Engineering Analysis Homework 3

October 4, 2023

[1]:
```
import matplotlib.pyplot as plt
import numpy as np
import sympy as sp

from eng_analysis import romberg

plt.style.use('../maroon_ipynb.mplstyle')
```

The source code is found here.

Contents

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1 Problem 1

The gamma function is an extension of the factorial function to the real numbers such that for all positive integers:

$$\Gamma(n) = (n-1)!$$

Defined by:

$$\Gamma(n) = \int_0^\infty t^{n-1} e^{-t} dt$$

Find:

a. $\Gamma(5)$ using the Gauss-Laguerre quadrature.
b. $\Gamma(1.5)$ to 6 decimal places using Romberg integration. You will need to select some finite but sufficiently large upper bound for the numerical integral.

1.1 Solution

I will first find the analytical solution to check the results from part a and b.

[2]:
```
sp.gamma(5)
```

[2]:

24

[3]:
```
sp.gamma(sp.S('1.5'))
```

[3]:

0.886226925452758

1.1.1 Part A

The Gauss-Laguerre quadrature is determined by finding weights that are dependent on the roots of the Laguerre polynomial up to n degrees:

$$\int_0^\infty e^{-x} f(x) dx \approx \sum_{i=0}^{n} A_i f(x_i)$$

We can use `sympy` to get the weights and roots of the polynomial. The expression for $f(x)$ above is $f(x) = t^4$.

[4]:
```
def get_weights(n):
    x_ = sp.Symbol("x")
    roots = sp.Poly(sp.laguerre(n + 1, x_)).all_roots()
    x_i_ = [rt.evalf(20) for rt in roots]
```

```python
    w_i_ = [(rt/((n + 2)*sp.laguerre((n + 2), rt))**2).evalf(20) for rt in␣
 ↪roots]
    return np.float64(x_i_), np.float64(w_i_)


x_i, A_i = get_weights(2)
x_i, A_i
```

[4]:
```
(array([0.41577456, 2.29428036, 6.28994508]),
 array([0.71109301, 0.27851773, 0.01038926]))
```

[5]:
```python
f = lambda x_: x_**4
sol = sum(f(x_i)*A_i)
sol
```

[5]:
```
24.000000000000004
```

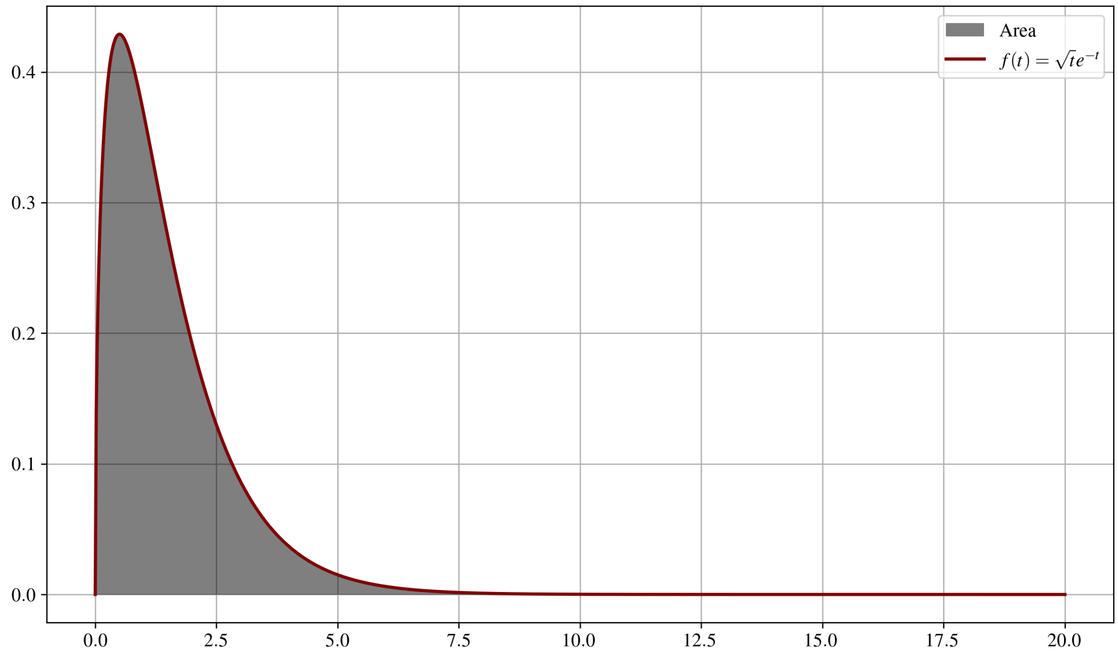
1.1.2 Part B

[6]:
```python
b = 16
f = lambda x_, n=1.5: x_**(n - 1)*np.exp(-x_)
sol, _ = romberg(f, 0, b)
sol
```

[6]:
```
0.8862262001064446
```

The above result is accurate to 6 decimal places. The selected bounds can be checked by graphing the result.

[7]:
```python
fig, ax = plt.subplots()
t = np.linspace(0, 20, 1000)
ax.fill_between(t[t <= b], f(t)[t <= b], zorder=2, color='black', alpha=0.5,␣
 ↪label='Area', lw=0)
ax.plot(t, f(t), label='$f(t)=\sqrt{t}e^{-t}$')
ax.legend()
plt.show()
```


Area
$f(t) = \sqrt{t}e^{-t}$
0.4
0.3
0.2
0.1
0.0
0.0
2.5
5.0
7.5
10.0
12.5
15.0
17.5
20.0

2 Problem 2

Evaluate the following integral numerically using Romberg integration. How many intervals do you need to calculate it to 6-digit precision? Is there a transformation you can use to improve this?

$$\int_0^{\pi/4} \frac{dx}{\sqrt{\sin x}}$$

2.1 Solution

The solution requires a transformation because the romberg algorithm will divide by zero as is because of the $\sqrt{\sin(x)}$ in the denominator. First, we will use `sympy` to get a solution for checking.

[8]:
```
x = sp.Symbol('x')
integral = sp.Integral(1/(sp.sqrt(sp.sin(x))), (x, 0, sp.pi/4))
integral
```

[8]:

$$\int\limits_{0}^{\frac{\pi}{4}} \frac{1}{\sqrt{\sin{\left(x \right)}}}\, dx$$

[9]:
```
integral.n()
```

[9]: 1.79116133811118

The Romberg integration can be found by simply transforming the integral to a form that is continuous on the limits of integration. Here are the steps:

$$u = \sqrt{x}$$
$$du = \frac{\cos x}{2\sqrt{\sin x}}$$
$$dx = \frac{2\sqrt{\sin x}}{\sqrt{1-u^4}}$$
$$\int_0^{\pi/4} \frac{dx}{\sqrt{\sin x}} = \int_0^{2^{-1/4}} \frac{2}{\sqrt{1-u^4}} du$$

[10]:
```
f = lambda u: 2/np.sqrt(1 - u**4)

sol, n_panel = romberg(f, 0, 2**(-1/4))
sol
```

[10]: 1.7911613389539645

[11]:
```
n_panel  # numer of panels/intervals
```

```
[11]: 64
```

3 Problem 3

Material toughness describes the ability of a material to absorb energy before fracture and is defined as the area under the stress-strain curve:

$$A = \int_0^{\epsilon_f} \sigma d\epsilon$$

Using the following stress-strain results, approximate the toughness of the specimen.

σ(MPa)	ε
586	0.001
662	0.025
765	0.045
841	0.068
814	0.089
122	0.122
150	0.150

3.1 Solution

This can be solved by using the trapezoid rule.

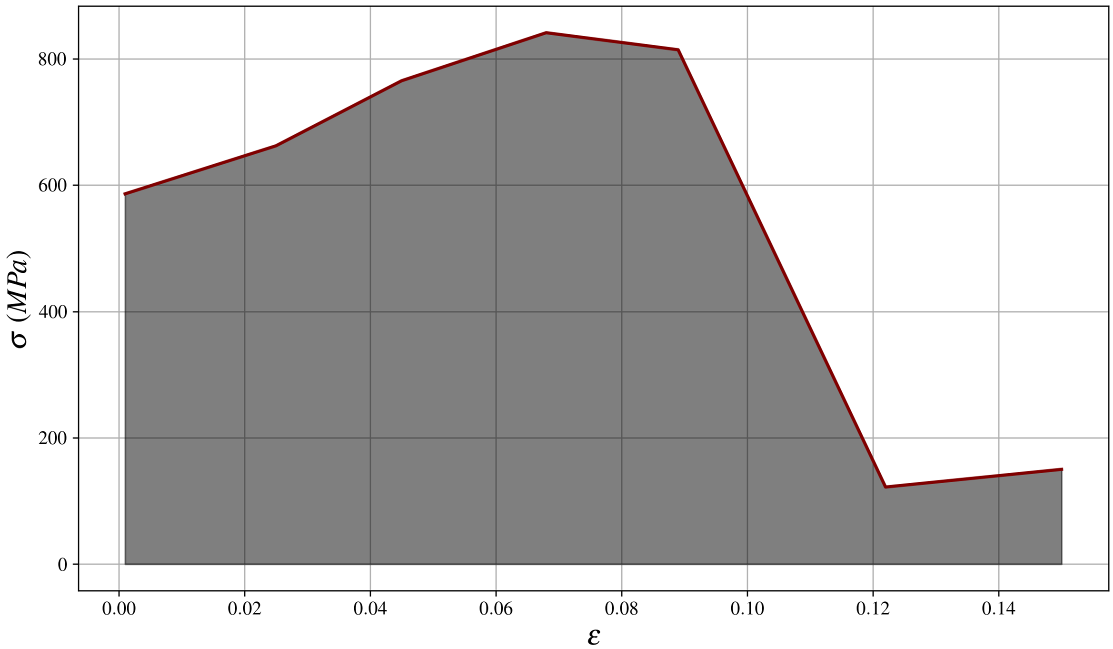
[12]:
```
sig = [586, 662, 765, 841, 814, 122, 150]   # MPa
eps = [0.001, 0.025, 0.045, 0.068, 0.089, 0.122, 0.150]

area = 0
for i in range(len(sig) - 1): area += (sig[i] + sig[i + 1])/2 * (eps[i + 1] -
 ↪eps[i])
area
```

[12]: 84.34450000000001

[13]:
```
np.trapz(sig, eps)   # Just to check
```

[13]: 84.34450000000001

[14]:
```
fig, ax = plt.subplots()
ax.plot(eps, sig, zorder=3)
ax.fill_between(eps, sig, color='black', alpha=0.5, zorder=2)
ax.set_xlabel('$\epsilon$')
ax.set_ylabel('$\sigma$ ($MPa$)')
plt.show()
```



This data is horrible, and I don't believe it for a second.