

# Finding the Contact Force

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```
[1]: import sympy as sp

# Defining symbols
p_s, v_s, a_s = sp.symbols(r'\vec{p}_s \vec{v}_s \vec{a}_s')
k, n = sp.symbols(r'k n')
del_t = sp.Symbol(r'\Delta t')
xi, eta = sp.symbols(r'\xi \eta')
p_k, v_k, a_k = sp.symbols(r'\vec{p}_k \vec{v}_k \vec{a}_k')
phi_k = sp.Function(r'\phi_k')(xi, eta)
F_s, f_c, R_s = sp.symbols(r'\vec{F}_s f_c \vec{R}_s')
F_k, N, R_k = sp.symbols(r'\vec{F}_k \vec{N} \vec{R}_k')
m_s, m_k = sp.symbols('m_s m_k')
```

Consider a contact pair (a patch and a single node) in which a force resolution needs to be acquired between these entities. The goal is to ensure that applied force moves the node and patch in such a way that node lies on the surface of the patch at the next time step. Such a condition can be achieved by solving the following equation:

$$\vec{p}_s + \vec{v}_s \Delta t + \frac{1}{2} \vec{a}_s \Delta t^2 = \sum_{k=0}^{n-1} \phi_k(\xi, \eta) \left[ \vec{p}_k + \vec{v}_k \Delta t + \frac{1}{2} \vec{a}_k \Delta t^2 \right]$$

where  $p$ ,  $v$ , and  $a$  are the position, velocity, and acceleration of the node and patch at the current time step, and subscript  $s$  denotes the slave node while  $k$  denotes the nodes that bound the master patch. Alternatively, the above equation can be represented as a matrix multiplication instead of a summation:

$$\vec{p}_s + \vec{v}_s \Delta t + \frac{1}{2} \vec{a}_s \Delta t^2 = \underbrace{\begin{bmatrix} p_{x0} + v_{x0} \Delta t + \frac{1}{2} a_{x0} \Delta t^2 & p_{x1} + v_{x1} \Delta t + \frac{1}{2} a_{x1} \Delta t^2 & \dots \\ p_{y0} + v_{y0} \Delta t + \frac{1}{2} a_{y0} \Delta t^2 & p_{y1} + v_{y1} \Delta t + \frac{1}{2} a_{y1} \Delta t^2 & \dots \\ p_{z0} + v_{z0} \Delta t + \frac{1}{2} a_{z0} \Delta t^2 & p_{z1} + v_{z1} \Delta t + \frac{1}{2} a_{z1} \Delta t^2 & \dots \end{bmatrix}}_A \begin{bmatrix} \phi_0(\xi, \eta) \\ \phi_1(\xi, \eta) \\ \vdots \\ \phi_n(\xi, \eta) \end{bmatrix}$$

The acceleration for the slave node and the master patch node can be written as:

$$\vec{a}_s = \frac{\vec{F}_s + f_c \vec{N} + \vec{R}_s}{m_s}$$

$$\vec{a}_k = \frac{\vec{F}_k - f_c \vec{N} \cdot \phi_k(\xi, \eta) + \vec{R}_k}{m_k}$$

where  $\vec{F}$  is the internal force known prior to the analysis,  $f_c$  is the incremental contact force between the current node and the patch, and  $\vec{R}$  is the force due to other contact pairs.  $\vec{N}$  is the unit normal at the contact point  $(\xi, \eta)$  and must be in the outward direction of the patch, facing the non-penetrated slave node.

$$N = \frac{\partial p / \partial \xi \times \partial p / \partial \eta}{|\partial p / \partial \xi \times \partial p / \partial \eta|}$$

[2]: 

```
eq1 = sp.Eq(p_s + v_s*del_t + sp.Rational(1, 2)*a_s*del_t**2, sp.Sum(phi_k*(p_k
    ↪ + v_k*del_t + sp.Rational(1, 2)*a_k*del_t**2), (k, 0, n-1)))
eq1
```

[2]: 
$$\frac{\Delta t^2 \vec{a}_s}{2} + \Delta t \vec{v}_s + \vec{p}_s = \sum_{k=0}^{n-1} \left( \frac{\Delta t^2 \vec{a}_k}{2} + \Delta t \vec{v}_k + \vec{p}_k \right) \phi_k(\xi, \eta)$$

[3]: 

```
eq2 = eq1.subs([
    (a_s, (F_s + fc*N + R_s)/m_s),
    (a_k, (F_k - fc*N*phi_k + R_k)/m_k)
])
eq2
```

[3]: 
$$\frac{\Delta t^2 (\vec{F}_s + \vec{N} f_c + \vec{R}_s)}{2m_s} + \Delta t \vec{v}_s + \vec{p}_s = \sum_{k=0}^{n-1} \left( \frac{\Delta t^2 (\vec{F}_k - \vec{N} f_c \phi_k(\xi, \eta) + \vec{R}_k)}{2m_k} + \Delta t \vec{v}_k + \vec{p}_k \right) \phi_k(\xi, \eta)$$

In the matrix form, this results in:

[4]: 

```
A = sp.Matrix([sp.Function('A')(xi, eta, fc)])
eq3 = sp.Eq(eq2.lhs, sp.MatMul(A, sp.Matrix([phi_k])), evaluate=False)
eq3
```

[4]: 
$$\frac{\Delta t^2 (\vec{F}_s + \vec{N} f_c + \vec{R}_s)}{2m_s} + \Delta t \vec{v}_s + \vec{p}_s = [A(\xi, \eta, f_c)] [\phi_k(\xi, \eta)]$$

To solve this with the Newton-Raphson method, we have the following

$$\begin{bmatrix} \xi_{i+1} \\ \eta_{i+1} \\ f_{ci+1} \end{bmatrix} = \begin{bmatrix} \xi_i \\ \eta_i \\ f_{ci} \end{bmatrix} - \mathbf{J}^{-1} \mathbf{F}$$

[5]: 

```
F = sp.Matrix([eq3.lhs]) - eq3.rhs
F
```

[5]:

$$\left[ \frac{\Delta t^2 (\vec{F}_s + \vec{N} f_c + \vec{R}_s)}{2m_s} + \Delta t \vec{v}_s + \vec{p}_s - A(\xi, \eta, f_c) \phi_k(\xi, \eta) \right]$$

[6]: `J = F.jacobian([xi, eta, fc])`  
`J`

[6]:  $\left[ -A(\xi, \eta, f_c) \frac{\partial}{\partial \xi} \phi_k(\xi, \eta) - \phi_k(\xi, \eta) \frac{\partial}{\partial \xi} A(\xi, \eta, f_c) \quad -A(\xi, \eta, f_c) \frac{\partial}{\partial \eta} \phi_k(\xi, \eta) - \phi_k(\xi, \eta) \frac{\partial}{\partial \eta} A(\xi, \eta, f_c) \quad \frac{\Delta t^2 \vec{N}}{2m_s} - \phi_k(\xi, \eta) \frac{\partial}{\partial f_c} A(\xi, \eta, f_c) \right]$

Here is a better look of the result:

$$\left[ -A(\xi, \eta, f_c) \frac{\partial}{\partial \xi} \phi_k(\xi, \eta) - \frac{\partial}{\partial \xi} A(\xi, \eta, f_c) \phi_k(\xi, \eta) \quad -A(\xi, \eta, f_c) \frac{\partial}{\partial \eta} \phi_k(\xi, \eta) - \frac{\partial}{\partial \eta} A(\xi, \eta, f_c) \phi_k(\xi, \eta) \quad \frac{\Delta t^2 \vec{N}}{2m_s} - \frac{\partial}{\partial f_c} A(\xi, \eta, f_c) \phi_k(\xi, \eta) \right]$$

Recall that for  $A$ , we have

$$\begin{bmatrix} p_{x0} + v_{x0} \Delta t + \frac{1}{2} \frac{F_{x0} - N_x f_c \cdot \phi_0(\xi, \eta) + R_{x0}}{m_0} \Delta t^2 & p_{x1} + v_{x1} \Delta t + \frac{1}{2} \frac{F_{x1} - N_x f_c \cdot \phi_1(\xi, \eta) + R_{x1}}{m_1} \Delta t^2 & \dots \\ p_{y0} + v_{y0} \Delta t + \frac{1}{2} \frac{F_{y0} - N_y f_c \cdot \phi_0(\xi, \eta) + R_{y0}}{m_0} \Delta t^2 & p_{y1} + v_{y1} \Delta t + \frac{1}{2} \frac{F_{y1} - N_y f_c \cdot \phi_1(\xi, \eta) + R_{y1}}{m_1} \Delta t^2 & \dots \\ p_{z0} + v_{z0} \Delta t + \frac{1}{2} \frac{F_{z0} - N_z f_c \cdot \phi_0(\xi, \eta) + R_{z0}}{m_0} \Delta t^2 & p_{z1} + v_{z1} \Delta t + \frac{1}{2} \frac{F_{z1} - N_z f_c \cdot \phi_1(\xi, \eta) + R_{z1}}{m_1} \Delta t^2 & \dots \end{bmatrix}$$

The following matrices are constructed using the outer product:

$$\begin{aligned} A &= \left( \vec{p}_k + \vec{v}_k \Delta t + \frac{1}{2} \Delta t^2 (\vec{F}_k + \vec{R}_k) \right) \otimes [1, 1, \dots, 1]_{1 \times n}^\top - \vec{N} \otimes \phi_k \frac{f_c}{2m_k} \Delta t^2 \\ \frac{\partial}{\partial \xi} A &= -\vec{N} \otimes \frac{\partial}{\partial \xi} \phi_k \frac{f_c}{2m_k} \Delta t^2 \\ \frac{\partial}{\partial \eta} A &= -\vec{N} \otimes \frac{\partial}{\partial \eta} \phi_k \frac{f_c}{2m_k} \Delta t^2 \end{aligned}$$