Gauss's Law

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Notes are online at `https://gabrielweredyk.com/papers/PhysicsC/Gauss.pdf`

Charge Density

Before we address Gauss's Law, it's important to understand what a charge density is. Much like how mass can have a density, charge can also have a density as it is an equally fundamental attribute of objects. Charge density has different symbols and units depending on how many dimensions it is distributed over. For objects with a uniform distribution of charge, the equation for charge density is simply

$$\lambda = \frac{q}{\ell} \qquad \sigma = \frac{q}{A} \qquad \rho = \frac{q}{V}$$

Note that ρ is also used for mass density in Physics 2 and that λ was used for linear mass density in Physics C: Mechanics, which demonstrates the similarities between charge density and mass density. When charge is uniformly distributed, we can easily get the charge of certain subpartitions of an object by using its charge density.

Say for example I had a sphere labeled Sphere A of radius R and uniform charge $+Q$. Now by using the fact that the sphere has a uniform distribution of charge, I can get the charge of a subpartition of Sphere A. If I wanted the charge of a smaller sphere that could be carved out of Sphere A, labeled Sphere B with radius r, I could do

$$\begin{aligned}
q &= \rho \cdot V_B \\
q &= \frac{Q}{V_A} V_B \\
q &= \frac{3Q}{4\pi R^3}\left(\frac{4}{3}\pi r^3\right) \\
q &= \frac{Qr^3}{R^3}
\end{aligned}$$

Although its rare to see on the test, charge does not always have to be uniformly distributed over an object. Just as we saw with mass, the charge of an object could instead be distributed according to a formula of position in any given dimension. In a more general definition of charge density, the variables can be defined as

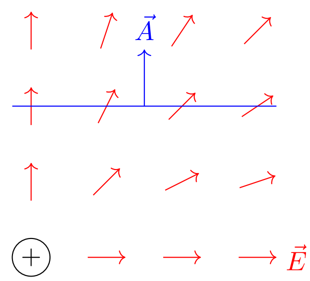
$$\lambda = \frac{\mathrm{d}q}{\mathrm{d}\ell} \qquad \sigma = \frac{\mathrm{d}q}{\mathrm{d}A} \qquad \rho = \frac{\mathrm{d}q}{\mathrm{d}V}$$

Electric Flux

As we learned from magnetism, flux is the total magnitude brought through a certain region by a vector field. In Physics 2, we only cared about magnetic flux and how that related to Faraday-Lenz's Law; however, in Physics C electric flux is a crucial component to Gauss's law. As one may assume, electric flux is the total amount of electric field passing through a given region. In a uniform electric field, the equation for flux is of course

$$\Phi_E = \vec{E} \cdot \vec{A} = |\vec{E}||\vec{A}| \cos\theta$$

But the electric field isn't always uniform, especially when considering electric fields being emitted from charged objects or point charges. To solve for the case when electric field isn't uniform, let's consider how we would determine the flux in the following diagram.



Although not all the vectors of the electric field are shown, we can "start" determining the electric field by considering the flux caused by each electric field vector shown in the diagram. The flux is trivially zero for any electric field vector that doesn't pass through the plane, so we only have to consider the dot product between the electric field vectors and the area vector for the electric field vectors in the plane. Doing so would result in an equation that looks like:

$$\Phi_E = \vec{E}(0,2) \cdot \vec{A} + \vec{E}(1,2) \cdot \vec{A} + \vec{E}(2,2) \cdot \vec{A} + \vec{E}(3,2) \cdot \vec{A}$$

$$\Phi_E = \sum_{k=0}^{3} \vec{E}(k,2) \cdot \vec{A}$$

But we don't want to use a discrete sum of select vectors in the electric field passing through the plane, we want a continuous sum of **all** the electric field vectors passing through the plane. So, we can use an integral instead of a summation to get:

$$\Phi_E = \oint_R \vec{E} \cdot \mathrm{d}\vec{A}$$

Note that $\mathrm{d}\vec{A}$ is equal to the area vector of an infinitesimally small area on the surface. If you're unfamiliar with the $\oint_R$ symbol, it's simply shorthand to denote that we are integrating for every vector in the closed region R. I know that this looks intimidating if you haven't studied vector calculus yet, but as you will see we can often circumvent having to perform actual line integrals.

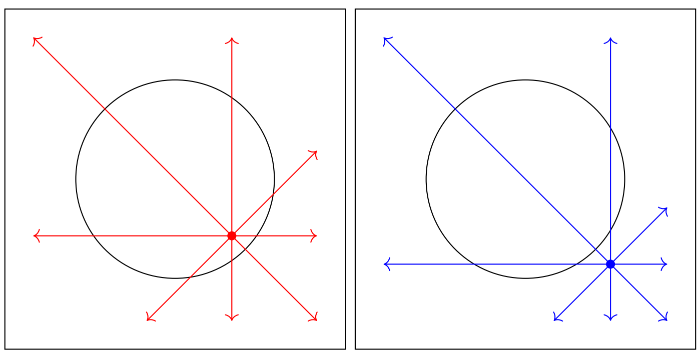
Gauss's Law

Gauss's law states that the flux of the electric field out of an arbitrary closed surface is proportional to the electric charge enclosed by the surface. Mathematically this notion is represented as:

$$\Phi_E = \oint_R \vec{E} \cdot \mathrm{d}\vec{A} = \frac{Q_{enc}}{\varepsilon_0}$$

What Gauss's Law allows us to do is that given the emitted electric field of a charged object or the charge of said object, we can determine the other attribute. The "arbitrary closed surfaces" that are used in Gauss's law are called **Gaussian Surfaces**. Often times Gaussian surfaces do not have to be a physical surface, but could instead represent a mathematical idea or just a general region. For example, if you wanted to determine the electric field emitted 5m away from a charged object, your Gaussian surface would resemble a sphere of radius of 5m because a sphere is the set of three-dimensional points that are a fixed distance from its center.

According to Gauss's law, the electric flux of the region is only affected by point charges **within** the region. To see why this is intuitively, imagine the number of times electric field lines emitted point charges penetrate the surface of a Gaussian surface.

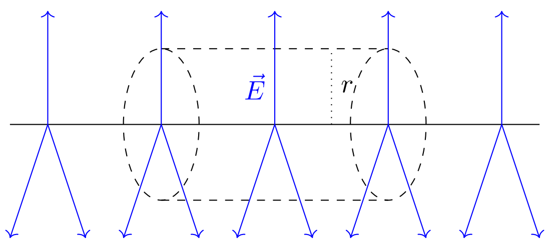


When a point charge is enclosed inside of a Gaussian surface, every electric field line it emits exits the region at one point, thus it contributes to the net amount of electric field traveling through the surface. But, when a point charge isn't enclosed by the Gaussian surface, every electric field line that penetrates the surface must penetrate the surface twice by the Borsuk–Ulam theorem[1]. Thus, the charges outside of the surface don't contribute to the net electric flux traveling through the Gaussian surface.

But if in the diagram on the left we tried to determine the magnitude of the point charge by integrating the electric field over the Gaussian surface, we'd have a very difficult time. This is because the electric field is not constant for every point on the Gaussian surface, the electric field lines are more dense towards the bottom left surface of the diagram, and less dense towards the top right. We very well could compute it with vector calculus, but this is completely unnecessary.

[1] You don't need to know this theorem, I just wanted to throw fun math jargon in here

In general, when we choose Gaussian surfaces to surround charged objects, we want to choose surfaces that guarantee that the electric field strength is constant at every value on the surface. When we choose the geometry of our Gaussian surface, we can also take advantage of the fact that electric field lines parallel to the surface don't contribute to the flux. Consider the following problem: An infinitely long non-conductive wire has a charge density of λ, derive an expression for the electric field strength a distance of r from the wire. When choosing the Gaussian surface to consider our integral on, we need a shape that has a set of points equidistant to a line. That shape would best be represented by a cylinder, as a cylinder can be thought of as a set of circles (which are sets of points equidistant to an origin) in a line. Sketching a quick diagram of our surface looks like



While our wire may be infinitely long, our Gaussian surface must be finite. While this looks we are not properly enclosing the wire, we can actually use the fact that we only care about the charge enclosed to help us derive our expression. Say our cylinder is a length of ℓ along the wire. We can start with Gauss's Law:

$$\oint_R \vec{E} \cdot \mathrm{d}\vec{A} = \frac{Q_{enc}}{\varepsilon_0}$$

Since we want to integrate over the entire surface of the cylinder, we must split our integral into two parts, one part for the two circles at the end of the cylinder, and one part for the plane wrapped around the wire.

$$2\oint_C \vec{E} \cdot \mathrm{d}\vec{A} + \oint_P \vec{E} \cdot \mathrm{d}\vec{A} = \frac{Q_{enc}}{\varepsilon_0}$$

We know that since the electric field is radially emitted from the wire, the electric flux traveling through the two ends of the wire will be equal to zero, which lets us eliminate those integrals. The reverse is true for the wrapped plane: Since the electric field vector is perpendicular, to the surface, the dot product between $\vec{E}$ and $\mathrm{d}\vec{A}$ will be equal to the product of their magnitudes. Thus, we can simplify our equation to

$$\oint_P E \, \mathrm{d}A = \frac{Q_{enc}}{\varepsilon_0}$$

Since every point on the wrapped plane is equidistant to the wire, we know that the electric field strength will be constant for every value on the plane. This allows us to pull the E out of the integral and have

$$E \oint_P \mathrm{d}A = \frac{Q_{enc}}{\varepsilon_0}$$

Now while the term $\oint_P \mathrm{d}A$ looks like its confusing to compute, if you think about what we'd have to do to compute it, the task becomes more sizable. We are adding up all of the infinitesimally small areas over the surface, otherwise known as computing the surface area. So, we can replace the integral with the equation for the surface area of a cylinder (without the two caps) to get

$$E \cdot 2\pi r\ell = \frac{Q_{enc}}{\varepsilon_0}$$

Now we can use our knowledge about charge densities to get the enclosed charge

$$E \cdot 2\pi r\ell = \frac{\lambda\ell}{\varepsilon_0}$$

Then we can algebraically manipulate our equation to finally get the magnitude of electric field at a distance r from the wire.

$$E = \frac{\lambda}{2\pi r\varepsilon_0}$$

At a first glance, the solution to this problem and choosing the Gaussian surface can seem like huge jumps in logic, but really the process can be boiled down to the following steps:

- Find the set of points that are equidistant to the charged object and determine what geometry they resemble
- Find the surface area of all the sides of the surface area that have electric field vectors going **through** them
- Determine the enclosed charge
- Use $E \cdot A = \frac{Q_{enc}}{\varepsilon_0}$

After-note

Often times, the voltage as a function of distance to the object is asked about on the exam. So, it's important to know how voltage relates to electric fields on a calculus level

$$\vec{E} \cdot \hat{\imath} = -\frac{\mathrm{d}V}{\mathrm{d}x}$$

$$\Delta V = -\int \vec{E} \cdot \mathrm{d}\vec{r}$$

Both of these equations are on the reference sheet, and help conceptualize that voltage is to electric fields what work is to force. It is also useful to remember the superposition formula for voltage:

$$V = \frac{1}{4\pi\varepsilon_0} \sum \frac{q_k}{r_k}$$

Its also worth noting that the electric properties of materials are altered when they can conduct electricity. All the physics we have been doing has been for non-conductive objects. If you remember from Physics 2, the electric field inside a conductor is always zero, and the voltage is the same everywhere in and on the conductor.

Exercises

1. Using Gauss's law, show that the electric field generated by a point charge is equal to $\frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}$.

2. Derive an expression for the magnitude of electric field a distance r away from an infinite plane of charge density $+\sigma$.

3. A conducting cube of height h has a linear charge density of λ up the y-axis, determine the electric field in the cube at a distance y from the bottom

4. A nonconducting sphere has a uniform charge density of $+\rho$ and a radius of R derive piecewise expressions for both the voltage and electric field at a distance of r away from the center of the sphere.

5. A nonconducting sphere of radius R has excess charge distributed throughout its volume so that the volume charge density ρ as a function of r (the distance from the sphere's center) is given by the equation $\rho(r) = \rho_0(r/R)^2$, where ρ_0 is a constant. Determine the electric field at points inside and outside the sphere.

6. Consider a very long conducting cylinder of radius R, which carries a uniform linear charge density λ. Determine a formula for the potential at a point outside of the cylinder relative to the potential on the cylinder.

7. A nonconducting sphere of radius R has an excess charge of $+Q$ distributed uniformly throughout its volume. Let r be a distance from the center of the sphere such that $r < R$. Write an expression for the work required to bring a charge of $+q$ from infinity to r.

8. Using the integral definition of work, show that $E = -\frac{\mathrm{d}V}{\mathrm{d}r}$

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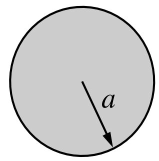
PHYSICS C

SECTION II, ELECTRICITY AND MAGNETISM

Time—45 minutes

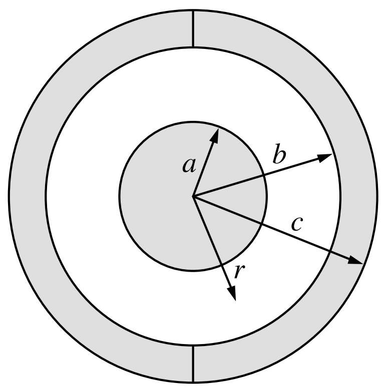
3 Questions

Directions: Answer all three questions. The suggested time is about 15 minutes for answering each of the questions, which are worth 15 points each. The parts within a question may not have equal weight. Show all your work in the pink booklet in the spaces provided after each part, NOT in this green insert.



E&M 1. An isolated conducting sphere of radius $a = 0.20\text{ m}$ is at a potential of $-2{,}000\text{ V}$.

(a) Determine the charge Q_0 on the sphere.



The charged sphere is then concentrically surrounded by two uncharged conducting hemispheres of inner radius $b = 0.40\text{ m}$ and outer radius $c = 0.50\text{ m}$, which are joined together as shown above, forming a spherical capacitor. A wire is connected from the outer sphere to ground, and then removed.

(b) Determine the magnitude of the electric field in the following regions as a function of the distance r from the center of the inner sphere.

i. $r < a$
ii. $a < r < b$
iii. $b < r < c$
iv. $r > c$

(c) Determine the magnitude of the potential difference between the sphere and the conducting shell.

(d) Determine the capacitance of the spherical capacitor.

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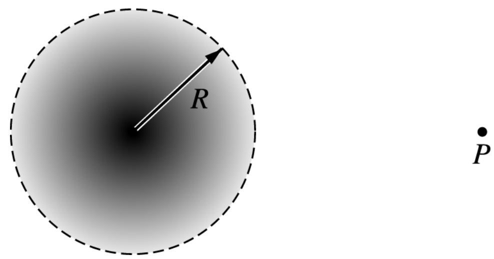

E&M. 1.

A spherical cloud of charge of radius R contains a total charge $+Q$ with a nonuniform volume charge density that varies according to the equation

$$\rho(r) = \rho_0\left(1 - \frac{r}{R}\right) \text{ for } r \le R \text{ and}$$
$$\rho = 0 \text{ for } r > R,$$

where r is the distance from the center of the cloud. Express all algebraic answers in terms of Q, R, and fundamental constants.

(a) Determine the following as a function of r for $r > R$.

i. The magnitude E of the electric field

ii. The electric potential V

(b) A proton is placed at point P shown above and released. Describe its motion for a long time after its release.

(c) An electron of charge magnitude e is now placed at point P, which is a distance r from the center of the sphere, and released. Determine the kinetic energy of the electron as a function of r as it strikes the cloud.

(d) Derive an expression for ρ_0.

(e) Determine the magnitude E of the electric field as a function of r for $r \le R$.

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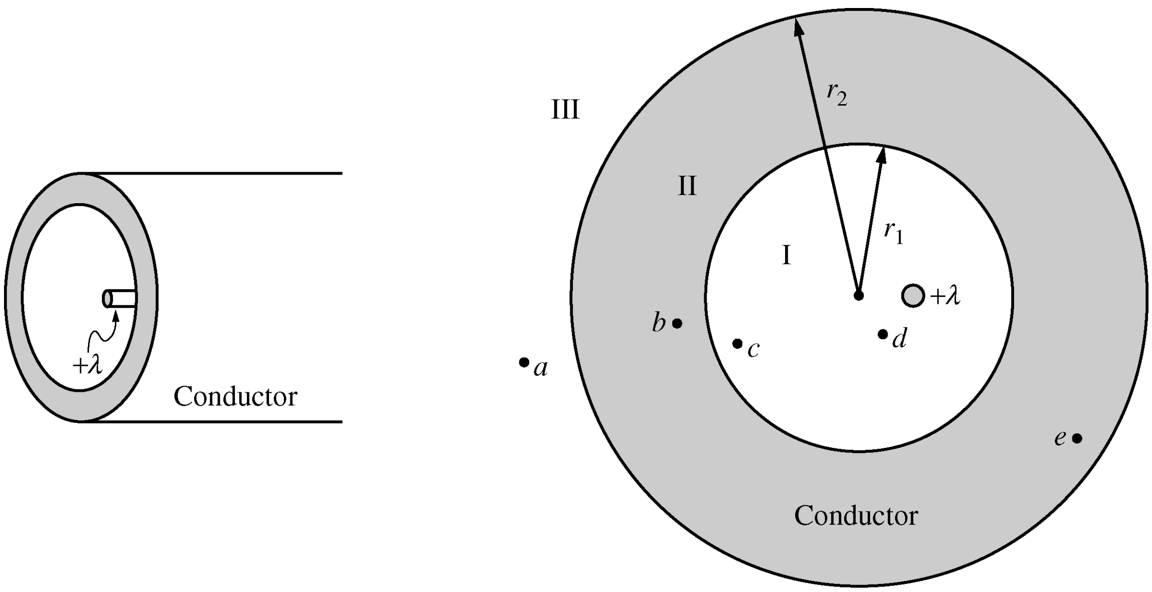

Cross Section

E&M. 1.

The figure above left shows a hollow, infinite, cylindrical, uncharged conducting shell of inner radius r_1 and outer radius r_2. An infinite line charge of linear charge density $+\lambda$ is parallel to its axis but off center. An enlarged cross section of the cylindrical shell is shown above right.

(a) On the cross section above right,

  i. sketch the electric field lines, if any, in each of regions I, II, and III and

  ii. use + and – signs to indicate any charge induced on the conductor.

(b) In the spaces below, rank the electric potentials at points a, b, c, d, and e from highest to lowest (1 = highest potential). If two points are at the same potential, give them the same number.

____ V_a ____ V_b ____ V_c ____ V_d ____ V_e

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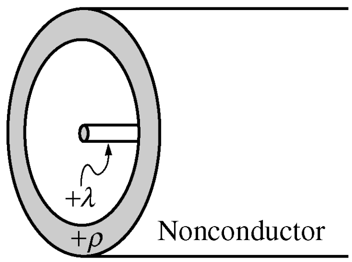
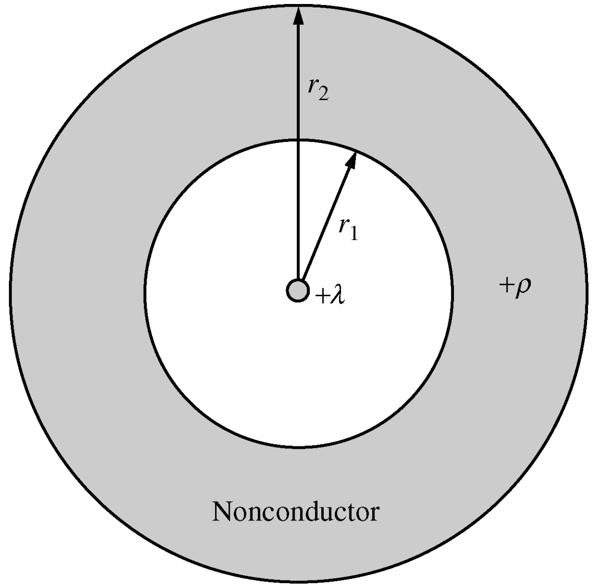




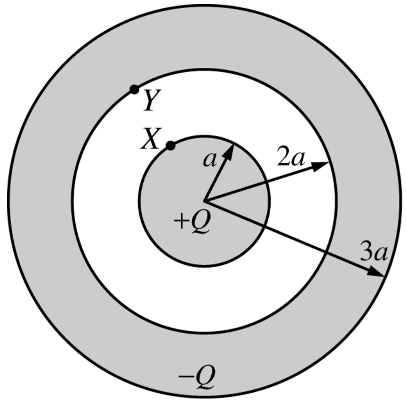
Cross Section

(c) The shell is replaced by another cylindrical shell that has the same dimensions but is nonconducting and carries a uniform volume charge density $+\rho$. The infinite line charge, still of charge density $+\lambda$, is located at the center of the shell as shown above. Using Gauss's law, calculate the magnitude of the electric field as a function of the distance r from the center of the shell for each of the following regions. Express your answers in terms of the given quantities and fundamental constants.

i. $r < r_1$

ii. $r_1 \le r \le r_2$

iii. $r > r_2$



E&M 2.

In the figure above, a nonconducting solid sphere of radius a with charge $+Q$ uniformly distributed throughout its volume is concentric with a nonconducting spherical shell of inner radius $2a$ and outer radius $3a$ that has a charge $-Q$ uniformly distributed throughout its volume. Express all answers in terms of the given quantities and fundamental constants.

(a) Using Gauss's law, derive expressions for the magnitude of the electric field as a function of radius r in the following regions.

i. Within the solid sphere ($r < a$)

ii. Between the solid sphere and the spherical shell ($a < r < 2a$)

iii. Within the spherical shell ($2a < r < 3a$)

iv. Outside the spherical shell ($r > 3a$)

(b) What is the electric potential at the outer surface of the spherical shell ($r = 3a$)? Explain your reasoning.

(c) Derive an expression for the electric potential difference $V_X - V_Y$ between points X and Y shown in the figure.

GO ON TO THE NEXT PAGE.

PHYSICS C: ELECTRICITY AND MAGNETISM
SECTION II
Time—45 minutes
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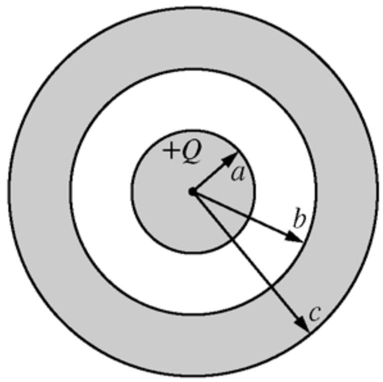


E&M. 1.

A metal sphere of radius a contains a charge $+Q$ and is surrounded by an uncharged, concentric, metallic shell of inner radius b and outer radius c, as shown above. Express all algebraic answers in terms of the given quantities and fundamental constants.

(a) Determine the induced charge on each of the following and explain your reasoning in each case.

i. The inner surface of the metallic shell

ii. The outer surface of the metallic shell

(b) Determine expressions for the magnitude of the electric field E as a function of r, the distance from the center of the inner sphere, in each of the following regions.

i. $r < a$

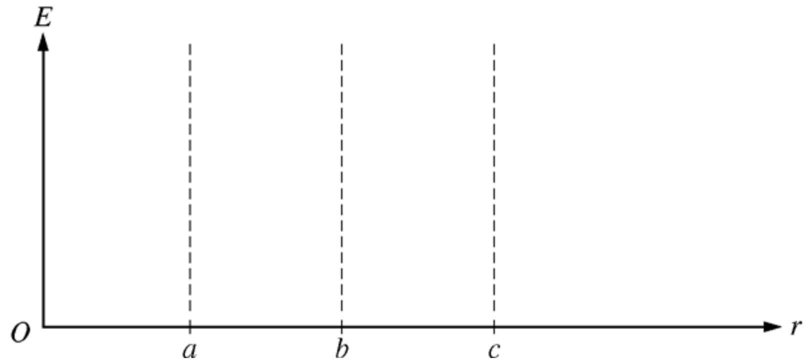
ii. $a < r < b$

iii. $b < r < c$

iv. $c < r$

GO ON TO THE NEXT PAGE.

(c) On the axes below, sketch a graph of E as a function of r.



(d) An electron of mass m_e carrying a charge $-e$ is released from rest at a very large distance from the spheres. Derive an expression for the speed of the particle at a distance $10r$ from the center of the spheres.

E&M.3.

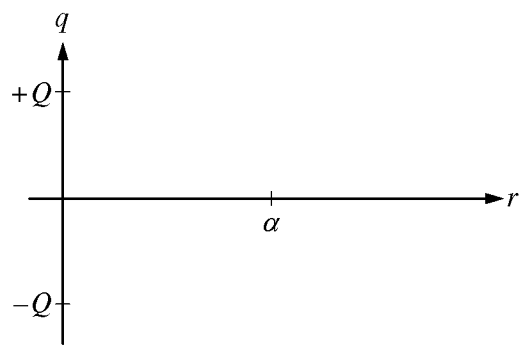
A scientist describes an electrically neutral atom with a model that consists of a nucleus that is a point particle with positive charge $+Q$ at the center of the atom and an electron volume charge density of the form

$$\rho(r) = \begin{cases} -\dfrac{\beta}{r^2} e^{-r/\alpha} & r < \alpha \\ 0 & r > \alpha \end{cases}$$

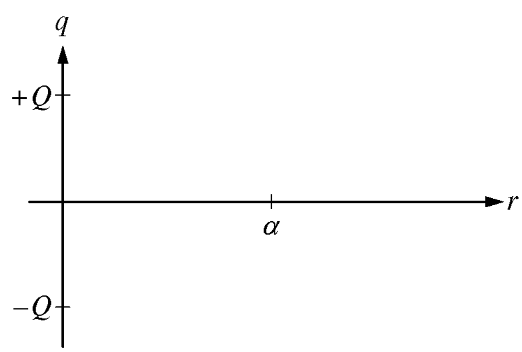
where α and β are positive constants and r is the distance from the center of the atom.

(a) On the axes below, let r stand for the radius of a Gaussian sphere. Sketch the graph for each of the following charges enclosed by the Gaussian sphere as a function of r. Explicitly label any intercepts, asymptotes, maxima, or minima with numerical values or algebraic expressions, as appropriate.

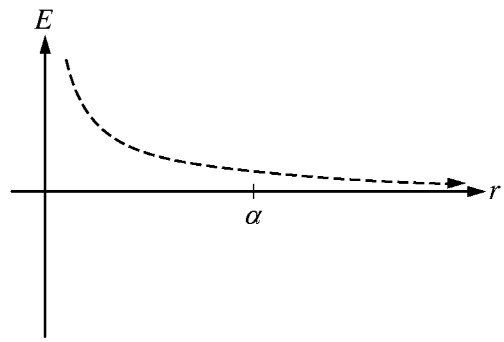
i. The nuclear charge only



ii. The electron charge only



(b) The dashed curve on the graph below represents the electric field as a function of distance r due to the positive nucleus of the atom without any electrons. The nucleus is modeled as a point particle of charge $+Q$. On the same graph, sketch the electric field as a function of distance r for the neutral atom as defined by the scientist's model, which includes the nucleus and the negative electrons surrounding it.



(c) Use Gauss's law to derive an expression for the electric field strength due to the neutral atom for the following positions in terms of $Q, \alpha, \beta, r,$ and fundamental constants, as appropriate.

i. $r > \alpha$

ii. $r < \alpha$

(d) Based on the model proposed by the scientist, what is the physical meaning of the constant α ?

STOP

END OF EXAM

PHYSICS C: ELECTRICITY AND MAGNETISM

SECTION II
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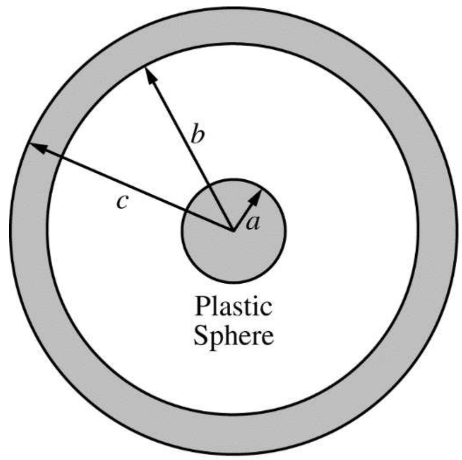


Conducting Spherical Shell

1. A solid plastic sphere of radius a and a conducting spherical shell of inner radius b and outer radius c are shown in the figure above. The shell has an unknown charge. The solid plastic sphere has a charge per unit volume given by $\rho(r) = \beta r$, where β is a positive constant and r is the distance from the center of the sphere. Express your answers to parts (a), (b), and (c) in terms of β , r, a, and physical constants, as appropriate.

   (a) Consider a Gaussian sphere of radius r concentric with the plastic sphere. Derive an expression for the charge enclosed by the Gaussian sphere for the following regions.

      i. $r < a$

      ii. $a < r < b$

   (b) Use Gauss's law to derive an expression for the magnitude of the electric field in the following regions.

      i. $r < a$

      ii. $a < r < b$

   (c) At any point outside of the conducting shell, it is observed that the magnitude of the electric field is zero.

      i. Determine the charge on the inner surface of the conducting shell.

      Justify your answer.

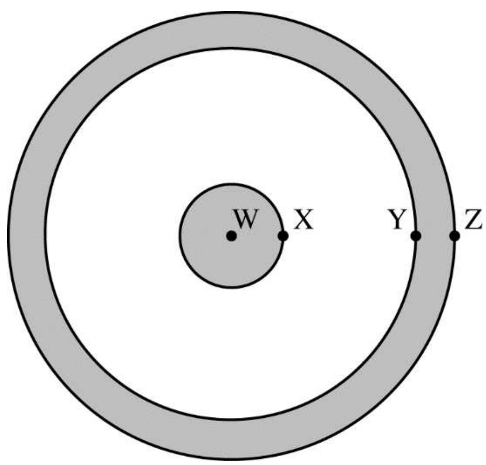
      ii. Determine the charge on the outer surface of the conducting shell.

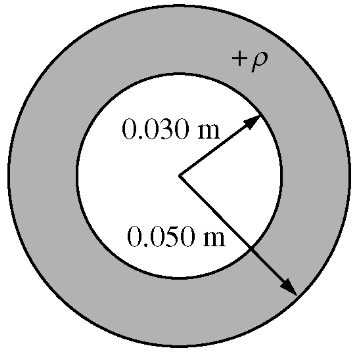
(d)

i. On the axes below, sketch the electric field E as a function of distance r from the center of the sphere. Sketch the graph for the range $r = 0$ at the center of the sphere to $r = c$ at the outside of the conducting shell.

ii. The figure below shows the sphere and shell with four points labeled W, X, Y, and Z. Point W is at the center of the sphere, point X is on the surface of the sphere, and points Y and Z are on the inner and outer surface of the shell, respectively. Rank the points according to the electric potential at that point, with 1 indicating the largest electric potential. If two points have the same electric potential, give them the same numerical ranking.

____ W   ____ X   ____ Y   ____ Z





2. A nonconducting hollow sphere of inner radius 0.030 m and outer radius 0.050 m carries a positive volume charge density ρ, as shown in the figure above. The charge density ρ of the sphere is given as a function of the distance r from the center of the sphere, in meters, by the following.

$$r < 0.030 \text{ m}: \quad \rho = 0$$

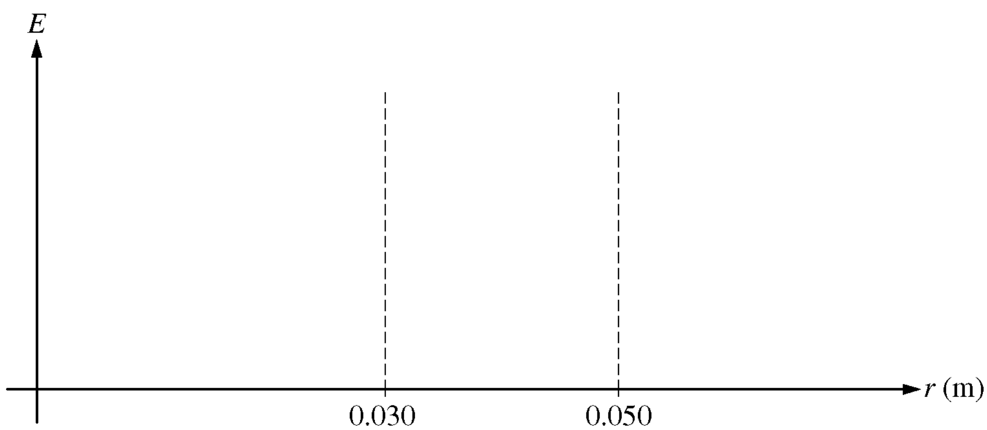
$$0.030 \text{ m} < r < 0.050 \text{ m}: \quad \rho = b/r, \text{ where } b = 1.6 \times 10^{-6} \text{ C/m}^2$$

$$r > 0.050 \text{ m}: \quad \rho = 0$$

(a) Calculate the total charge of the sphere.

(b) Using Gauss's law, calculate the magnitude of the electric field E at the outer surface of the sphere.

(c) On the axes below, sketch the magnitude of the electric field E as a function of distance r from the center of the sphere.



(d) Calculate the electric potential V at the outer surface of the sphere. Assume the electric potential to be zero at infinity.

(e) A proton is released from rest at the outer surface of the sphere at time $t = 0$ s.

i. Calculate the magnitude of the initial acceleration of the proton.

ii. Calculate the speed of the proton after a long time.