System Dynamics Final Exam

November 27, 2023

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[1]:
```
import control as ct
import numpy as np
import matplotlib.pyplot as plt
import sympy as sp
from scipy.optimize import fsolve

plt.style.use('../maroon_ipynb.mplstyle')
```

Contents

1 Problem 1

1.1 Given

$$T(s) = \frac{X(s)}{Y(s)} = \frac{10}{10s^2 + 15s + 17}$$

1.2 Find

For the transfer function above,

a. Plot the magnitude ($M(\omega)$ not in decibels) and phase response for $0.1 \leq \omega < 10\, rad/s$. You can use whichever method you prefer, but only use the `bode()` function for checking.
b. Find the resonant frequency ω_r.
c. If the input function is $y(t) = 11\sin(5t)$, find the steady state function $x_{ss}(t)$. Plot the result with the transient response up to 6 seconds.

1.3 Solution

1.3.1 Part A - Plotting the Frequency Response (20 Points)

[2]:
```
m, c, k = 10, 15, 17

T_jw = lambda om: 10/(m*(1j*om)**2 + c*1j*om + k)

T = ct.tf(10, [m, c, k])
T
```

[2]:

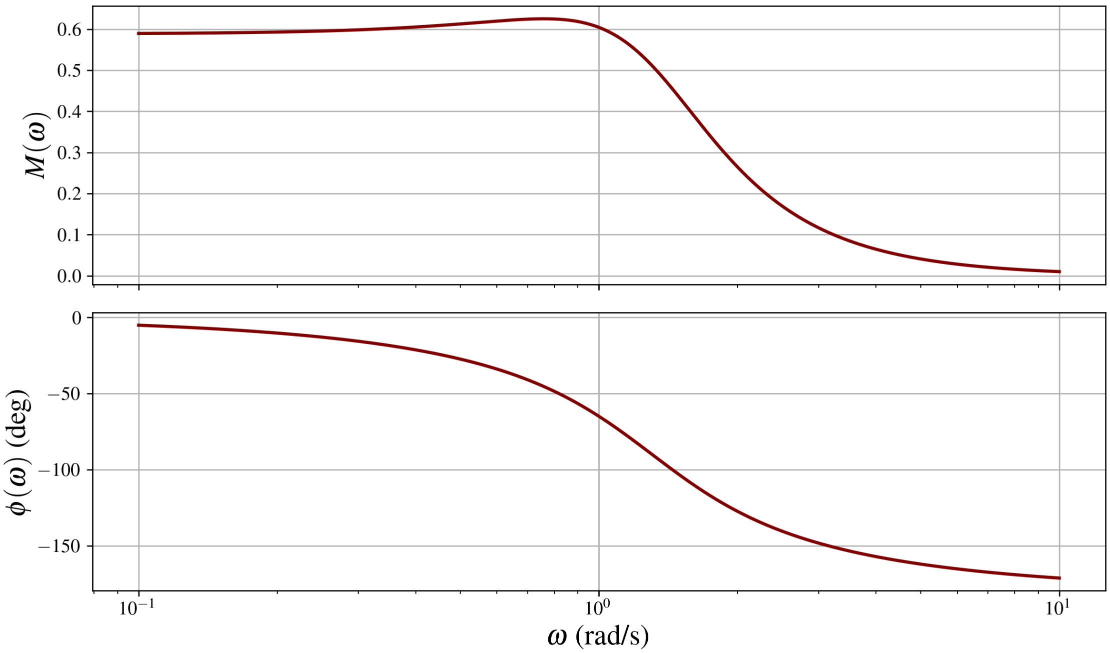
$$\frac{10}{10s^2 + 15s + 17}$$

[3]:
```
omegas = np.linspace(0.1, 10, 10_000)
mag = np.abs(T_jw(omegas))
phase = np.angle(T_jw(omegas))

fig, (ax1, ax2) = plt.subplots(nrows=2, ncols=1, sharex=True)
ax1.set_xscale('log')

ax1.plot(omegas, mag)
# ax1.plot(omegas, 20*np.log10(mag))
ax1.set_ylabel(r'$M(\omega)$')

ax2.plot(omegas, np.rad2deg(phase))
ax2.set_ylabel(r'$\phi(\omega)$ (deg)')
ax2.set_xlabel(r'$\omega$ (rad/s)')
plt.show()
```



1.3.2 Part B - Finding the Resonant Frequency (10 Points)

[4]:
```
wn = np.sqrt(k/m)
zeta = c/(2*np.sqrt(k*m))
wr = wn*np.sqrt(1 - 2*zeta**2)
wr  # rad/s
```

[4]: 0.7582875444051551

[5]:
```
omegas[max(mag) == mag][0]  # rad/s
```

[5]: 0.7584158415841584

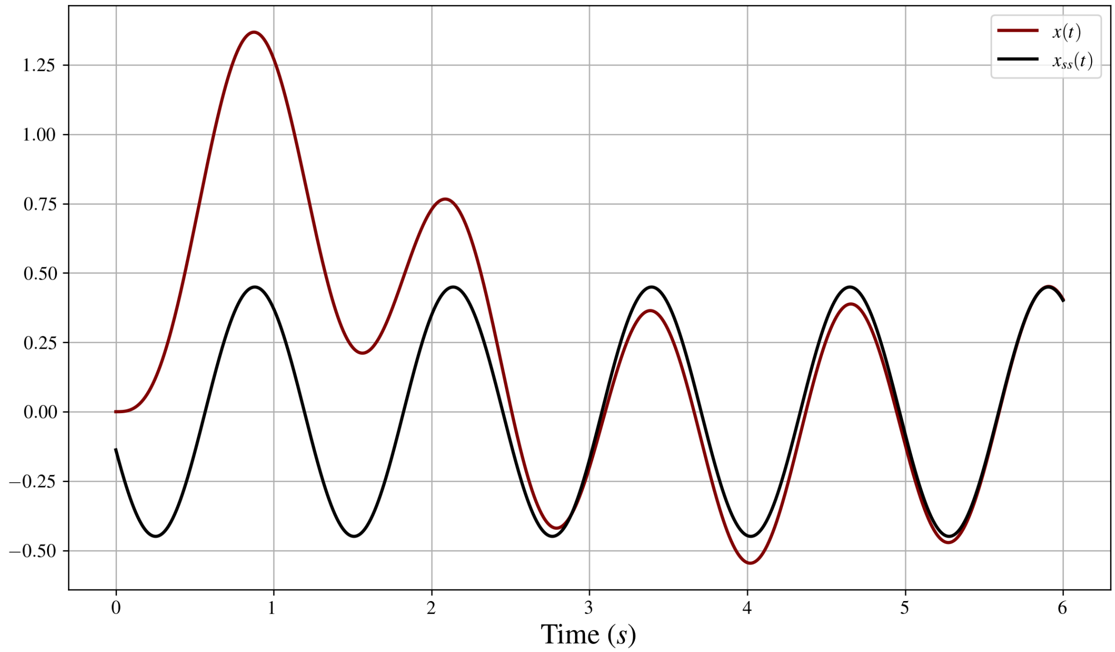
1.3.3 Part C - Getting the Transient and Steady State Response (20 Points)

[6]:
```
A = 11
w = 5

B = A*np.abs(T_jw(w))
phi = np.angle(T_jw(w))
x_ss = lambda t_: B*np.sin(w*t_ + phi)

t_array = np.linspace(0, 6, 1000)
_, x_t = ct.forced_response(T, T=t_array, U=A*np.sin(t_array*w))
```

```
plt.plot(t_array, x_t, label='$x(t)$')
plt.plot(t_array, x_ss(t_array), label='$x_{ss}(t)$')
plt.legend()
plt.xlabel('Time ($s$)')
plt.show()
```



2 Problem 2

2.1 Given

$$T(s) = \frac{1}{30s^2 + 30s + 40}$$

2.2 Find

For the transfer function above,

a. Find the analytical solution for the magnitude response ($M(\omega)$).
b. Plot the magnitude response from $0.1 \leq \omega < 10\, rad/s$.
c. Find the bandwidth (ω_1 to ω_2) and classify the filter type.

2.3 Solution

2.3.1 Part A - Finding the Magnitude Response Analytically (15 Points)

[7]:
```
m, c, k = 30, 30, 40
T = ct.tf(1, [m, c, k])
T
```

[7]:

$$\frac{1}{30s^2 + 30s + 40}$$

[8]:
```
s = sp.Symbol('s')
T_s = 1/(m*s**2 + c*s + k)
T_s
```

[8]:
$$\frac{1}{30s^2 + 30s + 40}$$

[9]:
```
omega = sp.Symbol(r'\omega', real=True, positive=True)
T_jw = T_s.subs(s, sp.I*omega)
T_jw
```

[9]:
$$\frac{1}{-30\omega^2 + 30i\omega + 40}$$

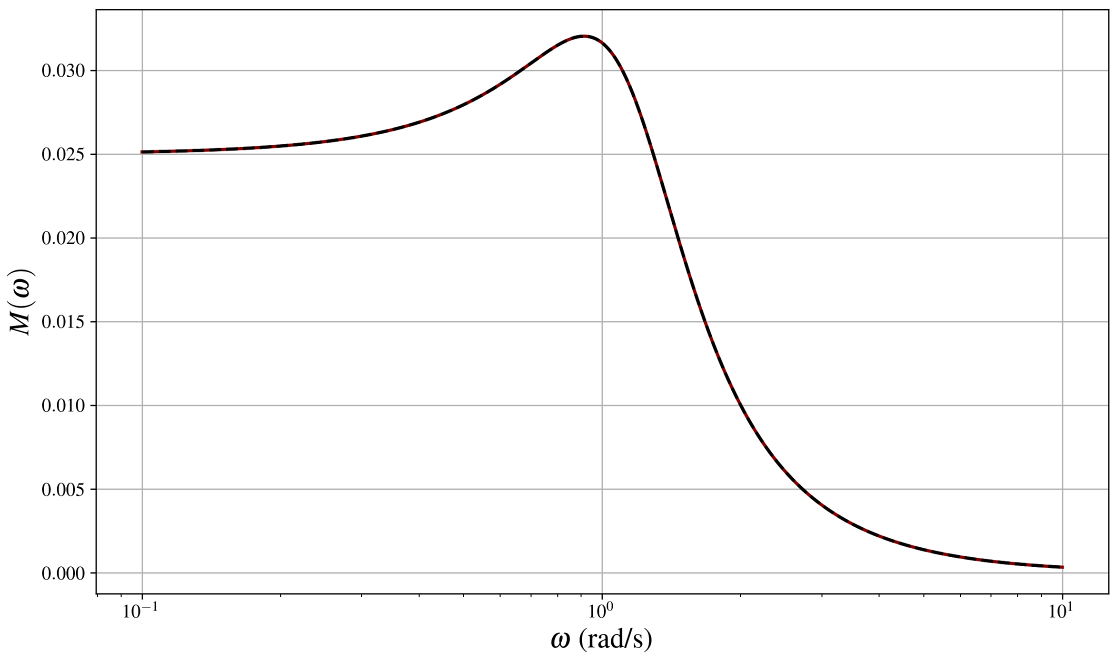
[10]:
```
mag = sp.Abs(T_jw)
mag_lamb = sp.lambdify(omega, mag, modules='numpy')
mag
```

[10]:
$$\frac{1}{\sqrt{900\omega^4 - 1500\omega^2 + 1600}}$$

2.3.2 Part B - Plotting the Magnitude Response (15 Points)

[11]:
```
mag, _, _ = ct.frequency_response(T, omegas)  # Using the same omegas from above

plt.xscale('log')
plt.plot(omegas, mag)
plt.plot(omegas, mag_lamb(omegas), ls='--')
plt.xlabel(r'$\omega$ (rad/s)')
plt.ylabel(r'$M(\omega)$')
plt.show()
```



2.3.3 Part C - Finding the Bandwidth (20 Points)

[12]:
```
wn = np.sqrt(k/m)
zeta = c/(2*np.sqrt(k*m))
wr = wn*np.sqrt(1 - 2*zeta**2)
M_peak, _, _ = ct.frequency_response(T, wr)
M_peak = M_peak[0]
wr
```

[12]: 0.9128709291752769

[13]:
```
omegas[max(mag) == mag][0]
```

[13]: 0.9128712871287129

[14]:
```
M_peak = max(mag)
wr = omegas[M_peak == mag][0]

def find_band(om):
    mag_, _, _ = ct.frequency_response(T, om)
    # mag_ = mag_lamb(om)[0]
    return mag_[0] - M_peak/np.sqrt(2)

mag0, _, _ = ct.frequency_response(T, 0)
mag0[0] > M_peak/np.sqrt(2)
```

[14]:
```
True
```

Because the magnitude at $\omega = 0$ is greater than $\frac{M_{peak}}{\sqrt{2}}$, this is a low-pass filter and $\omega_1 = 0$. Furthermore, the damping ratio is greater than 0.382.

[15]:
```
w2 = fsolve(find_band, np.array([wr + 1, ]))[0]
w2  # rad/s
```

[15]:
```
1.3690019477951116
```

Thus, the bandwidth is $0 \leq \omega \leq 1.36\, rad/s$.