Contact Point to Reference

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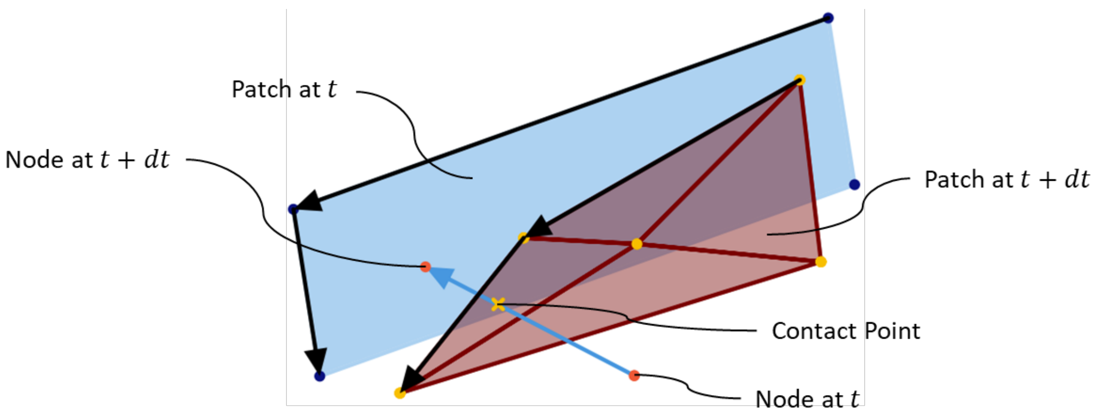
[1]:
```
import sympy as sp
import numpy as np

xc, yc = sp.symbols('x_c y_c')  # x and y values of contact point
xi, eta = sp.symbols(r'xi eta')  # reference coordinate variables
p, n = sp.symbols('p n')  # p is the index of an array and n is the total␣
 ↪number of points
xp, yp = sp.Function('x'), sp.Function('y')  # x and y components of a vector
xi_p, eta_p = sp.Function(r'\xi'), sp.Function(r'\eta')  # xi and eta points␣
 ↪that make up the boundary of the surface (either 1 or -1 always)
```



This demo is provided for constructing the set of non-linear functions to solve for the ξ and η reference coordinates of the contact point and construct a Newton-Raphson scheme.

For mapping a reference point (ξ, η) to the global/actual position point $(\vec{s})$, we use the following

$$\vec{s} = \sum_{p=0}^{n-1} \phi_p(\xi, \eta)\vec{s}_p$$

where $\phi_p(\xi, \eta) = \frac{1}{4}(1 + \xi_p\xi)(1 + \eta_p\eta)$ is the basis/shape function for 2D corresponding to a known reference point $\vec{s}_p$. The position point has components

$$\vec{s}_p = \begin{bmatrix} x_p \\ y_p \end{bmatrix}$$

At some contact point (ξ_c, η_c), we can set up the following equation below to be analyzed.

[2]:
```
phi_p = (sp.Rational(1, 4)*(1 + xi*xi_p(p))*(1 + eta*eta_p(p))).simplify()  #␣
↪shape function in 2D space
phi_p_func = sp.Function(r'\phi_p')(xi, eta)
s_p = sp.Matrix([xp(p), yp(p)])  # Position vector of surface point
b = sp.Matrix([xc, yc])

eq = sp.Eq(b, sp.Sum(phi_p_func*s_p, (p, 0, n - 1)), evaluate=False)
display(eq)
eq1 = sp.Eq(b, sp.Sum(phi_p*s_p, (p, 0, n - 1)), evaluate=False)
display(eq1)
```

$$\begin{bmatrix} x_c \\ y_c \end{bmatrix} = \sum_{p=0}^{n-1} \begin{bmatrix} \phi_p(\xi, \eta) x(p) \\ \phi_p(\xi, \eta) y(p) \end{bmatrix}$$

$$\begin{bmatrix} x_c \\ y_c \end{bmatrix} = \sum_{p=0}^{n-1} \begin{bmatrix} \frac{(\eta\eta(p)+1)(\xi\xi(p)+1)x(p)}{4} \\ \frac{(\eta\eta(p)+1)(\xi\xi(p)+1)y(p)}{4} \end{bmatrix}$$

The $x(p)$, $\xi(p)$, and so on should be interpreted as x_p, ξ_p, and so on. This is how we can use `sympy` to symbolically construct the Newton-Raphson scheme in terms of reference points. For the Newton-Raphson scheme, we have

$$\begin{bmatrix} \xi_{i+1} \\ \eta_{i+1} \end{bmatrix} = \begin{bmatrix} \xi_i \\ \eta_i \end{bmatrix} - \mathbf{J}^{-1}\mathbf{F}$$

[3]:
```
# Constructing the vector function F
F = eq1.rhs.doit() - eq1.lhs
F
```

[3]:
$$\begin{bmatrix} -x_c + \sum_{p=0}^{n-1} \left(\frac{\eta\xi\eta(p)\xi(p)x(p)}{4} + \frac{\eta\eta(p)x(p)}{4} + \frac{\xi\xi(p)x(p)}{4} + \frac{x(p)}{4} \right) \\ -y_c + \sum_{p=0}^{n-1} \left(\frac{\eta\xi\eta(p)\xi(p)y(p)}{4} + \frac{\eta\eta(p)y(p)}{4} + \frac{\xi\xi(p)y(p)}{4} + \frac{y(p)}{4} \right) \end{bmatrix}$$

[4]:
```
# Constructing the jacobian J
jac = F.jacobian([xi, eta])
jac
```

[4]:
$$\begin{bmatrix} \sum_{p=0}^{n-1} \left(\frac{\eta\eta(p)\xi(p)x(p)}{4} + \frac{\xi(p)x(p)}{4} \right) & \sum_{p=0}^{n-1} \left(\frac{\xi\eta(p)\xi(p)x(p)}{4} + \frac{\eta(p)x(p)}{4} \right) \\ \sum_{p=0}^{n-1} \left(\frac{\eta\eta(p)\xi(p)y(p)}{4} + \frac{\xi(p)y(p)}{4} \right) & \sum_{p=0}^{n-1} \left(\frac{\xi\eta(p)\xi(p)y(p)}{4} + \frac{\eta(p)y(p)}{4} \right) \end{bmatrix}$$

[5]:
```
# Construct the inverse jacobian J^{-1}
jac_inv = jac.inv()
jac_inv
```

[5]:
$$\begin{bmatrix} \frac{\sum_{p=0}^{n-1} \left(\frac{\xi\eta(p)\xi(p)y(p)}{4} + \frac{\eta(p)y(p)}{4} \right)}{\left(\sum_{p=0}^{n-1} \left(\frac{\eta\eta(p)\xi(p)x(p)}{4} + \frac{\xi(p)x(p)}{4} \right)\right) \sum_{p=0}^{n-1} \left(\frac{\xi\eta(p)\xi(p)y(p)}{4} + \frac{\eta(p)y(p)}{4} \right) - \left(\sum_{p=0}^{n-1} \left(\frac{\eta\eta(p)\xi(p)y(p)}{4} + \frac{\xi(p)y(p)}{4} \right)\right) \sum_{p=0}^{n-1} \left(\frac{\xi\eta(p)\xi(p)x(p)}{4} + \frac{\eta(p)x(p)}{4} \right)} & \cdots \\ -\frac{\sum_{p=0}^{n-1} \left(\frac{\eta\eta(p)\xi(p)y(p)}{4} + \frac{\xi(p)y(p)}{4} \right)}{\left(\sum_{p=0}^{n-1} \left(\frac{\eta\eta(p)\xi(p)x(p)}{4} + \frac{\xi(p)x(p)}{4} \right)\right) \sum_{p=0}^{n-1} \left(\frac{\xi\eta(p)\xi(p)y(p)}{4} + \frac{\eta(p)y(p)}{4} \right) - \left(\sum_{p=0}^{n-1} \left(\frac{\eta\eta(p)\xi(p)y(p)}{4} + \frac{\xi(p)y(p)}{4} \right)\right) \sum_{p=0}^{n-1} \left(\frac{\xi\eta(p)\xi(p)x(p)}{4} + \frac{\eta(p)x(p)}{4} \right)} & \cdots \end{bmatrix}$$

[6]:
```python
# The denominator of each item is messy
# To clean it up, I'm replacing it with a variable "d"
a11 = jac_inv[0, 0]
_, d = sp.fraction(a11)
jac_inv.subs(d, sp.Symbol("d"))
```

[6]:
$$\begin{bmatrix} \frac{\sum_{p=0}^{n-1}\left(\frac{\xi\eta(p)\xi(p)y(p)}{4}+\frac{\eta(p)y(p)}{4}\right)}{d} & -\frac{\sum_{p=0}^{n-1}\left(\frac{\xi\eta(p)\xi(p)x(p)}{4}+\frac{\eta(p)x(p)}{4}\right)}{d} \\ -\frac{\sum_{p=0}^{n-1}\left(\frac{\eta\eta(p)\xi(p)y(p)}{4}+\frac{\xi(p)y(p)}{4}\right)}{d} & \frac{\sum_{p=0}^{n-1}\left(\frac{\eta\eta(p)\xi(p)x(p)}{4}+\frac{\xi(p)x(p)}{4}\right)}{d} \end{bmatrix}$$

In summary, we have

$$F = \begin{bmatrix} -x_c + \sum_{p=0}^{n-1}\left(\frac{\eta\xi\eta_p\xi_p x_p}{4} + \frac{\eta\eta_p x_p}{4} + \frac{\xi\xi_p x_p}{4} + \frac{x_p}{4}\right) \\ -y_c + \sum_{p=0}^{n-1}\left(\frac{\eta\xi\eta_p\xi_p y_p}{4} + \frac{\eta\eta_p y_p}{4} + \frac{\xi\xi_p y_p}{4} + \frac{y_p}{4}\right) \end{bmatrix}$$

$$J = \begin{bmatrix} \sum_{p=0}^{n-1}\left(\frac{\eta\eta_p\xi_p x_p}{4} + \frac{\xi_p x_p}{4}\right) & \sum_{p=0}^{n-1}\left(\frac{\xi\eta_p\xi_p x_p}{4} + \frac{\eta_p x_p}{4}\right) \\ \sum_{p=0}^{n-1}\left(\frac{\eta\eta_p\xi_p y_p}{4} + \frac{\xi_p y_p}{4}\right) & \sum_{p=0}^{n-1}\left(\frac{\xi\eta_p\xi_p y_p}{4} + \frac{\eta_p y_p}{4}\right) \end{bmatrix}$$

$$J^{-1} = \begin{bmatrix} \frac{\sum_{p=0}^{n-1}\left(\frac{\xi\eta_p\xi_p y_p}{4}+\frac{\eta_p y_p}{4}\right)}{d} & -\frac{\sum_{p=0}^{n-1}\left(\frac{\xi\eta_p\xi_p x_p}{4}+\frac{\eta_p x_p}{4}\right)}{d} \\ -\frac{\sum_{p=0}^{n-1}\left(\frac{\eta\eta_p\xi_p y_p}{4}+\frac{\xi_p y_p}{4}\right)}{d} & \frac{\sum_{p=0}^{n-1}\left(\frac{\eta\eta_p\xi_p x_p}{4}+\frac{\xi_p x_p}{4}\right)}{d} \end{bmatrix}$$

where

$$d = \left(\sum_{p=0}^{n-1}\left(\frac{\eta\eta_p\xi_p x_p}{4} + \frac{\xi_p x_p}{4}\right)\right)\sum_{p=0}^{n-1}\left(\frac{\xi\eta_p\xi_p y_p}{4} + \frac{\eta_p y_p}{4}\right) - \left(\sum_{p=0}^{n-1}\left(\frac{\eta\eta_p\xi_p y_p}{4} + \frac{\xi_p y_p}{4}\right)\right)\sum_{p=0}^{n-1}\left(\frac{\xi\eta_p\xi_p x_p}{4} + \frac{\eta_p x_p}{4}\right)$$

The code below is the numerical implementation of this scheme.

[7]:
```python
class Node:
    def __init__(self, pos, ref_pos):
        """
        :param pos: Physical position array
        :param ref_pos: Reference position array
        """
        self.x, self.y, self.z = pos
        self.xi, self.eta, self.zeta = ref_pos

def get_F(reference_point, physical_point, nodes):
    xi_, eta_ = reference_point
    xc_, yc_ = physical_point
    return np.array([
        0.25*sum([eta_*xi_*p_.eta*p_.xi*p_.x + eta_*p_.eta*p_.x + xi_*p_.xi*p_.
 ↪x + p_.x for p_ in nodes]) - xc_,
        0.25*sum([eta_*xi_*p_.eta*p_.xi*p_.y + eta_*p_.eta*p_.y + xi_*p_.xi*p_.
 ↪y + p_.y for p_ in nodes]) - yc_
    ])
```

```
def get_jacobian_inverse(reference_point, nodes):
    xi_, eta_ = reference_point
    den = (sum([eta_*p_.eta*p_.xi*p_.x + p_.xi*p_.x for p_ in nodes])/
 ↪4*sum([xi_*p_.eta*p_.xi*p_.y + p_.eta*p_.y for p_ in nodes])/4 -
          sum([eta_*p_.eta*p_.xi*p_.y + p_.xi*p_.y for p_ in nodes])/
 ↪4*sum([xi_*p_.eta*p_.xi*p_.x + p_.eta*p_.x for p_ in nodes])/4)
    return np.array([
        [sum([xi_*p_.eta*p_.xi*p_.y + p_.eta*p_.y for p_ in nodes])/4/den,␣
 ↪-sum([xi_*p_.eta*p_.xi*p_.x + p_.eta*p_.x for p_ in nodes])/4/den],
        [-sum([eta_*p_.eta*p_.xi*p_.y + p_.xi*p_.y for p_ in nodes])/4/den,␣
 ↪sum([eta_*p_.eta*p_.xi*p_.x + p_.xi*p_.x for p_ in nodes])/4/den]
    ])

def newton_raphson(reference_point, physical_point, nodes, tol=1e-8,␣
 ↪max_iter=100):
    xi_, eta_ = reference_point
    for i in range(max_iter):
        F_ = get_F([xi_, eta_], physical_point, nodes)
        jac_inv_ = get_jacobian_inverse([xi_, eta_], nodes)
        xi_, eta_ = np.array([xi_, eta_]) - jac_inv_ @ F_
        if np.linalg.norm(F_) < tol:
            break
    # noinspection PyUnboundLocalVariable
    return np.array([xi_, eta_]), i
```

Consider a quadrilateral surface bound by the following points:

Label	ξ, η, ζ	x, y, z
0	$-1, -1, -1$	$0.51025339, 0.50683559, 0.99572776$
1	$1, -1, -1$	$1.17943427, 0.69225101, 1.93591633$
2	$1, 1, -1$	$0.99487331, 0.99743665, 2.97094874$
3	$-1, 1, -1$	$0.49444608, 0.99700943, 1.96411315$

The contact point is $(0.92088978, 0.74145551, 1.89717136)$. The analysis omits ζ because we already know that the contact point is on the exterior surface. For this case, $\zeta = -1$. **Note: The implemented procedure needs to use those reference points that are changing.** For example, if contact is on the reference plane $\eta = 1$, then the process needs to solve for ξ and ζ.

[8]:

```
patch_nodes = [
    Node([0.51025339, 0.50683559, 0.99572776], [-1, -1, -1]),
    Node([1.17943427, 0.69225101, 1.93591633], [1, -1, -1]),
    Node([0.99487331, 0.99743665, 2.97094874], [1, 1, -1]),
    Node([0.49444608, 0.99700943, 1.96411315], [-1, 1, -1])
]
```

```python
newton_raphson([0.5, -0.5], [0.92088978, 0.74145551], patch_nodes)
```

[8]: (array([0.34340497, -0.39835547]), 3)