"Normal modes" in Physics equations

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Maxwell's Equations:

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\varepsilon_0} \tag{1}$$

$$\nabla \cdot \boldsymbol{B} = 0 \tag{2}$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \tag{3}$$

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} + \mu_0 \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} \tag{4}$$

Maxwell's Equations:

$$\begin{aligned}
\boldsymbol{\nabla} \cdot \boldsymbol{E} &= \frac{\rho}{\varepsilon_0} && (1)\\
\boldsymbol{\nabla} \cdot \boldsymbol{B} &= 0 && (2)\\
\boldsymbol{\nabla} \times \boldsymbol{E} &= -\frac{\partial \boldsymbol{B}}{\partial t} && (3)\\
\boldsymbol{\nabla} \times \boldsymbol{B} &= \mu_0 \boldsymbol{J} + \mu_0 \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} && (4)
\end{aligned}$$

Consider no charges $\rho = 0$ and no currents $\boldsymbol{J} = \mathbf{0}$. With $c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$ this gives the vector wave equations:

$$\nabla^2 \mathbf{E} - \frac{1}{c^2}\, \partial_t^2 \mathbf{E} = \mathbf{0}, \qquad \nabla^2 \mathbf{B} - \frac{1}{c^2}\, \partial_t^2 \mathbf{B} = \mathbf{0}.$$

Separation of variables & Dirichlet boundary conditions

Consider a parallel-plate cavity on $x \in [0, L]$ and a transverse field component $E_y(x,t) = \varphi(t)\,\phi(x)$.

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Eigenmodes. (Normal modes) $\phi_n(x) = \sin\left(\frac{n\pi x}{L}\right)$, $\varphi_n(t) = \exp(\pm \mathrm{i}\omega_n t)$, $k_n = \frac{n\pi}{L}$, $\omega_n = ck_n$, $n = 1, 2, \ldots$.

Finite Differences

On the computer the derivative can be approximated as:

$$\frac{\partial \phi}{\partial x}(x_i) \approx \frac{\phi(x_{i+1}) - \phi(x_{i-1})}{2\Delta x} \tag{5}$$

$$\frac{\partial^2 \phi}{\partial x^2}(x_i) \approx \frac{\phi(x_{i+1}) - 2\phi(x_i) + \phi(x_{i-1})}{\Delta x^2} \tag{6}$$

where we assume to have a grid discretization $x_0, x_1, \cdots, x_N, x_{N+1}$, $\Delta x = x_i - x_{i-1}$ and the function evaluated on those points $\phi(x_i) = \phi_i$.

Finite Differences

Then considering the derivative approximation + BC we obtain:

$$\frac{1}{\Delta x^2}\begin{bmatrix} -2 & 1 & 0 & \cdots & 0 & 0 \\ 1 & -2 & 1 & \cdots & 0 & 0 \\ 0 & 1 & -2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -2 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \vdots \\ \phi_{N-1} \\ \phi_N \end{bmatrix} = -k^2 \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \vdots \\ \phi_{N-1} \\ \phi_N \end{bmatrix} \tag{7}$$

where we use the boundary condition $\phi_0 = 0$ and $\phi_{N+1} = 0$.

Wave Equation Eigensystem Code

```julia
function analytical_eigenstate(i_mode)
    return x -> sin(x * pi * i_mode/ p.L)
end

T = Float64
p = (L = 1,
    N = 100,
    c = 1,
    )
A = make_EM_wave_Dirichlet_matrix(T, p)
x = LinRange(T(0), T(p.L), p.N+2)
F = eigen(Matrix(A), sortby=abs)
```

Wave Equation solution

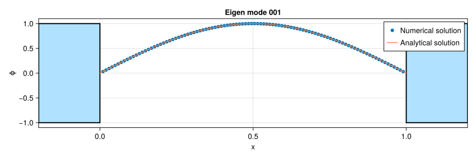

Figure 1: First eigen mode

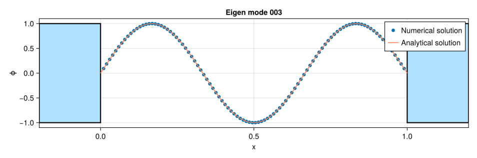

Figure 2: Third eigen mode

Wave Equation Eigenvalues

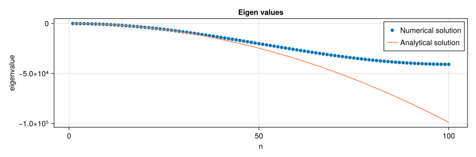

Figure 3: Wave Equation Eigenvalues

Wave Equation Eigenvalues

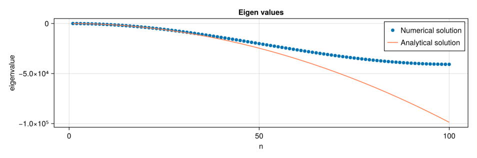

Figure 3: Wave Equation Eigenvalues

Can we do better?

Wave Equation Eigenvalues

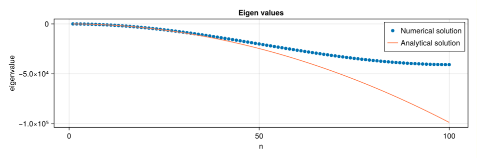

Figure 3: Wave Equation Eigenvalues

Can we do better?

- Higher order finite difference approx

Wave Equation Eigenvalues

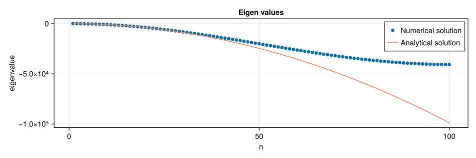

Figure 3: Wave Equation Eigenvalues

Can we do better?

- Higher order finite difference approx
- Finite Element Method

Wave Equation Eigenvalues

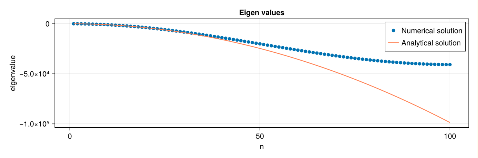

Figure 3: Wave Equation Eigenvalues

Can we do better?

- Higher order finite difference approx
- Finite Element Method
- Sine Fourier Transform

Diffusion Equation

Diffusion Equation

Diffusion Equation:

$$\frac{\partial \rho}{\partial t} = \nabla \cdot (D \nabla \rho) \tag{8}$$

- ρ particle density.
- $\boldsymbol{j} = -D\nabla\rho$ particle density current.

Separation of variables & Neumann boundary conditions

Consider a parallel-walls cavity on $x \in [0, L]$, a probability density $\rho(x,t) = \varphi(t)\,\phi(x)$, and D constant.

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The 1D diffusion equation becomes

$$\phi''(x) + k^2\phi(x) = 0, \qquad \dot{\varphi}(t) + Dk^2\varphi(t) = 0.$$

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Eigenmodes. (Normal modes) $\phi_n(x) = \cos\left(\frac{n\pi x}{L}\right),\ \ \varphi_n(t) = \exp(-\tau_n t), \quad k_n = \frac{n\pi}{L}$, $\tau_n = Dk_n^2,\ \ n = 0, 1, \ldots.$

Finite Differences

On the computer the derivative + Neumann BC can be approximated as:

$$\frac{1}{\Delta x^2}\begin{bmatrix} -1 & 1 & 0 & \cdots & 0 & 0 \\ 1 & -2 & 1 & \cdots & 0 & 0 \\ 0 & 1 & -2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -1 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \vdots \\ \phi_{N-1} \\ \phi_N \end{bmatrix} = -k^2 \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \vdots \\ \phi_{N-1} \\ \phi_N \end{bmatrix} \tag{9}$$

where we use the boundary condition $\nabla\phi_0 = 0$ and $\nabla\phi_{N+1} = 0$.

Difussion Equation Eigensystem Code

```julia
function analytical_eigenstate(i_mode)
    return x -> cos(x * pi * i_mode/ p.L)
end

T = Float64
p = (L = 1,
    N = 100,
    c = 1,
    )
A = make_Diffusion_eq_Neumann_matrix(T, p)
x = LinRange(T(0), T(p.L), p.N+2)
F = eigen(Matrix(A), sortby=abs)
```

Diffusion Equation solution

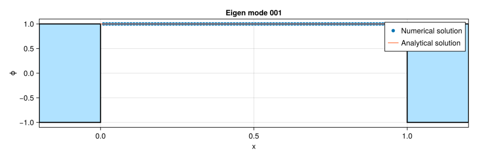

Figure 4: First eigen mode

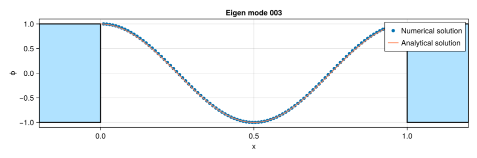

Figure 5: Third eigen mode

Schrödinger Equation

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Schrödinger Equation in position basis:

$$\mathrm{i}\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V(\boldsymbol{x})\psi \tag{10}$$

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- $-\frac{\hbar^2}{2m}\nabla^2\psi$ comes from momentum contribution to the Hamiltonian.
- $V(\boldsymbol{x})$ is the potential energy contribution to the Hamiltonian.

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- ψ gives probability amplitude.
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- $V(\boldsymbol{x})$ is the potential energy contribution to the Hamiltonian.

We will consider $V(\boldsymbol{x}) = V_0$ for $x \in [-L, L]$ and $V(\boldsymbol{x}) = 0$ elsewhere.

Finite well $V_0 < 0$

Bound states

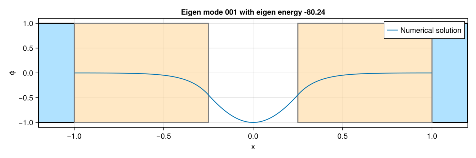

Figure 6: First bound state

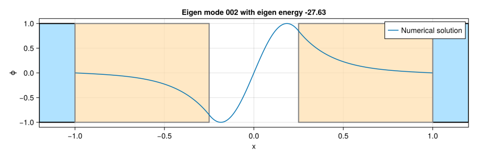

Figure 7: Second bound state

Finite well $V_0 < 0$

No bound states

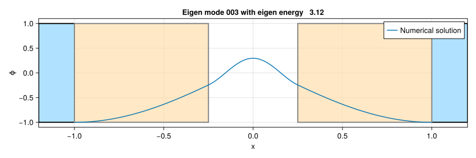

Figure 8: First no bound state

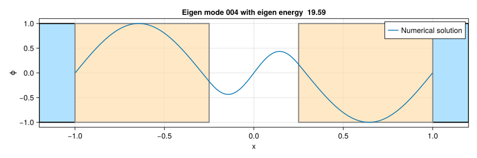

Figure 9: Second no bound state

Finite barrier $V_0 > 0$

Near degenerated eigen states

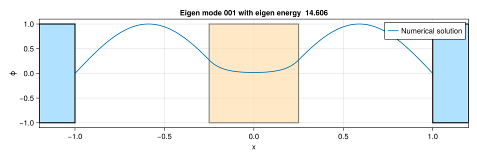

Figure 10: First eigen state

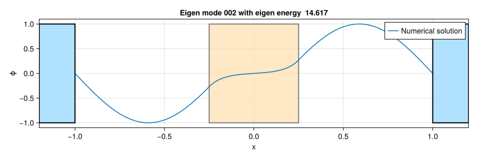

Figure 11: Second eigen state

Sum of near degenerated eigen states