System Dynamics Homework 3

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Gabe Morris

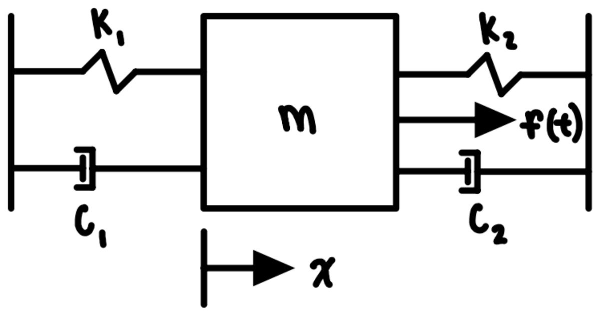
[1]:
```
import control as ct
import matplotlib.pyplot as plt
import numpy as np
import sympy as sp
import pandas as pd

plt.style.use('../maroon_ipynb.mplstyle')
```

Contents

ME 3613 Homework 3 Gabe Morris
gnm54

1 Problem 1

1.1 Given



$$m = 10\ kg$$

$$k_1 = 85\ N/m,\ k_2 = 30\ N/m$$

$$c_1 = 4\ N \cdot s/m,\ c_2 = 3\ N \cdot s/m$$

The input force $f(t)$ is in the `data.xlsx` file.

1.2 Find

Using the control package `tf()` function to get the forced response, find the following:

a. The equation of motion for the system.
b. The transfer function $X(s)/F(s)$.
c. The forced response for the no noise data. Plot the input force and the response on separate axes.
d. Repeat part c for the high frequency noise data.
e. Repeat part c for the low frequency noise data.

1.3 Solution

1.3.1 Part A

[2]:
```
t = sp.Symbol('t')
x = sp.Function('x')(t)
f = sp.Function('f')(t)

eq = sp.Eq(10*x.diff(t, 2), f - 85*x - 30*x - 4*x.diff() - 3*x.diff())
eq
```

[2]:

$$10\frac{d^2}{dt^2}x(t) = f(t) - 115x(t) - 7\frac{d}{dt}x(t)$$

1.3.2 Part B

[3]:
```
s = sp.Symbol('s')
eq_s = sp.Eq(
    sp.laplace_transform(eq.lhs, t, s)[0],
    sp.laplace_transform(eq.rhs, t, s)[0]
)
eq_s
```

[3]:

$$10s^2\mathcal{L}_t\left[x(t)\right](s) - 10sx(0) - 10\left.\frac{d}{dt}x(t)\right|_{t=0} = -7s\mathcal{L}_t\left[x(t)\right](s) + \mathcal{L}_t\left[f(t)\right](s) - 115\mathcal{L}_t\left[x(t)\right](s) + 7x(0)$$

[4]:
```
eq_s = eq_s.subs([
    (x.subs(t, 0), 0),
    (x.diff().subs(t, 0), 0)
])
eq_s
```

[4]:

$$10s^2\mathcal{L}_t\left[x(t)\right](s) = -7s\mathcal{L}_t\left[x(t)\right](s) + \mathcal{L}_t\left[f(t)\right](s) - 115\mathcal{L}_t\left[x(t)\right](s)$$

[5]:
```
X = sp.solve(eq_s, sp.laplace_transform(x, t, s)[0])[0]
X
```

[5]:

$$\frac{\mathcal{L}_t\left[f(t)\right](s)}{10s^2 + 7s + 115}$$

[6]:
```
# Transfer function
sys1 = ct.tf(1, [10, 7, 115])
sys1
```

[6]:

$$\frac{1}{10s^2 + 7s + 115}$$

1.3.3 Part C

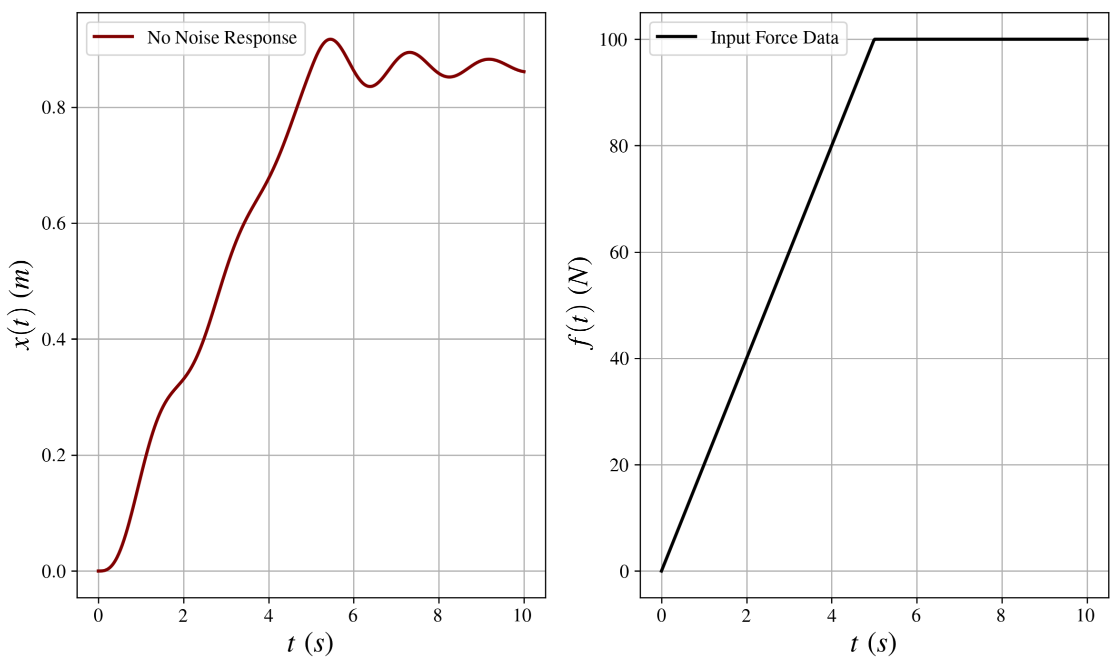
[7]:
```
data = pd.read_excel('data.xlsx', sheet_name='Input Force Data')
data
```

[7]:

	Time (s)	High (N)	Low (N)	No Noise (N)
0	0.00000	-4.849845	1.076581	0.000000
1	0.02004	-1.574263	1.076581	0.400802
2	0.04008	-2.771360	1.076581	0.801603
3	0.06012	2.071162	1.076581	1.202405
4	0.08016	3.660872	1.076581	1.603206
..	…	…	…	…
495	9.91984	95.248337	95.648604	100.000000
496	9.93988	101.726636	95.648604	100.000000

```
497   9.95992   95.527882  95.648604    100.000000
498   9.97996  102.458740  95.648604    100.000000
499  10.00000  102.326431  95.648604    100.000000

[500 rows x 4 columns]
```

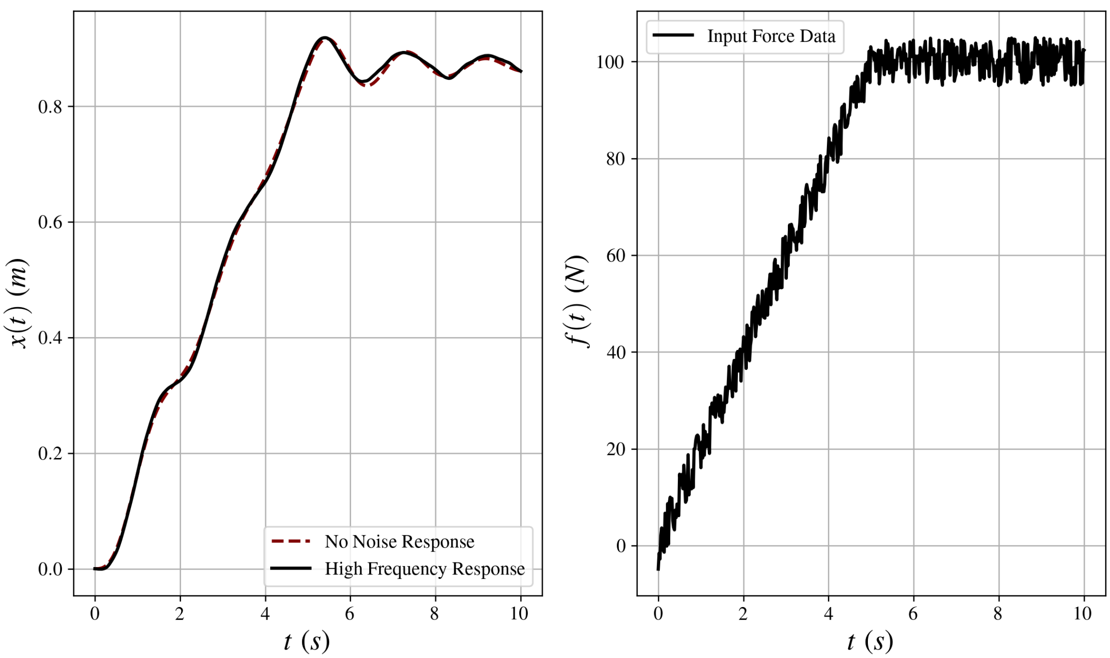
[8]:
```
t, f_high, f_low, f_none = np.array(data['Time (s)']), np.array(data['High␣
↪(N)']), np.array(data['Low (N)']), np.array(data['No Noise (N)'])
_, x_none = ct.forced_response(sys1, T=t, U=f_none)

fig, ax = plt.subplots(nrows=1, ncols=2)
ax[0].set_xlabel('$t$ ($s$)')
ax[1].set_xlabel('$t$ ($s$)')
ax[0].set_ylabel('$x(t)$ ($m$)')
ax[1].set_ylabel('$f(t)$ ($N$)')

ax[1].plot(t, f_none, label='Input Force Data', color='black')
ax[0].plot(t, x_none, label='No Noise Response')

ax[0].legend()
ax[1].legend()
plt.show()
```



1.3.4 Part D

[9]:

```
_, x_high = ct.forced_response(sys1, T=t, U=f_high)

fig, ax = plt.subplots(nrows=1, ncols=2)
ax[0].set_xlabel('$t$ ($s$)')
ax[1].set_xlabel('$t$ ($s$)')
ax[0].set_ylabel('$x(t)$ ($m$)')
ax[1].set_ylabel('$f(t)$ ($N$)')

ax[1].plot(t, f_high, label='Input Force Data', color='black')
ax[0].plot(t, x_none, label='No Noise Response', ls='--')
ax[0].plot(t, x_high, label='High Frequency Response')

ax[0].legend()
ax[1].legend()
plt.show()
```

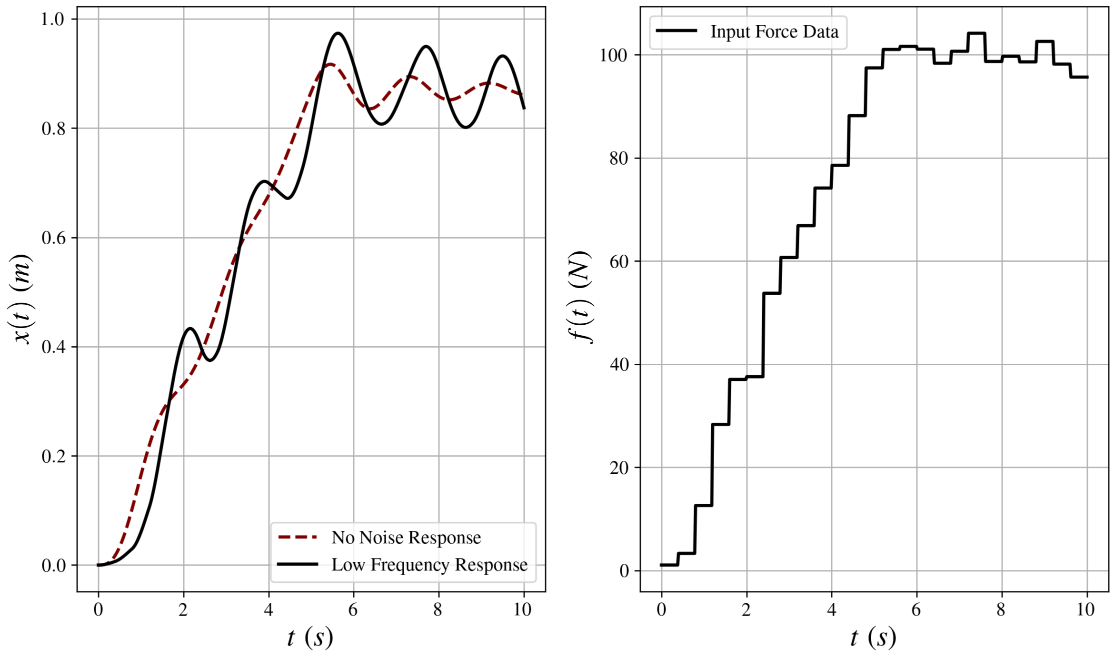


1.3.5 Part E

[10]:

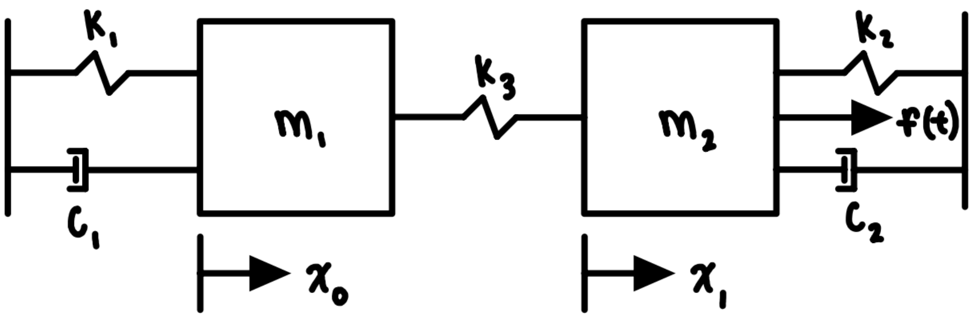
```
_, x_low = ct.forced_response(sys1, T=t, U=f_low)

fig, ax = plt.subplots(nrows=1, ncols=2)
ax[0].set_xlabel('$t$ ($s$)')
ax[1].set_xlabel('$t$ ($s$)')
ax[0].set_ylabel('$x(t)$ ($m$)')
```

```
ax[1].set_ylabel('$f(t)$ ($N$)')

ax[1].plot(t, f_low, label='Input Force Data', color='black')
ax[0].plot(t, x_none, label='No Noise Response', ls='--')
ax[0].plot(t, x_low, label='Low Frequency Response')

ax[0].legend()
ax[1].legend()
plt.show()
```



2 Problem 2

2.1 Given



$$m_1 = 5\ kg,\ m_2 = 10\ kg$$

$$k_1 = 85\ N/m,\ k_2 = 30\ N/m,\ k_3 = 500\ N/m$$

$$c_1 = 4\ N \cdot s/m,\ c_2 = 3\ N \cdot s/m$$

$$f(t) = 10^{-t}$$

The initial conditions are $x_0(0) = 0.5$, $\dot{x_0}(0) = 0$, $x_1(0) = 0$, $\dot{x_1}(0) = 0$.

2.2 Find

a. Determine the equations of motion.
b. Put the system in the state-variable form.
c. Define a state-space object using `ct.ss`.
d. Plot the response $x_0(t)$ and $x_1(t)$ (first 10 seconds) on the same plot using the `ct.forced_response` function. You may check the solution using the `odeint` function as well.

2.3 Solution

2.3.1 Part A

[11]:
```
t = sp.Symbol('t')
x0, x1 = sp.Function('x_0')(t), sp.Function('x_1')(t)

eq1 = sp.Eq(5*x0.diff(t, 2), -85*x0 - 4*x0.diff() + 500*(x1 - x0))
eq2 = sp.Eq(10*x1.diff(t, 2), 500*(x0 - x1) - 30*x1 - 3*x1.diff() + 10**(-t))
```

```
display(eq1, eq2)
```

$$5\frac{d^2}{dt^2}x_0(t) = -585x_0(t) + 500x_1(t) - 4\frac{d}{dt}x_0(t)$$

$$10\frac{d^2}{dt^2}x_1(t) = 500x_0(t) - 530x_1(t) - 3\frac{d}{dt}x_1(t) + 10^{-t}$$

2.3.2 Part B

[12]:
```
# Define new x2 and x3 functions
x2, x3 = sp.Function('x_2')(t), sp.Function('x_3')(t)

# Define two new equations
eq3 = sp.Eq(x0.diff(), x2)
eq4 = sp.Eq(x1.diff(), x3)

# Make substitutions to the first two equations
subs = [
    (x0.diff(t, 2), x2.diff()),
    (x0.diff(), x2),
    (x1.diff(t, 2), x3.diff()),
    (x1.diff(), x3)
]
eq1 = eq1.subs(subs)
eq2 = eq2.subs(subs)

# Solve it
state_sol = sp.solve([eq1, eq2, eq3, eq4], (x0.diff(), x1.diff(), x2.diff(), x3.
  ↪diff()), dict=True)[0]
for key, value in state_sol.items():
    display(sp.Eq(key, value))
```

$$\frac{d}{dt}x_0(t) = x_2(t)$$

$$\frac{d}{dt}x_1(t) = x_3(t)$$

$$\frac{d}{dt}x_2(t) = -117x_0(t) + 100x_1(t) - \frac{4x_2(t)}{5}$$

$$\frac{d}{dt}x_3(t) = 50x_0(t) - 53x_1(t) - \frac{3x_3(t)}{10} + \frac{10^{-t}}{10}$$

[13]:
```
f_lamb = lambda t_: 10**(-t_)
def state_vars(x_, t_):
    return [
        x_[2],
        x_[3],
        -117*x_[0] + 100*x_[1] - 4/5*x_[2],
```

```
        50*x_[0] - 53*x_[1] - 3/10*x_[3] + 1/10*f_lamb(t_)
    ]
```

Note: The state variable model should have no derivatives on the right hand side of each equation.

2.3.3 Part C

[14]:
```
A = [
    [0, 0, 1, 0],
    [0, 0, 0, 1],
    [-117, 100, -4/5, 0],
    [50, -53, 0, -3/10]
]

B = [
    [0],
    [0],
    [0],
    [1/10]
]

C = [
    [1, 0, 0, 0],
    [0, 1, 0, 0],
    [0, 0, 1, 0],
    [0, 0, 0, 1]
]

D = [
    [0],
    [0],
    [0],
    [0]
]

sys1 = ct.ss(A, B, C, D)
sys1
```

[14]:

$$\left(\begin{array}{cccc|c} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -117 & 100 & -0.8 & 0 & 0 \\ 50 & -53 & 0 & -0.3 & 0.1 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

2.3.4 Part D

[15]:
```
t_array = np.linspace(0, 10, 500)
_, x_response = ct.forced_response(sys1, T=t_array, U=f_lamb(t_array), X0=(0.5,␣
↪0, 0, 0))

# Just for checking
# from scipy.integrate import odeint
# sol = odeint(state_vars, (0.5, 0, 0, 0), t_array)

plt.plot(t_array, x_response[0], label='$x_0(t)$')
# plt.plot(t_array, sol[:, 0], ls='--')
plt.plot(t_array, x_response[1], label='$x_1(t)$')
# plt.plot(t_array, sol[:, 1], ls='--')

plt.xlabel('$t$ ($s$)')
plt.ylabel('Response ($m$)')
plt.legend()
plt.show()
```