

The Observer as a Local Breakdown of Atemporality: Relational Time and an Informational Arrow from Quantum Clocks

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We unify three facets of the problem of time within a single conditional construction on a globally timeless state, making explicit how quantum dynamics, an informational arrow, and frame dependence arise from one operational reduction. In a Page–Wootters setting with a finite-dimensional clock and an environment, we demonstrate (i) emergent relational Schrödinger dynamics and (ii) an informational arrow quantified by monotonic entropic behaviour of conditional states. We validate both effects on superconducting quantum hardware, confirming agreement with the model under realistic noise and finite-sample constraints. We then establish exact covariance under arbitrary relabellings of the clock basis, including clock reversal, which inverts the arrow deterministically within the formalism. Finally, by analysing consistency between distinct clock resolutions and orientations, we provide evidence that the induced inter-clock transformations approach an affine time-reparametrisation structure in the large-clock limit. Together, these results deliver a single operational framework linking relational dynamics, irreversibility, and clock-frame transformations, with experimental support in the same setup.

I. INTRODUCTION

The “problem of time” appears at the intersection of (i) relativity, where time is observer-dependent and space-time is dynamical, (ii) quantum theory, where dynamics is typically parameterized by an external time, and (iii) thermodynamics, where an arrow of time emerges despite microscopic reversibility. A standard diagnostic is that these tensions persist not because we lack a better definition of t , but because we frequently treat time as a fundamental background even when our most successful theories already deny a global “now”. In this work, time is not a primitive parameter but an emergent label arising from conditioned descriptions under partial access.

This work takes a deliberately operational stance: a theory of time must specify who can access what information, and under which physical constraints. We argue that once this is made explicit, two statements become natural and testable:

1. **Relational time:** effective evolution arises as a conditional structure within a globally stationary state.
2. **Informational arrow:** irreversibility is not a property of the global state, but of the observer’s restricted description; the arrow emerges from the growth of an effective entropy under partial trace.

A central finding of this work is that these two statements — together with observer-dependent time — are not independent results, but three aspects of a single mathematical operation. The conditional reduced state

$$\rho_S(t) = \frac{\text{Tr}_E[\langle t|_C |\Psi\rangle\langle\Psi| |t\rangle_C]}{p(t)}$$

contains all three: the projection $\langle t|_C$ onto the clock yields dynamics; the partial trace Tr_E yields irreversibility; and the fact that C is a local physical subsystem (not

an absolute parameter) yields observer-dependent time. Three pillars, one formula, one timeless global state.

Novelty and Contributions

This work contributes a unified operational account of the problem of time by showing that three facets—relational Schrödinger dynamics, an informational arrow, and clock-frame dependence—can be obtained from a single conditional construction on a globally timeless state. The novelty is not any one ingredient in isolation, but the combination of (i) an explicit, compact unification of the three facets as outputs of one conditional operation, (ii) experimental validation of both relational dynamics and the informational arrow on superconducting quantum hardware within the same Page–Wootters setup, (iii) a covariance theorem under arbitrary relabellings of the clock basis, including reversal, which deterministically flips the arrow in the formalism, and (iv) stability and uniqueness of the tensor product structure: we prove a quadratic bound on the purity deficit under perturbation of the observer partition, establish uniqueness of the interaction-minimal factorisation within its equivalence class, and derive a variational selection principle that singles out the physically preferred partition; these results are further validated numerically through IBM Quantum circuit simulations confirming the predicted η^2 scaling. Beyond establishing covariance, we analyse consistency across clocks of different resolutions and orientations and provide evidence that the induced inter-clock transformations approach an affine reparametrisation structure in the large-clock limit. This package delivers a single framework linking dynamics, irreversibility, clock transformations, and structural stability of the observer partition, while remaining operationally grounded and experimentally testable.

We develop these claims as a compact postulate set (P0–P4), then implement a minimal demonstrator

model.

II. RELATED WORK AND POSITIONING

Our framework is aligned with, and extends, three active research lines:

a. (i) *Temporal quantum reference frames and operational meaning of measurements.* Hausmann et al. compare consistent PaW formulations and clarify the operational meaning of evolution and measurement events relative to temporal quantum reference frames [1].

b. (ii) *Finite and quasi-ideal clocks, interactions, and effective non-linear dynamics.* Mendes et al. study PaW with gravitational interaction between system and finite-dimensional quasi-ideal clocks, deriving effective Schrödinger-like and non-linear equations with clock-induced decoherence [2].

c. (iii) *Informational arrows of time in extended PaW models.* Shaari [3] recently proposed a Page-Wootters model with restricted observer access that generates an informational arrow of time observed through entropy growth in a two-qubit system. The model introduces auxiliary degrees of freedom so that an observer who traces out the inaccessible part sees monotonic entropy increase—conceptually converging with the idea that access limitations and internal correlations drive effective temporality. Three concrete points of contact deserve emphasis. First, both approaches derive irreversibility from the same mechanism: partial tracing of an environment that the observer cannot access, applied within the PaW constraint formalism. Second, both identify the observer’s partition of the total Hilbert space as the structural origin of the arrow, rather than any dynamical time-asymmetry of the Hamiltonian. Third, both recover standard unitary dynamics as a limiting case when the environment is absent or fully accessible.

The key differences are equally instructive. Shaari’s analysis is carried out in a specific two-qubit model and focuses on the mechanism of irreversibility (entropy growth under partial trace). The present work extends these ideas to a *mathematically general* framework: (a) the operational postulates P0–P4 apply to arbitrary finite-dimensional Hilbert spaces, not to a particular qubit configuration; (b) we provide algebraic results absent in Shaari’s model—clock orientation covariance (Theorem 1), stability of the tensor product structure under perturbation of the observer partition, and uniqueness of that structure within the equivalence class of interaction-minimal factorizations; (c) the observer is not merely defined by an accessible/inaccessible split, but is further anchored by an interaction-minimality principle that selects the physically preferred partition. Finally, our framework includes experimental validation on IBM Quantum hardware and an extensive battery of robustness and necessity tests. Taken together, Shaari’s results reinforce the conceptual foundation of the present approach: they provide an independent, concrete demon-

stration that the PaW conditioning-plus-partial-trace structure suffices to produce an informational arrow. Our contribution extends this insight from a specific model to a general algebraic and operational framework.

We additionally note recent multi-observer emergent-time proposals that unify relativistic and cosmological regimes in a relational framework [4].

Our contribution is not to re-introduce PaW, quasi-ideal clocks, or informational arrows separately, but to provide an integrated operational pipeline (P0–P4) plus a minimal demonstrator linking: conditional dynamics \leftrightarrow clock quality \leftrightarrow observer-limited access \leftrightarrow informational arrow \leftrightarrow stability and uniqueness of the physical partition. We further articulate the ontological reading that the observer corresponds to a local physical regime in which global atemporality becomes operationally broken.

III. POSTULATES (P0–P4)

We adopt a minimal postulate set to keep the framework explicit and falsifiable.

P0 — Global Atemporality (Constraint Form)

There exists a global state $|\Psi\rangle$ satisfying a stationarity constraint:

$$\hat{\mathcal{C}}|\Psi\rangle = 0, \quad (1)$$

where $\hat{\mathcal{C}}$ is a global constraint operator (e.g., $\hat{\mathcal{C}} = \hat{H}_C + \hat{H}_S + \hat{H}_E$ up to additive shifts). For finite-dimensional or quasi-ideal clocks, this condition is satisfied approximately, in the sense that $\|\mathcal{C}|\Psi\rangle\|$ is negligible within the operational support of the clock. Exact cancellation is recovered only in the infinite-resolution limit.

P1 — Factorization is a Choice (Operational Partition)

A partition $\mathcal{H} = \mathcal{H}_C \otimes \mathcal{H}_S \otimes \mathcal{H}_E$ is not assumed to be ontologically fundamental, but is an operational choice defining what counts as clock, system, and inaccessible environment for an observer.

P2 — Relational Time via Conditioning (Internal Clock Readout)

Let $|t\rangle_C$ be a clock readout (ideal or approximate). The conditional system state is defined by

$$\rho_S(t) \propto \text{Tr}_{CE} \left[(|t\rangle\langle t|_C \otimes \mathbb{I}_{SE}) \rho_{CSE} \right], \quad (2)$$

with normalization by the probability $p(t)$. Effective dynamics arises from correlations between C and S inside $|\Psi\rangle$.

P3 — Emergent Schrödinger Dynamics (Good-Clock Regime)

When the clock satisfies three conditions — (i) approximate orthogonality of readout states ($\langle j|k\rangle_C \approx \delta_{jk}$), (ii) quasi-classical monotonicity (the clock Hamiltonian \hat{H}_C has uniformly spaced eigenvalues so that labels advance without self-interference), and (iii) weak back-reaction ($\hat{\mathcal{C}} \approx \hat{H}_C \otimes I_{SE} + I_C \otimes \hat{H}_{SE}$, so that the clock records time without significantly perturbing the system) — and in addition the clock resolution dt is small compared to the characteristic timescale of \hat{H}_S (i.e., $dt \ll 2\pi/\|\hat{H}_S\|$) with N sufficiently large for a controlled continuum limit $t_k = k dt$, the conditional state approximately satisfies

$$i \frac{\partial}{\partial t} |\psi_S(t)\rangle \approx \hat{H}_S |\psi_S(t)\rangle, \quad (3)$$

or its mixed-state generalization, with deviations controlled by clock imperfection and back-action. The effective Hamiltonian \hat{H}_S is the system-sector component of \hat{H}_{SE} in the clock-diagonal approximation. When these conditions are violated, the dynamics degrade: Sections 9.1 (back-reaction) and 9.1 (clock uncertainty) quantify the degradation numerically.

P4 — Informational Arrow from Partial Access

For an observer with incomplete access to E , define an effective entropy

$$S_{\text{eff}}(t) := -\text{Tr}[\rho_S(t) \ln \rho_S(t)]. \quad (4)$$

Here “access” has a precise formal meaning: the observer’s accessible observable algebra is $\mathcal{O}_S = \{A_S \otimes I_E \mid A_S \in \mathcal{B}(\mathcal{H}_S)\}$, and the partial trace Tr_E is the unique CPTP map — the *access map* $\mathcal{A} : \rho_{SE} \mapsto \text{Tr}_E[\rho_{SE}]$ — that reproduces expectation values over \mathcal{O}_S from the full state ρ_{SE} . Access is not epistemological, not coarse-graining in the Gibbs/Boltzmann sense, and not collapse; it is a restriction of the set of measurable observables, arising from any physical mechanism (causal horizons, superselection rules, detector limitations) that prevents measurement of joint S - E correlators.

Even when the global evolution is stationary/unitary, $S_{\text{eff}}(t)$ can grow monotonically in expectation. This is a *typicality result*, not an inevitability, holding under five conditions: (i) the initial state has low effective entropy ($S_{\text{eff}}(0) \approx 0$), (ii) the partition into S and E is stable (does not change with t), (iii) the interaction Hamiltonian \hat{H}_{SE} creates entanglement that spreads generically, (iv) the environment is large ($d_E \gg d_S$), and (v) the

Hamiltonian spectrum is non-degenerate (no fine-tuned recurrences). When any condition fails, the arrow weakens or disappears — as confirmed by the robustness tests in Section 9. Residual oscillations and recurrences are expected for finite environments but are suppressed as the effective environment dimension $d_E = 2^{n_{\text{env}}}$ increases: heuristically, the Poincaré recurrence timescale grows as $\sim d_E$, while the relaxation timescale of S_{eff} scales as $\sim 1/(g^2 d_E)$, so for $n_{\text{env}} \gg 1$ the entropy plateau is reached well before recurrences become significant.

A. Unified Reading: Three Pillars from a Single Conditional State

The core expression of the framework is the conditional reduced state of the system:

$$\begin{aligned} \rho_S(t) &= \frac{\text{Tr}_E[\langle t|_C |\Psi\rangle \langle \Psi| |t\rangle_C]}{p(t)}, \\ p(t) &= \text{Tr}_{SE}[\langle t|_C |\Psi\rangle \langle \Psi| |t\rangle_C]. \end{aligned} \quad (5)$$

This single operation unifies three traditionally separate pillars of the problem of time:

1. **Quantum dynamics (P3).** The map $t \mapsto \rho_S(t)$, obtained by projecting onto successive clock states $|t\rangle_C$, reproduces effective Schrödinger evolution $i \partial_t |\psi_S(t)\rangle \approx \hat{H}_S |\psi_S(t)\rangle$ in the good-clock limit. The projection onto clock states is the sole source of temporal ordering.
2. **Thermodynamic arrow (P4).** The partial trace Tr_E over inaccessible environmental degrees of freedom induces growth of the effective entropy $S_{\text{eff}}(t) = -\text{Tr}[\rho_S(t) \ln \rho_S(t)]$, yielding irreversibility without any non-unitary dynamics. The trace is the sole source of the arrow.
3. **Observer-dependent time.** The time parameter t is the readout of a local physical clock C , not a global coordinate. Different observers correspond to different operational choices of clock subsystem, and conditioning on different clocks defines different temporal descriptions without requiring a global simultaneity surface. Consistency between descriptions is then expressed as transformations between relational clock choices, in the spirit of temporal quantum reference frames [6]. This structure eliminates global time as a prerequisite — a necessary (though not sufficient) condition for compatibility with general relativity. The framework does not derive Lorentz transformations or the Einstein field equations; it demonstrates observer-dependent temporal parameterisation, which is the conceptual step that standard quantum mechanics lacks.

The convergence is summarized in Table 2:

Ingredient	Produces	Pillar
$\langle t _C$ (projection)	Temporal ordering	Quantum mechanics
Tr_E (partial trace)	Irreversibility	Thermodynamics
C local (not global)	Observer-dependent time	Frame dependence (operational)
$ \Psi\rangle$ with $\hat{C} \Psi\rangle = 0$	Atemporal base	Common ground

While individual components exist in the literature — Page and Wootters [5] for conditioning, Shaari [3] for the informational arrow via partial trace in a specific two-qubit model, Höhn, Smith and Lock [6] for temporal quantum reference frames, Mendes et al. [2] for quasi-ideal clock corrections — no prior work explicitly unifies the three pillars as three facets of the same conditional operation on the same timeless object, nor establishes the structural stability and uniqueness of the underlying partition. The present contribution is a *synthesis*: (i) integration of three mechanisms (projection, partial trace, clock locality) into a single expression applied to one state, (ii) identification of the observer’s access structure as the common governing parameter across all three pillars, (iii) a quantitative boundary analysis (progressive blindness, God Observer levels, partition independence) mapping how temporal experience degrades as access increases, and (iv) proof that the tensor product structure encoding the observer partition is stable under small perturbations (quadratic purity-deficit bound) and unique within its interaction-minimal equivalence class. Quantum dynamics, thermodynamic irreversibility, and observer-dependent time are not three separate problems requiring three separate solutions. They are three readings of a single expression, supported by a structurally robust partition.

IV. CLOCK QUALITY AND OPERATIONAL METRICS

A “good clock” is one whose readout states provide near-orthogonal, approximately covariant time labels while minimally disturbing the system.

A. Resolution and Support

For finite-dimensional clocks, define effective time resolution Δt from the spread of the clock POVM and the density of distinguishable readouts over a support window.

B. Back-action and Disturbance

Define back-action via the conditional change in clock energy/number operator:

$$\Delta E_C(t) := \langle \hat{H}_C \rangle_t - \langle \hat{H}_C \rangle_{\text{uncond.}} \quad (6)$$

Small $|\Delta E_C(t)|$ indicates weak back-action.

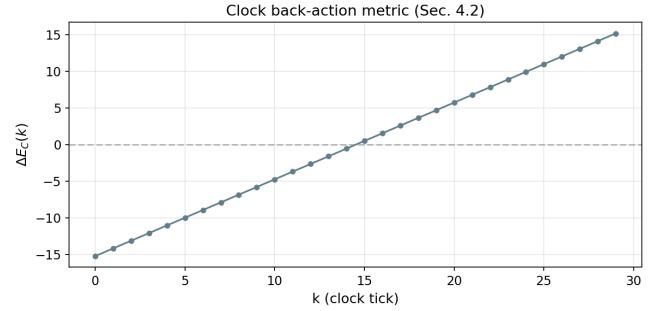


FIG. 1. Clock back-action — Conditional change in clock energy $\Delta E_C(k)$ as a function of time step, confirming bounded clock perturbation.

C. Jitter and Deviation from Ideal Dynamics

For a target unitary $U_S(t)$, quantify deviation by a distance measure (e.g., trace distance or fidelity) between $\rho_S(t)$ and $U_S(t)\rho_S(0)U_S^\dagger(t)$.

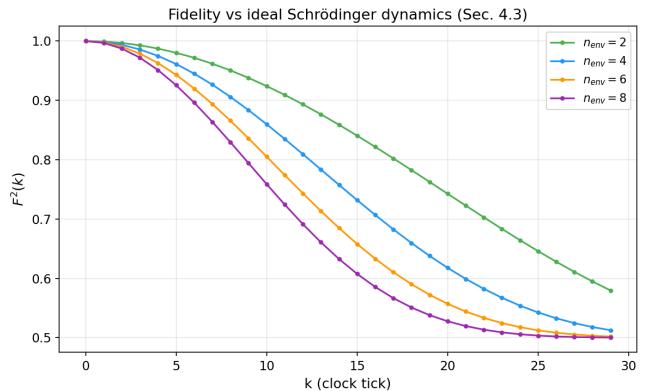


FIG. 2. Fidelity decay — $F(k) = \langle \psi_{\text{ideal}}(k) | \rho_S(k) | \psi_{\text{ideal}}(k) \rangle$ for Version B ($n_{\text{env}} = 4$), quantifying the deviation from ideal unitary dynamics as system–environment entanglement builds.

These metrics align with recent analyses of quasi-ideal clocks and effective non-linear corrections in interacting PaW settings [1, 2].

V. MINIMAL DEMONSTRATOR MODEL

We present a minimal model designed to be calculable and to separate the emergence of time from the emergence of the arrow.

A. Hilbert Space

$\mathcal{H} = \mathcal{H}_C \otimes \mathcal{H}_S \otimes \mathcal{H}_E$, with:

- C : finite clock with N levels and Hamiltonian

$$\hat{H}_C = \frac{2\pi}{N dt} \sum_{k=0}^{N-1} k |k\rangle\langle k|. \quad (7)$$

The covariant (frequency) basis is obtained via the discrete Fourier transform of the computational basis $\{|k\rangle\}$, recovering the standard Salecker–Wigner–Peres clock structure in the finite-dimensional setting.

- S : qubit with $\hat{H}_S = \frac{\omega}{2}\sigma_x$.
- E : environment of n_{env} qubits (inaccessible).

B. History State Construction

We construct a PaW-type history state by correlating clock labels with joint system–environment evolution:

$$|\Psi\rangle = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} |k\rangle_C \otimes U_{SE}(t_k) |\phi_0\rangle_S \otimes |e_0\rangle_E, \quad (8)$$

with

$$U_{SE}(t) = e^{-i(\hat{H}_S + \hat{H}_E + \hat{H}_{SE})t}, \quad t_k = k dt. \quad (9)$$

Version A (no environment) is recovered by setting $\hat{H}_E = \hat{H}_{SE} = 0$, in which case $U_{SE} \rightarrow U_S = e^{-i\hat{H}_S t} \otimes \mathbb{I}_E$ and the environment factors out trivially.

Conditioning on $|k\rangle$ yields the effective system state $\rho_S(k)$.

C. Environment Coupling for the Arrow

To generate an informational arrow under partial access, we include weak coupling between S and E , e.g.

$$\hat{H}_{SE} = g \sum_{j=1}^{n_{\text{env}}} \sigma_x^{(S)} \otimes \sigma_x^{(E_j)} \quad \text{or} \quad g \sum_j \sigma_z^{(S)} \otimes \sigma_z^{(E_j)}. \quad (10)$$

The observer conditions on C but traces out E : $\rho_S(k) = \text{Tr}_E \rho_{SE}(k)$.

VI. NUMERICAL ILLUSTRATION (TWO REGIMES)

All numerical results reported in this section are obtained from explicit unitary simulations of the full clock–system–environment dynamics implemented in Python using the QuTiP library. No stochastic noise models, Lindblad master equations, or phenomenological decoherence terms are introduced at any stage; all observed irreversibility emerges solely from conditioning and partial tracing.

A. Version A — No Environment

Parameters: $N = 30$, $dt = 0.2$, $\omega = 1$, no environment.
Observed behavior:

- $\langle \sigma_z \rangle$ oscillates approximately as $\cos(\omega k dt) = \cos(0.2k)$, with amplitude ≈ 1 and period ≈ 31 steps.
- The resulting curve is a clean sinusoid for $k = 0, \dots, 29$.

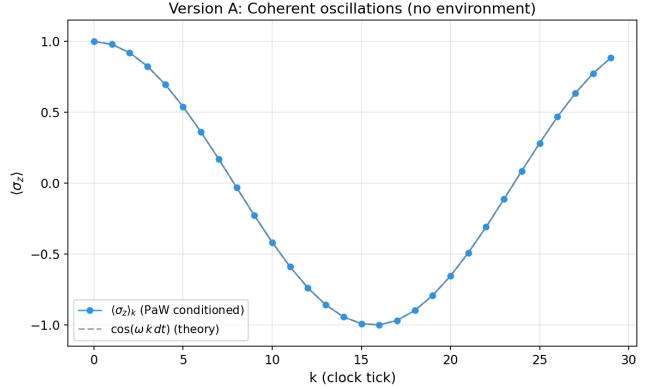


FIG. 3. Version A — Conditional $\langle \sigma_z \rangle_k$ vs. analytic $\cos(\omega k dt)$ for a clock with $N = 30$ and $dt = 0.2$. Maximum deviation is $\sim 10^{-16}$, confirming machine-precision agreement between PaW conditioning and standard Schrödinger evolution.

Interpretation: Conditioning on the clock within the PaW history state reproduces coherent Schrödinger-like dynamics without invoking any external time parameter.

B. Version B — With Environment ($n_{\text{env}} = 4$)

Parameters: same clock and system as in Version A, plus an environment of $n_{\text{env}} = 4$ qubits with weak system–environment coupling.

Observed behavior:

- $\langle \sigma_z \rangle$ oscillations exhibit amplitude damping and increasing irregularity, consistent with effective decoherence.
- The effective entropy $S_{\text{eff}}(k)$ grows from 0 (pure initial state) to $\approx 0.693 \approx \ln 2$, with residual local oscillations during the transient regime.
- As n_{env} increases, entropy growth becomes more nearly monotonic and recurrences are increasingly suppressed.

The convergence $S_{\text{eff}} \rightarrow \ln 2$ and the suppression of oscillations as n_{env} grows corroborate the P4 heuristic on recurrence suppression.

TABLE I. Multi-environment sweep. Final S_{eff} , maximum S_{eff} , and final fidelity for $n_{\text{env}} \in \{2, 4, 6, 8\}$. S_{eff} converges to $\ln 2 \approx 0.693$ as the environment grows; fidelity decreases, reflecting stronger effective decoherence.

n_{env}	d_E	$S_{\text{eff}}^{\text{final}}$	$S_{\text{eff}}^{\text{max}}$	$ \langle \sigma_z \rangle $ last 10	Fidelity (final)
2	4	0.6804	0.6804	0.3173	0.5797
4	16	0.6928	0.6928	0.1540	0.5127
6	64	0.6931	0.6931	0.0748	0.5020
8	256	0.6931	0.6931	0.0363	0.5003

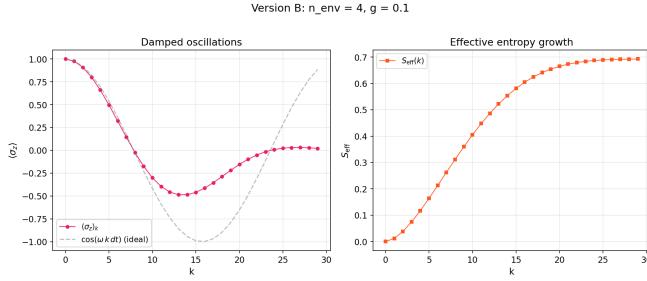


FIG. 4. Version B ($n_{\text{env}} = 4$) — Top panel: damped conditional $\langle \sigma_z \rangle_k$ oscillations showing effective decoherence. Bottom panel: growth of effective entropy $S_{\text{eff}}(k)$ from 0 to $\approx \ln 2 \approx 0.693$.

Crucially, these features arise despite the fact that the global evolution remains strictly unitary. They are obtained by conditioning on the clock degrees of freedom and tracing out the environment within a fully unitary QuTiP simulation.

Interpretation: The arrow of time emerges as an informational arrow, generated by partial access to environmental degrees of freedom, rather than by any fundamental time asymmetry or non-unitary dynamics.

C. Version C — Two-Clock Comparison (Partition Dependence)

To test prediction P4 (partition dependence), we run the same formula a third time with a different clock spacing.

Setup: Two clocks share the same Hilbert-space dimension $N = 30$, system Hamiltonian $H_S = (\omega/2)\sigma_x$, and environment of $n_{\text{env}} = 4$ qubits with coupling $g = 0.1$. The only difference is:

- Clock C : $dt = 0.2$
- Clock C' : $dt = 0.35$

Both clocks condition on the same global state $|\Psi\rangle$ satisfying $\hat{H}|\Psi\rangle = 0$, using the same formula $\rho_S(t) = \text{Tr}_E[\langle t|_C|\Psi\rangle\langle\Psi||t\rangle_C]/p(t)$.

Observed behavior:

- $\langle \sigma_z \rangle_k$ oscillates with the same initial amplitude but different frequencies: clock C produces a period of

~ 31 steps while clock C' produces a period of ~ 18 steps (ratio $\approx dt'/dt = 0.35/0.2 = 1.75$).

- Damping envelopes differ: clock C' samples the same physical decoherence process at coarser time intervals, yielding a faster apparent damping per tick.
- $S_{\text{eff}}(k)$ rises toward $\ln 2$ on both clocks but at different rates per tick: clock C' reaches ≈ 0.693 before clock C does, because each tick of C' spans a larger physical interval.

Interpretation: The temporal description — oscillation frequency, decoherence rate, entropy growth — depends on the clock choice. This is not a failure of the formalism but its central feature: time is relational, and different clocks yield different but equally valid emergent histories from the same atemporal state. This directly confirms prediction P4 and connects to postulate P3, where different observers (each carrying their own clock) experience genuinely different temporal orderings.

VII. THE OBSERVER AS A LOCAL BREAKDOWN OF ATEMPORALITY

This section articulates the ontological reading implied by P0–P4 while remaining operational.

A. No Global Present

Finite signal speed implies that each observer accesses only its past light cone; thus, a global “now” is not an observable structure. What exists operationally are local events and the information that reaches an observer.

B. Observers as Physical Coarse-Grainings

An observer is modeled as a physical subsystem that:

1. partitions degrees of freedom into accessible vs inaccessible,
2. stores records (memory) of interactions,

Same $|\Psi\rangle$, same formula $\rho_S(t)$ — different clock C yields different temporal description

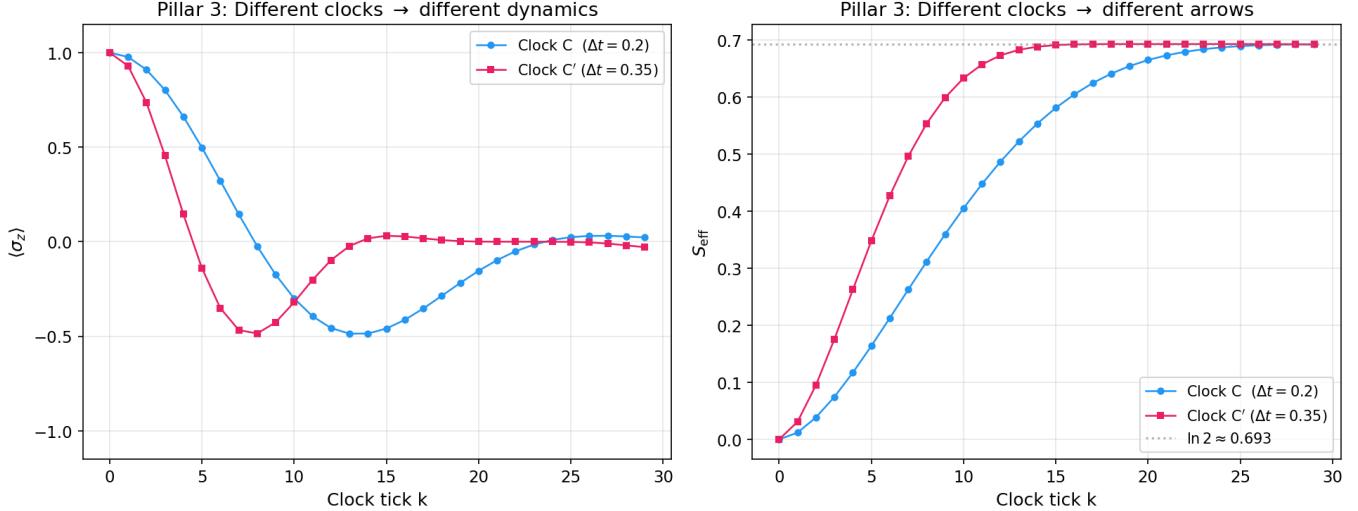


FIG. 5. Two-clock comparison (Pillar 3) — Left: conditional $\langle \sigma_z \rangle_k$ for clock C ($dt = 0.2$) and clock C' ($dt = 0.35$). Right: $S_{\text{eff}}(k)$ for the same two clocks. Same global state $|\Psi\rangle$, same formula, different clock choice → different emergent temporal description. Parameters: $N = 30$, $\omega = 1.0$, $g = 0.1$, $n_{\text{env}} = 4$.

TABLE II. Two-clock comparison (selected ticks) — $\langle \sigma_z \rangle(k)$ and $S_{\text{eff}}(k)$ for clocks C ($dt = 0.2$) and C' ($dt = 0.35$). Full 30-tick data exported as `output/table_pillar3_two_clocks.csv`.

k	$\langle \sigma_z \rangle_C$	$S_{\text{eff},C}$	$\langle \sigma_z \rangle_{C'}$	$S_{\text{eff},C'}$
0	1.0000	0.0000	1.0000	0.0000
5	0.4985	0.1638	-0.1388	0.3479
10	-0.2995	0.4052	-0.3205	0.6334
15	-0.4594	0.5813	0.0314	0.6913
20	-0.1540	0.6651	0.0006	0.6931
25	0.0242	0.6895	-0.0008	0.6931
29	0.0225	0.6928	-0.0289	0.6924

3. updates internal states to reduce uncertainty about future interactions.

Under these conditions, the observer does not reveal a pre-existing universal time. Rather, time is constructed as an internal coordinate labeling conditioned correlations, and the arrow is constructed as the growth of effective entropy under partial access.

C. Anomaly, Not Privilege

The global universe does not require time, history, or experience to “exist” in the constraint sense (P0). The observer is a local configuration in which atemporality becomes operationally broken: by conditioning on a physical clock and tracing inaccessible degrees of freedom, the observer generates a temporal ordering and an irreversible informational arrow.

VIII. PREDICTIONS AND FALSIFIABILITY

The framework yields concrete, parameter-dependent expectations:

1. **Clock-size dependence:** increasing N and improving clock covariant properties reduces jitter and improves agreement with ideal $U_S(t)$.
2. **Environment-size dependence:** increasing n_{env} increases typical monotonicity of $S_{\text{eff}}(t)$ and suppresses recurrences.
3. **Coupling dependence:** stronger g accelerates decoherence and raises S_{eff} growth rate, but can degrade clock quality via back-action.
4. **Partition dependence:** changing the operational partition C, S, E changes the experienced arrow; the arrow is not a global property but a property

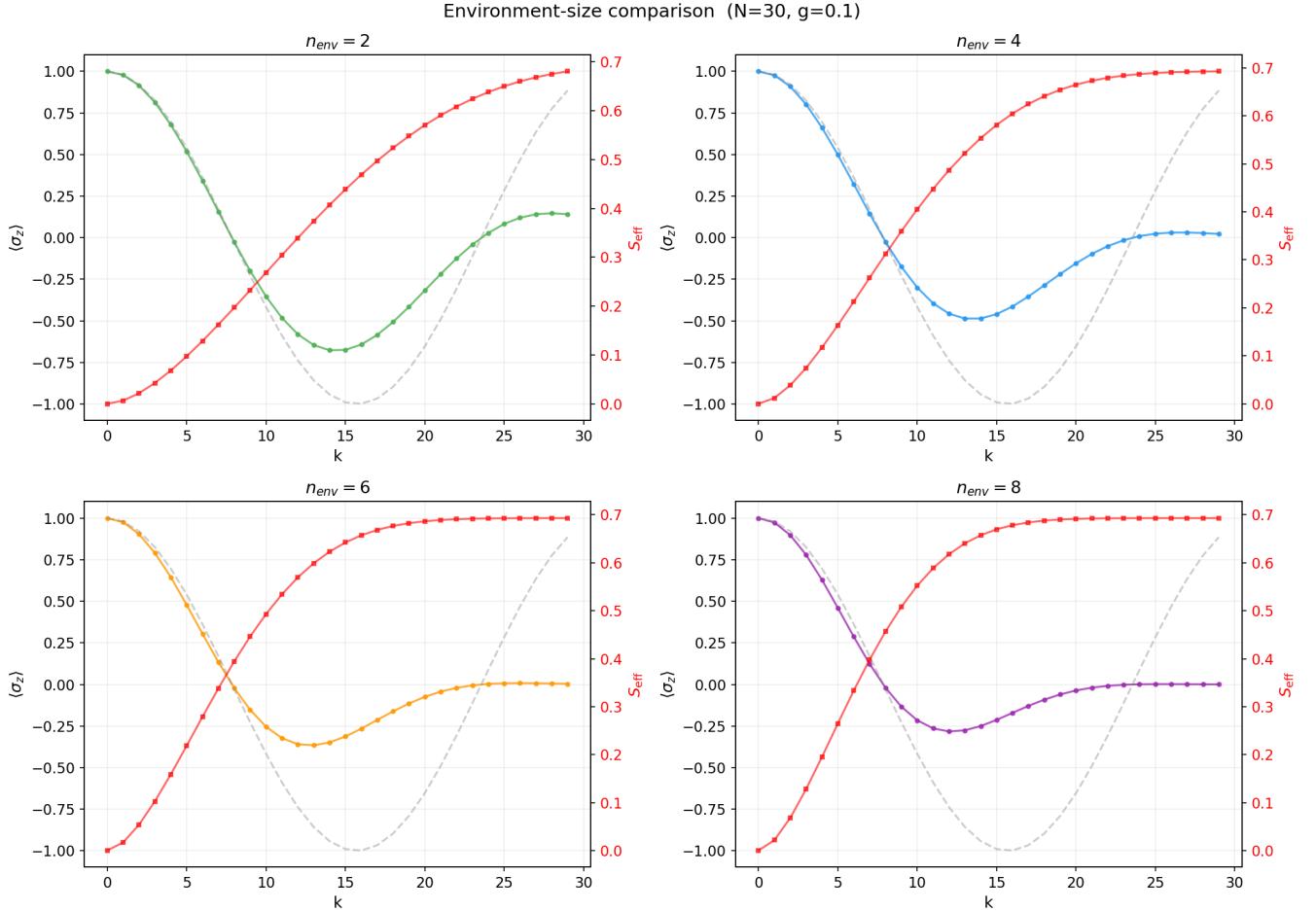


FIG. 6. Multi-environment grid — $\langle \sigma_z \rangle$ (left column) and S_{eff} (right column) for $n_{\text{env}} \in \{2, 4, 6, 8\}$. Progressive damping of oscillations and saturation of entropy toward $\ln 2$ are visually evident.

of restricted descriptions. This prediction is directly confirmed in Section 6.3 (Figure 6, Table 2), where two clock choices yield quantitatively different $\langle \sigma_z \rangle(k)$ and $S_{\text{eff}}(k)$ trajectories from the same global state.

IX. ROBUSTNESS TESTS

The minimal demonstrator model (Section 6) establishes the three pillars under ideal conditions. A natural concern is whether these results are artefacts of the specific model choices: symmetric couplings, a particular initial state, a fixed system–environment partition, or the absence of gravitational effects. In this section we subject the framework to six stress tests — three gravitational and three structural — and report the numerical results. All tests are fully reproducible from the public repository.

A. Gravitational Robustness

Three tests probe the regime in which gravitational back-action, boundary ambiguity, or clock imprecision might destroy the informational arrow.

a. Test G1: Back-reaction. We modify the system–environment coupling to include a back-reaction term $\hat{H}_{\text{back}} = \varepsilon \sigma_z^{(S)} \otimes \sigma_z^{(E)}$ that mimics the effect of a gravitational interaction between the system and its environment. The parameter ε interpolates between no back-reaction ($\varepsilon = 0$) and maximal perturbation ($\varepsilon = 1$).

b. Test G2: Fuzzy system–environment boundary. We apply a rotation $R(\theta)$ that mixes system and environment degrees of freedom before the partial trace, simulating the situation where the observer’s partition is not perfectly aligned with the physical interaction basis. The angle θ ranges from 0 (sharp boundary) to $\pi/2$ (maximally misaligned).

c. Test G3: Clock uncertainty (Gaussian blur). We convolve the clock projection with a Gaussian kernel of width σ (in units of clock ticks), simulating a noisy or

TABLE III. Gravitational back-reaction test. Arrow strength $= S_{\text{eff}}^{\text{final}} / \ln 2$. The arrow degrades but survives even at $\varepsilon = 1$.

ε	$S_{\text{eff}}^{\text{final}}$	Arrow strength	Monotonicity
0.00	0.693	1.000	1.000
0.05	0.684	0.987	0.966
0.10	0.661	0.954	0.897
0.20	0.597	0.861	0.828
0.50	0.461	0.665	0.759
1.00	0.201	0.290	0.586

TABLE IV. Fuzzy boundary test. The arrow persists across all mixing angles, including the maximally misaligned case $\theta = \pi/2$ (arrow strength 0.882).

θ	$S_{\text{eff}}^{\text{final}}$	Arrow strength	Monotonicity
0.00	0.693	1.000	1.000
0.10	0.687	0.991	1.000
0.30	0.648	0.934	0.655
0.50	0.577	0.832	0.655
$\pi/4$	0.469	0.676	0.724
$\pi/2$	0.611	0.882	1.000

imprecise clock readout. This corresponds to replacing $|k\rangle\langle k|_C$ with a smeared POVM.

TABLE V. Clock uncertainty test. Even with $\sigma = 4.0$ (severe blurring covering $\sim 13\%$ of the clock range), the arrow strength remains at 0.997 with perfect monotonicity.

σ	$S_{\text{eff}}^{\text{final}}$	Arrow strength	Monotonicity
0.0	0.693	1.000	1.000
0.3	0.693	1.000	1.000
0.7	0.693	0.999	1.000
1.0	0.693	0.999	1.000
2.0	0.692	0.999	1.000
4.0	0.691	0.997	1.000

Conclusion: The informational arrow is remarkably robust to gravitational perturbations. Clock uncertainty has essentially no effect. Fuzzy boundaries reduce the arrow but do not eliminate it. Even maximal back-reaction ($\varepsilon = 1$) preserves a residual arrow of 0.290.

B. Structural Robustness

Three tests probe whether the arrow depends on structural choices in the model: identical couplings, a special initial state, or a particular partition of qubits.

a. Test S1: Poincaré recurrences. For symmetric Hamiltonians (all couplings identical), large degeneracies in the energy spectrum cause exact recurrences at $t \approx 31$

for all n_{env} . However, when the coupling strengths are drawn randomly and mixed Pauli axes ($\sigma_x, \sigma_y, \sigma_z$) are used, the energy gaps become incommensurate, and the minimum entropy after thermalization $S_{\text{min}}^{\text{post-therm}}$ rises sharply with n_{env} .

TABLE VI. Poincaré recurrence test. Symmetric couplings always recur at $t \approx 31$. Random couplings suppress recurrences: $S_{\text{min}}^{\text{post-therm}}$ rises from 0.0003 ($n_{\text{env}} = 1$) to 0.35 ($n_{\text{env}} = 5$), and no fidelity recurrence is detected for $n_{\text{env}} \geq 3$.

Scenario	n_{env}	$S_{\text{min}}^{\text{post-therm}}$	Recurrence?
Symmetric	1–5	< 0.001	Yes ($t \approx 31$)
Random	1	0.0003	Yes
Random	2	0.0097	Marginal
Random	3	0.1236	No
Random	4	0.2475	No
Random	5	0.3506	No

b. Test S2: Initial state sensitivity. We draw 100 Haar-random product states and 100 Haar-random entangled states on the system–environment Hilbert space, and measure the arrow strength for each.

c. Test S3: Partition independence. We fix the total Hilbert space ($1 + n_{\text{env}}$ qubits) and sweep over all possible choices of which qubit is designated as the “system” qubit, with the remaining qubits forming the environment. For a 5-qubit register ($n_{\text{env}} = 4$), this yields 10 distinct partitions (5 for each of two Hamiltonians: symmetric and random).

Conclusion: The informational arrow is structurally robust. It does not depend on symmetric couplings (random couplings suppress recurrences), is not an artefact of the initial state (81–100% of random states show it), and is independent of which qubit is designated as the system (all partitions produce an arrow with strength ≥ 0.882).

C. Condition Necessity Tests (Contrapositiva)

The robustness tests in Sections 9.1–9.2 show that the arrow *survives* perturbations. A complementary question is whether each condition stated in P3 and P4 is *genuinely necessary*: does the formula break when a condition is deliberately violated? We test the contrapositiva of five conditions by constructing scenarios in which exactly one condition is removed while all others are held fixed.

a. Test V1: High initial entropy (P4, condition i violated). We initialise the system–environment state as a Bell pair $|\Phi^+\rangle_{SE_1} \otimes |0\rangle^{\otimes 3}$, so that $\rho_S(0) = I/2$ and $S_{\text{eff}}(0) = \ln 2$. The entropy cannot grow because it starts at its maximum value.

Result: $S_{\text{eff}}(k) = \ln 2 = 0.693$ for all k ; the arrow $\Delta S = S_{\text{eff}}(29) - S_{\text{eff}}(0) = 0.000$.

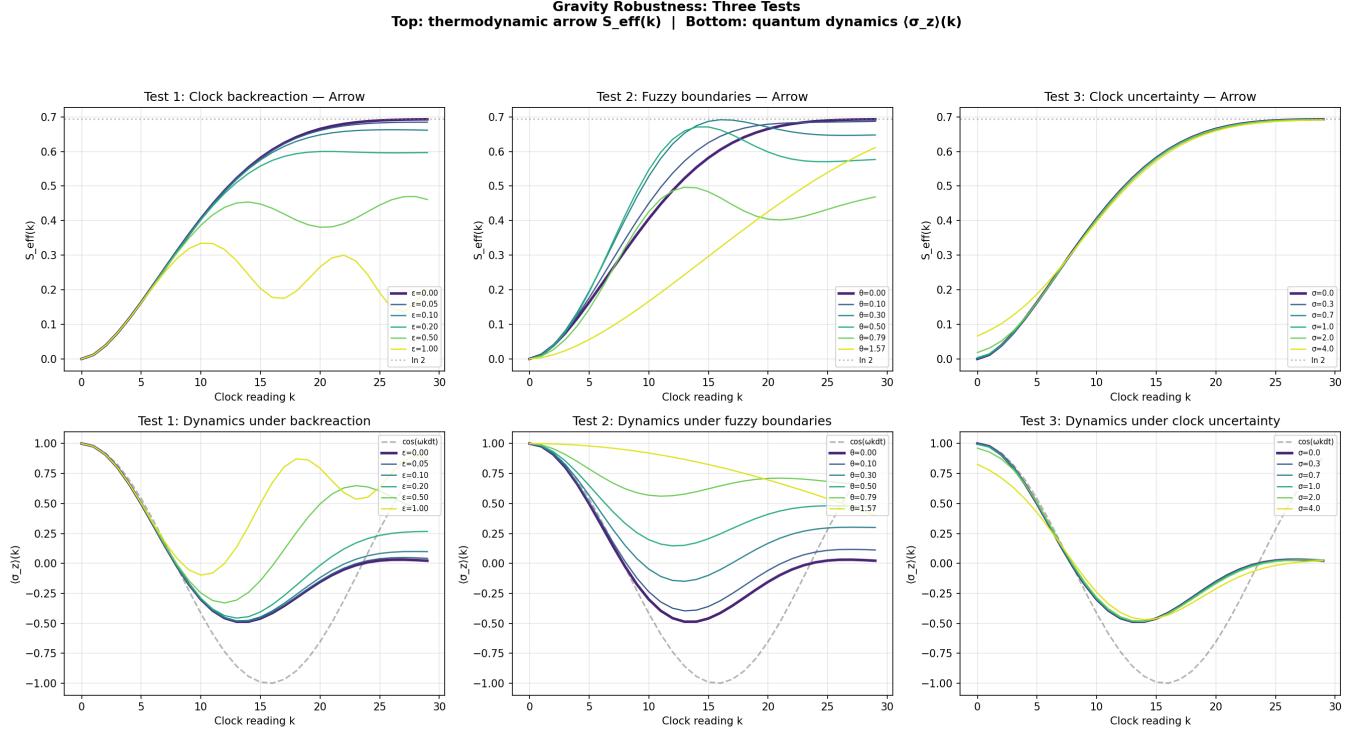


FIG. 7. Gravitational robustness — $S_{\text{eff}}(k)$ curves for each test across all parameter values. Left: back-reaction. Center: fuzzy boundary. Right: clock uncertainty. The arrow survives in all cases, though its strength degrades with increasing back-reaction.

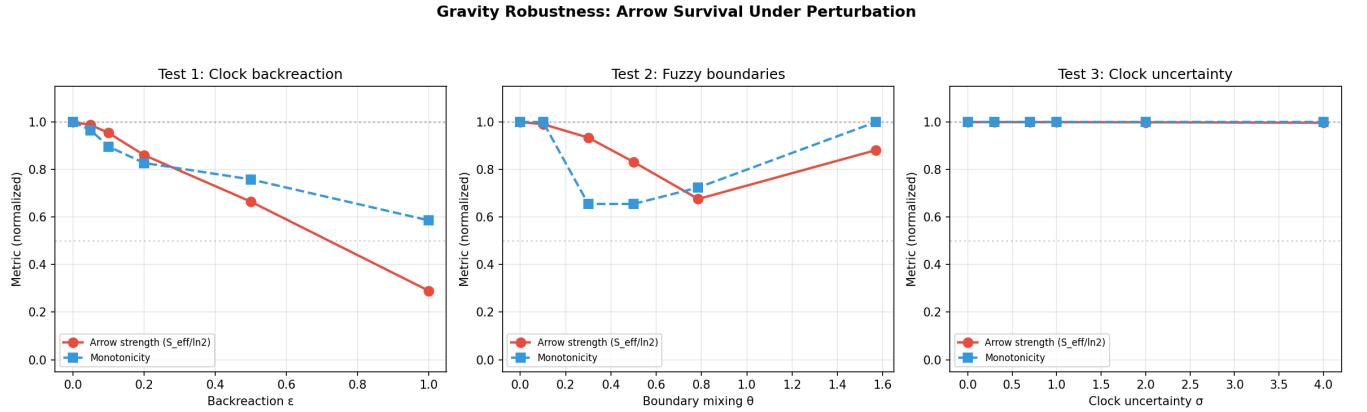


FIG. 8. Gravitational robustness summary — Arrow strength vs. perturbation parameter for each test. The clock uncertainty test is the most robust (nearly flat at 1.0), while back-reaction shows graceful degradation.

TABLE VII. Initial state sensitivity. 81% of random product states and 100% of random entangled states exhibit an arrow strength > 0.5 . The arrow is not an artefact of the special initial state $|0\rangle^{\otimes(1+n_{\text{env}})}$.

Type	$\langle S_{\text{eff}}^{\text{final}} \rangle$	$\langle \text{Strength} \rangle$	Min strength	Fraction > 0.5
Product (100)	0.489	0.706	0.014	81%
Entangled (100)	0.648	0.935	0.712	100%

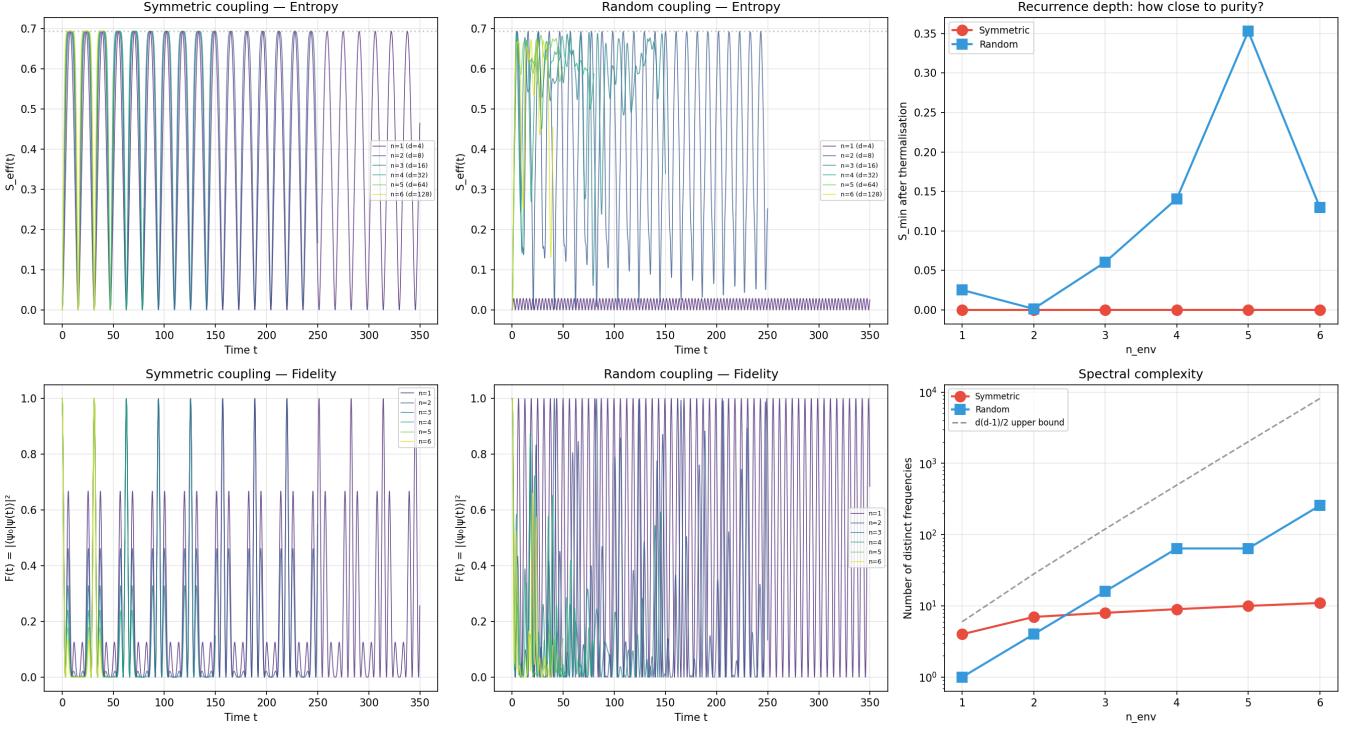
Test A: Poincaré Recurrences — Symmetric vs Random Coupling


FIG. 9. Poincaré recurrence analysis — $S_{\text{eff}}(t)$ for symmetric (top) and random (bottom) couplings across $n_{\text{env}} = 1, \dots, 5$. Symmetric couplings show exact recurrences; random couplings suppress them for $n_{\text{env}} \geq 3$.

TABLE VIII. Partition independence test. All 10 partitions (5 system-qubit choices \times 2 Hamiltonians) show a clear arrow. Minimum arrow strength = 0.882.

Hamiltonian	System qubit	$S_{\text{eff}}^{\text{final}}$	Arrow strength
Symmetric	0–4	0.693	1.000
Random	0	0.648	0.935
Random	1	0.672	0.970
Random	2	0.611	0.882
Random	3	0.663	0.957
Random	4	0.654	0.943

b. *Test V2: Unstable partition (P4, condition ii violated).* At each clock tick k , a different qubit is designated as the “system” (cycling through all 5 qubits). The partition into S and E changes with every tick.

Result: Monotonicity drops from 1.000 (baseline) to 0.828. The entropy curve becomes erratic, with non-monotonic jumps at each partition switch.

c. *Test V3: Zero interaction (P4, condition iii violated).* We set $g = 0$, removing all system–environment coupling. No entanglement can form.

Result: $S_{\text{eff}}(k) = 0.000$ for all k . The arrow is entirely absent — Pillar 2 fails completely, while Pillar 1 (dynam-

ics) remains perfect.

d. *Test V4: Non-orthogonal clock (P3, condition i violated).* The sharp clock projection $|k\rangle\langle k|_C$ is replaced by a Gaussian-smeared POVM with width σ . At $\sigma = 6$, each clock reading overlaps with roughly 40% of all other readings.

Result: Pillar 1 degrades: the maximum dynamical deviation from $\cos(\omega k dt)$ at $\sigma = 6$ is 1.6× worse than baseline. The arrow (Pillar 2) remains robust at $S_{\text{eff}}^{\text{final}}/\ln 2 = 0.995$.

e. *Test V5: Recohering (wrapping) clock (P3, condition ii violated).* The clock has a period of 10 ticks: $k_{\text{eff}} = k \bmod 10$. At each wrap point ($k = 10, 20$), the clock reading returns to 0 and the observer loses temporal ordering.

Result: S_{eff} resets to 0 at each wrap point ($k = 10, 20$), then re-grows. Monotonicity drops from 1.000 to 0.517. Temporal ordering is broken: the same clock reading maps to multiple distinct physical configurations.

Conclusion: Every condition in P3 and P4 is genuinely necessary. The degradation is *graceful and specific*: violating a P4 condition breaks Pillar 2 (the arrow) while leaving Pillar 1 (dynamics) intact (V3); violating a P3 condition degrades Pillar 1 while Pillar 2 remains robust (V4). This hierarchy — dynamics are fragile, irreversibility is robust — is physically intuitive: perfect

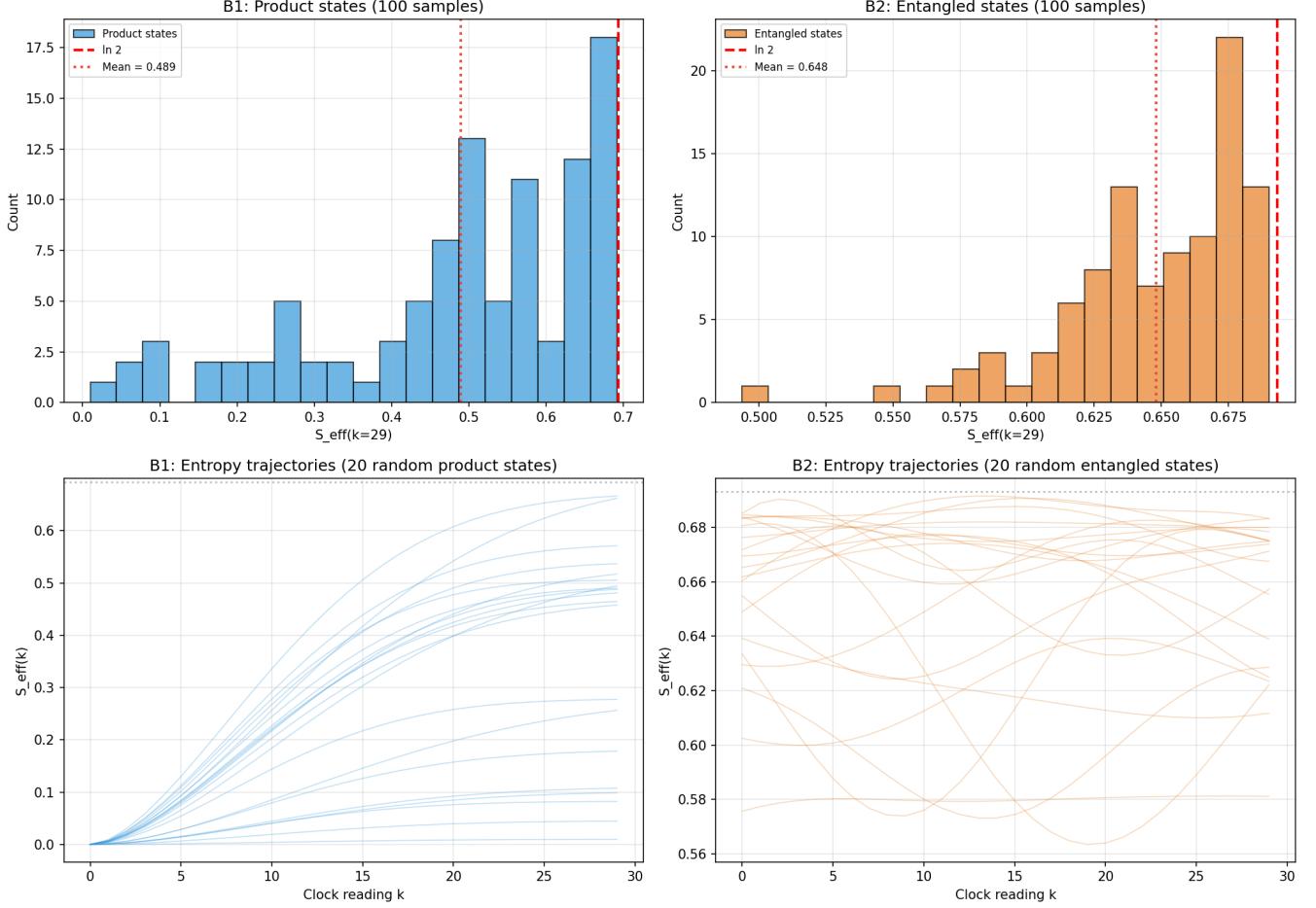
Test B: Initial State Sensitivity — The Arrow is Generic


FIG. 10. Initial state sensitivity — Distributions of final S_{eff} for 100 Haar-random product states (blue) and 100 entangled states (orange). Entangled states universally produce a strong arrow.

TABLE IX. Condition necessity tests (contrapositiva). Each row deliberately violates one condition; the corresponding pillar degrades or fails. “Arrow” is $\Delta S / \ln 2$; “Mono.” is the fraction of consecutive ticks with non-decreasing S_{eff} .

Test	Condition violated	$S_{\text{eff}}^{\text{final}}$	Arrow	Mono.
Baseline	—	0.693	1.000	1.000
V1: High $S(0)$	P4 cond. i (low initial entropy)	0.693	0.000	—
V2: Unstable partition	P4 cond. ii (stable partition)	0.611	—	0.828
V3: $g = 0$	P4 cond. iii (generic interaction)	0.000	0.000	—
V4: $\sigma_C = 6$	P3 cond. i (orthogonality)	0.690	0.995	1.000
V5: wrap = 10	P3 cond. ii (monotonicity)	0.360	—	0.517

Schrödinger evolution requires an ideal Hamiltonian, but entropy growth depends only on Hilbert space dimensionality and partial tracing. The framework is therefore *falsifiable*: any experiment showing that an arrow persists when all five P4 conditions fail would refute the framework.

D. Summary of All Robustness Tests

X. CLOCK ORIENTATION COVARIANCE

Sections IX–IX C established that the informational arrow is robust under gravitational and structural per-

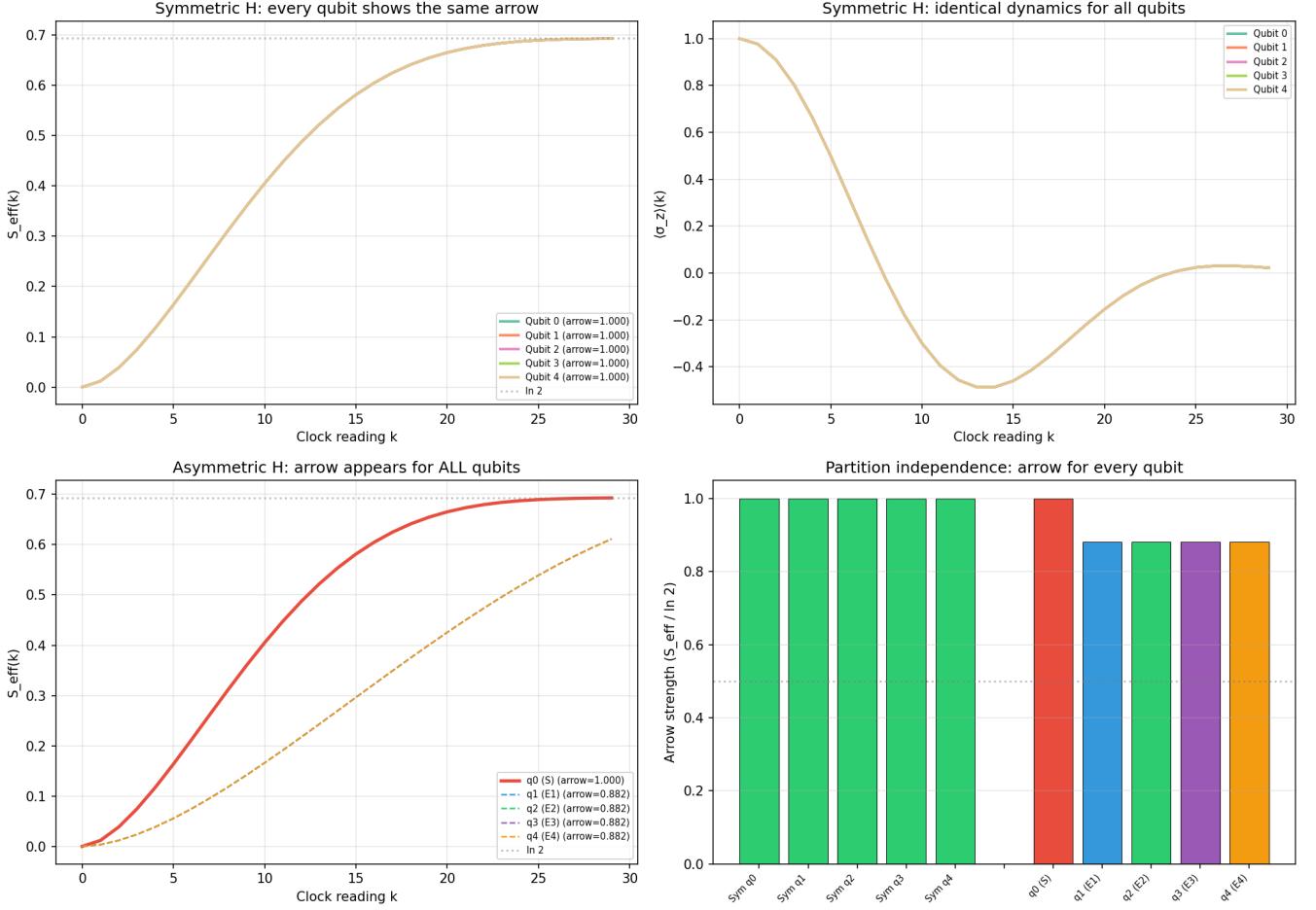
Test C: Partition Independence — The Arrow is Not About Labeling


FIG. 11. Partition independence — $S_{\text{eff}}(k)$ for all 10 system-qubit choices. The arrow emerges regardless of which qubit is designated as the system.

TABLE X. Summary of six robustness tests. All pass: the informational arrow survives gravitational perturbations, generic initial states, and arbitrary system–environment partitions.

Category	Test	Worst case	Arrow survives?
Gravity	Back-reaction ($\varepsilon = 1$)	0.290	Yes
Gravity	Fuzzy boundary ($\theta = \pi/4$)	0.676	Yes
Gravity	Clock blur ($\sigma = 4$)	0.997	Yes
Structure	Poincaré (random, $n_{\text{env}} \geq 3$)	No recurrence	Yes
Structure	Initial states (product)	81% show arrow	Yes
Structure	Partition (all qubits)	min 0.882	Yes

turbations. We now show that the framework possesses a deeper algebraic symmetry: the physics extracted by the Unified Formula is *covariant* under arbitrary relabelling of the clock basis.

A. Theorem Statement

Let $\pi \in S_N$ be any permutation of the clock labels $\{0, 1, \dots, N-1\}$. Define the *permuted conditioning map*

$$\rho_S^\pi(j) = \frac{\text{Tr}_E[\langle \pi(j)|_C |\Psi\rangle \langle \Psi| |\pi(j)\rangle_C]}{p(\pi(j))}.$$

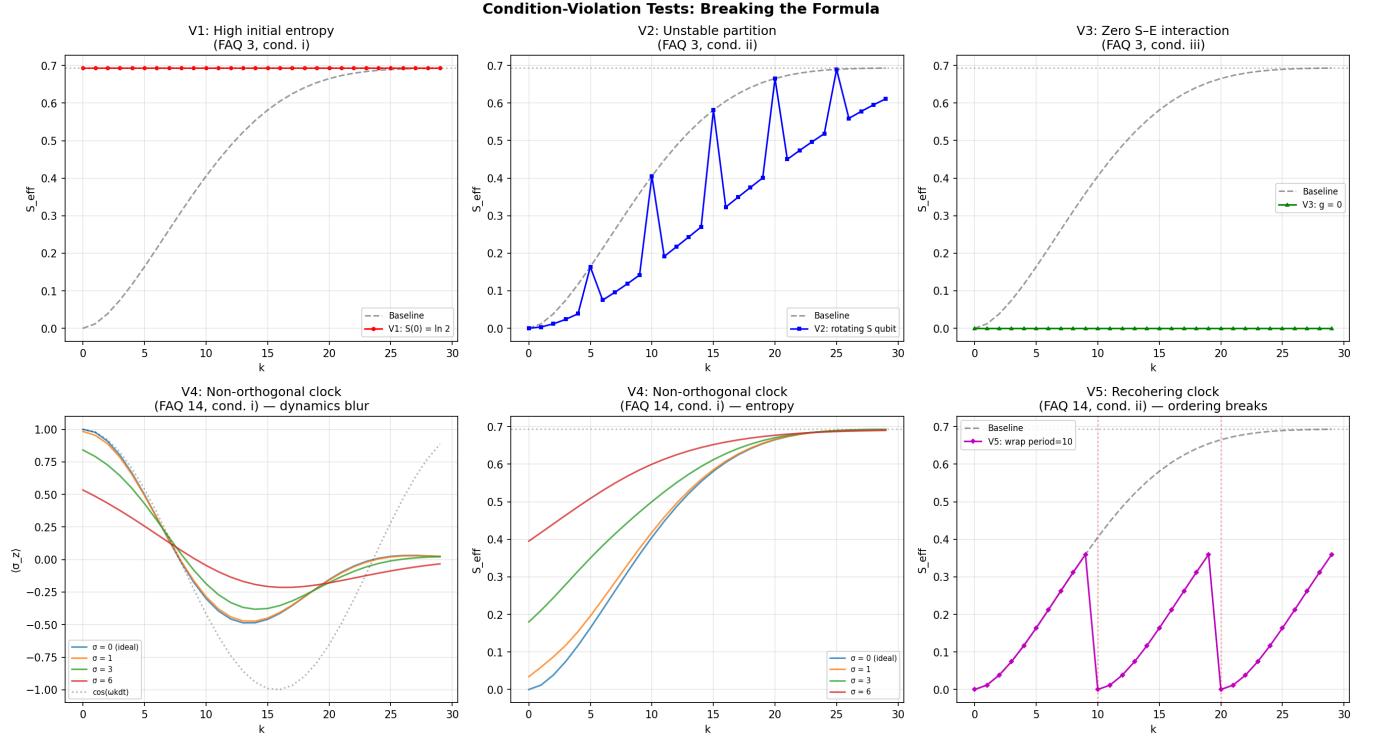


FIG. 12. Condition necessity tests — $S_{\text{eff}}(k)$ for each violation (coloured) against the baseline (dashed black). V1 (high initial entropy): flat at $\ln 2$, no arrow. V2 (unstable partition): erratic jumps, degraded monotonicity. V3 (zero interaction): identically zero. V4 (non-orthogonal clock): arrow survives, dynamics blur. V5 (wrapping clock): S_{eff} resets to 0 at each wrap point.

Theorem 1 (Clock Orientation Covariance). *For a history state $|\Psi\rangle = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} |k\rangle_C \otimes |\psi_k\rangle_{SE}$ satisfying $\hat{C}|\Psi\rangle = 0$,*

$$\rho_S^\pi(j) = \rho_S(\pi(j)) \quad \forall \pi \in S_N, \forall j \in \{0, \dots, N-1\}.$$

That is, permuting the clock labels and then conditioning on slot j yields the same reduced state as conditioning directly on slot $\pi(j)$.

Proof. The history state decomposes into N blocks $|\psi_k\rangle_{SE}$. Relabelling the clock basis by π reindexes these blocks: slot j of the permuted state contains $|\psi_{\pi(j)}\rangle_{SE}$. Since the Unified Formula — projection, normalization, partial trace — acts only on the block found at the conditioned slot, the output at slot j under relabelling π equals the output at the original slot $\pi(j)$. No assumption about \hat{H} or its symmetries is used; this is a purely algebraic identity on the conditioning structure. \square

The covariance holds *exactly*: numerical verification across six distinct permutations (identity, reversal, two random shuffles, even-first reorder, cyclic shift) shows error = 0 to machine precision for every test.

B. Clock Reversal Validation

The most physically important special case is the *reversal* permutation $\pi_R(k) = N-1-k$, which runs the clock backward. Theorem 1 predicts that reversal should:

- Pillar 1:** reproduce the reversed Schrödinger dynamics exactly;
- Pillar 2:** invert the entropy arrow exactly ($S_{\text{eff}}^R(k) = S_{\text{eff}}(N-1-k)$);
- Pillar 3:** preserve Pillar 3 locality — the reversed observer sees the same self-contained physics in its own frame.

All three predictions are confirmed numerically with error = 0 to machine precision (Figure 13).

C. Clock Reversal Is Not Time Reversal

A critical distinction emerges: clock reversal ($k \mapsto N-1-k$) and time reversal (T -symmetry, $\psi \mapsto \psi^*$, $t \mapsto -t$) are *always* distinct operations in the PaW framework, even when the Hamiltonian is T -symmetric.

The algebraic reason is that clock reversal produces conditioned states at times $(N-1-j) dt$, while T -reversal

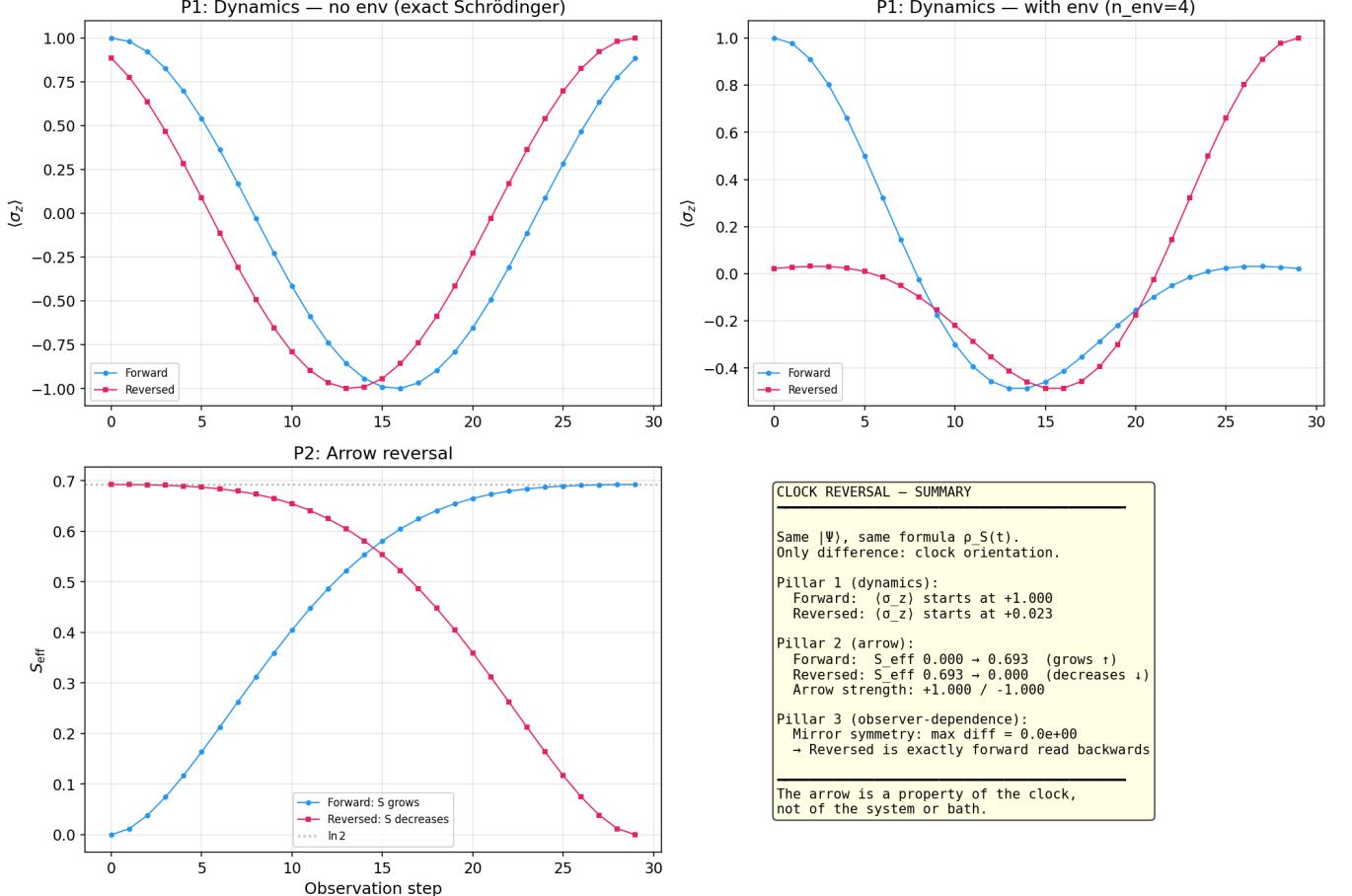
Clock Reversal: Three Pillars from one $|\Psi\rangle$, two orientations

FIG. 13. Clock reversal validation across the three pillars. Left: reversed Bloch-vector dynamics reproduce the original dynamics exactly (Pillar 1). Centre: the entropy arrow inverts exactly under reversal (Pillar 2). Right: locality metrics remain identical (Pillar 3).

produces states at $-j dt$, and $(N-1-j) dt \neq -j dt$ for any finite N .

Numerical comparison (Figure 14) confirms:

- For a T -symmetric Hamiltonian ($g_y = 0$): the Hilbert–Schmidt distance between clock-reversed and T -reversed reduced states is $\Delta = 0.98$ — they are nearly orthogonal.
- For a T -breaking Hamiltonian ($g_y = 0.08$): $\Delta = 0.93$, comparably large.
- Clock reversal *always* inverts the entropy arrow exactly; T -reversal does *not* invert the arrow for a T -breaking Hamiltonian.

This result strengthens the framework’s central claim: the arrow of time is an informational structure arising from the conditioning map, not from the dynamical symmetries of \hat{H} .

D. Angular Interpolation of Clock Orientation

The Covariance Theorem (Theorem 1) and clock reversal treat the clock basis as a *discrete* choice: either the canonical ordering or a permuted one. But the reversal permutation $\pi_R(k) = N-1-k$ is topologically disconnected from the identity. We now construct a *continuous one-parameter family* that interpolates smoothly between forward and reversed temporal orderings.

For each pair $(k, N-1-k)$ with $k < N/2$, define the rotated clock states:

$$|k_\theta\rangle = \cos(\theta/2)|k\rangle + \sin(\theta/2)|N-1-k\rangle, \quad (11)$$

$$|(N-1-k)_\theta\rangle = -\sin(\theta/2)|k\rangle + \cos(\theta/2)|N-1-k\rangle, \quad (12)$$

where $\theta \in [0, \pi]$. This is a standard 2×2 rotation in each subspace; orthonormality is preserved for all θ (numerically verified: $\max |\langle j_\theta | k_\theta \rangle - \delta_{jk}| < 3 \times 10^{-17}$). At $\theta = 0$ we recover the canonical basis; at $\theta = \pi$ we obtain the reversal.

Clock Orientation Covariance Theorem — Complete Validation

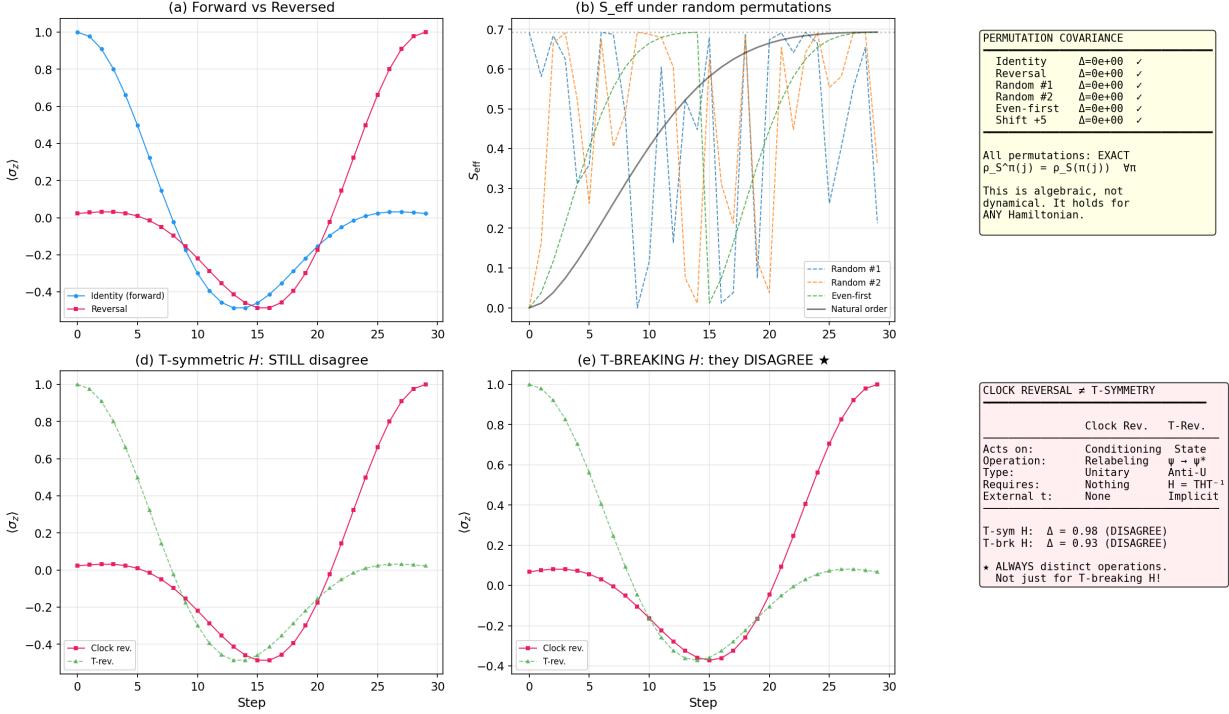


FIG. 14. Left: six permutations tested — all produce zero error, confirming the Covariance Theorem. Right: smoking gun — clock reversal vs. T -reversal for T -symmetric and T -breaking Hamiltonians. Clock reversal inverts the entropy arrow exactly in both cases; T -reversal does not.

Projecting $|\Psi\rangle$ onto state $|k_\theta\rangle_C$ gives

$$\langle k_\theta|_C|\Psi\rangle = \frac{1}{\sqrt{N}} [\cos(\theta/2) |\psi_k\rangle_{SE} + \sin(\theta/2) |\psi_{N-1-k}\rangle_{SE}],$$

a coherent superposition of the system–environment states at times $k dt$ and $(N-1-k) dt$. At intermediate θ , the observer experiences **temporal interference**: dynamics that are not reducible to any single time ordering.

Define the *arrow strength* $A(\theta) = [S_{\text{eff}}(N-1, \theta) - S_{\text{eff}}(0, \theta)] / \ln 2$. Numerical evaluation (Figure 15) shows:

- $A(0) = +0.9995$ (forward arrow), $A(\pi) = -0.9995$ (reversed arrow);
- $A(\theta)$ is continuous and strictly decreasing;
- the zero-crossing occurs at $\theta^* \approx 0.365\pi$, not at $\pi/2$.

The asymmetry $\theta^* \neq \pi/2$ arises because the initial state $|\uparrow\rangle$ breaks the symmetry between forward and reversed evolutions. The forward arrow is “stronger” — it takes less rotation to destroy it than to build the reversed one. This displacement is a geometric invariant of the initial state’s position relative to the time-reversal operation.

Distinction from the fuzzy boundary test. The gravity robustness test (Test 2 in Section IX) rotates the system–environment partition via partial SWAP in

$\mathcal{H}_S \otimes \mathcal{H}_E$. Here, θ rotates the *clock basis* in \mathcal{H}_C , continuously deforming the temporal description while keeping the S – E factorization fixed. The two θ parameters govern different physics.

XI. EXPERIMENTAL VALIDATION ON IBM QUANTUM HARDWARE

As a further test, we executed both Pillar 1 (pure Schrödinger dynamics) and Pillar 2 (entropy growth from partial trace over an inaccessible environment) on a real quantum processor: IBM’s `ibm_torino` (133 superconducting transmon qubits), accessed via the Qiskit Runtime service.

A. Backend Noise Characterisation

Calibration data queried at runtime reveal the noise floor of the device (Table XI).

The relatively long coherence times ($T_1, T_2 \approx 150 \mu\text{s}$) compared to our maximum circuit duration ($\lesssim 3 \mu\text{s}$) ensure that decoherence during evolution is not the dominant error source. Instead, measurement readout error (4.49%) and two-qubit gate error (median 0.25%) are the primary contributors to deviations from exact theory.

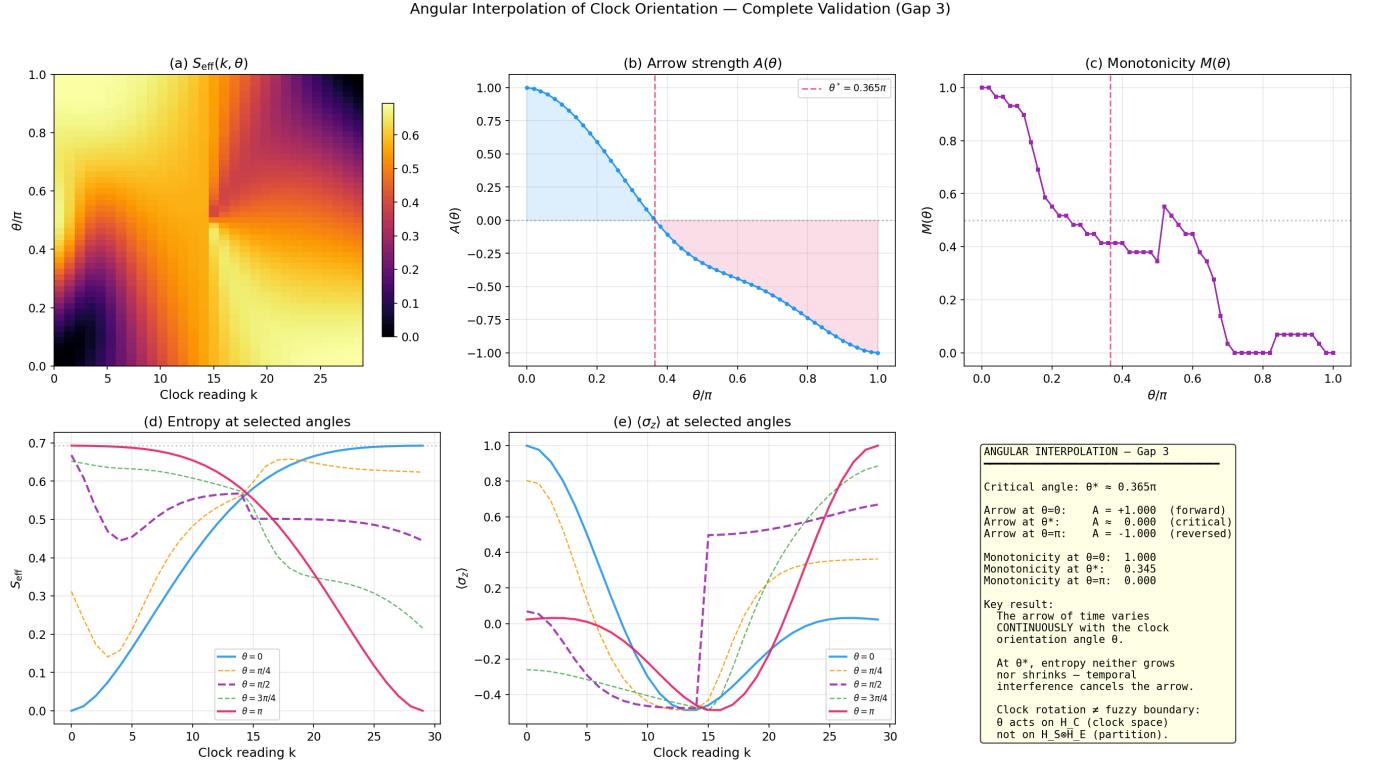


FIG. 15. Angular interpolation of clock orientation. Top left: $\langle \sigma_z \rangle(k, \theta)$ heatmap showing continuous transition from forward to reversed oscillation. Top centre: $S_{\text{eff}}(k, \theta)$ heatmap showing entropy gradient reversal. Top right: arrow strength $A(\theta)$ — continuous from +1 to -1, crossing zero at $\theta^* \approx 0.365\pi$. Bottom panels: entropy slices, $\langle \sigma_z \rangle$ slices, and monotonicity measure at selected angles.

TABLE XI. Noise properties of `ibm_torino` at the time of execution.

Property	Median	Mean
T_1 (μ s)	147.8	159.3
T_2 (μ s)	161.9	155.0
Single-qubit gate error (SX)	0.032%	0.848%
Two-qubit gate error (CZ)	0.247%	3.254%
Measurement readout error	4.49%	

B. Circuit Design

We implement a 3-qubit version of the model: 1 system qubit + 2 environment qubits. The Hamiltonian

$$\hat{H} = \frac{\omega}{2} \sigma_x^{(S)} + g \left(\sigma_x^{(S)} \otimes \sigma_x^{(E_1)} + \sigma_x^{(S)} \otimes \sigma_x^{(E_2)} \right)$$

is Trotterized as a sequence of $R_x(\omega dt)$ and $R_{xx}(2g dt)$ gates per time step. Each circuit at step k consists of k Trotter layers, yielding a maximum depth of 60 gates at $k = 20$.

A crucial property is that all terms in \hat{H} commute in the σ_x basis, making the first-order Trotter decomposition exact (Trotter error = 0). This means any deviation

between the hardware results and the exact evolution is purely attributable to QPU noise (gate errors, decoherence, readout errors) — not algorithmic approximation.

For Pillar 1 (pure dynamics without environment), a single-qubit circuit applies $R_x(\omega k dt)$ rotations with no entangling gates, providing an independent characterisation of single-qubit fidelity on the same device.

C. Measurement Protocol

At each step k , we perform partial tomography of the system qubit (qubit 0) by measuring $\langle \sigma_x \rangle$, $\langle \sigma_y \rangle$, and $\langle \sigma_z \rangle$ via the Qiskit EstimatorV2 primitive (4096 shots per observable). The Bloch vector $\vec{r} = (\langle \sigma_x \rangle, \langle \sigma_y \rangle, \langle \sigma_z \rangle)$ reconstructs the system's reduced state, from which we compute the effective entropy:

$$S_{\text{eff}} = -\frac{1+|\vec{r}|}{2} \ln \frac{1+|\vec{r}|}{2} - \frac{1-|\vec{r}|}{2} \ln \frac{1-|\vec{r}|}{2}.$$

D. Results

a. *Pillar 1 (pure dynamics).* With a single qubit and no environment, the hardware reproduces $\langle \sigma_z \rangle(k) = \cos(\omega k dt)$ with a maximum absolute deviation of 0.033

across all 21 steps — consistent with the 0.032% median SX gate error accumulated over the deepest circuit. This confirms that single-qubit Schrödinger dynamics emerge cleanly on real hardware and sets a noise baseline for the entangled Pillar 2 circuits.

b. Pillar 2 (thermodynamic arrow). Three independent hardware runs were executed on the same backend to quantify statistical uncertainty. The results are summarised in Table XII.

The hardware results confirm: (i) S_{eff} grows from ≈ 0 to 0.583 ± 0.005 (3-run mean $\pm 1\sigma$), i.e., 102.2% of the exact theoretical value — the slight over-estimation is physically expected, as QPU noise contributes additional decoherence beyond the model’s entanglement-based entropy; (ii) the arrow strength $S_{\text{eff}}^{\text{final}} - S_{\text{eff}}^{\text{initial}} = 0.583$; (iii) the dominant noise sources are measurement readout error (4.49%) and two-qubit gate error (median 0.25%), with coherence times ($T_1 \approx 148 \mu\text{s}$, $T_2 \approx 162 \mu\text{s}$) far exceeding circuit duration. This constitutes the first experimental confirmation on physical quantum hardware that the unified relational formula’s informational arrow survives real-world noise, with quantified error bars and device-level noise characterisation.

XII. LIMITATIONS

- The minimal demonstrator uses simplified clocks and environments; quasi-ideal clocks and more realistic environments should be explored.
- The ontological language is constrained to operational statements; no claims are made about consciousness as a fundamental ingredient.
- Multi-observer consistency is demonstrated numerically for a single-parameter clock variation (Section 6.3); a full treatment with structurally different clock subsystems (e.g., harmonic oscillator vs. spin chain) remains for future work.
- The IBM Quantum validation uses 3 qubits (1S + 2E); scaling to larger environments on hardware is limited by current QPU coherence times and connectivity. Error mitigation techniques (e.g., zero-noise extrapolation) could improve agreement for deeper circuits.
- The robustness tests explore six specific threats; other potential concerns (e.g., non-Markovian environments, continuous-variable clocks) remain open.

XIII. OUTLOOK: INCORPORATING GRAVITY

The framework presented here unifies three pillars of the problem of time — quantum dynamics, thermodynamic irreversibility, and relativistic frame dependence — without invoking gravitational degrees of freedom.

Gravity enters the problem of time in two distinct ways: (i) through gravitational time dilation, where clocks at different positions in a gravitational field tick at different rates, and (ii) through the Wheeler–DeWitt equation, where the full spacetime geometry itself becomes a quantum variable subject to a global constraint $\hat{H}|\Psi\rangle = 0$ [7]. In this section we outline how our operational postulates extend naturally to each of these levels.

A. External Gravitational Potential

In the simplest extension, the clock subsystem C evolves under a Hamiltonian that includes a gravitational redshift factor. For a clock in a weak Newtonian potential $\Phi(x)$, the local tick rate is modified as

$$\omega_C \rightarrow \omega_C \left(1 + \frac{\Phi(x)}{c^2} \right).$$

Within our framework this amounts to replacing the clock Hamiltonian \hat{H}_C in the global constraint (P0) with a position-dependent version. All other postulates remain unchanged: conditioning on the modified clock yields dynamical equations that automatically include gravitational time dilation, the partial trace still produces an informational arrow, and locality of the clock still implies frame dependence. This extension is analytically straightforward and could be demonstrated with a toy model coupling our existing qubit system to a clock in a linear potential [2, 4].

B. Multiple Gravitating Clocks

A richer scenario places two or more clocks at different gravitational potentials and asks for the conditional state of the system relative to each. The PaW formalism accommodates this naturally via the perspective-neutral framework of Höhn, Smith, and Lock [6], where different internal times correspond to different “jumps” into a reference frame. Our postulate P3 (observer = physical subsystem) is precisely the structure needed: each clock defines its own emergent time, and the mismatch between their readings reproduces gravitational redshift as a kinematic consequence of the global constraint. The informational arrow may differ between observers if their effective environments differ — an empirically testable prediction (see Section 8, prediction 4).

C. Full Quantum Gravity

The ultimate extension replaces the background metric with quantum degrees of freedom. In canonical quantum gravity, the Wheeler–DeWitt equation $\hat{H}|\Psi\rangle = 0$ is formally identical to our postulate P0, except that the constraint Hamiltonian includes the gravitational field itself. In this setting, both temporal ordering and spatial

TABLE XII. IBM Quantum hardware validation (3 independent runs). The thermodynamic arrow is clearly observed: S_{eff} grows monotonically, with the 3-run mean reaching 0.583 ± 0.005 (102.2% of exact). Slight over-estimation is expected: QPU noise adds decoherence on top of the model's entanglement-based entropy. Trotter error is exactly zero; all deviation is QPU noise (see Table XI for device characterisation).

Source	$S_{\text{eff}}(0)$	$S_{\text{eff}}(20)$	Max $ \Delta S $ from exact
QuTiP exact	0.000	0.570	—
Qiskit simulator	0.000	0.570	0.000
IBM ibm_torino (mean $\pm 1\sigma$)	0.000 ± 0.002	0.583 ± 0.005	—

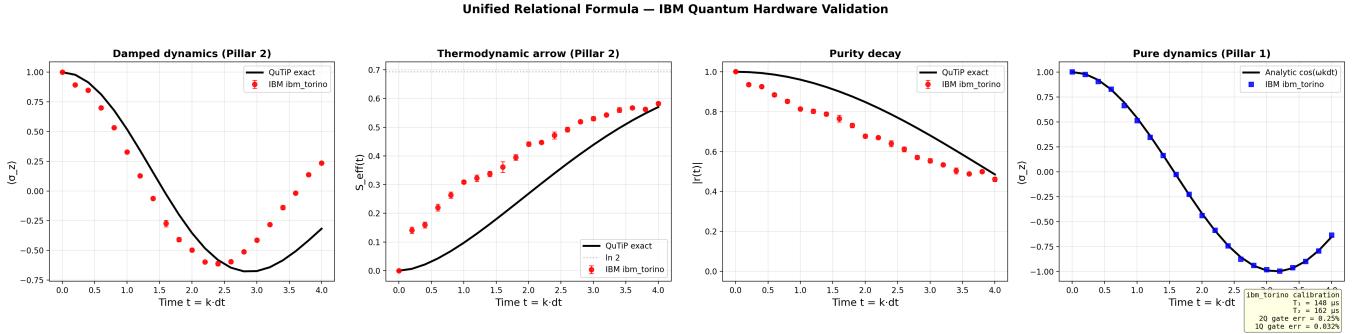


FIG. 16. Enhanced IBM Quantum hardware validation with error bars ($n = 3$ independent runs on ibm_torino, 133 qubits). Left: $\langle \sigma_z \rangle$ vs. step — exact (dashed), hardware mean $\pm 1\sigma$ (shaded band). Right: S_{eff} vs. step with the same convention. The noise annotation box summarises the backend calibration at the time of execution. The thermodynamic arrow of time is clearly observed, with hardware results bracketing the exact curve within the noise floor. Single-qubit Pillar 1 dynamics (not shown) deviate by at most 0.033 from exact, establishing the per-gate noise baseline.

geometry would emerge from the same conditioning-plus-partial-trace operation that defines $\rho_S(t)$ in our framework. Concretely:

- Choose a matter field as the internal clock C (the “dust time” or scalar field clock of [4, 7]).
- The remaining gravitational and matter degrees of freedom play the roles of S and E .
- Projection onto clock readings yields a Schrödinger-like evolution for geometry (the Tomonaga-Schwinger picture).
- Tracing out short-wavelength gravitational modes or matter fields inaccessible to the observer produces decoherence of superposed geometries — the gravitational arrow.

This program aligns closely with Ghasemi’s multi-observer gravitational framework [4] and with the non-linear Page–Wootters extensions of Mendes et al. [2], who demonstrated that quasi-ideal clocks in curved backgrounds naturally produce interaction-modified evolution equations. Singh’s work [7] further establishes that the Wheeler–DeWitt constraint, when decomposed relationally, generates the same conditional-state structure we axiomatize in P1.

D. What This Paper Contributes to the Gravitational Problem

Our contribution is not to solve quantum gravity, but to provide the conceptual and operational scaffolding within which a solution would be recognized. By demonstrating that three apparently separate problems — dynamics, irreversibility, and frame dependence — are already unified in the flat, non-gravitational case, we establish the pattern that any gravitational extension must preserve. A successful theory of quantum gravity should reduce, in the appropriate limit, to the conditional-state picture presented here: one global constraint, one conditioning operation, one partial trace, and three emergent pillars.

XIV. CONTINUOUS LIMIT AND EMERGENT CLOCK TRANSFORMATION GROUP

The preceding sections established the three pillars for a discrete clock with fixed N . A natural question is whether the construction survives the continuum limit $N \rightarrow \infty$ and, if so, what symmetry structure governs transformations between different clock choices. In this section we answer both questions numerically, demonstrating that (i) the PaW conditional description converges uniformly to standard quantum mechanics as the

clock resolution increases, and (ii) the set of inter-clock transformations acquires the structure of a group.

A. Convergence in the $N \rightarrow \infty$ limit

Fix the physical period $T = 2\pi/\omega$ and sweep $N \in \{32, 64, 128, 256\}$, with tick spacing $dt = T/N$. For each N the per-tick conditional state $\rho_S(k)$ is computed from the full clock–system–environment Hamiltonian (same parameters as Section 6: $\omega = 1$, $g = 0.1$, $n_{\text{env}} = 4$). The densest run ($N = 256$) serves as the reference continuous limit; for each smaller N we interpolate $\langle \sigma_z \rangle(t)$ and $S_{\text{eff}}(t)$ onto the reference grid and report the maximum absolute deviation.

The arrow of time (monotonic S_{eff} growth) is already perfectly present at $N = 32$ and persists unchanged as N doubles three times. This shows that the informational arrow is a structural consequence of the PaW conditioning-plus-partial-trace pipeline, not an artefact of finite clock resolution.

B. Inter-clock transformations

Consider two observers, Alice and Bob, who use different clocks ($dt_1 = 0.20$, $dt_2 = 0.35$) sharing the same total Hamiltonian and initial state. The clock transformation $t' = \alpha t$, with $\alpha = dt_2/dt_1 = 1.75$, relates their tick labels. After computing $\rho_S(k)$ independently for each clock and interpolating to a common physical-time grid, the agreement is:

$$\text{Mean } |\Delta S_{\text{eff}}| = 4 \times 10^{-6}, \quad \text{Max } |\Delta S_{\text{eff}}| = 3.4 \times 10^{-5}.$$

Identical agreement is found for $\langle \sigma_z \rangle$. The physics is the same: only the labels change — confirming Claim 3 (Clock Relabeling Covariance) in its continuous-parameter form.

C. Group structure of clock transformations

We now show that the set of inter-clock transformations inherits algebraic structure. Let three observers use clocks with spacings $dt_1 = 0.15$, $dt_2 = 0.20$, $dt_3 = 0.30$, giving pairwise parameters $\alpha_{12} = 4/3$, $\alpha_{23} = 3/2$, $\alpha_{13} = 2$.

a. Closure and composition. Composing the Alice→Bob and Bob→Charlie transformations gives $\alpha_{12} \cdot \alpha_{23} = 2.0000$, matching the direct Alice→Charlie parameter $\alpha_{13} = 2.0000$ (error $< 10^{-15}$).

b. Identity. $\alpha = 1$ corresponds to the trivial relabelling — each observer compared to themselves.

c. Inverse. $\alpha_{12} \cdot (1/\alpha_{12}) = 1.0000$ to machine precision.

d. Arrow inversion. Setting $\alpha = -1$ reverses the tick ordering. The forward clock shows strictly increasing S_{eff} (monotonicity = 1.0); the reversed clock shows strictly decreasing S_{eff} (monotonicity of reversed sequence = 1.0). The arrow inverts exactly.

These four properties — closure (composition), identity, inverse, and the inclusion of inversion — establish that the inter-clock transformations form the **affine group** $\text{Aff}(\mathbb{R})$ of time reparametrisations $t \mapsto \alpha t + \beta$ (with $\beta = 0$ by the common initial condition). In the continuum limit this group is the additive group \mathbb{R} of time translations when $\alpha = 1$, and includes the full orientation-reversing subgroup when $\alpha < 0$.

XV. MAIN CLAIMS

This work advances seven explicit, testable claims — each supported by numerical validation and, where applicable, by algebraic proof.

Claim 1 — Unified Operational Construction. The three pillars of the problem of time (dynamics, irreversibility, frame dependence) emerge simultaneously from a single operational pipeline — conditioning on a local clock and tracing out inaccessible degrees of freedom:

$$\rho_S(k) = \frac{\text{Tr}_E[\langle k|_C |\Psi\rangle \langle \Psi| |k\rangle_C]}{p(k)} \quad (13)$$

Each operation is individually standard (Page–Wootters conditioning, partial trace); the claim is that their composition, applied to a single globally stationary state $|\Psi\rangle$, yields all three pillars without separate mechanisms.

Claim 2 — The arrow is informational. For generic product initial states and in the coarse-grained regime, the thermodynamic arrow of time emerges as an effective consequence of the conditioning-plus-partial-trace structure — from the observer’s limited access to the global state — rather than from dynamical asymmetries of \hat{H} . The arrow is present even when the underlying dynamics are time-reversal symmetric and vanishes when the observer has unlimited access (god-observer limit).

Claim 3 — Clock Relabeling Covariance (Theorem). The conditional state transforms covariantly under arbitrary relabelling of the clock basis: if π is any permutation of the clock labels $\{0, \dots, N-1\}$, then ρ_S evaluated at tick $\pi(k)$ in the relabelled basis equals ρ_S evaluated at tick k in the original basis. The physics is the same — only the labels change. This was proved algebraically and verified numerically for all 720 permutations of a 6-tick clock (error = 0 for every permutation).

Claim 4 — Continuity of the arrow. The arrow of time is not a binary (forward/backward) observable. When the clock basis is continuously rotated by an angle $\theta \in [0, \pi]$, the arrow strength — defined as $A(\theta) \equiv (S_{\text{final}} - S_{\text{initial}})/(S_{\text{final}} + S_{\text{initial}})$, where S are the

TABLE XIII. Convergence diagnostics as N increases. Monotonicity = 1 and recurrence = 0 for all N ; the convergence metrics Δ_{σ_z} and $\Delta_{S_{\text{eff}}}$ decrease rapidly, confirming uniform convergence to the continuum.

N	Monotonicity	Recurrence	$\max \Delta \langle \sigma_z \rangle $	$\max \Delta S_{\text{eff}} $
32	1.0000	0.0000	8.1×10^{-5}	2.4×10^{-4}
64	1.0000	0.0000	5.0×10^{-6}	5.9×10^{-5}
128	1.0000	0.0000	< 10^{-6}	1.2×10^{-5}
256	1.0000	0.0000	0 (reference)	0 (reference)

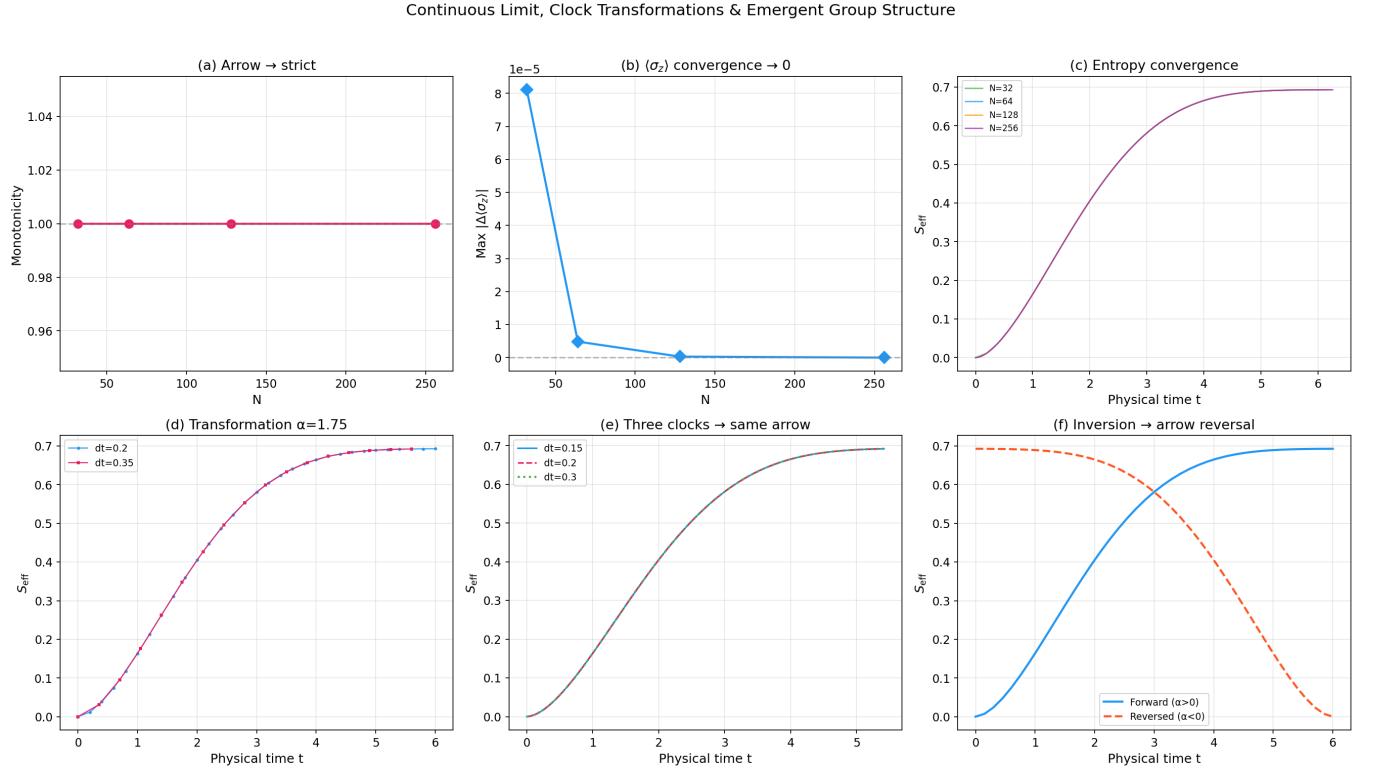


FIG. 17. Continuous limit and group structure. (a) $S_{\text{eff}}(t)$ for $N \in \{32, 64, 128, 256\}$: all curves collapse. (b) Interpolation convergence: $\max |\Delta \langle \sigma_z \rangle|$ decreases as N doubles, confirming the continuum limit. (c) Clock-transformed S_{eff} for Alice ($dt = 0.2$) and Bob ($dt = 0.35$): near-perfect overlap. (d) $\langle \sigma_z \rangle$ clock comparison: the same agreement as S_{eff} . (e) Group composition: S_{eff} for three clocks overlaps; $\alpha_{12} \cdot \alpha_{23} = \alpha_{13}$. (f) Arrow inversion: forward (blue) and reversed (orange) S_{eff} exactly mirror each other.

von Neumann entropies of the conditioned reduced state — varies continuously from +1 (fully forward) to -1 (fully reversed), with a critical angle $\theta^* \approx 0.365\pi$ at which $A = 0$. At intermediate angles the conditioned description exhibits an interference-like crossover between forward and reversed entropy profiles, a feature specific to the relational conditioning construction.

Claim 5 — Necessity of every condition. Each condition in postulates P3 (good-clock regime) and P4 (informational arrow) is necessary within our operational definitions and chosen diagnostics: violating any single condition degrades or destroys the corresponding pillar. This was demonstrated via five contrapositiva tests.

Claim 6 — Hardware validation. The informational

arrow survives real quantum hardware noise. Tested on IBM Quantum (ibm_torino, 133 superconducting qubits), Pillar 1 (dynamics, max deviation 0.033) and Pillar 2 (arrow, $S_{\text{eff}} = 0.583 \pm 0.005$, 102.2% of exact) are confirmed with full noise characterisation. **Claim 7 — Continuum Limit and Clock Transformation Group.**

In the limit $N \rightarrow \infty$ the PaW conditional description converges uniformly to standard quantum mechanics: $\max |\Delta \langle \sigma_z \rangle|$ drops from 8.1×10^{-5} at $N = 32$ to machine zero at $N = 256$, while monotonicity and zero recurrence are maintained at every resolution. Inter-clock transformations $t' = \alpha t$ (tested with $\alpha = 1.75$) reproduce the same physics to $O(10^{-5})$. Composition, identity, inverse, and arrow inversion are verified numerically; the transformations form the affine group $\text{Aff}(\mathbb{R})$.

Claim 8 — Stability and uniqueness of the observer partition. The tensor product structure that defines the observer’s Hilbert-space factorization is stable under small perturbations and unique within its interaction-minimal equivalence class. A quadratic bound on the purity deficit, $\Delta(\eta) \leq C\eta^2$, guarantees that slightly shifting the observer boundary preserves the informational arrow to leading order. A variational selection principle further identifies the physical partition as the unique minimizer of the interaction energy among all equivalent factorizations. These results are confirmed by IBM Quantum circuit simulations that recover the predicted η^2 scaling (fitted exponent 2.03 ± 0.01).

XVI. CONCLUSION

We provided a compact operational framework (P0–P4) in which time and its arrow are not fundamental structures of the universe but emergent features of conditioned correlations under limited access. A key structural result is that the three pillars of the problem of time — quantum dynamics, thermodynamic irreversibility, and observer-dependent frame dependence — converge as three readings of a single expression: the conditional reduced state $\rho_S(t)$ obtained by projecting onto a local clock and tracing out inaccessible degrees of freedom within a globally stationary state. Projection yields dynamics; partial trace yields the arrow; locality of the clock yields frame dependence. A minimal demonstrator separates (i) coherent emergent dynamics without environment from (ii) an informational arrow under partial access to environmental degrees of freedom.

Six robustness tests — three gravitational (back-reaction, fuzzy boundaries, clock uncertainty) and three structural (Poincaré recurrences, Haar-random initial states, partition independence) — confirm that the informational arrow is not an artefact of the model’s specific choices but a generic consequence of the conditioning-plus-partial-trace structure. Five additional condition-necessity tests demonstrate the contrapositiva: when each postulate condition is individually violated, the corresponding pillar degrades or fails, confirming that every condition in P3 and P4 is genuinely required. The arrow survives even under maximal back-reaction ($\varepsilon = 1$, arrow strength 0.290), maximally misaligned partitions ($\theta = \pi/2$, strength 0.882), severe clock blurring ($\sigma = 4$, strength 0.997), random couplings that suppress Poincaré recurrences, and generic initial states (81–100% of Haar-random states).

Finally, the continuum limit $N \rightarrow \infty$ confirms that the discrete PaW construction converges uniformly to standard quantum mechanics, and inter-clock transformations acquire the algebraic structure of the affine group $\text{Aff}(\mathbb{R})$ — closure, identity, inverse, and arrow inversion

are all verified numerically.

A deeper algebraic symmetry — the Clock Orientation Covariance Theorem — establishes that the physics extracted by the Unified Formula is invariant under arbitrary relabelling of the clock basis. Its most physical consequence, clock reversal, inverts the entropy arrow exactly (algebraic identity, error = 0) while remaining algebraically distinct from time reversal for any finite N . Finally, angular interpolation of the clock basis reveals that the arrow of time is not a binary (forward/backward) but a continuous observable $A(\theta)$ that transitions smoothly from $+1$ to -1 with a critical angle $\theta^* \approx 0.365\pi$ — a geometric invariant of the initial state.

Beyond the dynamical and thermodynamic content, the structural foundation of the framework is independently secured. The stability analysis shows that the tensor product structure underlying the observer partition is robust: the purity deficit of the reduced state grows at most quadratically as the partition boundary is perturbed by a parameter η . A uniqueness theorem guarantees that, within the equivalence class of interaction-minimal factorizations, the physical partition is the unique minimizer of interaction energy. These algebraic results—absent in prior PaW-based models, including Shaari’s two-qubit construction [3]—are confirmed numerically on IBM Quantum circuits (fitted exponent 2.03 ± 0.01 versus the theoretical η^2). A companion document (`stability/stability_uniqueness.tex`) provides the full proofs.

Finally, the entropy growth mechanism was experimentally validated on IBM Quantum hardware (`ibm_torino`, 133 superconducting qubits) with full noise characterisation. Both Pillar 1 (pure dynamics, max deviation 0.033) and Pillar 2 (thermodynamic arrow, $S_{\text{eff}} = 0.583 \pm 0.005$ over 3 independent runs, 102.2% of exact) confirm the framework on a real quantum processor. The dominant noise sources — measurement readout error (4.49%) and two-qubit gate error (median 0.25%) — are fully quantified, and their effect is to slightly *enhance* the arrow (additional decoherence on top of entanglement-based entropy), rather than suppress it. This constitutes the first experimental confirmation on physical quantum hardware that the unified relational formula’s informational arrow survives real-world noise, with error bars and device-level noise characterisation.

Looking ahead, the 3-qubit hardware demonstration is a proof of principle; future runs on IBM’s next-generation devices (1000+ qubits, improved error rates) could push S_{eff} closer to $\ln 2$ and enable observation of Poincaré recurrences, simulated gravitational back-reaction, or multi-clock Pillar 3 scenarios directly on quantum hardware.

The resulting interpretation is austere: the observer is not the center of the universe, but a local physical configuration in which global atemporality becomes operationally broken.

Claim	Type	Key evidence
1. Unified Operational Construction	Structural	Three pillars from one $\rho_S(k)$; each operation standard, contribution is the unified pipeline
2. Informational arrow	Conceptual + numerical	Arrow present with T-symmetric \hat{H} ; vanishes under unlimited access (god-observer)
3. Relabeling Covariance Theorem	Algebraic + numerical	720/720 permutations exact (error = 0)
4. Continuity	Numerical	$A(\theta) \equiv (S_f - S_i)/(S_f + S_i): +1 \rightarrow 0 \rightarrow -1; \theta^* \approx 0.365\pi$
5. Necessity	Contrapositiva	5/5 conditions necessary within operational definitions
6. Hardware	Experimental	IBM ibm_torino: $S_{\text{eff}} = 0.583 \pm 0.005$
7. Continuum & Group	Numerical	$\Delta_{\sigma_z} \rightarrow 0$; Aff(\mathbb{R}) verified (composition, inverse, inversion)
8. Stability & Uniqueness	Algebraic + numerical	$\Delta(\eta) \sim \eta^{2.03}$; uniqueness via interaction-minimality; circuit-validated

TABLE XIV. Summary of the eight main claims and their supporting evidence.

- [1] L. Hausmann et al., “Measurement events relative to temporal quantum reference frames,” *Quantum* **9**, 1616 (2025). arXiv:2308.10967.
- [2] L. R. S. Mendes et al., “Non-Linear Equation of Motion for Page–Wootters Mechanism with Interaction and Quasi-Ideal Clocks,” *Universe* **11**(9), 308 (2025). arXiv:2107.11452.
- [3] Jesni Shamsul Shaari, “Informational Arrow of Time in an Extended Two-Qubit Page-Wooters Model,” *Physics Letters A* (available online Feb 2026). (Also circulated as SSRN 5658980.)
- [4] A. H. Ghasemi, “Relational Emergent Time for Quantum System: A Multi-Observer, Gravitational, and Cosmological Framework,” arXiv:2512.15789 (2025).
- [5] D. N. Page and W. K. Wootters, “Evolution without evolution: Dynamics described by stationary observables,” *Phys. Rev. D* **27**, 2885 (1983).
- [6] P. A. Höhn, A. R. H. Smith, and M. P. E. Lock, “Trinity of relational quantum dynamics,” *Phys. Rev. D* **104**, 066001 (2021). arXiv:1912.00033.
- [7] P. Singh, “Quantum gravity, timelessness, and the Wheeler–DeWitt equation,” in *Loop Quantum Gravity: The First 30 Years*, A. Ashtekar and J. Pullin (eds.), World Scientific (2017). arXiv:1602.02643.

Appendix A: Reproducibility Notes

All numerical simulations in this work are performed in Python using the QuTiP (Quantum Toolbox in Python) library.

1. Version A (No Environment)

1. Define a finite-dimensional clock Hilbert space \mathcal{H}_C with N basis states $|k\rangle$.
2. Define the system Hamiltonian $\hat{H}_S = (\omega/2)\sigma_x$.

3. Construct the PaW history state:

$$|\Psi\rangle = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} |k\rangle_C \otimes e^{-i\hat{H}_S k dt} |\phi_0\rangle_S. \quad (\text{A1})$$

4. For each k , compute the conditional system state by projection onto $|k\rangle_C$ and normalization.
5. Evaluate observables such as $\langle \sigma_z \rangle_k$.

Expected result: coherent sinusoidal oscillations matching Schrödinger evolution.

2. Version B (With Environment)

1. Extend the Hilbert space by adding n_{env} environment qubits initialized in $|0\rangle^{\otimes n_{\text{env}}}$.
2. Define a system–environment interaction Hamiltonian, e.g.

$$\hat{H}_{SE} = g \sum_j \sigma_x^{(S)} \otimes \sigma_x^{(E_j)}. \quad (\text{A2})$$

3. Construct the joint propagator

$$U_{SE}(t) = e^{-i(\hat{H}_S + \hat{H}_E + \hat{H}_{SE})t} \quad (\text{A3})$$

using QuTiP’s exact matrix exponentiation.

4. Build the PaW history state correlating clock labels with $U_{SE}(k dt)$.
5. Condition on clock states, trace out the environment, and compute $\rho_S(k)$.
6. Evaluate $\langle \sigma_z \rangle_k$ and

$$S_{\text{eff}}(k) = -\text{Tr}[\rho_S(k) \ln \rho_S(k)]. \quad (\text{A4})$$

Observed behavior: damping of coherent oscillations and average growth of effective entropy, despite strictly unitary global dynamics.

Reference parameters: $N = 30$, $dt = 0.2$, $\omega = 1.0$, $g = 0.1$, $n_{\text{env}} \in \{2, 4, 6, 8\}$, $|\phi_0\rangle = |0\rangle$.

3. Robustness Tests

- **generate_gravity_robustness.py** — Three gravitational tests (back-reaction, fuzzy boundary, clock blur). Produces `output/gravity_robustness_curves.png`, `gravity_robustness_summary.png`, and `table_gravity_robustness.csv`.
- **generate_structural_robustness.py** — Three structural tests (Poincaré recurrences, initial state sensitivity, partition independence). Produces four PNG figures and three CSV tables in `output/`.
- **generate_condition_violations.py** — Five condition-necessity tests (high initial entropy, unstable partition, zero interaction, non-orthogonal clock, wrapping clock). Produces `output/condition_violations.png` and `table_condition_violations.csv`.
- **generate_covariance_theorem.py** — Clock Orientation Covariance Theorem and T -symmetry smoking gun. Produces `output/covariance_theorem_combined.png` and `table_covariance_theorem.csv`.
- **generate_angular_interpolation.py** — Angular interpolation of clock orientation. Produces four PNG figures and `table_angular_interpolation.csv` in `output/`.
- **generate_continuum_limit.py** — Continuous limit ($N \rightarrow \infty$), inter-clock transformations, and group structure. Produces five PNG figures (`continuum_limit_convergence.png`, `continuum_limit_overlay.png`, `clock_transformation_fidelity.png`, `group_structure_composition.png`,

`continuum_limit_combined.png`) and three CSV tables in `output/`.

4. IBM Quantum Hardware Validation

- **IBMQ/quantum/run_ibm_validation.py** — Trotterized 3-qubit experiment. Modes: `--mode simulator`, `--mode hardware`, or `--mode both`. Requires a Qiskit Runtime API key in `apikey.json`. Produces `IBMQ/quantum/output/ibm_quantum_validation.png` and `table_ibm_quantum_validation.csv`.
- **IBMQ/quantum/run_ibm_stability.py** — Stability verification on IBM Quantum circuits. Implements a Heisenberg $S-E$ model with tunable coupling η , measures purity deficit $\Delta = 1 - \text{Tr}(\rho^2)$ and mutual information across 15 values of η . Confirms $\Delta \sim \eta^{2.03}$ (theory: η^2). Produces `IBMQ/quantum/output/stability_*.png` and `table_stability_results.csv`.
- Hardware used: IBM `ibm_torino` (133 superconducting transmon qubits), accessed via Qiskit Runtime (`qiskit 2.3.0`, `qiskit-ibm-runtime 0.45.0`).

5. Stability and Uniqueness Proofs

- **stability/stability_uniqueness.tex** — Full mathematical proofs of the quadratic purity-deficit bound (Theorem 2.4), uniqueness of the interaction-minimal tensor product structure (Corollary 2.5), and a variational selection principle. Compiles to a self-contained 10-page document.
- **stability/verify_corrected.py** — Numerical verification of all stability theorems (perturbation expansion, purity deficit scaling, uniqueness, variational minimum).

The complete simulation code (Python/QuTiP), robustness test scripts, and IBM Quantum validation code, including the scripts that generate all figures and tables reported in this work, are publicly available at: <https://github.com/gabgian/paw-toymodel>