Lagrange Multipliers

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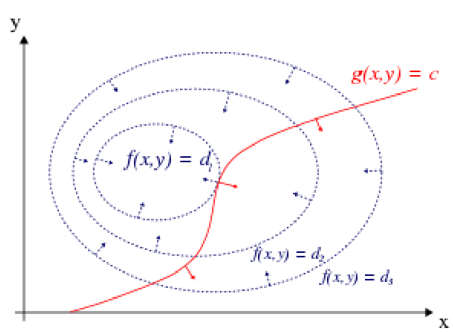
Abstract

Oftentimes one wants to find a minima or maxima of a (differentiable) function subject to one or more constraints. An elegant way to find an extremum is by using the so-called Lagrange Multipliers. Lagrange Multipliers are handy when solving optimization problems in Economics, Business, Computer Science, etc.



Contents

1 Lagrange Multipliers

1.1 Introduction

Theorem 1. *(Implicit Function Theorem)*
Suppose that $F : \mathbb{R}^{n+1} \to \mathbb{R}$ is C^1. We will denote points in $\mathbb{R}^{n+1}$ by ($\boldsymbol{x}$,z), where $\boldsymbol{x} \in \mathbb{R}^n$ and $z \in \mathbb{R}$. Assume

$$F(\boldsymbol{x}_0, z_0) = c \qquad \text{and} \qquad \nabla F(\boldsymbol{x}_0, z_0) \neq \mathbf{0}$$

Then there is a ball U that contains $\boldsymbol{x}_0$ and a neighborhood V of z_0 in $\mathbb{R}$ such that there is a function $z = g(\boldsymbol{x})$ defined for $\boldsymbol{x}$ in U and z in V that satisfies

$$F(\boldsymbol{x}, g(\boldsymbol{x})) = c$$

Theorem 2. *(Method of Lagrange Multipliers)*
Suppose that $f : U \subset \mathbb{R}^n \to \mathbb{R}$ and $g : U \subset \mathbb{R}^n \to \mathbb{R}$ are C^1. Let $\boldsymbol{x}_0 \in U$ and $g(\boldsymbol{x}_0) = c$, and let S be the level set for g with value c (i.e these are the set of points $\boldsymbol{x} \in \mathbb{R}^n$ that satisfy $g(\boldsymbol{x}) = c$). Assume $\nabla g(\boldsymbol{x}_0) \neq \mathbf{0}$.
If f achieves a local extremum at $\boldsymbol{x}_0$, then there exists λ such that

$$\nabla f(\boldsymbol{x}_0) = \lambda \nabla g(\boldsymbol{x}_0)$$

Proof. For the sake of simplicity, take n=3. Then we are dealing with a level surface of the function $g(x, y, x) = c$ though the point (x_0, y_0, z_0). By the Implicit Function Theorem we know that there is a function $z = \phi(x, y)$ satisfying $g(x, y, \phi(x, y)) = c$ for (x, y) near (x_0, y_0) and z near z_0. It follows that locally (near z_0), the surface S is the graoh of the function ϕ. For ϕ differentiable and continuous, the tangent plane at (x_0, y_0, z_0) to S is given by:

$$z = z_0 + \left[\frac{\partial \phi}{\partial x}(x_0, y_0)\right](x - x_0) + \left[\frac{\partial \phi}{\partial y}(x_0, y_0)\right](y - y_0) \tag{1}$$

We can substitute

$$\frac{\partial \phi}{\partial x} = -\frac{g_x}{g_z} \qquad\qquad \frac{\partial \phi}{\partial y} = -\frac{g_y}{g_z}$$

in (1) and obtain:

$$(x - x_0)g_x + (y - y_0)g_y + (z - z_0)g_z = 0$$

$$(x - x_0, y - y_0, z - z_0) \cdot \nabla g(x_0, y_0, z_0) = 0 \tag{2}$$

At (x_0, y_0, z_0) the tangent planet to the level surface g is orthogonal to $\nabla g(x_0, y_0, z_0)$. Now we need to show that every vector tangent to S at (x_0, y_0, z_0) is tangent to every curve in S. If $\boldsymbol{v} = (x - x_0, y - y_0, z - z_0)$ is tangent to S, then (v) is tangent to every path in S given by

$$c(t) = (x_0 + t(x - x_0), y_0 + t(y - y_0), \phi(x_0 + t(x - x_0), y_0 + t(y - y_0)))$$

at $t = 0$. Now, if (x_0, y_0, z_0) is an extremum, then $f(c(t))$ is an extremum when $t = 0$ and $c'(0)$ is a tangent vector to S at (x_0, y_0, z_0)

$$\frac{d}{dt} f(c(t))\bigg|_{t=0} = \nabla f(x_0) \cdot c'(0) = 0$$

Thus, $\nabla f(x_0)$ is orthogonal to every tangent vector to S at (x_0, y_0, z_0). Since the space orthogonal to this tangent space is a line, $\nabla f(x_0, y_0, z_0)$ and $\nabla g(x_0, y_0, z_0)$ are parallel and $\nabla f(x_0, y_0, z_0)$ is a multiple of $\nabla g(x_0, y_0, z_0)$

$$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0)$$

The multiple λ is called a Lagrange Multiplier. □

1.2 Single Constraint

Example 1. *Find the points closest to the origin on* $xy + 3x + z^2 = 9$

The distance from any point (x, y, z) to the origin can be expressed as $\sqrt{x^2 + y^2 + z^2}$. We want to minimize this distance subject to the constraint $xy + 3x + z^2 = 9$. We will denote the function we want to minimize as f. We will use the square of the distance formula as it does not change the result but makes the calculations simpler. We have

$$f(x, y, z) = x^2 + y^2 + z^2 \qquad g(x, y, z) = xy + 3x + z^2 - 9$$

The Method of Lagrange Multipliers tells us that at the critical points of a function

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z) \tag{1}$$

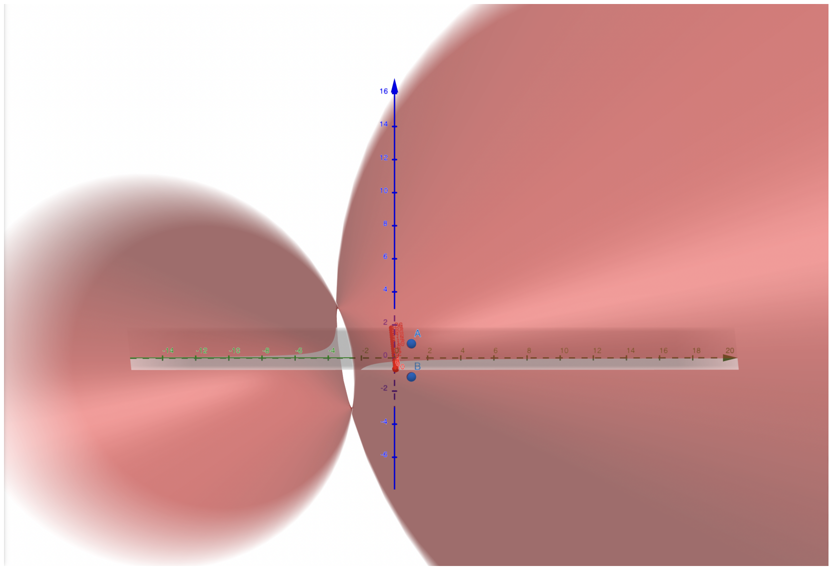
We find the gradient of f and g

$$\nabla f(x, y, x) =< 2x, 2y, 2z > \nabla g(x, y, x) =< y + 3, x, 2z >$$

After substituting in (1), we get a system of three equations and three variables. For the fourth equation we use the constraint.

$$\begin{cases} 2x = \lambda(y + 3) \\ 2y = \lambda x \\ 2z = \lambda 2z \\ xy + 3x + z^2 = 9 \end{cases}$$

From the third equation we see that $\lambda = 1$ and substituting in the first and second we get $x = 2$ and $y = 1$. Finally, substituting in the last equation gives us $z = \pm 1$. So, the two critical points are $A = (2, 1, 1)$ and $B = (2, 1, -1)$ and the distance to the origin is $\sqrt{6}$. We know that at this points on f we will be closest to the origin because any other point we choose on f will give us a bigger distance.



Example 2. *Find the ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ *that passes through* $(3, 1)$ *and has the smallest area.*

The function we are striving to minimize is the area of an ellipse function $A = \pi ab$ that is subject to the constraint $\frac{9}{a^2} + \frac{1}{b^2} = 1$. We need $a, b > 0$. We will denote the function we want to minimize $f(a, b) = \pi ab$ and the constraint $g(a, b) = \frac{9}{a^2} + \frac{1}{b^2} - 1$. For f and g C^1, for the critical points the method of Lagrange Multipliers gives us

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$$

$$< \pi a, \pi b >= \lambda < -18a^{-3}, -2b^{-3} >$$

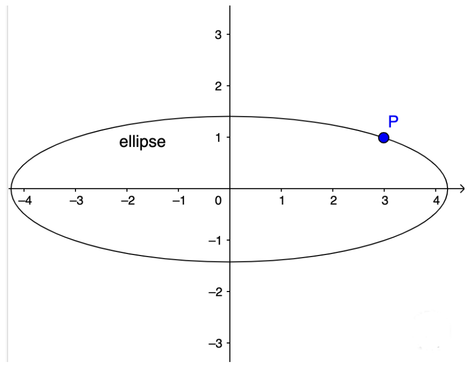
$$\begin{cases} \pi b = \lambda(-18)a^{-3} \\ \pi a = \lambda(-2)b^{-3} \\ \dfrac{9}{a^2} + \dfrac{1}{b^2} = 1 \end{cases}$$

Solving for λ in the first equation and substituting in the second we get

$$\begin{cases} \lambda = \dfrac{\pi b a^3}{-18} \\ a = \dfrac{a^3}{9b^2} \\ \dfrac{9}{a^2} + \dfrac{1}{b^2} = 1 \end{cases}$$

Solving the second equation gives us $a = 0$ and $9b^2 - a^2 = 0$. $a = 0$ is not in out domain. Solving for b gives us $b = \pm\frac{a}{3}$. Now, we solve the third equation

for a and obtain $a = \pm 3\sqrt{2}$.
For $a = \pm 3\sqrt{2}$ and $b = \pm \frac{a}{3}$ we get the following choices of (a, b): $(3\sqrt{2}, \sqrt{2})$, $(3\sqrt{2}, -\sqrt{2})$, $(-3\sqrt{2}, \sqrt{2})$, $(-3\sqrt{2}, -$
Since we are interested in $a, b > 0$ the only choice is $(3\sqrt{2}, \sqrt{2})$.



1.3 Multiple Constraints

1.4 Second Derivative Test

1.5 Lagrangian

2 Applications