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Finite difference discretization of the Kuramoto–Sivashinsky equation*

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Summary. We analyze a Crank–Nicolson–type finite difference scheme for the Kuramoto–Sivashinsky equation in one space dimension with periodic boundary conditions. We discuss linearizations of the scheme and derive second–order error estimates.

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1. Introduction

For T, $\nu > 0$, we consider the following periodic initial–value problem for the Kuramoto–Sivashinsky (KS) equation: We seek a real–valued function u defined on $\mathbb{R} \times [0, T]$, 1–periodic in the first variable and satisfying

$$u_t + u\,u_x + u_{xx} + \nu\,u_{xxxx} = 0 \qquad \text{in} \quad \mathbb{R} \times [0, T] \tag{1.1}$$

and

$$u(\cdot, 0) = u_0 \qquad \text{in} \quad \mathbb{R}\,, \tag{1.2}$$

where u_0 is a given 1–periodic function. We assume that (1.1)–(1.2) has a unique, sufficiently smooth solution, cf. Nicolaenko and Scheurer (1984).

For the mathematical theory and the physical significance of the KS equation as well as for related computational work we refer the reader to Kuramoto (1978), Sivashinsky (1980), Papageorgiou et al. (1990), Hyman and Nicolaenko (1986), Nicolaenko and Scheurer (1984), Nicolaenko et al. (1985), Temam (1988), Papageorgiou and Smyrlis (1991), and the references therein. Denoting by $\|\cdot\|$ the norm in $L^2(0, 1)$, it is easily seen that

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$$\|u(\cdot,t)\|^2 \le \|u_0\|^2\, \mathrm{e}^{\frac{t}{2\nu}}\,, \qquad 0 \le t \le T\,, \tag{1.3}$$

$$\|u(\cdot,t)\| \le \|u(\cdot,s)\|, \qquad 0 \le s \le t \le T, \qquad \text{for} \quad \nu \ge \frac{1}{4\pi^2}\,, \tag{1.4}$$

and

$$\int_0^1 u(x,t)\,dx = \int_0^1 u_0(x)\,dx\,, \qquad 0 \le t \le T\,, \tag{1.5}$$

see Sect. 2.

Let $J, N \in \mathbb{N}$, $h := \frac{1}{J}$, $x_i := ih$, $i \in \mathbb{Z}$, $k := \frac{T}{N}$, $t^n := nk$, $n = 0,\dots,N$, and

$$\mathbb{R}^J_{\text{per}} := \left\{ v = (v_i)_{i\in\mathbb{Z}} : \quad v_i \in \mathbb{R} \text{ and } \quad v_{i+J} = v_i\,, \quad i \in \mathbb{Z} \right\}.$$

For $v \in \mathbb{R}^J_{\text{per}}$ let $\Delta_h v_i := \frac{1}{h^2}(v_{i-1} - 2v_i + v_{i+1})$, $\Delta_h^2 v_i := \frac{1}{h^2}(\Delta_h v_{i-1} - 2\Delta_h v_i + \Delta_h v_{i+1})$, $i \in \mathbb{Z}$, and for $v^0,\dots,v^N \in \mathbb{R}^J_{\text{per}}$ set $\partial v^n := \frac{1}{k}(v^{n+1} - v^n)$, and $v^{n+1/2} := \frac{1}{2}(v^n + v^{n+1})$, $n = 0,\dots,N-1$.

We discretize problem (1.1)–(1.2) by the following Crank–Nicolson–type finite difference scheme: We approximate $u^n \in \mathbb{R}^J_{\text{per}}$, $u_i^n := u(x_i,t^n)$, by $U^n \in \mathbb{R}^J_{\text{per}}$, where $U^0 := u^0$, and for $n = 0,\dots,N-1$

$$\partial U_i^n + \frac{1}{6h}\left(U_{i-1}^{n+1/2} + U_i^{n+1/2} + U_{i+1}^{n+1/2}\right)\left(U_{i+1}^{n+1/2} - U_{i-1}^{n+1/2}\right) + \Delta_h U_i^{n+1/2} + \nu\Delta_h^2 U_i^{n+1/2} = 0\,, \qquad i = 1,\dots,J\,. \tag{1.6}$$

This kind of spatial discretization of the term $u\,u_x$ has often been used and analyzed in the literature, see, e.g., Richtmyer and Morton (1967), Zabusky and Kruskal (1965), Fornberg (1973). One possibility to get this discretization is to write the nonlinear term in the form $\frac{1}{3}\,uu_x + \frac{1}{3}\,(u^2)_x$ and discretize each one of these terms in the standard fashion for second–order schemes. Another more systematical way is provided by the finite element method when continuous and piecewise linear elements are used.

Introducing in $\mathbb{R}^J_{\text{per}}$ the discrete L^2 norm $\|\cdot\|_h$ by

$$\|v\|_h := \left(h\sum_{i=1}^J (v_i)^2\right)^{1/2}, \qquad v \in \mathbb{R}^J_{\text{per}}\,,$$

we show in Sect. 2 that

$$\|U^n\|_h^2 \le \|U^0\|_h^2\, \mathrm{e}^{\frac{\alpha}{2\nu}t^n}\,, \quad \alpha > 1\,, \quad k \le 8\nu\,\frac{\alpha-1}{\alpha}\,, \quad n = 1,\dots,N\,, \tag{1.7}$$

$$\|U^{n+1}\|_h \le \|U^n\|_h, \qquad n = 0,\dots,N-1, \qquad \text{for} \quad \nu \ge [\sigma(h)]^2\,, \tag{1.8}$$

where $\sigma(h) := \frac{h}{2\sin(\pi h)}$, and

$$h\sum_{i=1}^J U_i^n = h\sum_{i=1}^J U_i^0\,, \qquad n = 1,\dots,N\,, \tag{1.9}$$

which correspond to (1.3), (1.4) and (1.5), respectively. Note that $\lim_{h\to 0}\sigma(h) = \frac{1}{2\pi}$.

In Sect. 3 we show existence of the Crank–Nicolson approximations for $k < 8\nu$, and derive the optimal–order error estimate

$$\max_{0\le n\le N} \|u^n - U^n\|_h \le c(k^2 + h^2), \tag{1.10}$$

where here and in the sequel c and C denote generic constants independent of k and h, not necessarily the same at any two places unless indices are used. For $k\,h^{-1/5}$ sufficiently small the Crank–Nicolson approximations are uniquely defined by (1.6). In the last section we linearize the scheme (1.6) by Newton's method. We extrapolate from previous time levels to obtain starting values and perform one Newton iteration. For k and h sufficiently small and $k = o(h^{1/4})$ the scheme is well defined, and an estimate of the form (1.10) holds. A disadvantage of this method is that the matrix of the linear system to be solved at each time level t^n changes with n. We also discuss an efficient implementation which requires solving linear systems with the same matrix, which is symmetric and for $k < 8\nu$ positive definite, and prove second–order error estimates for this scheme as well.

2. Some properties of the Crank–Nicolson approximations

First, we shall discuss some properties of the solution u of problem (1.1)–(1.2). Multiplying the KS equation with u and integrating by parts over $[0,1]$, we obtain by periodicity

$$\frac{1}{2}\frac{d}{dt}\|u(\cdot,t)\|^2 = \|u_x(\cdot,t)\|^2 - \nu\,\|u_{xx}(\cdot,t)\|^2 . \tag{2.1}$$

Now, for v 1–periodic and smooth $\|v'\|^2 = -\int_0^1 v(x)\,v''(x)\,dx$, i.e., $\|v'\|^2 \le \|v\|\,\|v''\|$, and, therefore,

$$\|v'\|^2 \le \nu\,\|v''\|^2 + \frac{1}{4\nu}\,\|v\|^2 . \tag{2.2}$$

Thus, (2.1) yields

$$\frac{d}{dt}\|u(\cdot,t)\|^2 \le \frac{1}{2\nu}\,\|u(\cdot,t)\|^2 ,$$

and (1.3) follows easily, cf. Temam (1988), p. 141. Using the well–known Wirtinger inequality

$$\|v'\| \le \frac{1}{2\pi}\,\|v''\| \qquad v \text{ 1–periodic and smooth}, \tag{2.3}$$

cf. Osserman (1978), (2.1) yields

$$\frac{1}{2}\frac{d}{dt}\|u(\cdot,t)\|^2 \le \left(\frac{1}{4\pi^2} - \nu\right)\|u_{xx}(\cdot,t)\|^2 ,$$

from which (1.4) follows. Integrating the KS Eq., (1.5) follows easily by periodicity.

Before we proceed to derive some discrete analogs to (1.3)–(1.5) we introduce notation and give an auxiliary lemma. The discrete L^2 inner product $(\cdot,\cdot)_h$ in $\mathbb{R}^J_{\mathrm{per}}$ is given by

$$(v,w)_h := h \sum_{i=1}^{J} v_i\, w_i\,, \qquad v,w \in \mathbb{R}^J_{\mathrm{per}}\,.$$

In addition to the discrete L^2 norm $\|\cdot\|_h$, we shall use the discrete H^1 and H^2 seminorms, denoted by $|\cdot|_{1,h}$ and $|\cdot|_{2,h}$, respectively,

$$|v|_{1,h} := \left[h \sum_{i=1}^{J} \left(\frac{v_i - v_{i-1}}{h} \right)^2 \right]^{1/2}, \quad |v|_{2,h} := \left[h \sum_{i=1}^{J} (\Delta_h\, v_i)^2 \right]^{1/2}, \quad v \in \mathbb{R}^J_{\mathrm{per}}\,.$$

Let $\varphi,\psi\ :\ \mathbb{R}^J_{\mathrm{per}} \times \mathbb{R}^J_{\mathrm{per}} \to \mathbb{R}^J_{\mathrm{per}}$ $\big(\varphi(v,w)\big)_i := (v_{i-1} + v_i + v_{i+1})(w_{i+1} - w_{i-1})$, $\big(\psi(v,w)\big)_i := -(2v_{i-1} + v_i)\,w_{i-1} + (v_{i+1} - v_{i-1})\,w_i + (2v_{i+1} + v_i)\,w_{i+1}$. Note that $\psi(v,w) = \psi(w,v)$. In the next lemma we collect some auxiliary results.

Lemma 2.1. *For $v,w,r \in \mathbb{R}^J_{\mathrm{per}}$ we have*

(2.4) $\quad \big(\varphi(v,w),w\big)_h = -h \sum_{i=1}^{J} (v_{i+1} - v_{i-2})\, w_i\, w_{i-1}\,,$

(2.5) $\quad \big(\varphi(v,v),w\big)_h = -h \sum_{i=1}^{J} \big(v_i^2 + v_i v_{i+1} + v_{i+1}^2\big)\,(w_{i+1} - w_i)\,,$

(2.6) $\quad \big(\psi(v,w),r\big)_h = -h \sum_{i=1}^{J} [v_i(w_{i+1} + 2w_i) + v_{i+1}(2w_{i+1} + w_i)]\,(r_{i+1} - r_i)\,,$

(2.7) $\quad \big(\varphi(v,v),v\big)_h = 0\,,$

(2.8) $\quad \varphi(v,v) - \varphi(w,w) = \psi(w, v-w) + \varphi(v-w, v-w)\,,$

(2.9) $\quad -\big(\Delta_h\, v, v\big)_h = |v|_{1,h}^2\,,$

(2.10) $\quad \big(\Delta_h^2\, v, v\big)_h = |v|_{2,h}^2\,,$

(2.11) $\quad |v|_{1,h}^2 \le \|v\|_h\, |v|_{2,h}\,,$

(2.12) $\quad |v|_{1,h}^2 \le \nu\, |v|_{2,h}^2 + \frac{1}{4\nu}\, \|v\|_h^2,$

and

(2.13) $\quad |v|_{1,h} \le \sigma(h)\, |v|_{2,h}\,, \qquad \sigma(h) = \dfrac{h}{2\,\sin(\pi h)}\,.$

Proof. (2.4)–(2.6) can be easily proved by summation by parts; (2.7) follows from (2.4). (2.8) can be easily shown, by Taylor expanding φ around (w_{i-1}, w_i, w_{i+1}), say. Further

$$\big(\Delta_h^2\, v, v\big)_h = \frac{1}{h} \sum_{i=1}^{J} (v_{i-1} - 2v_i + v_{i+1})\, \Delta_h\, v_i$$

and (2.10) follows. (2.9) can be shown analogously, (2.11) follows immediately from (2.9), and (2.12) from (2.11). Hence, it remains to show (2.13). Let $\mathbb{R}^J_{\mathrm{per},0} := \big\{ v \in \mathbb{R}^J_{\mathrm{per}}\ :\ v_1 + v_2 + \ldots + v_J = 0 \big\}$, and $v^1, v^2, \ldots, v^{J-1} \in \mathbb{R}^J_{\mathrm{per},0}$ be given by

$$v_i^j := \sqrt{2}\,\sin(2j\pi x_i)\,, \qquad j = 1, \ldots, \left[\frac{J-1}{2}\right], \qquad i \in \mathbb{Z}\,,$$

$$v_i^{\left[\frac{J-1}{2}\right]+j} := \sqrt{2}\,\cos(2j\pi x_i)\,, \qquad j = 1, \ldots, \left[\frac{J}{2}\right], \qquad i \in \mathbb{Z}\,.$$

It is well–known that $v^1, v^2, \ldots, v^{J-1}$ are orthonormal with respect to $(\cdot,\cdot)_h$, cf. Hämmerlin and Hoffmann (1989), pp. 190, 191, 232. It is also easily seen that $v^1, v^2, \ldots, v^{J-1}$ are eigenvectors of $-\Delta_h$ with corresponding eigenvalues $\lambda_\ell := \frac{4}{h^2}\sin^2(\ell\pi h)$, $\ell = 1, \ldots, [\frac{J}{2}]$, cf. Samarskij (1984), pp. 76–77 or Thomée (1990), p. 26. Now, $\lambda_1 = [\sigma(h)]^{-2}$ is the smallest eigenvalue, and using (2.9) we conclude that

$$\|v\|_h \le \sigma(h)\,|v|_{1,h} \qquad v \in \mathbb{R}^J_{\text{per},0}\,.$$

(2.13) follows immediately from this inequality. $\square$

Taking in (1.6) the inner product with $U^{n+1/2}$, and using (2.7), (2.9) and (2.10), we obtain

$$\|U^{n+1}\|_h^2 - \|U^n\|_h^2 = 2k\left\{|U^{n+1/2}|_{1,h}^2 - \nu\,|U^{n+1/2}|_{2,h}^2\right\}. \tag{2.14}$$

Therefore, by (2.12)

$$\|U^{n+1}\|_h^2 - \|U^n\|_h^2 \le \frac{k}{2\nu}\,\|U^{n+1/2}\|_h^2\,,$$

i.e.,

$$\left(1 - \frac{k}{8\nu}\right)\|U^{n+1}\|_h \le \left(1 + \frac{k}{8\nu}\right)\|U^n\|_h\,, \qquad n = 0, \ldots, N-1\,. \tag{2.15}$$

Obviously, for $\alpha > 1$

$$\frac{8\nu + k}{8\nu - k} \le 1 + \frac{\alpha}{4\nu}\,k \qquad \text{for} \quad k \le 8\nu\,\frac{\alpha - 1}{\alpha}\,. \tag{2.16}$$

(1.7) follows immediately from (2.15), (2.16). Using (2.13) we obtain from (2.14)

$$\|U^{n+1}\|_h^2 - \|U^n\|_h^2 \le 2k\left\{[\sigma(h)]^2 - \nu\right\}|U^{n+1/2}|_{2,h}^2\,,$$

and conclude (1.8). Finally, summing in (1.6) from $j = 1$ to J we get (1.9).

3. Existence, convergence and uniqueness

In this section we show existence of the approximate solutions $U^1, \ldots, U^N \in \mathbb{R}^J_{\text{per}}$ satisfying (1.6) under the condition $k < 8\nu$, we derive second–order error estimates, and prove uniqueness of the Crank–Nicolson approximations for u smooth and $kh^{-1/5}$ sufficiently small.

Existence. We shall use the following well–known variant of the Brouwer fixed–point theorem, see, e.g., Browder (1965).

Lemma 3.1. *Let $(H, (\cdot,\cdot)_H)$ be a finite dimensional inner product space and denote by $\|\cdot\|_H$ the induced norm. Suppose that $g : H \to H$ is continuous and there exists* *an $\alpha > 0$ such that $\big(g(x), x\big)_H > 0$ for all $x \in H$ with $\|x\|_H = \alpha$. Then, there exists $x^* \in H$ such that $g(x^*) = 0$ and $\|x^*\|_H \le \alpha$.* □

The argument of existence of the Crank–Nicolson approximations proceeds in an inductive way. So assume that $U^0, \dots, U^n$, $n < N$, exist. Let $g : \mathbb{R}^J_{\mathrm{per}} \to \mathbb{R}^J_{\mathrm{per}}$ be defined by

$$g(V) := 2V - 2U^n + \frac{k}{6h}\,\varphi(V, V) + k\,\Delta_h\, V + \nu\, k\,\Delta_h^2\, V\,,$$

with φ as in Sect. 2. Then, g is obviously continuous. Taking the inner product with V, and using (2.7), (2.9) and (2.10), we have

$$\big(g(V), V\big)_h = 2\,\|V\|_h^2 - 2(U^n, V)_h - k\left\{\,|V|_{1,h}^2 - \nu\,|V|_{2,h}^2\,\right\}.$$

Therefore, by (2.12)

$$\big(g(V), V\big)_h \ge 2\,\|V\|_h \left\{ \left(1 - \frac{k}{8\nu}\right)\,\|V\|_h - \|U^n\|_h \right\}. \tag{3.1}$$

Hence, for $k < 8\nu$ and $\|V\|_h = \frac{8\nu}{8\nu - k}\,\|U^n\|_h + 1$ obviously $\big(g(V), V\big)_h > 0$, and via Lemma 3.1 we deduce existence of $V^* \in \mathbb{R}^J_{\mathrm{per}}$ such that $g(V^*) = 0$. It is easily seen that $U^{n+1} := 2V^* - U^n$ satisfies (1.6).

Convergence. The main result in this section is given in the following theorem.

Theorem 3.1. *Let the solution u of (1.1)–(1.2) be sufficiently smooth, $U^0 = u^0$, and $U^1, \dots, U^N \in \mathbb{R}^J_{\mathrm{per}}$ satisfy (1.6). Then, for k sufficiently small*

$$\max_{0 \le n \le N} \|u^n - U^n\|_h \;\le\; c\,(k^2 + h^2)\,. \tag{3.2}$$

Proof. Let $r^n \in \mathbb{R}^J_{\mathrm{per}}$ be the consistency error of method (1.6),

$$r^n := \partial u^n + \frac{1}{6h}\,\varphi\left(u^{n+1/2}, u^{n+1/2}\right) + \Delta_h\, u^{n+1/2} + \nu\, \Delta_h^2\, u^{n+1/2}\,. \tag{3.3}$$

Here $u^{n+1/2} = \frac{1}{2}\,(u^n + u^{n+1})$. It is easily seen that

$$\max_{i,n} |r_i^n| \;\le\; c\,(k^2 + h^2)\,. \tag{3.4}$$

Let $e^n := u^n - U^n$, $n = 0, \dots, N$. Then, (1.6) and (3.3) yield

$$\begin{aligned}
&\partial e^n + \Delta_h\, e^{n+1/2} + \nu\,\Delta_h^2\, e^{n+1/2} \\
&\quad = \frac{1}{6h}\left[\varphi\left(U^{n+1/2}, U^{n+1/2}\right) - \varphi\left(u^{n+1/2}, u^{n+1/2}\right)\right] + r^n\,,
\end{aligned}$$

which we write in a form more appropriate for our purposes

$$\begin{aligned}
\partial e^n + \Delta_h\, e^{n+1/2} + \nu\,\Delta_h^2\, e^{n+1/2} = \frac{1}{6h}\,&\Big[\varphi\left(e^{n+1/2}, e^{n+1/2}\right) - \varphi\left(e^{n+1/2}, u^{n+1/2}\right) \\
&-\varphi\left(u^{n+1/2}, e^{n+1/2}\right)\Big] + r^n\,,
\end{aligned}$$

cf. Baker et al. (1983). Taking the inner product with $e^{n+1/2}$, using (2.7), (2.9), (2.10), rewriting the third term on the right–hand side according to (2.4), using the boundedness of u_x and applying the Schwarz inequality, we obtain

$$\frac{1}{2k}\left(\|e^{n+1}\|_h^2-\|e^n\|_h^2\right)\le |e^{n+1/2}|_{1,h}^2-\nu\,|e^{n+1/2}|_{2,h}^2+C\,\|e^{n+1/2}\|_h^2+\|r^n\|_h\,\|e^{n+1/2}\|_h\,.$$

Therefore, using (2.12)

$$(1-ck)\,\|e^{n+1}\|_h\le(1+ck)\,\|e^n\|_h+C\,k\,\left(k^2+h^2\right)\,,\qquad n=0,\dots,N-1\,.$$

Then, (3.2) follows in view of Gronwall's discrete inequality. □

Uniqueness. Assuming smoothness of the solution u such that (3.2) holds, we shall show uniqueness of the Crank–Nicolson approximations for $kh^{-1/5}$ sufficiently small. Suppose $V^0=u^0$, and let $V^1,\dots,V^N\in\mathbb{R}^J_{\text{per}}$ satisfy

$$\partial V^n+\frac{1}{6h}\,\varphi\big(V^{n+1/2},V^{n+1/2}\big)+\Delta_h\,V^{n+1/2}+\nu\,\Delta_h^2\,V^{n+1/2}=0\,. \tag{3.5}$$

Letting $E^n:=V^n-U^n$, $n=0,\dots,N$, and using (2.8), we obtain from (1.6) and (3.5)

$$\partial E^n+\Delta_h\,E^{n+1/2}+\nu\,\Delta_h^2\,E^{n+1/2}=\\ -\frac{1}{6h}\left[\psi\big(U^{n+1/2},E^{n+1/2}\big)+\varphi\big(E^{n+1/2},E^{n+1/2}\big)\right]. \tag{3.6}$$

Now, (3.2) yields

$$\max_{i,n}\,|U_i^n|\le c\big(1+k^2\,h^{-1/2}\big)\,. \tag{3.7}$$

Taking in (3.6) the inner product with $E^{n+1/2}$, using (2.9), (2.10), (2.6), (2.7), (3.7), and applying the Schwarz inequality, we obtain

$$\frac{1}{2k}\left(\|E^{n+1}\|_h^2-\|E^n\|_h^2\right)-|E^{n+1/2}|_{1,h}^2+\nu\,|E^{n+1/2}|_{2,h}^2\\ \le C\left(1+k^2\,h^{-1/2}\right)|E^{n+1/2}|_{1,h}\,\|E^{n+1/2}\|_h\,.$$

Therefore

$$\frac{1}{2k}\left(\|E^{n+1}\|_h^2-\|E^n\|_h^2\right)\\ \le 2\,|E^{n+1/2}|_{1,h}^2-\nu\,|E^{n+1/2}|_{2,h}^2+C\,\left(1+k^2\,h^{-1/2}\right)^2\,\|E^{n+1/2}\|_h^2$$

i.e., by (2.11)

$$\|E^{n+1}\|_h^2-\|E^n\|_h^2\le C\,k\,\left(1+k^2\,h^{-1/2}\right)^2\,\|E^{n+1/2}\|_h^2\,,$$

from which, for $kh^{-1/5}$ sufficiently small, uniqueness follows easily by induction. We note moreover that for $k=O(h^{1/4})$ and k sufficiently small

$$\|E^{n+1}\|_h \le (1+c\,k)\,\|E^n\|_h\ , \tag{3.8}$$

which represents the stability of the scheme (1.6).

4. Linearization by Newton's method

Computing the Crank–Nicolson approximations $U^1,\dots,U^N$ requires solving at each time level a $J\times J$ nonlinear system. In this section we shall discuss the approximate solution of these systems by Newton's method.

In the rest of the paper, for $v^0,\dots,v^N\in\mathbb{R}^J_{\text{per}}$, we let $\hat v^0 := v^0$, $\hat v^1 := v^1$ unless explicitly otherwise stated, and $\hat v^{n+1} := 2v^n - v^{n-1}$, $n=1,\dots,N-1$. We approximate u^n by $W^n\in\mathbb{R}^J_{\text{per}}$ where $W^0 := u^0$, and for $n=0,\dots,N-1$

$$\begin{aligned}\partial W^n + \Delta_h\, W^{n+1/2} + \nu\, \Delta_h^2\, W^{n+1/2} + \frac{1}{24h}\,\psi\big(W^n+\hat W^{n+1},\ W^{n+1}-\hat W^{n+1}\big)&\\ = -\frac{1}{24h}\,\varphi\big(W^n+\hat W^{n+1},\ W^n+\hat W^{n+1}\big)\,,&\end{aligned}\tag{4.1}$$

where $\hat W^1$ is given by

$$\partial \hat W^0 + \Delta_h\, \hat W^{1/2} + \nu\,\Delta_h^2\,\hat W^{1/2} = -\frac{1}{6h}\,\varphi\left(u^0,u^0\right)\ . \tag{4.2}$$

It is easily seen that substituting W^n for U^n in (1.6), and letting $\hat W^{n+1}$ be a starting approximation to U^{n+1}, (4.1) describes the first Newton iteration to the nonlinear system. It can be shown that for k and h sufficiently small and $k=o(h^{1/4})$, $W^1,\dots,W^N$ are uniquely defined by (4.1)–(4.2), and

$$\max_{0\le n\le N}\|u^n - W^n\|_h \le C\,\left(k^2+h^2\right)\ . \tag{4.3}$$

To compute W^{n+1} by (4.1), we have to solve a $J\times J$ linear system whose matrix changes with n. To avoid this we next analyze an iterative scheme for the approximate solution of (4.1) that requires solving linear systems with the same coefficient matrix.

For $j_1,\dots,j_N\in\mathbb{N}$ we define approximations $U^{n(j)}\in\mathbb{R}^J_{\text{per}}$, $j=0,\dots,j_n$, $V^n := U^{n(j_n)}$, to u^n as follows: Let $V^0 := u^0$, $\hat V^1 := \hat W^1$, $U^{n(0)} := \hat V^n$, and for $n=0,\dots,N-1$

$$\begin{aligned}&\frac{1}{k}\left(U^{n+1(j+1)}-V^n\right)+\frac{1}{2}\,\Delta_h\left(U^{n+1(j+1)}+V^n\right)\\ &+\frac{\nu}{2}\,\Delta_h^2\left(U^{n+1(j+1)}+V^n\right) = -\frac{1}{24\,h}\,\psi\big(V^n+\hat V^{n+1},\ U^{n+1(j)}-\hat V^{n+1}\big)\\ &-\frac{1}{24\,h}\,\varphi\big(V^n+\hat V^{n+1},\ V^n+\hat V^{n+1}\big)\,,\quad j=0,\dots,j_{n+1}-1.\end{aligned}\tag{4.4}$$

Let $k<8\nu$. Then, the coefficient matrix of the linear systems (4.2), (4.4) is positive definite, cf. (3.1); in particular, $U^{n(j)}$, $j=0,\dots,j_n$, $n=1,\dots,N$, are well defined.

Theorem 4.1. *Let u be sufficiently smooth, k and h be sufficiently small and $k = o(h^{1/4})$. Then,*

$$\max_{0\le n\le N} \|u^n - V^n\|_h \le c(k^2 + h^2). \tag{4.5}$$

Proof. Let $\hat{e}^1 := u^1 - \hat{V}^1$. Using (2.8), we obtain from (4.2) and (3.3)

$$\begin{aligned}\hat{e}^1 + \frac{k}{2}\,\Delta_h\,\hat{e}^1 + \frac{\nu k}{2}\,\Delta_h^2\,\hat{e}^1 &= \\ &= -\frac{k}{24h}\left[\psi(2u^0,\, u^1 - u^0) + \varphi(u^1 - u^0,\, u^1 - u^0)\right] + k\,r^0 .\end{aligned}$$

Taking the inner product with $\hat{e}^1$, and using (2.9), (2.10), (2.12), and the fact that $\big(\psi(v,w)\big)_i = (v_i + 2v_{i+1})\,(w_{i+1} - w_{i-1}) + (2w_{i-1} + w_i)\,(v_{i+1} - v_{i-1})$, we obtain

$$\left(1 - \frac{k}{8\nu}\right)\,\|\hat{e}^1\|_h^2 \le C\,k^2\,\|\hat{e}^1\|_h + k\,\|r^0\|_h\,\|\hat{e}^1\|_h ,$$

and, therefore, for k small enough

$$\|\hat{e}^1\|_h^2 \le c\big(k^2 + h^2\big)^2 . \tag{4.6}$$

Next, we shall show inductively that

$$\|u^\ell - V^\ell\|_h^2 \le c_\ell\,\big(k^2 + h^2\big)^2, \qquad \ell = 0, \dots, N , \tag{4.7}$$

where $c_0 = 0$, $c_1 = 1$ say, and

$$\begin{aligned} c_\ell &= 2\big(\tilde{d}\,k\big)^{j_\ell}\,\big(D + 8c_{\ell-1} + 2c_{\ell-2}\big) \\ &\quad + (1 + dk)\,c_{\ell-1} + dk\,c_{\ell-2} + dk , \qquad \ell = 2, \dots, N . \end{aligned} \tag{4.8}$$

Here, D is such that $\|u^n - \hat{u}^n\|_h^2 \le D\,k^4$, and $\tilde{d}$ and d are as follows: let $s^n \in \mathbb{R}^J_{\text{per}}$ be the consistency error of method (4.1). Using (2.8) and (3.3), we easily see that

$$s^n = r^n - \frac{1}{24h}\,\varphi\left(u^{n+1} - \hat{u}^{n+1},\, u^{n+1} - \hat{u}^{n+1}\right),$$

and, therefore,

$$\max_{i,n} |s_i^n|^2 \le c(u)\,\big(k^2 + h^2\big)^2 . \tag{4.9}$$

Let $M := \max_{x,t} |u(x,t)| + 1$, $d_1 := \frac{\nu+1}{\nu}$, $d_2 := 3\,M^2$, $d_3 := d_1 + 30\,M^2 + \frac{135}{16}$, $d_4 := 3\,M^2 + \frac{15}{16}$, $d_5 := \frac{1}{2}\,c(u)$, and $\tilde{d}$ and d be such that for sufficiently small k

$$\frac{d_2\,k}{1 - d_1\,k} \le \tilde{d}\,k , \qquad \frac{\delta_{3j} + d_j\,k}{(1 - d_1\,k)\,(1 - \tilde{d}\,k)} \le \delta_{3j} + d\,k, \quad j = 3, \dots, 5,$$

where δ is the Kronecker symbol. It can be easily seen that $\max_{0\le n\le N} c_n \le c^*$ with a constant c^* independent of h and k. In the sequel we shall assume that h and k are small enough such that

$$c^* \, h^{-1} \left(k^2 + h^2\right)^2 \ \le \ 1 \, . \tag{4.10}$$

Note that (4.7) holds trivially for $\ell = 0$. Assume now that (4.7) holds for $\ell = 0, \dots, n$. Letting $e^{n(j)} := u^n - U^{n(j)}$, $j = 0, \dots, j_n$, and $e^n := u^n - V^n$, $n = 0, \dots, N$, we have

$$\begin{aligned} &\frac{1}{k}\left(e^{n+1(j+1)} - e^n\right) + \frac{1}{2}\,\Delta_h\left(e^{n+1(j+1)} + e^n\right) + \frac{\nu}{2}\,\Delta_h^2\left(e^{n+1(j+1)} + e^n\right) \\ &= -\frac{1}{24\,h}\left[\psi\big(V^n + \hat{V}^{n+1},\ e^{n+1(j)} + e^n\big) + \psi\big(u^{n+1} - \hat{u}^{n+1},\ e^n + \hat{e}^{n+1}\big)\right] \\ &\quad - \frac{1}{24\,h}\,\varphi\big(e^n + \hat{e}^{n+1},\ e^n + \hat{e}^{n+1}\big) + s^n \, . \end{aligned} \tag{4.11}$$

Taking the inner product with $e^{n+1(j+1)} + e^n$, and using (2.6), (2.5), (4.10) and the induction hypothesis, we obtain

$$\begin{aligned} &\frac{1}{k}\left(\|e^{n+1(j+1)}\|_h^2 - \|e^n\|_h^2\right) - \frac{1}{2}\left|e^{n+1(j+1)} + e^n\right|_{1,h}^2 + \frac{\nu}{2}\left|e^{n+1(j+1)} + e^n\right|_{2,h}^2 \\ &\le \left(M\,\|e^{n+1(j)} + e^n\|_h + M\,\|e^n + \hat{e}^{n+1}\|_h + \frac{1}{8}\,h^{-1/2}\,\|e^n + \hat{e}^{n+1}\|_h^2\right)\,|e^{n+1(j+1)} + e^n|_{1,h} \\ &+ \|s^n\|_h\,\|e^{n+1(j+1)} + e^n\|_h \, . \end{aligned}$$

Therefore, by the arithmetic–geometric mean inequality

$$\begin{aligned} &\frac{1}{k}\left(\|e^{n+1(j+1)}\|_h^2 - \|e^n\|_h^2\right) \le \left|e^{n+1(j+1)} + e^n\right|_{1,h}^2 - \frac{\nu}{2}\left|e^{n+1(j+1)} + e^n\right|_{2,h}^2 \\ &+ \frac{3}{2}\,M^2\left(\|e^{n+1(j)} + e^n\|_h^2 + \|e^n + \hat{e}^{n+1}\|_h^2\right) + \frac{3}{128}\,h^{-1}\,\|e^n + \hat{e}^{n+1}\|_h^4 \\ &+ \frac{1}{2}\,\|s^n\|_h^2 + \frac{1}{2}\,\|e^{n+1(j+1)} + e^n\|_h^2 \, . \end{aligned} \tag{4.12}$$

Thus, using (2.11), for $n \ge 1$ we have

$$\begin{aligned} (1 - d_1\,k)\,\|e^{n+1(j+1)}\|_h^2 \ &\le\ d_2\,k\,\|e^{n+1(j)}\|_h^2 + (1 + d_3\,k)\,\|e^n\|_h^2 \\ &\quad + d_4\,k\,\|e^{n-1}\|_h^2 + d_5\,k\left(k^2 + h^2\right)^2 , \end{aligned}$$

and conclude that (4.7) holds for $\ell = n + 1$.
Finally, for $n = 0$, (4.12) and (4.6) yield for k small enough

$$\|e^1\|_h^2 \ \le\ C\,k\left(\|\hat{e}^1\|_h^2 + \|s^0\|_h^2\right) .$$

Using (4.6) and (4.9), we see that (4.7) is valid for $\ell = 1$, and the proof is complete. □

Taking $j_n = 1$, $n = 1, \dots, N$, the scheme described above takes for $n \ge 1$ the form

$$\begin{aligned} &\partial V^n + \Delta_h\,V^{n+1/2} + \nu\,\Delta_h^2\,V^{n+1/2} = \\ &= -\frac{1}{24h}\,\varphi\big(V^n + \hat{V}^{n+1},\ V^n + \hat{V}^{n+1}\big) , \end{aligned} \tag{4.13}$$

which is the standard way for linearizing the second–order scheme (1.6) by extrapolating in the nonlinear term from previous time levels.

Remark. We assume here that the initial-value u^0 is an odd function. Then, $v, v(x,t) := -u(-x,t)$, is a solution of (1.1)-(1.2). Consequently, by uniqueness $v = u$, i.e., $u(\cdot,t)$ is odd for $0 \leq t \leq T$. This property is carried over to the approximations discussed above. For instance, $\tilde{U}^n, \tilde{U}^n_i := -U^n_{-i}, i \in \mathbb{Z}, n = 0, \ldots, N$, satisfy (1.6). Therefore, for $k\ h^{-1/5}$ sufficiently small $\tilde{U}^n = U^n$. This results to a reduction of the number of the nonlinear equations (1.6) from J to $\lceil \frac{J+1}{2} \rceil - 1$.

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