Objects of Nonuniform Density

Gabriel Weredyk

May 7, 2023

Notes are online at `https://gabrielweredyk.com/papers/PhysicsC/NonuniformDensity.pdf`

Introduction

Like many summations that can be found in physics, the superposition of inertia and center of mass can be added continually with integral calculus. However, contrary to the standard processes for transforming physics equations into integrals, we do not integrate with respect to position! Instead, we take the integral with respect to mass.

$$I_{tot} = \sum m_i r_i^2 \qquad \bar{x} = \frac{\sum m_i x_i}{\sum m_i}$$

$$I_{tot} = \int r^2 \,\mathrm{d}m \qquad \bar{x} = \frac{\int x \,\mathrm{d}m}{\int \mathrm{d}m}$$

The reason for this change can be explained both mathematically and physically. First off, integrating with respect to mass simply works nicer for the two equations in question, as the distance either squared or absent in some of the integrals. Secondly, physically when we measure these quantities discretely, we care primarily about the different masses attached to an object, not about their distance. But we cannot integrate with respect to mass, as we cannot graph it on a coordinate plane like distance due to it being a scalar. The linear mass density allows us to do exactly this. "Notated with the Greek letter λ, linear mass density maps out the object's mass at every point on the coordinate plane, serving as a conversion factor from mass to position. This function is important because it lets us substitute a mass differential for a position differential. At first glance this function may seem useless based off of its mathematical definition, but the λ of any given object can be calculated quite easily if not given to us.

$$\lambda := \frac{\mathrm{d}m}{\mathrm{d}x}$$

$$\lambda \mathrm{d}x = \mathrm{d}m$$

Although density doesn't come up typically in Physics 1, it can serve some uses here in these types of questions. Namely, instead of substituting $\mathrm{d}m$ with $\lambda \mathrm{d}x$, it could serve to benefit us instead if we used the following equation where ρ is the symbol for density.

$$\mathrm{d}m = \rho \cdot \mathrm{d}V$$

Examples

1. A thin rod lies along the x-axis from $x = 0$ to $x = 3\,\text{cm}$. The linear density of the rod, λ, is given by the equation $\lambda = 2 + x^3$ and is in units of g/cm.

   (a) Where is the center of mass of the rod?

   Solution: $\boxed{x = 2.19\,\text{cm}}$

$$\bar{x} = \frac{\int x\,\mathrm{d}m}{\int \mathrm{d}m}$$
$$\bar{x} = \frac{\int x\lambda\,\mathrm{d}x}{\int \lambda\,\mathrm{d}x}$$
$$\bar{x} = \frac{\int_0^3 x(2+x^3)\,\mathrm{d}x}{\int_0^3 (2+x^3)\,\mathrm{d}x}$$
$$\bar{x} = \frac{\int_0^3 2x + x^4\,\mathrm{d}x}{\int_0^3 2 + x^3\,\mathrm{d}x}$$
$$\bar{x} = \frac{[x^2 + \frac{1}{5}x^5]_0^3\,\mathrm{g}\cdot\mathrm{cm}}{[2x + \frac{1}{4}x^4]_0^3\,\mathrm{g}}$$
$$\bar{x} = \frac{(3)^2 + \frac{1}{5}(3)^5\,\mathrm{cm}}{2(3) + \frac{1}{4}(3)^4}$$
$$\bar{x} = 2.19\,\mathrm{cm}$$

   (b) What is the moment of inertia of the rod about the origin?

   Solution: $\boxed{139.5\,\mathrm{g\,cm^2}}$

$$I = \int r^2\,\mathrm{d}m$$
$$I = \int_0^3 x^2\lambda\,\mathrm{d}x$$
$$I = \int_0^3 x^2(2+x^3)\,\mathrm{d}x$$
$$I = \int_0^3 2x^2 + x^5\,\mathrm{d}x$$
$$I = \left[\frac{2}{3}x^3 + \frac{1}{6}x^6\right]_0^3\,\mathrm{g\,cm^2}$$
$$I = \frac{2}{3}(3)^3 + \frac{1}{6}(3)^6\,\mathrm{g\,cm^2}$$
$$I = 139.5\,\mathrm{g\,cm^2}$$

2. Show that the moment of inertia about the edge of a rod of uniform mass M and length L is given by $\frac{1}{3}ML^2$.

Solution: Let ρ be the density of the rod and A the cross-sectional area of the rod.

$$\begin{aligned}
I &= \int r^2 \,\mathrm{d}m \\
I &= \int_0^L x^2 \rho \,\mathrm{d}V \\
I &= \int_0^L x^2 \rho A \,\mathrm{d}x \\
I &= \rho A \int_0^L x^2 \,\mathrm{d}x \\
I &= \frac{1}{3}\rho A L^3 \\
I &= \frac{1}{3}(\rho \cdot AL)L^2 \\
I &= \frac{1}{3}(\rho V)L^2 \\
I &= \frac{1}{3}ML^2
\end{aligned}$$

3. Show that the center of mass for a quarter disk is $\frac{4r}{3\pi}$ away from its flat edges.

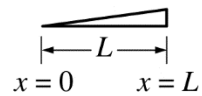
Solution: Let ρ be the density of the rod and M the mass of the quarter disk.

$$\begin{aligned}
\bar{x} &= \frac{\int x \,\mathrm{d}m}{\int \mathrm{d}m} \\
\bar{x} &= \frac{\int_0^r x\rho \,\mathrm{d}A}{M} \\
\bar{x} &= M^{-1} \int_0^r x\rho\sqrt{r^2 - x^2}\,\mathrm{d}x \\
\bar{x} &= \rho M^{-1} \int_0^r x\sqrt{r^2 - x^2}\,\mathrm{d}x \\
u(x) &= r^2 - x^2 \\
\bar{x} &= -\frac{1}{2}\rho M^{-1} \int_{r^2}^0 \sqrt{u}\,\mathrm{d}u \\
\bar{x} &= \frac{1}{2}\rho M^{-1} \left[\frac{2}{3}\sqrt{u}^3\right]_0^{r^2} \\
\bar{x} &= \frac{1}{3}\rho M^{-1} r^3 \\
\bar{x} &= \frac{4r}{3\pi} M^{-1} \left(\rho \cdot \frac{1}{4}\pi r^2\right) \\
\bar{x} &= \frac{4r}{3\pi} M^{-1} (\rho \cdot A) \\
\bar{x} &= \frac{4r}{3\pi}
\end{aligned}$$

Exercises

1. A rod of mass M and length L has a linear mass density given by the equation $\lambda = \frac{M^2}{3L}r^2$, where r is the distance from the leftmost point in the rod.
   (a) Find the distance between the left most point of the rod and the rod's center of mass in terms of the variables provided in the problem.
   (b) Find the moment of inertia about the leftmost point in the rod in terms of the variables provided in the problem.
   (c) Find the moment of inertia about the rightmost point in the rod in terms of the variables provided in the problem.
2. Show that the rotational inertia of a homogeneous cylinder of radius R and mass M, rotating about its central axis, is given by the equation $I = \frac{1}{2}MR^2$.
3. Show that the rotational inertia of a homogeneous **solid** sphere of radius R and mass M, rotating about its diameter, is given by the equation $I = \frac{2}{5}MR^2$.
4. Show that the rotational inertia of a homogeneous **hollow** sphere of radius R and mass M, rotating about its diameter, is given by the equation $I = \frac{2}{3}MR^2$.
5. Using integral calculus, find the center of mass for a homogeneous triangle on the xy-plane whose vertices lie on the points $(0, 0)$, $(w, 0)$ and (w, h).
6. A rod that leftmost point lies on the origin of the xy-plane with length L, mass M and a linear density λ such that $\exists k \in \mathbb{R}^+ | \lambda = kx$ will always have a moment of inertia of $\frac{1}{2}ML^2$ about the origin. Why is this so?

$$\lambda = \left(\frac{2M}{L^2}\right)x$$



3. A triangular rod, shown above, has length L, mass M, and a nonuniform linear mass density given by the equation $\lambda = \frac{2M}{L^2}x$, where x is the distance from one end of the rod.

(a) Using integral calculus, show that the rotational inertia of the rod about its left end is $ML^2/2$.

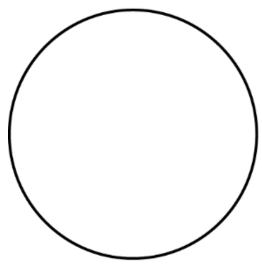


Figure 1

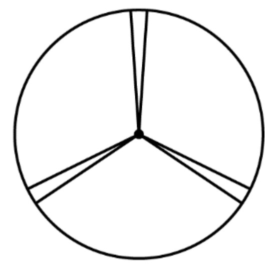


Figure 2

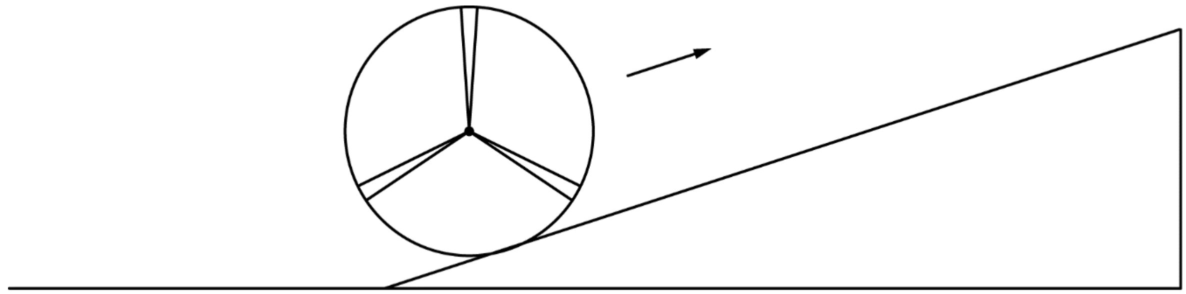
The thin hoop shown above in Figure 1 has a mass M, radius L, and a rotational inertia around its center of ML^2. Three rods identical to the rod from part (a) are now fastened to the thin hoop, as shown in Figure 2 above.

(b) Derive an expression for the rotational inertia I_{tot} of the hoop-rods system about the center of the hoop. Express your answer in terms of M, L, and physical constants, as appropriate.

The hoop-rods system is initially at rest and held in place but is free to rotate around its center. A constant force F is exerted tangent to the hoop for a time Δt.

(c) Derive an expression for the final angular speed ω of the hoop-rods system. Express your answer in terms of M, L, F, Δt, and physical constants, as appropriate.

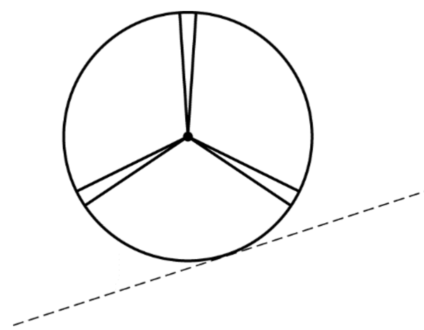
GO ON TO THE NEXT PAGE.



The hoop-rods system is rolling without slipping along a level horizontal surface with the angular speed ω found in part (c). At time $t = 0$, the system begins rolling without slipping up a ramp, as shown in the figure above.

(d)

i. On the figure of the hoop-rods system below, draw and label the forces (not components) that act on the system. Each force must be represented by a distinct arrow starting at, and pointing away from, the point at which the force is exerted on the system.



ii. Justify your choice for the direction of each of the forces drawn in part (d)i.

(e) Derive an expression for the change in height of the center of the hoop from the moment it reaches the bottom of the ramp until the moment it reaches its maximum height. Express your answer in terms of M, L, I_{tot}, ω, and physical constants, as appropriate.

STOP

END OF EXAM