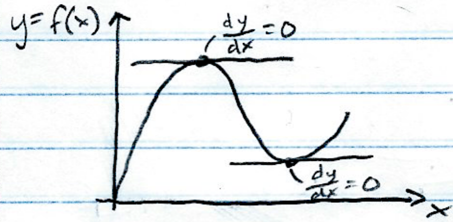
Multivariable Calculus

Recall: Derivative

With a function of 1 variable, the derivative is the instantaneous slope, or the slope of the tangent line

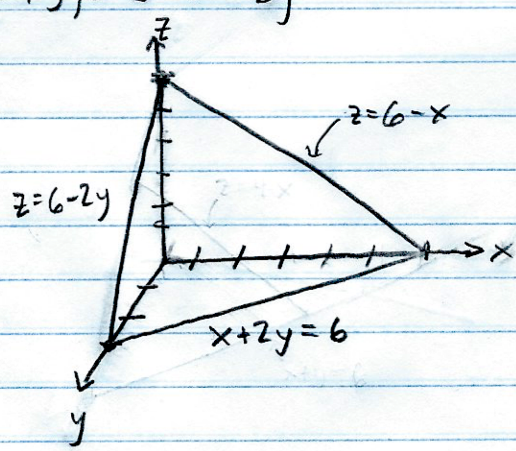


Thus, when the derivative is zero, the function has a local maximum or minimum.

How do we interpret the slope of a function of 2 or more variables? With the $\boxed{\text{partial derivative}}$

Consider the surface given by

$$z=f(x,y)=6-x-2y$$



A plane is like a straight line in higher dimensions

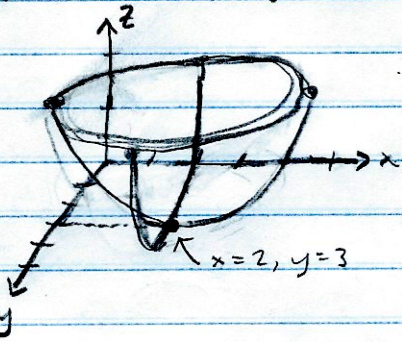
We take the partial derivative with respect to one variable by treating the others as constants.

$$\frac{\partial z}{\partial x}=-1 \qquad \frac{\partial z}{\partial y}=-2$$

Regardless of where we are on the surface, there is a constant slope in the x and y directions!

Additionally, we can have quadratics with more than one variable. Consider the function:

$$z = f(x,y) = (x-2)^2 + (y-3)^2$$



If we wanted to find the minimum of this function, we can find the partial derivatives and set them to 0.

$$\frac{\partial z}{\partial x} = 2(x-2) = 0 \longrightarrow 2x = 4 \longrightarrow \boxed{x=2}$$

$$\frac{\partial z}{\partial y} = 2(y-3) = 0 \longrightarrow 2y = 6 \longrightarrow \boxed{y=3}$$

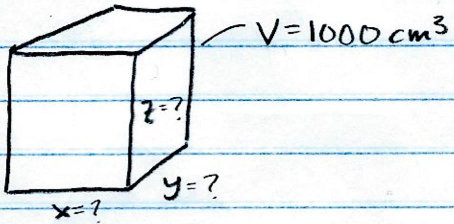
Recall, with a single variable, we determine whether a point is a max or min based on the second derivative. With more variables, it is slightly more complicated, as we need to consider multiple second order partial derivatives. When checking the point (x, y), there are 4 possible outcomes:

$$D = \frac{\partial^2 z}{\partial x^2} \cdot \frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2$$

1. $D > 0$ and $\frac{\partial^2 z}{\partial x^2} > 0 \Rightarrow (x, y)$ is a minimum
2. $D > 0$ and $\frac{\partial^2 z}{\partial x^2} < 0 \Rightarrow (x, y)$ is a maximum
3. $D < 0 \Rightarrow (x, y)$ is a saddle point
4. $D = 0 \Rightarrow$ no conclusion

Example: Minimizing Surface Area

We are tasked with choosing the dimensions of a box to minimize the surface area with the constraint that the volume of the box must be 1000 cm^3.



Thus, we are minimizing a function of 3 variables

$$A(x,y,z) = 2xy + 2xz + 2yz$$

By incorporating the volume constraint, we eliminate 1 variable

$$xyz = 1000 \longrightarrow z = \frac{1000}{xy}$$

$$A(x,y) = 2xy + 2x\left(\frac{1000}{xy}\right) + 2y\left(\frac{1000}{xy}\right)$$
$$= 2xy + \frac{2000}{y} + \frac{2000}{x}$$

Next, we take the partial derivatives and set to 0

$$\frac{\partial A}{\partial x} = 2y - \frac{2000}{x^2} = 0 \rightarrow yx^2 = 1000 \longrightarrow y = \frac{1000}{x^2}$$

$$\frac{\partial A}{\partial y} = 2x - \frac{2000}{y^2} = 0 \rightarrow xy^2 = 1000$$

Substituting $\frac{1000}{x}$ for y yields:

$$\frac{1000^2}{x^4}x = 1000 \rightarrow x^3 = 1000 \longrightarrow \boxed{x=10} \Rightarrow \boxed{y=10} \Rightarrow \boxed{z=10}$$

Finally, we test the second order partial derivatives

$$\frac{\partial^2 A(x,y)}{\partial x^2} = \frac{4000}{x^3} \rightarrow \frac{\partial^2 A(10,10)}{\partial x^2} = 4$$

$$\frac{\partial^2 A(x,y)}{\partial y^2} = \frac{4000}{y^3} \rightarrow \frac{\partial^2 A(10,10)}{\partial y^2} = 4$$

$$\frac{\partial^2 A(x,y)}{\partial x \partial y} = 2$$

Using the second order partial derivative rule

$\Rightarrow$ $4 \times 4 - 4 = 12 > 0$

$\boxed{x=10,\ y=10,\ z=10 \text{ is a minimum}}$

Recall: when taking the derivative of nested functions, we use the <u>chain rule</u>

$$F(x) = f(g(x)) \longrightarrow F'(x) = g'(x)\, f'(g(x))$$

By rewriting a function to use multiple variables, we can use partial derivatives to simplify calculation of the derivative. We begin with:

$$f(t) = h_1(g_1(t)) + h_2(g_2(t))$$

with derivative

$$f'(t) = g_1'(t)\, h_1'(g_1(t)) + g_2'(t)\, h_2'(g_2(t))$$

And rewrite as

$$f(x,y) = h_1(x) + h_2(y), \quad x = g_1(t), \quad y = g_2(t)$$

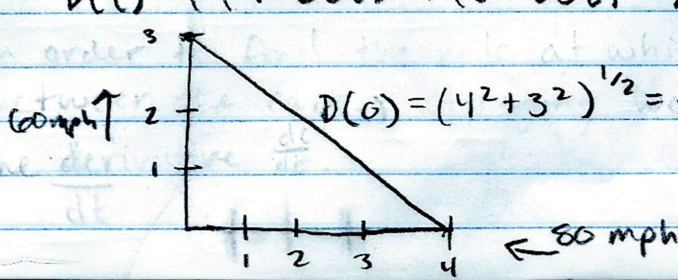
with derivative

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

<u>Example</u>: There are two cars. One is approaching an intersection from the east at 80 mph and is 4 miles from the intersection. The second car is 3 miles north of the intersection, driving away at 60 mph. Are the cars getting closer to each other or further away? At what rate is the distance between the cars (as the crow flies) changing?

<u>Solution</u>: We first formulate a function to represent the distance between the two cars:

$$D(t) = \left((4-80t)^2 + (3+60t)^2\right)^{1/2}$$



$$D(0) = (4^2+3^2)^{1/2} = \sqrt{25} = 5$$

In order to simplify calculation of the derivative, we rewrite the function as a multivariable function and use partial derivatives

$$D(x,y) = (x+y)^{1/2}, \quad x = (4-80t)^2, \quad y = (3+60t)^2$$

Using the chain rule, we calcutate the derivative with respect to t (hours)

$$\frac{dD}{dt} = \frac{\partial D}{\partial x}\frac{dx}{dt} + \frac{\partial D}{\partial y}\frac{dy}{dt}$$

$$= \frac{1}{2}(x+y)^{-1/2}\left(-80\cdot 2(4-80t)\right) + \frac{1}{2}(x+y)^{-1/2}\left(60\cdot 2(3+60t)\right)$$

$$= \frac{60(3+60t) - 80(4-80t)}{(x+y)^{1/2}}$$

$10000t - 140$

Plugging in for x and y yields

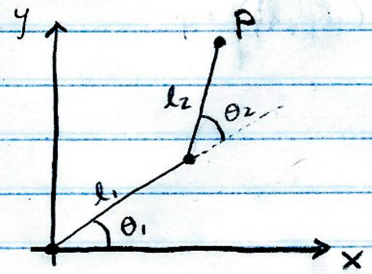
$$\frac{dD}{dt} = \frac{180 + 3600t - 3200 + 6400t}{\left((4-80t)^2 + (3+60t)^2\right)^{1/2}}$$

$$= \frac{10000t - 140}{\left((4-80t)^2 + (3+60t)^2\right)^{1/2}}$$

To find the rate they are approaching eachother, evaluate at $t=0$

$$\frac{dD(0)}{dt} = -\frac{140}{\sqrt{16+9}} = -\frac{140}{5} = \boxed{-28 \text{ mph}}$$

Example: Two-Link robot



The two-link robot above is controlled by two motors that change the angles θ_1 and θ_2. We can determine the coordinates of the point P with two functions of θ_1 and θ_2. We define the x-coordinate as $p_x(\theta_1, \theta_2)$ and the y-coordinate as $p_y(\theta_1, \theta_2)$:

$$p_x(\theta_1, \theta_2) = l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2)$$
$$p_y(\theta_1, \theta_2) = l_1 \sin(\theta_1) + l_2 \sin(\theta_1 + \theta_2)$$

a) Let $\theta_1 = \alpha(t) = 6t$ and $\theta_2 = \beta(t) = -3t$, Find the instaneous velocities $V_x(t)$ and $V_y(t)$ of the endpoint of the robot (note t is time in seconds)

Solution:
As in single variable problems, we find the velocity by taking the derivative of the position. Thus

$$V_x(t) = \frac{dP_x}{dt} \qquad V_y = \frac{dP_y}{dt}$$

Using the chain rule, we get

$$\frac{dP_x}{dt} = \frac{\partial P_x}{\partial \theta_1}\frac{d\theta_1}{dt} + \frac{\partial P_x}{\partial \theta_2}\frac{d\theta_2}{dt}$$

$$\frac{dP_y}{dt} = \frac{\partial P_y}{\partial \theta_1}\frac{d\theta_1}{dt} + \frac{\partial P_y}{\partial \theta_2}\frac{d\theta_2}{dt}$$

We can easily solve for each of the six values with techniques we already know!

$$\frac{\partial P_x}{\partial \theta_1} = -l_1 \sin(\theta_1) - l_2 \sin(\theta_1 + \theta_2)$$

$$\frac{\partial P_x}{\partial \theta_2} = -l_2 \sin(\theta_1 + \theta_2)$$

$$\frac{\partial P_y}{\partial \theta_1} = l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2)$$

$$\frac{\partial P_y}{\partial \theta_2} = l_2 \sin(\theta_1 + \theta_2)$$

$$\frac{d\theta_1}{dt} = \frac{d\alpha(t)}{dt} = 6$$

$$\frac{d\theta_2}{dt} = \frac{d\beta(t)}{dt} = -3$$

Plugging into the equations for V_x and V_y and substituting $\alpha(t)$ and $\beta(t)$ for θ_1 and θ_2 yields

$$V_x(t) = [-l_1 \sin(6t) - l_2 \sin(3t)]6 + [-l_2 \sin(3t)](-3)$$
$$V_y(t) = [l_1 \cos(6t) + l_2 \cos(3t)]6 + [l_2 \cos(3t)](-3)$$

Simplifying:

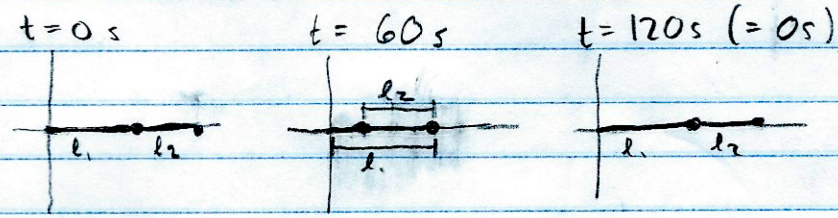
$$\boxed{\begin{aligned} V_x(t) &= -6 l_1 \sin(6t) - 3 l_2 \sin(3t) \\ V_y(t) &= 6 l_1 \cos(6t) + 3 l_2 \cos(3t) \end{aligned}}$$

b) When is $v_x(t) = 0$?

Solution:

We note that $v_x(t) = 0$ anytime $\sin(6t) = 0$ and $\sin(3t) = 0$.
Because $\sin(x) = 0$ when x is $0°, 180°, 360°, 540°, 720°$, etc.,
we see that $\sin(x) = 0 \Rightarrow \sin(2x) = 0$ (note $\sin(2x) = 0 \not\Rightarrow \sin(x) = 0$).
Thus, $V_x(t) = 0$ whenever $3t = 0, 180, 360, 540, \ldots$ or
whenever $\boxed{t = 0, 60, 120, 180, \ldots}$.

We can see this visually by drawing the configurations of the robot that result after each minute:



Note that depending on the values of l_1 and l_2 there may be more times t for which $v_x(t) = 0$...