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Numerical solutions of the generalized Kuramoto–Sivashinsky equation by Chebyshev spectral collocation methods

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ABSTRACT

Chebyshev spectral collocation methods (known as El-Gendi method [S.E. El-Gendi, Chebyshev solution of differential integral and integro-differential equations, Comput. J. 12 (1969) 282–287; B. Mihaila, I. Mihaila, Numerical approximation using Chebyshev polynomial expansions: El-gendi's method revisited, J. Phys. A 35 (2002) 731–746]) are extended to deal with the generalized Kuramoto–Sivashinsky equation. The problem is reduced to a system of ordinary differential equations that are solved by combinations of backward differential formula and appropriate explicit schemes (implicit–explicit BDF methods [G. Akrivis, Y.S. Smyrlis, Implicit–explicit BDF methods for the Kuramoto–Sivashinsky equation, Appl. Numer. Math. 51 (2004) 151–169]). Good numerical results have been obtained and compared with the exact solutions.

1. Introduction

The generalized Kuramoto–Sivashinsky (GKS) equation is a model of nonlinear partial differential equation (NLPDE) frequently encountered in the study of continuous media which exhibits a chaotic behavior form

$$u_t + uu_x + \alpha u_{xx} + \beta u_{xxx} + \gamma u_{xxxx} = 0, \tag{1}$$

where α, β and γ are nonzero constants.

For $\beta = 0$, Eq. (1) is called the Kuramoto–Sivashinsky (KS) equation which is a canonical nonlinear evolution equation arising in a variety of physical contexts, e.g. long waves on thin films, long waves on the interface between two viscous fluids [4], unstable drift waves in plasmas, reaction diffusion systems [5] and flame front instability [6]. For $\alpha = \gamma = 1$ and $\beta = 0$ it represents models of pattern formation on unstable flame fronts and thin hydrodynamic films [6], Eq. (1) has thus been studied extensively [7,8].

The KS equation has been studied numerically by many authors, see [9–17], while the space discretization has been consistently carried out via spectral methods, many different methods have been used for time discretization including Runge–Kutta methods of different orders, a split scheme of variable time, according to the Strang-split method [16,17] and implicit–explicit methods [18–20].

In recent years various methods have been presented to construct exact solutions of KS equation. For more details about these methods, we refer to [21,22]. Also many methods were developed for finding exact solutions of some nonlinear evolution equations. For more details about these methods, we refer to [23–27].

In this paper, we approximate the solution of GKS equation (1) in the region $-1 \leq x \leq 1, t \geq 0$ with $u(x, 0)$, $u(-1, t)$, $u_x(-1, t)$, $u_{xx}(-1, t)$ and $u_{xxx}(-1, t)$ prescribed, by discretizing in time through combinations of backward differential

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formula (BDF) and appropriate explicit schemes (implicit–explicit BDF methods [3]) leading to unconditionally stable, linearly implicit schemes and in space by Chebyshev spectral collocation methods [1,2].

Now, we present the following solitary wave solutions of GKS equation (1) computed by tanh-function method [21,22] for special values of α, β and γ.

(i) For $\alpha=\gamma=1$ and $\beta=4$

$$u(x,t)=9\pm 2c\pm 15\tanh\xi-15\tanh^2\xi\mp 15\tanh^3\xi \tag{2}$$

with $\xi=\mp(1/2)x+ct$.

(ii) For $\alpha=\gamma=1$ and $\beta=12/\sqrt{47}$

$$u(x,t)=\frac{45\mp 4418c}{47\sqrt{47}}+\frac{15}{47\sqrt{47}}[\pm 3\tanh\xi-3\tanh^2\xi\pm\tanh^3\xi], \tag{3}$$

where $\xi=\pm(1/2\sqrt{47})x+ct$.

(iii) For $\alpha=\gamma=1$ and $\beta=16/\sqrt{73}$

$$u(x,t)=\frac{2(30\mp 5329c)}{73\sqrt{73}}+\frac{15}{73\sqrt{73}}[\pm 5\tanh\xi-4\tanh^2\xi\pm\tanh^3\xi], \tag{4}$$

where $\xi=\pm(1/2\sqrt{73})x+ct$.

(iv) For $\beta=0$

$$u(x,t)=-\frac{c}{k}+\frac{60}{19}k(-38\gamma k^2+\alpha)\tanh\xi+120\gamma k^3\tanh^3\xi, \tag{5}$$

where $\xi=kx+ct$ and $k=(1/2)\sqrt{11\alpha/19\gamma}$.

In all solutions above c is an arbitrary parameter.

The present method is accomplished through starting with Chebyshev approximation for the highest-order derivative and generating approximations for the lower-order derivatives through successive integration of the highest-order derivative.

The paper is organized as follows: In Section 2 we describe numerical solutions of GKS equation, by discretizing in time by p-step implicit–explicit BDF scheme and in space by Chebyshev spectral collocation methods. The numerical results are presented in Section 3 and Section 4 gives the conclusions.

2. Numerical solutions of the GKS equation

2.1. Discretization in space

Consider Eq. (1), the highest-order derivative of the solution $u(x,t)$ can be approximated in the interval $[-1,1]$ by either one of the two formulas

$$u_{xxxx}=\sum_{j=0}^{N}{}'' a_j T_j(x) \tag{6}$$

or

$$u_{xxxx}=\sum_{j=0}^{N-1}{}' b_j T_j(x), \tag{7}$$

where $T_j(x)$ is the jth Chebyshev polynomial. A summation symbol with one prime denotes a sum with the first term halved and the summation symbol with double primes denotes a sum with the first and last term halved. The approximate formulas (6) and (7) are exact at x equal to x_j given by the extrema of the Chebyshev polynomial of order N, see Ref. [1]

$$x_j=\cos\left(\frac{\pi j}{N}\right),\quad j=0,1,\ldots,N \tag{8}$$

and at x equal to $\bar{x}_j$ given by the zeros of the Chebyshev polynomial of order N, see Ref. [2]

$$\bar{x}_j=\cos\left(\frac{\pi(j-1/2)}{N}\right),\quad j=1,2,\ldots,N \tag{9}$$

respectively.

Let $\phi(x,t) = u_{xxxx}$, and by successive integration, we get

$$u_{xxx} = \int_{-1}^{x} \phi \mathrm{d}x + c_1 \tag{10}$$

$$u_{xx} = \int_{-1}^{x}\int_{-1}^{x} \phi \mathrm{d}x\mathrm{d}x + c_1(1+x) + c_2 \tag{11}$$

$$u_x = \int_{-1}^{x}\int_{-1}^{x}\int_{-1}^{x} \phi \mathrm{d}x\mathrm{d}x\mathrm{d}x + \frac{1}{2}c_1(1+x)^2 + (1+x)c_2 + c_3 \tag{12}$$

$$u = \int_{-1}^{x}\int_{-1}^{x}\int_{-1}^{x}\int_{-1}^{x} \phi \mathrm{d}x\mathrm{d}x\mathrm{d}x\mathrm{d}x + \frac{1}{6}(1+x)^3 c_1 + \frac{1}{2}(1+x)^2 c_2 + (1+x)c_3 + c_4, \tag{13}$$

where $c_1 = u_{xxx}(-1,t)$, $c_2 = u_{xx}(-1,t)$, $c_3 = u_x(-1,t)$ and $c_4 = u(-1,t)$.

Now, we give approximations to the integrals (10)–(13), based on the Chebyshev expansions (6) and (7) in the matrix format as follows

$$U_{xxx} = B^{(1)}\Phi + F^{(1)} \tag{14}$$

$$U_{xx} = B^{(2)}\Phi + F^{(2)} \tag{15}$$

$$U_x = B^{(3)}\Phi + F^{(3)} \tag{16}$$

$$U = B^{(4)}\Phi + F^{(4)} \tag{17}$$

$$F_i^{(k)} = \sum_{j=1}^{k} \frac{(1+x_i)^{k-j}}{(k-j)!} \frac{\partial^{(4-j)}}{\partial x^{(4-j)}} u(x,t)\,|_{x=-1}, \quad i = 0, \ldots, N \tag{18}$$

$$B_{ij}^{(k)} = \frac{(x_i - x_j)^{k-1}}{(k-1)!} B_{ij}, \quad i, j = 0, 1, \ldots, N, k = 1(1)4, \tag{19}$$

also $B^{(k)}$ can be written in the following form

$$B^{(k)} = B^k, \quad k = 1(1)4, \tag{20}$$

for the case of grid (8) and

$$F_i^{(k)} = \sum_{j=1}^{k} \frac{(1+\bar{x}_i)^{k-j}}{(k-j)!} \frac{\partial^{(4-j)}}{\partial x^{(4-j)}} u(x,t)\,|_{x=-1}, \quad i = 1(1)N \tag{21}$$

$$B_{ij}^{(k)} = \frac{(\bar{x}_i - \bar{x}_j)^{k-1}}{(k-1)!} S_{ij}, \quad i, j = 1, 2, \ldots, N, k = 1(1)4 \tag{22}$$

or

$$B^{(k)} = S^k, \quad k = 1(1)4 \tag{23}$$

for the case of grid (9), respectively. The elements of the column matrices Φ, Φ_t, U, U_x, U_{xx}, U_{xxx} and U_{xxxx} are given by either $\phi(x_i,t)$, $\phi_t(x_i,t)$, $u(x_i,t)$, $u_x(x_i,t)$, $u_{xx}(x_i,t)$ and $u_{xxx}(x_i,t)$ or $\phi(\bar{x}_i,t)$, $\phi_t(\bar{x}_i,t)$, $u(\bar{x}_i,t)$, $u_x(\bar{x}_i,t)$, $u_{xx}(\bar{x}_i,t)$, and $u_{xxx}(\bar{x}_i,t)$. The values of the matrix elements B_{ij} and S_{ij} are given in [1,2], respectively.

Now the GKS equation (1) can be transformed to the following system of ordinary differential equations (ODEs)

$$D\dot{\Phi} + C(\Phi) + A\Phi + F = 0, \tag{24}$$

where $D = B^{(4)}$, $A = \alpha B^{(2)} + \beta B^{(1)} + \gamma I$, $C(\Phi) = [u]B^{(3)}\Phi$, $. = \mathrm{d}/\mathrm{d}t$, I is unit matrix, [ ] denotes a diagonal matrix and

$$F = \dot{F}^{(4)} + [u]F^{(3)} + \alpha F^{(2)} + \beta F^{(1)}. \tag{25}$$

2.2. Discretization in time

For $p = 1, \ldots, 6$, the polynomials $\bar{\alpha}$, $\bar{\beta}$ and $\bar{\gamma}$ are given by

$$\bar{\alpha}(\eta) := \sum_{j=1}^{p} \frac{1}{j}\eta^{p-j}(\eta-1)^j, \; \bar{\beta}(\eta) := \eta^p \quad \text{and} \quad \bar{\gamma}(\eta) := \eta^p - (\eta-1)^p.$$

Let $\bar{\alpha}_i$ and $\bar{\gamma}_i$ denote the coefficients of η^i, of the polynomials $\bar{\alpha}$ and $\bar{\gamma}$ respectively. The $(\bar{\alpha}, \bar{\beta})$-scheme described by the polynomials $\bar{\alpha}$ and $\bar{\beta}$ is the p-step BDF scheme ; these schemes will be used for the discretization of the linear part of (24). The explicit scheme $(\bar{\alpha}, \bar{\gamma})$ will be used for the discretization of the nonlinear part of (24). Let us note that this particular choice of the polynomial $\bar{\gamma}$ is motivated by the fact that, for given $(\bar{\alpha}, \bar{\beta})$-scheme, it is the only choice leading to a p-step implicit–explicit $(\bar{\alpha}, \bar{\beta}, \bar{\gamma})$-scheme of order p, see Ref. [18–20]

Let $T > 0$, Δt denote the time step, L is a positive integer such that $L\Delta t = T$, and $t^n := n\Delta t$, $n = 0, \ldots, L$. We use the $(\bar{\alpha}, \bar{\beta}, \bar{\gamma})$-scheme to define the approximations Φ^n by

$$D\sum_{i=0}^{p}\bar{\alpha}_i\Phi^{n+i} + \Delta tA\Phi^{n+p} + \Delta tF^{n+p} + \Delta t\sum_{i=0}^{p-1}\bar{\gamma}_iC(\Phi^{n+i}) = 0, \quad n = 0, \ldots, L-p. \tag{26}$$

for given starting approximations $\Phi^0, \ldots, \Phi^{p-1}$. Since $\bar{\alpha}_p > 0$, the approximations $\Phi^p, \ldots, \Phi^L$ are well defined by (26).

For $p = 1, \ldots, 6$, the scheme (26) takes the following forms, respectively,

$$D(\Phi^{n+1} - \Phi^n) + \Delta tA\Phi^{n+1} + \Delta tF^{n+1} + \Delta tC(\Phi^n) = 0, \tag{27}$$

$$D\left(\frac{3}{2}\Phi^{n+2} - 2\Phi^{n+1} + \frac{1}{2}\Phi^n\right) + \Delta tA\Phi^{n+2} + \Delta tF^{n+2} + \Delta t(2C(\Phi^{n+1}) - C(\Phi^n)) = 0, \tag{28}$$

$$D\left(\frac{11}{6}\Phi^{n+3} - 3\Phi^{n+2} + \frac{3}{2}\Phi^{n+1} - \frac{1}{3}\Phi^n\right) + \Delta tA\Phi^{n+3} + \Delta tF^{n+3} + \Delta t(3C(\Phi^{n+2}) - 3C(\Phi^{n+1}) + C(\Phi^n)) = 0, \tag{29}$$

$$D\left(\frac{25}{12}\Phi^{n+4} - 4\Phi^{n+3} + 3\Phi^{n+2} - \frac{4}{3}\Phi^{n+1} + \frac{1}{4}\Phi^n\right) + \Delta tA\Phi^{n+4} + \Delta tF^{n+4} + \Delta t(4C(\Phi^{n+3}) - 6C(\Phi^{n+2}) + 4C(\Phi^{n+1}) - C(\Phi^n)) = 0, \tag{30}$$

$$D\left(\frac{137}{60}\Phi^{n+5} - 5\Phi^{n+4} + 5\Phi^{n+3} - \frac{10}{3}\Phi^{n+2} + \frac{5}{4}\Phi^{n+1} - \frac{1}{5}\Phi^n\right) + \Delta tA\Phi^{n+5} + \Delta tF^{n+5} + \Delta t(5C(\Phi^{n+4}) - 10C(\Phi^{n+3}) + 10C(\Phi^{n+2}) - 5C(\Phi^{n+1}) + C(\Phi^n)) = 0, \tag{31}$$

$$D\left(\frac{147}{60}\Phi^{n+6} - 6\Phi^{n+5} + \frac{15}{2}\Phi^{n+4} - \frac{20}{3}\Phi^{n+3} + \frac{15}{4}\Phi^{n+2} - \frac{6}{5}\Phi^{n+1} + \frac{1}{6}\Phi^n\right) + \Delta tA\Phi^{n+6} + \Delta tF^{n+6} + \Delta t(6C(\Phi^{n+5}) - 15C(\Phi^{n+4}) + 20C(\Phi^{n+3}) - 15C(\Phi^{n+2}) + 6C(\Phi^{n+1}) - C(\Phi^n)) = 0. \tag{32}$$

Scheme (27) is obviously a combination of the implicit and the forward Euler methods.

Starting approximations. To maintain the order of accuracy of the p-step scheme, starting approximations $\Phi^0, \ldots, \Phi^{p-1}$, for $p = 2, \ldots, 6$, of the same order are required. We present here some choices leading to such approximations.

The first choice is based on linearly implicit Runge–Kutta schemes of order at least $p - 1$ for the computation of $\Phi^1, \ldots, \Phi^{p-2}$, see Refs. [28,29]; Φ^{p-1} can be computed by the $(p-1)$-step implicit–explicit scheme. We note that schemes of order $p - 1$, when applied a fixed number of steps, yield approximations of order p.

Let us also mention that a further, suitable and popular approach is to bootstrap by starting with low-order schemes but with sufficiently small time steps such that the overall accuracy is not diminished. As already mentioned, for the second-order scheme (28), we may compute Φ^1 by performing one step by the first-order scheme (27), i.e., we let Φ^1 be given by

$$D(\Phi^1 - \Phi^0) + \Delta tA\Phi^1 + \Delta tF^1 + \Delta tC(\Phi^0) = 0. \tag{33}$$

Let us also briefly discuss one convenient way leading to appropriate starting approximations for the third-order scheme. In this case, we need third-order starting approximations Φ^1 and Φ^2. Once Φ^1 has been calculated, we may perform one step with the second-order scheme (28) to get Φ^2. For Φ^1, we begin with a second-order approximation $\bar{\Phi}^1$, computed by the implicit–explicit Euler scheme (27),

$$D(\bar{\Phi}^1 - \Phi^0) + \Delta tA\bar{\Phi}^1 + \Delta tF^1 + \Delta tC(\Phi^0) = 0, \tag{34}$$

and correct it to a third-order approximation by the linearly implicit Crank–Nicolson scheme,

$$D(\Phi^1 - \Phi^0) + \Delta tA\Phi^1 + \Delta tF^1 + \Delta tC\left(\frac{1}{2}(\bar{\Phi}^1 + \Phi^0)\right) = 0. \tag{35}$$

The next starting approximation Φ^2 may then be calculated by

$$D\left(\frac{3}{2}\Phi^2 - 2\Phi^1 + \frac{1}{2}\Phi^0\right) + \Delta tA\Phi^2 + \Delta tF^2 + \Delta t(2C(\Phi^1) - C(\Phi^0)) = 0. \tag{36}$$

Table 1
$\alpha = .1, \beta = 0, c = .1, N = 21, t = 1$

Order	γ	$\Delta t = \frac{1}{10}$	$\Delta t = \frac{1}{20}$	$\Delta t = \frac{1}{40}$	$\Delta t = \frac{1}{80}$	$\Delta t = \frac{1}{160}$
1	0.5	3.19, −4	1.80, −4	1.10, −4	7.51, −5	5.75, −5
2		4.17, −7	1.79, −7	1.19, −7	1.79, −7	1.79, −7
3		2.98, −7	1.19, −7	1.19, −7	1.79, −7	1.79, −7
4		4.17, −7	1.19, −7	1.19, −7	1.79, −7	2.38, −7
5		3.70, −6	1.19, −7	1.19, −7	3.58, −7	1.37, −6
6		4.17, −6	1.79, −7	5.36, −7	2.33, −6	1.91, −6
1	0.9	6.93, −5	3.73, −5	2.14, −5	1.33, −5	9.24, −6
2		1.79, −7	1.19, −7	1.19, −7	1.19, −7	1.19, −7
3		1.19, −7	1.19, −7	1.19, −7	1.19, −7	1.19, −7
4		1.19, −7	1.19, −7	1.19, −7	1.19, −7	1.19, −7
5		2.38, −7	1.19, −7	1.19, −7	1.19, −7	1.19, −6
6		7.15, −7	1.19, −7	1.19, −7	1.19, −7	8.35, −7

2.3. Fully discrete schemes

We now present the fully discrete schemes. For $p \le 3$, we also give the starting approximations, except for the first one which is the initial value.

The first-order scheme is

$$D(\Phi^{n+1} - \Phi^n) + \Delta t A\Phi^{n+1} + \Delta t F^{n+1} + \Delta t C(\Phi^n) = 0. \tag{37}$$

The second-order scheme takes the form

$$D(\Phi^1 - \Phi^0) + \Delta t A\Phi^1 + \Delta t F^1 + \Delta t C(\Phi^0) = 0, \tag{38}$$

cf. (33) and

$$D\left(\frac{3}{2}\Phi^{n+2} - 2\Phi^{n+1} + \frac{1}{2}\Phi^n\right) + \Delta t A\Phi^{n+2} + \Delta t F^{n+2} + \Delta t(2C(\Phi^{n+1}) - C(\Phi^n)) = 0. \tag{39}$$

The approximation Φ^1 for the third-order scheme can be calculated from

$$D(\bar{\Phi}^1 - \Phi^0) + \Delta t A\bar{\Phi}^1 + \Delta t F^1 + \Delta t C(\Phi^0) = 0, \tag{40}$$

cf. (34) and

$$D(\Phi^1 - \Phi^0) + \Delta t A\Phi^1 + \Delta t F^1 + \Delta t C\left(\frac{1}{2}(\bar{\Phi}^1 + \Phi^0)\right) = 0, \tag{41}$$

cf. (35), the approximation Φ^2 by

$$D\left(\frac{3}{2}\Phi^2 - 2\Phi^1 + \frac{1}{2}\Phi^0\right) + \Delta t A\Phi^2 + \Delta t F^2 + \Delta t(2C(\Phi^1) - C(\Phi^0)) = 0, \tag{42}$$

cf. (36) and the others approximations by

$$D\left(\frac{11}{6}\Phi^{n+3} - 3\Phi^{n+2} + \frac{3}{2}\Phi^{n+1} - \frac{1}{3}\Phi^n\right) + \Delta t A\Phi^{n+3} + \Delta t F^{n+3} + \Delta t(3C(\Phi^{n+2}) - 3C(\Phi^{n+1}) + C(\Phi^n)) = 0. \tag{43}$$

The fourth, fifth, and sixth-order schemes can readily be obtained from (30)–(32), respectively.

3. Numerical results

Example 1. In Table 1, we give the maximum absolute error between the exact solution (5) and the results obtained by approximations (23) for $p = 1, \ldots, 6$, of the corresponding p-step scheme for various time steps with $\alpha = .1, \beta = 0, c = .1$, $N = 21, t = 1$, and different values of γ. These data for $p = 1, \ldots, 6$, are presented in Fig. 1. In both Table 1 and Fig. 1, the order of accuracy of the scheme for $p = 2, 3, 4$ is the same as the order of accuracy of the scheme $p = 5, 6$ in the case of the time steps $\frac{1}{20}, \frac{1}{40}, \frac{1}{80}$ but the error in the case of the time steps $\frac{1}{10}, \frac{1}{160}$ for 5 and 6-step methods has not decreased as expected, since that would be beyond machine accuracy. It is noteworthy that the computation cost of BDF methods is independent of the number of steps.

Example 2. The maximum absolute error between the obtained numerical solution and the exact solutions of GKS equation (1) is computed for different values of $\alpha, \beta, \gamma, c, N$ and Δt. The obtained results are summarized in the following tables and figures.

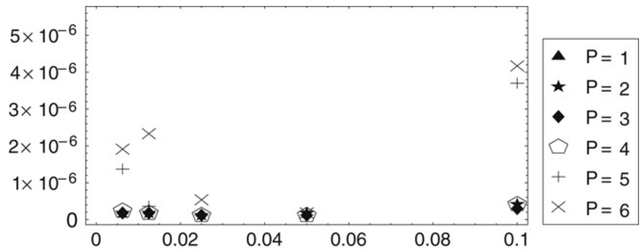

Fig. 1. Maximum absolute error for various time steps, $\alpha = .1$, $\beta = 0$, $c = .1$, $N = 21$, $t = 1$ and $\gamma = 0.5$.

Table 2
$N = 11$, $\Delta t = 0.05$, $\alpha = \gamma = 1$, $\beta = 4$

The methods	c	t			
		0.4	0.6	0.8	1.0
Method 1	0.1	4.19, −4	3.79, −4	3.23, −4	2.63, −4
Method 2		4.22, −4	3.78, −4	3.26, −4	2.68, −4
Method 3		4.27, −4	3.07, −4	3.37, −4	2.81, −4
Method 4		4.06, −4	3.65, −4	3.16, −4	2.60, −4
Method 1	0.01	3.20, −5	3.15, −5	3.34, −5	3.24, −5
Method 2		6.59, −6	6.35, −6	5.66, −6	7.69, −6
Method 3		5.67, −5	5.63, −5	5.72, −5	5.69, −5
Method 4		8.76, −6	3.99, −6	5.66, −6	4.65, −6
Method 1	0.001	3.29, −5	3.43, −5	3.34, −5	3.24, −5
Method 2		1.58, −4	3.61, −6	1.91, −6	3.55, −6
Method 3		5.25, −5	4.46, −5	5.44, −5	5.48, −5
Method 4		3.22, −6	3.10, −6	2.50, −6	3.34, −6

Table 3
$N = 11$, $\Delta t = 0.05$, $\alpha = \gamma = 1$, $\beta = 0$

The methods	c	t			
		0.4	0.6	0.8	1.0
Method 1	0.1	1.91, −6	4.29, −6	5.25, −6	5.72, −6
Method 2		2.03, −6	3.82, −6	5.25, −6	5.72, −6
Method 3		7.03, −6	1.79, −6	3.82, −6	5.01, −6
Method 4		1.67, −6	4.29, −6	5.25, −6	5.48, −6
Method 1	0.01	6.86, −7	6.56, −7	6.56, −7	7.45, −7
Method 2		2.38, −7	2.38, −7	4.77, −7	5.96, −7
Method 3		1.67, −6	1.19, −7	9.54, −7	1.43, −6
Method 4		3.58, −7	8.35, −7	2.38, −7	4.77, −7
Method 1	0.001	6.86, −7	5.96, −7	6.85, −7	7.45, −7
Method 2		4.77, −7	2.38, −7	2.38, −7	3.58, −7
Method 3		8.35, −7	9.54, −7	1.07, −6	1.07, −6
Method 4		4.77, −7	3.58, −7	4.77, −7	3.58, −7

In Table 2, we give the maximum absolute error between the exact solution (2) and the results obtained by methods 1–4 for $N = 11$, $\Delta t = 0.05$, $\alpha = \gamma = 1$, $\beta = 4$ in which the second-order scheme (28) is used. It is seen that the maximum absolute errors from methods 2, 4 are smaller than those from methods 1, 3 for $c = 0.01$ and $c = 0.001$.

Methods 1–4, refer to approximations (19), (20), (22) and (23), respectively.

In Table 3, we give the maximum absolute error between the exact solution (5) and the results obtained by methods 1–4 for $N = 11$, $\Delta t = 0.05$, $\alpha = \gamma = 1$, $\beta = 0$ in which the third-order scheme (29) is used. It is seen that the maximum absolute errors from methods 1, 2 and 4 are smaller than those from method 3.

In Table 4, we give the maximum absolute error between the exact solution (3) and the results obtained by methods 1–4 for $N = 11$, $\Delta t = 0.05$, $\alpha = \gamma = 1$, $\beta = 12/\sqrt{47}$ in which the second-order scheme (28) is used. It is seen that the order of accuracy of the methods 1–2 is the same as the order of accuracy of the method 3–4. Also we observe that the errors become smaller for decreasing parameter c.

In Table 5, we give the maximum absolute error between the exact solution (4) and the results obtained by methods 1–4 for $N = 11$, $\Delta t = 0.05$, $\alpha = \gamma = 1$, $\beta = 16/\sqrt{73}$ in which the second-order scheme (28) is used.

From Tables 2–5, we observe that the errors become smaller for decreasing parameter c.

Table 4
$N = 11, \Delta t = 0.05, \alpha = \gamma = 1, \beta = 12/\sqrt{47}$

The methods	c	t				
		0.2	0.4	0.6	0.8	1.0
Method 1	0.1	2.38, −7	2.38, −7	2.38, −7	1.19, −7	2.38, −7
Method 2		2.38, −7	1.19, −7	1.19, −7	1.19, −7	2.38, −7
Method 3		2.38, −7	1.19, −7	1.19, −7	2.38, −7	2.38, −7
Method 4		2.38, −7	2.38, −7	1.19, −7	2.38, −7	1.19, −7
Method 1	0.9	8.58, −6	1.91, −6	9.54, −7	9.54, −7	9.54, −7
Method 2		8.58, −6	9.54, −7	9.54, −7	9.54, −7	9.54, −7
Method 3		7.63, −6	9.54, −7	9.54, −7	9.54, −7	9.54, −7
Method 4		7.63, −6	9.54, −7	9.54, −7	9.54, −7	9.54, −7
Method 1	2.0	5.15, −5	3.82, −6	3.82, −6	1.91, −6	1.91, −6
Method 2		5.34, −5	3.82, −6	3.82, −6	1.91, −6	1.91, −6
Method 3		4.39, −5	1.91, −6	3.82, −6	1.91, −6	1.91, −6
Method 4		4.58, −5	1.91, −6	3.82, −6	1.91, −6	1.91, −6

Table 5
$N = 11, \Delta t = 0.05, \alpha = \gamma = 1, \beta = 16/\sqrt{73}$

The methods	c	t				
		0.2	0.4	0.6	0.8	1.0
Method 1	0.1	2.38, −7	1.19, −7	1.19, −7	2.38, −7	2.38, −7
Method 2		1.19, −7	1.19, −7	1.19, −7	2.38, −7	1.19, −7
Method 3		1.19, −7	1.19, −7	1.19, −7	1.19, −7	1.19, −7
Method 4		1.19, −7	2.38, −7	2.38, −7	1.19, −7	2.38, −7
Method 1	0.9	6.68, −6	1.91, −6	1.91, −6	1.91, −6	9.54, −7
Method 2		6.68, −6	1.91, −6	1.91, −6	1.91, −6	9.54, −7
Method 3		4.77, −6	1.91, −6	1.91, −6	9.54, −7	1.91, −6
Method 4		3.82, −6	1.91, −6	1.91, −6	9.54, −7	1.91, −6
Method 1	1.3	1.14, −5	1.91, −6	1.91, −6	1.91, −6	1.91, −6
Method 2		1.14, −5	1.91, −6	1.91, −6	1.91, −6	1.91, −6
Method 3		3.05, −5	3.82, −6	1.91, −6	3.82, −6	1.91, −6
Method 4		3.05, −5	1.91, −6	1.91, −6	1.91, −6	1.91, −6

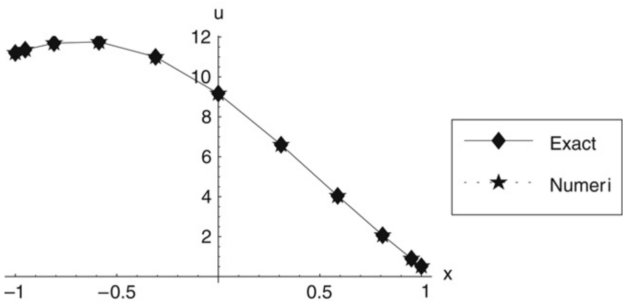

Fig. 2. $N = 10, \Delta t = 0.05, t = 1.0, c = 0.01, \beta = 4, p = 2$ and $\alpha = \gamma = 1$.

In Figs. 2 and 3, we present the numerical and the exact results of the solution u of the GKS equation (1) for different values of α, β, γ, c, N and t. In Fig. 4, we display Error (|Numerical − Exact|) of GKS equation (1) at time $t = 0.4$ for $N = 11$, $\Delta t = 0.05$, $c = 0.1$, $p = 3$, $\beta = 0$ and $\alpha = \gamma = 1$, we see that maximum error occurs at the right-hand boundary.

4. Conclusions

In this paper the Chebyshev spectral collocation methods (known as El-Gendi method [1,2]) are extended to obtain numerical solutions for GKS equation in a bounded domain. Using these methods the problem is reduced to a system of ODEs that are solved by implicit–explicit BDF methods. The obtained approximate numerical solutions maintains a good accuracy compared with the exact solution for the best choice of different values of parameters c and β.

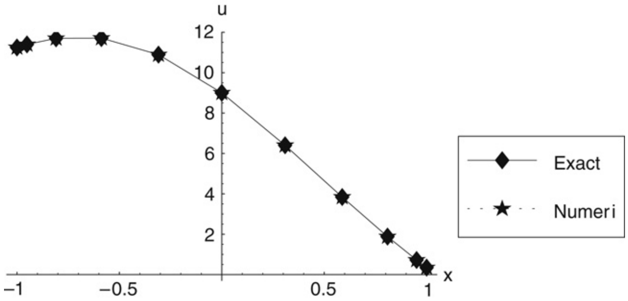

Fig. 3. $N = 10$, $\Delta t = 0.05$, $t = 0.5$, $c = 0.0001$, $\beta = 4$ and $\alpha = \gamma = 1$.

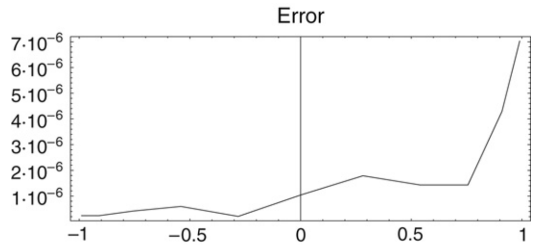

Fig. 4. Error (|Numerical − Exact|) of GKS equation (1) at time $t = 0.4$ for $N = 11$, $\Delta t = 0.05$, $c = 0.1$, $p = 3$, $\beta = 0$ and $\alpha = \gamma = 1$.

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