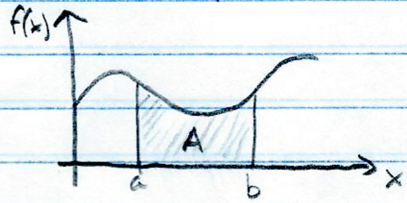
Week 7

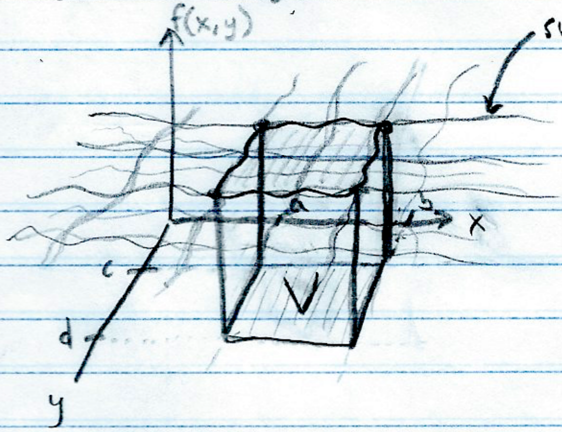
Multivariable Calculus

Recall: The integral can be viewed as the area underneath a line



$$\boxed{A = \int_a^b f(x)\,dx}$$

Suppose rather than a single line, we have a surface, given by a function of two variables. To find the volume under the surface, we can use a double integral

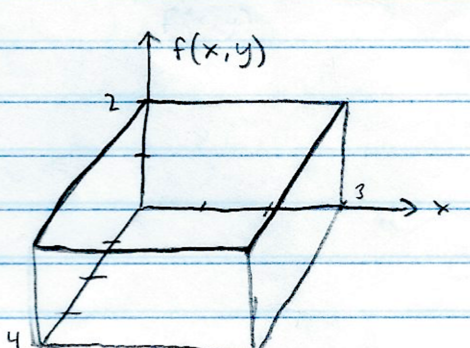


$$\boxed{\begin{aligned} V &= \int_c^d \int_a^b f(x,y)\,dx\,dy \\ &= \int_a^b \int_c^d f(x,y)\,dy\,dx \end{aligned}}$$

Example: Volume of a box

Consider the surface defined by the function

$$f(x,y) = \begin{cases} 2, & 0 \le x \le 3 \text{ and } 0 \le y \le 4 \\ 0, & \text{otherwise} \end{cases}$$



To be rigorous, we consider the area under the entire surface, i.e.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy$$

Because $f(x,y)$ is piecewise, we break up the integrals as follows

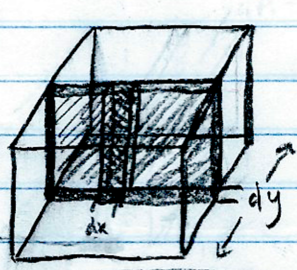
$$\cancel{\int_{-\infty}^{0}\int_{-\infty}^{0} 0\,dx\,dy} + \int_{0}^{4}\int_{0}^{3} 2\,dx\,dy + \cancel{\int_{4}^{\infty}\int_{3}^{\infty} 0\,dx\,dy}$$

To solve a double integral, work from the inside out

$$\int_{0}^{4}\int_{0}^{3} 2\,dx\,dy = \int_{0}^{4}\Big[2x\Big]_{0}^{3} dy$$
$$= \int_{0}^{4}(6-0)\,dy$$
$$= \int_{0}^{4} 6\,dy$$
$$= \Big[6y\Big]_{0}^{4}$$
$$= 24$$

In this simple case, we can confirm our answer by checking the volume of a box $2cm \times 3cm \times 4cm = 24\ cm^3$ ✓

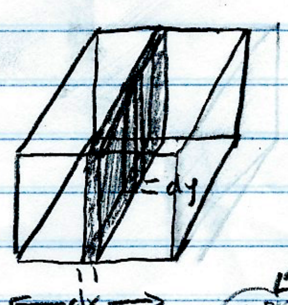
Note that we can swap the order of integration with x and y. These two methods yield two visualizations



fix on a single dy

$$\int_0^4 \int_0^3 f(x,y)\,dx\,dy$$

for given dy, sum all dx



fix on a single dx

$$\int_0^3 \int_0^4 f(x,y)\,dy\,dx$$

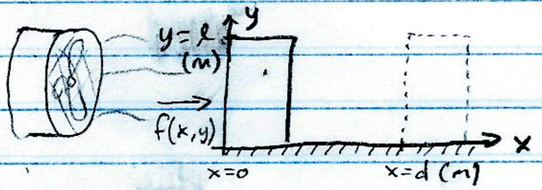
for given dx, sum all dy

Example: Work (2-dimensions)

Recall from lecture 4, that when a variable force $f(x)$ pushes a block a distance d, the work done is defined as:

$$W = \int_0^d f(x)\,dx$$

Now consider a force that varies along both the x and y dimension.



The force acting upon the block changes along the height of the block as well as along the position of the block on the x-axis. In this case, we define work as

$$W = \int_0^\ell \int_0^d f(x,y)\,dx\,dy \quad (N.m)$$

If $d = 1.0\,m$ and $\ell = 2.0\,m$

a) $f(x,y) = x^2 + y^2$

b) $f(x,y) = \sin(\frac{\pi}{2}x) + \cos(\frac{\pi}{2}y)$

Solution:

a) $$W = \int_0^\ell \int_0^d f(x,y)\,dx\,dy$$

$$= \int_0^2 \int_0^1 (x^2+y^2)\,dx\,dy$$

$$= \int_0^2 \left[\frac{1}{3}x^3 + xy^2\right]_{x=0}^{1} dy$$

$$= \int_0^2 \left(\frac{1}{3} + y^2\right) dy$$

$$= \left[\frac{1}{3}y + \frac{1}{3}y^3\right]_{y=0}^{2}$$

$$= \frac{2}{3} + \frac{8}{3}$$

$$\boxed{W = \frac{10}{3}\ N\cdot m}$$

b) $$W = \int_0^{\ell}\int_0^{d} f(x,y)\,dx\,dy$$

$$= \int_0^2\int_0^1 \left(\sin\left(\frac{\pi}{2}x\right) + \cos\left(\frac{\pi}{2}y\right)\right) dx\,dy$$

$$= \int_0^2 \left[-\frac{2}{\pi}\cos\left(\frac{\pi}{2}x\right) + x\cos\left(\frac{\pi}{2}y\right)\right]_{x=0}^{1} dy$$

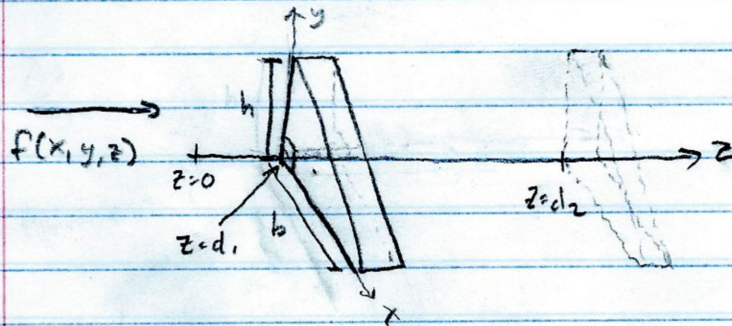
$$= \int_0^2 \left(\frac{2}{\pi} + \cos\left(\frac{\pi}{2}y\right)\right) dy$$

$$= \left[\frac{2}{\pi}y + \frac{2}{\pi}\sin\left(\frac{\pi}{2}y\right)\right]_{y=0}^{2}$$

$$= \boxed{\frac{4}{\pi}\ N\cdot m}$$

Example: Work (3 dimensions!)

Now consider a force on that varies along the face of a triangular block. The force also varies over distance as it pushes the block. Thus, the force is a function over 3 variables:



With $h = 4m$, $b = 3m$, $d_1 = 1m$, and $d_2 = 2m$, calculate the work done for the force:

$$f(x,y,z) = xy + z^2$$

Solution:

Before plugging in $f(x,y,z)$, let's begin by determining the limits of integration (which will be the same for any $f(x,y,z)$). Because our force varies in 3 dimensions, we will need a triple integral of the form:

$$\iiint f(x,y,z)\,dx\,dy\,dz$$

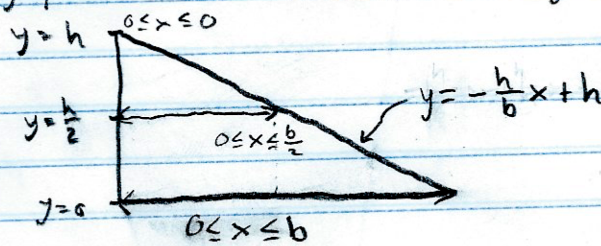
Working our way from the outside in, let's begin with the integration over z.

$$\int_{d_1}^{d_2}\iint f(x,y,z)\,dx\,dy\,dz$$

Next we consider the height of the triangle in integrating over y.

$$\int_{d_1}^{d_2}\int_{0}^{h}\int d(x,y,z)\,dx\,dy\,dz$$

Finally, we integrate over x, but note that our integration over x is for a constant value of y! Thus, the region we integrate is dependant on y. Looking at the block in the xy-plane we see different ranges of x for different y's



In general, we that x ranges from 0 to the line $y=-\frac{h}{b}x+h$. We can solve for x on the line by rewriting:

$$y=-\frac{h}{b}x+h$$
$$\frac{h}{b}x=h-y$$
$$x=b-\frac{b}{h}y$$

Thus, for a given y, x ranges from

$$0 \le x \le b-\frac{b}{h}y$$

Note that we can plug in the 3 values determined by inspection above to verify. Thus, our formula for work is:

$$\boxed{W=\int_{d_1}^{d_2}\int_0^h\int_0^{b-\frac{b}{h}y} f(x,y,z)\,dx\,dy\,dz}$$

Plugging in the given values gives:

$$W=\int_1^2\int_0^4\int_0^{3-\frac{3}{4}y}(xy+z^2)\,dx\,dy\,dz$$
$$=\int_1^2\int_0^4\left[\frac{1}{2}yx^2+xz^2\right]_{x=0}^{3-\frac{3}{4}y}dy\,dz$$
$$=\int_1^2\int_0^4\left(\frac{1}{2}y\left(3-\frac{3}{4}y\right)^2+\left(3-\frac{3}{4}y\right)z^2\right)dy\,dz$$
$$=\int_1^2\int_0^4\left(\frac{1}{2}y\left(\frac{9}{16}y^2-\frac{9}{2}y+9\right)+3z^2-\frac{3}{4}z^2y\right)dy\,dz$$
$$=\int_1^2\int_0^4\left(\frac{9}{32}y^3-\frac{9}{4}y^2+\frac{9}{2}y+3z^2-\frac{3}{4}z^2y\right)dy\,dz$$

$$= \int_1^2 \left[\frac{9}{128} y^4 - \frac{9}{12} y^3 + \frac{9}{4} y^2 + 3z^2 y - \frac{3}{8} z^2 y^2 \right]_{y=0}^{4} dz$$

$$= \int_1^2 \left(6 + 12z^2 - 6z^2 \right) dz$$

$$= \int_1^2 \left(6z^2 + 6 \right) dz$$

$$= \left[2z^3 + 6z \right]_1^2$$

$$= 16 - 2 + 12 - 6$$

$$\boxed{W = 20 \text{ N} \cdot \text{m}}$$

Question: what would the limits of integration be if we switched the order of integration for x and y?

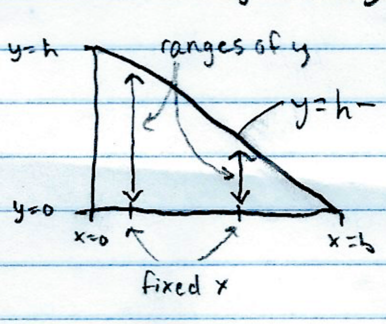
Solution: Note that the limits of integration on z will not change. Thus, we have:

$$W = \int_{d_1}^{d_2} \int \int f(x,y,z)\, dy\, dx\, dz$$

Again, we begin with the outer integral and set the limits of x to cover the whole base of the triangle.

$$W = \int_{d_1}^{d_2} \int_0^b \int f(x,y,z)\, dy\, dx\, dz$$

Finally, we look at the block in the xy-plane and note the range of y for fixed x values



$$\Rightarrow \boxed{W = \int_{d_1}^{d_2} \int_0^b \int_0^{h - \frac{h}{b}x} f(x,y,z)\, dy\, dx\, dz}$$

Example: Calculating mass with variable density.

Suppose we are building a base for a robot, and the manufacturer charges based on the mass of the part. We would like to create a tetrahedron given by the volume under the plane given by

$$2x+y+3z=6 \text{ (cm)}$$

in the positive octant of the xyz-space. Furthermore, we will create it with variable density given by density function

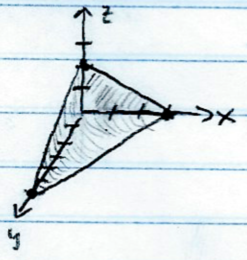
$$\rho(x,y,z)=x^2yz \;(\text{g}/\text{cm}^3)$$

Find a formula for the mass of the tetrahedron

Solution: The mass is found by integrating density over the volume of the object:

$$m=\iiint \rho(x,y,z)\,dx\,dy\,dz$$

The limits of integration are then determined by the shape of the tetrahedron:



The plane is given by $2x+y+3z=6$, so the tetrahedron is given by

$$0\le 2x+y+3z\le 6$$

Starting with z, we determine the limits from the outside in

$dz \to 0\le 3z\le 6 \to 0\le z\le 2$

y can't be negative!

$dy \to 0\le y+3z\le 6 \to -3z\le y\le 6-3z \to 0\le y\le 6-3z$

$dx \to 0\le 2x+y+3z\le 6 \to -y-3z\le 2x\le 6-3z-y \to 0\le x\le 3-\frac{3}{2}z-\frac{1}{2}y$

Thus, mass is given by

$$\boxed{M=\int_0^2\int_0^{6-3z}\int_0^{3-\frac{3}{2}z-\frac{1}{2}y} x^2yz\,dx\,dy\,dz \text{ (g)}}$$