Solenoids & Inductance

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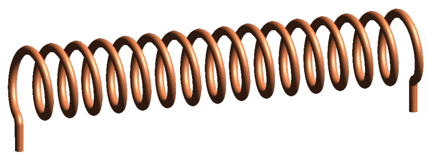
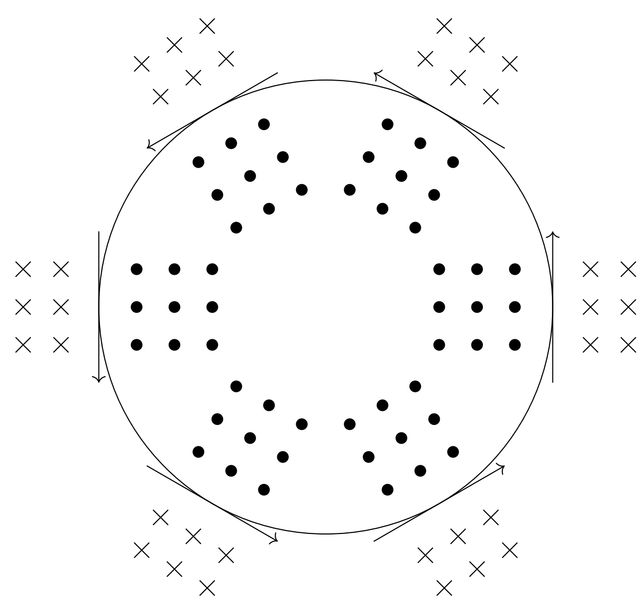
May 7, 2023



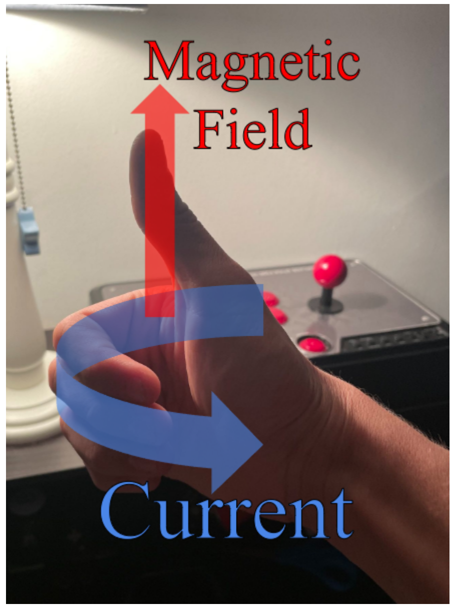
Notes are online at `https://gabrielweredyk.com/papers/PhysicsC/Inductors.pdf`

Solenoids & Inductors

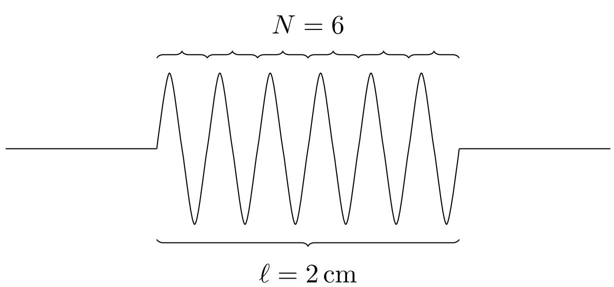
As discussed in Physics 2, straight wires carrying a current create a circular magnetic field flowing around the wire; however, thanks to the duality between electricity and magnetism, we can actually create a straight magnetic field using a circular wire. This is most commonly integrated within physical circuits by taking a long piece of wire and winding it in a helix around the length of a hollow tube it a device called a **solenoid**. Using our first right hand rule on each side of the coiled wire, we can see that if current travels in a circular path, it will create a nearly straight magnetic field. Notice how inside of the coil, the magnetic field lines are very concentrated going into the page, making a noticeable magnetic field. Yet, on the outside of the coil, despite the presence of magnetic field lines, they quickly become too dispersed to be significant.



Given this information, we can derive the all too elusive second right hand rule, where the curl of our fingers represent the direction current coils in and our thumb represents the direction of the net magnetic field.



Besides the current in the wire composing the solenoid, a solenoid's properties are determined by its number of coils, N, and its physical length in meters, ℓ. The ratio of twists to length, N/ℓ, is also a significant enough value such that we denote it with n. In the figure below, the n of the solenoid would be $300\,\mathrm{m}^{-1}$.



Then, using Ampere's Law, we can derive an expression for the magnitude of the magnetic field within the solenoid. The derivation itself is not necessary to know for the exam and is a little non-rigorous, but seeing the logic behind the reason could help you come to terms with the equation.

$$\oint \vec{B} \cdot \mathrm{d}\vec{s} = \mu_0 I$$

$$B\ell = \mu_0 N I$$

$$B = \frac{\mu_0 N I}{\ell}$$

$$\boxed{B = \mu_0 n I}$$

As the equation shows, the magnetic field in a solenoid can be increased by either increasing the current flowing through the wire or winding the coil tighter.

Solenoids are particularly useful in circuits because of their ability to resist changes in current. If you remember Faraday's law of inductance, when a closed loop undergoes a change in magnetic flux, emf will be induced in order to generate an opposing change in magnetic flux. So when the coiled wire produces a current and subsequentially a change in magnetic flux, it will produce a self-induced emf in order to counteract that change. When a solenoid is utilized in a circuit in this manner, it is called an **inductor**. The total amount of magnetic flux traveling through an inductor is equal to Φ_B, the magnetic flux going through one of the coils, times the total number of coils, N. This information is needed in determining and utilizing the inductance constant, L, of the inductor. The inductance of a coil is given by

$$LI = N\Phi_B$$

From this definition of inductance, we can get the induced emf (that fights the voltage in the circuit) by manipulating the equation to utilize Faraday-Lenz's Law.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}LI &= \frac{\mathrm{d}}{\mathrm{d}t}N\Phi_B \\
L\frac{\mathrm{d}I}{\mathrm{d}t} &= N\frac{\mathrm{d}\Phi_B}{\mathrm{d}t} \\
&\boxed{\varepsilon = -L\frac{\mathrm{d}I}{\mathrm{d}t}}
\end{aligned}$$

The inductance of an inductor is also important in determining the energy stored within a capacitor, because it is given by the equation

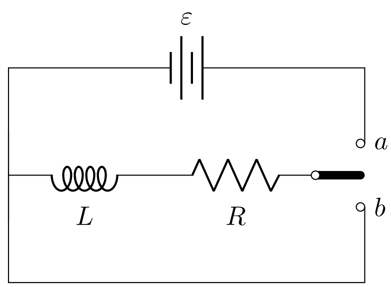
$$U_L = \frac{1}{2}LI^2$$

The derivation from this formula comes from integrating the power needed to be supplied over time with current I

$$\begin{aligned}
P &= IV \\
P &= I \cdot -\left(-L\frac{\mathrm{d}I}{\mathrm{d}t}\right) \\
P &= LI\frac{\mathrm{d}I}{\mathrm{d}t} \\
\int_0^t P\,\mathrm{d}t &= \int_0^t LI\frac{\mathrm{d}I}{\mathrm{d}t}\,\mathrm{d}t \\
\int_0^t P\,\mathrm{d}t &= \int_0^I LI\,\mathrm{d}I \\
U_L &= \frac{1}{2}LI^2
\end{aligned}$$

RL Circuits

Much like how capacitors don't instantaneously hold charge when placed in a RC circuit, inductors also take their sweet time in carrying a current because of their inductance. To mathematically express the current in an inductor, let's imagine a simple RL circuit with a battery of voltage ε, a resistor with a resistance R and an inductor with inductance L and a switch.



To the start the mechanism and charge the inductor, the switch is moved to point a at time $t = 0$. Then we can use Kirchhoff's Loop Law to derive an equation with current and time. Keep in mind that all components are in series, so they should share the same current.

$$
\begin{aligned}
\varepsilon &= IR - \left(-L\frac{\mathrm{d}I}{\mathrm{d}t}\right) \\
\varepsilon &= IR + L\frac{\mathrm{d}I}{\mathrm{d}t} \\
\frac{\varepsilon}{R} &= I + \frac{L}{R}\frac{\mathrm{d}I}{\mathrm{d}t} \\
\frac{\varepsilon}{R} - I &= \frac{L}{R}\frac{\mathrm{d}I}{\mathrm{d}t} \\
\int_0^t \frac{R\,\mathrm{d}t}{L} &= \int_0^I \frac{\mathrm{d}I}{\frac{\varepsilon}{R} - I} \\
\frac{Rt}{L} &= -\ln\left(\frac{\varepsilon}{R} - I\right) + \ln\left(\frac{\varepsilon}{R}\right) \\
\frac{Rt}{L} &= \ln\left(\frac{R}{\varepsilon - IR}\right) + \ln\left(\frac{\varepsilon}{R}\right) \\
\frac{Rt}{L} &= \ln\left(\frac{\varepsilon}{\varepsilon - IR}\right) \\
\exp\left(\frac{Rt}{L}\right) &= \frac{\varepsilon}{\varepsilon - IR} \\
\varepsilon\exp\left(\frac{Rt}{L}\right) - IR\exp\left(\frac{Rt}{L}\right) &= \varepsilon \\
IR\exp\left(\frac{Rt}{L}\right) &= \varepsilon\exp\left(\frac{Rt}{L}\right) - \varepsilon \\
I &= \frac{\varepsilon}{R}\left(\frac{\exp\left(\frac{Rt}{L}\right) - 1}{\exp\left(\frac{Rt}{L}\right)}\right) \\
I &= \frac{\varepsilon}{R}\left(1 - \exp\left(-\frac{Rt}{L}\right)\right)
\end{aligned}
$$

The ratio of L to R is called the inductive time constant, τ_L, so the current in an inductor as a monotonously increasing function of time is given by

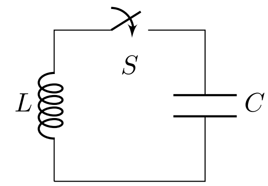
$$I(t) = \frac{\varepsilon}{R}\left(1 - \exp\left(-\frac{t}{\tau_L}\right)\right)$$

Normally if the switch were moved to point b and the battery is removed, the circuit would instantly lose all of its current. But, since the inductor resists changes in current, the current dissipates gradually and can be derived from Kirchhoff's Loop Law again.

$$\begin{aligned}
IR - \left(-L\frac{\mathrm{d}I}{\mathrm{d}t}\right) &= 0 \\
IR + L\frac{\mathrm{d}I}{\mathrm{d}t} &= 0 \\
\frac{\mathrm{d}I}{\mathrm{d}t} &= -\frac{R}{L}I \\
\frac{\mathrm{d}I}{\mathrm{d}t} &= -\frac{1}{\tau_L}I \\
I &= I_0 \exp\left(-\frac{t}{\tau_L}\right) \\
I &= \frac{\varepsilon}{R}\exp\left(-\frac{t}{\tau_L}\right)
\end{aligned}$$

LC Circuits

While RC and RL circuits are interesting due to the exponential nature of the buildup and release of charge or current, a combination of the inductor and capacitor leads to an interesting interaction in the circuit. Imagine we have a circuit with a fully charged capacitor of capacitance C at a charge of Q_{max} and an inductor with an inductance of L. The switch is initially open, but closes at time $t = 0$.



We can use our handy-dandy Kirchhoff's Loop law in order to determine how the charge in this circuit changes overtime.

$$\begin{aligned}\frac{Q}{C} - -L\frac{\mathrm{d}I}{\mathrm{d}t} &= 0\\ \frac{Q}{C} + L\frac{\mathrm{d}I}{\mathrm{d}t} &= 0\\ \frac{\mathrm{d}I}{\mathrm{d}t} &= -\frac{1}{LC}Q\\ \frac{\mathrm{d}^2Q}{\mathrm{d}t^2} &= -\frac{1}{LC}Q\end{aligned}$$

This differential equation models that of an object in oscillation like a spring, meaning that we can treat the coefficient in the differential equation as an angular velocity and get the equation

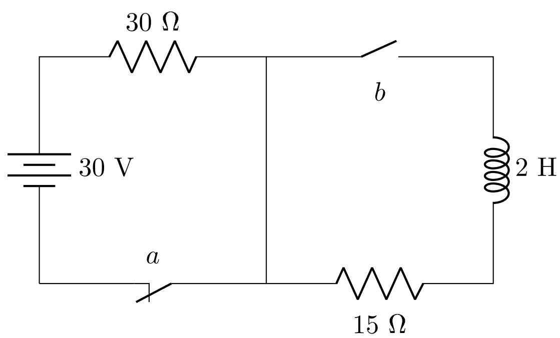
$$Q(t) = Q_{max}\cos\left(\frac{t}{\sqrt{LC}}\right)$$

Exercises

1. In a particular manufacturing process, a solenoid is wound such that there are 5 cm of 1 mm wiring in a singular turn of the coil. What is the magnetic field created by one of these solenoids when 5 amps of current are passed through them?

2. The cross-sectional area of an inductor with 20 coils is $0.1\,\mathrm{m}^2$. What is its inductance when 10 amps of current are passed through the inductors?

3. Consider the following circuit



Switch b closes at time $t = 0$.

(a) Calculate the current in the 15 Ω at time $t = 0$

(b) Calculate the current in the 15 Ω resistor after switch b has been left closed for a considerable amount of time.

(c) Calculate the current flowing through the capacitor after switch b has been closed for a second

After a considerable amount of time of switch b being closed, switch a is opened.

(d) Calculate the energy stored in the inductor

(e) Calculate the initial back-emf of the inductor

(f) Calculate the amount of time it takes for the circuit to have a current of less than 0.1 A.

4. An LC circuit consists of an inductor with inductance 5 H and a capacitor with capacitance of 4 µF and initial charge of +20 C. How long would it take for the capacitor to have a charge of 20 C on the other plate?

5. Given your knowledge on how inductors work, when inductors are added in parallel, is the total inductance added arithmetically ($L_T = \sum L_k$) or harmonically ($\frac{1}{L_T} = \sum \frac{1}{L_k}$)?

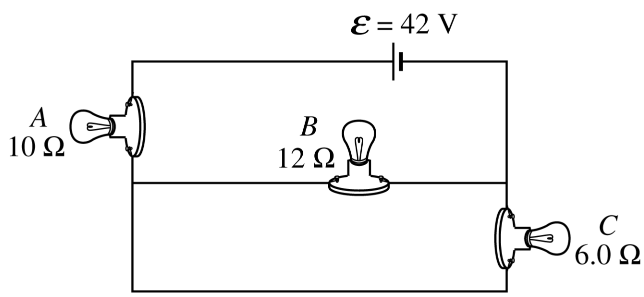
6. A circuit is situated such that a straight wire runs perpendicular 1 cm above a solenoid such that at the middle point of the solenoid, the magnetic field is zero. If a current of 10 A is running through the circuit, how tightly must the solenoid be wound?

2000 AP® PHYSICS C FREE-RESPONSE QUESTIONS

PHYSICS C

Section II, ELECTRICITY AND MAGNETISM
Time—45 minutes
3 Questions

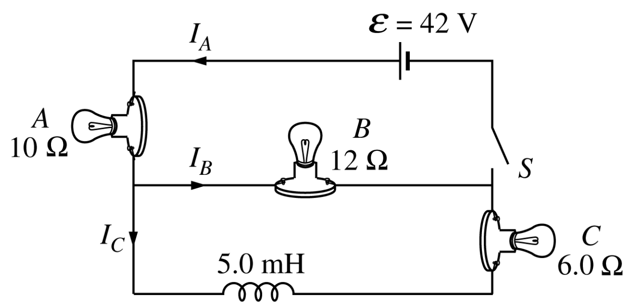
Directions: Answer all three questions. The suggested time is about 15 minutes for answering each of the questions, which are worth 15 points each. The parts within a question may not have equal weight. Show all your work in the pink booklet in the spaces provided after each part, NOT in this green insert.



E & M 1.

Lightbulbs A, B, and C are connected in the circuit shown above.

(a) List the bulbs in order of their brightness, from brightest to least bright. If any bulbs have the same brightness, state which ones. Justify your answer.

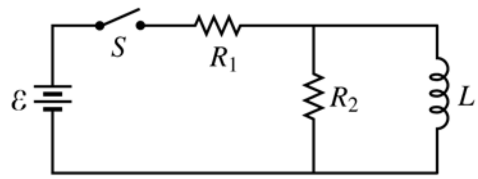


Now a switch S and a 5.0 mH inductor are added to the circuit, as shown above. The switch is closed at time $t = 0$.

(b) Determine the currents I_A, I_B, and I_C for the following times.

i. Immediately after the switch is closed

ii. A long time after the switch is closed



E&M. 2.

In the circuit shown above, resistors 1 and 2 of resistance R_1 and R_2, respectively, and an inductor of inductance L are connected to a battery of emf $\mathcal{E}$ and a switch S. The switch is closed at time $t = 0$. Express all algebraic answers in terms of the given quantities and fundamental constants.

(a) Determine the current through resistor 1 immediately after the switch is closed.

(b) Determine the magnitude of the initial rate of change of current, dI/dt, in the inductor.

(c) Determine the current through the battery a long time after the switch has been closed.

(d) On the axes below, sketch a graph of the current through the battery as a function of time.

Some time after steady state has been reached, the switch is opened.

(e) Determine the voltage across resistor 2 just after the switch has been opened.

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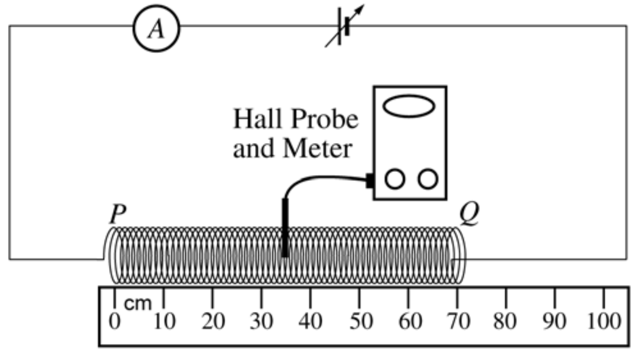


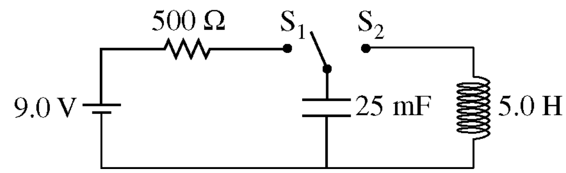
E&M. 3.

A student performs an experiment to measure the magnetic field along the axis of the long, 100-turn solenoid PQ shown above. She connects ends P and Q of the solenoid to a variable power supply and an ammeter as shown. End P of the solenoid is taped at the 0 cm mark of a meterstick. The solenoid can be stretched so that the position of end Q can be varied. The student then positions a Hall probe* in the center of the solenoid to measure the magnetic field along its axis. She measures the field for a fixed current of 3.0 A and various positions of the end Q. The data she obtains are shown below.

Trial	Position of End Q (cm)	Measured Magnetic Field (T) (directed from P to Q)	n (turns/m)
1	40	9.70×10^{-4}	
2	50	7.70×10^{-4}	
3	60	6.80×10^{-4}	
4	80	4.90×10^{-4}	
5	100	4.00×10^{-4}	

(a) Complete the last column of the table above by calculating the number of turns per meter.

* A Hall Probe is a device used to measure the magnetic field at a point.



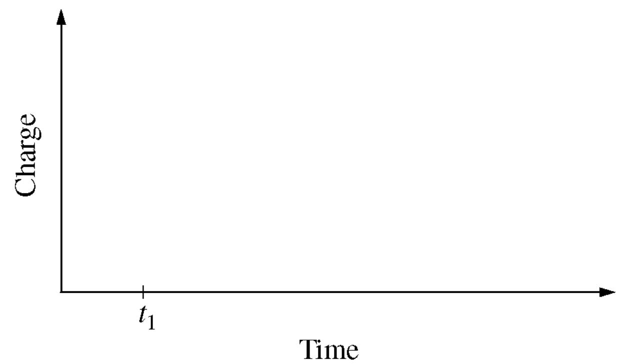
E&M. 2.

The circuit represented above contains a 9.0 V battery, a 25 mF capacitor, a 5.0 H inductor, a 500 Ω resistor, and a switch with two positions, S_1 and S_2. Initially the capacitor is uncharged and the switch is open.

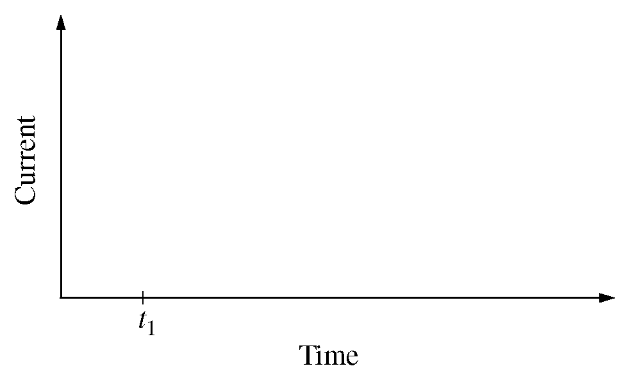
(a) In experiment 1 the switch is closed to position S_1 at time t_1 and left there for a long time.

i. Calculate the value of the charge on the bottom plate of the capacitor a long time after the switch is closed.

ii. On the axes below, sketch a graph of the magnitude of the charge on the bottom plate of the capacitor as a function of time. On the axes, explicitly label any intercepts, asymptotes, maxima, or minima with numerical values or algebraic expressions, as appropriate.



iii. On the axes below, sketch a graph of the current through the resistor as a function of time. On the axes, explicitly label any intercepts, asymptotes, maxima, or minima with numerical values or algebraic expressions, as appropriate.



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(b) In experiment 2 the capacitor is again uncharged when the switch is closed to position S_1 at time t_1. The switch is then moved to position S_2 at time t_2 when the magnitude of the charge on the capacitor plate is 105 mC, allowing electromagnetic oscillations in the LC circuit.

i. Calculate the energy stored in the capacitor at time t_2.

ii. Calculate the maximum current that will be present during the oscillations.

iii. Calculate the time rate of change of the current when the charge on the capacitor plate is 50 mC.

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