Lagrange Multipliers

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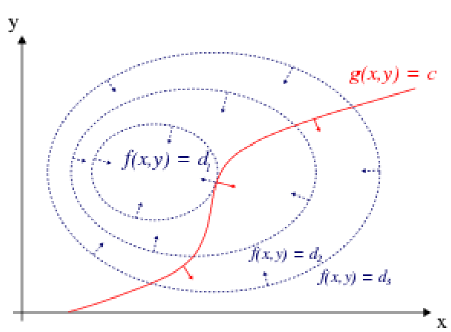
Abstract

Oftentimes one wants to find a minima or maxima of a (differentiable) function subject to one or more constraints. An elegant way to find an extremum is by using the so-called Lagrange Multipliers. Lagrange Multipliers are handy when solving optimization problems in Economics, Business, Computer Science, etc.



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1.1 Introduction

Theorem 1. *(Implicit Function Theorem)*
Suppose that $F : \mathbb{R}^{n+1} \to \mathbb{R}$ *is* C^1*. We will denote points in* $\mathbb{R}^{n+1}$ *by (*$\boldsymbol{x}$*,*z*), where* $\boldsymbol{x} \in \mathbb{R}^n$ *and* $z \in \mathbb{R}$*. Assume*

$$F(\boldsymbol{x}_0, z_0) = c \qquad \textit{and} \qquad \nabla F(\boldsymbol{x}_0, z_0) \neq \mathbf{0}$$

Then there is a ball U *that contains* $\boldsymbol{x}_0$ *and a neighborhood* V *of* z_0 *in* $\mathbb{R}$ *such that there is a function* $z = g(\boldsymbol{x})$ *defined for* $\boldsymbol{x}$ *in* U *and* z *in* V *that satisfies*

$$F(\boldsymbol{x}, g(\boldsymbol{x})) = c$$

Theorem 2. *(Method of Lagrange Multipliers)*
Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ *and* $g : U \subset \mathbb{R}^n \to \mathbb{R}$ *are* C^1*. Let* $\boldsymbol{x}_0 \in U$ *and* $g(\boldsymbol{x}_0) = c$*, and let* S *be the level set for* g *with value* c *(i.e these are the set of points* $\boldsymbol{x} \in \mathbb{R}^n$ *that satisfy* $g(\boldsymbol{x}) = c$*). Assume* $\nabla g(\boldsymbol{x}_0) \neq \mathbf{0}$*.*
If $(\boldsymbol{x}_0)$ *is a local extremum, then there exists* λ *such that*

$$\nabla f(\boldsymbol{x}_0) = \lambda \nabla g(\boldsymbol{x}_0)$$