Normal Modes

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Initial condition:

$$\begin{aligned} x(0) &= x_0 \\ \dot{x}(0) &= v_0 \end{aligned}$$

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Initial condition:

$$\begin{aligned} x(0) &= x_0 \\ \dot{x}(0) &= v_0 \end{aligned}$$

Solution:

$$\begin{aligned} x(t) &= x_0 \cos(\omega t) + \frac{v_0}{\omega} \sin(\omega t) \\ \dot{x}(t) &= -\omega x_0 \sin(\omega t) + v_0 \cos(\omega t) \end{aligned}$$

Simple Harmonic Oscillator Matrix form

On computer we can use first order ODE's

$$\dot{x} = v, \quad \dot{v} = -\omega^2 x.$$

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If we use matrix form:

$$\begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix}$$

Matrix exponential equation

Make the change $v \leftarrow \frac{v}{\omega}$ then we have

$$\begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} = \omega \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix}$$

with formal solution

$$\begin{bmatrix} x \\ v \end{bmatrix} = \exp\left(\omega t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right) \begin{bmatrix} x_0 \\ v_0 \end{bmatrix}$$

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Eigenvalues $\mathrm{i}[-\omega, \omega]$ and eigenvectors

$$\begin{bmatrix} 1 \\ -\mathrm{i} \end{bmatrix}; \begin{bmatrix} 1 \\ \mathrm{i} \end{bmatrix}$$

Matrix exponential equation

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$$\begin{bmatrix} 1 \\ -\mathrm{i} \end{bmatrix}; \begin{bmatrix} 1 \\ \mathrm{i} \end{bmatrix}$$

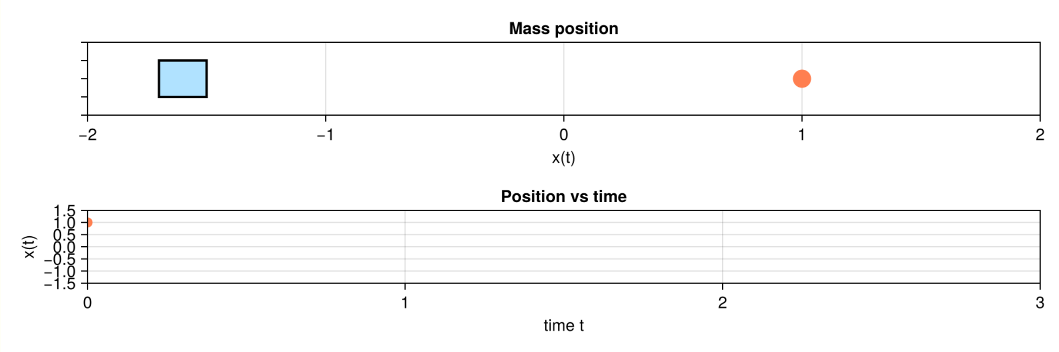
How to go from here to our previous solution?

Simple Harmonic Oscillator Code

```
function harmonic_oscillator!(du, u ,p, t)
    ω = p.ω
    du[1] = u[2]
    du[2] = - ω^2*u[1]
    return nothing
end

T = Float64
p = (ω = 2pi,
     τ = 3)
u0 = T[1, 0]
tspan = (T(0), T(2pi/p.ω * p.τ))
prob = ODEProblem(harmonic_oscillator!, u0, tspan, p)
sol = solve(prob)
```

Simple Harmonic Oscillator Animation



2 Oscillators

2 Oscillators

Linear Chain of two Oscillators Differential Equation:

$$\begin{aligned} \ddot{x}_1 &= -\omega^2 x_1 - \omega^2 (x_1 - x_2) \qquad (2) \\ \ddot{x}_2 &= -\omega^2 x_2 - \omega^2 (x_2 - x_1) \qquad (3) \end{aligned}$$

Linear Chain of two Oscillators Differential Equation:

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(Linear) change of variables to $y_1 = x_1 - x_2$ and $y_2 = x_1 + x_2$:

$$\begin{aligned} \ddot{y}_1 &= -(\sqrt{3}\omega)^2 y_1 && (4) \\ \ddot{y}_2 &= -\omega^2 y_2 && (5) \end{aligned}$$

2 Oscillators

Linear Chain of two Oscillators Differential Equation:

$$\ddot{x}_1 = -\omega^2 x_1 - \omega^2 (x_1 - x_2) \tag{2}$$
$$\ddot{x}_2 = -\omega^2 x_2 - \omega^2 (x_2 - x_1) \tag{3}$$

(Linear) change of variables to $y_1 = x_1 - x_2$ and $y_2 = x_1 + x_2$:

$$\ddot{y}_1 = -(\sqrt{3}\omega)^2 y_1 \tag{4}$$
$$\ddot{y}_2 = -\omega^2 y_2 \tag{5}$$

Solution:

$$x_1(t) - x_2(t) = (x_1(0) - x_2(0)) \cos\left(\sqrt{3}\omega t\right) + \frac{\dot{x}_1(0) - \dot{x}_2(0)}{\sqrt{3}\omega} \sin\left(\sqrt{3}\omega t\right)$$
$$x_1(t) + x_2(t) = (x_1(0) + x_2(0)) \cos(\omega t) + \frac{\dot{x}_1(0) + \dot{x}_2(0)}{\omega} \sin(\omega t)$$

2 Oscillators Matrix form

In matrix form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{v}_1 \\ \dot{v}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2\omega^2 & \omega^2 & 0 & 0 \\ \omega^2 & -2\omega^2 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ v_1 \\ v_2 \end{bmatrix} \tag{6}$$

2 Oscillators Matrix form

In matrix form:

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with eigenvalues $\mathrm{i}[-\omega, \omega, -\sqrt{3}\omega, \sqrt{3}\omega]$ and eigenvectors

$$\begin{bmatrix} -\mathrm{i} \\ -\mathrm{i} \\ -1 \\ -1 \end{bmatrix}; \begin{bmatrix} \mathrm{i} \\ \mathrm{i} \\ -1 \\ -1 \end{bmatrix}; \begin{bmatrix} \mathrm{i} \\ -\mathrm{i} \\ \sqrt{3} \\ -\sqrt{3} \end{bmatrix}; \begin{bmatrix} -\mathrm{i} \\ \mathrm{i} \\ \sqrt{3} \\ -\sqrt{3} \end{bmatrix}$$

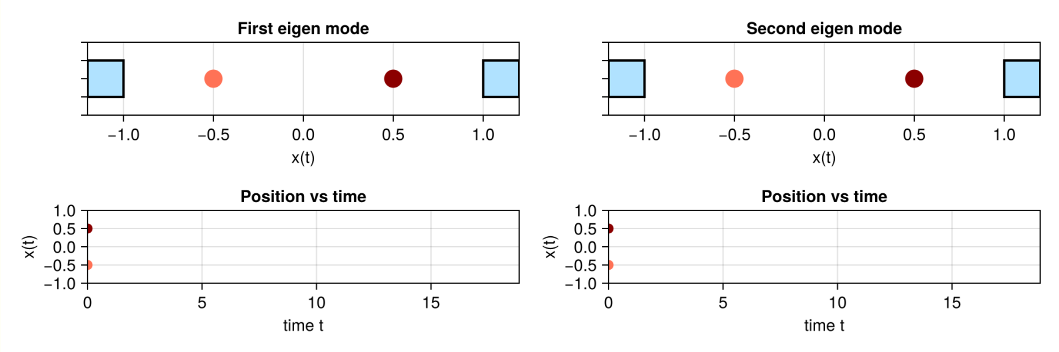
2 Oscillators Code

```
function two_harmonic_oscillator!(du, u, p, t)
    ω = p.ω
    du[1] = u[3]
    du[2] = u[4]
    du[3] = -ω^2 * (2*u[1] - u[2])
    du[4] = -ω^2 * (2*u[2] - u[1])
    return nothing
end

# eigen modes
ω = p.ω
A = T[
     0      0    1   0;
     0      0    0   1;
  -2ω^2   ω^2   0   0;
    ω^2 -2ω^2   0   0
]
```

2 Oscillators Code

```
F = eigen(A, sortby=abs)
min_omega = abs(imag(F.values[1]))
max_omega = abs(imag(F.values[end]))
@show min_omega # 1
@show max_omega # sqrt(3) ~ 1.73205...
tspan = (T(0), T(2pi/min_omega * p.τ))

# first eigen mode
u0_1 = T.(real.(F.vectors[:,1] + F.vectors[:,2]) )
prob_1 = ODEProblem(two_harmonic_oscillator!, u0_1, tspan, p)
sol_1 = solve(prob_1)
```

2 Oscillators Animation



N Oscillators

N Oscillators

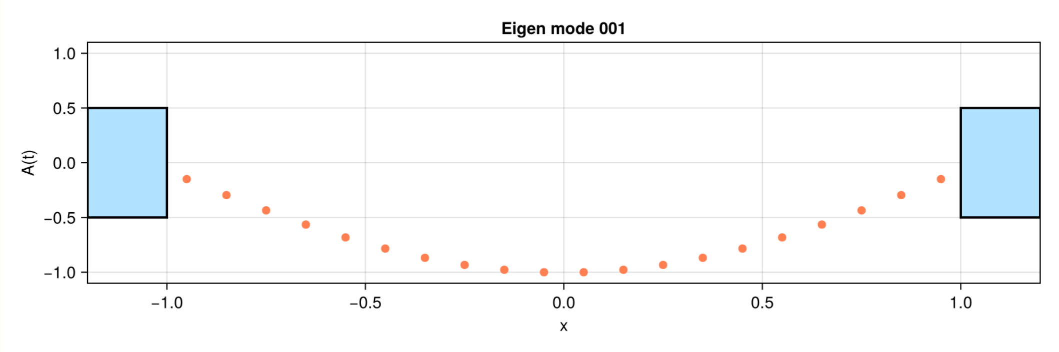
Linear Chain of N Oscillators Differential Equation:

$$\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \\ \vdots \\ \ddot{x}_{N-1} \\ \ddot{x}_N \end{bmatrix} = \omega^2 \begin{bmatrix} -2 & 1 & 0 & \cdots & 0 & 0 \\ 1 & -2 & 1 & \cdots & 0 & 0 \\ 0 & 1 & -2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{N-1} \\ x_N \end{bmatrix} \tag{7}$$

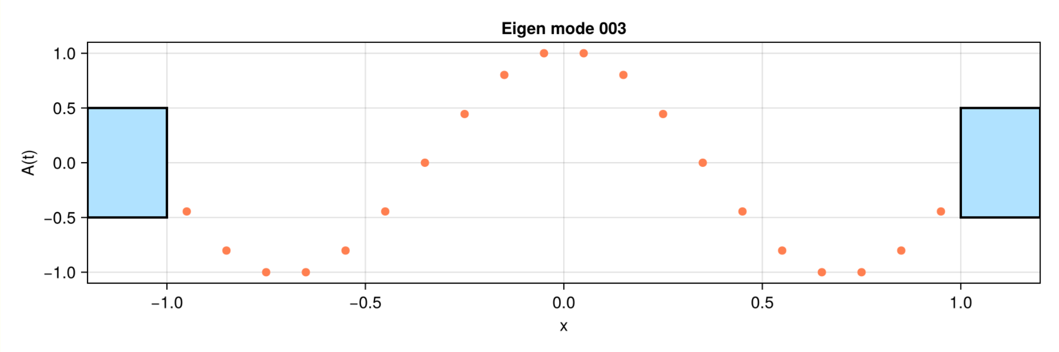
N Oscillators Code

```
function n_harmonic_oscillator(du, u, p, t)
    mul!(du, p.A, u)
    return nothing
end
T = Float64
p = (ω = 1,
     τ = 3,
     N = 20)

A = make_A_simple_matrix(T, p)
F = eigen(Matrix(A), sortby=abs)
min_omega = abs(imag(F.values[1]))
tspan = (T(0), T(2pi/min_omega * p.τ))
l_mode = 1
u0 = T.(imag.(F.vectors[:,l_mode] -  F.vectors[:,l_mode+1] ))
prob = ODEProblem(n_harmonic_oscillator, u0, tspan, (A=A,))
sol = solve(prob)
```

N Oscillators Animation



N Oscillators Animation



N Oscillators Eigen Energies and Frequencies

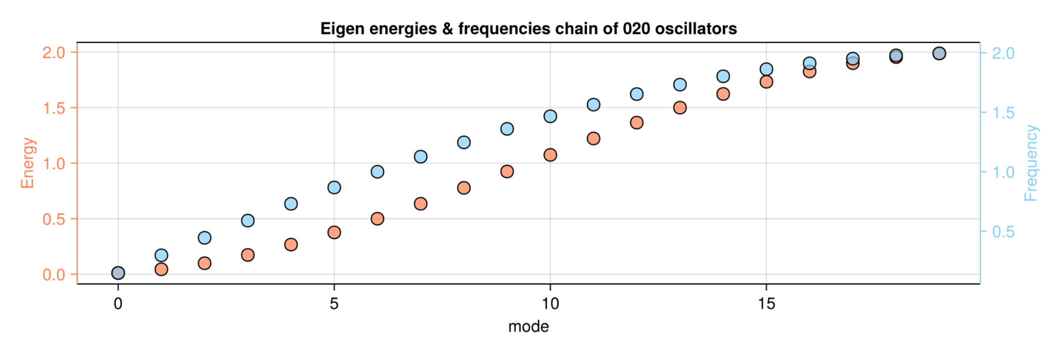


Figure 1: Eigen energies and frequencies for a chain of 20 oscillators.