Lagrange Multipliers

Gabriela Georgieva
Angelina Kuznetsova
Fedor Shafranov

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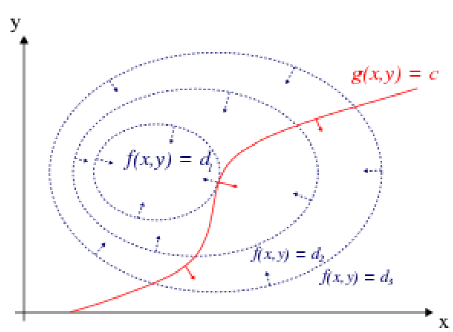
Abstract

Oftentimes one wants to find a minima or maxima of a (differentiable) function subject to one or more constraints. An elegant way to find an extremum is by using the so-called Lagrange Multipliers. Lagrange Multipliers are handy when solving optimization problems in Economics, Business, Computer Science, etc.



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1 Lagrange Multipliers

1.1 Introduction

Theorem 1. *(Implicit Function Theorem)*
Suppose that $F : \mathbb{R}^{n+1} \to \mathbb{R}$ is C^1. We will denote points in $\mathbb{R}^{n+1}$ by ($\boldsymbol{x}$,z), where $\boldsymbol{x} \in \mathbb{R}^n$ and $z \in \mathbb{R}$. Assume

$$F(\boldsymbol{x}_0, z_0) = c \qquad \text{and} \qquad \nabla F(\boldsymbol{x}_0, z_0) \neq \mathbf{0}$$

Then there is a ball U that contains $\boldsymbol{x}_0$ and a neighborhood V of z_0 in $\mathbb{R}$ such that there is a function $z = g(\boldsymbol{x})$ defined for $\boldsymbol{x}$ in U and z in V that satisfies

$$F(\boldsymbol{x}, g(\boldsymbol{x})) = c$$

Theorem 2. *(Method of Lagrange Multipliers)*
Suppose that $f : U \subset \mathbb{R}^n \to \mathbb{R}$ and $g : U \subset \mathbb{R}^n \to \mathbb{R}$ are C^1. Let $\boldsymbol{x}_0 \in U$ and $g(\boldsymbol{x}_0) = c$, and let S be the level set for g with value c (i.e these are the set of points $\boldsymbol{x} \in \mathbb{R}^n$ that satisfy $g(\boldsymbol{x}) = c$). Assume $\nabla g(\boldsymbol{x}_0) \neq \mathbf{0}$.
If f achieves a local extremum at $\boldsymbol{x}_0$, then there exists λ such that

$$\nabla f(\boldsymbol{x}_0) = \lambda \nabla g(\boldsymbol{x}_0)$$

Proof. For the sake of simplicity, take n=3. Then we are dealing with a level surface of the function $g(x, y, x) = c$ though the point (x_0, y_0, z_0). By the Implicit Function Theorem we know that there is a function $z = \phi(x, y)$ satisfying $g(x, y, \phi(x, y)) = c$ for (x, y) near (x_0, y_0) and z near z_0. It follows that locally (near z_0), the surface S is the graoh of the function ϕ. For ϕ differentiable and continuous, the tangent plane at (x_0, y_0, z_0) to S is given by:

$$z = z_0 + \left[\frac{\partial \phi}{\partial x}(x_0, y_0)\right](x - x_0) + \left[\frac{\partial \phi}{\partial y}(x_0, y_0)\right](y - y_0) \tag{1}$$

We can substitute

$$\frac{\partial \phi}{\partial x} = -\frac{g_x}{g_z} \qquad\qquad \frac{\partial \phi}{\partial y} = -\frac{g_y}{g_z}$$

in (1) and obtain:

$$(x - x_0)g_x + (y - y_0)g_y + (z - z_0)g_z = 0$$

$$(x - x_0, y - y_0, z - z_0) \cdot \nabla g(x_0, y_0, z_0) = 0 \tag{2}$$

At (x_0, y_0, z_0) the tangent planet to the level surface g is orthogonal to $\nabla g(x_0, y_0, z_0)$. Now we need to show that every vector tangent to S at (x_0, y_0, z_0) is tangent to every curve in S. If $\boldsymbol{v} = (x - x_0, y - y_0, z - z_0)$ is tangent to S, then (v) is tangent to every path in S given by

$$c(t) = (x_0 + t(x - x_0), y_0 + t(y - y_0), \phi(x_0 + t(x - x_0), y_0 + t(y - y_0)))$$

at $t = 0$. Now, if (x_0, y_0, z_0) is an extremum, then $f(c(t))$ is an extremum when $t = 0$ and $c'(0)$ is a tangent vector to S at (x_0, y_0, z_0)

$$\frac{d}{dt} f(c(t))\Big|_{t=0} = \nabla f(x_0) \cdot c'(0) = 0$$

Thus, $\nabla f(x_0)$ is orthogonal to every tangent vector to S at (x_0, y_0, z_0). Since the space orthogonal to this tangent space is a line, $\nabla f(x_0, y_0, z_0)$ and $\nabla g(x_0, y_0, z_0)$ are parallel and $\nabla f(x_0, y_0, z_0)$ is a multiple of $\nabla g(x_0, y_0, z_0)$

$$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0)$$

The multiple λ is called a Lagrange Multiplier. □

1.2 Single Constraint

Example 1. *Find the points closest to the origin on* $xy + 3x + z^2 = 9$

The distance from any point (x, y, z) to the origin can be expressed as $\sqrt{x^2 + y^2 + z^2}$. We want to minimize this distance subject to the constraint $xy + 3x + z^2 = 9$. We will denote the function we want to minimize as f. We will use the square of the distance formula as it does not change the result but makes the calculations simpler. We have

$$f(x, y, z) = x^2 + y^2 + z^2 \qquad\qquad g(x, y, z) = xy + 3x + z^2 - 9$$

The Method of Lagrange Multipliers tells us that at the critical points of a function

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z) \tag{1}$$

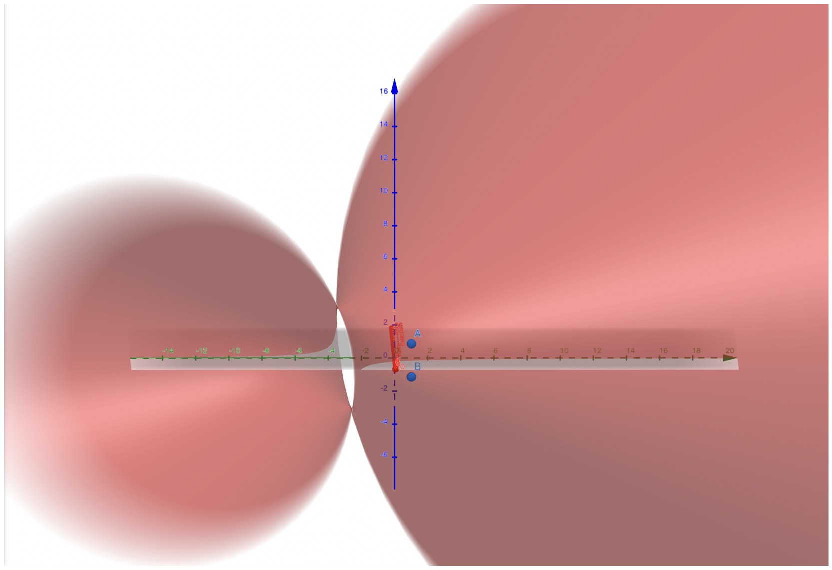
We find the gradient of f and g

$$\nabla f(x, y, x) =< 2x, 2y, 2z > \nabla g(x, y, x) =< y + 3, x, 2z >$$

After substituting in (1), we get a system of three equations and three variables. For the fourth equation we use the constraint.

$$\begin{cases} 2x =\lambda(y + 3) \\ 2y =\lambda x \\ 2z =\lambda 2z \\ xy{+}3x + z^2 = 9 \end{cases}$$

From the third equation we see that $\lambda = 1$ and substituting in the first and second we get $x = 2$ and $y = 1$. Finally, substituting in the last equation gives us $z = \pm 1$. So, the two critical points are $A = (2, 1, 1)$ and $B = (2, 1, -1)$ and the distance to the origin is $\sqrt{6}$. We know that at this points on f we will be closest to the origin because any other point we choose on f will give us a bigger distance.



Example 2. *Find the ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ *that passes through* $(3, 1)$ *and has the smallest area.*

The function we are striving to minimize is the area of an ellipse function $A = \pi ab$ that is subject to the constraint $\frac{9}{a^2} + \frac{1}{b^2} = 1$. We need $a, b > 0$. We will denote the function we want to minimize $f(a, b) = \pi ab$ and the constraint $g(a, b) = \frac{9}{a^2} + \frac{1}{b^2} - 1$. For f and g C^1, for the critical points the method of Lagrange Multipliers gives us

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$$

$$< \pi a, \pi b >= \lambda < -18a^{-3}, -2b^{-3} >$$

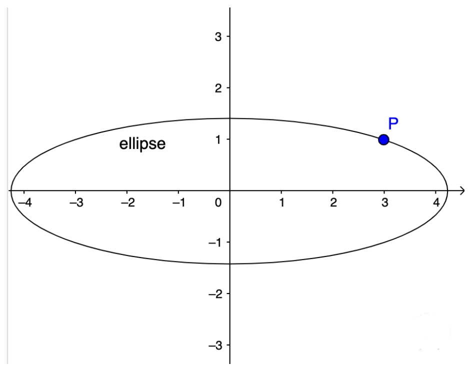
$$\begin{cases} \pi b = \lambda(-18)a^{-3} \\ \pi a = \lambda(-2)b^{-3} \\ \dfrac{9}{a^2} + \dfrac{1}{b^2} = 1 \end{cases}$$

Solving for λ in the first equation and substituting in the second we get

$$\begin{cases} \lambda = \dfrac{\pi b a^3}{-18} \\ a = \dfrac{a^3}{9b^2} \\ \dfrac{9}{a^2} + \dfrac{1}{b^2} = 1 \end{cases}$$

Solving the second equation gives us $a = 0$ and $9b^2 - a^2 = 0$. $a = 0$ is not in out domain. Solving for b gives us $b = \pm\frac{a}{3}$. Now, we solve the third equation

for a and obtain $a = \pm 3\sqrt{2}$.
For $a = \pm 3\sqrt{2}$ and $b = \pm\frac{a}{3}$ we get the following choices of (a, b): $(3\sqrt{2}, \sqrt{2})$, $(3\sqrt{2}, -\sqrt{2}), (-3\sqrt{2}, \sqrt{2}), (-3\sqrt{2}, -\sqrt{2})$. Since we are interested in $a, b > 0$ the only choice is $(3\sqrt{2}, \sqrt{2})$. The method of Lagrange Multipliers guarantees that the result we obtain is an extremum but does not tell us whether it is a minima or maxima. To find out, we pick another (a,b) that satisfies the constraint, for example $(\sqrt{\frac{27}{2}}, \sqrt{3})$. Since for this choice of (a,b) we get a bigger area, then we conclude that, indeed, for $(3\sqrt{2}, \sqrt{2})$ we achieve the smallest area considering the constraint.



1.3 Multiple Constraints

If we have n number of constraints, then the method of Lagrangian Multiplers tells us that if there is an extremum at point P on S, then there exist $\lambda_1, \lambda_2, ..., \lambda_n$ such that

$$\nabla f(x, y, z) = \lambda_1 \nabla g_1(x, y, z) + \lambda_2 \nabla g_2(x, y, z) + ... + \lambda_n \nabla g_n(x, y, z)$$

Example 3. *Find the minimum distance from the origin to the line of intersection of the two planes $x + y + z = 8$ and $2x - y + 3z = 28$*

We will again use the squared distance formula, so we set $f(x, y, x) = x^2 + y^2 + z^2$. We write the two constraints in the form $g(x, y, z) = x + y + z - 8$ and $h(x, y, z) = 2x - y + 3z - 28$. Then we have

$$\nabla f(x, y, z) = \lambda_1 \nabla g(x, y, z) + \lambda_2 \nabla h(x, y, z)$$

$$< 2x, 2y, 2z >= \lambda_1 < 1, 1, 1 > +\lambda_2 < 2, -1, 3 >$$

To find the point on the line closest to the origin we need to solve the system

$$\begin{cases} 2x = & \lambda_1 + 2\lambda_2 \\ 2y = & \lambda_1 - \lambda_2 \\ 2z = & \lambda_1 + 3\lambda_2 \\ x + y + z = 8 \\ 2x - y + 3z = 28 \end{cases}$$

From the first three equations we get $x = \frac{\lambda_1+2\lambda_2}{2}$, $y = \frac{\lambda_1-\lambda_2}{2}$ and $z = \frac{\lambda_1+3\lambda_2}{2}$. After substituing in the 4th and 5th equation we get $\lambda_1 = 0$ and $\lambda_2 = 4$. After substituting back for x, y, z we get the point $(4, -2, 6)$. Thus, the distance from $(4, -2, 6)$ to the origin is $\sqrt{56}$. We know that the point we got is closest to the origin because $max(\sqrt{x^2 + y^2 + z^2}) \to \infty$.

1.4 Second Derivative Test

1.5 Lagrangian

The idea behind the Lagnagian is to reduce constrained optimization to unconstrained optimization, and to take the (functional) constraints into account by augmenting the objective function with a weighted sum of them. If we have a function $f(x)$ and a constraint $g(x) = x$ the Lagrangian is defined as

$$L(x, \lambda) = f(x) - \lambda(g(x) - c)$$

2 Applications

2.1 Utility Maximization

Lagrange Multipliers are widely used in microeconomic theory to find the optimal consumption of production bundles under a certain constraint. Even though the examples are trivial, we see how basic understanding of Lagrange Multipliers can help solving Economics problems for optimization.

Let us take a utility function that will represent a utility for consuming a bundle of goods where x_1 and x_2 are some goods.

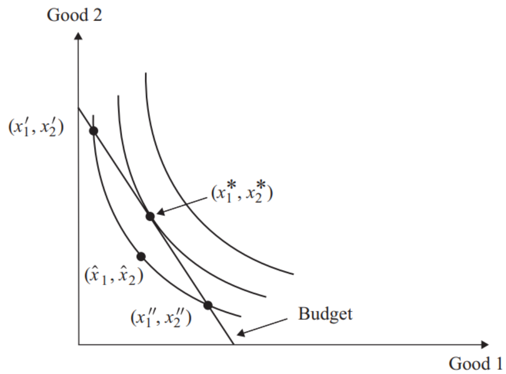
$$u = u(x_1, x_2)$$

The budget of the agent is limited, so we need to introduce a budget constraint. Where x_1, x_2 are goods and p_1, p_2 is their price

$$p_1x_1 + p_2x_2 = B$$

Now, we need to find an optimum point of consumption that lies on the edge of the intersection between budget and utility.

Here we introduce a Lagrangian multiplier to find a point where gradients are parallel.



Maximize

$$L = U(x_1, x_2) + \lambda[B - p_1 x_1 - p_2 x_2]$$

FOC:

$$\frac{\partial L}{\partial x_1} = 0 \implies \frac{\partial U}{\partial x_1} - \lambda p_1 = 0 \implies \frac{\partial U}{\partial x_1} = \lambda p_1 \tag{2}$$

$$\frac{\partial L}{\partial x_2} = 0 \implies \frac{\partial U}{\partial x_2} - \lambda p_2 = 0 \implies \frac{\partial U}{\partial x_2} = \lambda p_2 \tag{3}$$

$$\frac{\partial L}{\partial \lambda} = 0 \implies B = p_1 x_1 - p_2 x_2 \tag{4}$$

In Economics, a corresponding change in utility U over an increase in x_i is called marginal utility MU_i. Now, finishing calculations, we see

$$\frac{\frac{\partial U}{\partial x_1}}{\frac{\partial U}{\partial x_2}} = \frac{p_1}{p_2} \implies \frac{MU_1}{p_1} = \frac{MU_2}{p_2} \implies \frac{MU_1}{MU_2} = \frac{p_1}{p_2} \tag{5}$$

And we can see that it is exactly the point where the constraint touches the curve.

2.2 Cost Minimization

The next example is similar, but now we view the production decision of a firm that minimizes its costs. A firm fixes its output level, making it a constraint, and chooses a budget at which to produce.

$$L = w_1 x_1 + w_2 x_2 + \lambda[y - f(x, y)]$$

FOC:

$$w_1 - \lambda \frac{\partial f}{\partial x_1} = 0 \tag{6}$$

$$w_2 - \lambda \frac{\partial f}{\partial x_2} = 0 \tag{7}$$

$$\frac{w_1}{w_2} = \frac{\frac{\partial f}{\partial x_1}}{\frac{\partial f}{\partial x_2}} \tag{8}$$