# Finding the Contact Point

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[1]:
```python
# Imports
import sympy as sp

p_s, v_s, a_s = sp.symbols(r'\vec{p}_s \vec{v}_s \vec{a}_s')
k, n = sp.symbols(r'k n')
del_t = sp.Symbol(r'\Delta t')
xi, eta = sp.symbols(r'\xi \eta')
p_k, v_k, a_k = sp.symbols(r'\vec{p}_k \vec{v}_k \vec{a}_k')
phi_k = sp.Function(r'\phi_k')(xi, eta)
```

## 1 Introduction of the Problem

Consider a patch and node that moves with time. Our goal is to determine the contact point in the reference space of the patch/element. The mapping of a reference point $(\xi, \eta)$ to the global space is given by

$$\vec{s} = \sum_{p=0}^{n-1} \phi_p(\xi, \eta)\vec{s}_p$$

where $\vec{s}$ is the position in the global space and $\vec{s}_p$ is a basis vector of the patch in the global space. If we consider the contact point to be of interest ($\vec{s}_c$) and the fact that its position moves with time as well as the basis vectors, we can write

$$\vec{p}_s + \vec{v}_s\Delta t + \frac{1}{2}\vec{a}_s\Delta t^2 = \sum_{k=0}^{n-1} \phi_k(\xi, \eta)(\vec{p}_k + \vec{v}_k\Delta t + \frac{1}{2}\vec{a}_k\Delta t^2)$$

where $p$, $v$, and $a$ are the position, velocity, and acceleration vectors, respectively. The subscript $s$ denotes the slave node, and the subscript $k$ denotes the patch nodes. This relationship can be re-written as a matrix multiplication like so

$$\vec{p}_s + \vec{v}_s\Delta t + \frac{1}{2}\vec{a}_s\Delta t^2 = \underbrace{\begin{bmatrix} p_{x0} + v_{x0}\Delta t + \frac{1}{2}a_{x0}\Delta t^2 & p_{x1} + v_{x1}\Delta t + \frac{1}{2}a_{x1}\Delta t^2 & \cdots \\ p_{y0} + v_{y0}\Delta t + \frac{1}{2}a_{y0}\Delta t^2 & p_{y1} + v_{y1}\Delta t + \frac{1}{2}a_{y1}\Delta t^2 & \cdots \\ p_{z0} + v_{z0}\Delta t + \frac{1}{2}a_{z0}\Delta t^2 & p_{z1} + v_{z1}\Delta t + \frac{1}{2}a_{z1}\Delta t^2 & \cdots \end{bmatrix}}_{A} \begin{bmatrix} \phi_0(\xi, \eta) \\ \phi_1(\xi, \eta) \\ \vdots \\ \phi_{n-1}(\xi, \eta) \end{bmatrix}$$

The above results in a system of three equations and three unknowns - $\xi$, $\eta$, and $\Delta t$.

After the contact point is found as well as the $\Delta t$, the node is considered to be in contact with the patch if the following conditions are met:

1. All reference coordinates ($\xi$, $\eta$) are between -1 and 1.
2. The solution for $\Delta t$ is between 0 and $dt$ (the time step of the explicit analysis).

## 2 Sympy Solution

[2]:
```
eq1 = sp.Eq(p_s + v_s*del_t + sp.Rational(1, 2)*a_s*del_t**2, sp.Sum(phi_k*(p_k
 ↪+ v_k*del_t + sp.Rational(1, 2)*a_k*del_t**2), (k, 0, n-1)))
eq1
```

[2]:

$$\frac{\Delta t^2\vec{a}_s}{2} + \Delta t\vec{v}_s + \vec{p}_s = \sum_{k=0}^{n-1}\left(\frac{\Delta t^2\vec{a}_k}{2} + \Delta t\vec{v}_k + \vec{p}_k\right)\phi_k(\xi, \eta)$$

[3]:
```
# Working off the matrix form
A = sp.Matrix([sp.Function('A')(del_t)])
eq2 = sp.Eq(eq1.lhs, sp.MatMul(A, sp.Matrix([phi_k])), evaluate=False)
eq2
```

[3]: $\frac{\Delta t^2 \vec{a}_s}{2} + \Delta t \vec{v}_s + \vec{p}_s = \left[A(\Delta t)\right] \left[\phi_k(\xi, \eta)\right]$

In the vector form, the Newton-Raphson scheme is

$$\begin{bmatrix} \xi_{i+1} \\ \eta_{i+1} \\ \Delta t_{i+1} \end{bmatrix} = \begin{bmatrix} \xi_i \\ \eta_i \\ \Delta t_i \end{bmatrix} - \mathbf{J}^{-1}\mathbf{F}$$

[4]:
```
# Constructing the vector function F
F = eq2.rhs - sp.Matrix([eq2.lhs])
F
```

[4]: $\left[-\frac{\Delta t^2 \vec{a}_s}{2} - \Delta t \vec{v}_s - \vec{p}_s + A(\Delta t)\phi_k(\xi, \eta)\right]$

[5]:
```
# Constructing the Jacobian matrix J
J = F.jacobian([xi, eta, del_t])
J
```

[5]: $\left[A(\Delta t)\frac{\partial}{\partial \xi}\phi_k(\xi, \eta) \quad A(\Delta t)\frac{\partial}{\partial \eta}\phi_k(\xi, \eta) \quad -\Delta t \vec{a}_s - \vec{v}_s + \phi_k(\xi, \eta)\frac{d}{d\Delta t}A(\Delta t)\right]$

The above is correct, but the $\frac{\partial}{\partial \Delta t}A(\Delta_t)$ must be to the left of the $\phi_k$ vector. The $\frac{\partial}{\partial \Delta t}A$ matrix is constructed as

$$\frac{\partial}{\partial \Delta t}A = \begin{bmatrix} v_{x0} + a_{x0}\Delta t & v_{x1} + a_{x1}\Delta t & \cdots \\ v_{y0} + a_{y0}\Delta t & v_{y1} + a_{y1}\Delta t & \cdots \\ v_{z0} + a_{z0}\Delta t & v_{z1} + a_{z1}\Delta t & \cdots \end{bmatrix}$$