Nonlinear Interactions and Collective Dynamics

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Chain of Nonlinear Oscillators

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Periodic Linear Chain of N Nonlinear Oscillators Differential Equation:

$$\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \\ \vdots \\ \ddot{x}_{N-1} \\ \ddot{x}_N \end{bmatrix} = \omega^2 \begin{bmatrix} -2 & 1 & 0 & \cdots & 0 & 1 \\ 1 & -2 & 1 & \cdots & 0 & 0 \\ 0 & 1 & -2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{N-1} \\ x_N \end{bmatrix} - \begin{bmatrix} f(x_1) \\ f(x_2) \\ f(x_3) \\ \vdots \\ f(x_{N-1}) \\ f(x_N) \end{bmatrix} \tag{1}$$

Chain of Nonlinear Oscillators

Periodic Linear Chain of N Nonlinear Oscillators Differential Equation:

$$
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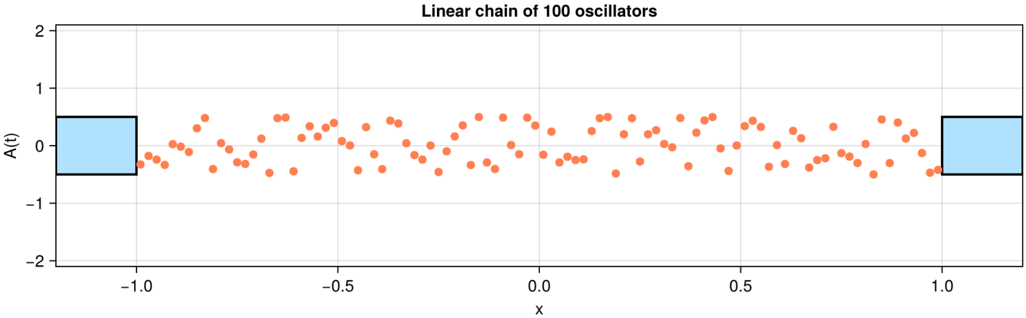
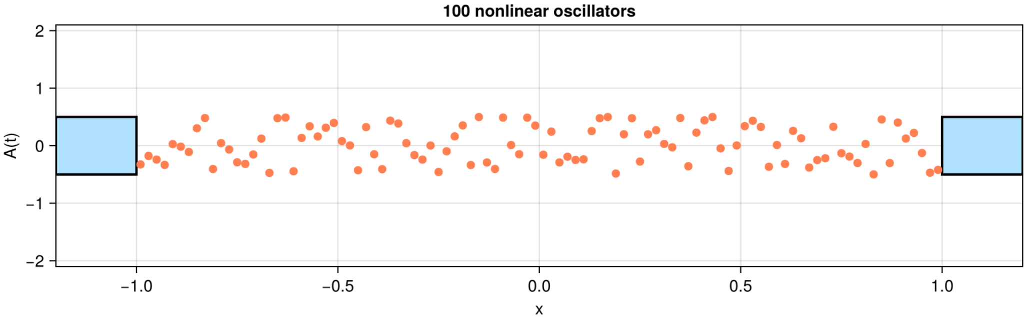
$$

$$
f(x) = ax^3 + bx \tag{2}
$$

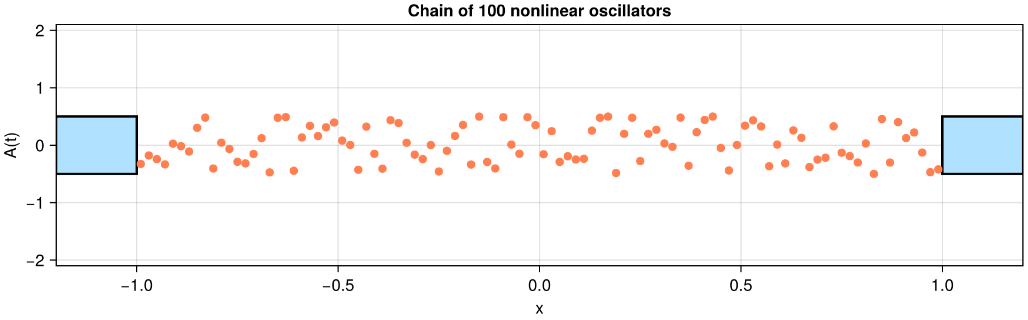
Chain N Linear Oscillators random position



N Nonlinear Oscillators random position



Chain N Nonlinear Oscillators random position



Nonlinear Schrödinger Equation

Nonlinear Schrödinger Equation

Nonlinear Schrödinger Equation in position basis (Gross–Pitaevskii Equation):

$$\mathrm{i}\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V(\boldsymbol{x})\psi + g|\psi|^2\psi \tag{3}$$

Nonlinear Schrödinger Equation

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- $g < 0$ gives focussing nonlinearity.
- $g > 0$ gives nonfocussing nonlinearity.

Solitons in NSLE

If we consider the 1D case $V(\boldsymbol{x}) = 0$ then a solution of NSLE is:

- Case $g < 0$, bright soliton:

$$\psi(x,t) \quad = \quad |\psi_0| \exp\left(-\frac{\mathrm{i}g|\psi_0|^2 t}{2\hbar}\right) \operatorname{sech}\left(\frac{\sqrt{-mg}|\psi_0|}{\hbar}x\right)$$

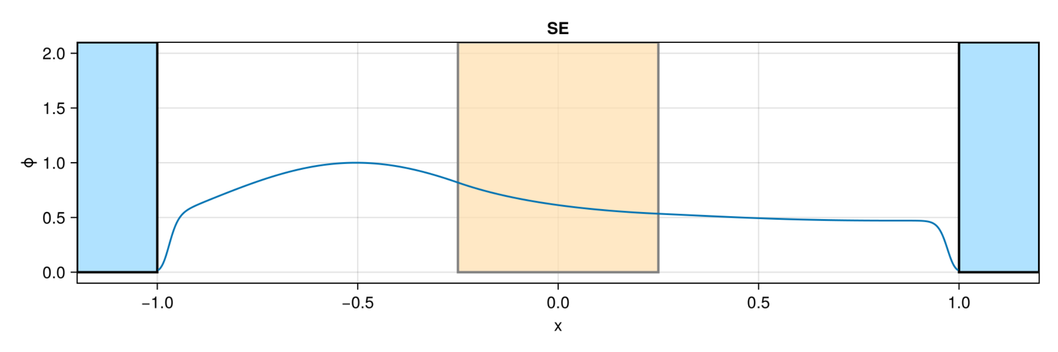
- Case $g > 0$, dark soliton:

$$\psi(x,t) \quad = \quad |\psi_0| \exp\left(-\frac{\mathrm{i}g|\psi_0|^2 t}{\hbar}\right) \tanh\left(\frac{\sqrt{mg}|\psi_0|}{\hbar}x\right)$$

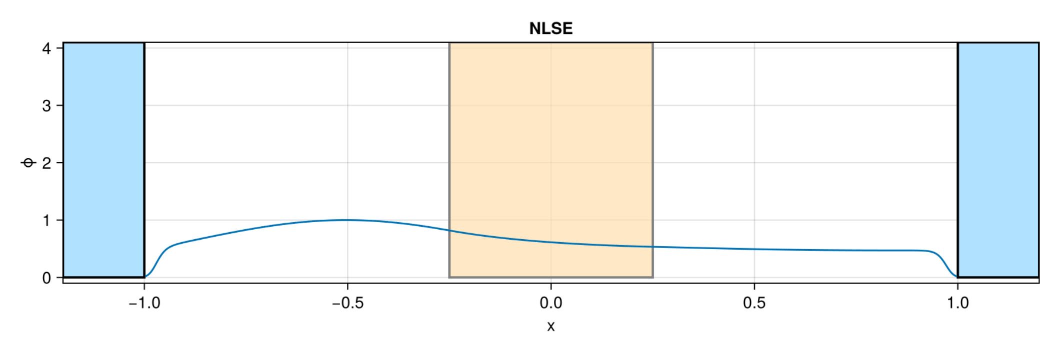
N Oscillators Code

```
function nlse!(du, u, p, t)
    A = p.A
    g = p.g
    r_du, i_du = eachslice(du, dims=1)
    r_u, i_u = eachslice(u, dims=1)

    # linear part
    mul!(r_du, A, i_u)
    mul!(i_du, A, -r_u)

    # non linear part
    map!((z,x,y)-> z + g*(x^2+y^2)*y, r_du, r_du, r_u, i_u)
    map!((z,x,y)-> z - g*(x^2+y^2)*x, i_du, i_du, r_u, i_u)
    return nothing
end
```

SE finite barrier



NSLE $g < 0$ finite barrier



Reaction Diffusion Barkley model

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The Barkley model of reaction diffusion

$$\frac{\partial u}{\partial t} = \frac{1}{\epsilon} u(1-u)\left(u - \frac{v+b}{a}\right) + \nabla^2 u \tag{4}$$

$$\frac{\partial v}{\partial t} = u - v + D_v \nabla^2 v \tag{5}$$

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- u activator, fast time scale.
- v inhibitor, slow time scale.

Reaction Diffusion Barkley model