1 Q1

1. Mr. Ludwig gives his students the chance to use a cheat sheet on one of his tests with a game of chance. Initially, the students are asked to pick a number between 1 and 12, then Mr. Ludwig rolls a fair 12-sided die. If the die lands on the student's number, they get to use the cheat sheet, otherwise they must take the current test without a cheat sheet. At the next test, if the student didn't win the last game of chance, they can choose one more number than before. What is the expected number of games it takes for a student to be able to use their cheat sheet?

1.1 Answer

The expected value of a discrete random distribution can be calculated by:

$$E = \sum np(n)$$

The probability of winning on the 1st, 2nd and 3rd time can be calculated intuitively as follows:

$$\begin{aligned}
p(1) &= \frac{1}{12} \\
p(2) &= \frac{11}{12} \cdot \frac{2}{12} \\
p(3) &= \frac{11}{12} \cdot \frac{10}{12} \cdot \frac{3}{12}
\end{aligned}$$

The same pattern can be followed for $p(n)$ up to $n = 12$, resulting in the general formula for $n > 1$:

$$\begin{aligned}
p(n) &= \frac{n}{12} \prod_{k=2}^{n} \frac{12 - (k-1)}{12} \\
p(n) &= \frac{n}{12} \prod_{k=2}^{n} \frac{13 - k}{12} \\
p(n) &= \frac{n}{12} \cdot \frac{12!}{(13-n)!12^{n-1}} \\
p(n) &= \frac{12!n}{(13-n)!12^{n}}
\end{aligned}$$

Thus the expected value is evaluated at:

$$\begin{aligned}
E &= \sum_{n=1}^{12} \frac{12!n^2}{(13-n)!12^n} \\
E &= 6.455
\end{aligned}$$

And students should expect to get access to a cheat sheet within 7 tests.