Dynamical Systems Homework 3

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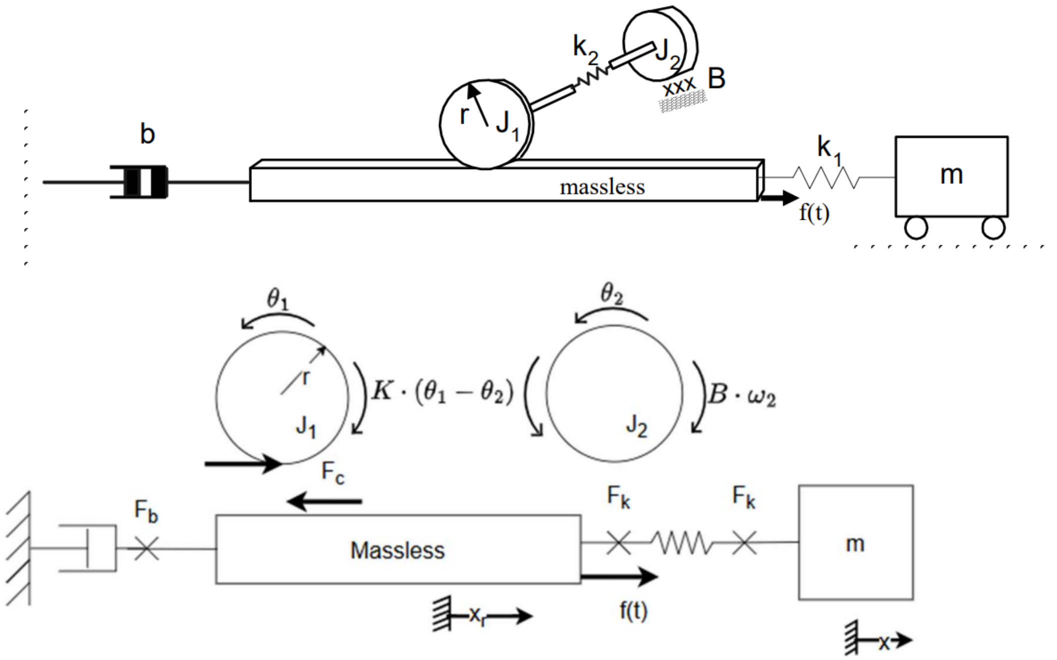
[1]:
```python
# toc
import control as ct
import matplotlib.pyplot as plt
import numpy as np
import sympy as sp
from scipy.integrate import solve_ivp, cumulative_trapezoid

plt.style.use('../maroon_ipynb.mplstyle')
```

Contents

Problem 1

Given



The constants from the above figure are given as follows:

$$
\begin{aligned}
r &= 0.05\,m \\
k_1 &= 1000\,N/m \\
k_2 &= 500\,N/m \\
b &= 100\,N \cdot s/m \\
B &= 2.5\,N \cdot m \cdot s \\
J_1 = J_2 &= 1\,kg \cdot m^2 \\
m &= 1\,kg
\end{aligned}
$$

Find

With the above system,

a. Develop a set of state variable equations. Define the state variable using the general matrix form: $\dot{S} = A \cdot S + B \cdot U$, where A and B are matrices and S is a vector of state variables and U is a vector of inputs. Be careful with labeling since one of the damping coefficients is also labeled B in the figure.
b. Solve the state variable equations using a 4th order Runge-Kutta numerical method for a total simulation time of 20 seconds and assuming that all initial velocities and spring forces are zero and that $f(t) = 10\,N$. Make sure the check that the number of points used in the simulation is adequate.

c. Verify your results using energy conservation. Show that at a given time, t, the total energy that has crossed the boundary through inputs and dampers is equal to the total energy stored in the system through mass/inertias and springs.
d. Use the output matrix form $Y = C \cdot S + D \cdot U$, where Y is the chosen output and C and D are matrices. Use this form to define the contact force between the rack and the pinion using state variable simulation results. Then plot the contact force.
e. Find the input-output equation between the input force and the contact force. One approach of doing this is the use the `ss2tf` function in the `scipy.signal` module.
f. Use the laplace transform approach to solve the input-output equation and compare with results from part d. Comment on how the roots of the characteristic polynomial affect the response.

Solution

Part A

From the above figure, the massless element is denoted as x_1 and the mass denoted by m is x_2.

[2]:
```
# Define symbols
r, k1, k2, b, B, J1, J2, m, t, s = sp.symbols('r k1 k2 b B J1 J2 m t s',␣
 ↪real=True, positive=True)
th1, th2 = sp.Function('theta_1')(t), sp.Function('theta_2')(t)
x1, x2 = sp.Function('x1')(t), sp.Function('x2')(t)
f, F_c = sp.Function('f')(t), sp.Function('F_c')(t)

# Construct equations
eq1 = sp.Eq(0, f - F_c - b*x1.diff() + k1*(x2 - x1))
eq2 = sp.Eq(m*x2.diff(t, 2), k1*(x1 - x2))
eq3 = sp.Eq(J1*th1.diff(t, 2), F_c*r + k2*(th2 - th1))
eq4 = sp.Eq(J2*th2.diff(t, 2), k2*(th1 - th2) - B*th2.diff())
eq5 = sp.Eq(x1.diff(t, 2), r*th1.diff(t, 2))
eqs = [eq1, eq2, eq3, eq4, eq5]
display(*eqs)
```

$$0 = -b\frac{d}{dt}x_1(t) + k_1\left(-x_1(t) + x_2(t)\right) - F_c(t) + f(t)$$

$$m\frac{d^2}{dt^2}x_2(t) = k_1\left(x_1(t) - x_2(t)\right)$$

$$J_1\frac{d^2}{dt^2}\theta_1(t) = k_2\left(-\theta_1(t) + \theta_2(t)\right) + rF_c(t)$$

$$J_2\frac{d^2}{dt^2}\theta_2(t) = -B\frac{d}{dt}\theta_2(t) + k_2\left(\theta_1(t) - \theta_2(t)\right)$$

$$\frac{d^2}{dt^2}x_1(t) = r\frac{d^2}{dt^2}\theta_1(t)$$

Let's clean this up a little by considering the velocities v_1, v_2, ω_1, and ω_2. Also, let's omit considering the coordinate positions in the system and only look to the displacements of the compliance elements. I will define Δx and $\Delta\theta$ as

$$\begin{cases} \Delta x = x_2 - x_1 \\ \Delta\theta = \theta_2 - \theta_1 \end{cases}$$

[3]:
```
d_theta = sp.Function(r'\Delta\theta')(t)
d_x = sp.Function(r'\Delta x')(t)

v1, v2 = sp.Function('v1')(t), sp.Function('v2')(t)
omega1, omega2 = sp.Function(r'\omega_1')(t), sp.Function(r'\omega_2')(t)

subs = [
    (x2 - x1, d_x),
    (th2 - th1, d_theta),
    (x1.diff(), v1),
    (x2.diff(), v2),
    (th1.diff(), omega1),
    (th2.diff(), omega2)
]

eqs = [eq.subs(subs) for eq in eqs]
display(*eqs)
```

$$0 = -bv_1(t) + k_1\Delta x(t) - F_c(t) + f(t)$$

$$m\frac{d}{dt}v_2(t) = -k_1\Delta x(t)$$

$$J_1\frac{d}{dt}\omega_1(t) = k_2\Delta\theta(t) + rF_c(t)$$

$$J_2\frac{d}{dt}\omega_2(t) = -B\omega_2(t) - k_2\Delta\theta(t)$$

$$\frac{d}{dt}v_1(t) = r\frac{d}{dt}\omega_1(t)$$

Now we can solve this system for $\dot{v}_1$, $\dot{v}_2$, $\dot{\omega}_1$ $\dot{\omega}_2$. Additionally, I will solve for F_c to be used later.

[4]:
```
state_sol = sp.solve(eqs, (v1.diff(), v2.diff(), omega1.diff(), omega2.diff(),␣
↪F_c), dict=True)[0]
for key, value in state_sol.items():
    display(sp.Eq(key, value))
```

$$F_c(t) = -bv_1(t) + k_1\Delta x(t) + f(t)$$

$$\frac{d}{dt}\omega_1(t) = -\frac{brv_1(t)}{J_1} + \frac{k_1r\Delta x(t)}{J_1} + \frac{k_2\Delta\theta(t)}{J_1} + \frac{rf(t)}{J_1}$$

$$\frac{d}{dt}\omega_2(t) = -\frac{B\omega_2(t)}{J_2} - \frac{k_2\Delta\theta(t)}{J_2}$$

$$\frac{d}{dt}v_1(t) = -\frac{br^2v_1(t)}{J_1} + \frac{k_1r^2\Delta x(t)}{J_1} + \frac{k_2r\Delta\theta(t)}{J_1} + \frac{r^2f(t)}{J_1}$$

$$\frac{d}{dt}v_2(t) = -\frac{k_1\Delta x(t)}{m}$$

Notice in the above solution that we don't have any terms that include ω_1. This is due to the direct algebraic relationship and the order in which `sympy` solved the system. Therefore, we can ignore the $\dot{\omega}_1$ equation. Also, the solution for F_c is not an ODE so it isn't directly a part of the state space solution.

[5]:

```python
# Getting the final five equations
eq1 = sp.Eq(d_x.diff(), v2 - v1)
eq2 = sp.Eq(d_theta.diff(), omega2 - v1/r)
eq3 = sp.Eq(v1.diff(), state_sol[v1.diff()])
eq4 = sp.Eq(v2.diff(), state_sol[v2.diff()])
eq5 = sp.Eq(omega2.diff(), state_sol[omega2.diff()])
eqs = [eq1, eq2, eq3, eq4, eq5]
display(*eqs)
```

$$\frac{d}{dt}\Delta x(t) = -v_1(t) + v_2(t)$$

$$\frac{d}{dt}\Delta\theta(t) = \omega_2(t) - \frac{v_1(t)}{r}$$

$$\frac{d}{dt}v_1(t) = -\frac{br^2v_1(t)}{J_1} + \frac{k_1r^2\Delta x(t)}{J_1} + \frac{k_2r\Delta\theta(t)}{J_1} + \frac{r^2f(t)}{J_1}$$

$$\frac{d}{dt}v_2(t) = -\frac{k_1\Delta x(t)}{m}$$

$$\frac{d}{dt}\omega_2(t) = -\frac{B\omega_2(t)}{J_2} - \frac{k_2\Delta\theta(t)}{J_2}$$

[6]:

```python
# Putting it in the state space form
S_dot = sp.Matrix([eq.lhs for eq in eqs])
S = S_dot.integrate(t)
rhs = [eq.rhs for eq in eqs]
A, b_sym = sp.linear_eq_to_matrix(rhs, list(S))
B_sym = -b_sym/f
U = sp.Matrix([f])
ans = sp.Eq(S_dot, sp.Add(sp.MatMul(A, S), sp.MatMul(B_sym, U)))  # doing this␣
↪makes it not simplify
ans
```

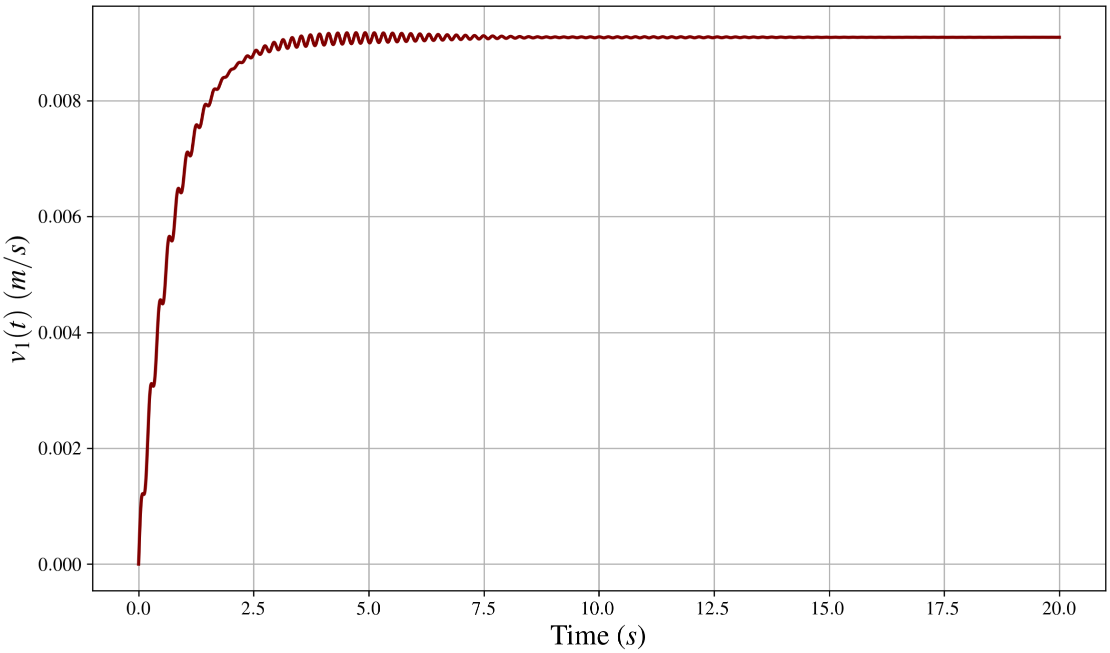
Answer

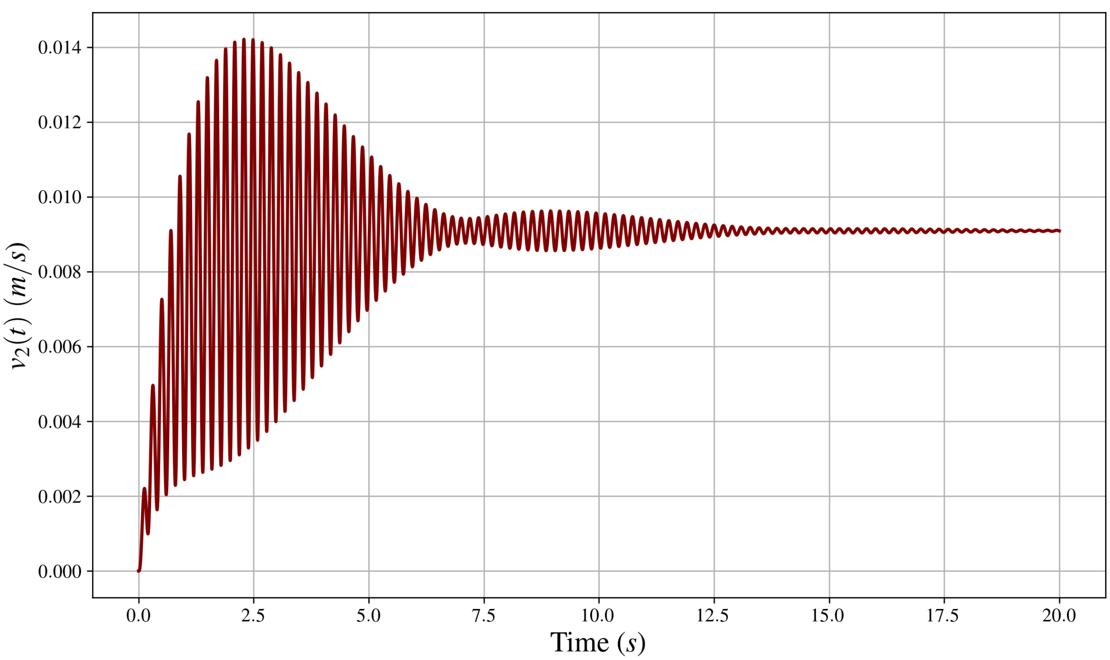
$$\begin{bmatrix} \frac{d}{dt}\Delta x(t) \\ \frac{d}{dt}\Delta\theta(t) \\ \frac{d}{dt}v_1(t) \\ \frac{d}{dt}v_2(t) \\ \frac{d}{dt}\omega_2(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \frac{r^2}{J_1} \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} f(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -\frac{1}{r} & 0 & 1 \\ \frac{k_1r^2}{J_1} & \frac{k_2r}{J_1} & -\frac{br^2}{J_1} & 0 & 0 \\ -\frac{k_1}{m} & 0 & 0 & 0 & 0 \\ 0 & -\frac{k_2}{J_2} & 0 & 0 & -\frac{B}{J_2} \end{bmatrix} \begin{bmatrix} \Delta x(t) \\ \Delta\theta(t) \\ v_1(t) \\ v_2(t) \\ \omega_2(t) \end{bmatrix}$$

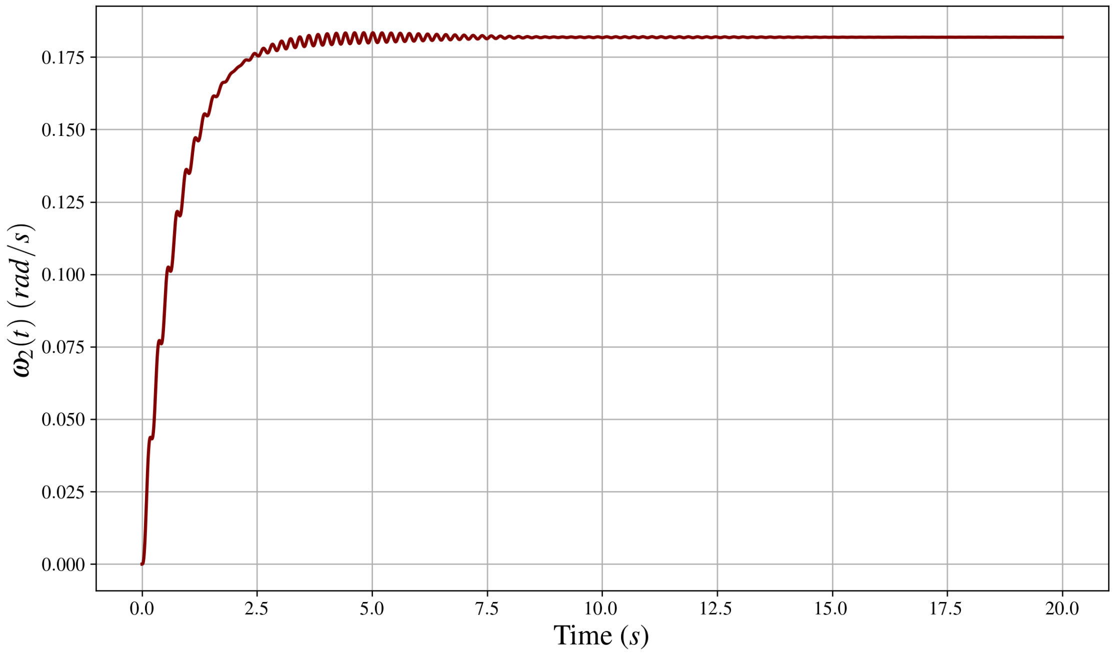
Part B

All we need to do is construct a python function that will return the right hand side of the equation above.

[7]:
```
r_ = 0.05
k1_ = 1000
k2_ = 500
b_ = 100
B_num = 2.5
J1_ = J2_ = 1
m_ = 1

sub_vals = [
    (r, sp.Rational(5, 100)),
    (k1, k1_),
    (k2, k2_),
    (b, b_),
    (B, sp.Rational(5, 2)),
    (J1, J1_),
    (J2, J2_),
    (m, m_)
]
f_lamb = lambda t_: 10

A_ = np.float64(A.subs(sub_vals))
B_ = np.float64(B_sym.subs(sub_vals)).flatten()

def state_vars(t_, y):
    return A_@y + B_*f_lamb(t_)

sol1 = solve_ivp(state_vars, (0, 20), [0]*5, method='RK45', rtol=1e-9,
 ↪atol=1e-12)

for i in range(2, 5):
    fig, ax = plt.subplots()
    ax.plot(sol1.t, sol1.y[i])

    unit = '$(m/s$)' if i != 4 else '$(rad/s)$'

    label = sp.latex(list(S)[i])
    ax.set_ylabel(f'${label}$ {unit}')
    ax.set_xlabel(f'Time ($s$)')
    plt.show()
```


0.000
0.002
0.004
0.006
0.008
$v_1(t)$ (m/s)
0.0
2.5
5.0
7.5
10.0
12.5
15.0
17.5
20.0
Time (s)


0.000
0.002
0.004
0.006
0.008
0.010
0.012
0.014
$v_2(t)$ (m/s)
0.0
2.5
5.0
7.5
10.0
12.5
15.0
17.5
20.0
Time (s)



The above uses `scipy`'s RK45 solver, which is more convenient for determining the correct step size. Also, I am only showing the solutions for the velocities as they have more meaning than the displacements. This is why I'm a fan of not combining the state variables into a displacement, but this method is easier to do by hand and has fewer equations.

Verification of Parts A-B

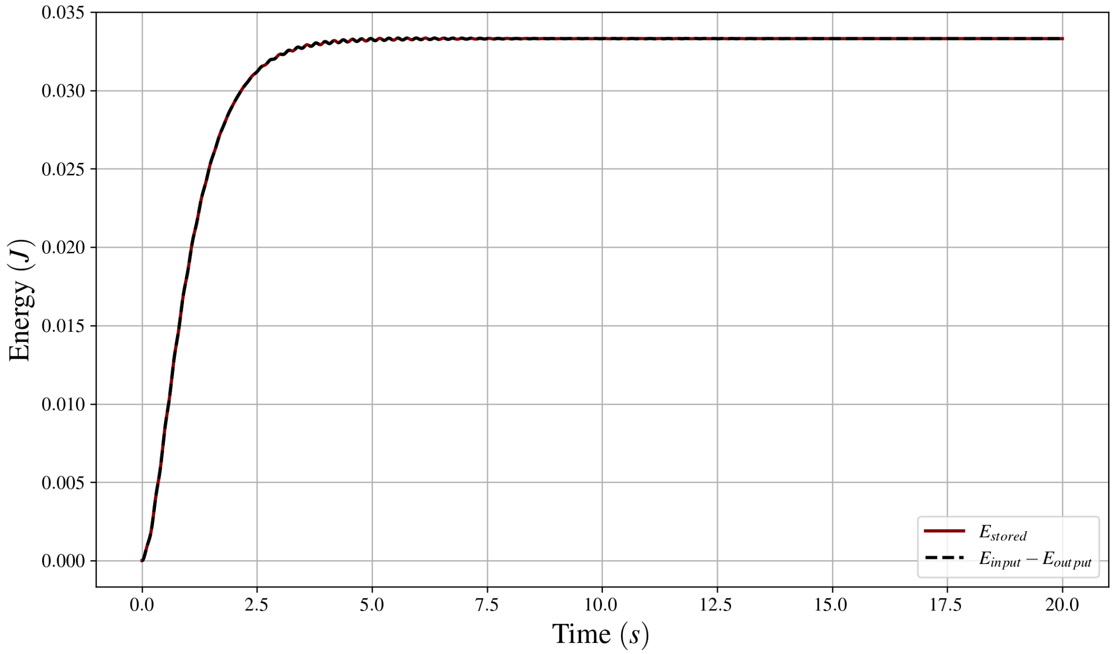
Part C

We can further verify the results by testing to see if the energy was conserved with the following relationship:

$$E_{stored}(t) = E_{input}(t) - E_{output}(t)$$

[8]:
```
d_x_, d_theta_, v1_, v2_, omega2_ = sol1.y

# Kinetic energy
KE = (
    1/2*(J1_/r_**2)*v1_**2 +
    1/2*m_*v2_**2 +
    1/2*J2_*omega2_**2
)

PE = (
    1/2*k1_*d_x_**2 +
    1/2*k2_*d_theta_**2
```

```
)
E_stored = KE + PE

# input power
P_input = f_lamb(sol1.t)*v1_
E_input = cumulative_trapezoid(P_input, sol1.t, initial=0)

# output power
P_output = b_*v1_**2 + B_num*omega2_**2
E_output = cumulative_trapezoid(P_output, sol1.t, initial=0)

fig, ax = plt.subplots()
ax.plot(sol1.t, E_stored, label=r'$E_{stored}$')
ax.plot(sol1.t, E_input - E_output, label=r'$E_{input}-E_{output}$', ls='--')
ax.set_xlabel('Time $(s)$')
ax.set_ylabel(r'Energy $(J)$')
ax.legend()
plt.show()
```



Part D

You can see from the previous work that the contact force is

[9]:
```
state_sol[F_c]
```

[9]: $-bv_1(t) + k_1\Delta x(t) + f(t)$

This means that you can represent C and D as

$$C = \begin{bmatrix} k_1 & 0 & -b & 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 1 \end{bmatrix}.$$

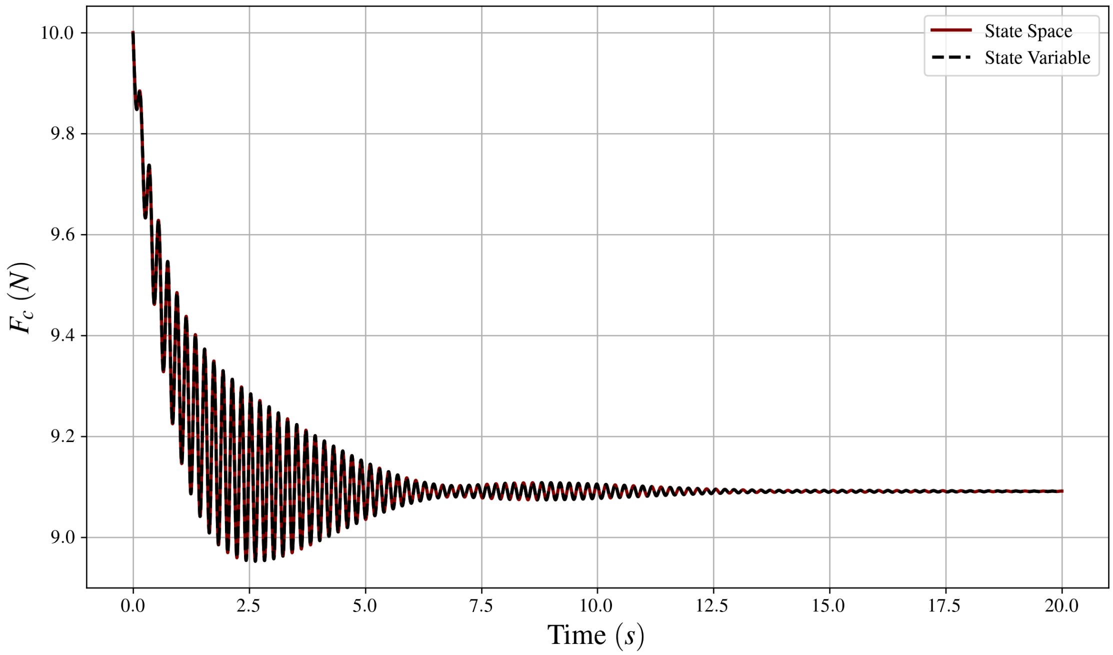
[10]:
```
C = np.array([[k1_, 0, -b_, 0, 0]])
D = np.array([[1]])

ss = ct.ss(A_, B_.reshape(-1, 1), C, D)
ss
```

[10]:

$$\left(\begin{array}{rllrrrr|l} 0 & & 0 & -1 & 1 & 0 & 0 \\ 0 & & 0 & -20 & 0 & 1 & 0 \\ 2.5 & & 25 & -0.25 & 0 & 0 & 0.0025 \\ -1 & \cdot 10^{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & & -500 & 0 & 0 & -2.5 & 0 \\ \hline 1 & \cdot 10^{3} & 0 & -100 & 0 & 0 & 1 \end{array}\right)$$

Here I used the `control` package to define the state space object. This library is more full-proof for this kind of work, and it offers an alternative to the matlab code in the undergraduate system dynamics book.

[11]:
```
# Getting the contact force using the control package
t_array = np.linspace(0, 20, sol1.t.size)
_, F_c_ = ct.forced_response(ss, T=t_array, U=f_lamb(t_array))

# Here is the solution using the original solution
F_c_original = -b_*v1_ + k1_*d_x_ + f_lamb(t_array)

fig, ax = plt.subplots()
ax.plot(t_array, F_c_, label='State Space')
ax.plot(sol1.t, F_c_original, label='State Variable', ls='--')
ax.set_xlabel('Time $(s)$')
ax.set_ylabel('$F_c$ $(N)$')
ax.legend()
plt.show()
```



Part E

The control package has a function to transform this to a transfer function. We can also find the transfer function algebraically with `sympy`.

[12]:
```
ct_tf = ct.ss2tf(ss)
ct_tf
```

Answer

$$\frac{s^5 + 2.5s^4 + 2000s^3 + 3750s^2 + 1 \times 10^6 s + 1.25 \times 10^6}{s^5 + 2.75s^4 + 2003s^3 + 4131s^2 + 1.002 \times 10^6 s + 1.375 \times 10^6}$$

[13]:
```
# Getting the symbolic solution
eqs_with_contact = eqs + [sp.Eq(F_c, state_sol[F_c])]

laplace = lambda expr: sp.laplace_transform(expr, t, s, noconds=True)

initial = [
    (d_x.subs(t, 0), 0),
    (d_theta.subs(t, 0), 0),
    (v1.subs(t, 0), 0),
    (v2.subs(t, 0), 0),
    (omega2.subs(t, 0), 0)
]
```

```
eqs_s = [sp.Eq(laplace(eq.lhs), laplace(eq.rhs)).subs(initial) for eq in
 eqs_with_contact]
S_s = [laplace(ex) for ex in list(S)]
sol_s = sp.solve(eqs_s, S_s + [laplace(F_c)], dict=True)[0]
for key, value in sol_s.items():
    display(sp.Eq(key, value.simplify()))
```

$$\mathcal{L}_t\left[F_c(t)\right](s) = \frac{(BJ_1k_1s^2 + BJ_1ms^4 + Bk_1k_2 + Bk_2ms^2 + J_1J_2k_1}{BJ_1k_1s^2 + BJ_1ms^4 + Bbk_1r^2s + Bbmr^2s^3 + Bk_1k_2 + Bk_1mr^2s^2 + Bk_2ms^2 + J_1J_2k_1s^3 + J_1J_2ms^5}$$

$$\mathcal{L}_t\left[\Delta x(t)\right](s) = \frac{mr^2s\,(-}{BJ_1k_1s^2 + BJ_1ms^4 + Bbk_1r^2s + Bbmr^2s^3 + Bk_1k_2 + Bk_1mr^2s^2 + Bk_2ms^2 + J_1J_2k_1s^3 + J_1J_2ms}$$

$$\mathcal{L}_t\left[\Delta\theta(t)\right](s) = \frac{r\,(-Bk_1 - B}{BJ_1k_1s^2 + BJ_1ms^4 + Bbk_1r^2s + Bbmr^2s^3 + Bk_1k_2 + Bk_1mr^2s^2 + Bk_2ms^2 + J_1J_2k_1s^3 + J_1J_2ms}$$

$$\mathcal{L}_t\left[\omega_2(t)\right](s) = \frac{k_2r}{BJ_1k_1s^2 + BJ_1ms^4 + Bbk_1r^2s + Bbmr^2s^3 + Bk_1k_2 + Bk_1mr^2s^2 + Bk_2ms^2 + J_1J_2k_1s^3 + J_1J_2ms^5}$$

$$\mathcal{L}_t\left[v_1(t)\right](s) = \frac{r^2\,(Bk_1s + Bms^3 + J_2}{BJ_1k_1s^2 + BJ_1ms^4 + Bbk_1r^2s + Bbmr^2s^3 + Bk_1k_2 + Bk_1mr^2s^2 + Bk_2ms^2 + J_1J_2k_1s^3 + J_1J_2ms^5}$$

$$\mathcal{L}_t\left[v_2(t)\right](s) = \frac{k_1r^2\,(B}{BJ_1k_1s^2 + BJ_1ms^4 + Bbk_1r^2s + Bbmr^2s^3 + Bk_1k_2 + Bk_1mr^2s^2 + Bk_2ms^2 + J_1J_2k_1s^3 + J_1J_2ms^5}$$

Very large expressions! Note that the denominator is the same across all results. This means we are doing good!

[14]:
```
# Only getting the contact force and substituting values
sp_tf = sol_s[laplace(F_c)]/laplace(f)
sp_tf = sp_tf.subs(sub_vals).simplify()
sp_tf
```

[14]:

$$\frac{4\left(2s^5 + 5s^4 + 4000s^3 + 7500s^2 + 2000000s + 2500000\right)}{8s^5 + 22s^4 + 16025s^3 + 33050s^2 + 8015000s + 11000000}$$

The results are a match.

Verification of Parts D-E

Part F

[15]:
```
# Taking the inverse laplace to solve for F_c
F_c_s = sp_tf*10/s
num, den = sp.fraction(F_c_s)
num = sp.Poly(num.expand(), s)
den = sp.Poly(den.expand(), s)
F_poly = num.as_expr()/den.as_expr()
# f_c_t = sp.inverse_laplace_transform(F_poly, s, t)
# f_c_t
F_poly
```

[15]:

$$\frac{80s^5 + 200s^4 + 160000s^3 + 300000s^2 + 80000000s + 100000000}{8s^6 + 22s^5 + 16025s^4 + 33050s^3 + 8015000s^2 + 11000000s}$$

That above cell will result in an error due to complexity. I tried getting this symbolically with MatLab as well. I guess we can do it manually.

[16]:
```python
# Find roots
r1, r2, r3, r4, r5, r6 = roots = sp.nroots(den)
display(*roots)
factor_den = sp.prod([(s - root) for root in roots])
F_c_s = num/8/factor_den
F_c_s.n(5)
```

-1.3750300358734

0

$-0.379308107105189 - 31.1793861747501i$

$-0.379308107105189 + 31.1793861747501i$

$-0.308176874958113 - 32.0682682599475i$

$-0.308176874958113 + 32.0682682599475i$

[16]:
$$\frac{10.0s^5 + 25.0s^4 + 20000.0s^3 + 37500.0s^2 + 1.0 \cdot 10^7 s + 1.25 \cdot 10^7}{s\,(s + 1.375)\,(s + 0.30818 - 32.068i)\,(s + 0.30818 + 32.068i)\,(s + 0.37931 - 31.179i)\,(s + 0.37931 + 31.179i)}$$

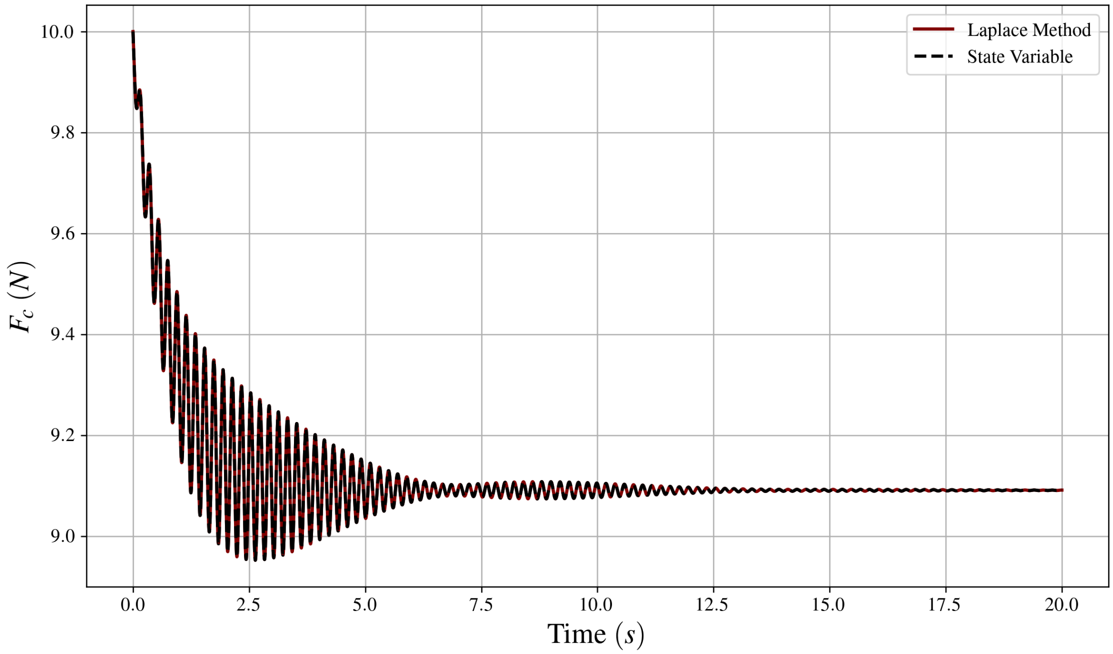
[17]:
```python
C1 = (F_c_s*s).subs(s, 0).simplify()
C2 = (F_c_s*(s + -r1)).subs(s, r1).simplify()
C3 = (F_c_s*(s + -r3)).subs(s, r3).simplify()
C4 = (F_c_s*(s + -r4)).subs(s, r4).simplify()
C5 = (F_c_s*(s + -r5)).subs(s, r5).simplify()
C6 = (F_c_s*(s + -r6)).subs(s, r6).simplify()
F_c_s_rhs = C1/s + C2/(s - r1) + C3/(s - r3) + C4/(s - r4) + C5/(s - r5) + C6/
  ↪(s - r6)
F_c_s_rhs.n(5)
```

[17]:
$$\frac{-0.1001 - 0.010002i}{s + 0.37931 + 31.179i} + \frac{-0.1001 + 0.010002i}{s + 0.37931 - 31.179i} + \frac{0.10772 - 0.01024i}{s + 0.30818 + 32.068i} + \frac{0.10772 + 0.01024i}{s + 0.30818 - 32.068i} + \frac{0.89384}{s + 1.375} + \frac{9.0909}{s}$$

The above representation is good enough for `sympy` to take the inverse Laplace.

[18]:
```python
f_c_t = sp.inverse_laplace_transform(F_c_s, s, t)
f_c_t_lamb = sp.lambdify(t, f_c_t, modules='numpy')

fig, ax = plt.subplots()
ax.plot(t_array, np.real(f_c_t_lamb(t_array)), label='Laplace Method')
ax.plot(sol1.t, F_c_original, label='State Variable', ls='--')
ax.set_xlabel('Time $(s)$')
ax.set_ylabel('$F_c$ $(N)$')
ax.legend()
```

```
plt.show()
```



[19]:
```
# Roots of cp
_, cp_den = sp.fraction(sp_tf)
cp_roots = sp.nroots(cp_den)
display(*cp_roots)
```

-1.3750300358734

$-0.379308107105189 - 31.1793861747501i$

$-0.379308107105189 + 31.1793861747501i$

$-0.308176874958113 - 32.0682682599475i$

$-0.308176874958113 + 32.0682682599475i$

We can get the time constant.

[20]:
```
# Getting time constant
cp_roots_ = np.complex64(cp_roots)
taus = -1/np.real(cp_roots_)
tau_max = np.max(taus)
float(tau_max)  # dominant root
```

[20]: 3.244889736175537

[21]:
```
float(4.6*tau_max)
```

[21]: 14.926492691040039

The analysis above shows us that the system should reach steady state in about 15 seconds and the graphs confirm this.

[22]:
```
# Getting peak to peak oscillation periods
1/np.imag(cp_roots_[1:])*2*np.pi  # seconds
```

[22]: array([-0.2015173 , 0.2015173 , -0.19593155, 0.19593155], dtype=float32)

Looking at the imaginary component shows us that we should have oscillations from peak to peak at 0.2 second time intervals. The graphs show this as well!