Updates: Insert/Delete (p): $O(B^{1+\varepsilon})$

1. Remove q from I, D.
2. Add p to I/D.

If overflow, $|I|, |D| > B$:

Extract points L in x-order (increasing), apply updates to L'

Put point in internal priority queue w/ key = y.

Split L' in $b_1 \dots b_{l'}$ blocks. $O(l')$

Fashion using prio. queue in $O(1)$ IO's

$|L| = O(B^{1+\varepsilon})$

Reconstruction of L $O(|L|/B) = O(B^{\varepsilon})$

$\Rightarrow$ $O(1/B^{1-\varepsilon})$ IOs pr. buffered update. ✓

3-sided reporting queries:

$Q = [x_1, x_2] \times [y, \infty]$.

Find t blocks intersected by sweep line at y.

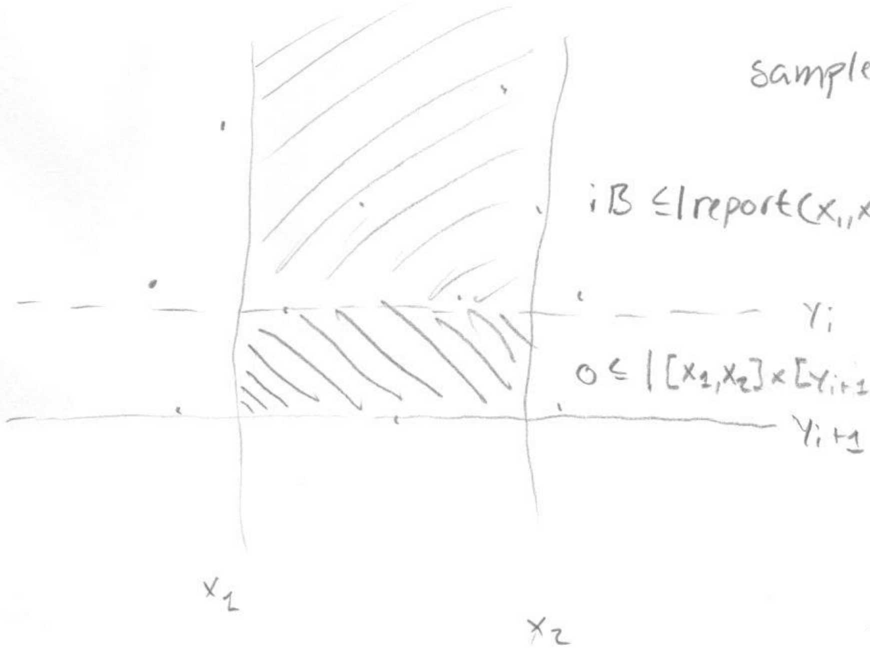
Can be found using catalog in $O(1)$ IOs.

Report using scan of t blocks in $O(1+t) = O(1+K/B)$ IOs.

From construction $K \geq B\lfloor (t-2)/2 \rfloor$. ✓

Sampling queries:

sample(x_1, x_2)



$iB \le |report(x_1, x_2, y_i)| \le iB + \alpha B$

$0 \le |[x_1, x_2] \times [y_{i+1}, y_i[| \le (1+\alpha)B$

y_3, y_4

$3B \le |report(x_1, x_2, y_3)| \le 3B + \alpha B$

$4B \le |report(x_1, x_2, y_4)| \le 4B + \alpha B$

$0 \le |[x_1, x_2] \times [y_4, y_3[| \le B + \alpha B$

$0 < \varepsilon \le 1/2$

updates: $O(\frac{1}{\varepsilon B^{1-\varepsilon}} \log_B N)$ IOs

Queries: $O(\frac{1}{\varepsilon} \log_B N + K/B)$ am. IOs.

Linear space.

Construction: $O(N/B)$ IOs.

$O(B^{1+\varepsilon})$ structure.

3-sided queries: $O(1 + K/B)$ IOs.

Bath updates of $s^{\le B}$ points: $O(1 + s/B^{1-\varepsilon})$ am. IOs.

Sample: $O(1)$ IOs.

Construction: $O(N/B)$ IOs assuming x-sorted points.

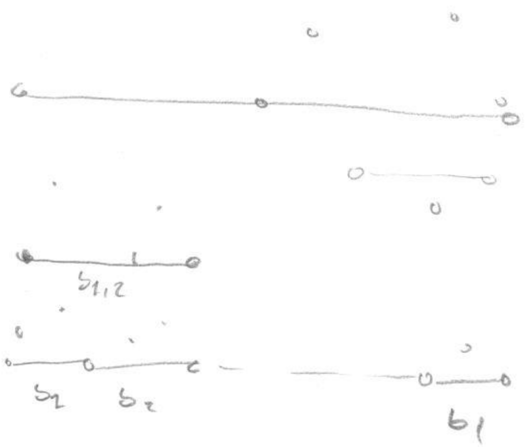
Linear space.

D.S. C:

- Static D.S. L storing $O(B^{1+\varepsilon})$ points.
- Buffers I, D of delayed deletions + insertions $|I|, |D| \le B$.
- Sampled y-values of size $O(B)$ S.

$I \cap D = \emptyset$

$l = |L|/B \Rightarrow 2l-1$ blocks.



Catalog:

	b_i	$b_{i,j}$	
	min max x	i,j min y	

$2l-1$

S:

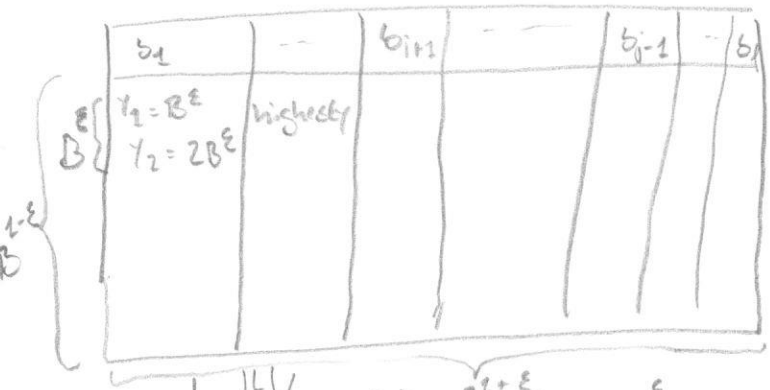
s_1		s_l
$y_1 = B^{\varepsilon}$ highest y $y_2 = 2B^{\varepsilon}$ ⋮ $y_i = i \cdot B^{\varepsilon}$ highest y		

$B^{1-\varepsilon}$

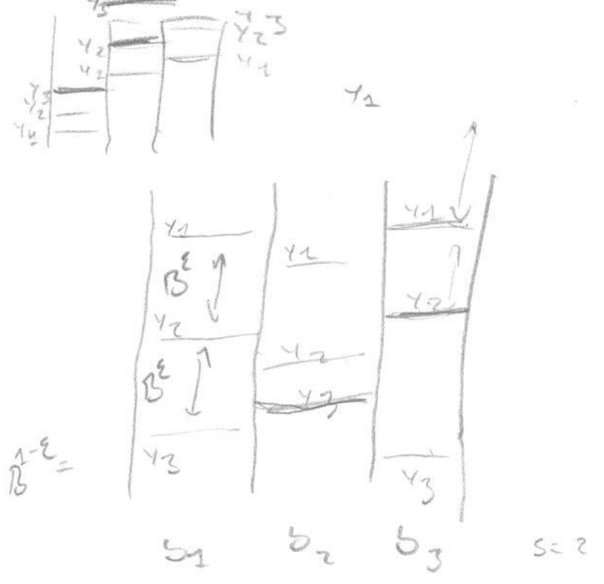
$\Rightarrow O(B^{\varepsilon} \cdot \overset{O(B^{\varepsilon})}{B^{1-\varepsilon}}) = O(B)$

Sampling queries:

S:



$t = |L|/B \Rightarrow B^{1+\varepsilon}/B = B^\varepsilon$



$s = 1$

$z \cdot B^{1-\varepsilon}$

Identify b_i, b_j spanning x_1, x_2.

Return $\lceil (s+1) \cdot B^{1-\varepsilon} \rceil$-th y-value for $s = 1, \ldots$

Bound number of points in $Q_s = [x_1, x_2] \times [y_s, \infty]$.

By construction $\lceil (s+1) \cdot B^{1-\varepsilon} \rceil$ y-values $\geq y_s$ in S from $b_{i+1} \cup \ldots \cup b_{j-1}$.

Assume n_t sampled y-values $\geq y_s$ in S from b_t

$\Rightarrow n_{i+1} + \ldots + n_{j-1} = \lceil (s+1) \cdot B^{1-\varepsilon} \rceil$.

$\lceil n_t \cdot B^\varepsilon \rceil \leq$ Num. points in b_t with y-value $\geq y_s \leq \lceil (n_t + 1) B^\varepsilon \rceil$

$$\Rightarrow |Q_s \cap (b_{i+1} \cup \ldots \cup b_{j-1})| \geq \sum_{t=i+1}^{j-1} \lceil n_t B^\varepsilon \rceil \geq B^\varepsilon \sum_{t=i+1}^{j-1} n_t = B^\varepsilon \cdot \lceil (s+1) \cdot B^{1-\varepsilon} \rceil \geq (s+1) B$$

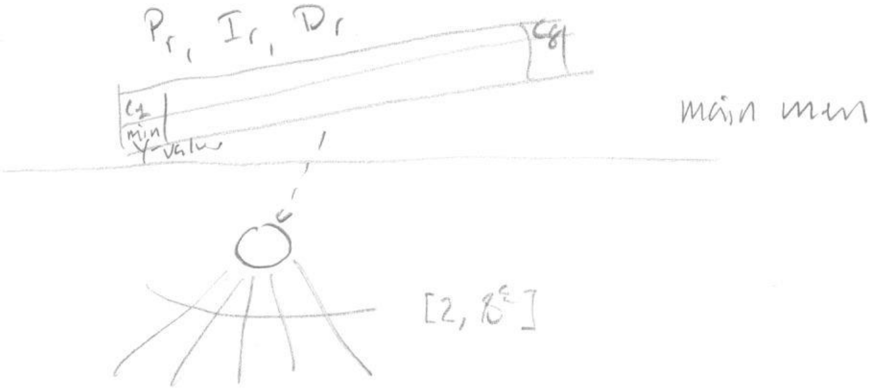
$$|Q_s \cap (b_{i+1} \cup \ldots \cup b_{j-1})| \leq \sum_{t=i+1}^{j-1} (n_t + 1) B^\varepsilon = (j-i-1) B^\varepsilon + B^\varepsilon \sum_{t=i+1}^{j-1} n_t$$

$$= (j-i-1) B^\varepsilon + B^\varepsilon \lceil (s+1) \cdot B^{1-\varepsilon} \rceil \leq (j-i) B^\varepsilon + (s+1) B$$

$$\xRightarrow{|O| \leq B \quad \cap (b_{i+1} \cup \ldots \cup b_{j-1})} |Q_s| \geq (s+1)B - B = sB$$

$$\xRightarrow{|I|, |b_i|, |b_j| \leq B} (j-i) B^\varepsilon + (s+1) B + 3B \overset{j-i = O(B^\varepsilon),\ \varepsilon \leq 1/2}{=} sB + O(B)$$

Main
Data structure



Point buffer P_v
Insert buffer I_v
Deletion buffer D_v

All points in P_v have larger value than points in decendants ~~children~~.

Invariants:

$P_v \cap I_v \cap D_v = \emptyset$

All points in ↑ have x-values spanned by subtree T_v.

$I_v \cup D_v$ y-values less than points in P_v.

Leaves: $I_v = D_v = \emptyset$.

Internal node also have C_v:

$O(B^{\varepsilon+1})$-structure of $\bigcup_{i=1}^{\delta} P_{c_i}$

if P_{c_i} empty.

P_v contains $B/2$ points or all insertions and deletion buffers in T_v empty and all points in T_v lies in P_v:

$\Rightarrow$

1.) $B/2 \le |P_v| \le B$, $|D_v| \le B/4$, $|I_v| \le B$ or

2.) $|P_v| < B/2$, $I_v = D_v = \emptyset$ and $P_w = I_w = D_w = \emptyset \;\forall w$ in T_v.

Updates:

Point $p = (p_x, p_y)$.

Remove p from P_r, I_r, D_r.

If $p_y <$ smallest y in P_r, insert into I_r / D_r.

$p_y \geq$ smallest y in P_r, insert into P_r.

If P_r overflows $(|P_r| = B+1)$, move smallest y-value from P_r to I_r

When deleting:

(1) handle overflowing deletion buffers i.e. $|D_r| > B/4$

Push $U \subseteq D_v$ of $\lceil |D_v| / B^\varepsilon \rceil$ deletions down to child c:

- Remove all points in U from D_v, I_c, D_c, P_c and C_v.
- Any point in U with y-value larger/equal than min y-value in U removed.
- If v is leaf, done.

  o/w add remaining points in U to D_c.

If $|D_c|$ overflows, recurse.

Worst case deletion buffers overflow all the way along root-to-leaf path: $\leq \lceil B/B^\varepsilon \rceil$ points are pushed down. Update of $\alpha(B/B^\varepsilon)$ takes $O(1 + B/B^\varepsilon \cdot B^\varepsilon/B) = O(1)$ IO's.

(2) Recursively fill underflowing point buffers

~~B^2~~ ~~$B^{\varepsilon} > B/4$~~

If $|P_v| < B/2$, we underflow.

Move $B/2$ top points into P_v from v's children:

If all subtrees below v do not store any points, remove all points from D_v, and move as many points as possible from I_v to P_v. $\Rightarrow |P_v| = B$ or $I_v = \emptyset$.

Else scan childrens point buffers $P_{c_1}, \dots, P_{c_\gamma}$ using $O(B^{\varepsilon})$ IO's to get $B/2$ points with largest y-values.

Delete from P_{c_i} using $O(B^{\varepsilon})$ IO's.

Delete from C_v in $O(B^{\varepsilon})$ IO's.

Remove all points in $X \cap D_v$ from X and D_v.

For all $p \in X \cap I_v$ replace $p \in X$ with $p \in I_v$.

While highest point in I_v has higher y-value than lowest point in X, we swap the two.

All remaining points in X are inserted into P_v in $O(1)$ io's.

and into $C_{Parent(v)}$ in $O(B^{\varepsilon})$ IO's.

i.e.
To pull $B/2$ points one level up: $O(B^{\varepsilon})$ IO's.

Insertions:

(ii) If $|I_v| > B$ overflows.

There must exist a child c where we can push $U \subseteq I_v$ of $\lceil |I_v|/B^{\varepsilon} \rceil$ insertions down to.

Remove all points in U from I_v, I_c, D_c, P_c and C_v. with $O(B/B^{\varepsilon})$ points in am. $O(1 + (B/B^{\varepsilon})/B^{1-\varepsilon}) = O(1)$ IO's.

Any point in U with y-value large than or equal to minimum y-value in P_c is inserted into P_c and C_v and removed from U.

If P_c overflows i.e. $|P_c| > B$ we move the points with smallest y-value from P_c to U until $|P_c| = B$.

If c is leaf all points in U are inserted into P_c.

else add remaining points in U to I_c

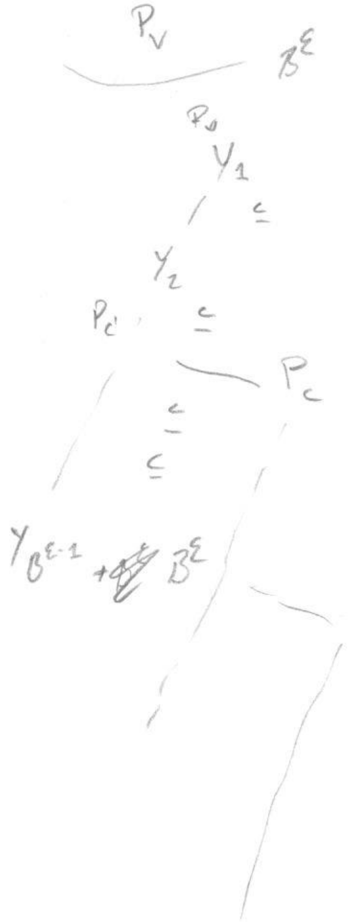
If I_c overflows, recurse.

(iii) If P_v at a leaf v overflows, we split v into v' and v'' and distribute evenly points in $O(1)$ IO's.

Splitting might cause parent(v) to get degree $B^{\varepsilon}+1$.

(iv) While node v have degree $B^{\varepsilon}+1$ split into v', v'' w.r.t x-value. Construct $C_{v'}$ and $C_{v''}$ from children point sets P_c in $O(B^{\varepsilon})$ IO's.

Top-k



$B^{\varepsilon-1}$ B^{ε}

$\bar{k} = 7t + 12k$

Construction