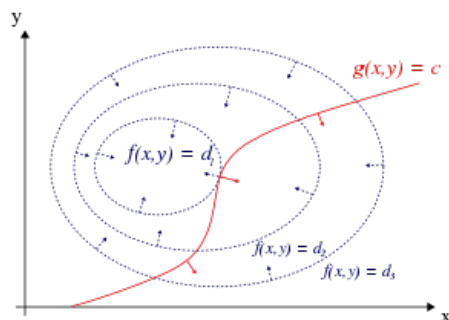# Lagrange Multipliers

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Abstract

Oftentimes one wants to find a minima or maxima of a (differentiable) function subject to one or more constraints. An elegant way to find an extremum is by using the so-called Lagrange Multipliers. Lagrange Multipliers are handy when solving optimization problems in Economics, Business, Computer Science, etc.

Contents

1 Lagrange Multipliers

1.1 Introduction

Theorem 1. *(Implicit Function Theorem)*
Suppose that $F : \mathbb{R}^{n+1} \to \mathbb{R}$ is C^1. We will denote points in $\mathbb{R}^{n+1}$ by $(\boldsymbol{x}, z)$, where $\boldsymbol{x} \in \mathbb{R}^n$ and $z \in \mathbb{R}$. Assume

$$F(\boldsymbol{x}_0, z_0) = c \qquad and \qquad \nabla F(\boldsymbol{x}_0, z_0) \neq \boldsymbol{0}$$

Then there is a ball U that contains $\boldsymbol{x}_0$ and a neighborhood V of z_0 in $\mathbb{R}$ such that there is a function $z = g(\boldsymbol{x})$ defined for $\boldsymbol{x}$ in U and z in V that satisfies

$$F(\boldsymbol{x}, g(\boldsymbol{x})) = c$$

Theorem 2. *(Method of Lagrange Multipliers)*
Suppose that $f : U \subset \mathbb{R}^n \to \mathbb{R}$ and $g : U \subset \mathbb{R}^n \to \mathbb{R}$ are C^1. Let $\boldsymbol{x}_0 \in U$ and $g(\boldsymbol{x}_0) = c$, and let S be the level set for g with value c (i.e these are the set of points $\boldsymbol{x} \in \mathbb{R}^n$ that satisfy $g(\boldsymbol{x}) = c$). Assume $\nabla g(\boldsymbol{x}_0) \neq \boldsymbol{0}$.
If f achieves a local extremum at $\boldsymbol{x}_0$, then there exists λ such that

$$\nabla f(\boldsymbol{x}_0) = \lambda \nabla g(\boldsymbol{x}_0)$$

Proof. For the sake of simplicity, take n=3. Then we are dealing with a level surface of the function $g(x, y, x) = c$ though the point (x_0, y_0, z_0). By the Implicit Function Theorem we know that there is a function $z = \phi(x, y)$

□

1.2 Single Constraint

1.3 Multiple Constraints

1.4 Second Derivative Test

1.5 Lagrangian

2 Applications