Vibrations and Controls Homework 4

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[1]:
```python
import sympy as sp
import matplotlib.pyplot as plt
from IPython.display import display, Latex
from scipy.integrate import odeint
import numpy as np

plt.style.use('maroon.mplstyle')


def display_latex(text):
    if isinstance(text, list):
        for thing in text:
            display(Latex(f'${sp.latex(thing)}$'))
    else:
        display(Latex(f'${sp.latex(text)}$'))


t, s = sp.symbols('t s')
```

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1 Problem 8.27

1.1 Given

A certain system has two coupled subsystems. One subsystem is a rotational system with the equation of motion:

$50\dot{\omega} + 10\omega = T(t)$

where $T(t)$ is the torque applied by an electric motor. The second subsystem is field-controlled motor. The model of the motor's field current, i_f in amperes is:

$0.001\dot{i_f} + 5i_f = v(t)$

where $v(t)$ is the voltage applied to the motor. The motor torque constant is $K_T = 25\,N \cdot m/A$.

1.2 Find

Obtain the damping ratio ζ, time constants, and undamped natural frequency ω_n of the combined system.

1.3 Solution

The system of differential equations can be related to each other from the following,

$T(t) = K_T i_f$

This leaves us with a system of differential equations with only one input ($v(t)$). We need to obtain the transfer function and analyze the characteristic equation.

$$\begin{cases} 50\dot{\omega} + 10\omega = 25i_f \\ 0.001\dot{i_f} + 5i_f = v(t) \end{cases}$$

[2]:
```
# Define symbols and put system in s domain
I_f, W, V = sp.Function('I_f')(s), sp.Function(r'\omega')(s), sp.
↪Function('V')(s)

eq1 = sp.Eq(50*s*W + 10*W, 25*I_f)
eq2 = sp.Eq(0.001*s*I_f + 5*I_f, V)

display_latex([eq1, eq2])
```

$50s\omega(s) + 10\omega(s) = 25\,\mathrm{I_f}(s)$

$0.001s\,\mathrm{I_f}(s) + 5\,\mathrm{I_f}(s) = V(s)$

[3]:
```
# Solve the system
solved = sp.solve([eq1, eq2], (W, I_f), dict=True)[0]
solved_list = [sp.Eq(key, value) for key, value in solved.items()]
display_latex(solved_list)
```

$\omega(s) = \frac{2500.0V(s)}{5.0s^2+25001.0s+5000.0}$

$\mathrm{I_f}(s) = \frac{1000.0V(s)}{s+5000.0}$

[4]:
```
# Grab the characteristic polynomial
poly = 1/solved[W]*2500*V
poly
```

[4]:

$5.0s^2 + 25001.0s + 5000.0$

The system is stable because all the signs of the equation are the same.

[5]:
```
# Find the roots
roots = list(sp.roots(poly))
display_latex(roots)
```

-5000.0

-0.2

There is no imaginary part of either root indicating that the system is overdamped. This means that the damping ratio should be greater than 1.

[6]:
```
# Calculating the undamped natural frequency (shaft equation)
sp.sqrt(5000/5)
```

[6]:

31.6227766016838

[7]:
```
# Calculating the damping ratio (shaft equation)
(25001/(2*sp.sqrt(5*5000))).n()
```

[7]:

79.0601037818697

[8]:
```
# The time constants are the negative of the reciprocal of the roots because␣
↪the damping ratio is greater than 1
# This means that we cannot do the 1/(zeta*undamped frequency)
display_latex([-1/root for root in roots])
```

0.0002

5.0

1.4 Answer

$\omega_n = 31.6\,\frac{rad}{s}$, $\zeta = 79.1$, and $\tau_1 = 5\,s;\ \tau_2 = 0.0002\,s$

2 Problem 8.29

2.1 Given

$\ddot{x} + 4\dot{x} + 8x = 2u_s(t)$

2.2 Find

Compute the maximum percent overshoot, the maximum overshoot, the peak time, the 100% rise time, the delay time, and the 2% settling time for the following model. The initial conditions are zero. Time is measured in seconds.

2.3 Solution

Check to see if the solution is underdamped before proceding.

[9]:
```
display_latex(list(sp.roots(t**2 + 4*t + 8)))
```

$-2 - 2i$

$-2 + 2i$

The system is underdamped because the roots have imaginary components.

[10]:
```
# Define the equations
M_percent, M_p, t_p, t_r, t_d, t_s = sp.symbols(r'M_{\%} M_p t_p t_r t_d t_s')
zeta, phi, w_n, k = sp.symbols(r'\zeta \phi \omega_n k')

equations = [
    sp.Eq(M_percent, 100*sp.E**(-sp.pi*zeta/sp.sqrt(1 - zeta**2))),
    sp.Eq(M_p, 1/k*sp.E**(-sp.pi*zeta/sp.sqrt(1 - zeta**2))),
    sp.Eq(t_p, sp.pi/(w_n*sp.sqrt(1 - zeta**2))),
    sp.Eq(t_r, (2*sp.pi - phi)/(w_n*sp.sqrt(1 - zeta**2))),
    sp.Eq(t_d, (1 + 0.7*zeta)/w_n),
    sp.Eq(t_s, 4/(zeta*w_n)),
    sp.Eq(phi, sp.atan(sp.sqrt(1 - zeta**2)/zeta) + sp.pi)
]
display_latex(equations)
```

$M_{\%} = 100e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}$

$M_p = \frac{e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}}{k}$

$t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$

$t_r = \frac{-\phi+2\pi}{\omega_n\sqrt{1-\zeta^2}}$

$t_d = \frac{0.7\zeta+1}{\omega_n}$

$t_s = \frac{4}{\omega_n \zeta}$

$\phi = \operatorname{atan}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right) + \pi$

[11]:
```python
# Write a quick algorithm for substituting in values
m, c, k_ = 1/2, 2, 4  # The formulas are based on the forced response being the
↪unit step function alone
zeta_ = c/(2*sp.sqrt(m*k_))
w_n_ = sp.sqrt(k_/m)
phi_ = sp.atan(sp.sqrt(1 - zeta_**2)/zeta_) + sp.pi

eval_equations = []
for eq in equations:
    expr = eq.rhs
    if zeta in expr.free_symbols:
        expr = expr.subs(zeta, zeta_)

    if w_n in expr.free_symbols:
        expr = expr.subs(w_n, w_n_)

    if phi in expr.free_symbols:
        expr = expr.subs(phi, phi_)

    if k in expr.free_symbols:
        expr = expr.subs(k, k_)

    eval_equations.append(sp.Eq(eq.lhs, expr.n()))

display_latex(eval_equations)
```

$M_{\%} = 4.32139182637723$

$M_p = 0.0108034795659431$

$t_p = 1.5707963267949$

$t_r = 1.17809724509617$

$t_d = 0.528553390593274$

$t_s = 2.0$

$\phi = 3.92699081698724$

2.4 Answer

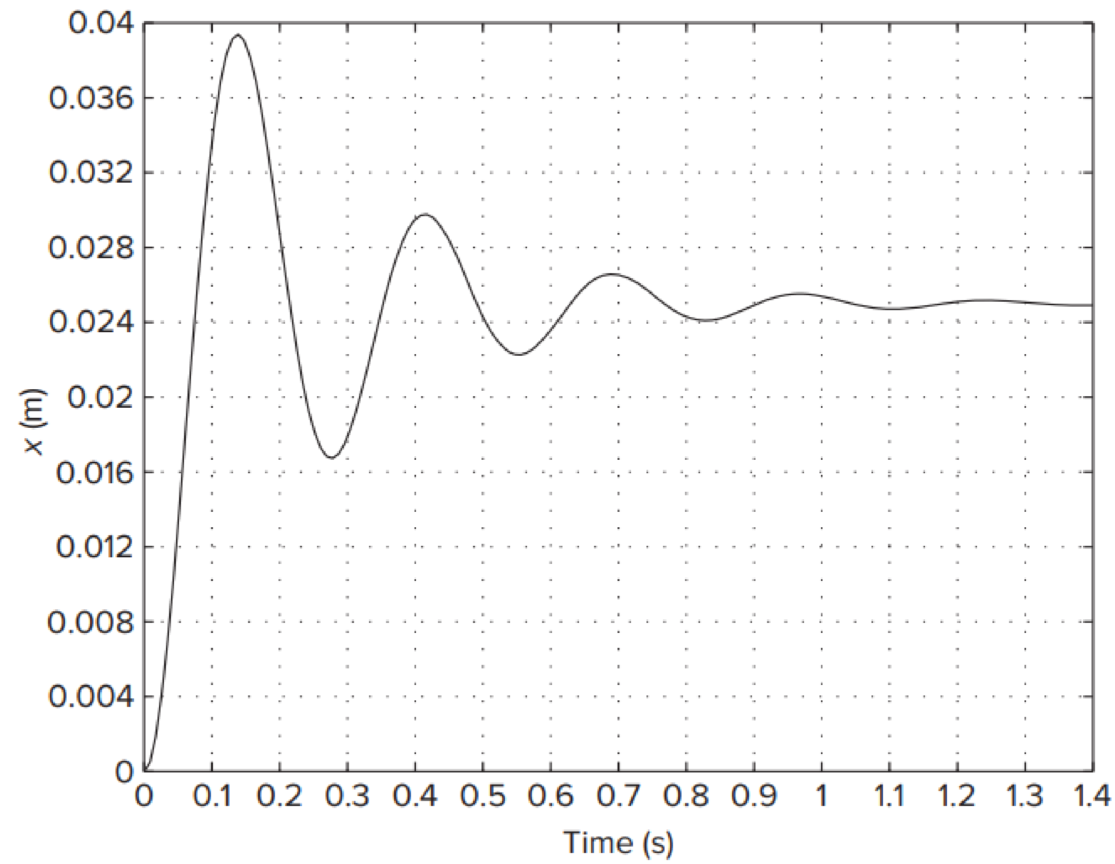
See above cell

3 Problem 8.35

3.1 Given

The figure below shows the response of a system to a step input of magnitude 1000 N. The equation of motion is,

$m\ddot{x} + c\dot{x} + kx = f(t)$



3.2 Find

Estimate the values of m, c, and k.

3.3 Solution

The following values may be obtained by looking at the figure,

$x_{ss} = 0.025\,m$
$t_p = 0.125\,s$
$x_{max} = 0.0395\,m$

And the steady state force is,

$f_{ss} = 1000\,N$

At steady state, the system is static. That means we can directly apply Hooke's Law to obtain the stiffness:

$k = \frac{f_{ss}}{x_{ss}} = 40,000\,\frac{N}{m}$

The maximum percent overshoot is defined as,

$M_\% = \frac{x_{max} - x_{ss}}{x_{ss}}(100) = 58\%$
$R = \ln(\frac{100}{M_\%}) = 0.545$

The damping ratio is,

$\zeta = \frac{R}{\sqrt{\pi^2 + R^2}} = 0.171$

The undamped natural frequency is,

$\omega_n = \frac{\pi}{t_p\sqrt{1-\zeta^2}} = 25.51\,\frac{rad}{s}$

The value of m,

$m = \frac{k}{\omega_n^2} = 61.48\,kg$

The value of c,

$c = 2\zeta\sqrt{mk} = 535.8$

3.4 Answer

$m = 61.48\,kg$, $c = 535.8\,\frac{N\,s}{m}$, and $k = 40,000\,\frac{N}{m}$

3.5 Verification

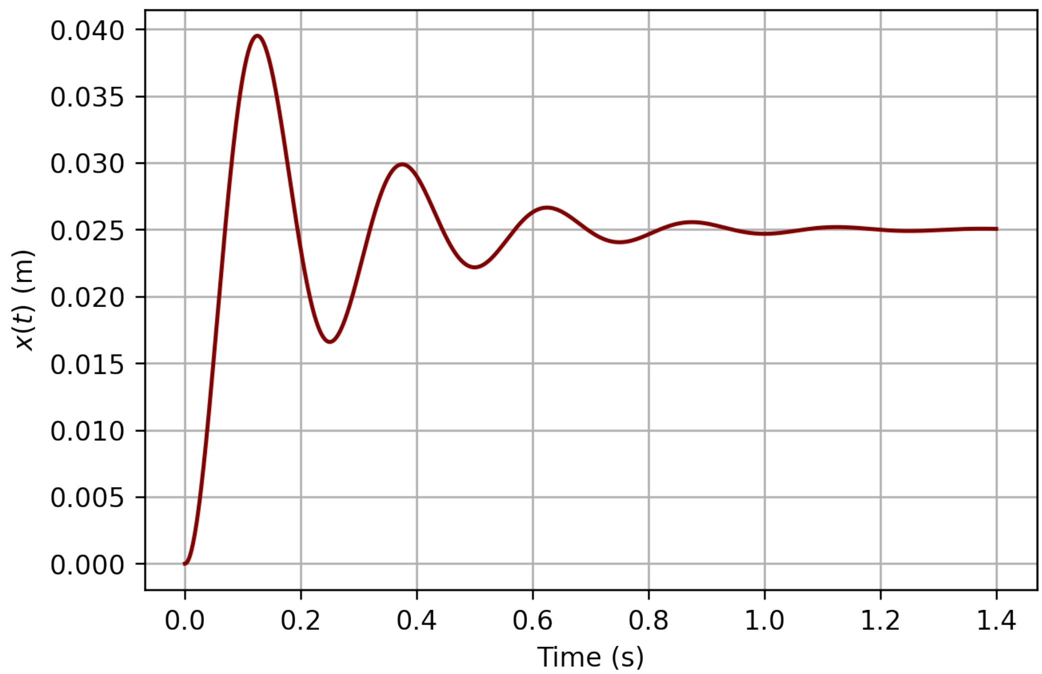
Here is a quick numerical solution.

[12]:
```
m, c, k = 61.48, 535.8, 40_000

def diffs(x, _):
    return [
        x[1],
        (1000 - c*x[1] - k*x[0])/m
    ]

time = np.linspace(0, 1.4, 1000)
solution = odeint(diffs, [0, 0], time)

plt.plot(time, solution[:, 0])
plt.xlabel('Time (s)')
plt.ylabel('$x(t)$ (m)')
plt.show()
```


0.040
0.035
0.030
0.025
0.020
0.015
0.010
0.005
0.000
x(t) (m)
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
Time (s)