MC2 Reflection 2

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Is it possible to construct a sequence (a_n) such that $\sum a_n$ converges but $\sum (a_n)^3$ diverges? To answer this question, we first define the necessary building blocks.

Definition 0.1 (Series Convergence)**.** A series

$$\sum_{n=1}^{\infty} a_n$$

converges if and only if the sequence of partial sums (s_m) converges (that is, $\lim_{m\to\infty} s_m = c$ for some constant c), where

$$s_m = \sum_{n=1}^{m} a_n.$$

If a series does not converge then it is divergent.

We also introduce the idea of infinite limits for a stronger case of divergence.

Definition 0.2 (Limit approaching infinity)**.** Let $f : \mathbb{N} \to \mathbb{R}$. $\lim_{x\to\infty} f(x) = \infty$ if for every $M > 0$, there exists $n \in \mathbb{N}$ such that $x > n$ implies $f(x) > M$.

Consider the famous harmonic series: $\sum \frac{1}{n}$.

Theorem 0.1.

$$\sum_{n=1}^{\infty} \frac{1}{n} = \infty.$$

Proof. To show $\lim_{n\to\infty} s_n \to \infty$, given $M > 0$ we want to show that there exists $N \in \mathbb{N}$ such that $n \geq N$ implies

$$s_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \cdots + \frac{1}{n} > M.$$

Notice that

$$\sum_{n=2}^{8} a_n < \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2} + \frac{1}{2} + \frac{1}{2}.$$

This motivates the following choice: given $M \in \mathbb{R}$, let $m = 2\lceil M \rceil$, twice the ceiling of M. pick

$$N = 1 + \sum_{k=0}^{m} 2^k.$$

$n \geq N$ implies $n \geq 1 + \sum 2^k$. Since none of the terms in the series are negative, it is sufficient to show that with $n = 1 + \sum 2^k$, $s_n > M$. We partition s_n into multiple sums whose bounds come from the sum that makes up n as follows:

$$s_n = a_1 + \sum_{n=2}^{2} a_n + \sum_{n=3}^{4} a_n + \sum_{n=5}^{8} a_n + \cdots + \sum_{n=2^{m-1}+1}^{2^m} a_n.$$

For any natural number k, the sum

$$\sum_{n=2^{k-1}+1}^{2^k} a_n$$

contains 2^{k-1} terms, each of which are greater than or equal to $\frac{1}{2^k}$, so

$$\sum_{n=2^{k-1}+1}^{2^k} a_n \geq 2^{k-1} \frac{1}{2^k} = \frac{1}{2}.$$

s_n contains m such sums, along with $a_1 = 1$, so

$$s_n \geq 1 + m \cdot \frac{1}{2} = 1 + \lceil M \rceil > M.$$

□

On the other hand, the same approach does not work for $b_n = (a_n)^3 = \frac{1}{n^3}$. That is,

$$\sum_{n=1}^{\infty} \frac{1}{n^3} = 1 + \frac{1}{8} + \frac{1}{27} + ... \neq \infty.$$

Every convergent series must have terms that get closer and closer to 0, so it is tempting to think that cubing each term in the sequence always results in a convergent series - or at the very least, a bounded series. In fact, this is true in the case that each term is non-negative.

Theorem 0.2. If $a_n \geq 0$ for all $n \in \mathbb{N}$ and $\sum a_n$ converges, then $\sum (a_n)^3$ converges.

Proof. Assume $a_n \geq 0$ for all $n \in \mathbb{N}$ and $\sum a_n$ converges, meaning $\lim_{m\to\infty} s_m = c$ for some $c \in \mathbb{R}$ where

$$s_m = \sum_{n=1}^{m} a_n.$$

The infinite limit of s_m being equal to c guarantees that there are finitely many values of n for which $a_n > 1$, so for some $N \in \mathbb{N}$, $n \geq N$ implies $a_n \leq 1$.

$$\sum_{n=1}^{\infty}(a_n)^3 = \sum_{n=1}^{N}(a_n)^3 + \sum_{n=N+1}^{\infty}(a_n)^3.$$

The sum up to N is the finite sum of finite terms so it is equal to a constant and $\sum(a_n)^3$ converges if the second sum on the right converges. For each of these terms in the sum starting from $n = N+1$, $0 \leq a_n \leq 1$ so $0 \leq (a_n)^3 \leq a_n$. Therefore

$$\sum_{n=N+1}^{\infty}(a_n)^3$$

converges by comparison test with

$$\sum_{n=N+1}^{\infty} a_n$$

(which is convergent since it is a subsequence of a non-negative convergent sequence), so $\sum(a_n)^3$ converges. □

However, this cannot be generalized to all convergent sequences (a_n). Motivated by the divergence of the harmonic series, we construct the following sequence.

Lemma 0.3. Let

$$(a_n) = b_1 - \frac{b_1}{2} - \frac{b_1}{2} + b_2 - \frac{b_2}{2} - \frac{b_2}{2} + \dots$$

where

$$b_n = \sqrt[3]{\frac{4}{3n}}.$$

$$\sum_{n=1}^{\infty} a_n = 0.$$

Proof. We will prove $\lim_{m\to\infty} s_m = 0$, where

$$s_m = \sum_{n=1}^{m} a_n.$$

Note that if m is a multiple of 3, $m = 3t$ for some $t \in \mathbb{N} \cup \{0\}$ so

$$s_m = b_1 - \frac{b_1}{2} - \frac{b_1}{2} + b_2 - \frac{b_2}{2} - \frac{b_2}{2} + \dots + b_t - \frac{b_t}{2} - \frac{b_t}{2} = 0 + 0 + \dots + 0 = 0.$$

Given $\varepsilon > 0$, pick N to be a natural number greater than $\frac{4}{3\varepsilon^3}$. Then $n \geq N$ implies

$$n > \frac{4}{3\varepsilon^3}.$$

Doing some algebraic manipulation yields

$$\sqrt[3]{\frac{4}{3n}} < \varepsilon.$$

Let m be the greatest multiple of 3 less than or equal to n; $m = 3t$ for some $t \in \mathbb{N} \cup \{0\}$. If $m = n$ (in the case that n is a multiple of 3), then the sum $s_n = 0$ as shown above. Otherwise

$$s_n = s_m + \sum_{k=m+1}^{n} a_k.$$

$s_m = 0$ for the same reason, and the sum on the right either looks like $b_{t+1} - \frac{b_{t+1}}{2}$ or b_{t+1} (both of which are positive) so

$$0 \leq s_n \leq b_{t+1} = \sqrt[3]{\frac{4}{3n}}$$

$$|s_n - 0| \leq \sqrt[3]{\frac{4}{3n}} < \varepsilon.$$

□

On the other hand, for the above sequence (a_n),

$$\sum_{n=1}^{\infty} (a_n)^3 = (b_1)^3 - \frac{(b_1)^3}{8} - \frac{(b_1)^3}{8} + (b_2)^3 - \frac{(b_2)^3}{8} - \frac{(b_2)^3}{8} + \dots.$$

Combining the fractions and plugging in the b_n terms we get

$$\frac{3(b_1)^3}{4} + \frac{3(b_2)^3}{4} + \frac{3(b_3)^3}{4} + \cdots = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots.$$

We recognize this as the harmonic series from earlier, which we proved diverges. Therefore, it is possible to construct a sequence (a_n) such that $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} (a_n)^3$ diverges. The trick that allowed us to do so was breaking a_n up into an unequal number of positive and negative terms - we used one positive and two negative - so that the sum still equals zero but the negative terms were made much smaller in magnitude relative to the positive terms when cubed. This method generalizes to higher odd exponents as well, and certainly other "targets" could be chosen besides the harmonic series to still get a divergent series from the sum of the cubed terms.

Overall, this problem felt quite difficult at first because it is rare to deal with sequences that have an number of positive and negative terms without being all one or the other, but once you see the solution it makes a lot of sense.