RC Circuits

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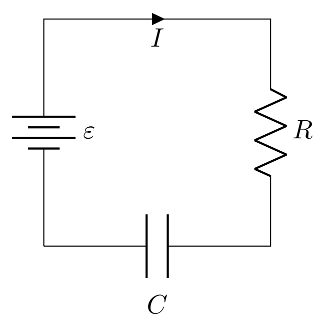
Notes are online at `https://gabrielweredyk.com/papers/PhysicsC/RCcircuits.pdf`

Charge Build-Up

In Physics 2: Algebra-Based, RC Circuits are only examined under two conditions: When the circuit is initially turned on, and when the circuit has reached a steady state. However, in Physics C, by solving the differential equation created by Kirchhoff's Loop Law, we can examine how a capacitor acts in a circuit over time. Consider the following circuit:



As stated, we can start with Kirchhoff's Loop Law to get an expression with the current of the circuit (and therefore the current of any individual component in series). From there we can solve for the charge of the capacitor as a function of time and using measurable values (R, ε & C).

$$
\begin{aligned}
V_T &= V_R + V_C \\
\varepsilon &= IR + \frac{Q}{C} \\
\varepsilon &= \frac{\mathrm{d}Q}{\mathrm{d}t}R + \frac{Q}{C} \\
\frac{\mathrm{d}Q}{\mathrm{d}t}R &= \varepsilon - \frac{Q}{C} \\
\frac{\mathrm{d}Q}{\mathrm{d}t} &= \frac{\varepsilon C - Q}{RC} \\
\frac{\mathrm{d}Q}{\varepsilon C - Q} &= \frac{\mathrm{d}t}{RC}
\end{aligned}
$$

$$
\begin{aligned}
\int_0^Q \frac{\mathrm{d}Q}{\varepsilon C - Q} &= \int_0^t \frac{\mathrm{d}t}{RC} \\
-\left[\ln(\varepsilon C - Q)\right]_0^Q &= \left[\frac{t}{RC}\right]_0^t \\
-\ln(\varepsilon C - Q) + \ln(\varepsilon C) &= \frac{t}{RC} \\
\ln(\varepsilon C - Q) - \ln(\varepsilon C) &= -\frac{t}{RC} \\
\ln\left(\frac{\varepsilon C - Q}{\varepsilon C}\right) &= -\frac{t}{RC} \\
\frac{\varepsilon C - Q}{\varepsilon C} &= e^{-\frac{t}{RC}} \\
\varepsilon C - Q &= \varepsilon C e^{-\frac{t}{RC}} \\
Q(t) &= \varepsilon C - \varepsilon C e^{-\frac{t}{RC}} \\
Q(t) &= \varepsilon C\left(1 - e^{-\frac{t}{RC}}\right)
\end{aligned}
$$

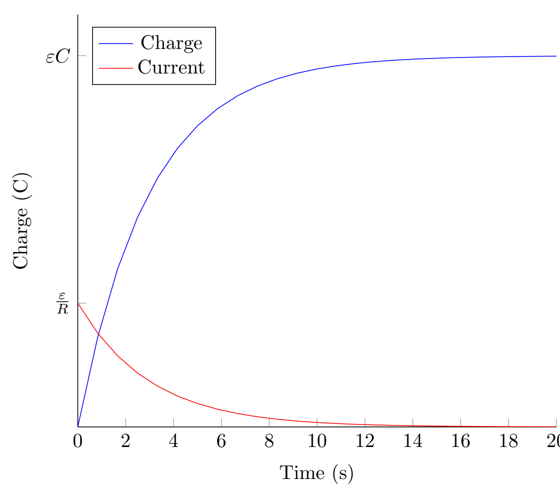
The quantity RC that appears in the exponent is called the **time constant** for the circuit and is represented by τ. As the time constant increases, the time it takes for the circuit to reach a steady state increases.

$$Q(t) = \varepsilon C\left(1 - e^{-\frac{t}{\tau}}\right)$$

Note that this function for the charge in the capacitor is consistent with our concepts used in Physics 2: Algebra-Based. Note that $Q(0) = 0$ and that $\lim_{t\to\infty} Q(t) = \varepsilon C$, which is consistent with the equation in our reference sheet. From this function of charge, we can derive functions for similar quantities in the capacitor.

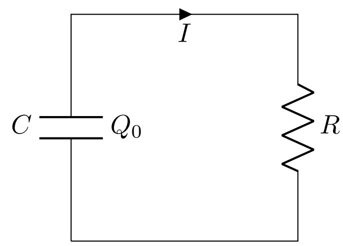
$$
\begin{aligned}
V(t) &= \varepsilon\left(1 - e^{-\frac{t}{\tau}}\right) \\
I(t) &= \frac{\varepsilon}{R} e^{-\frac{t}{\tau}}
\end{aligned}
$$

We can graph these functions to get a better sense of their nature



Charge Dissipation

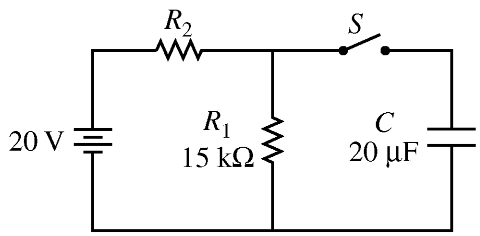
Now imagine we have a fully charged capacitor at charge Q_0 and it is inserted into a circuit with some total resistance R. We can use Kirchhoff's Loop Law again and solve the differential to get an equation for charge over time.

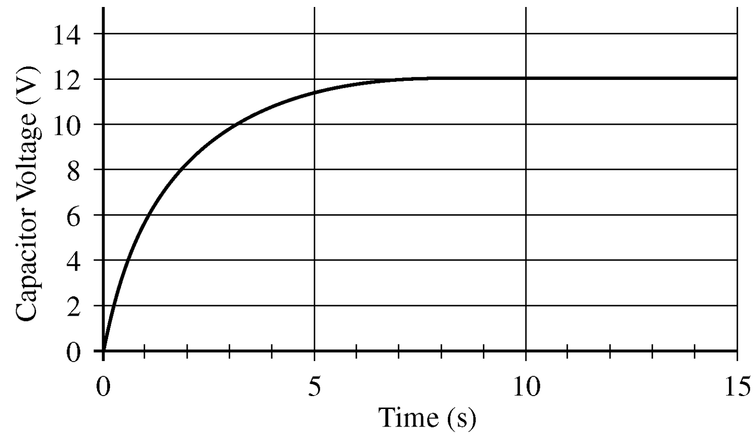


$$\begin{aligned}
V_R &= -V_C \\
IR &= \frac{Q}{C} \\
-\frac{\mathrm{d}Q}{\mathrm{d}t}R &= \frac{Q}{C} \\
-\frac{\mathrm{d}Q}{Q} &= \frac{\mathrm{d}t}{RC} \\
-\int_{Q_0}^{Q} \frac{\mathrm{d}Q}{Q} &= \int_0^t \frac{\mathrm{d}t}{RC} \\
\ln Q - \ln Q_0 &= -\frac{t}{RC} \\
\ln \frac{Q}{Q_0} &= \frac{t}{RC} \\
\frac{Q}{Q_0} &= e^{-\frac{t}{RC}} \\
Q(t) &= Q_0 e^{-\frac{t}{\tau}}
\end{aligned}$$

Note that this function for Q is also consistent with our findings from Physics 2: Algebra-Based. Both $Q(0) = Q_0$ and $\lim_{t\to\infty} Q(t) = 0$. We can get related variables to the charge once again.

$$\begin{aligned}
I(t) &= -\frac{Q_0}{\tau} e^{-\frac{t}{\tau}} \\
V(t) &= \frac{Q_0}{C} e^{-\frac{t}{\tau}}
\end{aligned}$$





E&M. 2.

In the circuit shown above left, the switch S is initially in the open position and the capacitor C is initially uncharged. A voltage probe and a computer (not shown) are used to measure the potential difference across the capacitor as a function of time after the switch is closed. The graph produced by the computer is shown above right. The battery has an emf of 20 V and negligible internal resistance. Resistor R_1 has a resistance of 15 kΩ and the capacitor C has a capacitance of 20 μF.

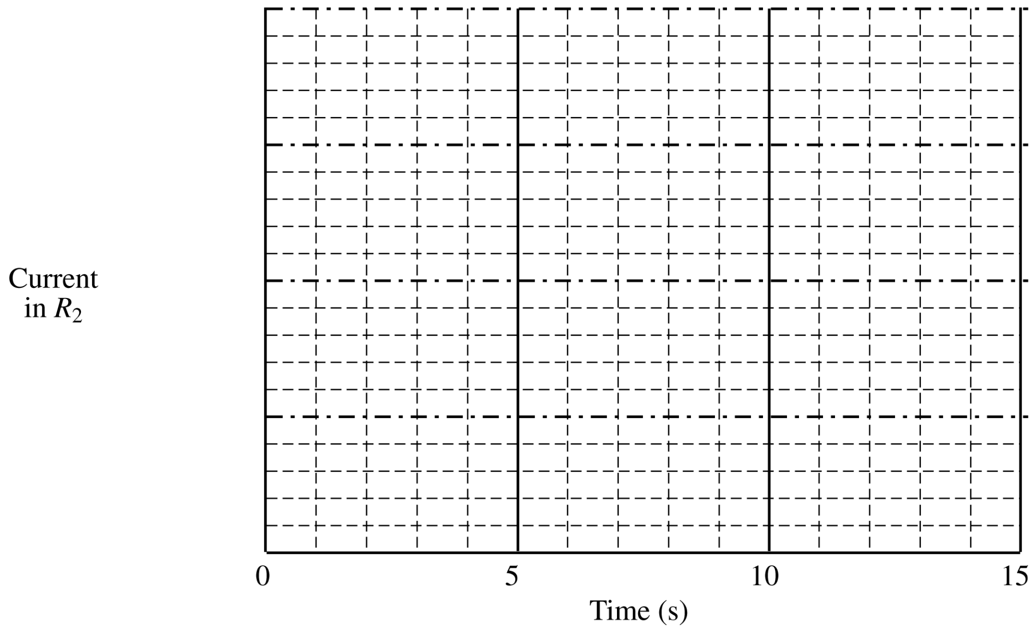
(a) Determine the voltage across resistor R_2 immediately after the switch is closed.

(b) Determine the voltage across resistor R_2 a long time after the switch is closed.

(c) Calculate the value of the resistor R_2.

(d) Calculate the energy stored in the capacitor a long time after the switch is closed.

(e) On the axes below, graph the current in R_2 as a function of time from 0 to 15 s. Label the vertical axis with appropriate values.

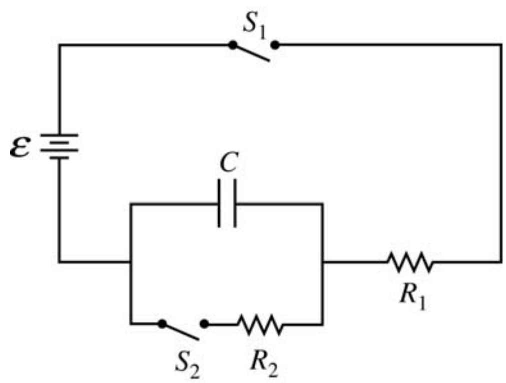


Resistor R_2 is removed and replaced with another resistor of lesser resistance. Switch S remains closed for a long time.

(f) Indicate below whether the energy stored in the capacitor is greater than, less than, or the same as it was with resistor R_2 in the circuit.

____ Greater than ____ Less than ____ The same as

Explain your reasoning.



E&M 2.

The circuit above contains a capacitor of capacitance C, a power supply of emf $\mathcal{E}$, two resistors of resistances R_1 and R_2, and two switches, S_1 and S_2. Initially, the capacitor is uncharged and both switches are open. Switch S_1 then gets closed at time $t = 0$.

(a) Write a differential equation that can be solved to obtain the charge on the capacitor as a function of time t.

(b) Solve the differential equation in part (a) to determine the charge on the capacitor as a function of time t.

Numerical values for the components are given as follows:

$\mathcal{E} = 12$ V
$C = 0.060$ F
$R_1 = R_2 = 4700\ \Omega$

(c) Determine the time at which the capacitor has a voltage 4.0 V across it.

After switch S_1 has been closed for a long time, switch S_2 gets closed at a new time $t = 0$.

(d) On the axes below, sketch graphs of the current I_1 in R_1 versus time and of the current I_2 in R_2 versus time, beginning when switch S_2 is closed at new time $t = 0$. Clearly label which graph is I_1 and which is I_2.

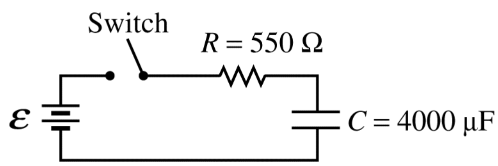
PHYSICS C: ELECTRICITY AND MAGNETISM

SECTION II

Time—45 minutes

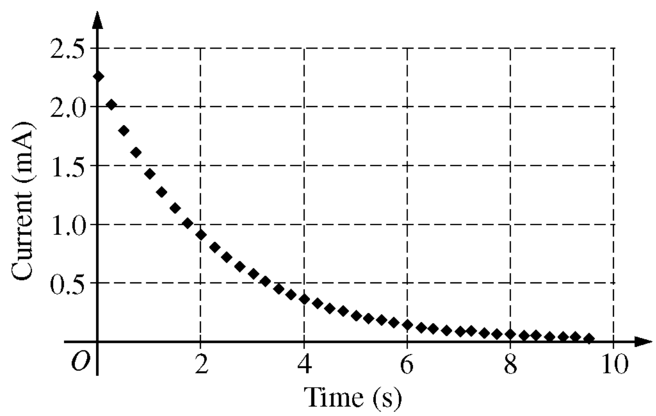
3 Questions

Directions: Answer all three questions. The suggested time is about 15 minutes for answering each of the questions, which are worth 15 points each. The parts within a question may not have equal weight. Show all your work in the pink booklet in the spaces provided after each part, NOT in this green insert.



E&M 1.

A student sets up the circuit above in the lab. The values of the resistance and capacitance are as shown, but the constant voltage $\mathcal{E}$ delivered by the ideal battery is unknown. At time $t = 0$, the capacitor is uncharged and the student closes the switch. The current as a function of time is measured using a computer system, and the following graph is obtained.



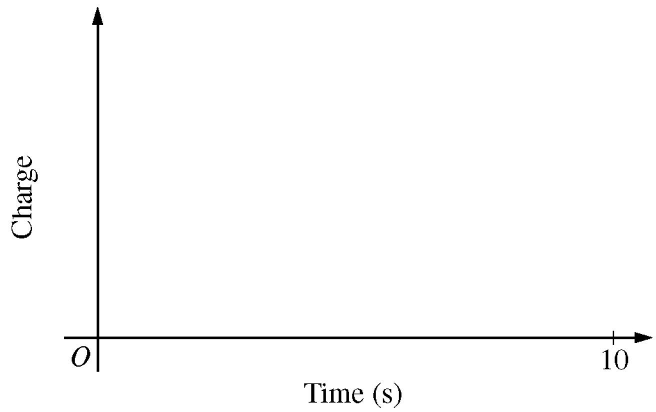
(a) Using the data above, calculate the battery voltage $\mathcal{E}$.

(b) Calculate the voltage across the capacitor at time $t = 4.0$ s.

(c) Calculate the charge on the capacitor at $t = 4.0$ s.

GO ON TO THE NEXT PAGE.

(d) On the axes below, sketch a graph of the charge on the capacitor as a function of time.



(e) Calculate the power being dissipated as heat in the resistor at $t = 4.0$ s.

(f) The capacitor is now discharged, its dielectric of constant $\kappa = 1$ is replaced by a dielectric of constant $\kappa = 3$, and the procedure is repeated. Is the amount of charge on one plate of the capacitor at $t = 4.0$ s now greater than, less than, or the same as before? Justify your answer.

_____Greater than _____Less than _____The same

GO ON TO THE NEXT PAGE.

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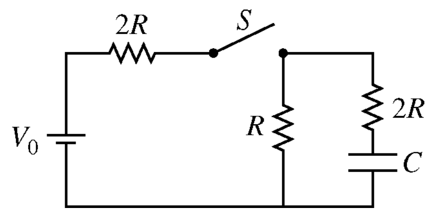

1. The circuit represented above is composed of three resistors with the resistances shown, a battery of voltage V_0, a capacitor of capacitance C, and a switch S. The switch is closed, and after a long time, the circuit reaches steady-state conditions. Answer the following questions in terms of V_0, R, C, and fundamental constants, as appropriate.

   (a) Derive an expression for the steady-state current supplied by the battery.

   (b) Derive an expression for the charge on the capacitor.

   (c) Derive an expression for the energy stored in the capacitor.

   Now the switch is opened at time $t = 0$.

   (d) Write, but do NOT solve, a differential equation that could be used to solve for the charge $q(t)$ on the capacitor as a function of the time t after the switch is opened.

(e)

i. Calculate the current in resistor R immediately after the switch is opened.

ii. On the axes below, sketch the current in the circuit as a function of time from time $t = 0$ to a long time after the switch is opened. Explicitly label the maxima with numerical values or algebraic expressions, as appropriate.

(f) Is the total amount of energy dissipated in the resistors after the switch is opened greater than, less than, or equal to the amount of energy stored in the capacitor calculated in part (c) ?

____ Greater than ____ Less than ____ Equal to

Justify your answer.