Engineering Analysis Homework 4

October 21, 2023

[1]:
```
import sympy as sp
import numpy as np
import matplotlib.pyplot as plt
from eng_analysis import gauss_legendre2, runge_kutta, CubicSpline,␣
 ↪newton_raphson

plt.style.use('../maroon_ipynb.mplstyle')
```

Additions to the source code can be found here.

Contents

1 Problem 1

Compute the approximate value of

$$\int_{-1}^{1}\int_{-1}^{1} e^{-(x^2+y^2)} dxdy$$

using a Gauss-Legendre quadrature. What integration order is necessary to get the answer accurate to 6 decimal places.

1.1 Solution

The analytical solution can be found using `sympy`.

[2]:
```python
x, y = sp.symbols('x y')
prob = sp.Integral(sp.exp(-x**2 - y**2), (x, -1, 1), (y, -1, 1))
prob
```

[2]:

$$\int\limits_{-1}^{1}\int\limits_{-1}^{1} e^{-x^2-y^2}\, dx\, dy$$

[3]:
```python
ana = prob.doit().n()
ana
```

[3]: 2.23098514140413

The Gauss-Legendre quadrature is determined by getting the nodes and weights up to some order and applying the following relationship:

$$I = \int_{-1}^{1} \sum_{i=0}^{n} A_i f(\xi_i, \eta)\, d\eta = \sum_{j=0}^{n} A_j \left[\sum_{i=0}^{n} A_i f(\xi_i, \eta_j)\right]$$

This process is shown in the `gauss_legendre2` function.

[4]:
```python
f = lambda x_, y_: np.exp(-x_**2 - y_**2)
x = [-1, 1, 1, -1]
y = [-1, -1, 1, 1]
m = 2
sol = 0

while abs(sol - float(ana)) >= 1e-6:
    m += 1
    sol = gauss_legendre2(f, x, y, m)

sol
```

[4]: 2.230985210842258

[5]:
```
m
```

[5]: 7

As seen above, an order of $m = 7$ will result in a value within 6 decimal places.

2 Problem 2

A ball of mass $m = 0.25\,kg$ is launched with an initial velocity $v_0 = 50\,m/s$ at an angle of 30° above the horizontal. Assuming the drag force acting on the ball is given by $F_D = C_D v^{3/2}$, the equations describing the motion in the horizontal and vertical directions are

$$\begin{cases} \ddot{x} = -\frac{C_D}{m}\dot{x}\sqrt{v} \\ \ddot{y} = -\frac{C_D}{m}\dot{y}\sqrt{v} - g \end{cases}$$

where v is the magnitude of the velocity ($v = \sqrt{\dot{x}^2 + \dot{y}^2}$). Using the Runge-Kutta method, determine the time before the ball hits the ground and how far away it lands from where it was launched. Use $C_d = 0.03\,\frac{kg}{\sqrt{ms}}$ and $g = 9.80665\,m/s^2$.

2.1 Solution

We need to first put this system in the state variable form by transforming it into a system of only first order differential equations. This can all be done symbolically through `sympy`. The initial velocity will need to be broken down into its vector components, and the greatest way to this is through Euler's Identity.

[6]:
```
v0 = 50*np.exp(1j*np.deg2rad(30))  # complex number ... real part is the x␣
↪component and the imaginary is the y component

t, CD, m, g = sp.symbols('t C_D m g')
x0, y0 = sp.Function('x0')(t), sp.Function('y0')(t)

sub_values = [
    (CD, sp.S('0.03')),
    (m, sp.S('0.25')),
    (g, sp.S('9.80665'))
]

v = sp.sqrt(x0.diff()**2 + y0.diff()**2)
eq1 = sp.Eq(x0.diff(t, 2), -CD/m*x0.diff()*sp.sqrt(v))
eq2 = sp.Eq(y0.diff(t, 2), -CD/m*y0.diff()*sp.sqrt(v) - g)
display(eq1, eq2)
```

$$\frac{d^2}{dt^2}x_0(t) = -\frac{C_D\sqrt[4]{\left(\frac{d}{dt}x_0(t)\right)^2 + \left(\frac{d}{dt}y_0(t)\right)^2}\frac{d}{dt}x_0(t)}{m}$$

$$\frac{d^2}{dt^2}y_0(t) = -\frac{C_D\sqrt[4]{\left(\frac{d}{dt}x_0(t)\right)^2 + \left(\frac{d}{dt}y_0(t)\right)^2}\frac{d}{dt}y_0(t)}{m} - g$$

[7]:

```
# Putting it into the state variable form
x1, y1 = sp.Function('x1')(t), sp.Function('y1')(t)

eq3 = sp.Eq(x0.diff(), x1)
eq4 = sp.Eq(y0.diff(), y1)

sub_states = [
    (x0.diff(t, 2), x1.diff()),
    (x0.diff(), x1),
    (y0.diff(t, 2), y1.diff()),
    (y0.diff(), y1)
]

eq1 = eq1.subs(sub_states)
eq2 = eq2.subs(sub_states)

state_sol = sp.solve([eq1, eq2, eq3, eq4], [x0.diff(), x1.diff(), y0.diff(), y1.
  ↪diff()], dict=True)[0]
for key, value in state_sol.items(): display(sp.Eq(key, value))
```

$$\frac{d}{dt}x_0(t) = x_1(t)$$

$$\frac{d}{dt}x_1(t) = -\frac{C_D \sqrt[4]{x_1^2(t) + y_1^2(t)}\, x_1(t)}{m}$$

$$\frac{d}{dt}y_0(t) = y_1(t)$$

$$\frac{d}{dt}y_1(t) = -\frac{C_D \sqrt[4]{x_1^2(t) + y_1^2(t)}\, y_1(t)}{m} - g$$

[8]:

```
# With values substituted
for key, value in state_sol.items():
    v = value.subs(sub_values).simplify()
    display(sp.Eq(key, v))
```

$$\frac{d}{dt}x_0(t) = x_1(t)$$

$$\frac{d}{dt}x_1(t) = -0.12 \sqrt[4]{x_1^2(t) + y_1^2(t)}\, x_1(t)$$

$$\frac{d}{dt}y_0(t) = y_1(t)$$

$$\frac{d}{dt}y_1(t) = -0.12 \sqrt[4]{x_1^2(t) + y_1^2(t)}\, y_1(t) - 9.80665$$

We can now implement the Runge-Kutta method to solve this problem.

[9]:

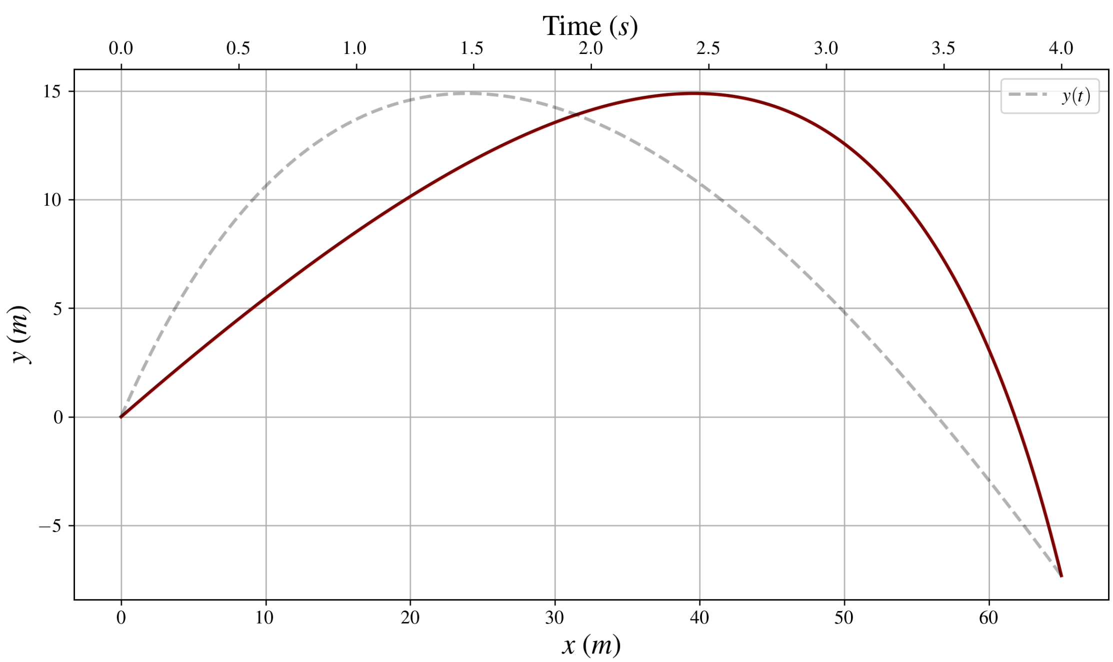
```
def state_vars(z, _):
    x0_, x1_, y0_, y1_ = z
```

```
    return [
        x1_,
        -0.12*(x1_**2 + y1_**2)**0.25*x1_,
        y1_,
        -0.12*(x1_**2 + y1_**2)**0.25*y1_ - 9.80665
    ]

t_array = np.linspace(0, 4, 500)
sol = runge_kutta(state_vars, (0, np.real(v0), 0, np.imag(v0)), t_array)
x_points, y_points = sol[0], sol[2]

fig, ax = plt.subplots()
ax2 = ax.twiny()
ax2.grid(False)
ax.set_xlabel('$x$ ($m$)')
ax.set_ylabel('$y$ ($m$)')
ax2.set_xlabel('Time ($s$)')

ax.plot(x_points, y_points)
ax2.plot(t_array, y_points, ls='--', color='black', alpha=0.3, label='$y(t)$')
ax2.legend()
plt.show()
```



The time is takes is close to 3.5 seconds as seen in the graph. Since we have the time points and the y points, we can make a cubic spline function, then use the Newton Raphson method to determine the actual value.

[10]:
```
spline_y = CubicSpline(t_array, y_points)
t_ground = newton_raphson(spline_y, [3.5])[0]
t_ground
```

[10]: 3.473241346676332

It takes $t = 3.47$ seconds till it hits the ground.

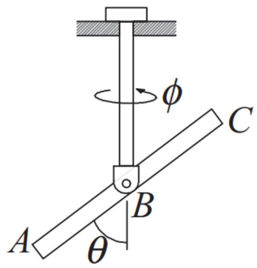
[11]:
```
# Finding the horizontal distance (close to 60)
spline_x = CubicSpline(t_array, x_points)
spline_x(t_ground)
```

[11]: 61.80007065961908

The horizontal distance is $61.8\,m$.

3 Problem 3

The bar ABC is attached to the vertical rod with a horizontal pin.



The assembly is free to rotate about the axis of the rod. Ignoring friction, the equations of motion for the system are:

$$\begin{cases} \ddot{\theta} = \dot{\phi}^2 \sin(\theta)\cos(\theta) \\ \ddot{\phi} = -2\dot{\theta}\dot{\phi}\cot(\theta) \end{cases}$$

The system has initial conditions as follows: $\theta(0) = \frac{\pi}{12}\,rad$, $\dot{\theta} = 0$, $\phi(0) = 0$, and $\dot{\phi}(0) = 20\,\frac{rad}{s}$.

Find a numerical solution for the system using the adaptive Runge-Kutta method for $t = 0$ to $t = 1.5$ seconds and plot $\dot{\phi}(t)$.

3.1 Solution

Just as before, the system needs to be placed in the state variable form.

[12]:
```
th0, phi0 = sp.Function(r'\theta_0')(t), sp.Function(r'\phi_0')(t)

eq1 = sp.Eq(th0.diff(t, 2), phi0.diff()**2*sp.sin(th0)*sp.cos(th0))
eq2 = sp.Eq(phi0.diff(t, 2), -2*th0.diff()*phi0.diff()*(1/sp.tan(th0)))
display(eq1, eq2)
```

$$\frac{d^2}{dt^2}\theta_0(t) = \sin\left(\theta_0(t)\right)\cos\left(\theta_0(t)\right)\left(\frac{d}{dt}\phi_0(t)\right)^2$$

$$\frac{d^2}{dt^2}\phi_0(t) = -\frac{2\frac{d}{dt}\phi_0(t)\frac{d}{dt}\theta_0(t)}{\tan\left(\theta_0(t)\right)}$$

[13]:
```
th1, phi1 = sp.Function(r'\theta_1')(t), sp.Function(r'\phi_1')(t)

eq3 = sp.Eq(th0.diff(), th1)
eq4 = sp.Eq(phi0.diff(), phi1)
```

```python
sub_states = [
    (th0.diff(t, 2), th1.diff()),
    (th0.diff(), th1),
    (phi0.diff(t, 2), phi1.diff()),
    (phi0.diff(), phi1)
]

eq1 = eq1.subs(sub_states)
eq2 = eq2.subs(sub_states)

state_sol = sp.solve([eq1, eq2, eq3, eq4], [th0.diff(), th1.diff(), phi0.
 ↪diff(), phi1.diff()], dict=True)[0]
for key, value in state_sol.items(): display(sp.Eq(key, value))
```

$$\frac{d}{dt}\phi_0(t) = \phi_1(t)$$

$$\frac{d}{dt}\phi_1(t) = -\frac{2\phi_1(t)\theta_1(t)}{\tan\left(\theta_0(t)\right)}$$

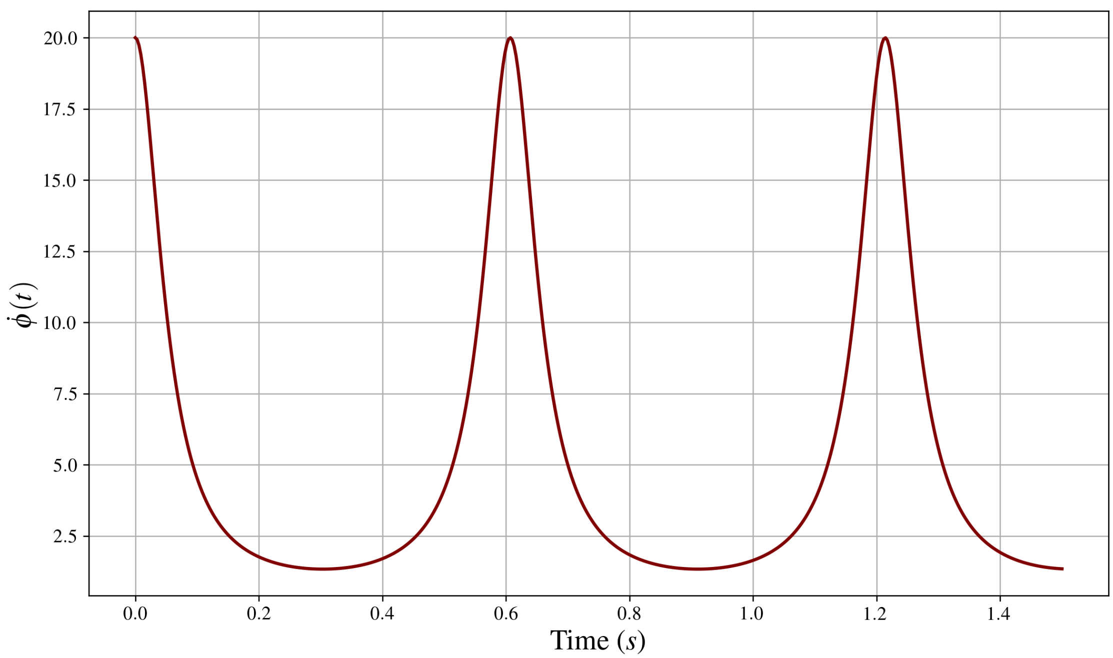
$$\frac{d}{dt}\theta_0(t) = \theta_1(t)$$

$$\frac{d}{dt}\theta_1(t) = \phi_1^2(t)\sin\left(\theta_0(t)\right)\cos\left(\theta_0(t)\right)$$

[14]:
```python
def state_vars(z, _):
    p0, p1, t0, t1 = z
    return [
        p1,
        -2*p1*t1/np.tan(t0),
        t1,
        p1**2*np.sin(t0)*np.cos(t0)
    ]

t_array = np.linspace(0, 1.5, 500)
sol = runge_kutta(state_vars, (0, 20, np.pi/12, 0), t_array)

plt.plot(t_array, sol[1])
plt.xlabel('Time ($s$)')
plt.ylabel('$\dot{\phi}(t)$')
plt.show()
```


20.0
17.5
15.0
12.5
10.0
7.5
5.0
2.5
$\dot{\phi}(t)$
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
Time (s)