Határozza meg a következő komplex számok trigonometrikus alakját.

(a) $1+i$

$1+i = r\cdot(\cos\varphi + i\cdot\sin\varphi)$ // $z := 1+i = a+bi \ (a,b\in\mathbb{R})$

$r = |z| = \sqrt{a^2+b^2} = \sqrt{1^2+1^2} = \sqrt{2}$

$\left.\begin{array}{l}\cos\varphi = \frac{a}{|z|} = \frac{1}{\sqrt{2}} = \sqrt{2} \\ b = 1 > 0\end{array}\right\} \varphi = 45^\circ$

$1+i = \sqrt{2}\cdot(\cos 45^\circ + i\cdot\sin 45^\circ)$

(b) $-\sqrt{3}+i$

$z = -\sqrt{3} + i = r\cdot(\cos\varphi + i\cdot\sin\varphi)$

$r = |z| = \sqrt{\sqrt{3}^2 + 1^2} = 2$

$\left.\begin{array}{l}\cos\varphi = \frac{-\sqrt{3}}{2} \\ b = 1 > 0\end{array}\right\} \Rightarrow \varphi = 150^\circ$

$-\sqrt{3}+i = 2\cdot(\cos 150^\circ + i\cdot\sin 150^\circ)$

(c) $\frac{9}{2} - \frac{9\sqrt{3}}{2}i$

$w = \frac{9}{2} - \frac{9\cdot\sqrt{3}}{2}i = r(\cos\varphi + i\cdot\sin\varphi)$

$r = |w| = \sqrt{\left(\frac{9}{2}\right)^2 + \left(-\frac{9\cdot\sqrt{3}}{2}\right)^2} = \sqrt{\frac{81+3\cdot 81}{4}} = \sqrt{\frac{4\cdot 81}{4}} = 9$

$\left.\begin{array}{l}\cos\varphi = {}^{9}/_{2} / 9 = {}^{1}/_{2} \\ b < 0\end{array}\right\} \Rightarrow \varphi' = 60^\circ,\ \varphi = 2\cdot 180^\circ - 60^\circ = 300^\circ$

$\frac{9}{2} - \frac{9\sqrt{3}}{2}i = 9\cdot(\cos 300^\circ + i\cdot\sin 300^\circ)$

(f) i

$i = r(\cos\varphi + i\cdot\sin\varphi) = z$

$r = |z| = \sqrt{0^2 + 1^2} = 1$

$\left.\begin{array}{l}\cos\varphi = \frac{0}{1} = 0 \\ b \geq 0\end{array}\right\} \Rightarrow \varphi = 90^\circ$

$i = \cos 90^\circ + i\cdot\sin 90^\circ$

(g) 10

$10 = r\cdot(\cos\varphi + i\cdot\sin\varphi) = w$

$r = |w| = \sqrt{10^2 + 0^2} = 10$

$\left.\begin{array}{l}\cos\varphi = \frac{10}{10} = 1 \\ b \geq 0\end{array}\right\} \Rightarrow \varphi = 0^\circ$

$10 = 10\cdot(\cos 0^\circ + i\cdot\sin 0^\circ)$

Végezze el a következő műveleteket a trigonometrikus alak felhasználásával.

(a) $\left(\frac{9}{2} - \frac{9\sqrt{3}}{2}i\right)\left(-\frac{\sqrt{14}}{2} - \frac{\sqrt{14}}{2}i\right)$ (i) $\left(1 - \frac{\sqrt{3}-i}{2}\right)^{24}$ (h) $\left(\frac{\frac{3}{2} + \frac{3\sqrt{3}}{2}i}{-\frac{5\sqrt{3}}{2} + \frac{5}{2}i}\right)^{12}$

Tétel (Moivre-azonosságok)

Legyenek $z, w \in \mathbb{C}$ *nemnulla komplex számok:* $z = |z|(\cos\varphi + i\sin\varphi)$, $w = |w|(\cos\psi + i\sin\psi)$, *és legyen* $n \in \mathbb{N}^+$. *Ekkor*

1. $zw = |z||w|(\cos(\varphi+\psi) + i\sin(\varphi+\psi))$;
2. $\frac{z}{w} = \frac{|z|}{|w|}\cdot(\cos(\varphi-\psi) + i\sin(\varphi-\psi))$;
3. $z^n = |z|^n(\cos n\varphi + i\sin n\varphi)$.

$$\underbrace{\left(\frac{9}{2}-\frac{9\sqrt{3}}{2}i\right)}_{9\cdot(\cos 300^\circ+i\cdot\sin 300^\circ)}\underbrace{\left(-\frac{\sqrt{14}}{2}-\frac{\sqrt{14}}{2}i\right)}_{\sqrt{7}\cdot(\cos 225^\circ+i\times\sin 225^\circ)}= 9\times\sqrt{7}\times(\cos\underbrace{(300+225)^\circ}_{525^\circ=1\times 360^\circ+165^\circ}+i\times\sin(300+225)^\circ) = 9\times\sqrt{7}\times(\cos 165^\circ+i\times\sin 165^\circ)$$

$$\left(1-\frac{\sqrt{3}-i}{2}\right)^{24}=\left(\left(1-\frac{\sqrt{3}}{2}\right)-\frac{1}{2}\cdot i\right)^{24}=\left(\sqrt{\left(1-\frac{\sqrt{3}}{2}\right)^2+\left(\frac{1}{2}\right)^2}\cdot(\cos\varphi+i\cdot\sin\varphi)\right)^{24}=$$

$$=\left(\sqrt{2-\sqrt{3}}\cdot(\cos 285^\circ+i\cdot\sin 285^\circ)\right)^{24}=\sqrt{2-\sqrt{3}}^{24}\cdot(\cos 0^\circ+i\cdot\sin 0^\circ)$$

$$\frac{24\cdot 285}{360}=\frac{285}{15}=19$$

$$\left(\frac{\frac{3}{2}+\frac{3\sqrt{3}}{2}i}{-\frac{5\sqrt{3}}{2}+\frac{5}{2}i}\right)^{12}=\left(\frac{3(\cos 60^\circ+i\cdot\sin 60^\circ)}{5\cdot(\cos 150^\circ+i\cdot\sin 150^\circ)}\right)^{12}$$

$$\frac{3}{2}+\frac{3\sqrt{3}}{2}i=\underbrace{\sqrt{\frac{9}{4}+\frac{9\cdot 3}{4}}}_{3}\cdot(\cos 60^\circ+i\sin 60^\circ)$$

$$\cos\varphi=\frac{3/2}{3}=\frac{3}{2}\cdot\frac{1}{3}=\frac{1}{2}\Rightarrow\varphi=60^\circ\ (b>0)$$

$$-\frac{5\sqrt{3}}{2}+\frac{5}{2}i=\underbrace{\sqrt{\frac{25\cdot 3+25}{4}}}_{5}\cdot(\cos 150^\circ+i\cdot\sin 150^\circ)$$

$$\sin\varphi=\frac{5/2}{5}=\frac{5}{2}\cdot\frac{1}{5}=\frac{1}{2}\Rightarrow\varphi=150^\circ\ (b>0)$$

$$\frac{3}{5}\cdot(\cos\underbrace{(60-150)^\circ}_{-90^\circ+360^\circ}+i\sin(60-150)^\circ)$$

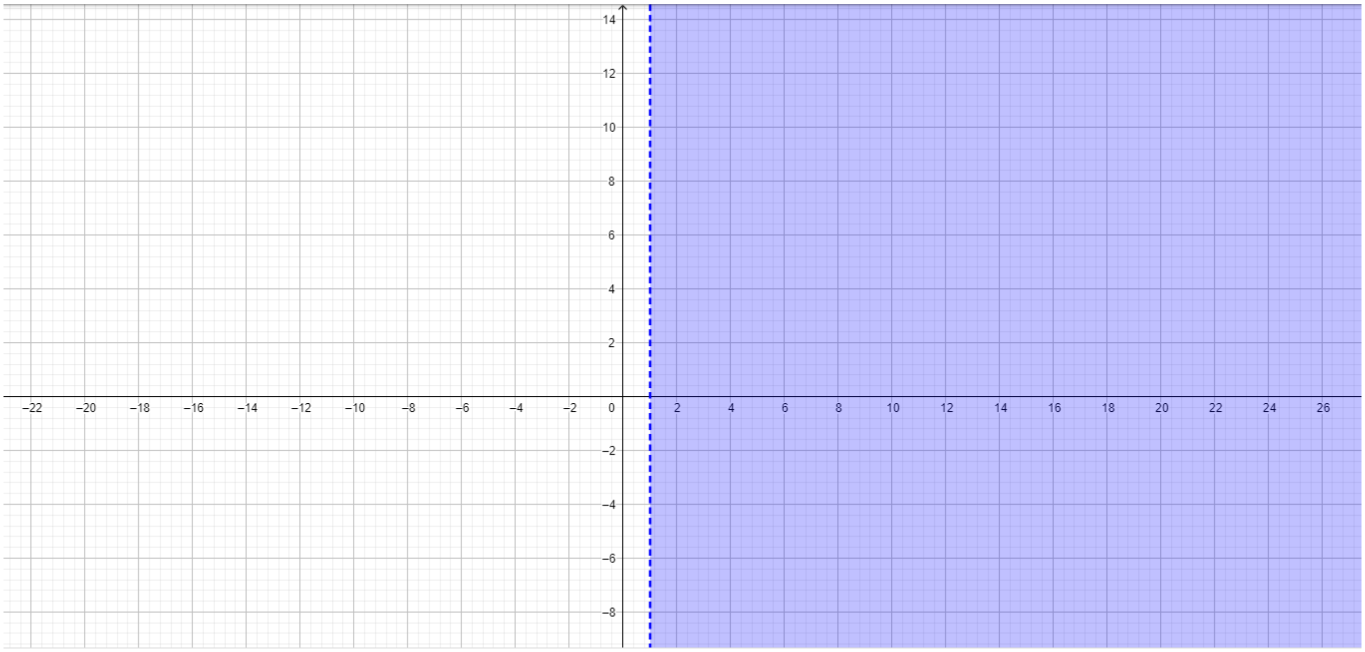
$$\left(\frac{\frac{3}{2}+\frac{3\sqrt{3}}{2}i}{-\frac{5\sqrt{3}}{2}+\frac{5}{2}i}\right)^{12}=\left(\frac{3(\cos 60^\circ+i\cdot\sin 60^\circ)}{5\cdot(\cos 150^\circ+i\cdot\sin 150^\circ)}\right)^{12}=\left(\frac{3}{5}\cdot(\cos 270^\circ+i\cdot\sin 270^\circ)\right)^{12}=$$

$$=\left(\frac{3}{5}\right)^{12}\cdot(\cos(12\cdot 270)^\circ+i\cdot\sin(12\cdot 270)^\circ)=\left(\frac{3}{5}\right)^{12}\cdot(\cos 0^\circ+i\cdot\sin 0^\circ)$$

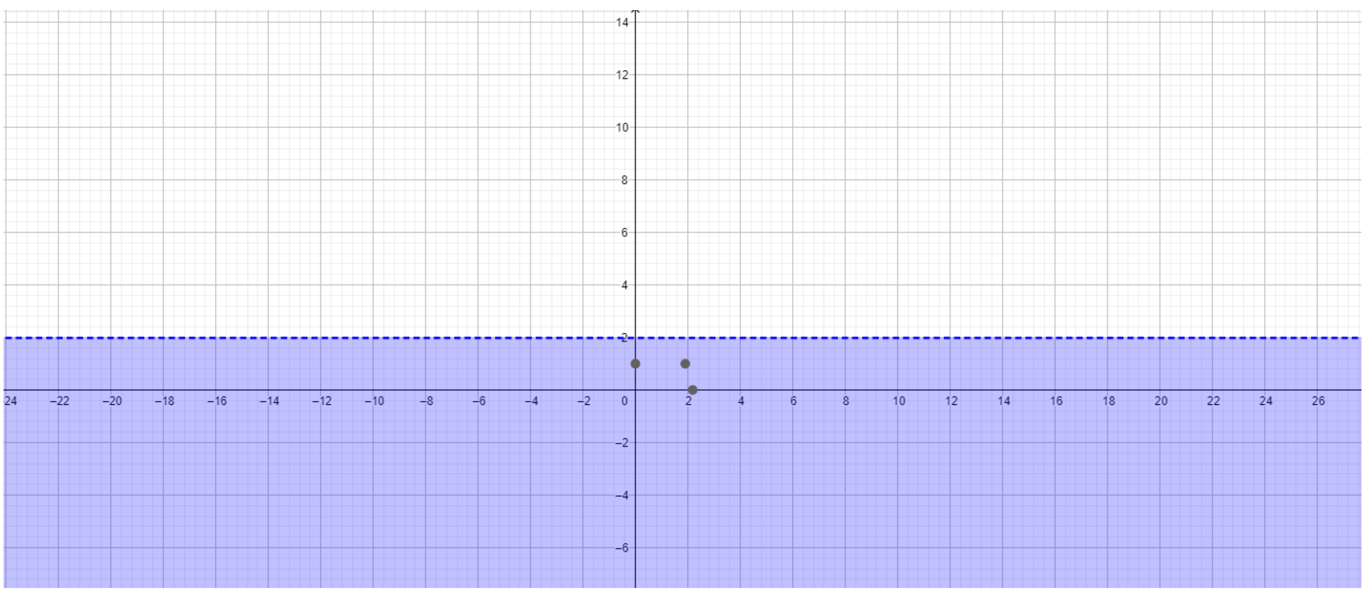
$$\frac{12\cdot 270}{360}=\frac{\cancel{12}\cdot 30\cdot 9}{\cancel{360}}=9x$$

Ábrázolja a következő halmazokat a Gauss-számsíkon.

$A = \{z \in \mathbb{C} \mid \operatorname{Re} z > 1\}$



$B = \{z \in \mathbb{C} \mid \operatorname{Im} z < 2\}$



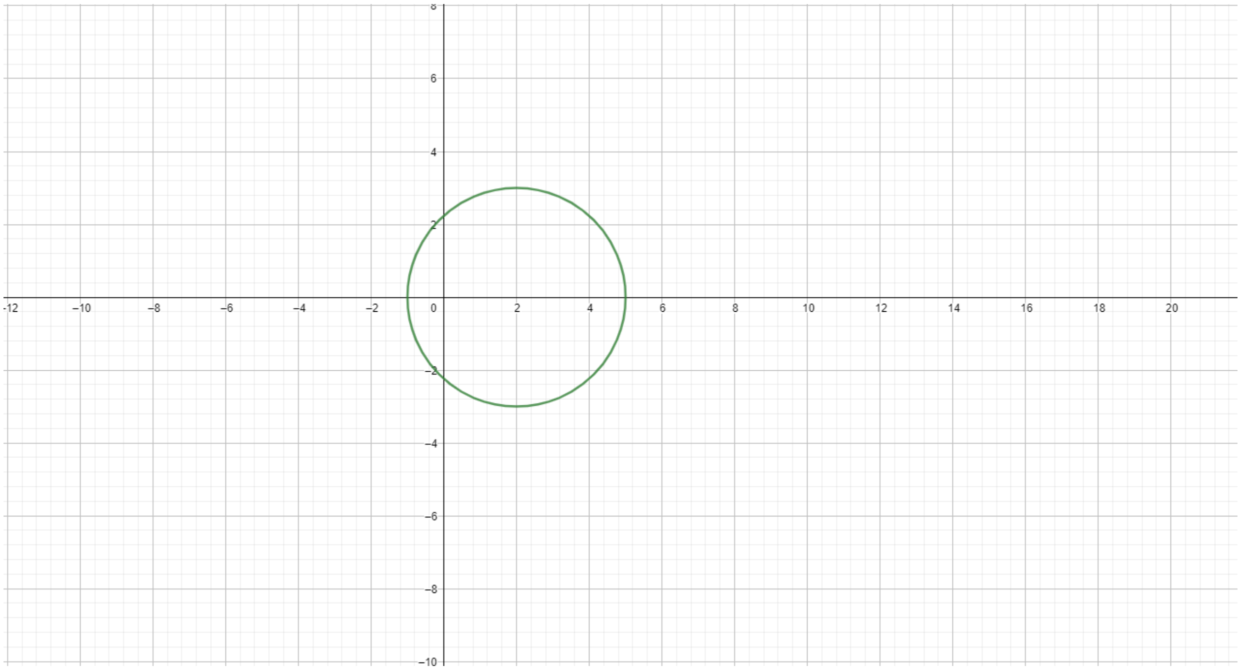
$C = \{z \in \mathbb{C} \mid |z - 2| = 3\}$

$|z - 2| = |a + bi - 2| = |(a - 2) + bi| = \sqrt{(a - 2)^2 + b^2}$

$\sqrt{(a - 2)^2 + b^2} = 3 \Longleftrightarrow (a - 2)^2 + b^2 = 9 = (a - 2)^2 + (b - 0)^2$

//köregyenlet – koordináta-geometria: $(x-u)^2 + (y-v)^2 = r^2$, ahol (u,v) a kör középpontja

$C = \{z \in \mathbb{C} \mid |z - 2| = 3\}$

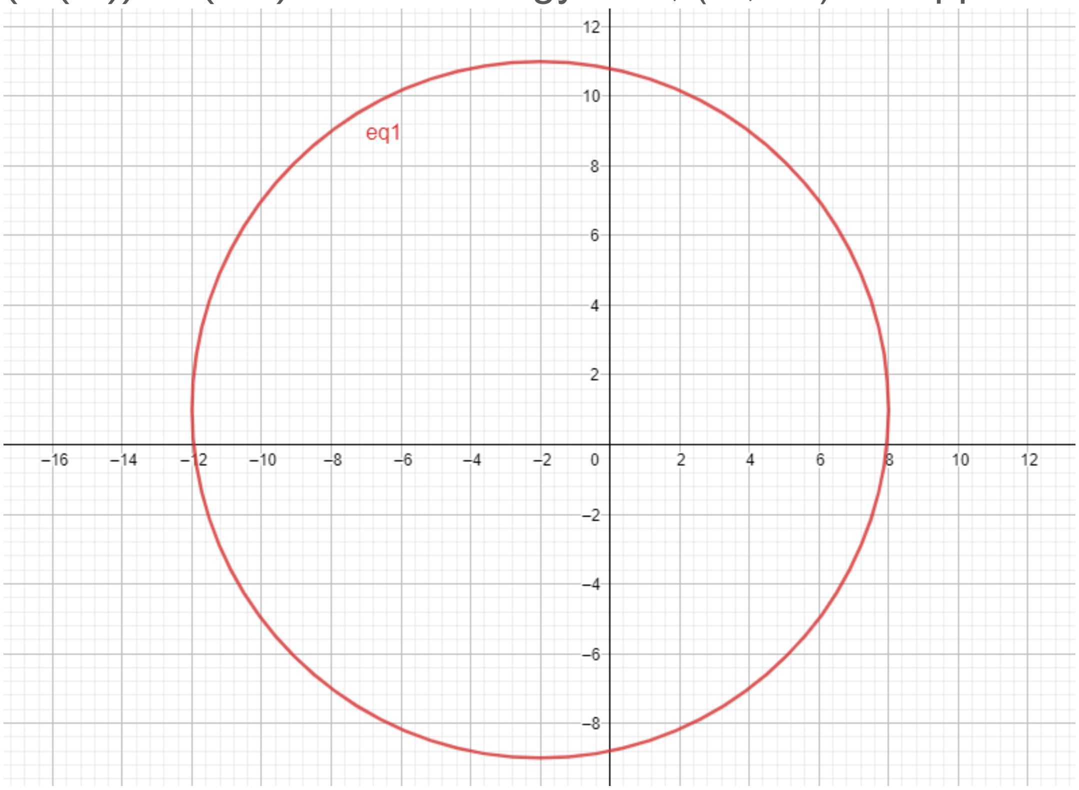


$\{z \in \mathbb{C} \mid |z - i + 2| = 10\}$

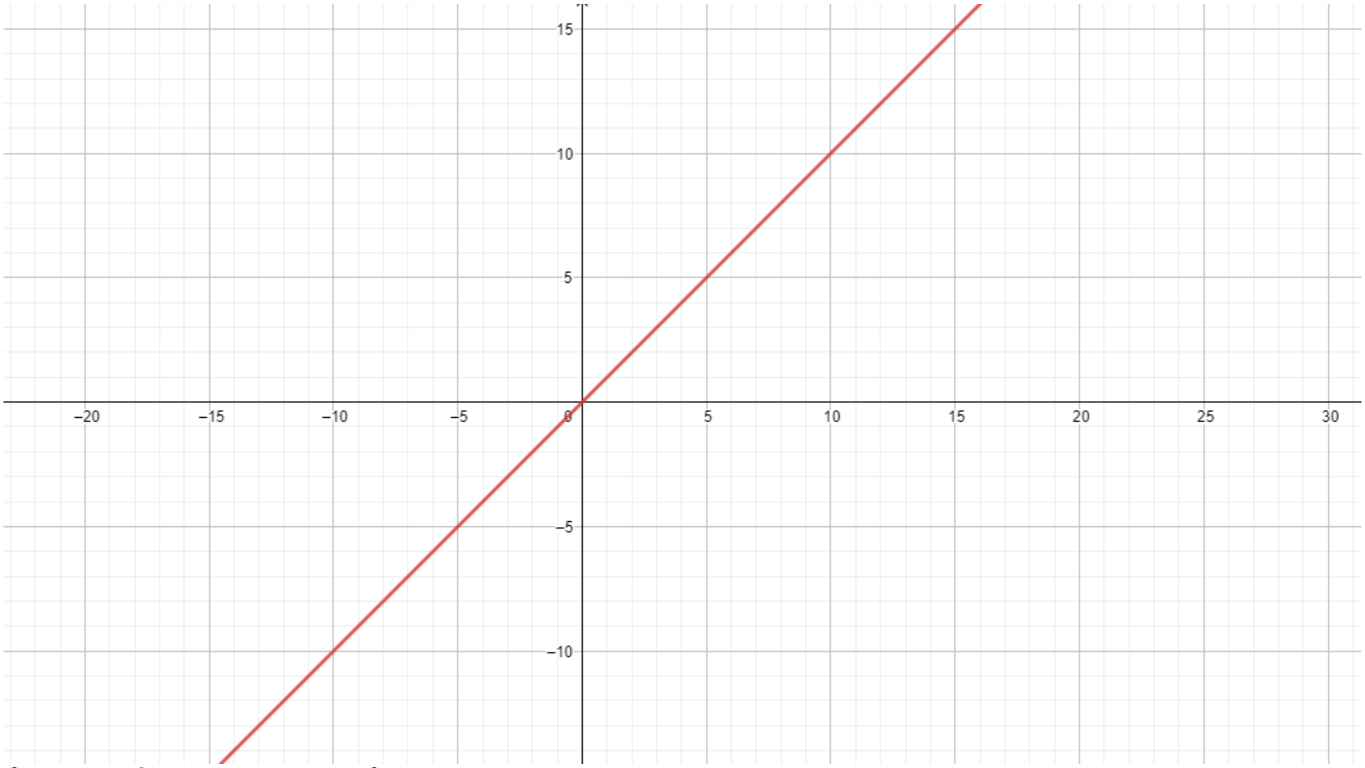
$|z - i + 2| = |a + bi - i + 2| = |(a + 2) + i \times (b - 1)| = \sqrt{(a+2)^2 + (b-1)^2}$

$\sqrt{(a+2)^2 + (b-1)^2} = 10 \iff (a+2)^2 + (b-1)^2 = 100 \Rightarrow$

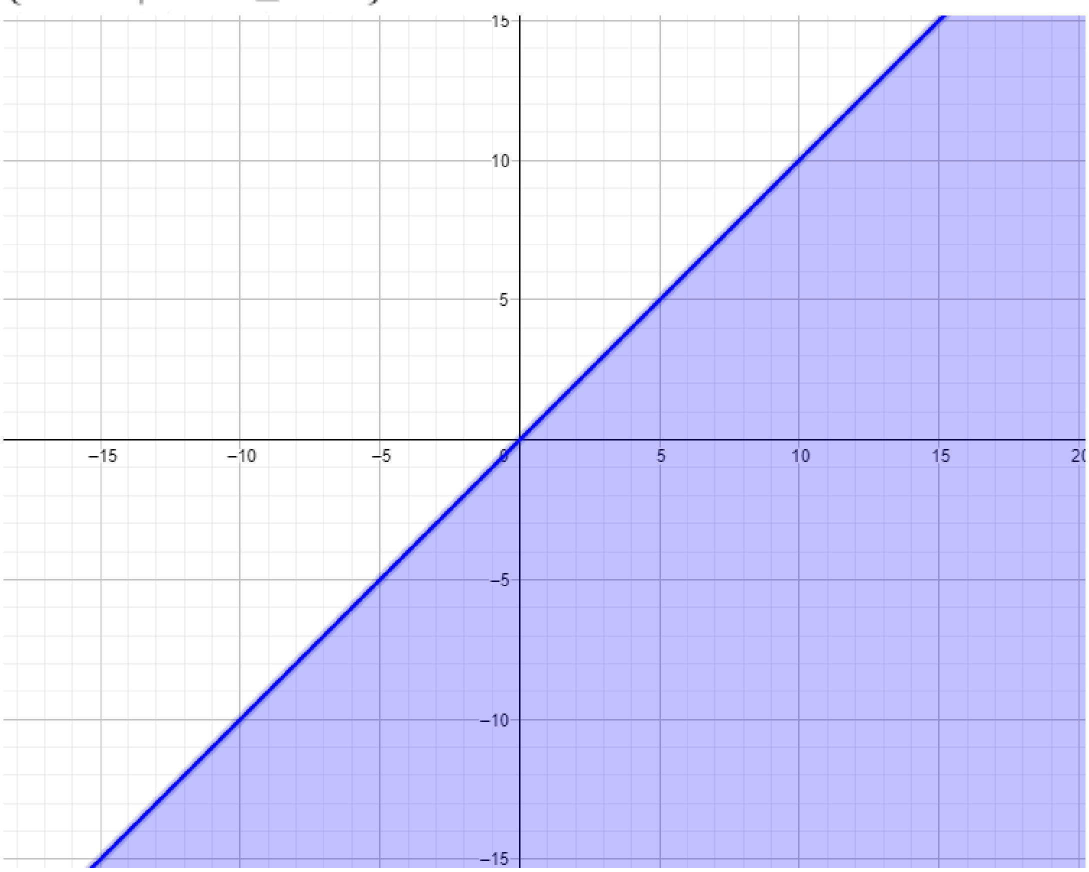
$(a-(-2))^2 + (b-1)^2 = 10^2$ //köregyenlet, (-2, +1) középponttal



$\{z \in \mathbb{C} \mid \operatorname{Re} z = \operatorname{Im} z\}$



$\{z \in \mathbb{C} \mid \operatorname{Re} z \geq \operatorname{Im} z\}$



$\{z \in \mathbb{C} \mid |z-2| \leq |z+3|\}$

$|z - 2| = |a + bi - 2| = |(a - 2) + bi| = \sqrt{(a - 2)^2 + b^2}$

$|z + 3| = |a + bi + 3| = |(a + 3) + bi| = \sqrt{(a + 3)^2 + b^2}$

$\sqrt{(a - 2)^2 + b^2} \leq \sqrt{(a + 3)^2 + b^2} \Leftrightarrow (a - 2)^2 + b^2 \leq (a + 3)^2 + b^2 \Leftrightarrow$

$(a - 2)^2 \leq (a + 3)^2 \Leftrightarrow a^2 - 2\times2\times a + 4 \leq a^2 + 2\times3\times a + 9 \Leftrightarrow$

$a^2 - 4\times a + 4 \leq a^2 + 6\times a + 9 \Leftrightarrow -4\times a + 4 \leq 6\times a + 9 \Leftrightarrow -5 \leq 10\times a \Leftrightarrow$

$\underline{-0.5 \leq a}$