System Dynamics Homework 4

October 19, 2023

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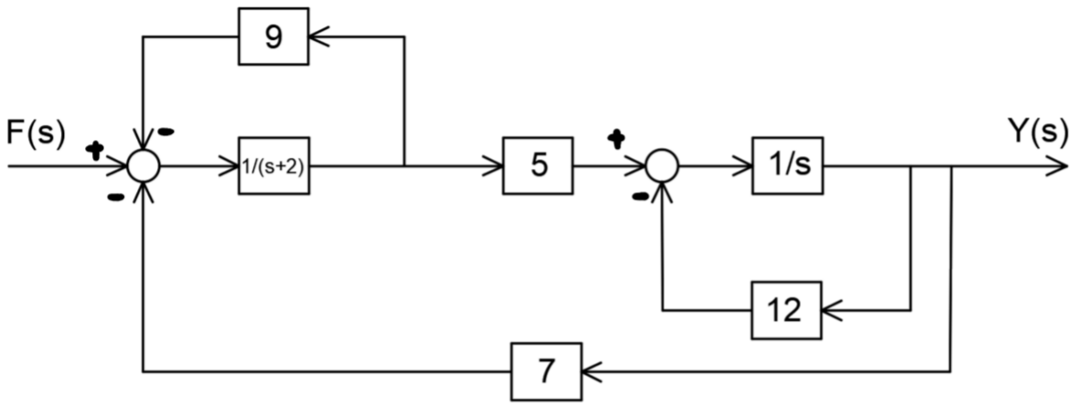
[1]:
```python
import sympy as sp
import control as ct
import matplotlib.pyplot as plt
import numpy as np
from scipy.integrate import odeint

plt.style.use('../maroon_ipynb.mplstyle')
```

Contents

1 Problem 1

1.1 Given



1.2 Find

Find the transfer function $\frac{Y(s)}{F(s)}$ for the block diagram.

1.3 Solution

The solution can be determined using two different methods. The first is an algebraic solution where B is the expression after the first block seen above. The second can be determined using the feedback and series functions.

[2]:
```python
# Using algebra
from sympy.abc import F, Y, B, s

eq1 = sp.Eq((F - 9*B - 7*Y)*1/(s + 2), B)
eq2 = sp.Eq((B*5 - 12*Y)*1/s, Y)
display(eq1, eq2)
```

$$\frac{-9B + F - 7Y}{s + 2} = B$$

$$\frac{5B - 12Y}{s} = Y$$

[3]:
```python
sol = sp.solve([eq1, eq2], [Y, B], dict=True)[0]
sol[Y]/F
```

[3]:
$$\frac{5}{s^2 + 23s + 167}$$

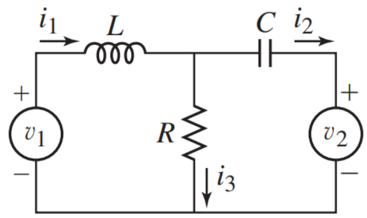
[4]:
```python
sys1 = ct.feedback(ct.tf(1, [1, 0]), 12)
sys2 = ct.series(5, sys1)
sys3 = ct.feedback(ct.tf(1, [1, 2]), 9)
sys4 = ct.series(sys3, sys2)
sys5 = ct.feedback(sys4, 7)
sys5
```

[4]:

$$\frac{5}{s^2 + 23s + 167}$$

2 Problem 2

2.1 Given



$$L = 500\, mH,\ C = 100\, \mu F,\ R = 300\, \Omega$$

$$v_1 = 5e^{-t}\sin(6t)\, V,\ v_2 = 10\sin(t)\, V$$

All initial conditions are zero.

2.2 Find

a. The system of ODE's (should be two equations if using mesh currents).
b. Solve the system for i_1, i_2, and i_3 as seen the figure above. Use any method to find the result and plot up to 6 seconds.

2.3 Solution

2.3.1 Part A

[5]:
```
L_, C_, R_ = 500e-3, 100e-6, 300

L, C, R = sp.symbols('L C R')
t = sp.Symbol('t')
V1 = sp.Function('v_1')(t)
V2 = sp.Function('v_2')(t)
iA = sp.Function('i_A')(t)  # Mesh currents
iB = sp.Function('i_B')(t)  # Mesh currents

eq1 = sp.Eq(L*iA.diff() + R*(iA - iB), V1)  # KVL around iA
eq2 = sp.Eq(1/C*iB + R*(iB.diff() - iA.diff()) + V2.diff(), 0)  # KVL around iB
↪(with derivative across each term)
display(eq1, eq2)
```

$$L\frac{d}{dt}i_A(t) + R\left(i_A(t) - i_B(t)\right) = v_1(t)$$

$$R\left(-\frac{d}{dt}i_A(t) + \frac{d}{dt}i_B(t)\right) + \frac{d}{dt}v_2(t) + \frac{i_B(t)}{C} = 0$$

2.3.2 Part B

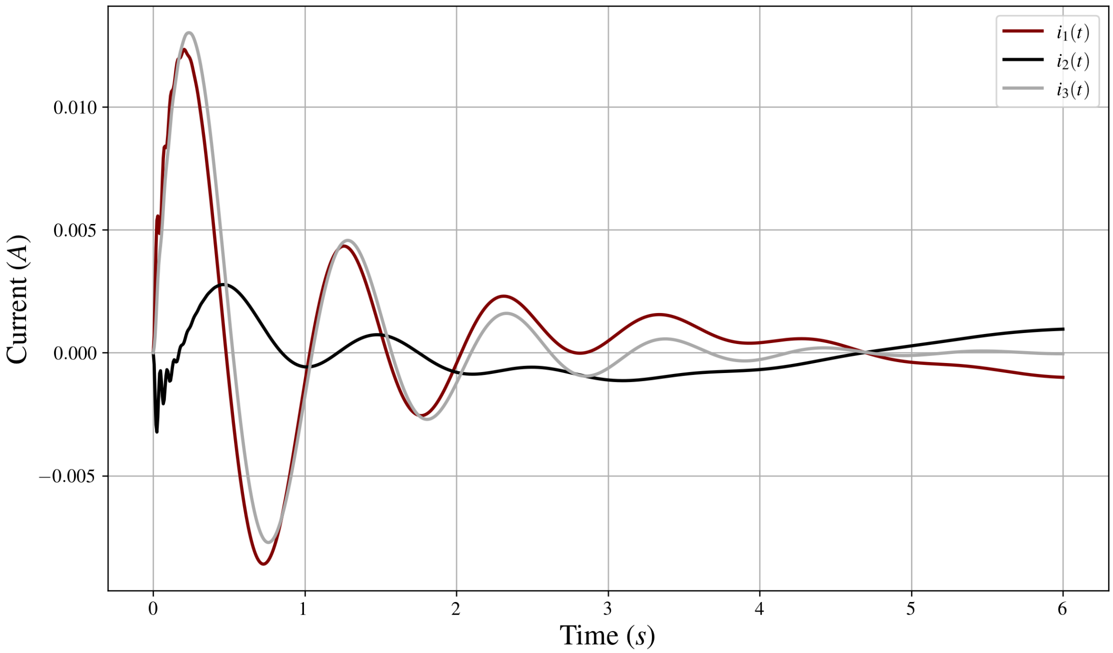
The state variable solution is the easiest since we already have a system of first order ODE's.

[6]:
```python
# Solving using state variables
state_sol = sp.solve([eq1, eq2], [iA.diff(), iB.diff()], dict=True)[0]
for key, value in state_sol.items(): display(sp.Eq(key, value))
```

$$\frac{d}{dt} i_A(t) = -\frac{R i_A(t)}{L} + \frac{R i_B(t)}{L} + \frac{v_1(t)}{L}$$

$$\frac{d}{dt} i_B(t) = -\frac{\frac{d}{dt} v_2(t)}{R} - \frac{R i_A(t)}{L} + \frac{R i_B(t)}{L} + \frac{v_1(t)}{L} - \frac{i_B(t)}{CR}$$

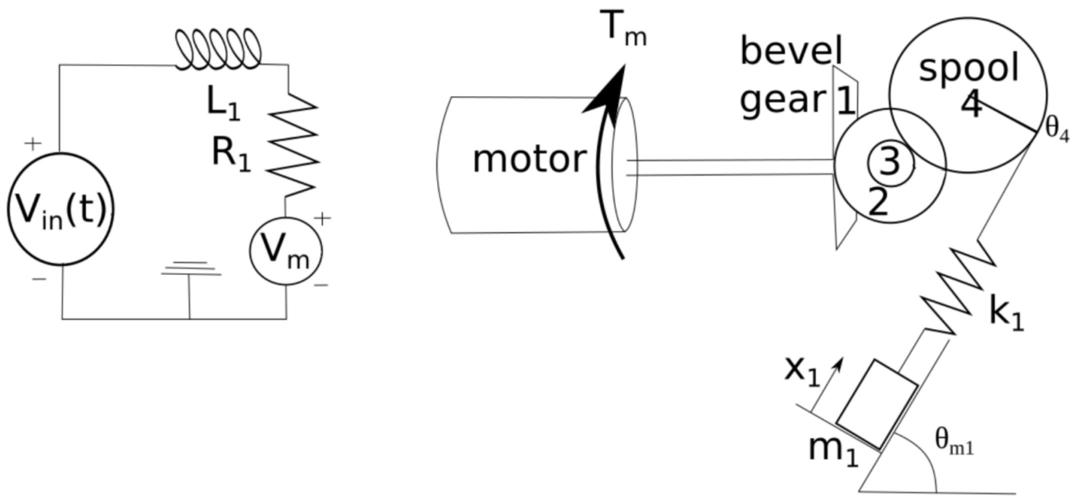
[7]:
```python
v1 = lambda t_: 5*np.exp(-t_)*np.sin(6*t_)
v2_diff = lambda t_: 10*np.cos(t_)

def state_vars(i_, t_):
    return [
        (v1(t_) + R_*i_[1] - R_*i_[0])/L_,
        -v2_diff(t_)/R_ - R_/L_*i_[0] + R_/L_*i_[1] + v1(t_)/L_ - i_[1]/(C_*R_)
    ]


t_array = np.linspace(0, 6, 1000)
sol = odeint(state_vars, (0, 0), t_array)
iA, iB = sol[:, 0], sol[:, 1]

plt.plot(t_array, iA, label='$i_1(t)$')
plt.plot(t_array, -iB, label='$i_2(t)$')
plt.plot(t_array, iA - iB, label='$i_3(t)$')
plt.legend()
plt.xlabel('Time ($s$)')
plt.ylabel('Current ($A$)')
plt.show()
```



$i_1(t)$
$i_2(t)$
$i_3(t)$
0.010
0.005
0.000
−0.005
Current (A)
0
1
2
3
4
5
6
Time (s)

3 Problem 3

3.1 Given



$$K_T = 0.01\,\frac{Nm}{A},\ R_1 = 0.5\,\Omega,\ L_1 = 0.002\,H$$
$$I_1 = 9\cdot 10^{-5}\,kg\,m^2,\ I_2 = 4\cdot 10^{-5}\,kg\,m^2,\ I_3 = 1\cdot 10^{-5}\,kg\,m^2,\ I_4 = 25\cdot 10^{-5}\,kg\,m^2$$
$$r_1 = 10\,cm,\ r_2 = 7\,cm,\ r_3 = 2.2\,cm,\ r_4 = 13\,cm$$
$$k_1 = 2\,N/m,\ m_1 = 100\,kg,\ \theta_{m1} = 70^{\circ}$$
$$V_{in} = 30\,V$$

[8]:
```
# Constants
KT, R1, L1, = sp.symbols('K_T R1 L1')
I1, I2, I3, I4 = sp.symbols('I1 I2 I3 I4')  # sympy automatically makes the␣
 ↪numbers a subscript
r1, r2, r3, r4 = sp.symbols('r1 r2 r3 r4')
k1, m1, thm1 = sp.symbols(r'k1 m1 \theta_{m1}')

# Other items you should consider using
t = sp.Symbol('t')
th1, th2, th3, th4 = sp.Function(r'\theta_1')(t), sp.Function(r'\theta_2')(t),␣
 ↪sp.Function(r'\theta_3')(t), sp.Function(r'\theta_4')(t),
i = sp.Function('i')(t)
x1 = sp.Function('x_1')(t)
Vin = sp.Function('V_{in}')(t)  # Input function

# The sp.S() function ensures that there is no floating point error
sub_values = [
```

```
    (KT, sp.S('0.01')),
    (R1, sp.S('0.5')),
    (L1, sp.S('0.002')),
    (I1, sp.S('9e-5')),
    (I2, sp.S('4e-5')),
    (I3, sp.S('1e-5')),
    (I4, sp.S('25e-5')),
    (r1, sp.S('0.1')),
    (r2, sp.S('0.07')),
    (r3, sp.S('0.022')),
    (r4, sp.S('0.13')),
    (k1, 2),
    (m1, 100),
    (thm1, sp.rad(70))
]

Vin_lamb = lambda t_: 10*(1 - np.exp(-10*t_))
```

3.2 Find

a. Determine the equivalent inertia (I_{eq}) of the gear train involving I_1, I_2, I_3, and I_4 as seen by the output shaft 1 of the motor.
b. Find the governing ODE's of the system.
c. Solve for and plot $x_1(t)$, $\theta_1(t)$, $\omega_1(t)$, and $i(t)$ up to 20 seconds.
d. Comment on the results. Is the motor able to raise the mass?

3.3 Solution

3.3.1 Part A

Use the concept of velocity ratios to relate everything back to ω_1. We know from this that

$$\frac{\omega_2}{\omega_1} = \frac{r_1}{r_2}$$

$$\omega_3 = \omega_2$$

$$\frac{\omega_4}{\omega_2} = \frac{r_3}{r_4}$$

[9]:
```
half = sp.Rational(1, 2)
KE = half*I1*th1.diff()**2 + half*I2*th2.diff()**2 + half*I3*th3.diff()**2 +
 ↪half*I4*th4.diff()**2
KE = KE.subs(th3.diff(), th2.diff())  # Same shaft, so same angular velocity
KE = KE.subs(th4.diff(), th2.diff()*r3/r4)
KE = KE.subs(th2.diff(), r1/r2*th1.diff())
KE.simplify()
```

[9]:
$$\frac{\left(I_{1} r_{2}^{2} r_{4}^{2} + I_{4} r_{1}^{2} r_{3}^{2} + r_{1}^{2} r_{4}^{2} \left(I_{2} + I_{3}\right)\right) \left(\frac{d}{dt} \theta_{1}(t)\right)^{2}}{2 r_{2}^{2} r_{4}^{2}}$$

[10]:
```
# Dividing the expression by 1/2*omega_1^2 will leave the leave only the␣
↪equivalent mass behind
Ie = KE/(half*th1.diff()**2)
Ie = Ie.simplify()
Ie.expand()
```

[10]:
$$I_{1} + \frac{I_{2} r_{1}^{2}}{r_{2}^{2}} + \frac{I_{3} r_{1}^{2}}{r_{2}^{2}} + \frac{I_{4} r_{1}^{2} r_{3}^{2}}{r_{2}^{2} r_{4}^{2}}$$

3.3.2 Part B

The force of the spring exerts a resistive torque on the equivalent mass that is equal to $F_s r_4$. You can find the output torque on the shaft by implementing the torque ratio for gears, which is $\frac{T_{out}}{T_{in}} = \frac{r_{out}}{r_{in}}$.

$$T_{out} = \frac{r_3 r_1}{r_4 r_2} F_s = \frac{r_3 r_1}{r_2} F_s$$

The force of the spring is $F_s = k_1(r_4\theta_4 - x_1) = k_1\left(\frac{r_1 r_3}{r_2}\theta_1 - x_1\right)$ because the distance that the spool expands/contracts the spring is equal to the arc length of a point on the edge of the spool.

[11]:
```
Tm = KT*i
Fs = k1*(x1 - r3*r1/r2*th1)
Tout = Fs.simplify()*r3*r1/r2

eq1 = sp.Eq(Ie.expand()*th1.diff(t, 2), Tm + Tout)
eq2 = sp.Eq(m1*x1.diff(t, 2), -Fs - m1*sp.S('9.81')*sp.sin(thm1))
eq3 = sp.Eq(L1*i.diff() + R1*i + KT*th1.diff(), Vin)
display(eq1, eq2, eq3)
```

$$\left(I_{1} + \frac{I_{2} r_{1}^{2}}{r_{2}^{2}} + \frac{I_{3} r_{1}^{2}}{r_{2}^{2}} + \frac{I_{4} r_{1}^{2} r_{3}^{2}}{r_{2}^{2} r_{4}^{2}}\right) \frac{d^{2}}{dt^{2}} \theta_{1}(t) = K_{T} i(t) + \frac{k_{1} r_{1} r_{3} \left(- r_{1} r_{3} \theta_{1}(t) + r_{2} x_{1}(t)\right)}{r_{2}^{2}}$$

$$m_{1} \frac{d^{2}}{dt^{2}} x_{1}(t) = - k_{1} \left(- \frac{r_{1} r_{3} \theta_{1}(t)}{r_{2}} + x_{1}(t)\right) - 9.81 m_{1} \sin\left(\theta_{m1}\right)$$

$$K_{T} \frac{d}{dt} \theta_{1}(t) + L_{1} \frac{d}{dt} i(t) + R_{1} i(t) = V_{in}(t)$$

3.3.3 Part C

The best way to solve this is to put it in the state variable form.

[12]:
```
# re-defining the names to keep track more easily and adding two state variables
y0, y1, y2, y3, y4 = sp.Function('y0')(t), sp.Function('y1')(t), sp.
↪Function('y2')(t), sp.Function('y3')(t), sp.Function('y4')(t)
```

```python
eq4 = sp.Eq(y0.diff(), y3)
eq5 = sp.Eq(y1.diff(), y4)

sub_states = [
    (x1, y0),
    (th1, y1),
    (i, y2),
    (y1.diff(t, 2), y4.diff()),
    (y1.diff(), y4),
    (y0.diff(t, 2), y3.diff()),
    (y0.diff(), y3)
]

eq1 = eq1.subs(sub_states)
eq2 = eq2.subs(sub_states)
eq3 = eq3.subs(sub_states)

display(eq1, eq2, eq3, eq4, eq5)
```

$$\left(I_1 + \frac{I_2 r_1^2}{r_2^2} + \frac{I_3 r_1^2}{r_2^2} + \frac{I_4 r_1^2 r_3^2}{r_2^2 r_4^2}\right) \frac{d}{dt} y_4(t) = K_T y_2(t) + \frac{k_1 r_1 r_3 \left(-r_1 r_3 y_1(t) + r_2 y_0(t)\right)}{r_2^2}$$

$$m_1 \frac{d}{dt} y_3(t) = -k_1 \left(-\frac{r_1 r_3 y_1(t)}{r_2} + y_0(t)\right) - 9.81 m_1 \sin\left(\theta_{m1}\right)$$

$$K_T y_4(t) + L_1 \frac{d}{dt} y_2(t) + R_1 y_2(t) = V_{in}(t)$$

$$\frac{d}{dt} y_0(t) = y_3(t)$$

$$\frac{d}{dt} y_1(t) = y_4(t)$$

[13]:
```python
state_sol = sp.solve([eq1, eq2, eq3, eq4, eq5], [y0.diff(), y1.diff(), y2.
 ↪diff(), y3.diff(), y4.diff()], dict=True)[0]
for key, value in state_sol.items(): display(sp.Eq(key, value))
```

$$\frac{d}{dt} y_0(t) = y_3(t)$$

$$\frac{d}{dt} y_1(t) = y_4(t)$$

$$\frac{d}{dt} y_2(t) = -\frac{K_T y_4(t)}{L_1} - \frac{R_1 y_2(t)}{L_1} + \frac{V_{in}(t)}{L_1}$$

$$\frac{d}{dt} y_3(t) = \frac{k_1 r_1 r_3 y_1(t)}{m_1 r_2} - \frac{k_1 y_0(t)}{m_1} - 9.81 \sin\left(\theta_{m1}\right)$$

$$\frac{d}{dt} y_4(t) = \frac{K_T r_2^2 r_4^2 y_2(t)}{I_1 r_2^2 r_4^2 + I_2 r_1^2 r_4^2 + I_3 r_1^2 r_4^2 + I_4 r_1^2 r_3^2} - \frac{k_1 r_1^2 r_3^2 r_4^2 y_1(t)}{I_1 r_2^2 r_4^2 + I_2 r_1^2 r_4^2 + I_3 r_1^2 r_4^2 + I_4 r_1^2 r_3^2} +$$

$$\frac{k_1 r_1 r_2 r_3 r_4^2 y_0(t)}{I_1 r_2^2 r_4^2 + I_2 r_1^2 r_4^2 + I_3 r_1^2 r_4^2 + I_4 r_1^2 r_3^2}$$

[14]:
```python
# Solution with substituted values
funcs = []
for key, value in state_sol.items():
    display(sp.Eq(key, value.subs(sub_values)))
    args = (y0, y1, y2, y3, y4, Vin)
    funcs.append(sp.lambdify(args, value.subs(sub_values), modules='numpy'))
```

$$\frac{d}{dt} y_0(t) = y_3(t)$$

$$\frac{d}{dt} y_1(t) = y_4(t)$$

$$\frac{d}{dt} y_2(t) = 500.0 V_{in}(t) - 250.0 y_2(t) - 5.0 y_4(t)$$

$$\frac{d}{dt} y_3(t) = -\frac{y_0(t)}{50} + 0.000628571428571429 y_1(t) - 9.81 \sin\left(\frac{7\pi}{18}\right)$$

$$\frac{d}{dt} y_4(t) = 304.16820059721 y_0(t) - 9.55957201876947 y_1(t) + 48.3903955495562 y_2(t)$$

[15]:
```python
# funcs now has the lambdified version of each equation
funcs[2](_, _, 1, _, 1, 1)  # Testing values on third equation
```

[15]: 245.0

[16]:
```python
def state_vars(y, t_):
    return [func(y[0], y[1], y[2], y[3], y[4], Vin_lamb(t_)) for func in funcs]

# Solving
t_array = np.linspace(0, 20, 1000)
sol = odeint(state_vars, (0, 0, 0, 0, 0), t_array)
x1_ = sol[:, 0]; th1_ = sol[:, 1]
omega_1 = sol[:, 4]; current = sol[:, 2]

fig, ax = plt.subplots(nrows=4, ncols=1)
fig.set_figheight(12)  # 12 inches

ax[0].plot(t_array, x1_)
ax[0].set_ylabel('$x_1(t)$ ($m$)')
ax[1].plot(t_array, np.rad2deg(th1_))
ax[1].set_ylabel(r'$\theta_1(t)$ ($deg$)')
ax[2].plot(t_array, omega_1*30/np.pi)
ax[2].set_ylabel(r'$\omega_1(t)$ ($RPM$)')
ax[3].plot(t_array, current)
ax[3].set_ylabel('$i(t)$ ($A$)')
```

```
ax[-1].set_xlabel('Time ($s$)')
plt.show()
```