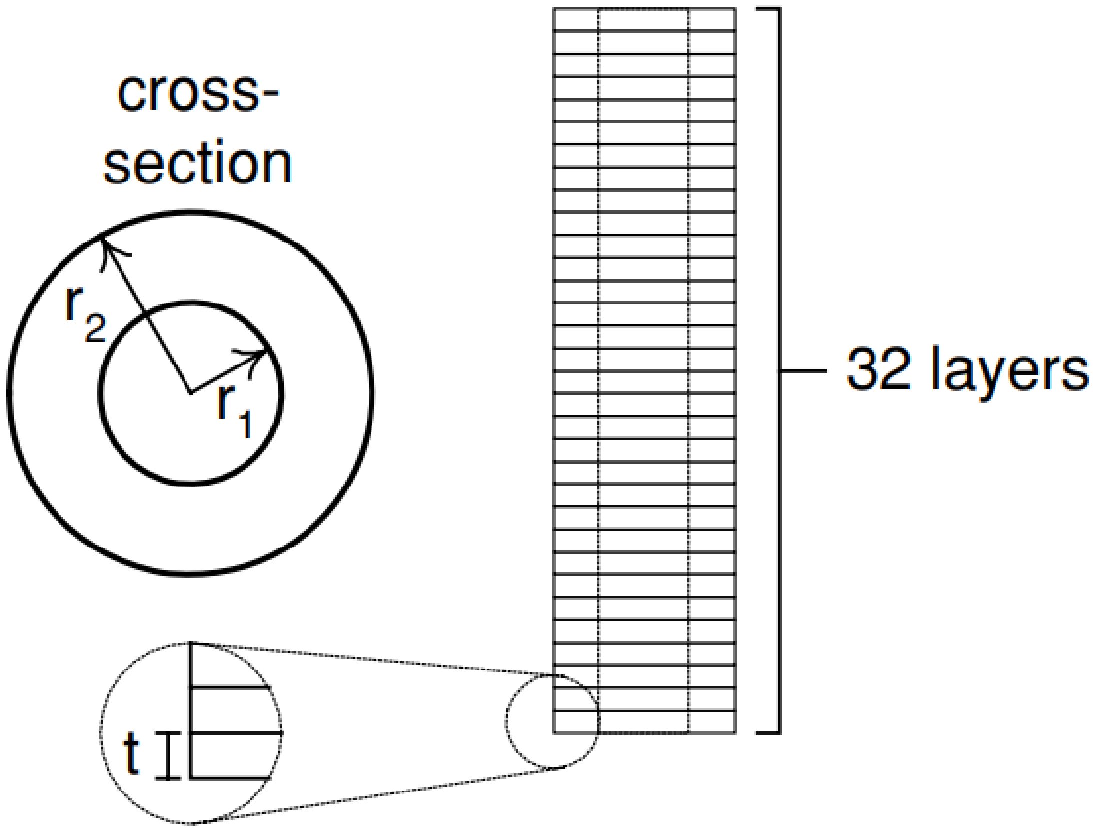
This is the area under the line in the graph above. The $\frac{1}{2} Y_3^E d_{33}^2$ term is a good metric for showing that a PZT material can provide more mechanical work for a given electric field.

[19]:
```
E_v = 0.5*Y33*d33**2*E**2
E_v/1000  # kJ/m^3
```

[19]:
```
array([ 2.96116883,  2.80820513,  1.68013158, 11.13881215,  4.36907035])
```

Note that the soft PZT material exhibits the highest volumetric energy density.

Problem 4.11



The geometry of a piezoelectric stack with annular cross section is shown in the figure above. Compute the stiffness, free displacement, and blocked force of the stack assuming that $r_1 = 5\ mm$, $r_2 = 15\ mm$, $t = 0.25\ mm$, and the applied voltage is 150 V. Assume the material properties of PZT-5H.

Solution

The stiffness of the stack is given by

$$k_s^E = \frac{Y_3^E A}{n t_p}$$

[20]:
```
Y = 1/20e-12  # Pa
A = np.pi*(0.015**2 - 0.005**2)  # m^2
n, tp = 32, 0.25e-3  # number of layers, m
k = Y*A/(n*tp)
k/1e6  # N/um
```

[20]:
```
3926.9908169872415
```

The free displacement is given by

$$\delta_o = n d_{33} v$$

[21]:
```
d33 = 650e-12  # m/V
delta = n*d33*150
delta*1e6  # um
```

[21]: 3.12

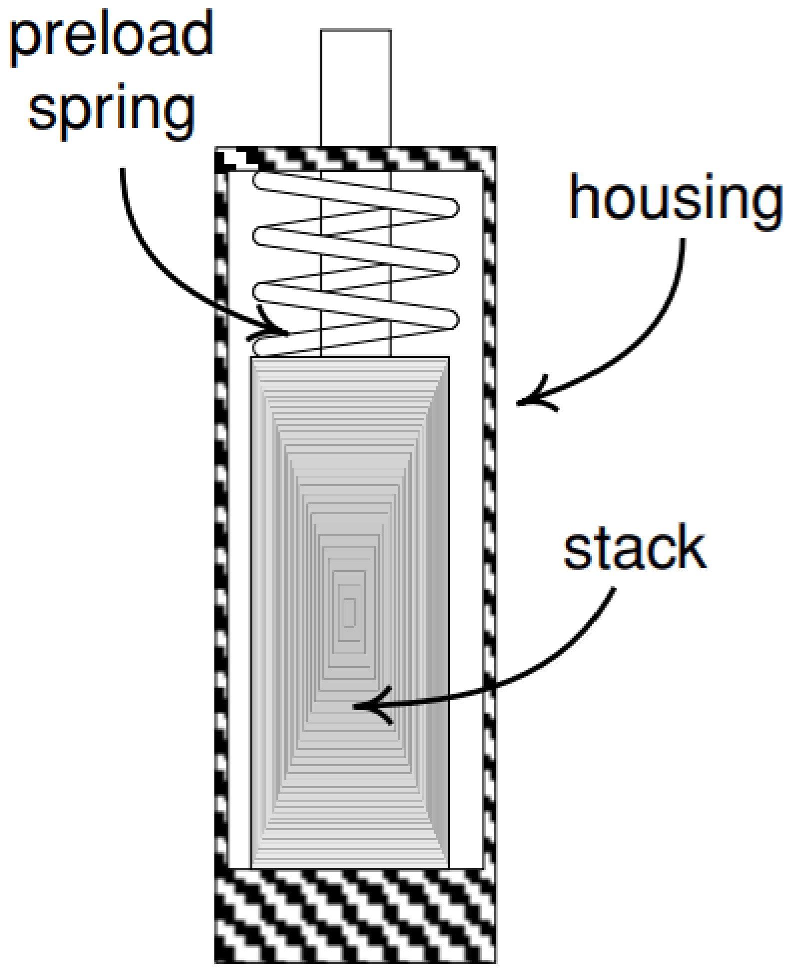
The blocked force is given by

$$f_{bl} = d_{33} Y_3^E \frac{A_p}{t_p} v$$

[22]:
```
f_bl = d33*Y*A/tp*150
f_bl  # N
```

[22]: 12252.211349000192

Problem 4.13



Piezoelectric stack actuators are often preloaded to reduce the risk of placing the brittle ceramic in tension. A piezoelectric stack with a free displacement of 30 μm and a stiffness of 40 $N/\mu m$ is placed in a housing as shown in the figure above.

a. Compute the stiffness of the preload spring such that the output displacement is 20 μm.
b. Compute the output force of the preloaded stack.

Solution

Part A

From equation 4.86,

$$u = \frac{\delta_o}{1 + k_l/k_s^E}$$

[23]:
```python
delta_o, kl, ks, u = sp.symbols(r'\delta_o k_l k_s u')
eq = sp.Eq(u, delta_o/(1 + kl/ks))
eq
```

[23]:

$$u = \frac{\delta_o}{\frac{k_l}{k_s} + 1}$$

[24]:
```python
eq_subs = eq.subs({delta_o: 30e-6, ks: 40e6, u: 20e-6})
eq_subs
```

[24]:

$$2.0 \cdot 10^{-5} = \frac{3.0 \cdot 10^{-5}}{2.5 \cdot 10^{-8} k_l + 1}$$

[25]:
```python
kl_solved = sp.solve(eq_subs, kl)[0]
kl_solved/1e6  # N/um
```

[25]:

20.0

Part B

[26]:
```python
f = kl_solved*20e-6
f  # N
```

[26]:

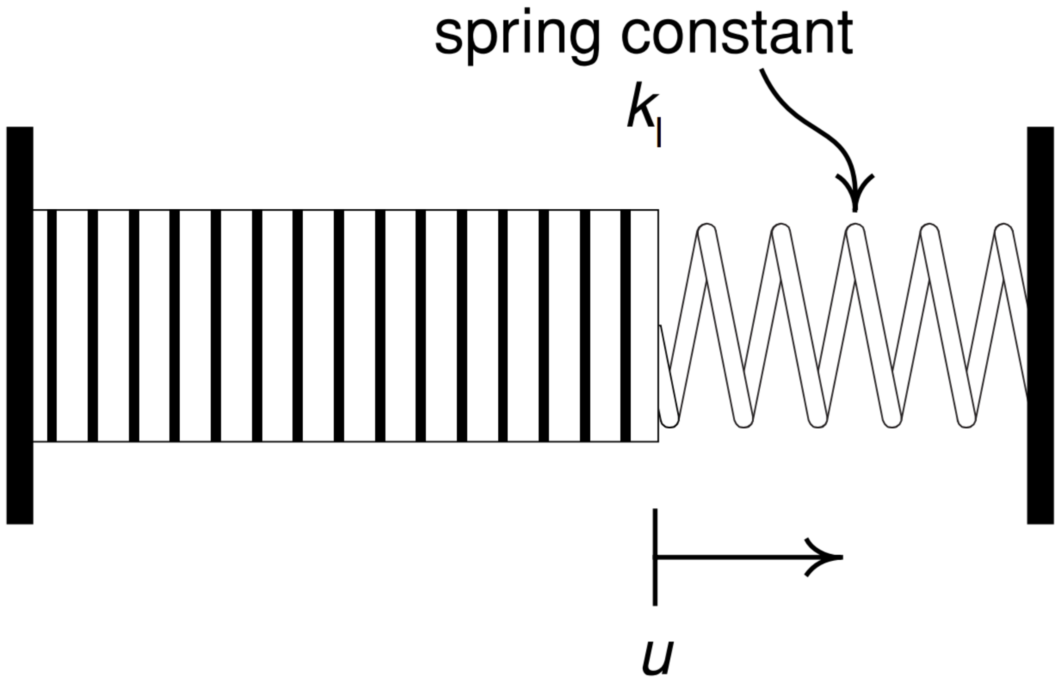
400.0

Problem 4.14

A piezoelectric stack with a free displacement of 50 μm and a blocking force of 750 N is being used for a static positioning application. The load is modeled as a linear elastic spring.

a. Determine the load stiffness that will maximize the work performed by the stack.
b. Compute the maximum work output of the stack.

Solution



Applying the constitutive equations and the conditions in the above image, we have

$$u = \frac{nt_p}{Y_3^E A_p} f + nd_{33}V$$

$$k_s^E = \frac{Y_3^E A_p}{nt_p}$$

$$\delta_o = nd_{33}V$$

With these equations defined above, we can simplify the model to the following equation

$$u = \frac{1}{k_s^E} f + \delta_o$$

Noting that work is the product of force and displacement, we can determine the work as a function of the load stiffness and displacement.

[27]:
```
W, f, kl, ks, delta_o, u = sp.symbols(r'W f k_l k_s \delta_o u')
eq1 = sp.Eq(u, 1/ks*f + delta_o)
eq2 = eq1.subs([
    (u, -W/f),  # work on the stack from the load is negative
    (f, -kl*u)
])
display(eq1, eq2)
```

$u = \delta_o + \frac{f}{k_s}$

$\frac{W}{k_l u} = \delta_o - \frac{k_l u}{k_s}$

[28]:
```
W_sol = sp.solve(eq2, W)[0]
W_sol.expand()  # Work as a function of displacement and load stiffness
```

[28]:

$\delta_o k_l u - \frac{k_l^2 u^2}{k_s}$

The work above can be maximized by finding the critical points of the multi-variable function.

[29]:
```
dW_dkl = W_sol.diff(kl).expand()
dW_du = W_sol.diff(u).expand()
display(dW_dkl, dW_du)
```

$\delta_o u - \frac{2k_l u^2}{k_s}$

$\delta_o k_l - \frac{2k_l^2 u}{k_s}$

[30]:
```
critical_points = sp.solve([dW_dkl, dW_du], [kl, u], dict=True)
for sol in critical_points:
    for key, value in sol.items():
        display(sp.Eq(key, value))
```

$u = \frac{\delta_o k_s}{2k_l}$

$k_l = 0$

$u = 0$

Part A

In the above analysis, it shows that work is maximized when we have

$$u = \frac{\delta_o k_s}{2k_l}$$

If we assume that the displacement is one half the free displacement, then the load stiffness is equal to the stiffness of the stack.

[31]:
```python
# Finding the stack stiffness
ks_ = 750/50e-6
ks_  # N/m
```

[31]: 15000000.0

Therefore, the load stiffness will be taken to be 15,000,000 N/m.

Part B

The maximum work is found by substituting in the critical point value into the work equation.

[32]:
```python
W_max = W_sol.subs(u, critical_points[0][u])
W_max
```

[32]: $\frac{\delta_o^2 k_s}{4}$

[33]:
```python
W_max.subs([
    (ks, ks_),
    (delta_o, 50e-6)
])  # J
```

[33]: 0.009375

Note that $W_{max} = \frac{f_{bl}\delta_o}{4}$ is also true.