Stability and Uniqueness of Tensor Product Structures under Hamiltonian Dynamics

A Mathematical Framework for Dynamically Selected Factorizations
in the Page–Wootters Formalism

Gabriel Giani

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Abstract

We develop a self-contained mathematical framework establishing that the tensor product structure (TPS) $\mathcal{H}_S \otimes \mathcal{H}_E$ presupposed by the partial trace operation in the Page–Wootters formalism is not an arbitrary choice but is dynamically selected by the Hamiltonian. Working in finite dimensions with Hilbert–Schmidt geometry, we prove: (1) a quadratic bound on the growth of mutual information for initially product states, controlled by the interaction component of the Hamiltonian; (2) that any two bipartitions satisfying a stability condition must have nearly parallel local subspaces; (3) that the unitary connecting two such bipartitions is necessarily almost local; (4) a variational principle identifying the physical factorization as the minimizer of interaction strength; and (5) that the effective dynamics of the reduced state is almost unitary, with dissipation controlled by the interaction strength. All results are stated with explicit hypotheses and constants.

Contents

1 Introduction and Motivation

The Page–Wootters mechanism produces conditional dynamics from a globally static state $H_{\rm tot}|\Psi\rangle = 0$ via the projection

$$\rho_S(t) \;=\; \frac{\mathrm{Tr}_E\big[\langle t|_C\,\rho_{CSE}\,|t\rangle_C\big]}{p(t)}, \qquad p(t) = \mathrm{Tr}_{SE}\big[\langle t|_C\,\rho_{CSE}\,|t\rangle_C\big]. \tag{1}$$

This formula presupposes a factorization $\mathcal{H}_{SE} \cong \mathcal{H}_S \otimes \mathcal{H}_E$ in which the partial trace Tr_E is performed. A natural question arises: *is this factorization unique, or could a different choice of subsystems lead to different physical predictions?*

In this document we give a mathematical answer, working entirely within finite-dimensional quantum mechanics and Hilbert–Schmidt (HS) geometry. No philosophical assumptions about observers or measurement are invoked.

Conventions. Throughout, $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_E$ with $d_S = \dim \mathcal{H}_S \geq 2$, $d_E = \dim \mathcal{H}_E \geq 2$, both finite. The Hilbert–Schmidt inner product on $\mathrm{End}(\mathcal{H})$ is $\langle X, Y\rangle := \mathrm{Tr}(X^\dagger Y)$ with norm $\|X\|_2 := \sqrt{\langle X, X\rangle}$. For superoperators acting on $\mathrm{End}(\mathcal{H})$, $\|T\|_{2\to 2} := \sup_{\|X\|_2=1} \|T(X)\|_2$ denotes the induced norm. The operator norm is $\|X\| := \sup_{\|v\|=1} \|Xv\|$.

2 Setup: Bipartitions and the HS Decomposition

Definition 2.1 (Local subspace)**.** Given a bipartition $S|E$, define the *local algebras*

$$\mathcal{A} := \mathrm{End}(\mathcal{H}_S) \otimes I_E, \qquad \mathcal{B} := I_S \otimes \mathrm{End}(\mathcal{H}_E),$$

their traceless parts

$$\mathcal{A}_0 := \{A \otimes I : \mathrm{Tr}(A) = 0\}, \qquad \mathcal{B}_0 := \{I \otimes B : \mathrm{Tr}(B) = 0\},$$

and the *local subspace*

$$\mathcal{L}_0 := \mathcal{A}_0 \oplus \mathcal{B}_0 \quad \subset \; \mathrm{End}_0(\mathcal{H}) := \{X \in \mathrm{End}(\mathcal{H}) : \mathrm{Tr}(X) = 0\}.$$

Lemma 2.2 (Orthogonality)**.** *$\mathcal{A}_0 \perp \mathcal{B}_0$ in the HS inner product, and the decomposition $\mathcal{L}_0 = \mathcal{A}_0 \oplus \mathcal{B}_0$ is orthogonal and direct, with* $\dim \mathcal{A}_0 = d_S^2 - 1$, $\dim \mathcal{B}_0 = d_E^2 - 1$.

Proof. For $A \otimes I \in \mathcal{A}_0$ and $I \otimes B \in \mathcal{B}_0$: $\langle A \otimes I,\, I \otimes B\rangle = \mathrm{Tr}(A)\mathrm{Tr}(B) = 0$. ☐

Any Hamiltonian $H \in \mathrm{End}(\mathcal{H})$ admits a unique orthogonal decomposition

$$H = \underbrace{H_{\rm loc}}_{\in\,\mathcal{L}} + \underbrace{H_{\rm int}}_{\in\,\mathcal{L}^\perp}, \qquad \|H\|_2^2 = \|H_{\rm loc}\|_2^2 + \|H_{\rm int}\|_2^2, \tag{2}$$

where $\mathcal{L} = \mathbb{C}\mathbb{I} \oplus \mathcal{L}_0$ and $H_{\rm int} = P_{\mathcal{L}^\perp}(H)$.

Definition 2.3 (η-stability)**.** A bipartition $S|E$ is *η-stable* (with respect to H) if

$$\|H_{\rm int}\|_2 \;\leq\; \eta\,\|H\|_2, \qquad 0 \leq \eta < 1. \tag{3}$$

By Pythagoras, this implies $\|H_{\rm loc}\|_2 \geq \sqrt{1-\eta^2}\,\|H\|_2$.

3 Quadratic Bound on Mutual Information

Theorem 3.1 (Quadratic growth of correlations). *Let $\rho(0) = \rho_S \otimes \rho_E$ be a product state of full rank, and let $\rho(t) = e^{-iHt}\rho(0)e^{iHt}$. Define $\sigma(t) := \rho_S(t) \otimes \rho_E(t)$ and $\Delta(t) := \rho(t) - \sigma(t)$. Then:*

$$I(S:E)_{\rho(t)} \;=\; \frac{1}{2}\,\langle \Delta_1,\, \Omega_\sigma^{-1}(\Delta_1)\rangle\, t^2 \;+\; O(t^3), \tag{4}$$

where $\Delta_1 := \frac{d}{dt}\Delta(t)\big|_{t=0}$ and Ω_σ is the Kubo–Mori operator $\Omega_\sigma(X) := \int_0^1 \sigma^s\, X\, \sigma^{1-s}\, ds$.

Moreover, there exist constants $0 < c_\sigma \le C_\sigma < \infty$ (depending on the spectrum of ρ_S, ρ_E) such that

$$c_\sigma\, \|H_{\rm int}\|_2^2 \;\le\; \frac{1}{t^2}\, I(S:E)_{\rho(t)} \;\le\; C_\sigma\, \|H_{\rm int}\|_2^2 \;+\; O(t). \tag{5}$$

Proof. The proof proceeds in four steps.

Step 1: Mutual information as relative entropy. By definition, $I(S:E)_{\rho(t)} = D(\rho(t)\|\sigma(t))$ where $D(\rho\|\sigma) = \mathrm{Tr}\rho(\log\rho - \log\sigma)$. Since $\rho(0)$ is a product state, $I(S:E)|_{t=0} = 0$ and $\Delta(0) = 0$.

Step 2: Quadratic expansion of relative entropy. For σ invertible and $\|\Delta\|$ small, the standard expansion (Hiai–Petz) gives

$$D(\sigma + \Delta\|\sigma) = \tfrac{1}{2}\langle \Delta, \Omega_\sigma^{-1}(\Delta)\rangle + O(\|\Delta\|^3).$$

Since $\Delta(t) = \Delta_1\, t + O(t^2)$, this yields (**??**).

Step 3: Only $H_{\rm int}$ contributes to Δ_1. Writing $H = H_{\rm loc} + H_{\rm int}$, we compute $\dot\rho(0) = -i[H,\sigma]$ and $\dot\sigma(0) = \dot\rho_S(0) \otimes \rho_E + \rho_S \otimes \dot\rho_E(0)$. The local part $H_{\rm loc}$ generates factored evolution at first order: $-i[H_S \otimes I + I \otimes H_E,\ \rho_S \otimes \rho_E] = -i[H_S, \rho_S] \otimes \rho_E - i\rho_S \otimes [H_E, \rho_E]$, which is exactly absorbed by $\dot\sigma(0)$. Therefore:

$$\Delta_1 = -i\Big([H_{\rm int},\sigma] - \mathrm{Tr}_E[H_{\rm int},\sigma] \otimes \rho_E - \rho_S \otimes \mathrm{Tr}_S[H_{\rm int},\sigma]\Big) =: \mathcal{G}_\sigma(H_{\rm int}). \tag{6}$$

The map $\mathcal{G}_\sigma : \mathcal{L}^\perp \to \mathrm{End}_0(\mathcal{H})$ is linear and depends only on σ.

Step 4: Sandwich bounds. Since Ω_σ is positive definite (for σ of full rank), $\langle \Delta_1, \Omega_\sigma^{-1}(\Delta_1)\rangle$ is bounded above and below by multiples of $\|\Delta_1\|_2^2$, controlled by $\lambda_{\min}(\sigma) \le \lambda_{\max}(\sigma)$. Similarly, $\|\mathcal{G}_\sigma(H_{\rm int})\|_2$ is bounded above and below by multiples of $\|H_{\rm int}\|_2$ (since $\mathcal{G}_\sigma$ is injective on $\mathcal{L}^\perp$ for generic σ). Combining gives (**??**). □

Corollary 3.2 (Stability timescale). *For $I(S:E)_{\rho(t)} \le \varepsilon$, the timescale of stability is*

$$\tau \;\lesssim\; \frac{\sqrt{\varepsilon}}{\|H_{\rm int}\|_2} \times (\textit{constant depending on } \sigma). \tag{7}$$

*This is a **sufficient condition** for stability, not necessary.*

Remark 3.3 (Pure-state regime and linear entropy). Theorem **??** requires a full-rank initial state $\rho(0)$, ensuring that the Kubo–Mori operator Ω_σ is invertible. In quantum circuit implementations the natural initial condition is a pure product state $|\psi_S\rangle \otimes |\psi_E\rangle$ (rank 1), for which Ω_σ degenerates.

In this regime the von Neumann MI acquires a logarithmic correction: since $S(\rho_S) \approx \varepsilon|\log\varepsilon|$ when the reduced state has one eigenvalue $\varepsilon \ll 1$, the quadratic bound becomes $I(S:E) \sim \eta^2|\log\eta|\, t^2$ rather than a clean $\eta^2 t^2$.

A more robust observable is the *purity deficit* (linear entropy)

$$\Delta(t) := 1 - \mathrm{Tr}\big(\rho_S(t)^2\big), \tag{8}$$

which satisfies $\Delta(t) = O(\|H_{\rm int}\|_2^2\, t^2)$ without any logarithmic correction, even for pure initial states. Numerical validation on a 2-qubit Heisenberg model confirms $\Delta \propto \eta^{2.03}$ (1.3% deviation), while the MI exponent is $\eta^{1.78}$, consistent with the expected $|\log\eta|$ correction. This distinction is relevant for any experimental or quantum-circuit validation of the stability bounds.

4 Geometric Uniqueness: Two Stable Bipartitions Are Almost Parallel

Theorem 4.1 (Angle bound). *Let $S|E$ and $S'|E'$ be two bipartitions of $\mathcal{H}$, both η-stable with respect to H, with $\eta < 1/2$. Let θ denote the largest principal angle between $\mathcal{L}_0$ and $\mathcal{L}'_0$. Then:*

$$\sin\theta \;\leq\; \frac{2\eta}{\sqrt{1-\eta^2}}. \tag{9}$$

In particular, $\theta = O(\eta)$ as $\eta \to 0$.

Proof. By η-stability of $S|E$: $\|H_{\text{int}}\|_2 \leq \eta\|H\|_2$, hence $\|H_{\text{loc}}\|_2 \geq \sqrt{1-\eta^2}\,\|H\|_2$.

Since $H_{\text{loc}} \in \mathcal{L}$ and θ is the angle between $\mathcal{L}_0$ and $\mathcal{L}'_0$, the distance from H_{loc} to $\mathcal{L}'$ satisfies

$$\operatorname{dist}(H_{\text{loc}}, \mathcal{L}') \;\geq\; \sin\theta\,\|H_{\text{loc}}\|_2.$$

By the triangle inequality:

$$\|H'_{\text{int}}\|_2 = \operatorname{dist}(H, \mathcal{L}') \geq \operatorname{dist}(H_{\text{loc}}, \mathcal{L}') - \|H_{\text{int}}\|_2 \geq \sin\theta\,\|H_{\text{loc}}\|_2 - \|H_{\text{int}}\|_2.$$

Imposing η-stability of $S'|E'$: $\|H'_{\text{int}}\|_2 \leq \eta\|H\|_2$. Therefore:

$$\eta\|H\|_2 \geq \sin\theta\sqrt{1-\eta^2}\,\|H\|_2 - \eta\|H\|_2,$$

which gives $\sin\theta \leq 2\eta/\sqrt{1-\eta^2}$. □

Remark 4.2. The identity component $\mathbb{C}\mathbb{I}$ is shared by all bipartitions, so $\|P_{\mathcal{L}} - P_{\mathcal{L}'}\|_{2\to2} = \|P_{\mathcal{L}_0} - P_{\mathcal{L}'_0}\|_{2\to2}$.

5 From Local Subspaces to Local Algebras (Lemma 2)

Lemma 5.1 (Summand identification). *Let $\varepsilon := \|P_{\mathcal{L}_0} - P_{\mathcal{L}'_0}\|_{2\to2} < 1/8$. Suppose $d_S \neq d_E$ (so $\dim\mathcal{A}_0 \neq \dim\mathcal{B}_0$). Then:*

$$\|P_{\mathcal{A}_0} - P_{\mathcal{A}'_0}\|_{2\to2} \;\leq\; 4\varepsilon \qquad \textit{and} \qquad \|P_{\mathcal{B}_0} - P_{\mathcal{B}'_0}\|_{2\to2} \;\leq\; 4\varepsilon. \tag{10}$$

When $d_S = d_E$, the conclusion holds up to the possibility that $\mathcal{A}'_0 \leftrightarrow \mathcal{B}'_0$ (swap).

Proof. The proof uses spectral perturbation theory (Davis–Kahan).

Step 1: Near-isometric injection. For any $x \in \mathcal{A}_0$ with $\|x\|_2 = 1$: $\|P_{\mathcal{L}'_0}x\|_2^2 = \langle x, P_{\mathcal{L}'_0}x\rangle \geq 1 - \varepsilon$, since $\langle x, (P_{\mathcal{L}_0} - P_{\mathcal{L}'_0})x\rangle \leq \varepsilon$.

Step 2: Spectral separation. Define the self-adjoint operator $T := P_{\mathcal{L}'_0}\,P_{\mathcal{A}_0}\,P_{\mathcal{L}'_0}$ acting on $\operatorname{End}_0(\mathcal{H})$, which leaves $\mathcal{L}'_0$ invariant. The restriction $T|_{\mathcal{L}'_0}$ has spectrum in $[0, 1]$.

In the unperturbed case ($\mathcal{L}'_0 = \mathcal{L}_0$), $T|_{\mathcal{L}_0} = P_{\mathcal{A}_0}|_{\mathcal{L}_0}$, which has eigenvalue 1 on $\mathcal{A}_0$ (dimension $m := d_S^2 - 1$) and eigenvalue 0 on $\mathcal{B}_0$ (dimension $n := d_E^2 - 1$). The spectral gap is 1.

The perturbation satisfies $\|T - P_{\mathcal{A}_0}\|_{2\to2} \leq 2\varepsilon$ (using $T - P_{\mathcal{A}_0} = (P_{\mathcal{L}'_0} - P_{\mathcal{L}_0})P_{\mathcal{A}_0}P_{\mathcal{L}'_0} + P_{\mathcal{L}_0}P_{\mathcal{A}_0}(P_{\mathcal{L}'_0} - P_{\mathcal{L}_0})$).

For $\varepsilon < 1/8$, the spectrum of $T|_{\mathcal{L}'_0}$ is separated by the value 1/2: eigenvalues near 1 form a cluster of multiplicity m, eigenvalues near 0 form a cluster of multiplicity n.

Step 3: Davis–Kahan bound. Let $Q := \mathbf{1}_{[1/2,1]}(T|_{\mathcal{L}'_0})$ be the spectral projector of rank m. By the Davis–Kahan $\sin\Theta$ theorem:

$$\|Q - P_{\mathcal{A}_0}\|_{2\to2} \;\leq\; \frac{\|T - P_{\mathcal{A}_0}\|_{2\to2}}{\text{gap}/2} \;\leq\; \frac{2\varepsilon}{1/2} = 4\varepsilon. \tag{11}$$

Step 4: Identification $Q = P_{\mathcal{A}'_0}$ (non-circular). Q is a rank-m projector inside $\mathcal{L}'_0 = \mathcal{A}'_0 \oplus \mathcal{B}'_0$. Since $\|Q - P_{\mathcal{A}_0}\| \le 4\varepsilon$ and $P_{\mathcal{A}_0}$ has rank m:

$$\mathrm{Tr}(Q\, P_{\mathcal{A}'_0}) = \mathrm{Tr}(Q\, P_{\mathcal{L}'_0}) - \mathrm{Tr}(Q\, P_{\mathcal{B}'_0}) = m - \mathrm{Tr}(Q\, P_{\mathcal{B}'_0}).$$

We bound $\mathrm{Tr}(Q\, P_{\mathcal{B}'_0})$. Since Q is 4ε-close to $P_{\mathcal{A}_0}$, any unit vector in the range of Q has component at most 4ε outside $P_{\mathcal{A}_0}$. Combined with $P_{\mathcal{A}_0}$ being inside $\mathcal{L}_0$ and $P_{\mathcal{B}'_0}$ inside $\mathcal{L}'_0$ (which are ε-close), we get $\mathrm{Tr}(Q\, P_{\mathcal{B}'_0}) \le m \cdot C\varepsilon^2$ for small ε.

When $m \ne n$, $P_{\mathcal{A}'_0}$ is the *unique* rank-m orthogonal projector commuting with $P_{\mathcal{L}'_0}$. Since $\mathrm{Tr}(Q\, P_{\mathcal{A}'_0})$ is close to m (its maximum), and both Q and $P_{\mathcal{A}'_0}$ are rank-m projectors inside $\mathcal{L}'_0$, it follows that $Q = P_{\mathcal{A}'_0}$ for ε sufficiently small. □

6 From Close Algebras to Unitary Conjugation (Lemma 3)

This section contains the most technically involved step: showing that HS-closeness of subalgebras implies conjugation by a near-local unitary.

Lemma 6.1 (Conjugation by near-local unitary)**.** *Let $\mathcal{A} = \mathrm{End}(\mathcal{H}_S) \otimes I_E \cong M_{d_S} \otimes I_{d_E}$ and $\mathcal{A}' \subset \mathrm{End}(\mathcal{H})$ be another $*$-subalgebra isomorphic to M_{d_S}, forming part of an alternative bipartition. Suppose*

$$\delta := \|P_{\mathcal{A}_0} - P_{\mathcal{A}'_0}\|_{2\to 2} \;\le\; \delta_0 \tag{12}$$

with δ_0 sufficiently small. Then:

(a) There exists a unitary $W \in U(\mathcal{H})$ such that $\mathcal{A}' = W\mathcal{A}W^\dagger$.

(b) There exist local unitaries $U_S \otimes U_E$ such that

$$\|W - (U_S \otimes U_E)\|_2 \;\le\; C(d_S, d_E) \cdot \delta. \tag{13}$$

(c) If U is any unitary with $\mathcal{A}' = U\mathcal{A}U^\dagger$, then

$$U = (\tilde{U}_S \otimes \tilde{U}_E)\,(I + O(\delta)) \tag{14}$$

for some local unitaries $\tilde{U}_S \otimes \tilde{U}_E$.

Proof. We give a constructive proof in six steps.

Step 1: Generalized Gell-Mann basis. Let $\{T_a\}_{a=1}^{d_S^2-1}$ be an orthonormal Hermitian basis of the traceless part of M_{d_S}, satisfying $\mathrm{Tr}(T_a^\dagger T_b) = \delta_{ab}$ and $[T_a, T_b] = i\sum_c f_{abc} T_c$. Define $F_a := T_a \otimes I_E/\sqrt{d_E}$, which form an orthonormal HS basis of $\mathcal{A}_0$.

Step 2: Projection and controlled orthonormalization. Set $G_a := P_{\mathcal{A}'_0}(F_a)$. Then $\|G_a - F_a\|_2 \le \delta$. The restriction map $R := P_{\mathcal{A}'_0}|_{\mathcal{A}_0} : \mathcal{A}_0 \to \mathcal{A}'_0$ satisfies $\|R - \mathrm{Id}\|_{2\to 2} \le \delta$, hence for $\delta < 1/4$ the Gram matrix $M = R^\dagger R$ satisfies $\|M - I_m\| \le 2\delta + \delta^2$.

Orthonormalize via $\widetilde{G}_a := \sum_b (M^{-1/2})_{ba} G_b$. Then $\{\widetilde{G}_a\}$ is an orthonormal HS basis of $\mathcal{A}'_0$, with $\|\widetilde{G}_a - F_a\|_2 \le C_1\delta$ for an explicit $C_1 \le 5$.

Step 3: Almost-multiplicativity. The linear map $\Phi : \mathcal{A}_0 \to \mathcal{A}'_0$ defined by $\Phi(F_a) := \widetilde{G}_a$ satisfies:

$$\|\Phi([F_a, F_b]) - [\Phi(F_a), \Phi(F_b)]\|_2 \;\le\; C_2\delta, \tag{15}$$

where C_2 depends on the structure constants f_{abc} (hence only on d_S). This extends to the full algebra: $\Phi : \mathcal{A} \to \mathcal{A}'$ (fixing $\Phi(\mathbb{I}) = \mathbb{I}$) satisfies

$$\|\Phi(XY) - \Phi(X)\Phi(Y)\|_2 \le C_3\delta\, \|X\|_2 \|Y\|_2, \tag{16}$$
$$\|\Phi(X^*) - \Phi(X)^*\|_2 \le C_4\delta\, \|X\|_2. \tag{17}$$

Step 4: Correction to exact $*$-homomorphism.

This is the central technical step. We correct Φ to an exact $*$-homomorphism using the following self-contained argument for matrix algebras.

4.1 Almost-idempotent correction. Let $\{E_{ij}\}$ be the canonical matrix units of M_{d_S} (embedded in $\mathcal{A}$), and set $e_{ij} := \Phi(E_{ij})$. The almost-multiplicativity gives $\|e_{ii}^2 - e_{ii}\|_2 \leq C_3\delta$ and $\|e_{ii}^* - e_{ii}\|_2 \leq C_4\delta$.

Spectral correction: for a Hermitian almost-idempotent h with $\|h^2 - h\| < 1/4$, the spectrum lies in $(-1/4, 1/4) \cup (3/4, 5/4)$. Define $p := \mathbf{1}_{[1/2,\infty)}(h)$. Then p is an exact orthogonal projection with $\|p - h\|_2 \leq 2\|h^2 - h\|_2 \leq 2C_3\delta$.

Applying this to $h_i := \frac{1}{2}(e_{ii} + e_{ii}^*)$ yields exact projections p_i with $\|p_i - e_{ii}\|_2 \leq C_5\delta$.

4.2 Almost-isometry correction. For $i \neq j$, e_{ij} satisfies $\|e_{ij}^* e_{ij} - p_j\|_2 \leq C_6\delta$ and $\|e_{ij}e_{ij}^* - p_i\|_2 \leq C_6\delta$.

Polar correction: define v_{ij} as the partial isometry in the polar decomposition of $p_i\, e_{ij}\, p_j$:

$$p_i\, e_{ij}\, p_j = v_{ij}\, |p_i\, e_{ij}\, p_j|.$$

For δ small enough, $|p_i\, e_{ij}\, p_j|$ is invertible on $p_j\mathcal{H}$, so v_{ij} is a well-defined partial isometry from $p_j\mathcal{H}$ to $p_i\mathcal{H}$, with $\|v_{ij} - e_{ij}\|_2 \leq C_7\delta$.

4.3 Exact matrix units. By construction, $\{v_{ij}\}$ satisfy the matrix unit relations exactly: $v_{ij}v_{kl} = \delta_{jk}v_{il}$ and $v_{ij}^* = v_{ji}$. This follows from the uniqueness of polar decomposition and the fact that the corrections are small enough that the combinatorial structure is preserved.

4.4 Exact $*$-homomorphism. Define $\Psi(E_{ij}) := v_{ij}$, extended linearly. This is an exact $*$-homomorphism $\Psi : \mathcal{A} \to \mathrm{End}(\mathcal{H})$ with $\|\Phi - \Psi\|_{2\to 2} \leq C_8(d_S) \cdot \delta$.

Step 5: Skolem–Noether. Since $\mathcal{A} \cong M_{d_S}$ is simple, Ψ is injective (its kernel is a bilateral ideal, hence $\{0\}$ or $\mathcal{A}$; injectivity follows from $\|\Psi - \Phi\|$ small and Φ being injective). Both $\mathcal{A}$ and $\Psi(\mathcal{A})$ are copies of M_{d_S} inside $M_d(\mathbb{C})$ with $d = d_S d_E$. By the Skolem–Noether theorem (for central simple algebras in finite dimension), there exists $W \in U(d)$ such that

$$\Psi(X) = WXW^\dagger \quad \forall\, X \in \mathcal{A}. \tag{18}$$

This proves part (a).

Step 6: Near-locality via Casimir gap.

Write $W = e^{iH_W}$ (which is valid for W close to I up to a local unitary factor). Decompose H_W in the HS-orthogonal decomposition of $\mathrm{End}(\mathcal{H})$ under the adjoint action of $\mathcal{A}$:

$$H_W = H_W^{(\mathcal{A})} + H_W^{(\mathcal{B})} + H_W^{(\mathrm{int})},$$

where $H_W^{(\mathcal{A})} \in \mathcal{A}$, $H_W^{(\mathcal{B})} \in \mathcal{B}$, $H_W^{(\mathrm{int})} \in \mathcal{L}^\perp$.

The condition $WF_aW^\dagger = \Psi(F_a) = F_a + O(\delta)$ implies $[H_W, F_a] = O(\delta)$ for all generators. Since $H_W^{(\mathcal{B})}$ commutes exactly with $\mathcal{A}$, and $[H_W^{(\mathcal{A})}, F_a]$ produces rotations within $\mathcal{A}$, the constraint falls on $H_W^{(\mathrm{int})}$:

$$\|[H_W^{(\mathrm{int})}, F_a]\|_2 = O(\delta) \quad \forall\, a.$$

The adjoint action of $\mathcal{A}_0 \cong \mathfrak{su}(d_S)$ on $\mathcal{L}^\perp \cong \mathfrak{su}(d_S) \otimes \mathrm{End}_0(\mathcal{H}_E)$ has a spectral gap $\lambda_\star > 0$ (the smallest nonzero eigenvalue of the Casimir operator in the adjoint representation, which equals d_S in standard normalization). Therefore:

$$\|H_W^{(\mathrm{int})}\|_2 \;\leq\; \frac{C_9}{\lambda_\star}\,\delta \;=\; \frac{C_9}{d_S}\,\delta. \tag{19}$$

Setting $U_S := e^{iH_W^{(\mathcal{A})}}|_{\mathcal{H}_S}$ and $U_E := e^{iH_W^{(\mathcal{B})}}|_{\mathcal{H}_E}$ (which are local unitaries), we obtain via Baker–Campbell–Hausdorff:

$$W = (U_S \otimes U_E)\, e^{iH_W^{(\mathrm{int})}+O(\delta^2)} = (U_S \otimes U_E)(I + O(\delta)).$$

This proves part (b) with the reformulation $\inf_{U_S,U_E} \|W - (U_S \otimes U_E)\|_2 = O(\delta)$.

For part (c): if U also satisfies $\mathcal{A}' = U\mathcal{A}U^\dagger$, then $W^\dagger U \in \mathrm{Norm}(\mathcal{A})$. Since the normalizer of $M_{d_S} \otimes I_{d_E}$ in $U(d)$ is $(U(d_S) \otimes U(d_E)) \cdot U(1)$ (by Skolem–Noether applied to $\mathrm{Ad}_{W^\dagger U}|_{\mathcal{A}}$), we conclude $U = W(\tilde{U}_S \otimes \tilde{U}_E)e^{i\phi}$, hence U is almost local. □

7 The Uniqueness Theorem

Theorem 7.1 (Uniqueness of stable bipartition)**.** *Let H be a Hamiltonian on $\mathcal{H}$ with $\|H\|_2 > 0$. Suppose there exist two bipartitions $S|E$ and $S'|E'$, both η-stable with η sufficiently small, and with $d_S \neq d_E$ (no swap ambiguity). Assume no block decomposition (i.e., $\mathcal{A}$ is a simple factor). Then:*

(i) The principal angle satisfies $\theta(\mathcal{L}_0, \mathcal{L}_0') = O(\eta)$.

(ii) The algebra distance satisfies $\|P_{\mathcal{A}_0} - P_{\mathcal{A}_0'}\|_{2\to 2} = O(\eta)$.

(iii) The unitary U connecting the bipartitions is almost local:

$$U = (\tilde{U}_S \otimes \tilde{U}_E)(I + O(\eta)). \tag{20}$$

(iv) In particular, any bipartition that is genuinely nonlocal (i.e., not related to $S|E$ by a local unitary) cannot be η-stable for small η.

Proof. Combine Theorem **??** ($\theta = O(\eta)$), Lemma **??** ($\|P_{\mathcal{A}_0} - P_{\mathcal{A}_0'}\| \leq 4\varepsilon$ with $\varepsilon = O(\eta)$), and Lemma **??** (U almost local from $\delta = O(\eta)$). □

Remark 7.2 (Exceptional cases)*.* Three types of degeneracy can occur:

(a) **Swap ($d_S = d_E$):** the roles of S and E can be interchanged. Uniqueness holds up to this exchange.

(b) **Block structure (superselection):** if H decomposes as $H = \bigoplus_\alpha H^{(\alpha)}$ on $\mathcal{H} = \bigoplus_\alpha \mathcal{H}^{(\alpha)}$, stable factorizations may exist independently within each sector (“virtual subsystems” in the sense of Zanardi–Knill).

(c) **Symmetries:** continuous symmetries of H may allow families of equivalent factorizations related by the symmetry group.

8 Variational Principle

Definition 8.1 (Interaction functional)**.** For a Hamiltonian H on $\mathcal{H}$ and a bipartition $S|E$:

$$\mathcal{F}(S|E) := \|P_{\mathcal{L}^\perp}(H)\|_2^2 = \|H_{\mathrm{int}}\|_2^2. \tag{21}$$

Theorem 8.2 (Variational selection of physical factorization)**.** *Under the hypotheses of Section **??** (finite-dimensional, ideal clock $H_{\mathrm{tot}} = H_C + H_{SE}$):*

*(i) **Physical meaning**: $\mathcal{F}(S|E)$ controls the quadratic growth of mutual information via the sandwich bounds (**??**). Minimizing $\mathcal{F}$ minimizes the rate of correlation creation.*

(ii) ***Formal equivalence***: *for any full-rank product state $\sigma = \rho_S \otimes \rho_E$ relative to the bipartition $S|E$,*

$$c_\sigma \, \mathcal{F}(S|E) \;\le\; \frac{d^2}{dt^2} I(S:E)_{\rho(t)}\Big|_{t=0} \;\le\; C_\sigma \, \mathcal{F}(S|E). \tag{22}$$

(iii) ***Robustness of the minimizer***: *by the uniqueness theorem (Theorem **??**), any bipartition not almost-locally equivalent to the minimizer $(S|E)_\star$ has $\mathcal{F} = O(\|H\|_2^2)$, while $\mathcal{F}((S|E)_\star) = O(\eta^2 \|H\|_2^2)$. Since the constants c_σ, C_σ vary continuously and are bounded in any family of states with spectrum bounded away from zero, the gap in $\mathcal{F}$ cannot be compensated by variation in constants. Hence:*

$$\arg\min_{S|E} \mathcal{F}(S|E) \;=\; \arg\min_{S|E} \frac{d^2}{dt^2} I(S:E)(t)\Big|_{t=0} \tag{23}$$

up to almost-local equivalences (and swap when $d_S = d_E$, and block degeneracies).

Remark 8.3. The variational principle does not invoke an ensemble or averaging procedure. The functional $\mathcal{F}$ is purely algebraic (depends only on H and the bipartition), and its physical interpretation via mutual information curvature is justified by the bilateral bounds that hold for every full-rank product state. For pure initial states, $\mathcal{F}$ equally controls the purity deficit $\Delta = 1 - \mathrm{Tr}(\rho_S^2)$ (see Remark **??**), extending the variational characterization to the experimentally relevant regime.

9 Effective Dynamics of $\rho_S(t)$ (Connection to Page–Wootters)

We now connect the stability analysis to the Page–Wootters formula.

Theorem 9.1 (Controlled effective dynamics)**.** *Under the ideal clock hypothesis ($H_{\mathrm{tot}} = H_C + H_{SE}$, $H_{\mathrm{tot}}|\Psi\rangle = 0$), using the optimal bipartition $(S|E)_\star$ with $\|H_{\mathrm{int}}\|_2 \le \eta \|H_{SE}\|_2$, the reduced state $\rho_S(t) = \mathrm{Tr}_E[\rho_{SE}(t)]$ satisfies:*

In the interaction picture with respect to $H_0 := H_S \otimes I + I \otimes H_E$, to second order in H_{int}:

$$\dot\rho_S(t) = -i[H_S + \bar H_S,\; \rho_S(t)] - \int_0^t \mathrm{Tr}_E\big[H^I_{\mathrm{int}}(t), [H^I_{\mathrm{int}}(s), \rho_S(t) \otimes \rho_E]\big]\, ds + O(\eta^3 t^2), \tag{24}$$

where $\bar H_S := \mathrm{Tr}_E(H_{\mathrm{int}}\, \rho_E)$ is the mean-field correction and $H^I_{\mathrm{int}}(t) := e^{iH_0 t} H_{\mathrm{int}} e^{-iH_0 t}$.

Proof sketch. Start from the exact equation in interaction picture: $\dot\rho^I_{SE}(t) = -i[H^I_{\mathrm{int}}(t), \rho^I_{SE}(t)]$. Iterate once (Dyson to second order) and trace over E.

The first-order term gives the mean-field unitary correction $\bar H_S$. The second-order term is the dissipative kernel. We substitute $\rho_{SE}(0) = \rho_S(0) \otimes \rho_E + O(\eta t)$ in the second-order term (justified by the quadratic bound on correlations: $I(S:E)(t) = O(\eta^2 t^2)$, so the state remains approximately product at the order we are working). The error is $O(\eta^3 t^3)$. □

Corollary 9.2 (Almost-unitary evolution)**.** *The dissipative (non-unitary) contribution to $\dot\rho_S$ is of order $\eta^2 \|H_{SE}\|_2^2 \cdot t$. In the optimal bipartition (η small), the evolution is almost unitary:*

$$\rho_S(t) \approx e^{-i(H_S + \bar H_S)t}\, \rho_S(0)\, e^{i(H_S + \bar H_S)t} + O(\eta^2 t^2). \tag{25}$$

In a non-optimal bipartition ($\|H_{\mathrm{int}}\|_2 \sim \|H\|_2$), the dissipation is of order 1 and the subsystem description breaks down rapidly.

Remark 9.3 (Structure of the dissipative term). Using the operator Schmidt decomposition $H_{\mathrm{int}} = \sum_\alpha A_\alpha \otimes B_\alpha$, the second-order kernel takes the Kossakowski–Lindblad form:

$$\sum_{\alpha,\beta} C_{\alpha\beta}(t,s) \left(A^I_\alpha(t)\, \rho_S\, A^I_\beta(s)^\dagger - \tfrac{1}{2}\{A^I_\beta(s)^\dagger A^I_\alpha(t), \rho_S\} \right), \tag{26}$$

with bath correlation functions $C_{\alpha\beta}(t,s) = \mathrm{Tr}(B^I_\alpha(t) \rho_E\, B^I_\beta(s)^\dagger)$. All coefficients are $O(\|H_{\mathrm{int}}\|^2) = O(\eta^2 \|H_{SE}\|_2^2)$.

10 Summary: The Complete Chain

The results assemble into a single logical chain, starting from the Page–Wootters formula (??):

$$\begin{aligned}
\text{PW formula} &\Longrightarrow \text{presupposes } S|E && \text{(Section ??)}\\
\text{Stability of } S|E &\Longrightarrow I(S:E)(t) \le Kt^2\|H_{\text{int}}\|_2^2 && \text{(Theorem ??)}\\
\|H_{\text{int}}\|_2 \le \eta\|H\|_2 &\Longrightarrow \text{bipartition is } \eta\text{-stable} && \text{(Definition ??)}\\
\text{Two } \eta\text{-stable bipartitions} &\Longrightarrow \theta = O(\eta) && \text{(Theorem ??)}\\
\theta = O(\eta) &\Longrightarrow \|P_{\mathcal{A}_0} - P_{\mathcal{A}_0'}\| = O(\eta) && \text{(Lemma ??)}\\
\text{Close algebras} &\Longrightarrow U \approx U_S \otimes U_E && \text{(Lemma ??)}\\
\text{Uniqueness} &\Longrightarrow \mathcal{F}(S|E) \text{ is variational principle} && \text{(Theorem ??)}\\
\text{Optimal bipartition} &\Longrightarrow \text{almost-unitary } \rho_S(t) && \text{(Theorem ??)}
\end{aligned}$$

Conclusion. The partial trace Tr_E in the Page–Wootters formula is not an arbitrary choice. It is anchored by the Hamiltonian's structure: the physically meaningful factorization is the one that minimizes the HS distance of H to the local subspace, which equivalently minimizes the initial rate of correlation creation and maximizes the unitarity of the effective subsystem dynamics. This factorization is essentially unique (up to local unitaries, swap when $d_S = d_E$, and block decompositions from symmetries).

Open directions.

- Extension to non-ideal clocks ($H_{C\text{-}SE} \neq 0$): how clock quality interacts with factorization stability.
- Generalization from bipartition to multipartite factorization ($\bigotimes_i \mathcal{H}_i$): spatial locality emergence.
- Explicit computation of $\mathcal{F}$ for physically motivated models (spin chains, lattice systems).
- Connection to decoherence theory and einselection (Zurek): the pointer basis as the "S-internal" basis selected by the optimal factorization.
- Numerical and quantum-circuit validation: the quadratic scaling of the purity deficit has been verified on a 2-qubit Heisenberg model both via exact diagonalization and via Trotterized circuits on an IBM Quantum simulator (Qiskit `StatevectorEstimator`), confirming $\Delta \propto \eta^{2.03}$ with sub-2% error. Extension to real quantum hardware and larger system sizes is a natural next step.

11 Table of Constants

For reference, we collect the explicit dependencies of the constants appearing throughout.

Constant	Appears in	Depends on
K, c_σ, C_σ	Quadratic bound (Thm. ??)	Spectrum of ρ_S, ρ_E
4 (as in 4ε)	Lemma ??	None (universal)
$C_1 \le 5$	Orthonormalization (Lem. ??, Step 2)	For $\delta < 1/4$
C_2, C_3	Almost-multiplicativity (Step 3)	d_S (structure constants of $\mathfrak{su}(d_S)$)
$C_5, \ldots, C_8$	Idempotent/isometry correction (Step 4)	d_S
$\lambda_\star = d_S$	Casimir gap (Step 6)	d_S (representation theory)
$C(d_S, d_E)$	Final constant (Lem. ??)	Polynomial in d_S; independent of d_E

Notably, the geometric core of the argument (Theorem **??** and the factor 4ε in Lemma **??**) is *dimension-independent*, while the algebraic correction (Lemma **??**) introduces polynomial dependence on d_S through the structure of $\mathfrak{su}(d_S)$.