Derivadas de ordem superior

Em geral, se $z = f(x,y)$ então f_x e f_y são também funções de duas variáveis. Definimos as derivadas parciais de segunda-ordem de f como as derivadas parciais de f_x e f_y.

Logo,

$$(f_x)_x = f_{xx} = f_{11} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$

$$(f_y)_y = f_{yy} = f_{22} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}$$

Temos também as parciais mistas:

$$(f_x)_y = f_{xy} = f_{12} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 z}{\partial y \partial x}$$

$$(f_y)_x = f_{yx} = f_{21} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 z}{\partial x \partial y}$$

Exemplo: Calcule as derivadas parciais de segunda ordem de $f(x,y) = x^3 + y^2 e^x$.

$$\Rightarrow f_x(x,y) = \frac{\partial}{\partial x}(x^3 + y^2 e^x) = 3x^2 + y^2 e^x$$

$$\Rightarrow f_y(x,y) = \frac{\partial}{\partial y}(x^3 + y^2 e^x) = e^x \cdot 2y = 2y e^x$$

<u>Derivadas de segunda ordem</u> $\Rightarrow \underbrace{(f_x)_x \, ; (f_x)_y \, ; (f_y)_y \, ; (f_y)_x}$

$$f_{xx}(x,y) = \frac{\partial}{\partial x} f_x = \frac{\partial}{\partial x}(3x^2 + y^2 e^x) = 6x + y^2 e^x$$

$$f_{yy}(x,y) = \frac{\partial}{\partial y} f_y = 2y e^x = 2e^x$$

$$\vec{f_{xy}}(x,y) = \frac{\partial}{\partial y} f_x = \frac{\partial}{\partial y}(3x^2 + y^2 e^x) = e^x 2y = \boxed{2y e^x}$$

$$\vec{f_{yx}}(x,y) = \frac{\partial}{\partial x} f_y = \frac{\partial}{\partial x} 2y e^x = \boxed{2y e^x}$$

<u>Teorema</u>: <u>Igualdade de Parciais Mistas</u> (Teorema de Clairaut)

Se f_{xy} e f_{yx} ambas existem e são contínuas em um disco D, então $f_{xy}(a,b) = f_{yx}(a,b)$ $\forall (a,b) \in D$. Portanto, em D,

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) \Longleftrightarrow f_{xy} = f_{yx}$$

Exemplo: Verifique que $\dfrac{\partial^2 w}{\partial U \partial T} = \dfrac{\partial^2 w}{\partial T \partial U}$ para

$$W = e^{U/T}$$

$$\frac{\partial w}{\partial T} = \frac{\partial}{\partial T} e^{U/T} = e^{U/T} \cdot \frac{\partial}{\partial T}\left(\frac{U}{T}\right) = U e^{U/T} \cdot \frac{\partial}{\partial T}\left(\frac{1}{T}\right)$$

$$= U \cdot e^{U/T} \cdot (-T^{-2}) = \boxed{-U T^{-2} e^{U/T}}.$$

$$\frac{\partial w}{\partial U} = \frac{\partial}{\partial U} e^{U/T} = e^{U/T} \cdot \frac{\partial}{\partial U}\left(\frac{U}{T}\right) = \frac{1}{T} e^{U/T} \cdot \frac{\partial}{\partial U}(U)$$

$$= \boxed{\frac{1}{T} e^{U/T}} = T^{-1} e^{U/T}.$$

Derivadas parciais mistas:

$$\underbrace{\frac{\partial}{\partial U}\left(\frac{\partial w}{\partial T}\right)}_{W_{TU}} = \frac{\partial}{\partial U}\left(-U T^{-2} e^{U/T}\right) = -T^{-2} \frac{\partial}{\partial U}\left(U \cdot e^{U/T}\right)$$

$$= -T^{-2} \cdot \left(1 \cdot e^{U/T} + U \cdot \underbrace{\frac{\partial}{\partial U} e^{U/T}}\right)$$

$$= -T^{-2}\left(e^{U/T} + U \cdot T^{-1} e^{U/T}\right)$$

$$= \boxed{-T^{-2} e^{U/T} - U T^{-3} e^{U/T}}$$

$$\overbrace{\frac{\partial}{\partial T}\left(\frac{\partial w}{\partial U}\right)}^{W_{UT}} = \frac{\partial}{\partial T}\left(T^{-1} \cdot e^{U/T}\right) = \left(-T^{-2} \cdot e^{U/T} + T^{-1} \cdot \frac{\partial}{\partial T} e^{U/T}\right)$$

$$= -T^{-2} \cdot e^{U/T} + T^{-1} \cdot \left(-U T^{-2} e^{U/T}\right)$$

$$= \boxed{-T^{-2} e^{U/T} - U T^{-3} e^{U/T}}$$

Obs.: Em geral, diferenciação parcial pode ser realizada em qualquer ordem, desde que as derivadas em questão sejam contínuas. Por exemplo,

$$f_{xyxy} = f_{xxyy} = f_{yyxx} = f_{yxyx} = f_{xyyx}$$
$$= f_{yxxy}.$$

Exemplo: Calcule a derivada parcial g_{zzwx}, onde

$$g(x, y, z, w) = x^3 w^2 z^2 + \operatorname{sen}\left(\frac{xy}{z^2}\right)$$

$\Rightarrow$ Diferenciando primeiro em relação a w, temos:

$$g_w = \frac{\partial}{\partial w}\left(x^3 w^2 z^2 + \underbrace{\operatorname{sen}\left(\frac{xy}{z^2}\right)}\right) = x^3 z^2 \cdot 2w$$
$$= \boxed{2x^3 w z^2}$$

Agora, calculemos g_{wz}, g_{wzz} e g_{wzzx}:

(*) $$g_{wz} = \frac{\partial}{\partial z} g_w = \frac{\partial}{\partial z}\left(2x^3 w z^2\right) = 2x^3 w \cdot 2z$$
$$= 4x^3 w z$$

(*) $g_{wzz} = \frac{\partial}{\partial z} g_{wz} = \frac{\partial}{\partial z}(4x^3wz) = 4x^3w$.

(*) $g_{wzzx} = \frac{\partial}{\partial x} g_{wzz} = \frac{\partial}{\partial x}(4x^3w) = 4w.3x^2$
$= \underbrace{12x^2w}$.

$$\underbrace{g_{zzwx}} = g_{wzzx} = \boxed{12x^2w}$$

<u>Exemplo</u>: Seja $f(x,y) = \begin{cases} xy\,\frac{x^2-y^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

(1.) Calcule $f_x(x,y)$ e $f_y(x,y)$ para $(x,y) \neq (0,0)$

$$f_x(x,y) = \frac{\partial}{\partial x}\left(\underbrace{xy}_{1} . \overbrace{\frac{x^2-y^2}{x^2+y^2}}^{2}\right) =$$

$$= \frac{\partial}{\partial x}(xy) . \frac{x^2-y^2}{x^2+y^2} + (xy) . \frac{\partial}{\partial x}\left(\frac{x^2-y^2}{x^2+y^2}\right)$$

$$= y . \frac{x^2-y^2}{x^2+y^2} + xy . \frac{2x.(x^2+y^2)-(x^2-y^2)2x}{(x^2+y^2)^2}$$

$$= y\,\frac{x^2-y^2}{x^2+y^2} + xy\,\frac{\cancel{2x^3}+2xy^2-\cancel{2x^3}+2xy^2}{(x^2+y^2)^2}$$

$$= y\,\frac{x^2-y^2}{\underbrace{x^2+y^2}} + xy\cdot\frac{4xy^2}{(x^2+y^2)^2}$$

$$= \frac{y(x^2-y^2)\cdot(x^2+y^2)+4x^2y^3}{(x^2+y^2)^2}$$

$(a-b)(a+b) = a^2-b^2$

$$= \frac{y(x^4-y^4)+4x^2y^3}{(x^2+y^2)^2} = \boxed{\frac{y(x^4+4x^2y^2-y^4)}{(x^2+y^2)^2}}$$

$\hookrightarrow \dfrac{\partial f}{\partial x} = f_x$ //

(*) $$f_y(x,y) = \frac{\partial}{\partial y}\left(xy\cdot\overbrace{\frac{x^2-y^2}{x^2+y^2}}\right)$$

$$= \frac{\partial}{\partial y}(xy)\cdot\frac{x^2-y^2}{x^2+y^2} + xy\cdot\frac{\partial}{\partial y}\left(\frac{x^2-y^2}{x^2+y^2}\right)$$

$$= x\cdot\frac{x^2-y^2}{x^2+y^2} + xy\cdot\frac{(-2y)\cdot(x^2+y^2)-(x^2-y^2)\cdot 2y}{(x^2+y^2)^2}$$

$$= x\cdot\frac{x^2-y^2}{x^2+y^2} + xy\cdot\frac{-2yx^2-\cancel{2y^3}-2yx^2+\cancel{2y^3}}{(x^2+y^2)^2}$$

$$= x \cdot \frac{x^2-y^2}{x^2+y^2} + xy \cdot \frac{(-4yx^2)}{(x^2+y^2)^2}$$

$$= \frac{x\cdot(x^2-y^2)}{x^2+y^2} - \frac{4y^2x^3}{(x^2+y^2)^2}$$

$$= \frac{x\cdot(x^2-y^2)(x^2+y^2) - 4y^2x^3}{(x^2+y^2)^2}$$

$$= \frac{x\cdot(x^4-y^4) - 4y^2x^3}{(x^2+y^2)^2} = \boxed{\frac{x\cdot(x^4-4y^2x^2-y^4)}{(x^2+y^2)^2}}$$

$$\hookrightarrow \frac{\partial f}{\partial y} = f_y.$$

2. Mostrar que $f_x(0,0) = f_y(0,0) = 0$ e que $f_{yx}(0,0)$ e $f_{xy}(0,0)$ ambas existem mas não são iguais.

$$f_x(a,b) = \lim_{h\to 0} \frac{f(a+h,b)-f(a,b)}{h} = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h}$$

$a = 0$, $b = 0$

$$f(x,y) = xy\,\frac{x^2-y^2}{x^2+y^2}, \quad (x,y) \neq (0,0)$$

$$= \lim_{h\to 0} \frac{h \cdot 0 \cdot \frac{h^2-0^2}{h^2+0^2} - 0}{h} = \lim_{h\to 0} \frac{0}{h} = 0$$

$$\boxed{f_x(0,0) = 0}$$

$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,h+0) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{0-0}{h} = \lim_{h\to 0} \frac{0}{h} = 0$$

$$\boxed{f_y(0,0) = 0}$$

$$\circledast\ f_{yx}(0,0) = \frac{\partial}{\partial x}\boxed{f_y}\Big|_{(0,0)} = \lim_{h\to 0} \frac{f_y(h,0) - f_y(0,0)}{h}$$

$$\frac{\partial f}{\partial x} = f_x(a,b) = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

$$= \lim_{h\to 0} \frac{h-0}{h} = \lim_{h\to 0} 1 = 1 \text{ //}$$

$$f_{xy}(0,0) = \frac{\partial}{\partial y} f_x \Big|_{(0,0)} = \lim_{h\to 0} \frac{f_x(0,h) - f_x(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{-h-0}{h} = \lim_{h\to 0} -1 = -1$$

$\Rightarrow$ Vimos que as parciais mistas no ponto $(0,0)$ existem mas não são iguais.

3. Mostre que f_{xy} não é contínua em $(0,0)$

$$\boxed{\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)}$$

↳ Continuidade da função f

Mostrar: $\lim_{(x,y)\to(0,0)} f_{xy}(x,y)$ não existe.