Exercise solutions for
Logic
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Chapter One: How to Think Logically

EXERCISE 1.1

1 A Not an argument. There are no premises and there is no conclusion.

B Argument. First two sentences are premises, and the "so" in the third sentence indicates it is a conclusion.

C Argument. First two sentences are premises, and the "therefore" in the third sentence indicates it is a conclusion.

D Not an argument. There are no premises and there is no conclusion.

2 A **Premises**
Professor Plum was in the drawing room and Miss Scarlet was in the conservatory. If Professor Plum was in the drawing room and the murder weapon was found in the drawing room then Professor Plum is in big trouble.
Conclusion
So, if the murder weapon was found in the drawing room then Professor Plum really is in big trouble
Reasons
The conclusion follows from the premises, and the "so" word in the last sentence indicates a conclusion.

B **Premises**
All human beings are mortal. After all, he is a human being.
Conclusion
So, it stands to reason that Socrates is mortal.
Reasons
The conclusion follows from the premises, and the "so" word in the second sentence indicates a conclusion.

C **Premises**
Very few elephants can fly. Very few elephants are pink. For fewer pink elephants than ordinary elephants can actually fly.
Conclusion
So, the pink flying elephant is truly a rare creature.
Reasons
The conclusion follows from the premises, and these sentences are in the form of an argument with the words "so" and "for".

D **Premises**
For the murderer used the knife and Professor Plum had the knife. And the murder was comitted in the hall and Professor Plum was certainly in the hall earlier.
Conclusion
Professor Plum was obviously the murderer in this instance.
Reasons
The conclusion follows from the premises not deductively, but inductively. These sentences are in the form of an argument with the word "for".

3 A An argument is valid if it is impossible that it's premises be true and its conclusion false. It follows, that if it is not a valid argument, then it is possible, that it's premises be true, and its conclusion false. Also, it is a valid argument only if it is impossible that it's premises be true and its conclusion false. It follows, that if it is possible that it's premises be true and its conclusion false, then it is an invalid argument. All in all, an argument is invalid if and only if it is possible that it's premises be true and its conclusion false.

B Yes, a valid argument can have false conclusion if and only if it has a false premise. In that case, the premise is false, which means that it is impossible that it's premises are true, and it's conclusion false.

C No, a for a valid argument it is not possible (impossible) to have true premises and false conclusion.

D No. It is impossible that a true premise be true, and a true conclusion to be false, therefore, it is a valid argument.

E No. A sound argument is both valid and has true premises.

F Yes. A sound argument is valid and has true premises. It is valid, so it is not true, that it's premises are true and it's conclusion false. The premises are true, therefore this last sentence can only be true if the conclusion is true.

G An argument form is valid if and only if every substitution-instance of that form is valid.

H An argument form is invalid if and only if there is any substitution-instance of that form that is invalid.

4 A Valid, not sound.

B Invalid. We do not know whether "better than" in this context is transitive.

C Invalid.

D Invalid.

5 (i) C If p then q. q. Therefore, p. D If p then q. Not p. Therefore, not q.

(ii) Yes, because both premises are true, but both conclusions are not true.

6 If I clean my room, then it is not the case, that there is dirt on the floor. It is not the case, that I clean my room. Therefore, there is no dirt on the floor.

7 By the definition given in this chapter, sentential variable is a variable, whose value is a well-formed sentence. Both 1 and 2 are one whole sentence which cannot be divided. The whole argument then is: p therefore, q.

(i) No, it is not a valid form. Counterexample is where p is "snow is white", q is "there are pink elephants". Here p is true, and q is not, then it is not not true, that p is true and q is false.

(ii) Yes, intuitively (or modally) it is a valid argument, because it is not true, that the premise be true and the conclusion be false.

Chapter Two: How to Prove that You Can Argue Logically #1

EXERCISE 2.1

1 The main connective in each case are

(i) conjunction,
(ii) negation,
(iii) conjunction,
(iv) conjunction,
(v) disjunction,
(vi) disjunction,
(vii) conditional,
(viii) conjunction,
(ix) biconditional,
(x) biconditional.

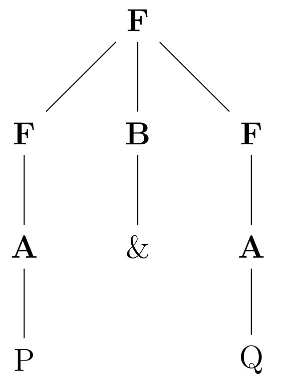
2 For (ii) and (x) it's the whole formula. For (iii) it's $\sim$P and $\sim$Q. For (iv) it's $\sim$(P & Q) and $\sim$Q.

3 The key is
P: Blind Lemon Jefferson is the only bluesman.
Q: Dr Strangely Strange is a bluesman.
R: Mr Oddly Normal is a bluesman.
S: Blind Lemon Jefferson is a milkman.
T: Blind Lemon Jefferson is a bluesman.
U: Blind Lemon recorded it.
V: It's a blues album.
The sentences translated to PL are

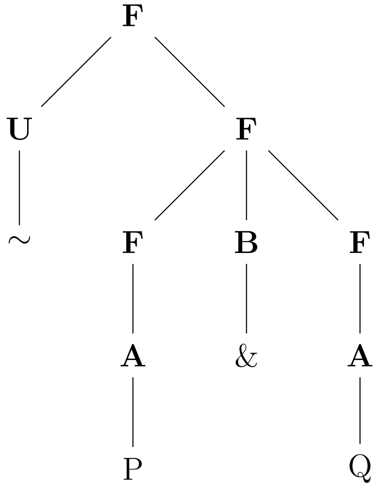
(i) $\sim$P,
(ii) $\sim$Q & $\sim$R,
(iii) S $\rightarrow$ $\sim$T,
(iv) $\sim$(U $\rightarrow$ $\sim$V),
(v) S v T,
(vi) $\sim$(S v T).

4 The trees are the followings.

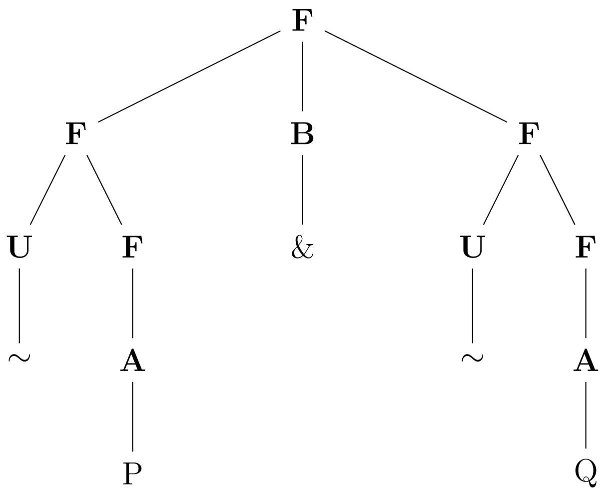
(i) P & Q



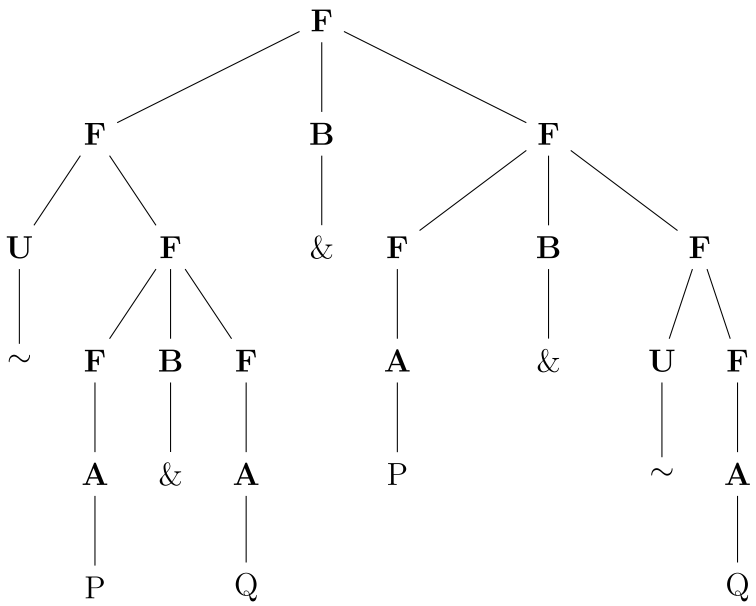
(ii) ∼(P & Q)



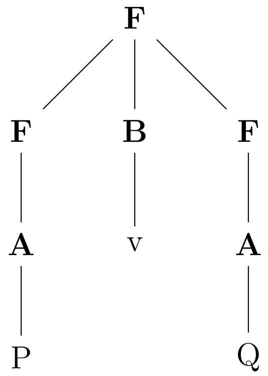
(iii) ∼P & ∼Q



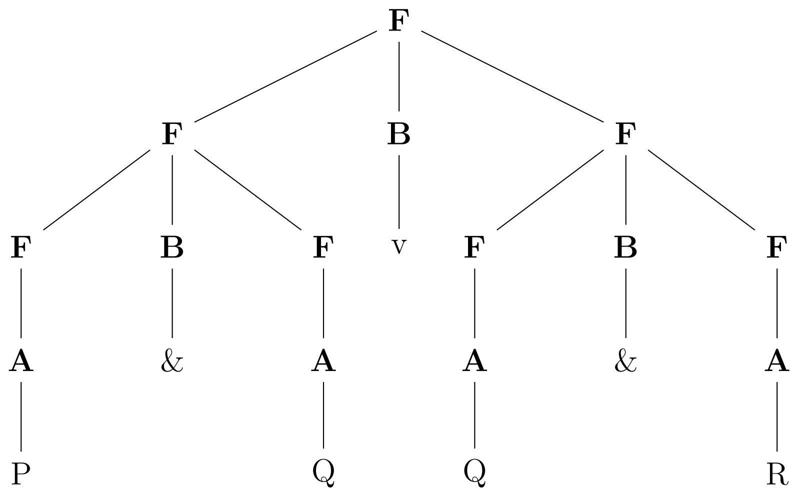
(iv) ∼(P & Q) & (P & ∼Q)



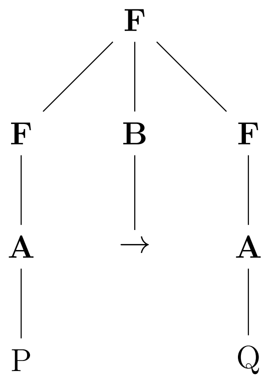
(v) P v Q



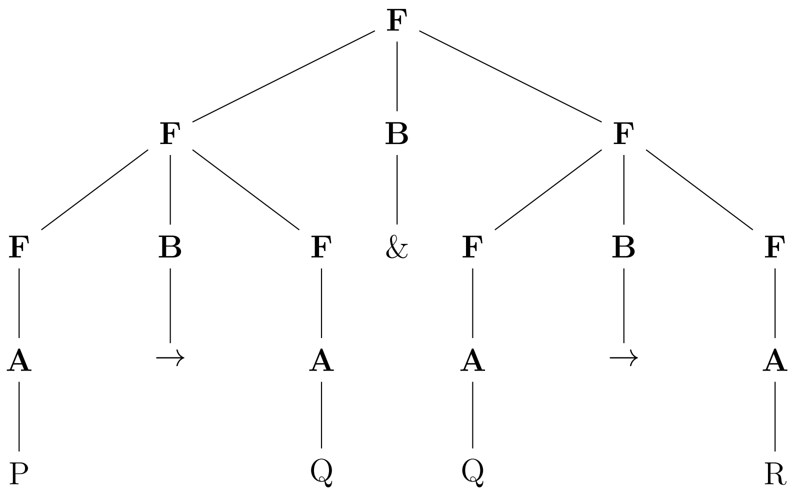
(vi) (P & Q) v (Q & R)



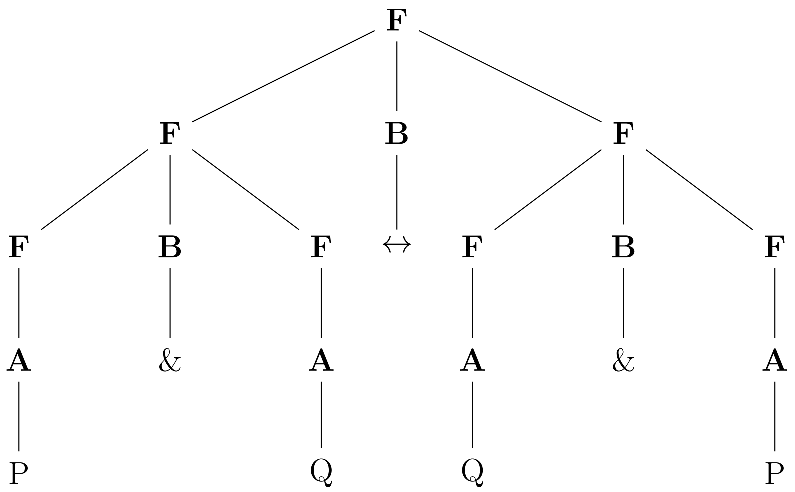
(vii) P $\rightarrow$ Q



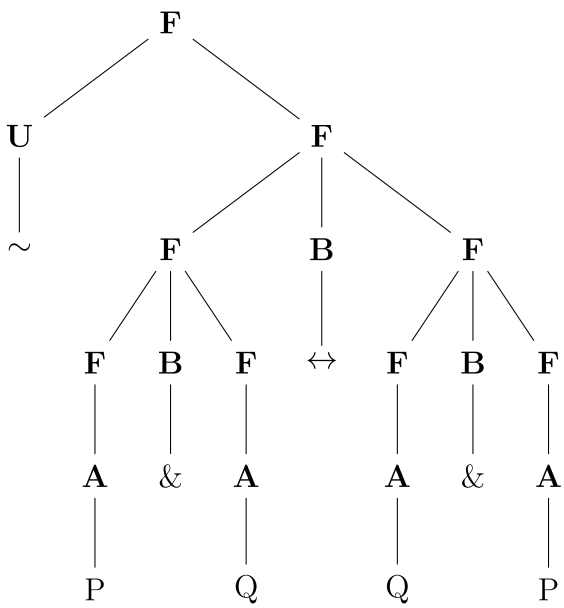
(viii) (P → Q) & (Q → R)



(ix) (P & Q) ↔ (Q & P)



(x) ∼((P & Q) ↔ (Q & P))



EXERCISE 2.2

1 $P \vee Q, \sim R \rightarrow Q, \sim R : \sim P$

(i) P: Big Bill Broonzy is a Delta bluesman.

(ii) Q: Big Bill Broonzy is a Chicago bluesman.

(iii) R: Big Bill Broonzy was born in Mississippi.

This is an invalid sequent.

2 $\sim(P \ \& \ Q), P : Q$

(i) P: Etta James was an angel.

(ii) Q: Robert Johnson sold his soul to the devil.

This is a valid sequent.

3 $\sim P \rightarrow \sim Q, \sim(P \ \& \sim R), R : Q$

(i) P: There's light on in the Venue.

(ii) Q: The band are on stage already.

(iii) R: It's going to be a great night.

This is an invalid sequent.

4 $P \vee Q, P \rightarrow \sim R, Q \rightarrow \sim R : \sim R$

(i) P: There's a punk rock band playing at the Venue tonight.

(ii) Q: The music is strictly classical.

(iii) R: There will be no blues at the Venue tonight.

This is a valid sequent.

5 $(P \ \& \ Q) \rightarrow \sim R : R \rightarrow (P \rightarrow Q)$

(i) P: There's a band on stage.

(ii) Q: The music is groovy.

(iii) R: It is the Nasal Flute Orchestra.

This is an invalid sequent.

EXERCISE 2.3

1 Dependencies, line number, formula, rule which was used in the line.

2 The proofs are the followings.

1. P, Q : P & Q

{1}	1.	P	Premise
{2}	2.	Q	Premise
{1, 2}	3.	P & Q	1, 2 &I

2. P, Q, R : (P & Q) & R

{1}	1.	P	Premise
{2}	2.	Q	Premise
{3}	3.	R	Premise
{1, 2}	4.	P & Q	1, 2 &I
{1, 2, 3}	5.	(P & Q) & R	3, 4 &I

3. P, Q, R, S : (P & Q) & (R & S)

{1}	1.	P	Premise
{2}	2.	Q	Premise
{3}	3.	R	Premise
{4}	4.	S	Premise
{1, 2}	5.	P & Q	1, 2 &I
{3, 4}	6.	R & S	3, 4 &I
{1, 2, 3, 4}	7.	(P & Q) & (R & S)	5, 6 &I

4. P & Q : P

{1}	1.	(P & Q)	Premise
{1}	2.	P	1 &E

5. P & Q : Q

{1}	1.	(P & Q)	Premise
{1}	2.	Q	1 &E

6. (P & Q) & R : P

{1}	1.	(P & Q) & R	Premise
{1}	2.	P & Q	1 &E
{1}	3.	P	2 &E

7. (P & Q) & (R & S) : P

{1}	1.	(P & Q) & (R & S)	Premise
{1}	2.	P & Q	1 &E
{1}	3.	P	2 &E

8. (Q & R), P : (P & Q) & R

{1}	1.	Q & R	Premise
{2}	2.	P	Premise
{1}	3.	Q	1 &E
{1}	4.	R	1 &E
{1, 2}	5.	P & Q	2, 3 &I
{1, 2}	6.	(P & Q) & R	4, 5 &I

EXERCISE 2.4

1 The proofs are the followings.

1. $P \rightarrow Q$, P : Q

{1}	1.	$P \rightarrow Q$	Premise
{2}	2.	P	Premise
{1, 2}	3.	Q	1, 2 MP

2. $P \rightarrow (P \rightarrow Q)$, P : Q

{1}	1.	$P \rightarrow (P \rightarrow Q)$	Premise
{2}	2.	P	Premise
{1, 2}	3.	$P \rightarrow Q$	1, 2 MP
{1, 2}	4.	Q	2, 3 MP

3. $P \rightarrow (P$ & $Q)$, P : Q

{1}	1.	$P \rightarrow (P$ & $Q)$	Premise
{2}	2.	P	Premise
{1, 2}	3.	P & Q	1, 2 MP
{1, 2}	4.	Q	3 &E

4. $P \rightarrow (Q \rightarrow R)$, $P \rightarrow Q$, P : R

{1}	1.	$P \rightarrow (Q \rightarrow R)$	Premise
{2}	2.	$P \rightarrow Q$	Premise
{3}	3.	P	Premise
{1, 3}	4.	$Q \rightarrow R$	1, 3 MP
{2, 3}	5.	Q	2, 3 MP
{1, 2, 3}	6.	R	4, 5 MP

2 The proofs are the followings.

(i) $P \to (Q \,\&\, R), P \,\&\, Q : P \,\&\, R$

{1}	1.	$P \to (Q \,\&\, R)$	Premise
{2}	2.	$P \,\&\, Q$	Premise
{2}	3.	P	2 &E
{1, 2}	4.	$Q \,\&\, R$	1, 3 MP
{1, 2}	5.	R	4 &E
{1, 2}	6.	$P \,\&\, R$	3, 5 &I

(ii) $(P \to Q) \to (R \to S), P \to Q, P \,\&\, R : S$

{1}	1.	$(P \to Q) \to (R \to S)$	Premise
{2}	2.	$P \to Q$	Premise
{3}	3.	$P \,\&\, R$	Premise
{1, 2}	4.	$R \to S$	1, 2 MP
{3}	5.	R	3 &E
{1, 2, 3}	6.	S	4, 5 MP

(iii) $P, P \to Q : P \,\&\, Q$

{1}	1.	P	Premise
{2}	2.	$P \to Q$	Premise
{1, 2}	3.	Q	1, 2 MP
{1, 2}	4.	$P \,\&\, Q$	1, 3 &I

(iv) $P, P \to Q, P \to (Q \to R) : P \,\&\, R$

{1}	1.	P	Premise
{2}	2.	$P \to Q$	Premise
{3}	3.	$P \to (Q \to R)$	Premise
{1, 2}	4.	Q	1, 2 MP
{1, 3}	5.	$Q \to R$	1, 3 MP
{1, 2, 3}	6.	R	4, 5 MP
{1, 2, 3}	7.	$P \,\&\, R$	1, 6 &I

(v) $P, P \to Q, P \to (Q \to R), R \to S : (P \,\&\, Q) \,\&\, (R \,\&\, S)$

{1}	1.	P	Premise
{2}	2.	$P \to Q$	Premise
{3}	3.	$P \to (Q \to R)$	Premise
{4}	4.	$R \to S$	Premise
{1, 2}	5.	Q	1, 2 MP
{1, 3}	6.	$Q \to R$	1, 3 MP
{1, 2, 3}	7.	R	5, 6 MP
{1, 2, 3}	8.	S	4, 7 MP
{1, 2}	9.	$P \,\&\, Q$	1, 5 &I
{1, 2, 3}	10.	$R \,\&\, S$	7, 8 &I
{1, 2, 3}	11.	$(P \,\&\, Q) \,\&\, (R \,\&\, S)$	9, 10 &I

EXERCISE 2.5

1 The proofs are the followings.

1. $P \to (Q \,\&\, R) : P \to Q$

{1}	1.	$P \to (Q \,\&\, R)$	Premise
{2}	2.	P	Assumption for CP
{1, 2}	3.	$Q \,\&\, R$	1, 2 MP
{1, 2}	4.	Q	3 &E
{1}	5.	$P \to Q$	2, 4 CP

2. $(P \,\&\, Q) \to R, P : Q \to R$

{1}	1.	$(P \,\&\, Q) \to R$	Premise
{2}	2.	P	Premise
{3}	3.	Q	Assumption for CP
{2, 3}	4.	$P \,\&\, Q$	2, 3 &I
{1, 2, 3}	5.	R	1, 4 MP
{1, 2}	6.	$Q \to R$	3, 5 CP

3. $(P \,\&\, Q), (P \,\&\, R) \to S : R \to S$

{1}	1.	$P \,\&\, Q$	Premise
{2}	2.	$(P \,\&\, R) \to S$	Premise
{3}	3.	R	Assumption for CP
{1}	4.	P	1 &E
{1, 3}	5.	$P \,\&\, R$	3, 4 &I
{1, 2, 3}	6.	S	2, 5 MP
{1, 2}	7.	$R \to S$	3, 6 CP

4. $(P \,\&\, Q) \to R : P \to (Q \to R)$

{1}	1.	$(P \,\&\, Q) \to R$	Premise
{2}	2.	P	Assumption for CP
{3}	3.	Q	Assumption for CP
{2, 3}	4.	$P \,\&\, Q$	2, 3 &I
{1, 2, 3}	5.	R	1, 4 MP
{1, 2}	6.	$Q \to R$	3, 5 CP
{1}	7.	$P \to (Q \to R)$	2, 6 CP

5. $P \to Q : (P \,\&\, R) \to (R \,\&\, Q)$

{1}	1.	$P \to Q$	Premise
{2}	2.	$P \,\&\, R$	Assumption for CP
{2}	3.	P	2 &E
{1, 2}	4.	Q	1, 3 MP
{2}	5.	R	2 &E
{1, 2}	6.	$R \,\&\, Q$	4, 5 &I
{1}	7.	$(P \,\&\, R) \to (R \,\&\, Q)$	2, 6 CP

6. $P \to Q : (Q \to R) \to (P \to R)$

{1}	1.	$P \to Q$	Premise
{2}	2.	$Q \to R$	Assumption for CP
{3}	3.	P	Assumption for CP
{1, 3}	4.	Q	1, 3 MP
{1, 2, 3}	5.	R	2, 4 MP
{1, 2}	6.	$P \to R$	3, 5 CP
{1}	7.	$(Q \to R) \to (P \to R)$	2, 6 CP

7. $R \to P, Q \to S : (P \to Q) \to (R \to S)$

{1}	1.	$R \to P$	Premise
{2}	2.	$Q \to S$	Premise
{3}	3.	$P \to Q$	Assumption for CP
{4}	4.	R	Assumption for CP
{1, 4}	5.	P	1, 4 MP
{1, 3, 4}	6.	Q	3, 5 MP
{1, 2, 3, 4}	7.	S	2, 6 MP
{1, 2, 3}	8.	$R \to S$	4, 7 CP
{1, 2}	9.	$(P \to Q) \to (R \to S)$	3, 8 CP

8. $P \to Q : (P \to R) \to (P \to (Q \& R))$

{1}	1.	$P \to Q$	Premise
{2}	2.	$P \to R$	Assumption for CP
{3}	3.	P	Assumption for CP
{1, 3}	4.	Q	1, 3 MP
{1, 2}	5.	R	2, 3 MP
{1, 2, 3}	6.	$Q \& R$	4, 5 &I
{1, 2}	7.	$P \to (Q \& R)$	3, 6 CP
{1}	8.	$(P \to R) \to (P \to (Q \& R))$	2, 7 CP

9. $P \to (Q \to R) : (S \to Q) \to (P \to (S \to R))$

{1}	1.	$P \to (Q \to R)$	Premise
{2}	2.	$S \to Q$	Assumption for CP
{3}	3.	P	Assumption for CP
{4}	4.	S	Assumption for CP
{2, 4}	5.	Q	2, 4 MP
{1, 3}	6.	$Q \to R$	1, 3 MP
{1, 2, 3, 4}	7.	R	5, 6 MP
{1, 2, 3}	8.	$S \to R$	4, 7 CP
{1, 2}	9.	$P \to (S \to R)$	3, 8 CP
{1}	10.	$(S \to Q) \to (P \to (S \to R))$	2, 9 CP

10. $P \to Q : ((R \& Q) \to S) \to ((R \& P) \to S)$

{1}	1.	$P \to Q$	Premise
{2}	2.	$(R \& Q) \to S$	Assumption for CP
{3}	3.	$R \& P$	Assumption for CP
{3}	4.	P	3 &E
{3}	5.	R	3 &E
{1, 3}	6.	Q	1, 4 MP
{1, 3}	7.	$R \& Q$	5, 6 &I
{1, 2, 3}	8.	S	2, 7 MP
{1, 2}	9.	$(R \& P) \to S$	3, 8 CP
{1}	10.	$((R \& Q) \to S) \to ((R \& P) \to S)$	2, 9 CP

EXERCISE 2.6

1 The proofs are the followings.

1. $: ((P \to P) \to Q) \to Q$

{1}	1.	P	Assumption for CP
–	2.	$P \to P$	1, 1 CP
{3}	3.	$(P \to P) \to Q$	Assumption for CP
{3}	4.	Q	2, 3 MP
–	5.	$((P \to P) \to Q) \to Q$	3, 4 CP

2. $: (P \to Q) \to ((Q \to R) \to (P \to R))$

{1}	1.	$P \to Q$	Assumption for CP
{2}	2.	$Q \to R$	Assumption for CP
{3}	3.	P	Assumption for CP
{1, 3}	4.	Q	1, 3 MP
{1, 2, 3}	5.	R	2, 4 MP
{1, 2}	6.	$P \to R$	3, 5 CP
{1}	7.	$(Q \to R) \to (P \to R)$	2, 6 CP
–	8.	$(P \to Q) \to ((Q \to R) \to (P \to R))$	1, 7 CP

3. $: (Q \to R) \to ((P \to Q) \to (P \to R))$

{1}	1.	$Q \to R$	Assumption for CP
{2}	2.	$P \to Q$	Assumption for CP
{3}	3.	P	Assumption for CP
{2, 3}	4.	Q	2, 3 MP
{1, 2, 3}	5.	R	1, 4 MP
{1, 2}	6.	$P \to R$	3, 5 CP
{1}	7.	$(P \to Q) \to (P \to R)$	2, 6 CP
–	8.	$(Q \to R) \to ((P \to Q) \to (P \to R))$	1, 7 CP

4. $: P \to (Q \to (P \,\&\, Q))$

{1}	1.	P	Assumption for CP
{2}	2.	Q	Assumption for CP
{1, 2}	3.	$P \,\&\, Q$	1, 2 &I
{1}	4.	$Q \to (P \,\&\, Q)$	2, 3 CP
–	5.	$P \to (Q \to (P \,\&\, Q))$	1, 4 CP

EXERCISE 2.7

1 The proofs are the followings.

1. P, P $\leftrightarrow$ Q : Q

{1}	1.	P	Premise
{2}	2.	P $\leftrightarrow$ Q	Premise
{2}	3.	(P $\rightarrow$ Q) & (Q $\rightarrow$ P)	$\leftrightarrow$ E
{2}	4.	P $\rightarrow$ Q	3 &E
{1, 2}	5.	Q	1, 4 MP

2. P & (P $\leftrightarrow$ Q) : P & Q

{1}	1.	P & (P $\leftrightarrow$ Q)	Premise
{1}	2.	P	1 &E
{1}	3.	P $\leftrightarrow$ Q	1 &E
{1}	4.	(P $\rightarrow$ Q) & (Q $\rightarrow$ P)	3 $\leftrightarrow$ E
{1}	5.	P $\rightarrow$ Q	4 &E
{1}	6.	Q	2, 5 MP
{1}	7.	P & Q	2, 6 &I

3. (P & Q) $\leftrightarrow$ P : P $\rightarrow$ Q

{1}	1.	(P & Q) $\leftrightarrow$ P	Premise
{1}	2.	((P & Q) $\rightarrow$ P) & (P $\rightarrow$ (P & Q))	1 $\leftrightarrow$ E
{1}	3.	P $\rightarrow$ (P & Q)	2 &E
{4}	4.	P	Assumption for CP
{1, 4}	5.	P & Q	3, 4 MP
{1, 4}	6.	Q	5 &E
{1}	7.	P $\rightarrow$ Q	4, 6 CP

4. P $\rightarrow$ Q : (Q $\rightarrow$ P) $\rightarrow$ (P $\leftrightarrow$ Q)

{1}	1.	P $\rightarrow$ Q	Premise
{2}	2.	Q $\rightarrow$ P	Assumption for CP
{1, 2}	3.	Q $\leftrightarrow$ P	1, 2 $\leftrightarrow$ I
{1}	4.	(Q $\rightarrow$ P) $\rightarrow$ (Q $\leftrightarrow$ P)	2, 3 CP

5. $P \rightarrow (Q \leftrightarrow R) : (P \& Q) \rightarrow R$

{1}	1.	$P \rightarrow (Q \leftrightarrow R)$	Premise
{2}	2.	$P \& Q$	Assumption for CP
{2}	3.	P	2 &E
{2}	4.	Q	2 &E
{1, 2}	5.	$Q \leftrightarrow R$	1, 3 MP
{1, 2}	6.	$(Q \rightarrow R) \& (R \rightarrow Q)$	5 $\leftrightarrow$ E
{1, 2}	7.	$Q \rightarrow R$	6 &E
{1, 2}	8.	R	4, 7 MP
{1}	9.	$(P \& Q) \rightarrow R$	2, 8 CP

6. $P \leftrightarrow Q, Q \leftrightarrow R : P \leftrightarrow R$

{1}	1.	$P \leftrightarrow Q$	Premise
{2}	2.	$Q \leftrightarrow R$	Premise
{1}	3.	$(P \rightarrow Q) \& (Q \rightarrow P)$	1 $\leftrightarrow$ E
{1}	4.	$P \rightarrow Q$	3 &E
{1}	5.	$Q \rightarrow P$	3 &E
{2}	6.	$(Q \rightarrow R) \& (R \rightarrow Q)$	2 $\leftrightarrow$ E
{2}	7.	$Q \rightarrow R$	6 &E
{2}	8.	$R \rightarrow Q$	6 &E
{9}	9.	P	Assumption for CP
{1, 9}	10.	Q	4, 9 MP
{1, 2, 9}	11.	R	7, 10 MP
{1, 2}	12.	$P \rightarrow R$	9, 11 CP
{13}	13.	R	Assumption for CP
{2, 13}	14.	Q	8, 13 MP
{1, 2, 13}	15.	P	5, 14 MP
{1, 2}	16.	$R \rightarrow P$	13, 15 CP
{1, 2}	17.	$P \leftrightarrow R$	12, 16 $\leftrightarrow$ I

EXERCISE 3.1

1 $P, P \rightarrow (Q \rightarrow R), \sim R : \sim Q$

{1}	1.	P	Premise
{2}	2.	$P \rightarrow (Q \rightarrow R)$	Premise
{3}	3.	$\sim R$	Premise
{1, 2}	4.	$Q \rightarrow R$	1, 2 MP
{1, 2, 3}	5.	$\sim Q$	3, 4 MT

EXERCISE 3.2

1 The proofs are the followings.

1. $\sim\sim(P \,\&\, Q) : \sim\sim(Q \,\&\, P)$

{1}	1.	$\sim\sim(P \,\&\, Q)$	Premise
{1}	2.	$P \,\&\, Q$	1 DNE
{1}	3.	P	2 &E
{1}	4.	Q	2 &E
{1}	5.	$Q \,\&\, P$	3, 4 &I
{1}	6.	$\sim\sim(Q \,\&\, P)$	5 DNI

2. $\sim P \to \sim Q : Q \to P$

{1}	1.	$\sim P \to \sim Q$	Premise
{2}	2.	Q	Assumption for CP
{2}	3.	$\sim\sim Q$	2 DNI
{1, 2}	4.	$\sim\sim P$	1, 3 MT
{1, 2}	5.	P	4 DNE
{1}	6.	$Q \to P$	2, 5 CP

3. $: (P \to Q) \to (\sim Q \to \sim P)$

{1}	1.	$P \to Q$	Assumption for CP
{2}	2.	$\sim Q$	Assumption for CP
{1, 2}	3.	$\sim P$	1, 2 MT
{1}	4.	$\sim Q \to \sim P$	2, 3 CP
–	5.	$(P \to Q) \to (\sim Q \to \sim P)$	1, 4 CP

4. $Q \to R : (\sim Q \to \sim P) \to (P \to R)$

{1}	1.	$Q \to R$	Premise
{2}	2.	$\sim Q \to \sim P$	Assumption for CP
{3}	3.	P	Assumption for CP
{3}	4.	$\sim\sim P$	3 DNI
{2, 3}	5.	$\sim\sim Q$	2, 4 MT
{2, 3}	6.	Q	5 DNE
{1, 2, 3}	7.	R	1, 6 MP
{1, 2}	8.	$P \to R$	3, 7 CP
{1}	9.	$(\sim Q \to \sim P) \to (P \to R)$	2, 8 CP

5. $(P \,\&\, Q) \rightarrow \sim R : R \rightarrow (P \rightarrow \sim Q)$

{1}	1.	$(P \,\&\, Q) \rightarrow \sim R$	Premise
{2}	2.	R	Assumption for CP
{3}	3.	P	Assumption for CP
{4}	4.	Q	Assumption for CP
{2}	5.	$\sim\sim R$	2 DNI
{1, 2}	6.	$\sim(P \,\&\, Q)$	1, 5 MT
{3, 4}	7.	$P \,\&\, Q$	3, 4 &I
{3}	8.	$Q \rightarrow (P \,\&\, Q)$	4, 7 CP
{1, 2, 3}	9.	$\sim Q$	6, 8 MT
{1, 2}	10.	$P \rightarrow \sim Q$	3, 9 CP
{1}	11.	$R \rightarrow (P \rightarrow \sim Q)$	2, 10 CP

6. $P : [(\sim(Q \rightarrow R) \rightarrow \sim P)] \rightarrow [(\sim R \rightarrow \sim Q)]$

{1}	1.	P	Premise
{2}	2.	$\sim(Q \rightarrow R) \rightarrow \sim P$	Assumption for CP
{1}	3.	$\sim\sim P$	1 DNI
{1, 2}	4.	$\sim\sim(Q \rightarrow R)$	2, 3 MT
{1, 2}	5.	$Q \rightarrow R$	4 DNE
{6}	6.	$\sim R$	Assumption for CP
{1, 2, 6}	7.	$\sim Q$	5, 6 MT
{1, 2}	8.	$\sim R \rightarrow \sim Q$	6, 7
{1}	9.	$(\sim(Q \rightarrow R) \rightarrow \sim P) \rightarrow (\sim R \rightarrow \sim Q)$	2, 8 CP

7. $P, \sim Q : \sim(P \rightarrow Q)$

{1}	1.	P	Premise
{2}	2.	$\sim Q$	Premise
{3}	3.	$P \rightarrow Q$	Assumption for CP
{1, 3}	4.	Q	1, 3 MP
{1}	5.	$(P \rightarrow Q) \rightarrow Q$	3, 4 CP
{1, 2}	6.	$\sim(P \rightarrow Q)$	2, 5 MT

8. $P, \sim P : Q$

{1}	1.	P	Premise
{2}	2.	$\sim P$	Premise
{3}	3.	$\sim Q$	Assumption for CP
{1, 3}	4.	$P \,\&\, \sim Q$	1, 3 &I
{1, 3}	5.	P	4 &E
{1}	6.	$\sim Q \rightarrow P$	3, 5 CP
{1, 2}	7.	$\sim\sim Q$	2, 6 MT
{1, 2}	8.	Q	7 DNE

9. : ~P → (P → Q)

{1}	1.	P	Assumption for CP
{2}	2.	~P	Assumption for CP
{3}	3.	~Q	Assumption for CP
{1, 3}	4.	P & ~Q	1, 3 &I
{1, 3}	5.	P	4 &E
{1}	6.	~Q → P	3, 5 CP
{1, 2}	7.	~~Q	2, 6 MT
{1, 2}	8.	Q	7 DNE
{2}	9.	P → Q	1, 8 CP
–	10.	~P → (P → Q)	2, 9 CP

10. P → ~P : ~P

{1}	1.	P → ~P	Premise
{2}	2.	P	Assumption for CP
{1, 2}	3.	~P	1, 2 MP
{4}	4.	P → ~P	Assumption for CP
{2, 4}	5.	P & (P → ~P)	2, 4 &I
{2, 4}	6.	P	5 &E
{2}	7.	(P → ~P) → P	4, 6 CP
{1, 2}	8.	~(P → ~P)	3, 7 MT
{1}	9.	P → ~(P → ~P)	2, 8 CP
{1}	10.	~(~(P → ~P))	1 DNI
{1}	11.	~P	9, 10 MT

EXERCISE 3.3

1 (P v Q) → R ⊢ (P → R) & (Q → R)

{1}	1.	(P v Q) → R	Premise
{2}	2.	P	Assumption for CP
{2}	3.	P v Q	2 vI (right-hand)
{1, 2}	4.	**R**	**1, 3 MP**
{1}	5.	**P → R**	2, 4 CP
{6}	6.	**Q**	Assumption for CP
{6}	7.	P v Q	**6 vI (left-hand)**
{1, 6}	8.	**R**	1, 7 MP
{1}	9.	**Q → R**	6, 8 CP
{1}	10.	(P → R) & (Q → R)	**5, 9 &I**

EXERCISE 3.4

1 They refer to (1) the disjunction, (2) first disjunct as an assumption, (3) the conclusion assuming the fist disjunct, (4) the second disjunct as an assumption, (5) the same conclusion assuming the second disjunct.

2 The proofs are the followings.

1. P v Q : (P v R) v (Q v R)

{1}	1.	P v Q	Premise
{2}	2.	P	Assumption for vE
{2}	3.	P v R	2 vI
{2}	4.	(P v R) v (Q v R)	3 vI
{5}	5.	Q	Assumption for vE
{5}	6.	Q v R	5 vI
{5}	7.	(P v R) v (Q v R)	6 vI
{1}	8.	(P v R) v (Q v R)	1, 2, 4, 5, 7 vE

2. (P & Q) v (P & R) : P & (Q v R)

{1}	1.	(P & Q) v (P & R)	Premise
{2}	2.	P & Q	Assumption for vE
{2}	3.	P	2 &E
{2}	4.	Q	2 &E
{2}	5.	Q v R	4 vI
{2}	6.	P & (Q v R)	3, 5 &I
{7}	7.	P & R	Assumption for vE
{7}	8.	P	7 &E
{7}	9.	R	7 &E
{7}	10.	Q v R	9 vI
{7}	11.	P & (Q v R)	8, 10 &I
{1}	12.	P & (Q v R)	1, 2, 6, 7, 11 vE

3. P v (P & Q) : P

{1}	1.	P v (P & Q)	Premise
{2}	2.	P	Assumption for vE
{3}	3.	P & Q	Assumption for vE
{3}	4.	P	4 &E
{1}	5.	P	1, 2, 2, 3, 4 vE

4. P v P : P

{1}	1.	P v P	Premise
{2}	2.	P	Assumption for vE
{1}	3.	P v P	1, 2, 2, 2, 2 vE

EXERCISE 3.5

1 R v S, ~Q → ~R, S → Q : Q v P

{1}	1.	R v S	Premise
{2}	2.	~Q → ~R	Premise
{3}	3.	S → Q	Premise
{4}	4.	R	Assumption for vE
{4}	5.	~~R	4 DNI
{2, 4}	6.	~~Q	2, 5 MT
{2, 4}	7.	Q	6 DNE
{9}	9.	S	Assumption for vE
{9, 3}	10.	Q	3, 9 MP
{1, 2, 3}	11.	Q	1, 2, 7, 9, 10 vE
{1, 2, 3}	12.	Q v P	11 vI

EXERCISE 3.6

1 The proofs are the followings.

1. (P v Q) & (P v R) : P v (Q & R)

{1}	1.	(P v Q) & (P v R)	Premise
{1}	2.	P v Q	1 &E
{1}	3.	P v R	1 &E
{4}	4.	P	Assumption for vE
{4}	5.	P v (Q & R)	4 vI
{6}	6.	Q	Assumption for vE
{7}	7.	R	Assumption for vE
{6, 7}	8.	Q & R	6, 7 &I
{6, 7}	9.	P v (Q & R)	8 vI
{1, 7}	10.	P v (Q & R)	2, 4, 5, 6, 9 vE
{1}	11.	P v (Q & R)	3, 4, 5, 7, 10 vE

2. P v (Q v R) : Q v (P v R)

{1}	1.	P v (Q v R)	Premise
{2}	2.	P	Assumption for vE
{2}	3.	P v R	2 vI
{2}	4.	Q v (P v R)	3 vI
{5}	5.	Q v R	Assumption for vE
{6}	6.	Q	Assumption for vE
{6}	7.	Q v (P v R)	6 vI
{8}	8.	R	Assumption for vE
{8}	9.	P v R	8 vI
{8}	10.	Q v (P v R)	9 vI
{5}	11.	Q v (P v R)	5, 6, 7, 8, 10 vE
{1}	12.	Q v (P v R)	1, 2, 4, 5, 11 vE

EXERCISE 3.7

1 The proofs are the followings.

1. P → (Q v R), R → S : P → (Q v S)

{1}	1.	P → (Q v R)	Premise
{2}	2.	R → S	Premise
{3}	3.	P	Assumption for CP
{1, 3}	4.	Q v R	1, 3 MP
{5}	5.	Q	Assumption for vE
{5}	6.	Q v S	5 vI
{7}	7.	R	Assumption for vE
{2, 7}	8.	S	2, 7 MP
{2, 7}	9.	Q v S	8 vI
{1, 2, 3}	10.	Q v S	4, 5, 6, 7, 9 vE
{1, 2}	11.	P → (Q v S)	3, 10 CP

2. Q → R : (P v Q) → (P v R)

{1}	1.	Q → R	Premise
{2}	2.	P v Q	Assumption for CP
{3}	3.	P	Assumption for vE
{3}	4.	P v R	3 vI
{5}	5.	Q	Assumption for vE
{1, 5}	6.	R	1, 5 MP
{1, 5}	7.	P v R	6 vI
{1, 2}	8.	P v R	2, 3, 4, 5, 7 vE
{1}	9.	(P v Q) → (P v R)	2, 8 CP

EXERCISE 3.8

1 The proofs are the followings.

1. P & (Q v R) : (P & Q) v (P & R)

{1}	1.	P & (Q v R)	Premise
{1}	2.	P	1 &E
{1}	3.	Q v R	1 &E
{4}	4.	Q	Assumption for vE
{1, 4}	5.	P & Q	2, 4 &I
{1, 4}	6.	(P & Q) v (P & R)	5 vI
{7}	7.	R	Assumption for vE
{1, 7}	8.	P & R	2, 7 &I
{1, 7}	9.	(P & Q) v (P & R)	8 vI
{1}	10.	(P & Q) v (P & R)	3, 4, 6, 7, 9 vE

2. (P v Q) → R : (P → R) & (Q → R)

{1}	1.	(P v Q) → R	Premise
{2}	2.	P	Assumption for CP
{2}	3.	P v Q	2 vI
{1, 2}	4.	R	1, 3 MP
{1}	5.	P → R	2, 4 CP
{6}	6.	Q	Assumption for CP
{6}	7.	P v Q	6 vI
{1, 6}	8.	R	1, 7 MP
{1}	9.	Q → R	6, 8 CP
{1}	10.	(P → R) & (Q → R)	5, 9 &I

EXERCISE 3.9

1 The proofs are the followings.

1. : ∼(P & ∼P)

{1}	1.	P & ∼P	Assumption for RAA
–	2.	∼(P & ∼P)	1, 1 RAA

2. P → ∼P : ∼P

{1}	1.	P → ∼P	Premise
{2}	2.	P	Assumption for RAA
{1, 2}	3.	∼P	1, 2 MP
{1, 2}	4.	P & ∼P	2, 3 &I
{1}	5.	∼P	2, 4 RAA

3. $P \to Q$, $Q \to \sim P$: $\sim P$

{1}	1.	$P \to Q$	Premise
{2}	2.	$Q \to \sim P$	Premise
{3}	3.	P	Assumption for RAA
{1, 3}	4.	Q	1, 3 MP
{1, 2, 3}	5.	$\sim P$	2, 4 MP
{1, 2, 3}	6.	P & $\sim P$	3, 5 &I
{1, 2}	7.	$\sim P$	3, 6 RAA

4. $P \to Q$, $\sim P \to Q$: Q

{1}	1.	$P \to Q$	Premise
{2}	2.	$\sim P \to Q$	Premise
{3}	3.	$\sim Q$	Assumption for RAA
{1, 3}	4.	$\sim P$	1, 3 MT
{2, 3}	5.	$\sim\sim P$	2, 3 MT
{1, 2, 3}	6.	$(\sim P)$ & $\sim(\sim P)$	4, 5 &I
{1, 2}	7.	$\sim\sim Q$	3, 6 RAA
{1, 2}	8.	Q	7 DNE

5. $\sim$(P v Q) : $\sim P$

{1}	1.	$\sim$(P v Q)	Premise
{2}	2.	P	Assumption for RAA
{2}	3.	P v Q	2 vI
{1, 2}	4.	(P v Q) & $\sim$(P v Q)	1, 3 &I
{1}	5.	$\sim P$	2, 4 RAA

6. $\sim$(P v Q), $R \to P$: $\sim R$

{1}	1.	$\sim$(P v Q)	Premise
{2}	2.	$R \to P$	Premise
{3}	3.	R	Assumption for RAA
{2, 3}	4.	P	2, 3 MP
{2, 3}	5.	P v Q	4 vI
{1, 2, 3}	6.	(P v Q) & $\sim$(P v Q)	1, 5 &I
{1, 2}	7.	$\sim R$	3, 6 RAA

7. (P & Q) → ∼R : R → (P → ∼Q)

{1}	1.	(P & Q) → ∼R	Premise
{2}	2.	R	Assumption for CP
{3}	3.	P	Assumption for CP
{4}	4.	Q	Assumption for RAA
{3, 4}	5.	P & Q	3, 4 &I
{1, 3, 4}	6.	∼R	1, 5 MP
{1, 2, 3, 4}	7.	R & ∼R	2, 6 &I
{1, 2, 3}	8.	∼Q	4, 7 RAA
{1, 2}	9.	P → ∼Q	3, 8 CP
{1}	10.	R → (P → ∼Q)	2, 9 CP

8. P → (Q → (R & ∼R)) : P → ∼Q

{1}	1.	P → (Q → (R & ∼R))	Premise
{2}	2.	P	Assumption for CP
{3}	3.	Q	Assumption for RAA
{1, 2}	4.	Q → (R & ∼R)	1, 2 MP
{1, 2, 3}	5.	R & ∼R	3, 4 MP
{1, 2}	6.	∼Q	3, 5 RAA
{1}	7.	P → ∼Q	2, 6 CP

9. ∼(P & ∼Q) : P → Q

{1}	1.	∼(P & ∼Q)	Premise
{2}	2.	P	Assumption for CP
{3}	3.	∼Q	Assumption for RAA
{2, 3}	4.	P & ∼Q	2, 3 &I
{1, 2, 3}	5.	(P & ∼Q) & ∼(P & ∼Q)	1, 4 &I
{1, 2}	6.	∼∼Q	3, 5 RAA
{1, 2}	7.	Q	6 DNE
{1}	8.	P → Q	2, 7 CP

10. P → Q : ∼(P & ∼Q)

{1}	1.	P → Q	Premise
{2}	2.	P & ∼Q	Assumption for RAA
{2}	3.	P	2 &E
{2}	4.	∼Q	2 &E
{1, 2}	5.	Q	1, 3 MP
{1, 2}	6.	Q & ∼Q	4, 5 &I
{1}	7.	∼(P & ∼Q)	2, 6 RAA

11. $\sim(P \rightarrow Q) : P \;\&\; \sim Q$

{1}	1.	$\sim(P \;\&\; \sim Q)$	Assumption for CP
{2}	2.	P	Assumption for CP
{3}	3.	$\sim Q$	Assumption for RAA
{2, 3}	4.	$P \;\&\; \sim Q$	2, 3 &I
{1, 2, 3}	5.	$(P \;\&\; \sim Q) \;\&\; \sim(P \;\&\; \sim Q)$	1, 4 &I
{1, 2}	6.	$\sim\sim Q$	3, 5 RAA
{1, 2}	7.	Q	6 DNE
{1}	8.	$P \rightarrow Q$	2, 7 CP
–	9.	$(\sim(P \;\&\; \sim Q)) \rightarrow (P \rightarrow Q)$	1, 8 CP
{10}	10.	$\sim(P \rightarrow Q)$	Premise
{10}	11.	$\sim\sim(P \;\&\; \sim Q)$	9, 10 MT
{10}	12.	$P \;\&\; \sim Q$	11 DNE

12. $P \rightarrow Q : (Q \rightarrow \sim P) \rightarrow \sim P$

{1}	1.	$P \rightarrow Q$	Premise
{2}	2.	$Q \rightarrow \sim P$	Assumption for CP
{3}	3.	P	Assumption for RAA
{1, 3}	4.	Q	1, 3 MP
{1, 2, 3}	5.	$\sim P$	2, 4 MP
{1, 2, 3}	6.	$P \;\&\; \sim P$	3, 5 &I
{1, 2}	7.	$\sim P$	3, 6 RAA
{1}	8.	$(Q \rightarrow \sim P) \rightarrow \sim P$	2, 7 CP

13. $P \rightarrow R,\; Q \rightarrow \sim R : \sim(P \;\&\; Q)$

{1}	1.	$P \rightarrow R$	Premise
{2}	2.	$Q \rightarrow \sim R$	Premise
{3}	3.	$P \;\&\; Q$	Assumption for RAA
{3}	4.	P	3 &E
{1, 3}	5.	R	1, 4 MP
{3}	6.	Q	3 &E
{2, 3}	7.	$\sim R$	2, 6 MP
{1, 2, 3}	8.	$R \;\&\; \sim R$	5, 7 &I
{1, 2}	9.	$\sim(P \;\&\; Q)$	3, 8 RAA

14. $\sim P : P \rightarrow Q$

{1}	1.	$\sim P$	Premise
{2}	2.	P	Assumption for CP
{3}	3.	$\sim Q$	Assumption for RAA
{2, 3}	4.	$P \ \& \sim Q$	2, 3 &I
{2, 3}	5.	P	4 &E
{1, 2, 3}	6.	$P \ \& \sim P$	1, 5 &I
{1, 2}	7.	$\sim\sim Q$	3, 6 RAA
{1, 2}	8.	Q	7 DNE
{1}	9.	$P \rightarrow Q$	2, 8 CP

15. $P, \sim P : Q$

{1}	1.	$\sim P$	Premise
{2}	2.	P	Premise
{3}	3.	$\sim Q$	Assumption for RAA
{2, 3}	4.	$P \ \& \sim Q$	2, 3 &I
{2, 3}	5.	P	4 &E
{1, 2, 3}	6.	$P \ \& \sim P$	1, 5 &I
{1, 2}	7.	$\sim\sim Q$	3, 6 RAA
{1, 2}	8.	Q	7 DNE

16. $: P \text{ v} \sim P$

{1}	1.	$\sim(P \text{ v} \sim P)$	Assumption for RAA
{2}	2.	P	Assumption for CP
{2}	3.	$P \text{ v} \sim P$	2 vI
–	4.	$P \rightarrow (P \text{ v} \sim P)$	2, 3 CP
{1}	5.	$\sim P$	1, 4 MT
{1}	6.	$P \text{ v} \sim P$	5 vI
{1}	7.	$(P \text{ v} \sim P) \ \& \ (\sim(P \text{ v} \sim P))$	1, 6 &I
–	8.	$\sim\sim(P \text{ v} \sim P)$	1, 7 RAA
–	9.	$P \text{ v} \sim P$	8 DNE

17. $P \text{ v } Q : \sim(\sim P \ \& \sim Q)$

{1}	1.	$P \text{ v } Q$	Premise
{2}	2.	$\sim P \ \& \sim Q$	Assumption for RAA
{2}	3.	$\sim P$	2 &E
{2}	4.	$\sim Q$	2 &E
{5}	5.	P	Assumption for vE
{2, 5}	6.	$P \ \& \sim P$	3, 5 &I
{5}	7.	$\sim(\sim P \ \& \sim Q)$	2, 6 RAA
{8}	8.	Q	Assumption for vE
{2, 8}	9.	$Q \ \& \sim Q$	4, 8 &I
{8}	10.	$\sim(\sim P \ \& \sim Q)$	2, 9 RAA
{1}	11.	$\sim(\sim P \ \& \sim Q)$	1, 5, 7, 8, 10 vE

18. ~(P v Q) : ~P & ~Q

{1}	1.	~(P v Q)	Premise
{2}	2.	P	Assumption for RAA
{2}	3.	P v Q	2 vI
{1, 2}	4.	(P v Q) & (~(P v Q))	1, 3 &I
{1}	5.	~P	2, 4 RAA
{6}	6.	Q	Assumption for RAA
{6}	7.	P v Q	6 vI
{1, 6}	8.	(P v Q) & (~(P v Q))	1, 7 &I
{1}	9.	~Q	6, 8 RAA
{1}	10.	~P & ~Q	5, 9 &I

19. ~(~P & ~Q) : P v Q

{1}	1.	~(P v Q)	Assumption for RAA
{2}	2.	P	Assumption for RAA
{2}	3.	P v Q	2 vI
{1, 2}	4.	(P v Q) & (~(P v Q))	1, 3 &I
{1}	5.	~P	2, 4 RAA
{6}	6.	Q	Assumption for RAA
{6}	7.	P v Q	6 vI
{1, 6}	8.	(P v Q) & (~(P v Q))	1, 7 &I
{1}	9.	~Q	6, 8 RAA
{1}	10.	~P & ~Q	5, 9 &I
{11}	11.	~(~P & ~Q)	Premise
{1, 11}	12.	(~P & ~Q) & (~(~P & ~Q))	10, 11 &I
{11}	13.	~~(P v Q)	1, 12 RAA
{11}	14.	P v Q	13 DNE

20. : $((P \to Q) \vee (Q \to R))$

{1}	1.	$\sim((P \to Q) \vee (Q \to R))$	Assumption for RAA
{2}	2.	Q	Assumption for RAA
{3}	3.	P	Assumption for CP
{2, 3}	4.	$P \& Q$	2, 3 &I
{2, 3}	5.	Q	4 &E
{2}	6.	$P \to Q$	3, 5 CP
{2}	7.	$(P \to Q) \vee (Q \to R)$	6 vI
{1, 2}	8.	$((P \to Q) \vee (Q \to R)) \& \sim((P \to Q) \vee (Q \to R))$	1, 7 &I
{1}	9.	$\sim Q$	1, 8 RAA
{10}	10.	Q	Assumption for CP
{11}	11.	$\sim R$	Assumption for RAA
{10, 11}	12.	$Q \& \sim R$	10, 11 &I
{10, 11}	13.	Q	12 &E
{1, 10, 11}	14.	$Q \& \sim Q$	1, 13 &I
{1, 10}	15.	$\sim\sim R$	11, 14 RAA
{1, 10}	16.	R	15 DNE
{1}	17.	$Q \to R$	10, 16 CP
{1}	18.	$(P \to Q) \vee (Q \to R)$	17 vI
{1}	19.	$((P \to Q) \vee (Q \to R)) \& \sim((P \to Q) \vee (Q \to R))$	1, 18 &I
–	20.	$\sim\sim((P \to Q) \vee (Q \to R))$	1, 19 RAA
–	21.	$(P \to Q) \vee (Q \to R)$	20 DNE

(Maybe the book says it can be done in 20 steps, because it uses only one Q assumption, and uses the assumption on line 2. for the assumption which is written in line 10. in my proof.)

REVISION EXERCISE I

1 The proofs are the followings.

1. $P \to Q : ((R \,\&\, Q) \to S) \to ((R \,\&\, P) \to S)$

{1}	1.	$P \to Q$	Premise
{2}	2.	$(R \,\&\, Q) \to S$	Assumption for CP
{3}	3.	$R \,\&\, P$	Assumption for CP
{3}	4.	R	3 &E
{3}	5.	P	3 &E
{1, 3}	6.	Q	1, 5 MP
{1, 3}	7.	$R \,\&\, Q$	4, 6 &I
{1, 2, 3}	8.	S	2, 7 MP
{1, 2}	9.	$(R \,\&\, P) \to S$	3, 8 CP
{1}	10.	$((R \,\&\, Q) \to S) \to ((R \,\&\, P) \to S)$	2, 9 CP

2. $(P \,\&\, Q) \to \sim R : R \to (P \to \sim Q)$

{1}	1.	$(P \,\&\, Q) \to \sim R$	Premise
{2}	2.	R	Assumption for CP
{3}	3.	P	Assumption for CP
{2}	4.	$\sim\sim R$	2 DNI
{1, 2}	5.	$\sim(P \,\&\, Q)$	1, 4 MT
{6}	6.	Q	Assumption for RAA
{3, 6}	7.	$P \,\&\, Q$	3, 6 &I
{1, 2, 3, 6}	8.	$(P \,\&\, Q) \,\&\, \sim(P \,\&\, Q)$	5, 7 &I
{1, 2, 3}	9.	$\sim Q$	6, 8 RAA
{1, 2}	10.	$P \to \sim Q$	3, 9 CP
{1}	11.	$R \to (P \to \sim Q)$	2, 10 CP

3. $: (P \to Q) \to (\sim Q \to \sim P)$

{1}	1.	$P \to Q$	Assumption for CP
{2}	2.	$\sim Q$	Assumption for CP
{1, 2}	3.	$\sim P$	1, 2 MT
{1}	4.	$\sim Q \to \sim P$	2, 3 CP
–	5.	$(P \to Q) \to (\sim Q \to \sim P)$	1, 4 CP

4. P v Q : (P v R) v (Q v R)

{1}	1.	P v Q	Premise
{2}	2.	P	Assumption for vE
{2}	3.	P v R	2 vI
{2}	4.	(P v R) v (Q v R)	3 vI
{5}	5.	Q	Assumption for vE
{5}	6.	Q v R	5 vI
{5}	7.	(P v R) v (Q v R)	6 vI
{1}	8.	(P v R) v (Q v R)	1, 2, 4, 5, 7 vE

5. P $\rightarrow$ R, Q $\rightarrow$ S : (P v Q) $\rightarrow$ (R v S)

{1}	1.	P $\rightarrow$ R	Premise
{2}	2.	Q $\rightarrow$ S	Premise
{3}	3.	P v Q	Assumption for CP
{4}	4.	P	Assumption for vE
{1, 4}	5.	R	1, 4 MP
{1, 4}	6.	R v S	5 vI
{7}	7.	Q	Assumption for vE
{2, 7}	8.	S	2, 7 MP
{2, 7}	9.	R v S	8 vI
{1, 2, 3}	10.	R v S	3, 4, 6, 7, 9 vE
{1, 2}	11.	(P v Q) $\rightarrow$ (R v S)	3, 10 CP

6. P $\rightarrow$ (Q v R), Q $\rightarrow$ R : P $\rightarrow$ R

{1}	1.	P $\rightarrow$ (Q v R)	Premise
{2}	2.	Q $\rightarrow$ R	Premise
{3}	3.	P	Assumption for CP
{1, 3}	4.	Q v R	1, 3 MP
{5}	5.	Q	Assumption for vE
{5, 2}	6.	R	2, 5 MP
{7}	7.	R	Assumption for vE
{1, 2, 3}	8.	R	4, 5, 6, 7, 7 vE
{1, 2}	9.	P $\rightarrow$ R	3, 8 CP

7. (P v Q) → R : (P → R) & (Q → R)

{1}	1.	(P v Q) → R	Premise
{2}	2.	P	Assumption for CP
{2}	3.	P v Q	2 vI
{1, 2}	4.	R	1, 3 MP
{1}	5.	P → R	2, 4 CP
{6}	6.	Q	Assumption for CP
{6}	7.	P v Q	6 vI
{1, 6}	8.	R	1, 7 CP
{1}	9.	Q → R	6, 8 CP
{1}	10.	(P → R) & (Q → R)	5, 9 &I

8. ∼(P & ∼Q) : P → Q

{1}	1.	∼(P & ∼Q)	Premise
{2}	2.	P	Assumption for CP
{3}	3.	∼Q	Assumption for RAA
{2, 3}	4.	P & ∼Q	2, 3 &I
{1, 2, 3}	5.	(P & ∼Q) & ∼(P & ∼Q)	1, 4 &I
{1, 2}	6.	∼∼Q	3, 5 RAA
{1, 2}	7.	Q	6 DNE
{1}	8.	P → Q	4, 7 CP

9. P → (Q ↔ R) : (P & Q) → R

{1}	1.	P → (Q ↔ R)	Premise
{2}	2.	P & Q	Assumption for CP
{2}	3.	P	2 &E
{2}	4.	Q	2 &E
{1, 2}	5.	Q ↔ R	1, 3 MP
{1, 2}	6.	(Q → R) & (R → Q)	5 ↔ E
{1, 2}	7.	Q → R	7 &E
{1, 2}	8.	R	4, 7 MP
{1}	9.	(P & Q) → R	2, 8 CP

10. : ∼P → (P → Q)

{1}	1.	∼P	Assumption for CP
{2}	2.	P	Assumption for CP
{3}	3.	∼Q	Assumption for RAA
{2, 3}	4.	P & ∼Q	2, 3 &I
{2, 3}	5.	P	4 &E
{1, 2, 3}	6.	P & ∼P	1, 5 &I
{1, 2}	7.	∼∼Q	3, 6 RAA
{1, 2}	8.	Q	7 DNE
{1}	9.	P → Q	2, 8 CP
–	10.	∼P → (P → Q)	1, 9 CP

REVISION EXERCISE II

1 The proofs are the followings.

1. P v P : P

{1}	1.	P v P	Premise
{2}	2.	P	Assumption for vE
{1}	3.	P	1, 2, 2, 2, 2 vE

2. P : (P → Q) → Q

{1}	1.	P	Premise
{2}	2.	P → Q	Assumption for CP
{1, 2}	3.	Q	1, 2 MP
{1}	4.	(P → Q) → Q	2, 3 CP

3. P : (∼(Q → R) → ∼P) → ((∼R → ∼Q))

{1}	1.	P	Premise
{2}	2.	∼(Q → R) → ∼P	Assumption for CP
{1}	3.	∼∼P	1 DNI
{1, 2}	4.	∼∼(Q → R)	2, 3 MT
{1, 2}	5.	Q → R	4 DNE
{6}	6.	∼R	Assumption for CP
{1, 2, 6}	7.	∼Q	5, 6 MT
{1, 2}	8.	∼R → ∼Q	6, 7 CP
{1}	9.	(∼(Q → R) → ∼P) → (∼R → ∼Q)	2, 8 CP

4. P → (Q v R), R → S : P → (Q v S)

{1}	1.	P → (Q v R)	Premise
{2}	2.	R → S	Premise
{3}	3.	P	Assumption for CP
{1, 3}	4.	Q v R	1, 3 MP
{5}	5.	Q	Assumption for vE
{5}	6.	Q v S	5 vI
{7}	7.	R	Assumption for vE
{2, 7}	8.	S	2, 7 MP
{2, 7}	9.	Q v S	8 vI
{1, 2, 3}	10.	Q v S	4, 5, 6, 7, 9 vE
{1, 2}	11.	P → (Q v S)	3, 10 CP

5. $\sim Q \rightarrow \sim R$, R v S, $S \rightarrow Q$: Q v P

{1}	1.	$\sim Q \rightarrow \sim R$	Premise
{2}	2.	R v S	Premise
{3}	3.	$S \rightarrow Q$	Premise
{4}	4.	R	Assumption for vE
{4}	5.	$\sim\sim R$	4 DNI
{1, 4}	6.	$\sim\sim Q$	1, 5 MT
{1, 4}	7.	Q	6 DNE
{1, 4}	8.	Q v P	7 vI
{9}	9.	S	Assumption for vE
{3, 9}	10.	Q	3, 9 MP
{3, 9}	11.	Q v P	10 vI
{1, 2, 3}	12.	Q v P	2, 4, 8, 9, 11 vE

6. $\sim$(P v Q) : $\sim P$ & $\sim Q$

{1}	1.	$\sim$(P v Q)	Premise
{2}	2.	P	Assumption for RAA
{2}	3.	P v Q	2 vI
{1, 2}	4.	(P v Q) & $\sim$(P v Q)	1, 3 &I
{1}	5.	$\sim P$	2, 4 RAA
{6}	6.	Q	Assumption for RAA
{6}	7.	P v Q	6 vI
{1, 6}	8.	(P v Q) & $\sim$(P v Q)	1, 7 &I
{1}	9.	$\sim Q$	6, 8 RAA
{1}	10.	$\sim P$ & $\sim Q$	5, 9 &I

7. $\sim\sim$(P v $\sim Q$) : $(P \rightarrow \sim Q)$ v $(\sim Q \rightarrow P)$

{1}	1.	$\sim\sim$(P v $\sim Q$)	Premise
{1}	2.	P v $\sim Q$	1 DNE
{3}	3.	P	Assumption for vE
{4}	4.	$\sim Q$	Assumption for CP
{3, 4}	5.	P & $\sim Q$	3, 4 &I
{3, 4}	6.	P	5 &E
{3}	7.	$\sim Q \rightarrow P$	4, 6 CP
{3}	8.	$(P \rightarrow \sim Q)$ v $(\sim Q \rightarrow P)$	7 vI
{9}	9.	$\sim Q$	Assumption for vE
{10}	10.	P	Assumption for CP
{9, 10}	11.	P & $\sim Q$	9, 10 &I
{9, 10}	12.	$\sim Q$	11 &E
{9}	13.	$P \rightarrow \sim Q$	10, 12 CP
{9}	14.	$(P \rightarrow \sim Q)$ v $(\sim Q \rightarrow P)$	13 vI
{1}	15.	$(P \rightarrow \sim Q)$ v $(\sim Q \rightarrow P)$	2, 3, 8, 9, 14 vE

(Maybe the book says it can be done in 11 steps because it uses the same steps 3, 4, 5 instead of introducing steps 9, 10 11, which would shorten this proof to 12.)

8. (P v Q) $\leftrightarrow$ P : Q $\rightarrow$ P

{1}	1.	(P v Q) $\leftrightarrow$ P	Premise
{1}	2.	((P v Q) $\rightarrow$ P) & (P $\rightarrow$ (P v Q))	1 $\leftrightarrow$ E
{1}	3.	(P v Q) $\rightarrow$ P	2 &E
{4}	4.	Q	Assumption for CP
{4}	5.	P v Q	4 vI
{1, 4}	6.	P	3, 5 MP
{1}	7.	Q $\rightarrow$ P	4, 6 CP

9. (P & Q) v (P & R) : P & (Q v R)

{1}	1.	(P & Q) v (P & R)	Premise
{2}	2.	P & Q	Assumption for vE
{2}	3.	P	2 &E
{2}	4.	Q	2 &E
{2}	5.	Q v R	4 vI
{2}	6.	P & (Q v R)	3, 5 &I
{7}	7.	P & R	Assumption for vE
{7}	8.	P	7 &E
{7}	9.	R	7 &E
{7}	10.	Q v R	9 vI
{7}	11.	P & (Q v R)	8, 10 &I
{1}	12.	P & (Q v R)	1, 2, 6, 7, 11 vE

10. : P v ~P

{1}	1.	~(P v ~P)	Assumption for RAA
{2}	2.	P	Assumption for RAA
{2}	3.	P v ~P	2 vI
{1, 2}	4.	(P v ~P) & ~(P v ~P)	1, 3 &I
{1}	5.	~P	2, 4 RAA
{1}	6.	P v ~P	5 vI
{1}	7.	(P v ~P) & ~(P v ~P)	1, 6 &I
–	8.	~~(P v ~P)	1, 7 RAA
–	9.	P v ~P	8 DNE

REVISION EXERCISE III

1 The proofs are the followings.

1\. : $((P \to P) \to Q) \to Q$

{1}	1.	$(P \to P) \to Q$	Assumption for CP
{2}	2.	P	Assumption for CP
–	3.	$P \to P$	2, 2 CP
{1}	4.	Q	1, 3 CP
–	5.	$((P \to P) \to Q) \to Q$	1, 4 CP

2\. $\sim(P \to Q)$: $P \& \sim Q$

{1}	1.	$\sim(P \to Q)$	Premise
{2}	2.	$\sim(P \& \sim Q)$	Assumption for RAA
{3}	3.	P	Assumption for CP
{4}	4.	$\sim Q$	Assumption for RAA
{3, 4}	5.	$P \& \sim Q$	3, 4 &I
{2, 3, 4}	6.	$(P \& \sim Q) \& \sim(P \& \sim Q)$	2, 5 &I
{2, 3}	7.	$\sim\sim Q$	4, 6 RAA
{2, 3}	8.	Q	7 DNE
{2}	9.	$P \to Q$	3, 8 CP
{1, 2}	10.	$(P \to Q) \& \sim(P \to Q)$	1, 9 &I
{1}	11.	$\sim\sim(P \& \sim Q)$	2, 10 RAA
{1}	12.	$P \& \sim Q$	11 DNE

3. (P v Q) & (R v S) : ((P & R) v (P & S)) v ((Q & R) v (Q & S))

{1}	1.	(P v Q) & (R v S)	Premise
{1}	2.	P v Q	1 &E
{1}	3.	R v S	1 &E
{4}	4.	P	Assumption for vE
{5}	5.	Q	Assumption for vE
{6}	6.	R	Assumption for vE
{7}	7.	S	Assumption for vE
{4, 6}	8.	P & R	4, 6 &I
{4, 6}	9.	(P & R) v (P & S)	8 vI
{4, 7}	10.	P & S	4, 7 &I
{4, 7}	11.	(P & R) v (P & S)	10 vI
{1, 4}	12.	(P & R) v (P & S)	3, 6, 9, 7, 11 vE
{1, 4}	13.	((P & R) v (P & S)) v ((Q & R) v (Q & S))	12 vI
{5, 6}	14.	Q & R	5, 6 &I
{5, 6}	15.	(Q & R) v (Q & S)	14 vI
{5, 7}	16.	Q & S	5, 7 &I
{5, 7}	17.	(Q & R) v (Q & S)	16 vI
{1, 5}	18.	(Q & R) v (Q & S)	3, 6, 15, 7, 17 vE
{1, 5}	20.	((P & R) v (P & S)) v ((Q & R) v (Q & S))	18 vI
{1}	21.	((P & R) v (P & S)) v ((Q & R) v (Q & S))	2, 4, 13, 14, 20 vE

4. P v Q, ∼Q : P

{1}	1.	P v Q	Premise
{2}	2.	∼Q	Premise
{3}	3.	P	Assumption for vE
{4}	4.	Q	Assumption for vE
{5}	5.	∼P	Assumption for RAA
{2, 4}	6.	Q & ∼Q	2, 4 &I
{2, 4, 5}	7.	(Q & ∼Q) & ∼P	5, 6 &I
{2, 4, 5}	8.	Q & ∼Q	7 &E
{2, 4}	9.	∼∼P	5, 8 RAA
{2, 4}	10.	P	9 DNE
{1, 2}	11.	P	1, 3, 3, 4, 10 vE

5. P v Q, ∼P : Q

{1}	1.	P v Q	Premise
{2}	2.	∼P	Premise
{3}	3.	Q	Assumption for vE
{4}	4.	P	Assumption for vE
{5}	5.	∼Q	Assumption for RAA
{2, 4}	6.	P & ∼P	2, 4 &I
{2, 4, 5}	7.	(P & ∼P) & ∼Q	5, 6 &I
{2, 4, 5}	8.	P & ∼P	7 &E
{2, 4}	9.	∼∼Q	5, 8 RAA
{2, 4}	10.	Q	9 DNE
{1, 2}	11.	Q	1, 3, 3, 4, 10 vE

6. : ((∼P → R) & (∼Q → R)) → (∼(P & Q) → R)

{1}	1.	(∼P → R) & (∼Q → R)	Assumption for CP
{1}	2.	∼P → R	1 &E
{1}	3.	∼Q → R	1 &E
{4}	4.	∼(P & Q)	Assumption for CP
{5}	5.	∼R	Assumption for CP
{1, 5}	6.	∼∼P	2, 5 MT
{1, 5}	7.	P	6 DNE
{1, 5}	8.	∼∼Q	3, 5 MT
{1, 5}	9.	Q	8 DNE
{1, 5}	10.	P & Q	7, 9 &I
{1}	11.	∼R → (P & Q)	5, 10 CP
{1, 4}	12.	∼∼R	4, 11 MT
{1, 4}	13.	R	12 DNE
{1}	14.	∼(P & Q) → R	4, 13 CP
–	15.	((∼P → R) & (∼Q → R)) → (∼(P & Q) → R)	1, 14 CP

7. P v Q, P v R : P v (Q & R)

{1}	1.	P v Q	Premise
{2}	2.	P v R	Premise
{3}	3.	P	Assumption for vE
{4}	4.	Q	Assumption for vE
{5}	5.	R	Assumption for vE
{3}	6.	P v (Q & R)	3 vI
{4, 5}	7.	Q & R	4, 5 &I
{4, 5}	8.	P v (Q & R)	7 vI
{1, 5}	9.	P v (Q & R)	1, 3, 6, 4, 8 vE
{1}	10.	P v (Q & R)	2, 3, 6, 5, 9 vE

8. P ↔ Q, Q ↔ R : P ↔ R

{1}	1.	P ↔ Q	Premise
{2}	2.	Q ↔ R	Premise
{1}	3.	(P → Q) & (Q → P)	1 ↔ E
{1}	4.	P → Q	3 &E
{1}	5.	Q → P	3 &E
{2}	6.	(Q → R) & (R → Q)	2 ↔ E
{2}	7.	Q → R	6 &E
{2}	8.	R → Q	6 &E
{9}	9.	P	Assumption for CP
{1, 9}	10.	Q	4, 9 MP
{1, 2, 9}	11.	R	7, 10 MP
{1, 2}	12.	P → R	9, 11 CP
{13}	13.	R	Assumption for CP
{2, 13}	14.	Q	8, 13 MP
{1, 2, 13}	15.	P	5, 14 MP
{1, 2}	16.	R → P	11, 15 CP
{1, 2}	17.	P ↔ R	12, 16 ↔ I

9. : P v (P → Q)

{1}	1.	~(P v (P → Q))	Assumption for RAA
{2}	2.	P	Assumption for RAA
{2}	3.	P v (P → Q)	2 vI
{1, 2}	4.	(P v (P → Q)) & ~(P v (P → Q))	1, 3 &I
{1}	5.	~P	2, 4 RAA
{2}	6.	P	Assumption for CP
{7}	7.	~Q	Assumption for RAA
{1, 2}	8.	P & ~P	6, 7 &I
{1, 2, 7}	9.	(P & ~P) & ~Q	7, 8 &I
{1, 2, 7}	10.	P & ~P	8 &E
{1, 2}	11.	~~Q	7, 10 RAA
{1, 2}	12.	Q	11 DNE
{1}	13.	P → Q	6, 12 CP
{1}	14.	P v (P → Q)	13 vI
{1}	15.	(P v (P → Q)) & ~(P v (P → Q))	1, 14 &I
–	16.	~~(P v (P → Q))	1, 15 RAA
–	17.	P v (P → Q)	16 DNE

(Maybe the book says it can be done in 16 steps because it uses the same step for 2 and 6.)

10. : $((P \to Q) \vee (Q \to R))$

{1}	1.	$\sim((P \to Q) \vee (Q \to R))$	Assumption for RAA
{2}	2.	Q	Assumption for RAA
{3}	3.	P	Assumption for CP
{2, 3}	4.	P & Q	2, 3 &I
{2, 3}	5.	Q	4 &E
{2}	6.	$P \to Q$	3, 5 CP
{2}	7.	$(P \to Q) \vee (Q \to R)$	6 vI
{1, 2}	8.	$((P \to Q) \vee (Q \to R))$ & $\sim((P \to Q) \vee (Q \to R))$	1, 7 &I
{1}	9.	$\sim$Q	2, 8 RAA
{10}	10.	Q	Assumption for CP
{11}	11.	$\sim$R	Assumption for RAA
{1, 10}	12.	Q & $\sim$Q	9, 10 &I
{1, 10, 11}	13.	(Q & $\sim$Q) & $\sim$R	12, 13 &I
{1, 10, 11}	14.	Q & $\sim$Q	13 &E
{1, 10}	15.	$\sim\sim$R	11, 14 RAA
{1, 10}	16.	R	15 DNE
{1}	17.	$Q \to R$	10, 16 CP
{1}	18.	$(P \to Q) \vee (Q \to R)$	17 vI
{1}	19.	$((P \to Q) \vee (Q \to R))$ & $\sim((P \to Q) \vee (Q \to R))$	1, 18 &I
–	20.	$\sim\sim((P \to Q) \vee (Q \to R))$	1, 19 RAA
–	21.	$(P \to Q) \vee (Q \to R)$	20 DNE

(Maybe the book says it can be done in 20 steps, because it uses only one Q assumption, and uses the assumption on line 2. for the assumption which is written in line 10. in my proof.)

REVISION EXERCISE IV

1 The proofs are the followings.

1. : (P v Q) → (Q v P)

{1}	1.	P v Q	Assumption for CP
{2}	2.	P	Assumption for vE
{2}	3.	Q v P	2 vI
{4}	4.	Q	Assumption for vE
{4}	5.	Q v P	4 vI
–	6.	Q v P	1, 2, 3, 4, 5 vE

2. : ∼(P v Q) → ∼P

{1}	1.	∼(P v Q)	Assumption for CP
{2}	2.	P	Assumption for RAA
{2}	3.	P v Q	2 vI
{1, 2}	4.	(P v Q) & ∼(P v Q)	1, 3 &I
{1}	5.	∼P	2, 4 RAA
–	6.	∼(P v Q) → ∼P	1, 5 CP

3. ∼(P & Q), P : ∼Q

{1}	1.	∼(P & Q)	Premise
{2}	2.	P	Premise
{3}	3.	Q	Assumption for RAA
{2, 3}	4.	P & Q	2, 3 &I
{1, 2, 3}	5.	(P & Q) & ∼(P & Q)	1, 4 &I
{1, 2}	6.	∼Q	3, 5 RAA

4. ∼(P & Q) : ∼P v ∼Q

{1}	1.	∼(P & Q)	Premise
{2}	2.	∼(∼P v ∼Q)	Assumption for RAA
{3}	3.	∼P	Assumption for RAA
{3}	4.	∼P v ∼Q	3 vI
{2, 3}	5.	(∼P v ∼Q) & ∼(∼P v ∼Q)	2, 4 &I
{2}	6.	∼∼P	3, 5 RAA
{2}	7.	P	6 DNE
{8}	8.	Q	Assumption for RAA
{2, 8}	9.	P & Q	7, 8 &I
{1, 2, 8}	10.	(P & Q) & ∼(P & Q)	1, 9 &I
{1, 2}	11.	∼Q	8, 10 RAA
{1, 2}	12.	∼P v ∼Q	11 vI
{1, 2}	13.	(∼P v ∼Q) & ∼(∼P v ∼Q)	2, 12 &I
{1}	14.	∼∼(∼P v ∼Q)	2, 13 RAA
{1}	15.	∼P v ∼Q	14 DNE

5. P v (Q v R) : (P v R) v Q

{1}	1.	P v (Q v R)	Premise
{2}	2.	P	Assumption for vE
{2}	3.	P v R	2 vI
{2}	4.	(P v R) v Q	3 vI
{5}	5.	Q v R	Assumption for vE
{6}	6.	Q	Assumption for vE
{6}	7.	(P v R) v Q	6 vI
{8}	8.	R	Assumption for vE
{8}	9.	P v R	8 vI
{8}	10.	(P v R) v Q	9 vI
{5}	11.	(P v R) v Q	6, 7, 8, 10 vE
{1}	12.	(P v R) v Q	1, 2, 4, 5, 11 vE

6. ∼P, ∼Q : ∼(P v Q)

{1}	1.	∼P	Premise
{2}	2.	∼Q	Premise
{3}	3.	P v Q	Assumption for RAA
{4}	4.	P	Assumption for vE
{1, 4}	5.	P & ∼P	1, 4 &I
{6}	6.	Q	Assumption for vE
{2, 6}	7.	Q & ∼Q	2, 6 &I
{8}	8.	∼(P & ∼P)	Assumption for RAA
{2, 6, 8}	9.	(Q & ∼Q) & ∼(P & ∼P)	7, 8 &I
{2, 6, 8}	10.	Q & ∼Q	9 &E
{2, 6}	11.	∼∼(P & ∼P)	8, 10 RAA
{2, 6}	12.	P & ∼P	11 DNE
{1, 2, 3}	13.	P & ∼P	3, 4, 5, 6, 12 vE
{1, 2}	14.	∼(P v Q)	3, 13 RAA

7. $P \to (Q \vee R) : (P \to Q) \vee (P \to R)$

{1}	1.	$P \to (Q \vee R)$	Premise
{2}	2.	$\sim((P \to Q) \vee (P \to R))$	Assumption for RAA
{3}	3.	Q	Assumption for RAA
{4}	4.	P	Assumption for CP
{3, 4}	5.	$P \& Q$	3, 4 &I
{3, 4}	6.	Q	5 &E
{3}	7.	$P \to Q$	4, 6 CP
{3}	8.	$(P \to Q) \vee (P \to R)$	7 vI
{2, 3}	9.	$((P \to Q) \vee (P \to R))$ & $\sim((P \to Q) \vee (P \to R))$	2, 8 &I
{2}	10.	$\sim Q$	3, 9 RAA
{1, 4}	11.	$Q \vee R$	1, 4 MP
{12}	12.	Q	Assumption for vE
{13}	13.	$\sim R$	Assumption for RAA
{2, 12}	14.	$Q \& \sim Q$	10, 12 &I
{2, 12, 13}	15.	$(Q \& \sim Q) \& \sim R$	13, 14 &I
{2, 12, 13}	16.	$Q \& \sim Q$	15 &E
{2, 12}	17.	$\sim\sim R$	13, 16 RAA
{2, 12}	18.	R	17 DNE
{19}	19.	R	Assumption for vE
{1, 2, 4}	20.	R	11, 12, 18, 18 vE
{1, 2}	22.	$P \to R$	4, 20 CP
{1, 2}	23.	$(P \to Q) \vee (P \to R)$	22 vI
{1, 2}	24.	$((P \to Q) \vee (P \to R))$ & $\sim((P \to Q) \vee (P \to R))$	2, 23 &I
{1}	25.	$\sim\sim((P \to Q) \vee (P \to R))$	2, 24 RAA
{1}	26.	$(P \to Q) \vee (P \to R)$	25 DNE

8. $P \to \sim Q, P \to \sim R, Q \vee R : P \to (\sim Q \vee R)$

{1}	1.	$P \to \sim Q$	Premise
{2}	2.	$P \to \sim R$	Premise
{3}	3.	$Q \vee R$	Premise
{4}	4.	P	Assumption for CP
{1, 4}	5.	$\sim Q$	1, 4 MP
{1, 4}	6.	$\sim Q \vee R$	5 vI
{1}	7.	$P \to (\sim Q \vee R)$	4, 6 CP

9. P v ∼Q, P v ∼R, Q v R : P

{1}	1.	P v ∼Q	Premise
{2}	2.	P v ∼R	Premise
{3}	3.	Q v R	Premise
{4}	4.	P	Assumption for vE
{5}	5.	∼Q	Assumption for vE
{6}	6.	∼R	Assumption for vE
{7}	7.	Q	Assumption for vE
{8}	8.	R	Assumption for vE
{9}	9.	∼P	Assumption for RAA
{5, 7}	10.	Q & ∼Q	5, 7 &I
{5, 7, 9}	11.	(Q & ∼Q) & ∼P	9, 10 &I
{5, 7, 9}	12.	Q & ∼Q	11 &E
{5, 7}	13.	∼∼P	9, 12 RAA
{5, 7}	14.	P	13 DNE
{8, 6}	15.	R & ∼R	8, 6 &I
{8, 6, 9}	16.	(R & ∼R) & ∼P	9, 15 &I
{8, 6, 9}	17.	R & ∼R	16 &E
{8, 6}	18.	∼∼P	9, 17 RAA
{8, 6}	19.	P	18 DNE
{3, 5, 6}	20.	P	3, 7, 14, 8, 19 vE
{2, 3, 5}	21.	P	2, 4, 4, 6, 20 vE
{1, 2, 3}	22.	P	1, 4, 4, 5, 21 vE

10. $(P \& Q) \to R : (P \to R) \vee (Q \to R)$

{1}	1.	(P & Q) → R	Premise
{2}	2.	∼(Q v ∼Q)	Assumption for RAA
{3}	3.	Q	Assumption for RAA
{3}	4.	Q v ∼Q	3 vI
{2, 3}	5.	(Q v ∼Q) & ∼(Q v ∼Q)	2, 4 &I
{2}	6.	∼Q	3, 5 RAA
{2}	7.	Q v ∼Q	6 vI
{2}	8.	(Q v ∼Q) & ∼(Q v ∼Q)	2, 7 &I
–	9.	∼∼(Q v ∼Q)	2, 8 RAA
–	10.	Q v ∼Q	9 DNE
{11}	11.	Q	Assumption for vE
{12}	12.	P	Assumption for CP
{11, 12}	13.	P & Q	11, 12 &I
{1, 11, 12}	14.	R	1, 13 MP
{1, 11}	15.	P → R	12, 14 CP
{1, 11}	16.	(P → R) v (Q → R)	15 vI
{17}	17.	∼Q	Assumption for vE
{18}	18.	Q	Assumption for CP
{19}	19.	∼R	Assumption for RAA
{17, 18}	20.	Q & ∼Q	17, 18 &I
{17, 18, 19}	21.	(Q & ∼Q) & ∼R	19, 20 &I
{17, 18, 19}	22.	Q & ∼Q	21 &E
{17, 18}	23.	∼∼R	19, 22 RAA
{17, 18}	24.	R	23 DNE
{17}	25.	Q → R	18, 24 CP
{17}	26.	(P → R) v (Q → R)	25 vI
{1}	27.	(P → R) v (Q → R)	10, 11, 16, 17, 26 vE

EXERCISE 3.10

1 The proofs are the followings.

1. P v Q, ∼P : Q

{1}	1.	P v Q	Premise
{2}	2.	∼P	Premise
{3}	3.	P	Assumption for vE
{4}	4.	∼Q	Assumption for RAA
{2, 3}	5.	P & ∼P	2, 3 &I
{2, 3, 4}	6.	(P & ∼P) & ∼Q	4, 5 &I
{2, 3, 4}	7.	P & ∼P	6 &E
{2, 3}	8.	∼∼Q	4, 7 RAA
{2, 3}	9.	Q	8 DNE
{10}	10.	Q	Assumption for vE
{1, 2}	11.	Q	1, 3, 9, 10 10 vE

2. P → Q, Q → R : P → R

{1}	1.	P → Q	Premise
{2}	2.	Q → R	Premise
{3}	3.	P	Assumption for CP
{1, 3}	4.	Q	1, 3 MP
{1, 2, 3}	5.	R	2, 4 MP
{1, 2}	6.	P → R	3, 5 CP

3. P → Q, ∼P → Q : Q

{1}	1.	P → Q	Premise
{2}	2.	∼P → Q	Premise
{3}	3.	∼Q	Assumption for RAA
{1, 3}	4.	∼P	1, 3 MT
{1, 2, 3}	5.	Q	2, 4 MP
{1, 2, 3}	6.	Q & ∼Q	3, 5 &I
{1, 2}	7.	∼∼Q	3, 6 RAA
{1, 2}	8.	Q	7 DNE

4. P → Q, P → ∼Q : ∼P

{1}	1.	P → Q	Premise
{2}	2.	P → ∼Q	Premise
{3}	3.	P	Assumption for RAA
{1, 3}	4.	Q	1, 3 MP
{2, 3}	5.	∼Q	2, 3 MP
{1, 2, 3}	6.	Q & ∼Q	4, 5 &I
{1, 2}	7.	∼P	3, 6 RAA

5. ~P → P : P

{1}	1.	~P → P	Premise
{2}	2.	~P	Assumption for RAA
{1, 2}	3.	P	1, 2 MP
{1, 2}	4.	P & ~P	2, 3 &I
{1}	5.	~~P	2, 4 RAA
{1}	6.	P	5 DNE

6. P → ~P : ~P

{1}	1.	P → ~P	Premise
{2}	2.	P	Assumption for RAA
{1, 2}	3.	~P	1, 2 MP
{1, 2}	4.	P & ~P	2, 3 &I
{1}	5.	~P	2, 4 RAA

Examination 1 in Formal Logic

1 The proofs are the followings.

1. P → Q : ((R & Q) → S) → ((R & P) → S)

{1}	1.	P → Q	Premise
{2}	2.	(R & Q) → S	Assumption for CP
{3}	3.	R & P	Assumption for CP
{3}	4.	R	3 &E
{3}	5.	P	3 &E
{1, 3}	6.	Q	1, 5 MP
{1, 3}	7.	R & Q	4, 6 &I
{1, 2, 3}	8.	S	2, 7 MP
{1, 2}	9.	(R & P) → S	3, 8 CP
{1}	10.	((R & Q) → S) → ((R & P) → S	2, 9 CP

2. P v Q : ~(~P & ~Q)

{1}	1.	P v Q	Premise
{2}	2.	~P & ~Q	Assumption for RAA
{2}	3.	~P	2 &E
{2}	4.	~Q	2 &E
{5}	5.	P	Assumption for vE
{2, 5}	6.	P & ~P	3, 5 &I
{5}	7.	~(~P & ~Q)	2, 6 RAA
{8}	8.	Q	Assumption for vE
{2, 8}	9.	Q & ~Q	4, 8 &I
{8}	10.	~(~P & ~Q)	2, 9 RAA
{1}	11.	~(~P & ~Q)	1, 5, 7, 8, 10 vE

3. P v ∼P

{1}	1.	∼(P v ∼P)	Assumption for RAA
{2}	2.	P	Assumption for RAA
{2}	3.	P v ∼P	2 vI
{1, 2}	4.	(P v ∼P) & ∼(P v ∼P)	1, 3 &I
{1}	5.	∼P	2, 4 RAA
{1}	6.	P v ∼P	5 vI
{1}	7.	(P v ∼P) & ∼(P v ∼P)	1, 6 &I
–	8.	∼∼(P v ∼P)	1, 7 RAA
–	9.	P v ∼P	8 DNE

2 The proofs are the followings.

1. P $\rightarrow$ Q : ∼P v Q

{1}	1.	P $\rightarrow$ Q	Premise
{2}	2.	∼(∼P v Q)	Assumption for RAA
{3}	3.	P	Assumption for RAA
{1, 3}	4.	Q	1, 3 MP
{1, 3}	5.	∼P v Q	4 vI
{1, 2, 3}	6.	(∼P v Q) & ∼(∼P v Q)	2, 5 &I
{1, 2}	7.	∼P	3, 6 RAA
{1, 2}	8.	∼P v Q	7 vI
{1, 2}	9.	(∼P v Q) & ∼(∼P v Q)	2, 8 &I
{1}	10.	∼∼(∼P v Q)	2, 9 RAA
{1}	11.	∼P v Q	10 DNE

2. ∼P v Q : P $\rightarrow$ Q

{1}	1.	∼P v Q	Premise
{2}	2.	P	Assumption for CP
{3}	3.	∼P	Assumption for vE
{4}	4.	∼Q	Assumption for RAA
{2, 3}	5.	P & ∼P	2, 3 &I
{2, 3, 4}	6.	(P & ∼P) & ∼Q	4, 5 &I
{2, 3, 4}	7.	P & ∼P	6 &E
{2, 3}	8.	∼∼Q	4, 7 RAA
{2, 3}	9.	Q	8 DNE
{3}	10.	P $\rightarrow$ Q	2, 9 CP
{11}	11.	Q	Assumption for vE
{2, 11}	12.	P & Q	2, 11 &I
{2, 11}	13.	Q	12 &E
{11}	14.	P $\rightarrow$ Q	2, 13 CP
{1}	15.	P $\rightarrow$ Q	1, 3, 10, 11, 14 vE

3. P v Q : ∼P → Q

{1}	1.	P v Q	Premise
{2}	2.	∼P	Assumption for CP
{3}	3.	P	Assumption for vE
{4}	4.	∼Q	Assumption for RAA
{2, 3}	5.	P & ∼P	2, 3 &I
{2, 3, 4}	6.	(P & ∼P) & ∼Q	4, 5 &I
{2, 3, 4}	7.	P & ∼P	6 &E
{2, 3}	8.	∼∼Q	4, 7 RAA
{2, 3}	9.	Q	8 DNE
{3}	10.	∼P → Q	2, 9 CP
{11}	11.	Q	Assumption for vE
{2, 11}	12.	∼P & Q	2, 11 &I
{2, 11}	13.	Q	12 &E
{11}	14.	∼P → Q	2, 13 CP
{1}	15.	∼P → Q	1, 3, 10, 11, 14 vE

4. ∼P → Q : P v Q

{1}	1.	∼P → Q	Premise
{2}	2.	∼(P v Q)	Assumption for RAA
{3}	3.	P	Assumption for RAA
{3}	4.	P v Q	3 vI
{2, 3}	5.	(P v Q) & ∼(P v Q)	2, 4 &I
{2}	6.	∼P	3, 5 RAA
{1, 2}	7.	Q	1, 6 MP
{1, 2}	8.	P v Q	7 vI
{1, 2}	9.	(P v Q) & ∼(P v Q)	2, 8 &I
{1}	10.	∼∼(P v Q)	2, 9 RAA
{1}	11.	P v Q	10 DNE

3 The proof is the following.

{1}	1.	$\sim$P $\leftrightarrow$ Q	Premise
{2}	2.	**P $\leftrightarrow$ Q**	Assumption for RAA
{3}	3.	Q	**Assumption for RAA**
{1}	4.	**($\sim$P $\rightarrow$ Q) & (Q $\rightarrow$ $\sim$P)**	1 $\leftrightarrow$ E
{2}	5.	**(P $\rightarrow$ Q) & (Q $\rightarrow$ P)**	2 $\leftrightarrow$ E
{2}	6.	Q $\rightarrow$ P	5 &E
{1}	7.	**Q $\rightarrow$ $\sim$P**	4 &E
{2, 3}	8.	P	**3, 6 MP**
{1, 3}	9.	$\sim$P	**3, 7 MP**
{1, 2, 3}	10.	**P & $\sim$P**	**8, 9 &I**
{1, 2}	11.	$\sim$Q	**3, 10 RAA**
{12}	12.	P	**Assumption for RAA**
{2}	13.	**P $\rightarrow$ Q**	5 &E
{2, 12}	14.	Q	12, 13 MP
{1, 2, 12}	15.	**Q & $\sim$Q**	11, 14 &I
{1, 2}	16.	$\sim$P	**12, 15 RAA**
{1}	17.	$\sim$P $\rightarrow$ Q	**4 &E**
{1, 2}	18.	**Q**	**16, 17 MP**
{1, 2}	19.	**Q & $\sim$Q**	11, 18 &I
{1}	20.	**$\sim$(P $\leftrightarrow$ Q)**	2, 19 RAA

4 The proofs are the followings.

(i) P $\rightarrow$ Q, $\sim$Q $\vdash$ $\sim$P

{1}	1.	P $\rightarrow$ Q	Premise
{2}	2.	$\sim$Q	Premise
{3}	3.	P	Assumption for RAA
{1, 3}	4.	Q	1, 3 MP
{1, 2, 3}	5.	Q & $\sim$Q	2, 4 &I
{1, 2}	6.	$\sim$P	3, 5 RAA

(ii) For this to work, instead of assuming P for RAA, P should be assumed for CP, and then conclude, that $P \to (Q \,\&\, {\sim}Q)$. Then the following needs to be proven.
$P \to (Q \,\&\, {\sim}Q) \vdash {\sim}P$

{1}	1.	$P \to (Q \,\&\, {\sim}Q)$	Premise
{2}	2.	P	Assumption for CP
{1, 2}	3.	$Q \,\&\, {\sim}Q$	1, 2 MP
{1, 2}	4.	Q	3 &E
{1, 2}	5.	${\sim}Q$	3 &E
{6}	6.	$P \to (Q \,\&\, {\sim}Q)$	Assumption for CP
{1, 2, 6}	7.	$(P \to (Q \,\&\, {\sim}Q)) \,\&\, Q$	4, 6 &I
{1, 2, 6}	8.	Q	7 &E
{1, 2}	9.	$(P \to (Q \,\&\, {\sim}Q)) \to Q$	6, 8 CP
{1, 2}	10.	${\sim}(P \to (Q \,\&\, {\sim}Q))$	5, 9 MT
{1}	11.	$P \to {\sim}(P \to (Q \,\&\, {\sim}Q))$	2, 10 CP
{1}	12.	${\sim}{\sim}(P \to (Q \,\&\, {\sim}Q))$	1 DNI
{1}	13.	${\sim}P$	11, 12 MT

5 $P, {\sim}P \vdash Q$

{1}	1.	P	Premise
{2}	2.	${\sim}P$	Premise
{3}	3.	${\sim}Q$	Assumption for RAA
{1, 2}	4.	$P \,\&\, {\sim}P$	1, 2 &I
{1, 2, 3}	5.	$(P \,\&\, {\sim}P) \,\&\, {\sim}Q$	3, 4 &I
{1, 2, 3}	6.	$P \,\&\, {\sim}P$	5 &E
{1, 2}	7.	${\sim}{\sim}Q$	3, 6 RAA
{1, 2}	8.	Q	7 DNE

Chapter Four: Formal Logic and Formal Semantics #1

EXERCISE 4.1

1 The complete truth-tables are the followings.

1. P → (P & P)

P	P	→	(	P	&	P	)
F	F	**T**		F	F	F	
T	T	**T**		T	T	T	

2. P & ~P

P	P	&	~	P
F	F	**F**	T	F
T	T	**F**	F	T

3. P v ~P

P	P	v	~	P
F	F	**T**	T	F
T	T	**T**	F	T

4. P → (Q → P)

P	Q	P	→	(	Q	→	P	)
F	F	F	**T**		F	T	F	
F	T	F	**T**		T	F	F	
T	F	T	**T**		F	T	T	
T	T	T	**T**		T	T	T	

5. (P & Q) ↔ (Q & P)

P	Q	(	P	&	Q	)	↔	(	Q	&	P	)
F	F		F	F	F		**T**		F	F	F	
F	T		F	F	T		**T**		T	F	F	
T	F		T	F	F		**T**		F	F	T	
T	T		T	T	T		**T**		T	T	T	

6. (P v Q) ↔ ~Q

P	Q	(	P	v	Q	)	↔	~	Q
F	F		F	F	F		**F**	T	F
F	T		F	T	T		**F**	F	T
T	F		T	T	F		**T**	T	F
T	T		T	T	T		**F**	F	T

7. (P → Q) → (∼Q → ∼P)

P	Q	(	P	→	Q	)	→	(	∼	Q	→	∼	P	)
F	F		F	T	F		**T**		T	F	T	T	F	
F	T		F	T	T		**T**		F	T	T	T	F	
T	F		T	F	F		**T**		T	F	F	F	T	
T	T		T	T	T		**T**		F	T	T	F	T	

8. ∼(P v Q) ↔ (∼P & ∼Q)

P	Q	∼	(	P	v	Q	)	↔	(	∼	P	&	∼	Q	)
F	F	T		F	F	F		**T**		T	F	T	T	F	
F	T	F		F	T	T		**F**		T	F	F	F	T	
T	F	F		T	T	F		**F**		F	T	F	T	F	
T	T	F		T	T	T		**T**		F	T	F	F	T	

9. ∼P & (Q v R)

P	Q	R	∼	P	&	(	Q	v	R	)
F	F	F	T	F	**F**		F	F	F	
F	F	T	T	F	**T**		F	T	T	
F	T	F	T	F	**T**		T	T	F	
F	T	T	T	F	**T**		T	T	T	
T	F	F	F	T	**F**		F	F	F	
T	F	T	F	T	**F**		F	T	T	
T	T	F	F	T	**F**		T	T	F	
T	T	T	F	T	**F**		T	T	T	

10. ∼(P & (Q v R))

P	Q	R	∼	(	P	&	(	Q	v	R	)	)
F	F	F	**T**		F	F		F	F	F		
F	F	T	**T**		F	F		F	T	T		
F	T	F	**T**		F	F		T	T	F		
F	T	T	**T**		F	F		T	T	T		
T	F	F	**T**		T	F		F	F	F		
T	F	T	**F**		T	T		F	T	T		
T	T	F	**F**		T	T		T	T	F		
T	T	T	**F**		T	T		T	T	T		

11. ∼∼(P & ∼P)

P	∼	∼	(	P	&	∼	P	)
F	**F**	T		F	F	T	F	
T	**F**	T		T	F	F	T	

12. ((P → Q) → (Q → R)) → (P → R)

P	Q	R	(	(	P	→	Q	)	→	(	Q	→	R	)	)	→	(	P	→	R	)
F	F	F			F	T	F		T		F	T	F			**T**		F	T	F	
F	F	T			F	T	F		T		F	T	T			**T**		F	T	T	
F	T	F			F	T	T		F		T	F	F			**T**		F	T	F	
F	T	T			F	T	T		T		T	T	T			**T**		F	T	T	
T	F	F			T	F	F		T		F	T	F			**F**		T	F	F	
T	F	T			T	F	F		T		F	T	T			**T**		T	T	T	
T	T	F			T	T	T		F		T	F	F			**T**		T	F	F	
T	T	T			T	T	T		T		T	T	T			**T**		T	T	T	

13. ∼((P → Q) → (Q → R)) → (P → R)

P	Q	R	∼	(	(	P	→	Q	)	→	(	Q	→	R	)	)	→	(	P	→	R	)
F	F	F	F			F	T	F		T		F	T	F			**T**		F	T	F	
F	F	T	F			F	T	F		T		F	T	T			**T**		F	T	T	
F	T	F	T			F	T	T		F		T	F	F			**T**		F	T	F	
F	T	T	F			F	T	T		T		T	T	T			**T**		F	T	T	
T	F	F	F			T	F	F		T		F	T	F			**T**		T	F	F	
T	F	T	F			T	F	F		T		F	T	T			**T**		T	T	T	
T	T	F	T			T	T	T		F		T	F	F			**F**		T	F	F	
T	T	T	F			T	T	T		T		T	T	T			**T**		T	T	T	

14. (P → ((Q → R) v ∼R)) v ∼Q

P	Q	R	(	P	→	(	(	Q	→	R	)	v	∼	R	)	)	v	∼	Q
F	F	F		F	T			F	T	F		T	T	F			**T**	T	F
F	F	T		F	T			F	T	T		T	F	T			**T**	T	F
F	T	F		F	T			T	F	F		T	T	F			**T**	F	T
F	T	T		F	T			T	T	T		T	F	T			**T**	F	T
T	F	F		T	T			F	T	F		T	T	F			**T**	T	F
T	F	T		T	T			F	T	T		T	F	T			**T**	T	F
T	T	F		T	T			T	F	F		T	T	F			**T**	F	T
T	T	T		T	T			T	T	T		T	F	T			**T**	F	T

15. $((P \& Q) \rightarrow (R \vee \sim S)) \rightarrow T$

T	S	R	Q	P	(	(	P	&	Q	)	→	(	R	v	~	S	)	)	→	T
F	F	F	F	F			F	F	F		T		F	T	T	F			**F**	F
F	F	F	F	T			T	F	F		T		F	T	T	F			**F**	F
F	F	F	T	F			F	F	T		T		F	T	T	F			**F**	F
F	F	F	T	T			T	T	T		T		F	T	T	F			**F**	F
F	F	T	F	F			F	F	F		T		T	T	T	F			**F**	F
F	F	T	F	T			T	F	F		T		T	T	T	F			**F**	F
F	F	T	T	F			F	F	T		T		T	T	T	F			**F**	F
F	F	T	T	T			T	T	T		T		T	T	T	F			**F**	F
F	T	F	F	F			F	F	F		T		F	F	F	T			**F**	F
F	T	F	F	T			T	F	F		T		F	F	F	T			**F**	F
F	T	F	T	F			F	F	T		T		F	F	F	T			**F**	F
F	T	F	T	T			T	T	T		F		F	F	F	T			**T**	F
F	T	T	F	F			F	F	F		T		T	T	F	T			**F**	F
F	T	T	F	T			T	F	F		T		T	T	F	T			**F**	F
F	T	T	T	F			F	F	T		T		T	T	F	T			**F**	F
F	T	T	T	T			T	T	T		T		T	T	F	T			**F**	F
T	F	F	F	F			F	F	F		T		F	T	T	F			**T**	T
T	F	F	F	T			T	F	F		T		F	T	T	F			**T**	T
T	F	F	T	F			F	F	T		T		F	T	T	F			**T**	T
T	F	F	T	T			T	T	T		T		F	T	T	F			**T**	T
T	F	T	F	F			F	F	F		T		T	T	T	F			**T**	T
T	F	T	F	T			T	F	F		T		T	T	T	F			**T**	T
T	F	T	T	F			F	F	T		T		T	T	T	F			**T**	T
T	F	T	T	T			T	T	T		T		T	T	T	F			**T**	T
T	T	F	F	F			F	F	F		T		F	F	F	T			**T**	T
T	T	F	F	T			T	F	F		T		F	F	F	T			**T**	T
T	T	F	T	F			F	F	T		T		F	F	F	T			**T**	T
T	T	F	T	T			T	T	T		F		F	F	F	T			**T**	T
T	T	T	F	F			F	F	F		T		T	T	F	T			**T**	T
T	T	T	F	T			T	F	F		T		T	T	F	T			**T**	T
T	T	T	T	F			F	F	T		T		T	T	F	T			**T**	T
T	T	T	T	T			T	T	T		T		T	T	F	T			**T**	T

EXERCISE 4.2

1 The formulas are the following kinds:

1 tautologous,
2 inconsistent,
3 tautologous,
4 tautologous,
5 tautologous,
6 contingent,
7 tautologous,
8 contingent,
9 contingent,
10 contingent,
11 inconsistent,
12 contingent,
13 contingent,
14 tautologous,
15 contingent.

2 (i) The negation of any tautologous formula is an inconsistent formula, (ii) and the negation of any contingent formula is a contingent formula.

3 The answer depends on what we mean by "have to test". There are simple ways to reduce the number of tests. We need to find if there is a case where the formula is false. We only need to test in case T is false, and the antecedent is true. The antecedentis true when either

1 (P & Q) is false or
2 (P & Q) is true and (R v ∼S) is true.

The first case can be false when P is false or Q is false. In effect, we only tested one case: T is false and P is false.

EXERCISE 4.3

1 The complete truth-tables are the followings. For each of these formulas, an IPLI is constructed by substituting "0=0" for true atomic formulas, and "0=1"for false

atomic formulas in the case all of the premises are true, and the conclusion is false.

1. P v Q : P
   Semantically invalid.

P	Q	P	v	Q	P
F	F	F	**F**	F	**F**
F	T	F	**T**	T	**F**
T	F	T	**T**	F	**T**
T	T	T	**T**	T	**T**

2. P v ~Q, ~Q : ~P
   Semantically invalid.

P	Q	P	v	~	Q	~	Q	~	P
F	F	F	**T**	T	F	**T**	F	**T**	F
F	T	F	**F**	F	T	**F**	T	**T**	F
T	F	T	**T**	T	F	**T**	F	**F**	T
T	T	T	**T**	F	T	**F**	T	**F**	T

3. Q : ~(~P & ~Q)
   Semantically valid.

P	Q	Q	~	(	~	P	&	~	Q	)
F	F	**F**	**F**		T	F	T	T	F	
F	T	**T**	**T**		T	F	F	F	T	
T	F	**F**	**T**		F	T	F	T	F	
T	T	**T**	**T**		F	T	F	F	T	

4. ~(P v ~Q) : ~P & Q
   Semantically valid.

P	Q	~	(	P	v	~	Q	)	~	P	&	Q
F	F	**F**		F	T	T	F		T	F	**F**	F
F	T	**T**		F	F	F	T		T	F	**T**	T
T	F	**F**		T	T	T	F		F	T	**F**	F
T	T	**F**		T	T	F	T		F	T	**F**	T

5. P → Q, Q → R : R → P
Semantically invalid.

P	Q	R	P	→	Q	Q	→	R	R	→	P
F	F	F	F	**T**	F	F	**T**	F	F	**T**	F
F	F	T	F	**T**	F	F	**T**	T	T	**F**	F
F	T	F	F	**T**	T	T	**F**	F	F	**T**	F
F	T	T	F	**T**	T	T	**T**	T	T	**F**	F
T	F	F	T	**F**	F	F	**T**	F	F	**T**	T
T	F	T	T	**F**	F	F	**T**	T	T	**T**	T
T	T	F	T	**T**	T	T	**F**	F	F	**T**	T
T	T	T	T	**T**	T	T	**T**	T	T	**T**	T

6. P → (Q → R) : Q → (P → R)
Semantically valid.

P	Q	R	P	→	(	Q	→	R	)	Q	→	(	P	→	R	)
F	F	F	F	**T**		F	T	F		F	**T**		F	T	F	
F	F	T	F	**T**		F	T	T		F	**T**		F	T	T	
F	T	F	F	**T**		T	F	F		T	**T**		F	T	F	
F	T	T	F	**T**		T	T	T		T	**T**		F	T	T	
T	F	F	T	**T**		F	T	F		F	**T**		T	F	F	
T	F	T	T	**T**		F	T	T		F	**T**		T	T	T	
T	T	F	T	**F**		T	F	F		T	**F**		T	F	F	
T	T	T	T	**T**		T	T	T		T	**T**		T	T	T	

7. P & ~Q : ~(P → Q)
Semantically valid.

P	Q	P	&	~	Q	~	(	P	→	Q	)
F	F	F	**F**	T	F	**F**		F	T	F	
F	T	F	**F**	F	T	**F**		F	T	T	
T	F	T	**T**	T	F	**T**		T	F	F	
T	T	T	**F**	F	T	**F**		T	T	T	

8. Q → P, P v Q : P v R
Semantically valid.

P	Q	R	Q	→	P	P	v	Q	P	v	R
F	F	F	F	**T**	F	F	**F**	F	F	**F**	F
F	F	T	F	**T**	F	F	**F**	F	F	**T**	F
F	T	F	T	**F**	F	F	**T**	T	F	**F**	F
F	T	T	T	**F**	F	F	**T**	T	F	**T**	T
T	F	F	F	**T**	T	T	**T**	F	T	**T**	F
T	F	T	F	**T**	T	T	**T**	F	T	**T**	F
T	T	F	T	**T**	T	T	**T**	T	T	**T**	F
T	T	T	T	**T**	T	T	**T**	T	T	**T**	T

9. : ((~P → Q)→ ~P) → ~P
Semantically valid.

P	Q	(	(	~	P	→	Q	)	→	~	P	)	→	~	P
F	F			T	F	F	F		T	T	F		**T**	T	F
F	T			T	F	T	T		T	T	F		**T**	T	F
T	F			F	T	T	F		F	F	T		**T**	F	T
T	T			F	T	T	T		F	F	T		**T**	F	T

10. : ~(P v ~Q) → (~P & Q)
Semantically valid.

P	Q	~	(	P	v	~	Q	)	→	(	~	P	&	Q	)
F	F	F		F	T	T	F		**T**		T	F	F	F	
F	T	T		F	F	F	T		**T**		T	F	T	T	
T	F	F		T	T	T	F		**T**		F	T	F	F	
T	T	F		T	T	F	T		**T**		F	T	F	T	

11. ~R → Q : (P v Q) → (~R → P)
Semantically invalid.

P	Q	R	~	R	→	Q	(	P	v	Q	)	→	(	~	R	→	P	)
F	F	F	T	F	**F**	F		F	F	F		**T**		T	F	F	F	
F	F	T	F	T	**T**	F		F	F	F		**T**		F	T	T	F	
F	T	F	T	F	**T**	T		F	T	T		**F**		T	F	F	F	
F	T	T	F	T	**T**	T		F	T	T		**T**		F	T	T	F	
T	F	F	T	F	**F**	F		T	T	F		**F**		T	F	F	F	
T	F	T	F	T	**T**	F		T	T	F		**T**		F	T	T	F	
T	T	F	T	F	**T**	T		T	T	T		**F**		T	F	F	F	
T	T	T	F	T	**T**	T		T	T	T		**T**		F	T	T	F	

12. ~P → (Q v R), ~P → ~R : Q
Semantically invalid.

P	Q	R	~	P	→	(	Q	v	R	)	~	P	→	~	R	**Q**
F	F	F	T	F	**F**		F	F	F		T	F	**T**	T	F	**F**
F	F	T	T	F	**T**		F	T	T		T	F	**F**	F	T	**F**
F	T	F	T	F	**T**		T	T	F		T	F	**T**	T	F	**T**
F	T	T	T	F	**T**		T	T	T		T	F	**F**	F	T	**T**
T	F	F	F	T	**T**		F	F	F		F	T	**T**	T	F	**F**
T	F	T	F	T	**T**		F	T	T		F	T	**T**	F	T	**F**
T	T	F	F	T	**T**		T	T	F		F	T	**T**	T	F	**T**
T	T	T	F	T	**T**		T	T	T		F	T	**T**	F	T	**T**

13. P & (Q v (Q → R)) : (P & Q) v ((P & ~Q) v (P & R))
Semantically valid.

P	Q	R	P	&	(	Q	v	(	Q	→	R	)	)	(	P	&	Q	)	v	(	(	P	&	~	Q	)	v	(	P	&	R	)	)
F	F	F	F	**F**		F	T		F	T	F				F	F	F		**F**			F	F	T	F		F		F	F	F		
F	F	T	F	**F**		F	T		F	T	T				F	F	F		**F**			F	F	T	F		F		F	F	T		
F	T	F	F	**F**		T	T		T	F	F				F	F	T		**F**			F	F	F	T		F		F	F	F		
F	T	T	F	**F**		T	T		T	T	T				F	F	T		**F**			F	F	F	T		F		F	F	T		
T	F	F	T	**T**		F	T		F	T	F				T	F	F		**T**			T	T	T	F		T		T	F	F		
T	F	T	T	**T**		F	T		F	T	T				T	F	F		**T**			T	T	T	F		T		T	T	T		
T	T	F	T	**T**		T	T		T	F	F				T	T	T		**T**			T	F	F	T		F		T	F	F		
T	T	T	T	**T**		T	T		T	T	T				T	T	T		**T**			T	F	F	T		T		T	T	T		

14. : (P v Q) ↔ ~(~(Q & ~P) & ~(P → Q))
Semantically invalid.

P	Q	(	P	v	Q	)	↔	~	(	~	(	Q	&	~	P	)	&	~	(	P	→	Q	)	)
F	F		F	F	F		**F**	T		T		F	F	T	F		F	F		F	T	F		
F	T		F	T	T		**F**	T		F		T	T	T	F		F	F		F	T	T		
T	F		T	T	F		**F**	F		T		F	F	F	T		T	T		T	F	F		
T	T		T	T	T		**T**	T		T		T	F	F	T		F	F		T	T	T		

15. (P v Q) ↔ (~R v S) : R → (P ↔ ~(R & ~S))
Semantically invalid.

P	Q	R	S	(	P	v	Q	)	↔	(	~	R	v	S	)	R	→	(	P	↔	~	(	R	&	~	S	)	)
F	F	F	F		F	F	F		**F**		T	F	T	F		F	**T**		F	F	T		F	F	T	F		
F	F	F	T		F	F	F		**F**		T	F	T	T		F	**T**		F	F	T		F	F	F	T		
F	F	T	F		F	F	F		**T**		F	T	F	F		T	**T**		F	T	F		T	T	T	F		
F	F	T	T		F	F	F		**F**		F	T	T	T		T	**F**		F	F	T		T	F	F	T		
F	T	F	F		F	T	T		**T**		T	F	T	F		F	**T**		F	F	T		F	F	T	F		
F	T	F	T		F	T	T		**T**		T	F	T	T		F	**T**		F	F	T		F	F	F	T		
F	T	T	F		F	T	T		**F**		F	T	F	F		T	**T**		F	T	F		T	T	T	F		
F	T	T	T		F	T	T		**T**		F	T	T	T		T	**F**		F	F	T		T	F	F	T		
T	F	F	F		T	F	F		**F**		T	F	T	F		F	**T**		T	T	T		F	F	T	F		
T	F	F	T		T	F	F		**F**		T	F	T	T		F	**T**		T	T	T		F	F	F	T		
T	F	T	F		T	F	F		**T**		F	T	F	F		T	**F**		T	F	F		T	T	T	F		
T	F	T	T		T	F	F		**F**		F	T	T	T		T	**T**		T	T	T		T	F	F	T		
T	T	F	F		T	T	T		**T**		T	F	T	F		F	**T**		T	T	T		F	F	T	F		
T	T	F	T		T	T	T		**T**		T	F	T	T		F	**T**		T	T	T		F	F	F	T		
T	T	T	F		T	T	T		**F**		F	T	F	F		T	**F**		T	F	F		T	T	T	F		
T	T	T	T		T	T	T		**T**		F	T	T	T		T	**T**		T	T	T		T	F	F	T		

EXERCISE 4.4

1 The truth-tables are the followings.

1. ~(P v Q) : ~P & Q

P	Q	~	(	P	v	~	Q	)	&	~	(	~	P	&	Q	)
F	F	F		F	T	T	F		**F**	T		T	F	F	F	
F	T	T		F	F	F	T		**F**	F		T	F	T	T	
T	F	F		T	T	T	F		**F**	T		F	T	F	F	
T	T	F		T	T	F	T		**F**	T		F	T	F	T	

2. P → Q, Q → R, P : R

P	Q	R	(	(	(	P	→	Q	)	&	(	Q	→	R	)	)	&	P	)	&	~	R
F	F	F				F	T	F		T		F	T	F			F	F		**F**	T	F
F	F	T				F	T	F		T		F	T	T			F	F		**F**	F	T
F	T	F				F	T	T		F		T	F	F			F	F		**F**	T	F
F	T	T				F	T	T		T		T	T	T			F	F		**F**	F	T
T	F	F				T	F	F		F		F	T	F			F	T		**F**	T	F
T	F	T				T	F	F		F		F	T	T			F	T		**F**	F	T
T	T	F				T	T	T		F		T	F	F			F	T		**F**	T	F
T	T	T				T	T	T		T		T	T	T			T	T		**F**	F	T

3. ~Q → (~P → Q), ~Q : (~P → Q)

P	Q	(	(	~	Q	→	(	~	P	→	Q	)	)	&	~	Q	)	&	~	(	~	P	→	Q	)
F	F			T	F	F		F	F	T	F			F	T	F		**F**	T		T	F	F	F	
F	T			F	T	T		T	F	T	T			F	F	T		**F**	F		T	F	T	T	
T	F			T	F	T		F	T	T	F			T	T	F		**F**	F		F	T	T	F	
T	T			F	T	T		F	T	T	T			F	F	T		**F**	F		F	T	T	T	

4. P → (Q → R), P, ~R : ~Q

P	Q	R	(	(	(	P	→	(	Q	→	R	)	)	&	P	)	&	~	R	)	&	~	~	Q
F	F	F				F	T		F	T	F			F	F		F	T	F		**F**	F	T	F
F	F	T				F	T		F	T	T			F	F		F	F	T		**F**	F	T	F
F	T	F				F	T		T	F	F			F	F		F	T	F		**F**	T	F	T
F	T	T				F	T		T	T	T			F	F		F	F	T		**F**	T	F	T
T	F	F				T	T		F	T	F			T	T		T	T	F		**F**	F	T	F
T	F	T				T	T		F	T	T			T	T		F	F	T		**F**	F	T	F
T	T	F				T	F		T	F	F			F	T		F	T	F		**F**	T	F	T
T	T	T				T	T		T	T	T			T	T		F	F	T		**F**	T	F	T

5. : (P → Q) v (Q → R)

P	Q	R	~	(	(	P	→	Q	)	v	(	Q	→	R	)	)
F	F	F	**F**			F	T	F		T		F	T	F		
F	F	T	**F**			F	T	F		T		F	T	T		
F	T	F	**F**			F	T	T		T		T	F	F		
F	T	T	**F**			F	T	T		T		T	T	T		
P	F	F	**F**			T	F	F		T		F	T	F		
P	F	T	**F**			T	F	F		T		F	T	T		
P	T	F	**F**			T	T	T		T		T	F	F		
P	T	T	**F**			T	T	T		T		T	T	T		

2 The truth-tables are the followings.

1. P → Q, Q → R : P → R

P	Q	R	(	(	P	→	Q	)	&	(	Q	→	R	)	)	→	(	P	→	R	)
F	F	F			F	T	F		T		F	T	F			**T**		F	T	F	
F	F	T			F	T	F		T		F	T	T			**T**		F	T	T	
F	Q	F			F	T	T		F		T	F	F			**T**		F	T	F	
F	Q	T			F	T	T		T		T	T	T			**T**		F	T	T	
P	F	F			T	F	F		F		F	T	F			**T**		T	F	F	
P	F	T			T	F	F		F		F	T	T			**T**		T	T	T	
P	Q	F			T	T	T		F		T	F	F			**T**		T	F	F	
P	Q	T			T	T	T		T		T	T	T			**T**		T	T	T	

2. P → Q, Q → R : P → R

P	Q	R	(	P	→	(	Q	→	R	)	)	→	(	Q	→	(	P	→	R	)	)
F	F	F		F	T		F	T	F			**T**		F	T		F	T	F		
F	F	T		F	T		F	T	T			**T**		F	T		F	T	T		
F	T	F		F	T		T	F	F			**T**		T	T		F	T	F		
F	T	T		F	T		T	T	T			**T**		T	T		F	T	T		
T	F	F		T	T		F	T	F			**T**		F	T		T	F	F		
T	F	T		T	T		F	T	T			**T**		F	T		T	T	T		
T	T	F		T	F		T	F	F			**T**		T	F		T	F	F		
T	T	T		T	T		T	T	T			**T**		T	T		T	T	T		

3. P → (Q → R), P, ∼R : ∼Q

P	Q	R	(	(	(	P	→	(	Q	→	R	)	)	&	P	)	&	∼	R	)	→	∼	Q
F	F	F				F	T		F	T	F			F	F		F	T	F		**T**	T	F
F	F	T				F	T		F	T	T			F	F		F	F	T		**T**	T	F
F	T	F				F	T		T	F	F			F	F		F	T	F		**T**	F	T
F	T	T				F	T		T	T	T			F	F		F	F	T		**T**	F	T
T	F	F				T	T		F	T	F			T	T		T	T	F		**T**	T	F
T	F	T				T	T		F	T	T			T	T		F	F	T		**T**	T	F
T	T	F				T	F		T	F	F			F	T		F	T	F		**T**	F	T
T	T	T				T	T		T	T	T			T	T		F	F	T		**T**	F	T

4. P, (Q & R) : (P & Q) v (P & R)

P	Q	R	(	P	&	(	Q	&	R	)	)	→	(	(	P	&	Q	)	v	(	P	&	R	)	)
F	F	F		F	F		F	F	F			**T**			F	F	F		F		F	F	F		
F	F	T		F	F		F	F	T			**T**			F	F	F		F		F	F	T		
F	T	F		F	F		T	F	F			**T**			F	F	T		F		F	F	F		
F	T	T		F	F		T	T	T			**T**			F	F	T		F		F	F	T		
T	F	F		T	F		F	F	F			**T**			T	F	F		v		T	F	F		
T	F	T		T	F		F	F	T			**T**			T	F	F		T		T	T	T		
T	T	F		T	F		T	F	F			**T**			T	T	T		T		T	F	F		
T	T	T		T	T		T	T	T			**T**			T	T	T		T		T	T	T		

5. P ↔ Q, Q ↔ R : P ↔ R

P	Q	R	(	(	P	↔	Q	)	&	(	Q	↔	R	)	)	→	(	P	↔	R	)
F	F	F			F	T	F		T		F	T	F			**T**		F	T	F	
F	F	T			F	T	F		F		F	F	T			**T**		F	F	T	
F	T	F			F	F	T		F		T	F	F			**T**		F	T	F	
F	T	T			F	F	T		F		T	T	T			**T**		F	F	T	
T	F	F			T	F	F		F		F	T	F			**T**		T	F	F	
T	F	T			T	F	F		F		F	F	T			**T**		T	T	T	
T	T	F			T	T	T		F		T	F	F			**T**		T	F	F	
T	T	T			T	T	T		T		T	T	T			**T**		T	T	T	

3 The truth-tables are the followings.

1. ∼(P v ∼Q) : ∼P & Q

P	Q	(	∼	(	P	v	∼	Q	)	)	→	(	∼	P	&	Q	)
F	F		F		F	T	T	F			**T**		T	F	F	F	
F	T		T		F	F	F	T			**T**		T	F	T	T	
T	F		F		T	T	T	F			**T**		F	T	F	F	
T	T		F		T	T	F	T			**T**		F	T	F	T	

2. $P \rightarrow Q,\ Q \rightarrow R,\ P : R$

P	Q	R	(	P	$\rightarrow$	Q	)	$\rightarrow$	(	(	Q	$\rightarrow$	R	)	$\rightarrow$	(	P	$\rightarrow$	R	)	)
F	F	F		F	T	F		**T**			F	T	F		T		F	T	F		
F	F	T		F	T	F		**T**			F	T	T		T		F	T	T		
F	T	F		F	T	T		**T**			T	F	F		T		F	T	F		
F	T	T		F	T	T		**T**			T	T	T		T		F	T	T		
T	F	F		T	F	F		**T**			F	T	F		F		T	F	F		
T	F	T		T	F	F		**T**			F	T	T		T		T	T	T		
T	T	F		T	T	T		**T**			T	F	F		T		T	F	F		
T	T	T		T	T	T		**T**			T	T	T		T		T	T	T		

3. $\sim Q \leftrightarrow (\sim P \rightarrow Q),\ \sim Q : (\sim P \rightarrow Q)$

P	Q	(	$\sim$	Q	$\rightarrow$	(	$\sim$	P	$\rightarrow$	Q	)	)	$\rightarrow$	(	$\sim$	Q	$\rightarrow$	(	$\sim$	P	$\rightarrow$	Q	)	)
F	F		T	F	F		T	F	F	F			**T**		T	F	F		T	F	F	F		
F	T		F	T	T		T	F	T	T			**T**		F	T	T		T	F	T	T		
T	F		T	F	T		F	T	T	F			**T**		T	F	T		F	T	T	F		
T	T		F	T	T		F	T	T	T			**T**		F	T	T		F	T	T	T		

4. $P \rightarrow (Q \rightarrow R),\ P,\ \sim R : \sim Q$

P	Q	(	$\sim$	Q	$\rightarrow$	(	$\sim$	P	$\rightarrow$	Q	)	)	$\rightarrow$	(	$\sim$	Q	$\rightarrow$	(	$\sim$	P	$\rightarrow$	Q	)	)
F	F		T	F	F		T	F	F	F			**T**		T	F	F		T	F	F	F		
F	T		F	T	T		T	F	T	T			**T**		F	T	T		T	F	T	T		
T	F		T	F	T		F	T	T	F			**T**		T	F	T		F	T	T	F		
T	T		F	T	T		F	T	T	T			**T**		F	T	T		F	T	T	T		

5. $: (P \rightarrow Q) \text{ v } (Q \rightarrow R)$

P	Q	R	(	P	$\rightarrow$	Q	)	v	(	Q	$\rightarrow$	R	)
F	F	F		F	T	F		**T**		F	T	F	
F	F	T		F	T	F		**T**		F	T	T	
F	T	F		F	T	T		**T**		T	F	F	
F	T	T		F	T	T		**T**		T	T	T	
P	F	F		T	F	F		**T**		F	T	F	
P	F	T		T	F	F		**T**		F	T	T	
P	T	F		T	T	T		**T**		T	F	F	
P	T	T		T	T	T		**T**		T	T	T	

4 The interpretations are the followings.

1. $P,\ \sim(P \ \&\ Q) : \sim Q$
   Valid sequent.

P	$\sim$	(	P	&	Q	)	$\sim$	Q
1	2		5	4	6		3	
T	T		T	F	F		F	

2. P → (Q → R) : Q → (P → R)
   Valid sequent.

P	→	(	Q	→	R	)	Q	→	(	P	→	R	)
7	1		9	8			3	2		5	4	6	
T	T		T	T			T	F		T	F	F	

3. Q → R : (∼Q → ∼P) → (P → R)
   Valid sequent.

Q	→	R	(	∼	Q	→	∼	P	)	→	(	P	→	R	)
8	1	7	(	10	9	3	11	P	)	2	(	5	4	6	)
F	T	F	(	T	F	T	T		)	F	(	T	F	F	)

4. ∼(P → Q), Q v (R & S) : R & S
   Valid sequent.

∼	(	P	→	Q	)	Q	v	(	R	&	S	)	R	&	S
1		5	4	6		7	2		9	8	10			3	11
T		T	F	F		F	T		T	T	T			F	T

5. (P & ∼Q) v (Q & ∼P) : P ↔ Q
   Invalid sequent.

(	P	&	∼	Q	)	v	(	Q	&	∼	P	)	P	↔	Q
	5	8	7	6		1		10	9	11	12		3	2	4
	F	F	F	T		T		T	T	T	F		F	F	T

EXERCISE 4.5

1 The truth-tables are the followings.

1. P & Q, ∼(∼P v ∼Q)

P	Q	P	&	Q	∼	(	∼	P	v	∼	Q	)
F	F	F	**F**	F	**F**		T	F	T	T	F	
F	T	F	**F**	T	**F**		T	F	T	F	T	
T	F	T	**F**	F	**F**		F	T	T	T	F	
T	T	T	**T**	T	**T**		F	T	F	F	T	

2. P v Q, ∼(∼P & ∼Q)

P	Q	P	v	Q	∼	(	∼	P	&	∼	Q	)
F	F	F	**F**	F	**F**		T	F	T	T	F	
F	T	F	**T**	T	**T**		T	F	F	F	T	
T	F	T	**T**	F	**T**		F	T	F	T	F	
T	T	T	**T**	T	**T**		F	T	F	F	T	

3. ∼(P & Q), ∼P v ∼Q

P	Q	∼	(	P	&	Q	)	∼	P	v	∼	Q
F	F	**T**		F	F	F		T	F	**T**	T	F
F	T	**T**		F	F	T		T	F	**T**	F	T
T	F	**T**		T	F	F		F	T	**T**	T	F
T	T	**F**		T	T	T		F	T	**F**	F	T

4. ∼(P v Q), ∼P & ∼Q

P	Q	∼	(	P	v	Q	)	∼	P	&	∼	Q
F	F	**T**		F	F	F		T	F	**T**	T	F
F	T	**F**		F	T	T		T	F	**F**	F	T
T	F	**F**		T	T	F		F	T	**F**	T	F
T	T	**F**		T	T	T		F	T	**F**	F	T

2 The truth-tables are the followings.

(i)

P: Professor Cameron's car is in the car park.
Q: Professor Cameron is in his office.

P	Q	P	→	Q	∼	Q	→	∼	P
F	F	F	**T**	F	T	F	**T**	T	F
F	T	F	**T**	T	F	T	**T**	T	F
T	F	T	**F**	F	T	F	**F**	F	T
T	T	T	**T**	T	F	T	**T**	F	T

(ii)

P: You eat your cake.
Q: You still have your cake.

P	Q	P	→	∼	Q	∼	(	P	&	Q	)
F	F	F	**T**	T	F	**T**		F	F	F	
F	T	F	**T**	F	T	**T**		F	F	T	
T	F	T	**T**	T	F	**T**		T	F	F	
T	T	T	**F**	F	T	**F**		T	T	T	

(iii)

P: Students love logic exams.
Q: Students are very enlightened.

P	Q	P	↔	Q	(	P	→	Q	)	&	(	Q	→	P	)
F	F	F	**T**	F		F	T	F		**T**		F	T	F	
F	T	F	**F**	T		F	T	T		**F**		T	F	F	
P	F	P	**F**	F		T	F	F		**F**		F	T	T	
P	T	P	**T**	T		T	T	T		**T**		T	T	T	

(iv)

P: The sun is shining.
Q: Everything in the garden is coming up roses.

P	Q	P	&	Q	~	(	~	P	v	~	Q	)
F	F	F	**F**	F	**F**		T	F	T	T	F	
F	T	F	**F**	T	**F**		T	F	T	F	T	
T	F	T	**F**	F	**F**		F	T	T	T	F	
T	T	T	**T**	T	**T**		F	T	F	F	T	

3 The truth-tables are the followings.

(i) ~P v ~Q, P → ~Q
Semantically equivalent.

P	Q	~	P	v	~	Q	P	→	~	Q
F	F	T	F	**T**	T	F	F	**T**	T	F
F	T	T	F	**T**	F	T	F	**T**	F	T
T	F	F	T	**T**	T	F	T	**T**	T	F
T	T	F	T	**F**	F	T	T	**F**	F	T

(ii) ~(P → Q), P & ~Q
Semantically equivalent.

P	Q	~	(	P	→	Q	)	P	&	~	Q
F	F	**F**		F	T	F		F	**F**	T	F
F	T	**F**		F	T	T		F	**F**	F	T
T	F	**T**		T	F	F		T	**T**	T	F
T	T	**F**		T	T	T		T	**F**	F	T

(iii) P → (P → Q), P → Q
Semantically equivalent.

P	Q	P	→	(	P	→	Q	)	P	→	Q
F	F	F	**T**		F	T	F		F	**T**	F
F	T	F	**T**		F	T	T		F	**T**	T
T	F	T	**F**		T	F	F		T	**F**	F
T	T	T	**T**		T	T	T		T	**T**	T

(iv) P v Q, ~(~P & ~Q)
Semantically equivalent.

P	Q	P	v	Q	~	(	~	P	&	~	Q	)
F	F	F	**F**	F	**F**		T	F	T	T	F	
F	Q	F	**T**	T	**T**		T	F	F	F	T	
T	F	T	**T**	F	**T**		F	T	F	T	F	
T	Q	T	**T**	T	**T**		F	T	F	F	T	

(v) P v (∼∼Q & R), Q → (P & ∼R)
Semantically inequivalent.

P	Q	R	P	v	(	∼	∼	Q	&	R	)	Q	→	(	P	&	∼	R	)
F	F	F	F	**F**		F	T	F	F	F		F	**T**		F	F	T	F	
F	F	T	F	**F**		F	T	F	F	T		F	**T**		F	F	F	T	
F	T	F	F	**F**		T	F	T	F	F		T	**F**		F	F	T	F	
F	T	T	F	**T**		T	F	T	T	T		T	**F**		F	F	F	T	
T	F	F	T	**T**		F	T	F	F	F		F	**T**		T	T	T	F	
T	F	T	T	**T**		F	T	F	F	T		F	**T**		T	F	F	T	
T	T	F	T	**T**		T	F	T	F	F		T	**T**		T	T	T	F	
T	T	T	T	**T**		T	F	T	T	T		T	**F**		T	F	F	T	

(vi) (P v Q) & ∼(P & Q), ∼(P ↔ Q)
Semantically equivalent.

P	Q	(	P	v	Q	)	&	∼	(	P	&	Q	)	∼	(	P	↔	Q	)
F	F		F	F	F		**F**	T		F	F	F		**F**		F	T	F	
F	T		F	T	T		**T**	T		F	F	T		**T**		F	F	T	
T	F		T	T	F		**T**	T		T	F	F		**T**		T	F	F	
T	T		T	T	T		**F**	F		T	T	T		**F**		T	T	T	

(vii) (P → Q) v ∼(∼R → S), (∼P v Q) v (R v S)
Semantically inequivalent.

P	Q	R	S	(	P	→	Q	)	v	∼	(	∼	R	→	S	)	(	∼	P	v	Q	)	v	(	R	v	S	)
F	F	F	F		F	T	F		**T**	T		T	F	F	F			T	F	T	F		**T**		F	F	F	
F	F	F	T		F	T	F		**T**	F		T	F	T	T			T	F	T	F		**T**		F	T	T	
F	F	T	F		F	T	F		**T**	F		F	T	T	F			T	F	T	F		**T**		T	T	F	
F	F	T	T		F	T	F		**T**	F		F	T	T	T			T	F	T	F		**T**		T	T	T	
F	T	F	F		F	T	T		**T**	T		T	F	F	F			T	F	T	T		**T**		F	F	F	
F	T	F	T		F	T	T		**T**	F		T	F	T	T			T	F	T	T		**T**		F	T	T	
F	T	T	F		F	T	T		**T**	F		F	T	T	F			T	F	T	T		**T**		T	T	F	
F	T	T	T		F	T	T		**T**	F		F	T	T	T			T	F	T	T		**T**		T	T	T	
T	F	F	F		T	F	F		**T**	T		T	F	F	F			F	T	F	F		**F**		F	F	F	
T	F	F	T		T	F	F		**F**	F		T	F	T	T			F	T	F	F		**T**		F	T	T	
T	F	T	F		T	F	F		**F**	F		F	T	T	F			F	T	F	F		**T**		T	T	F	
T	F	T	T		T	F	F		**F**	F		F	T	T	T			F	T	F	F		**T**		T	T	T	
T	T	F	F		T	T	T		**T**	T		T	F	F	F			F	T	T	T		**T**		F	F	F	
T	T	F	T		T	T	T		**T**	F		T	F	T	T			F	T	T	T		**T**		F	T	T	
T	T	T	F		T	T	T		**T**	F		F	T	T	F			F	T	T	T		**T**		T	T	F	
T	T	T	T		T	T	T		**T**	F		F	T	T	T			F	T	T	T		**T**		T	T	T	

4 The truth-tables are the followings.

(i) A → B, ∼A v B

A	B	A	→	B	∼	A	v	B
F	F	F	**T**	F	T	F	**T**	F
F	T	F	**T**	T	T	F	**T**	T
T	F	T	**F**	F	F	T	**F**	F
T	T	T	**T**	T	F	T	**T**	T

(ii) ∼(A → B), A & ∼B

A	B	∼	(	A	→	B	)	A	&	∼	B
F	F	**F**		F	T	F		F	**F**	T	F
F	T	**F**		F	T	T		F	**F**	F	T
T	F	**T**		T	F	F		T	**T**	T	F
T	T	**F**		T	T	T		T	**F**	F	T

(iii) ∼(A & B), ∼A v ∼B

A	B	∼	(	A	&	B	)	∼	A	v	∼	B
F	F	**T**		F	F	F		T	F	**T**	T	F
F	T	**T**		F	F	T		T	F	**T**	F	T
T	F	**T**		T	F	F		F	T	**T**	T	F
T	T	**F**		T	T	T		F	T	**F**	F	T

(iv) ∼(A v B), ∼A & ∼B

A	B	∼	(	A	v	B	)	∼	A	&	∼	B
F	F	**T**		F	F	F		T	F	**T**	T	F
F	T	**F**		F	T	T		T	F	**F**	F	T
T	F	**F**		T	T	F		F	T	**F**	T	F
T	T	**F**		T	T	T		F	T	**F**	F	T

(v) A ↔ B, (A & B) v (∼A & ∼B)

A	B	A	↔	B	(	A	&	B	)	v	(	∼	A	&	∼	B	)
F	F	F	**T**	F		F	F	F		**T**		T	F	T	T	F	
F	T	F	**F**	T		F	F	T		**F**		T	F	F	F	T	
T	F	T	**F**	F		T	F	F		**F**		F	T	F	T	F	
T	T	T	**T**	T		T	T	T		**T**		F	T	F	F	T	

(vi) ∼(A ↔ B), (A & ∼B) v (∼A & B)

A	B	∼	(	A	↔	B	)	(	A	&	∼	B	)	v	(	∼	A	&	B	)
F	F	**F**		F	T	F			F	F	T	F		**F**		T	F	F	F	
F	T	**T**		F	F	T			F	F	F	T		**T**		T	F	T	T	
A	F	**T**		A	F	F			T	T	T	F		**T**		F	T	F	F	
A	T	**F**		A	T	T			T	F	F	T		**F**		F	T	F	T	

EXERCISE 4.6

1 The consistency-trees are the followings.

1. P & Q, R & ∼S, P v S
   Consistent.

1.	P & Q		
2.	R & ∼S		
3.	P v S		
4.	P		From line 1.
5.	Q		From line 1.
6.	R		From line 2.
7.	∼S		From line 2.
8.	S ✗ 7, 8	P ✓	From line 3.

2. P & Q, ∼P v ∼Q, ∼Q
   Inconsistent.

1.	P & Q	
2.	∼P v ∼Q	
3.	∼Q	
4.	P	From line 1.
5.	Q ✗ 3, 5	From line 1.

3. P & (Q v R), ∼Q v ∼R, ∼R
   Consistent.

1.			P & (Q v R)		
2.			∼Q v ∼R		
3.			∼R		
4.			P		From line 1.
5.			Q v R		From line 1.
6.	Q			R ✗ 3, 6	From line 5.
7.	∼Q ✗ 6, 7	∼R ✓			From line 2.

4. (∼P v ∼Q) v R, ∼P & ∼Q, R
   Consistent.

1.			(∼P v ∼Q) v R		
2.			∼P & ∼Q		
3.			R		
4.			∼P		From line 2.
5.			∼Q		From line 2.
6.	∼P v ∼Q			R ✓	From line 1.
7.	∼P ✓	∼Q ✓			From line 6.

5. (∼P v ∼Q) v R, P & Q, ∼R
Inconsistent.

1.	(∼P v ∼Q) v R		
2.	P & Q		
3.	∼R		
4.	P		From line 2.
5.	Q		From line 2.
6.	∼P v ∼Q	R ✗ 3, 6	From line 1.
7.	∼P ✗ 4, 7 ∼Q ✗ 5, 7		From line 6.

EXERCISE 4.7

1 The consistency-trees are the followings. For actual counterexamples, each false atomic formula can be "0=1", and each true atomic formula can be "0=0".

1. $P \to Q, \sim P : \sim Q$
   Invalid. IPLI: P: F Q: T.

1.	$P \to Q$		Premise.
2.	$\sim P$		Premise.
3.	$\sim\sim Q$		Negated conclusion.
4.	Q		From line 3.
5.	$\sim P$ ✓	Q ✓	From line 1.

2. $\sim P \to Q : Q \to P$
   Invalid. IPLI: P: F, Q: T.

1.	$\sim P \to Q$		Premise.
2.	$\sim(Q \to P)$		Negated conclusion.
3.	Q		From line 2.
4.	$\sim P$		From line 2.
5.	$\sim\sim P$	Q ✓	From line 1.
6.	P ✗ 4, 6		From line 6.

3. P → Q, Q → R : P → R
Valid.

1.	P → Q			Premise.
2.	Q → R			Premise.
3.	~(P → R)			Negated conclusion.
4.	P			From line 3.
5.	~R			From line 3.
6.	~P ✗ 4, 6	Q		From line 1.
7.		~Q ✗ 6, 7	R ✗ 5, 7	From line 2.

4. (P → Q) → P : P
Valid.

1.	(P → Q) → P		Premise.
2.	~P		Negated conclusion.
3.	~(P → Q)	P ✗ 2, 3	From line 1.
4.	P ✗ 2, 4		From line 3.

5. ~(P v ~Q) : (~P & Q)
Valid.

1.	~(P v ~Q)		Premise.
2.	~(~P & Q)		Negated conclusion.
3.	~P		From line 1.
4.	~~Q		From line 1.
5.	Q		From line 4.
6.	~~P	~Q ✗ 5, 6	From line 2.
7.	P ✗ 3, 7		From line 6.

6. : (P v P) → P
Valid.

1.	∼((P v P) → P)		Negated conclusion.
2.	∼P		From line 1.
3.	P v P		From line 1.
4.	P ✗ 2, 4	P ✗ 2, 4	From line 3.

7. : ((∼P → Q) → ∼P) → ∼P
Valid.

1.	∼(((∼P → Q) → ∼P) → ∼P)		Negated conclusion.
2.	(∼P → Q) → ∼P		From line 1.
3.	∼∼P		From line 1.
4.	P		From line 3.
5.	∼(∼P → Q)	∼P ✗ 4, 5	From line 2.
6.	∼P ✗ 4, 6		From line 5.

8. (P → Q) → R : ∼R → P
Valid.

1.	(P → Q) → R		Premise.
2.	∼(∼R → P)		Negated conclusion.
3.	∼R		From line 2.
4.	∼P		From line 2.
5.	∼(P → Q)	R ✗ 3, 5	From line 1.
6.	P ✗ 4, 6		From line 5.

9\. $: (P \to Q) \to (\sim Q \to \sim P)$
Valid.

1.	$\sim((P \to Q) \to (\sim Q \to \sim P))$		Negated conclusion.
2.	$P \to Q$		From line 1.
3.	$\sim(\sim Q \to \sim P)$		From line 1.
4.	$\sim Q$		From line 3.
5.	$\sim\sim P$		From line 3.
6.	P		From line 5.
7.	$\sim P$ ✗ 6, 7	Q ✗ 4, 7	From line 2.

10\. $P \to \sim Q, \sim R \to P : Q \to R$
Valid.

1.	$P \to \sim Q$			Premise.
2.	$\sim R \to P$			Premise.
3.	$\sim(Q \to R)$			Negated conclusion.
4.	Q			From line 3.
5.	$\sim R$			From line 3.
6.	$\sim Q$ ✗ 4, 6	$\sim P$		From line 1.
7.		P ✗ 6, 7	$\sim\sim R$	From line 2.
8.			R ✗ 5, 8	From line 7.

11. : (P v ~P) → (Q v ~(Q v R))
Invalid. IPLI: P: T, Q: F, R: T.

1.		P v ~P		Premise.
2.		~(Q v ~(Q v R))		Negated conclusion.
3.		~Q		From line 2.
4.		~~(Q v R)		From line 2.
5.		Q v R		From line 4.
6.	Q ✗ 3, 6		R	From line 5.
7.		P ✓	~P ✓	From line 1.

12. : (P ↔ Q) ↔ ~(P & ~Q)
Invalid. IPLI: P: F, Q: T.

1.			~((P ↔ Q) ↔ ~(P & ~Q))				Negated conclusion.
2.	P ↔ Q			~(P ↔ Q)			From line 1.
3.	~~(P & ~Q)			~(P & ~Q)			
4.	P & ~Q		~P		~~Q		From line 3.
5.	P				Q		From line 4.
6.	~Q						From line 4.
7.	Q ✗ 6, 7	~P ✗ 5, 7	P ✗ 4, 7	~P	Q	~Q ✗ 5, 7	From line 2.
8.				Q ✓	~P ✓		

13. ∼R → Q : (P v Q) → (∼R → P)
Invalid. IPLI: P: F, Q: T, R: F.

1.	∼R → Q			Premise.
2.	∼((P v Q) → (∼R → P))			Negated conclusion.
3.	P v Q			From line 2.
4.	∼(∼R → P)			From line 2.
5.	∼R			From line 4.
6.	∼P			From line 4.
7.	P ✗ 6, 7	Q		From line 3.
8.		∼∼R	Q ✓	From line 1.
9.		R ✗ 5, 9		From line 8.

14. (P & Q) → R, ∼P → S : Q → (R v S)
Valid.

1.	(P & Q) → R				Premise.
2.	∼P → S				Premise.
3.	∼(Q → (R v S))				Negated conclusion.
4.	Q				From line 3.
5.	∼(R v S)				From line 3.
6.	∼R				From line 5.
7.	∼S				From line 5.
8.	R ✗ 6, 8	∼(P & Q)			From line 1.
9.		∼Q ✗ 4, 9	∼P		From line 8.
10.			S ✗ 7, 10	∼∼P	From line 2.
11.				P ✗ 9, 11	from line 10.

15. (P v Q) & (R v ∼S) : ((∼P v ∼R) & (∼P v S)) → ((Q & R) v (P & ∼S))
Invalid. IPLI: P: F, Q: T, R: F, S: F.

1.	(P v Q) & (R v ∼S)		Premise.
2.	∼(((∼P v ∼R) & (∼P v S)) → ((Q & R) v (P & ∼S)))		Negated conclusion.
3.	(∼P v ∼R) & (∼P v S)		From line 2.
4.	∼P v ∼R		From line 3.
5.	∼P v S		From line 3.
6.	∼((Q & R) v (P & ∼S))		From line 2.
7.	∼(Q & R)		From line 6.
8.	∼(P & ∼S)		From line 6.
9.	P v Q		From line 1.
10.	R v ∼S		From line 1.
11.	∼P	∼R	from line 4.
12.	∼P	S	from line 5.
13.	∼R	∼Q	from line 7.
14.	∼P	∼∼S	from line 8.
15.	Q	P	from line 9.
16.	∼S ✓	R	from line 10.

EXERCISE 4.8

1 We can easily see that

(i) (P → Q) is equivalent to (∼P v Q);
(P ↔ Q) is equivalent to (∼(∼(P → Q) v ∼(Q → P))).

(ii) (P & Q) is equivalent to ∼(∼P v ∼Q). For others, see above.

2 The substitutions are the followings.

(i) $P \vee Q$

1. $P \vee Q$
2. $\sim((P \mid P) \,\&\, (Q \mid Q))$
3. $(P \mid P) \mid (Q \mid Q)$

(ii) $\sim P \vee Q$

1. $\sim P \vee Q$
2. $\sim(\sim\sim P \,\&\, \sim Q)$
3. $\sim(P \,\&\, \sim Q)$
4. $\sim(P \,\&\, (Q \mid Q))$
5. $P \mid (Q \mid Q)$

(iii) $(\sim P \vee \sim Q) \vee \sim R$

1. $(\sim P \vee \sim Q) \vee \sim R$
2. $\sim(\sim(\sim P \vee \sim Q) \,\&\, \sim\sim R)$
3. $\sim(\sim(\sim P \vee \sim Q) \,\&\, R)$
4. $\sim(\sim\sim(P \,\&\, Q) \,\&\, R)$
5. $\sim(((P \mid Q) \mid (P \mid Q)) \,\&\, R)$
6. $((P \mid Q) \mid (P \mid Q)) \mid R$

3 $(\sim P)$ is equivalent to $(P \mid P)$, and $(P \,\&\, Q)$ is equivalent to $((P \mid Q) \mid (P \mid Q))$. Therefore, every other connective can be expressed by Sheffer's stroke.

4 $(\sim P)$ is equivalent to $(P \mid P)$.
To express $\rightarrow$ first see, that $(P \rightarrow Q)$ is equivalent $(\sim P \vee Q)$
$(\sim P \vee Q)$ is equivalent to $(P \mid (Q \mid Q))$ (see exercise 2, (ii) above).

Examination 2 in Formal Logic

1 The keys and the proofs are the followings.

(i) P → (Q & R) : P → Q

P: Pigs have wings.
Q: Pigs fly.
R: Air traffic controllers have nightmares.
Valid.

P	Q	R	P	→	(	Q	&	R	)	P	→	Q
F	F	F	F	**T**		F	F	F		F	**T**	F
F	F	T	F	**T**		F	F	T		F	**T**	F
F	T	F	F	**T**		T	F	F		F	**T**	T
F	T	T	F	**T**		T	T	T		F	**T**	T
T	F	F	T	**F**		F	F	F		T	**F**	F
T	F	T	T	**F**		F	F	T		T	**F**	F
T	T	F	T	**F**		T	F	F		T	**T**	T
T	T	T	T	**T**		T	T	T		T	**T**	T

{1}	1.	P → (Q & R)	Premise
{2}	2.	P	Assumption for CP
{1, 2}	3.	Q & R	1, 2 MP
{1, 2}	4.	Q	3 &E
{1}	5.	P → Q	2, 4 CP

(ii) P & Q, (P & R) → S : R → S

P: Professor Plum was in the drawing room.
Q: Miss Scarlet was in the kitchen.
R: Murder weapon was found in the drawing room.
S: Professor Plum is in big trouble.
Valid.

P	Q	R	S	P	&	Q	(	P	&	R	)	→	S	R	→	S
F	F	F	F	F	**F**	F		F	F	F		**T**	F	F	**T**	F
F	F	F	T	F	**F**	F		F	F	F		**T**	T	F	**T**	T
F	F	T	F	F	**F**	F		F	F	T		**T**	F	T	**F**	F
F	F	T	T	F	**F**	F		F	F	T		**T**	T	T	**T**	T
F	T	F	F	F	**F**	T		F	F	F		**T**	F	F	**T**	F
F	T	F	T	F	**F**	T		F	F	F		**T**	T	F	**T**	T
F	T	T	F	F	**F**	T		F	F	T		**T**	F	T	**F**	F
F	T	T	T	F	**F**	T		F	F	T		**T**	T	T	**T**	T
T	F	F	F	T	**F**	F		T	F	F		**T**	F	F	**T**	F
T	F	F	T	T	**F**	F		T	F	F		**T**	T	F	**T**	T
T	F	T	F	T	**F**	F		T	T	T		**F**	F	T	**F**	F
T	F	T	T	T	**F**	F		T	T	T		**T**	T	T	**T**	T
T	T	F	F	T	**T**	T		T	F	F		**T**	F	F	**T**	F
T	T	F	T	T	**T**	T		T	F	F		**T**	T	F	**T**	T
T	T	T	F	T	**T**	T		T	T	T		**F**	F	T	**F**	F
T	T	T	T	T	**T**	T		T	T	T		**T**	T	T	**T**	T

{1}	1.	P & Q	Premise
{2}	2.	(P & R) $\rightarrow$ S	Premise
{3}	3.	R	Assumption
{1}	4.	P	1 &E
{1, 3}	5.	P & R	3, 4 &I
{1, 2, 3}	6.	S	2, 5 MP
{1, 2}	7.	R $\rightarrow$ S	3, 6 CP

(iii) (P & Q) → R, ∼R → (Q & S) : ∼R → (P & U)
P: Professor Plum was in the study.
Q: Miss Scarlet was in the conservatory.
R: Reverend Green was the murderer.
S: Colonel Mustard was in the conservatory.
U: Colonel Mustard was in the study.
Invalid. IPLI: P: F, Q: T, R: F, S: T, U: F.

P	Q	R	S	U	(	P	&	Q	)	→	R	∼	R	→	(	Q	&	S	)	∼	R	→	(	P	&	U	)
F	F	F	F	F		F	F	F		**T**	F	T	F	**F**		F	F	F		T	F	**F**		F	F	F	
T	F	F	F	F		T	F	F		**T**	F	T	F	**F**		F	F	F		T	F	**F**		T	F	F	
F	T	F	F	F		F	F	T		**T**	F	T	F	**F**		T	F	F		T	F	**F**		F	F	F	
T	T	F	F	F		T	T	T		**F**	F	T	F	**F**		T	F	F		T	F	**F**		T	F	F	
F	F	T	F	F		F	F	F		**T**	T	F	T	**T**		F	F	F		F	T	**T**		F	F	F	
T	F	T	F	F		T	F	F		**T**	T	F	T	**T**		F	F	F		F	T	**T**		T	F	F	
F	T	T	F	F		F	F	T		**T**	T	F	T	**T**		T	F	F		F	T	**T**		F	F	F	
T	T	T	F	F		T	T	T		**T**	T	F	T	**T**		T	F	F		F	T	**T**		T	F	F	
F	F	F	T	F		F	F	F		**T**	F	T	F	**F**		F	F	T		T	F	**F**		F	F	F	
T	F	F	T	F		T	F	F		**T**	F	T	F	**F**		F	F	T		T	F	**F**		T	F	F	
F	T	F	T	F		F	F	T		**T**	F	T	F	**T**		T	T	T		T	F	**F**		F	F	F	
T	T	F	T	F		T	T	T		**F**	F	T	F	**T**		T	T	T		T	F	**F**		T	F	F	
F	F	T	T	F		F	F	F		**T**	T	F	T	**T**		F	F	T		F	T	**T**		F	F	F	
T	F	T	T	F		T	F	F		**T**	T	F	T	**T**		F	F	T		F	T	**T**		T	F	F	
F	T	T	T	F		F	F	T		**T**	T	F	T	**T**		T	T	T		F	T	**T**		F	F	F	
T	T	T	T	F		T	T	T		**T**	T	F	T	**T**		T	T	T		F	T	**T**		T	F	F	
F	F	F	F	T		F	F	F		**T**	F	T	F	**F**		F	F	F		T	F	**F**		F	F	T	
T	F	F	F	T		T	F	F		**T**	F	T	F	**F**		F	F	F		T	F	**T**		T	T	T	
F	T	F	F	T		F	F	T		**T**	F	T	F	**F**		T	F	F		T	F	**F**		F	F	T	
T	T	F	F	T		T	T	T		**F**	F	T	F	**F**		T	F	F		T	F	**T**		T	T	T	
F	F	T	F	T		F	F	F		**T**	T	F	T	**T**		F	F	F		F	T	**T**		F	F	T	
T	F	T	F	T		T	F	F		**T**	T	F	T	**T**		F	F	F		F	T	**T**		T	T	T	
F	T	T	F	T		F	F	T		**T**	T	F	T	**T**		T	F	F		F	T	**T**		F	F	T	
T	T	T	F	T		T	T	T		**T**	T	F	T	**T**		T	F	F		F	T	**T**		T	T	T	
F	F	F	T	T		F	F	F		**T**	F	T	F	**F**		F	F	T		T	F	**F**		F	F	T	
T	F	F	T	T		T	F	F		**T**	F	T	F	**F**		F	F	T		T	F	**T**		T	T	T	
F	T	F	T	T		F	F	T		**T**	F	T	F	**T**		T	T	T		T	F	**F**		F	F	T	
T	T	F	T	T		T	T	T		**F**	F	T	F	**T**		T	T	T		T	F	**T**		T	T	T	
F	F	T	T	T		F	F	F		**T**	T	F	T	**T**		F	F	T		F	T	**T**		F	F	T	
T	F	T	T	T		T	F	F		**T**	T	F	T	**T**		F	F	T		F	T	**T**		T	T	T	
F	T	T	T	T		F	F	T		**T**	T	F	T	**T**		T	T	T		F	T	**T**		F	F	T	
T	T	T	T	T		T	T	T		**T**	T	F	T	**T**		T	T	T		F	T	**T**		T	T	T	

2 The consistency-trees are the followings. For actual counterexamples, each false atomic formula can be "0=1", and each true atomic formula can be "0=0".

1. R → Q, P v Q : P v R
   Invalid. IPLI: P: F, Q: T, R: F.

1.	R → Q		Premise.
2.	P v Q		Premise.
3.	∼(P v R)		Negated conclusion.
4.	∼P		From line 3.
5.	∼R		From line 3.
6.	Q	∼R	From line 1.
7.	Q ✓	P	From line 2.

2. : (∼P & (P → Q)) → Q
   Invalid. IPLI: P: F, Q: F.

1.	∼((∼P & (P → Q)) → Q)		Negated conclusion.
2.	∼P & (P → Q)		From line 1.
3.	∼Q		From line 1.
4.	∼P		From line 2.
5.	P → Q		From line 2.
6.	∼P ✓	Q	From line 5.

3. : (P → Q) ↔ ∼(P & ∼Q)
Valid.

1.	∼((P → Q) ↔ ∼(P & ∼Q))				Negated conclusion.
2.	P → Q		∼(P → Q)		From line 1.
3.	∼∼(P & ∼Q)		∼(P & ∼Q)		From line 1.
4.	P & ∼Q				From line 3.
5.	P		P		From line 2 and 4.
6.	∼Q		∼Q		From line 2 and 4.
7.	∼P ✗ 5, 7	Q ✗ 6, 7	∼P ✗ 5, 7	∼∼Q ✗ 6, 7	From line 2 and 3.

4. P & (Q v R) : (P & Q) v (P & R)
Valid.

1.	P & (Q v R)				Premise.
2.	∼((P & Q) v (P & R))				Negated conclusion.
3.	∼(P & Q)				From line 2.
4.	∼(P & R)				From line 2.
5.	P				From line 1.
6.	Q v R				From line 1.
7.	Q		R		From line 6.
8.	∼P ✗ 5, 8	∼Q ✗ 7, 8	∼P ✗ 5, 8	∼R ✗ 7, 8	From line 3 and 4.

5. (P v Q) → ∼R : ((P & ∼R) → ∼R) & ((Q & R) → ∼R)
   Valid.

1.	(P v Q) → ∼R			Premise.
2.	∼(((P & ∼R) → ∼R) & ((Q & R) → ∼R))			Negated conclusion.
3.	∼((Q & R) → ∼R)		∼((P & ∼R) → ∼R)	From line 2.
4.	Q & R		P & ∼R	From line 3.
5.	∼∼R		∼∼R	From line 3.
6.	R		R	From line 5.
7.	R		∼R ✗ 6, 7	From line 4.
8.	Q			From line 4.
9.	∼(P v Q)	∼R ✗ 7, 9		From line 1.
10.	∼Q ✗ 8, 10			From line 9.

6. (P & Q) → (R & S) : (P → (P → R)) & (P → (Q → S))
   Invalid. IPLI: P: T, Q: F, R: F, S: F.

1.	(P & Q) → (R & S)			Premise.
2.	∼((P → (P → R)) & (P → (Q → S)))			Negated conclusion.
3.	∼(P → (P → R))		∼(P → (Q → S))	From line 2.
4.	P			From line 3.
5.	∼(P → R)			From line 3.
6.	P			From line 5.
7.	∼R			From line 5.
8.	∼(P & Q)	R & S		From line 1.
9.	∼Q ✓	∼P		From line 8.

7. For each combination of truth-values for P and Q in the sentence "P unless Q" results the same truth-value as (P v Q).

Chapter Five: An Introduction to First Order Predicate Logic

EXERCISE 5.1

1 (i) $\forall x[Fx \mathbin{\&} Gx]$

(ii) $\forall x[Fx \text{ v } Gx]$

(iii) $\forall x[Fx \rightarrow Gx]$

(iv) $\forall x[Fx \leftrightarrow Gx]$

(v) $\forall x[\sim Fx]$

2 (i) $\forall x[Fx] \mathbin{\&} \forall x[Gx]$

(ii) $\forall x[Fx] \text{ v } \forall x[Gx]$

(iii) $\forall x[Fx] \rightarrow \forall x[Gx]$

(iv) $\forall x[Fx] \leftrightarrow \forall x[Gx]$

(v) $\sim \forall x[Fx]$

3 (i) $Fx \mathbin{\&} Gx$

(ii) $Fx \text{ v } Gx$

(iii) $Fx \rightarrow Gx$

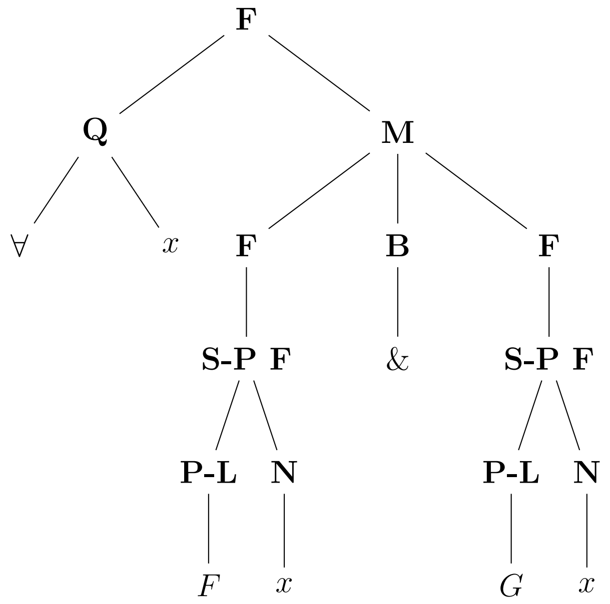
(iv) $Fx \leftrightarrow Gx$

(v) $\sim Fx$

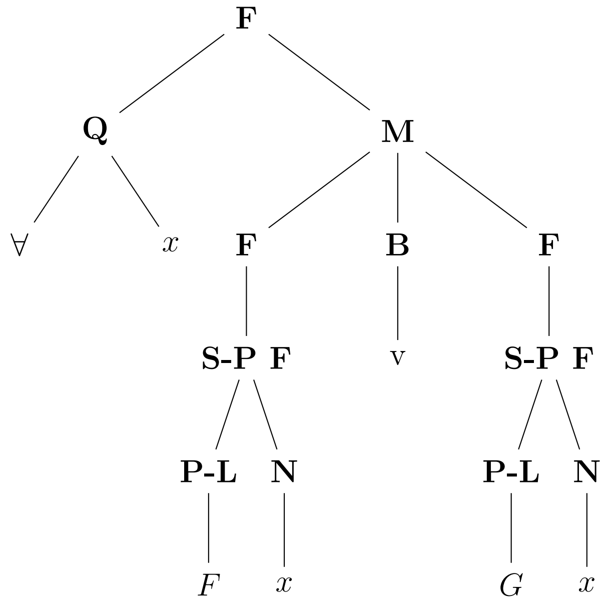
EXERCISE 5.2

1 The trees are the followings.

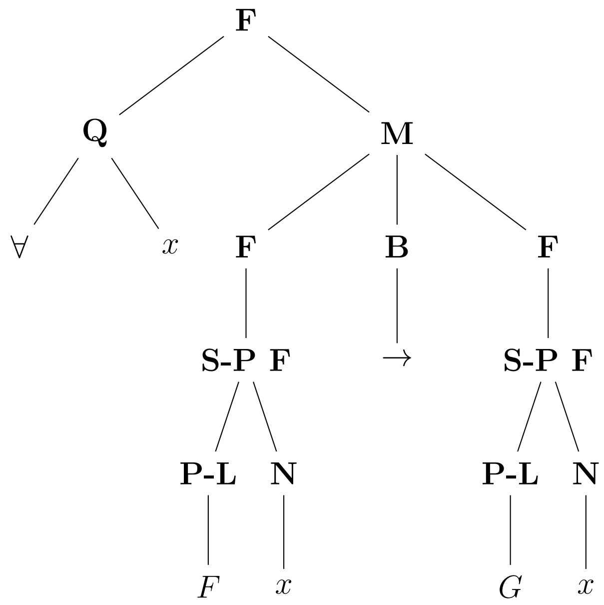
(i) $\forall x[Fx \;\&\; Gx]$



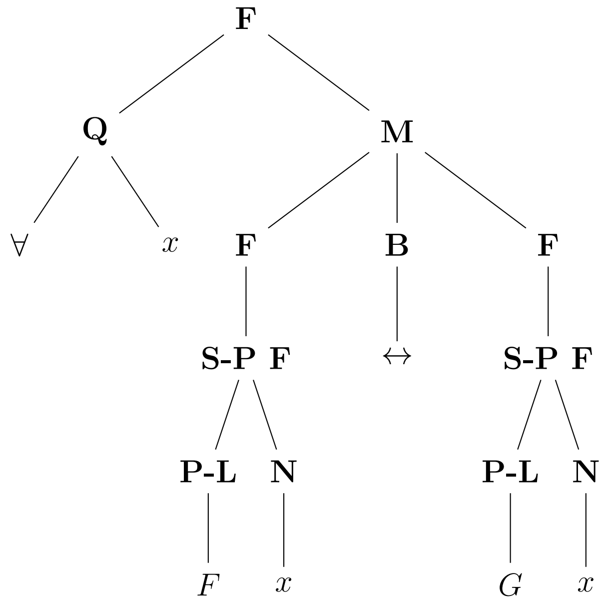
(ii) $\forall x[Fx \text{ v } Gx]$



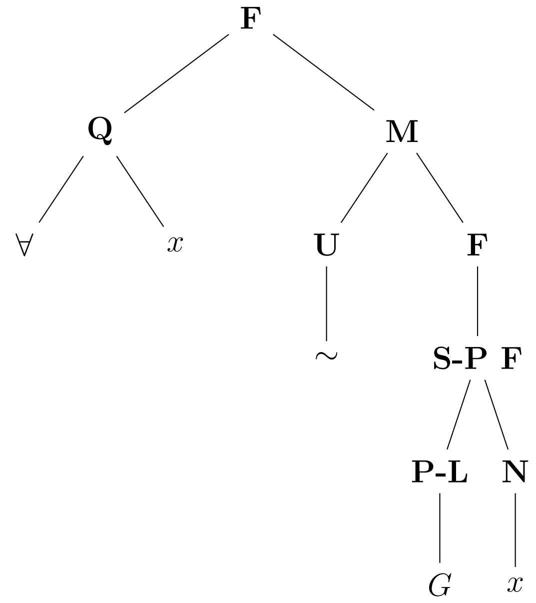
(iii) $\forall x[Fx \to Gx]$



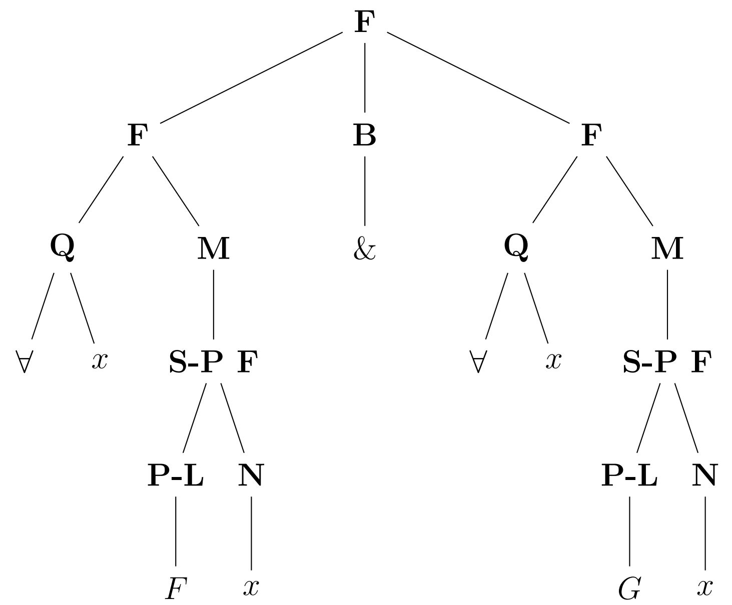
(iv) $\forall x[Fx \leftrightarrow Gx]$



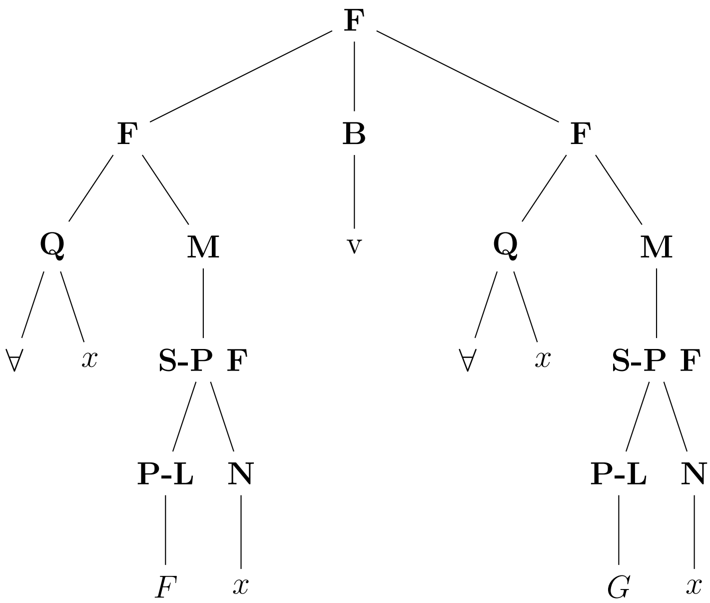
(v) $\forall x[\sim Fx]$



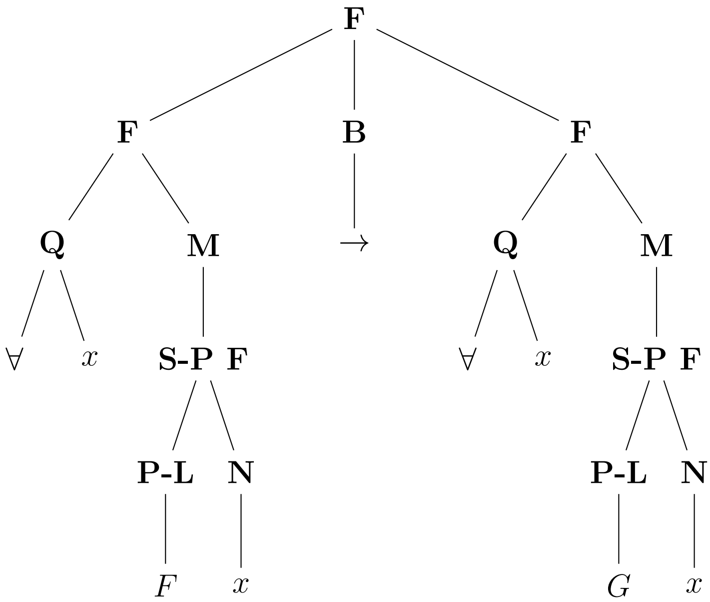
(vi) $\forall x[Fx]$ & $\forall x[Gx]$



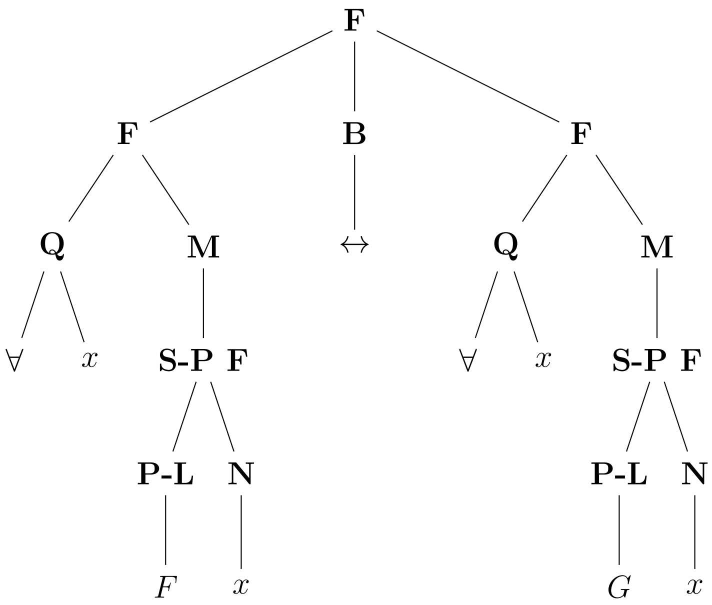
(vii) $\forall x[Fx]$ v $\forall x[Gx]$



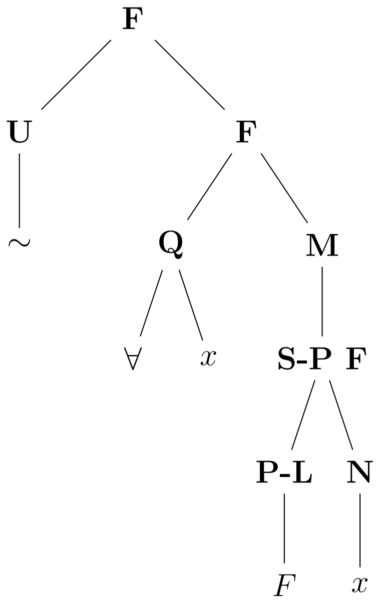
(viii) $\forall x[Fx] \to \forall x[Gx]$



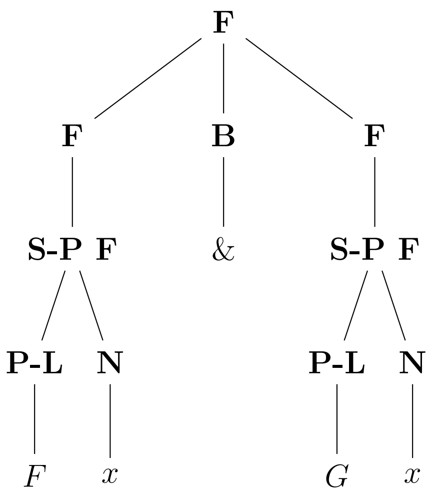
(ix) $\forall x[Fx] \leftrightarrow \forall x[Gx]$



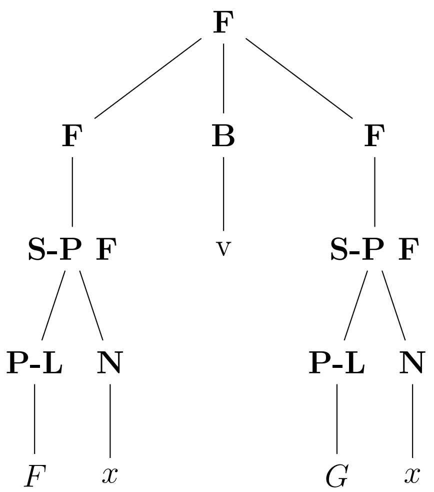
(x) $\sim \forall x[Fx]$



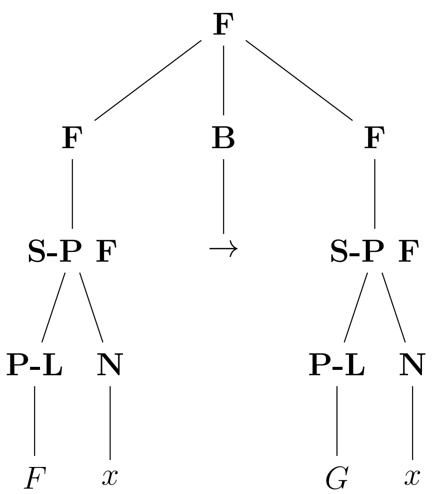
(xi) Fx & Gx



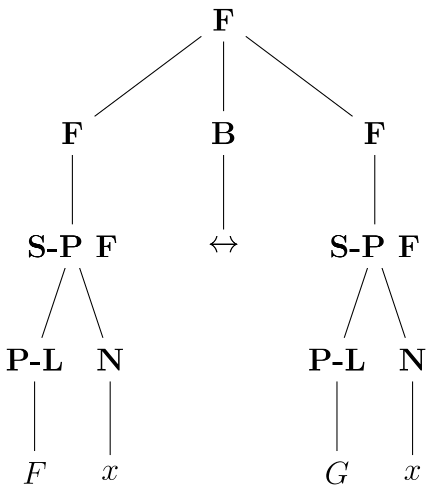
(xii) Fx v Gx



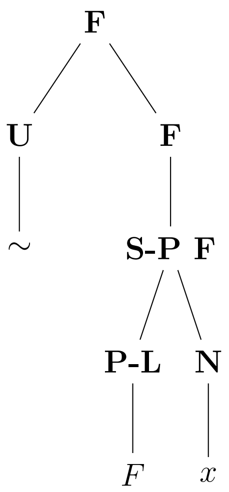
(xiii) $Fx \rightarrow Gx$



(xiv) $Fx \leftrightarrow Gx$



(xv) $\sim Fx$



2 (i) The scope of the negation connective is the quantified formula $\forall x\ [Fx \rightarrow Gx]$. The scope of the universal quantifier is the formula $Fx \rightarrow Gx$. The scope of the implication connective are the formulas Fx and Gx.

(ii) The scope of the universal quantifier is the formula $\sim (Fx \rightarrow Gx)$. The scope of the negation connective is the formula $Fx \rightarrow Gx$. The scope of the implication connective are the formulas Fx and Gx.

EXERCISE 5.3

1 In all of the following QL-interpretations, the domain is all human beings.

(i) $\exists x\ [Fx\ \&\ Gx]$,
F: ... is a florist,
G: ... is a greengrocer.

(ii) $\forall x\ [Fx \rightarrow Gx]$,
F: ... is a greengrocer,
G: ... is happy.

(iii) $\exists x\ [Fx\ \&\ Gx]$,
F: ... is a folk singer,
G: ... is groovy.

(iv) Fa,
F: ... is a folk singer.
a: Sandy Denny.

(v) $Fa\ \&\ Fb$,
F: ... is a folk singer.
a: Sandy Denny,
b: Julie Felix.

(vi) $\forall x\ [(Fx\ \&\ Gx) \rightarrow Hx]$,
F: ... is a folk singer,
G: ... is groovy,
H: ... plays guitar.

(vii) $\forall x\ [Fx \rightarrow Gx]$ v $\exists x\ [Fx\ \&\ Hx]$,
F: ... is a folk singer,
G: ... is groovy,
H: ... is dreadful.

(viii) $\exists x\ [Fx\ \&\ Gx]\ \&\ \forall x\ [Hx \rightarrow Ix]$,
F: ... is a folk singer,
G: ... is a florist,
H: ... is a greengrocer,
I: ... is groovy.

(ix) $\forall x\ [(Fx\ \&\ Gx) \rightarrow (Hx\ \&\ Ix\ \&\ Jx)]$,
F: ... is a folk singer,
G: ... is a florist,
H: ... is happy,
I: ... is generous,
J: ... is interesting.

(x) $\forall x\ [(Fx\ \&\ Gx) \rightarrow (Hx\ \&\ Ix)] \rightarrow \exists x\ [(Hx\ \&\ Ix)\ \&\ (Jx\ \&\ Kx)]$,
F: ... is a folk singer,
G: ... is a florist,
H: ... is groovy,
I: ... is a greengrocer,

J: ... is fearless,
K: ... is a firefighter.

(xi) $Fa \leftrightarrow Gb$,
F: ... is a folk singer,
G: ... is a flamenco dancer,
a: Sandy Denny,
b: Julie Felix.

(xii) $\forall x\ [Fx\ \&\ Gx] \rightarrow (Fa\ \&\ Ga)$,
F: ... is a folk singer,
G: ... is a flamenco dancer,
a: Sandy Denny.

(xiii) $((Fa\ \&\ Fb)\ \&\ Fc)$ v $((Ga\ \&\ Gb)\ \&\ Gc)$,
F: ... is a folk singer,
G: ... is a flamenco dancer,
a: Sandy Denny,
b: Julie Felix,
c: Tom Paxton.

(xiv) $\exists x\ [Fx\ \&\ Gx]$ v $\forall x\ (Fa$ v $Ga)$,
F: ... is a folk singer,
G: ... is a flamenco dancer.

(xv) $\forall x\ [Fx] \leftrightarrow\sim (Fa \rightarrow Gb)$,
F: ... is a folk singer,
G: ... is a flamenco dancer,
a: Tom Paxton,
b: Julie Felix.

EXERCISE 5.4

1 In all of the following QL-interpretations, the domain is all human beings.

(i) $\exists x\ [Fx] : \forall x\ [Fx]$, invalid,
F: ... is groovy.

(ii) $\forall x\ [Fx \rightarrow Gx], \forall x\ [Gx \rightarrow Hx] : \forall x\ [Fx \rightarrow Hx]$, valid,
F: ... is a florist,
G: ... is generous,
H: ... is happy.

(iii) $\exists x\ [Fx\ \&\ Gx], \exists x\ [Gx\ \&\ Hx] : \exists x\ [Fx\ \&\ Hx]$, invalid,
F: ... is a greengrocer,
G: ... is a folk singer,
H: ... is a haberdasher.

(iv) $\forall x\ [Fx \rightarrow Gx], \exists x\ [Fx\ \&\ Hx] : \exists x\ [Hx\ \&\ Gx]$, valid,
F: ... is a philosopher,
G: ... is absent-minded,
H: ... is a logician.

EXERCISE 5.5

1 In all of the following QL-interpretations, the domain is "all things". Unfortunately for some of these sentences (especially for "Everything is beautiful.") it's very difficult to determine what it applies to, and it's not even important for the exercises.

(i) $\sim \exists x \, [Fx]$,
F: ... is a unicorn.

(ii) $\sim \exists x \, [Fx \;\&\; Gx]$,
F: ... is free,
G: ... is a lunch.

(iii) $\forall x \, [Fx]$,
F: ... is beautiful.

(iv) $\forall x \, [Fx \rightarrow Gx]$,
F: ... is a logic student,
G: ... is a genius.

(v) $\sim \exists x \, [Fx \;\&\; Gx]$,
F: ... is a folk singer,
G: ... is a grunge fan.

(vi) $\forall x \, [Fx \text{ v } Gx]$,
F: ... is a folk fan,
G: ... is a jazz person.

(vii) $\forall x \, [Fx \text{ v } \sim Gx]$,
F: ... is a fan of traditional folk music,
G: ... is a person of taste.

(viii) $\sim \exists x \, [((Fx \;\&\; Gx) \;\&\; Hx) \;\&\; Ix]$,
F: ... is folk-singing,
G: ... is a logic studing,
H: ... is happy-go-lucky,
I: ... is a haberdasher.

(ix) $\forall x \, [(Fx \;\&\; Gx) \rightarrow (Hx \text{ v } Ix)]$,
F: ... is a folk fan,
G: ... is a jazz person,
H: ... is a person of taste,
I: ... is eccentric.

(x) $\forall x \, [(Fx \;\&\; Gx) \rightarrow (Hx \text{ v } Ix)]$,
F: ... is a folk fan,
G: ... is a jazz person,
H: ... is a person of taste,
I: ... is eccentric.

2 In all of the following QL-interpretations, the domain is "all things".

(i) $\exists x\ [Fx]\ \&\ \exists x\ [Gx] : \exists x\ [Fx\ \&\ Gx]$, invalid,
F: ... is a folk-singer,
G: ... is groovy.

(ii) $\forall x\ [Fx \rightarrow Gx], \forall x\ [Gx \rightarrow Hx] : \forall x\ [Fx \rightarrow Hx]$, valid,
F: ... is a formal logician,
G: ... is generous,
H: ... is happy.

(iii) $\exists x\ [(Fx\ \&\ Gx)\ \&\ Hx], \exists x\ [Fx\ \&\ Ix] : \exists x\ [(Ix\ \&\ Gx)\ \&\ Hx]$, invalid,
F: ... is a veggieburger,
G: ... is wholesome,
H: ... is tasty,
I: ... is a cheeseburger.

(iv) $\forall x\ [Fx \rightarrow (Gx\ \&\ Hx)], \forall x\ [Ix \rightarrow \sim (Gx\ \&\ Hx)] : \forall x\ [Fx \rightarrow \sim Ix]$, valid,
F: ... is a fire-fighter,
G: ... is fit,
H: ... is fearless,
I: ... is a folk singer.

(v) $\exists x\ [Fx]$ v $\forall x\ [Gx], \sim \forall x\ [Gx] : \exists x\ [Fx]$, valid,
F: ... is a florist,
G: ... is greengrocer.

(vi) $\exists x\ [Fx]$ v $\forall x\ [Gx], \forall x\ [\sim Gx] : \exists x\ [Fx]$, valid,
F: ... is a florist,
G: ... is greengrocer.

EXERCISE 5.6

1 Let's spell out the alternative formalizations for the cases 3-8 from BOX 5.7.

3 $\sim \exists x\ [\sim Rxa]$

4 $\sim \forall x\ [\sim Rxa]$

5 $\forall x\ [\sim Rxa]$

6 $\sim \exists x\ [\sim Rax]$

7 $\sim \forall x\ [\sim Rax]$

8 $\forall x\ [\sim Rax]$

EXERCISE 5.7

1 The translated sentences are the following. Note: I consider "Bob" as a typo in this exercise which should have been "Bill".

(i) $\forall x\ [xLd]$

(ii) $\forall x\ [xLx]$

(iii) $\sim \exists x\ [\forall y\ [xLy]]$

(iv) $\forall x\ [\exists y\ [xLy]]$

(v) $\exists x\ [\forall y\ [xLy]]$

(vi) $(bLa\ \&\ bLc)\ \&\ \exists x\ [xMc\ \&\ bLx]$

(vii) $((aLb\ \&\ cLb)\ \&\ dLb)\ \&\ \exists x\ [\exists y\ [(yMd \text{ v } yFd)\ \&\ yBx]\ \&\ xLd]$

(viii) $\exists x\ [\exists y\ [(yMa \text{ v } yFa)\ \&\ xFy]\ \&\ dLx]$

(ix) $\exists x\ [\exists y\ [(yWb \text{ v } yHb)\ \&\ xSy]\ \&\ cLx]$

(x) $(\exists x\ [\exists y\ [(yMd \text{ v } yFd)\ \&\ xMy]\ \&\ aLx]) \text{ v } (\exists x\ [\exists y\ [(yWa \text{ v } yHa)\ \&\ xSy]\ \&\ cLx])$

(xi) $\forall x\ [\exists y\ [\exists z\ [(yMz \text{ v } yFz)\ \&\ (xMy \text{ v } xFy)]] \rightarrow \exists y\ [xMy \text{ v } xFy]]$

(xii) $\exists x\ [\exists y\ [yFb\ \&\ xFy]\ \&\ dLx]$

(xiii) $\sim \exists x\ [\exists y\ [xFy]\ \&\ \exists y\ [xMy]]$

(xiv) $\forall x\ [\exists y\ [xSy] \rightarrow \exists y\ [xSy \text{ v } xBy]]$

(xv) $\forall x\ [\exists y\ [xMy \text{ v } xFy] \rightarrow \forall y\ [(xMy \text{ v } xFy) \rightarrow xLy]]$

2 (i) $\forall x\ [\exists y\ [xLy]] : \exists y\ [\forall x\ [xLy]]$, invalid,
D: {human beings},
L: ... loves ....

(ii) $\forall x\ [\exists y\ [yCx]] : \exists y\ [\forall x\ [xLy]]$, invalid,
D: {events},
C: ... is the cause of ....

(iii) $\forall x\ [\exists y\ [xTy]] : \exists y\ [\forall x\ [yTx]]$, invalid,
D: {human beings},
T: ... is taller than ....

(iv) $\forall x\ [Fx \rightarrow xLa], \exists x\ [Bx\ \&\ Fx] : \exists x\ [Bx\ \&\ xLa]$, valid,
D: {human beings},
F: ... is a folk fan,
L: ... likes ...,
B: ... is a bluesman,
a: the Amazing Blondel.

(v) $\forall x\ [\forall y\ [(Fx\ \&\ (Gy\ \&\ Wy)) \rightarrow xAy]], Gg, Wg : \forall x\ [Fx \rightarrow xAg]$, valid,
D: {human beings},
F: ... is a formal logician,
A: ... admires ...,
G: ... is a German philosopher,
W: ... wrote a logic text,
g: Gottlob Frege.

(vi) $\forall x\ [\forall y\ [(Cx\ \&\ My) \rightarrow xHy]], \exists x\ [Dx\ \&\ \exists y\ [My\ \&\ xLy]] : \forall x\ [Cx \rightarrow \exists y\ [Dy\ \&\ xHy]]$, invalid,
D: {animals},
C: ... is a cat,
M: ... is a mouse,

H: ... hates ...,
D: ... is a dog,
L: ... likes ....

(vii) $\exists x\ [Fx\ \&\ \forall y\ [Gy \rightarrow xRy]], \sim \exists x\ [Fx\ \&\ \exists y\ [Hy\ \&\ xRy]] : \sim \exists x\ [Gx\ \&\ Hx]$, valid,
D: {human beings},
F: ... is a florist,
G: ... is a greengrocer,
H: ... is a hitman,
R: ... respects ....

(viii) $\forall x\ [Fx \rightarrow \forall y\ [Ny \rightarrow xLy]], Nr : \forall x\ [Fx \rightarrow xLr]$, valid,
D: {human beings and flowers},
F: ... is a florist,
N: ... is a nice flower,
L: ... likes ...,
r: rose.

3 (i) $\forall x\ [\forall y\ [xAy]]$
D: {objects},
A: ... attracts ....

(ii) $\forall x\ [\forall y\ [xRy \rightarrow xBy]]$
D: {days},
R: ... brighter ...,
B: ... better ....

(iii) $\forall x\ [\forall y\ [(Cx\ \&\ (yMx\ \&\ yFx)) \rightarrow xLy]]$
D: {human beings and animals},
C: ... is a cat,
M: ... is the master of ...,
F: ... feeds ... generously,
L: ... loves ....

(iv) $\forall x\ [\forall y\ [(Px\ \&\ Ey) \rightarrow xHy]]$
D: {subatomic particles},
P: ... is a proton,
E: ... is an electron,
H: ... is heavier then ....

(v) $\forall x\ [Mx \rightarrow \exists y\ [\exists z\ [Gz\ \&\ yCz]\ \&\ xRy]]$
D: {biological processes and molecules},
M: ... is a mutation,
G: ... is a gene,
C: ... is a change in ...,
R: ... results from ....

(vi) $\forall x\ [xLa] \rightarrow \sim \exists x\ [xHb]$
D: {human beings},
L: ... loves ...,
H: ... hates ...,

a: Arlo Guthrie,
b: Blind Lemon Jefferson.

(vii) $\forall x\ [\forall y\ [(Sxry\ \&\ Cy) \rightarrow \exists z\ [Bzry]]]$
D: {events and animals},
S: ... is such an event that ... sees ...,
B: ... is such an event that ... barks at ...,
C: ... is a cat,
r: Rover.

(viii) $\forall x\ [((Bx\ \&\ Dx)\ \&\ Hs) \rightarrow \exists y\ [Fy\ \&\ yAx]]$
D: {bodies and forces and natural laws},
B: ... is a body,
D: ... decelerates,
H: ... holds,
F: ... is a force,
A: ... acts on ...,
s: Newton's Second Law.

(ix) $\sim \exists x\ [\forall y\ [yFx]]$
D: {human beings},
F: ... has fooled ....

(x) $\sim \exists x\ [\forall y\ [xFy]]$
D: {human beings},
F: ... has fooled ....

(xi) $\forall x\ [\exists y\ [xFy]\ \text{v}\ \exists y\ [yFx]]$
D: {human beings},
F: ... has fooled ....

(xii) $\forall x\ [\exists y\ [xFy] \rightarrow xFx]$
D: {human beings},
F: ... has fooled ....

(xiii) $\forall x\ [\exists y\ [yFx] \rightarrow xFx]$
D: {human beings},
F: ... has fooled ....

(xiv) $\forall x\ [\exists y\ [yFx\ \&\ xFy]]$
D: {human beings},
F: ... has fooled ....

(xv) $\exists x\ [\exists y\ [xFy]] \leftrightarrow \sim\sim \exists x\ [\exists y\ [xFy]]$
D: {human beings},
F: ... has fooled ....

EXERCISE 5.8

1 The explanation intuitively for these are properties can be found in the chapter. BOX 5.10 lists formally all of these properties.

2 An equivalence relation is defined as a relation that is reflexive, symmetrical and transitive. Therefore (i), (ii) and (iv) defines an equivalence relation from 1.

3 Each of these relations respectively are

(i) reflexive, symmetrical, transitive,

(ii) irreflexive, non-symmetrical, non-transitive (examples: Rxy and Ryx, but nobody is their own sister; but with three sisters Rxy and Ryz, and Rxz),

(iii) irreflexive, asymmetrical, intratransitive,

(iv) non-reflexive (but this is questionable), non-symmetrical, non-transitive,

(v) reflexive, symmetrical, transitive,

(vi) reflexive, non-symmetrical (if a=b then Rab and Rba, but if they're not equal, then only Rab or Rba is true), transitive.

EXERCISE 5.9

1 (i) $\exists x\ [x = b]$
D: {human beings},
b: Blind Lemon Jefferson.

(ii) $m = m$
D: {stars},
m: the Morning Star.

(iii) $m = e$
D: {stars},
m: the Morning Star,
e: the Evening Star.

(iv) $m = v \ \&\ e = v$
D: {astronomical objects},
m: the Morning Star
e: the Evening Star,
v: the planet Venus.

(v) $\forall x\ [x = x]$
D: {things}.

(vi) $\sim \exists x\ [x = x]$
D: {things}.

(vii) $\exists x\ [\sim x = x] \leftrightarrow \sim \forall x\ [x = x]$
D: {things}.

(viii) $\forall x\ [xFx \rightarrow (xFy \ \&\ \sim x = y)]$
D: {human beings},
F: ... has fooled ....

(ix) $\forall x\ [xFx \rightarrow (yFx \ \&\ \sim x = y)]$
D: {human beings},
F: ... has fooled ....

(x) $\forall x\ [xFx \rightarrow \forall y\ [\exists z\ [yFz \ \&\ \sim y = z]]]$
D: {human beings},
F: ... has fooled ....

2 (i) $m = e, e = v : m = v$, valid,
D: {astronomical objects},
m: the Morning Star,
e: the Evening Star,
v: the planet Venus.

(ii) $: (e = m \ \& \ m = v) \to e = v$, valid,
D: {astronomical objects},
m: the Morning Star,
e: the Evening Star,
v: the planet Venus.

(iii) $Gw, \sim \exists x\ [Gx \ \& \ Bx] : \sim Bw \text{ v } \sim Gw$, valid,
D: {human beings},
G: ... is a German philosopher,
B: ... is badly behaved,
w: Wittgenstein.

(iv) $\sim \exists x\ [xFx], pFz : \sim p = z$, valid,
D: {human beings and animals},
F: ... is the father of ...,
p: Paul,
z: Zebedee.

(v) $\forall x\ [Fx \to (Gx \text{ v } x = s)] : Gs \to \forall x\ [Fx \to Gx]$, valid,
D: {human beings},
F: ... is a folk singer,
G: ... is groovy,
s: Dr Strangely Strange.

(vi) $\forall x\ [Fx \to Gx] : \forall x\ [\forall y\ [(Fx \ \& \ \sim Gy) \to \sim x = y]]$, valid,
D: {human beings},
F: ... is a folk singer,
G: ... is groovy.

EXERCISE 5.10

1 In all of these formulas, the domain is all supernatural creatures, and the predicate G means "... is a God".

(i) $\exists x\ [Gx]$

(ii) $\sim \exists x\ [Gx \ \& \ \forall y\ [Gy \to x = y]]$

(iii) $\exists x\ [\exists y\ [(Gx \ \& \ Gy) \ \& \ (\sim x = y)] \ \& \ \forall z\ [Gz \to (z = x \text{ v } z = y)]]$

(iv) $\exists x\ [\exists y\ [\exists z\ [((Gx \ \& \ Gy) \ \& \ Gz) \ \& \ ((\sim x = y \ \& \ \sim x = z) \ \& \ \sim y = z)]]]$

(v) $\forall x\ [\forall y\ [((Gx \ \& \ Gy) \ \& \ (\sim x = y)) \to \forall z\ [Gz \to (z = x \text{ v } z = y)]]]$
(Note: I believe the restriction $(\sim x = y)$ is crucial.
Without this, it is possible that if there were 2 Gods,
and we choose x, y such that $(x = y)$,
to choose z such that z is not x nor y.
I believe the book is misleading in BOX 5.12.)

(vi) $\sim \exists x\ [Gx]$ v $\exists x\ [Gx \to \forall y\ [Gy \to x = y]]$ v $\exists x\ [\exists y\ [(Gx\ \&\ Gy)\ \&\ (\sim x = y)]\ \&\ \forall z\ [Gz \to (z = x \text{ v } z = y)]]$

(vii) $\exists x\ [\exists y\ [\exists z\ [((Gx\ \&\ Gy)\ \&\ Gz)\ \&\ ((\sim x = y\ \&\ \sim x = z)\ \&\ \sim y = z)\ \&\ \forall w\ [Gw \to ((w = x \text{ v } w = y) \text{ v } w = z)]]]]$

(viii) $\exists x\ [\exists y\ [\exists z\ [\exists w\ [(((Gx\ \&\ Gy)\ \&\ Gz)\ \&\ Gw)\ \&\ (((\sim x = y\ \&\ \sim x = z)\ \&\ \sim x = w)\ \&\ (\sim y = z\ \&\ \sim y = w)\ \&\ \sim z = w)]]]]$

(ix) $\forall x\ [\forall y\ [\forall z\ [\forall w\ [((((Gx\ \&\ Gy)\ \&\ Gz)\ \&\ Gw)\ \&\ (((\sim x = y\ \&\ \sim x = z)\ \&\ \sim x = w)\ \&\ (\sim y = z\ \&\ \sim y = w)\ \&\ \sim z = w)) \to \forall u\ [Gu \to (((u = x \text{ v } u = y) \text{ v } u = z) \text{ v } u = w)]]]]]$

(x) $\exists x\ [\exists y\ [\exists z\ [\exists w\ [((((Gx\ \&\ Gy)\ \&\ Gz)\ \&\ Gw)\ \&\ (((\sim x = y\ \&\ \sim x = z)\ \&\ \sim x = w)\ \&\ (\sim y = z\ \&\ \sim y = w)\ \&\ \sim z = w))\ \&\ \forall u\ [Gu \to (((u = x \text{ v } u = y) \text{ v } u = z) \text{ v } u = w)]]]]]$

2 $\forall x\ [\forall y\ [(xRy\ \&\ yRx) \to x = y]]$

EXERCISE 5.11

1 In each formula, the domain is all human beings.

(i) $\exists x\ [(Fx\ \&\ \forall y\ [Fy \to x = y])\ \&\ Gx]$,
F: ... is the King of rick'n'roll,
G: ... is dead.

(ii) $\exists x\ [(Fx\ \&\ \forall y\ [Fy \to x = y])\ \&\ Gx]$,
F: ... is the King of the blues,
G: ... was a genteel Delta bluesman.

(iii) $\exists x\ [((Fx\ \&\ Gx)\ \&\ \forall y\ [(Fy\ \&\ Gy) \to x = y])\ \&\ Hx]$,
F: ... is a Blind Lemon Jefferson album,
G: ... is on the table,
H: ... is Groovy.

(iv) $\exists x\ [(Fx\ \&\ \forall y\ [Fy \to x = y])\ \&\ ((Gx\ \&\ Hx)\ \&\ Ix)]$,
F: ... is the head of the philosophy department,
G: ... is cool,
H: ... is calm,
I: ... is collected.

(v) $\exists x\ [((Fx\ \&\ Gx)\ \&\ \forall y\ [(Fy\ \&\ Gy) \to x = y])\ \&\ Hx]$,
F: ... is release by Bob Dylan recently,
G: ... is the Greatest Hits album,
H: ... is preferred by Paul.

(vi) $\exists x\ [(Fx\ \&\ \forall y\ [Fy \to x = y])\ \&\ \forall y\ [yGx\ \&\ xGy]]$,
F: ... is the bluesman,
G: ... rated ....

Examination 3 in Formal Logic

1 In each formula, the domain is all human beings and all creatures.

(i) $\exists x\ [(Fx\ \&\ \forall y\ [Fy \to x = y])\ \&\ x = b]$,
F: ... is the President of the United States,
b: Bill Clinton.

(ii) $\exists x\ [(Fx\ \&\ \forall y\ [Fy \to x = y])\ \&\ Gx]$,
F: ... is the President of the United States,
G: ... is male.

(iii) $\exists x\ [(Fx\ \&\ \forall y\ [Fy \to x = y])\ \&\ Gx]$,
F: ... is the purple people-eater,
G: ... is a monster.

(iv) $\exists x\ [(Fx\ \&\ \forall y\ [Fy \to x = y])\ \&\ Gx]$,
F: ... is the Santa Claus,
G: ... is a charming fellow.

(v) $\sim \exists x\ [Fx]\ \&\ \sim \exists x\ [Gx]$,
F: ... is a creature named Santa Claus,
G: ... is a creature named Pegasus.

(vi) $\sim \exists x\ [Fx]$,
F: ... is a creature named Flubjob.

2 In each formula, the domain is all human beings.

(i) $\exists x\ [(Fx\ \&\ \forall y\ [Fy \to x = y])\ \&\ Gx]$,
$\forall x\ [Gx \to \sim \exists y\ [yHx]]$
$\therefore \forall x\ [Fx \to \sim \exists y\ [yHx]]$,
F: ... is the man in the iron mask,
G: ... is a bore,
H: ... likes ....

(ii) $\exists x\ [(Fx\ \&\ \forall y\ [Fy \to x = y])\ \&\ Gx]$,
$\exists x\ [(Ix\ \&\ \forall y\ [(Iy\ \&\ \sim x = y) \to xHy])\ \&\ (Jx\ \&\ \forall y\ [Jy \to x = y])]$,
$\forall x\ [\forall y\ [xHy \to \sim yHx]]$,
$\exists x\ [Fx\ \&\ Jx]$
$\therefore \exists x\ [(Fx\ \&\ \forall y\ [Fy \to x = y])\ \&\ (Ix\ \&\ Jx)]$,
F: ... is Alice,
G: ... sat in the logic exam that Professor Frege had recently devised,
H: ... is happier than ...,
I: ... is in the room,
J: ... passed with flying colours.
(Note: the statement "the happiest student" can be translated to $\exists x\ [\forall y\ [\sim x = y \to xHy]], \forall x\ [\forall y\ [xHy \to \sim yHx]]$, because if we assume that there are two a and b things ($\sim a = b$) who are the happiest, then it would be true, that aHb and bHa, which is impossible, therefore, in this case $a = b$.)

3 The domain is all animals and their body parts.
$\forall x\ [Px \to Ax] : \forall x\ [\forall y\ [(yHx\ \&\ Px) \to (yHx\ \&\ Ax)]]$
P: ... is a horse,
A: ... is an animal,
H: ... is an head of ....

4 1. $\sim \exists x\ [xBb]$
2. $\forall x\ [\forall y\ [((xDa\ \&\ \forall z\ [\sim z = a \to \sim xDz])\ \&\ \sim \exists z\ [yDz]) \to xCy]]$
3. aBb

(Note: premise 2. means the following.
For all two anything (particularly two human beings) it is true,
that if one (x) owns a cheese sandwich and nothing else,
and the other person (y) owns nothing,
than x is better off than y.)

Chapter Six: How to Argue Logically in QL

EXERCISE 6.1

1 The proofs are the followings.

1. $\forall x\ [Fx] : Fa$

{1}	1.	$\forall x\ [Fx]$	Premise
{1}	2.	Fa	1 UE

2. $\forall x\ [Fx] : (Fa\ \&\ Fb)\ \&\ (Fx\ \&\ Fd)$

{1}	1.	$\forall x\ [Fx]$	Premise
{1}	2.	Fa	1 UE
{1}	3.	Fb	1 UE
{1}	4.	Fc	1 UE
{1}	5.	Fd	1 UE
{1}	6.	$Fa\ \&\ Fb$	2, 3 &I
{1}	7.	$Fc\ \&\ Fd$	4, 5 &I
{1}	8.	$(Fa\ \&\ Fb)\ \&\ (Fc\ \&\ Fd)$	6, 7 &I

3. $\forall x\ [Fx\ \&\ Gx] : (Ga\ \&\ Fa)$

{1}	1.	$\forall x\ [Fx\ \&\ Gx]$	Premise
{1}	2.	$Fa\ \&\ Ga$	1 UE
{1}	3.	Fa	2 &E
{1}	4.	Ga	2 &E
{1}	5.	$Ga\ \&\ Fa$	3, 4 &I

4. $\forall x\ [Fx \to Gx], Fa : Ga$

{1}	1.	$\forall x\ [Fx \to Gx]$	Premise
{2}	2.	Fa	Premise
{1}	3.	$Fa \to Ga$	1 UE
{1, 2}	4.	Ga	2, 3 MP

5. $\forall x\ [Fx \to Gx], Fb : (Fb\ \&\ Gb)$

{1}	1.	$\forall x\ [Fx \to Gx]$	Premise
{2}	2.	Fb	Premise
{1}	3.	$Fb \to Gb$	1 UE
{1, 2}	4.	Gb	2, 3 MP
{1, 2}	5.	$Fb\ \&\ Gb$	2, 4 &I

6. $\forall x\ [Fx \to Gx], \sim Gc :\sim Fc$

{1}	1.	$\forall x\ [Fx \to Gx]$	Premise
{2}	2.	$\sim Gc$	Premise
{1}	3.	$Fc \to Gc$	1 UE
{1, 2}	4.	$\sim Fc$	2, 3 MT

7. $\forall x\ [Fx \to Gx], \forall x\ [Gx \to Hx] : (Fa \to Ha)$

{1}	1.	$\forall x\ [Fx \to Gx]$	Premise
{2}	2.	$\forall x\ [Gx \to Hx]$	Premise
{1}	3.	$Fa \to Ga$	1 UE
{2}	4.	$Ga \to Ha$	1 UE
{5}	5.	Fa	Assumption
{1, 5}	6.	Ga	3, 5 MP
{1, 2, 5}	7.	Ha	4, 6 MP
{1, 2}	8.	$Fa \to Ha$	5, 7 CP

8. $\sim Fa :\sim \forall x\ [Fx]$

{1}	1.	$\sim Fa$	Premise
{2}	2.	$\forall x\ [Fx]$	Assumption
{2}	3.	Fa	2 UE
{1, 2}	4.	$Fa\ \&\ \sim Fa$	1, 3 &I
{1}	5.	$\sim \forall x\ [Fx]$	2, 4 RAA

9. $\sim (Fa \;\&\; Fb) :\sim \forall x\ [Fx]$

{1}	1.	$\sim (Fa \;\&\; Fb)$	Premise
{2}	2.	$\forall x\ [Fx]$	Assumption
{2}	3.	Fa	2 UE
{2}	4.	Fb	2 UE
{2}	5.	$Fa \;\&\; Fb$	3, 4 &I
{1, 2}	6.	$(Fa \;\&\; Fb) \;\&\; \sim (Fa \;\&\; Fb)$	1, 5 &I
{1}	7.	$\sim \forall x\ [Fx]$	2, 6 RAA

10. $\forall x\ [Fx \rightarrow Gx] : \forall y\ [Fy] \rightarrow Gb$

{1}	1.	$\forall x\ [Fx \rightarrow Gx]$	Premise
{2}	2.	$\forall x\ [Fx]$	Assumption
{1}	3.	$Fb \rightarrow Gb$	1 UE
{2}	4.	Fb	2 UE
{1, 2}	5.	Gb	3, 4 MP
{1}	6.	$\forall x\ [Fx] \rightarrow Gb$	2, 5 CP

EXERCISE 6.2

1 The proofs are the followings.

1. $\forall x\ [Fx] \;\&\; \forall y\ [Gy] : \forall z\ [Fz \;\&\; Gz]$

{1}	1.	$\forall x\ [Fx] \;\&\; \forall y\ [Gy]$	Premise
{1}	2.	$\forall x\ [Fx]$	1 &E
{1}	3.	Fa	2 UE
{1}	4.	$\forall x\ [Gx]$	1 &E
{1}	5.	Ga	4 UE
{1}	6.	$Fa \;\&\; Ga$	3, 5 &I
{1}	7.	$\forall z\ [Fz \;\&\; Gz]$	6 UI

2. $\forall x\ [Fx] : \forall x\ [Fx \text{ v } Gx]$

{1}	1.	$\forall x\ [Fx]$	Premise
{1}	2.	Fa	1 UE
{1}	3.	$Fa \text{ v } Ga$	2 vI
{1}	4.	$\forall x\ [Fx \text{ v } Gx]$	3 UI

3. $\forall x\ [Fx \to Gx] : \forall x\ [(Fx\ \&\ Hx) \to Gx]$

{1}	1.	$\forall x\ [Fx \to Gx]$	Premise
{1}	2.	$Fa \to Ga$	1 UE
{3}	3.	$Fa\ \&\ Ha$	Assumption
{3}	4.	Fa	3 &E
{1, 3}	5.	Ga	2, 4 MP
{1}	6.	$(Fa\ \&\ Ha) \to Ga$	3, 5 CP
{1}	7.	$\forall x\ [(Fa\ \&\ Ha) \to Ga]$	6 UI

4. $\forall x\ [Fx \to Gx], \forall x\ [Gx \to Hx] : \forall x\ [Fx \to Hx]$

{1}	1.	$\forall x\ [Fx \to Gx]$	Premise
{2}	2.	$\forall x\ [Gx \to Hx]$	Premise
{1}	3.	$Fa \to Ga$	1 UE
{2}	4.	$Ga \to Ha$	2 UE
{5}	5.	Fa	Assumption
{1, 5}	6.	Ga	3, 5 MP
{1, 2, 5}	7.	Ha	4, 6 MP
{1, 2}	8.	$Fa \to Ha$	5, 7 CP
{1, 2}	9.	$\forall x\ [Fx \to Hx]$	8 UI

5. $\forall x\ [Fx \to Gx], \forall x\ [Hx \to\sim Gx] : \forall x\ [Fx \to\sim Hx]$

{1}	1.	$\forall x\ [Fx \to Gx]$	Premise
{2}	2.	$\forall x\ [Hx \to\sim Gx]$	Premise
{1}	3.	$Fa \to Ga$	1 UE
{2}	4.	$Ha \to\sim Ga$	2 UE
{5}	5.	Fa	Assumption
{1, 5}	6.	Ga	3, 5 MP
{1, 5}	7.	$\sim\sim Ga$	6 DNI
{1, 2, 5}	8.	$\sim Ha$	4, 7 MT
{1, 2}	9.	$Fa \to\sim Ha$	5, 8 CP
{1, 2}	10.	$\forall x\ [Fx \to\sim Hx]$	9 UI

6. $\mathrm{P} \to \forall x\ [Fx] : \forall x\ [\mathrm{P} \to Fx]$

{1}	1.	$\mathrm{P} \to \forall x\ [Fx]$	Premise
{2}	2.	P	Assumption
{1, 2}	3.	$\forall x\ [Fx]$	1, 2 MP
{1, 2}	4.	Fa	3 UE
{1}	5.	$\mathrm{P} \to Fa$	2, 4 CP
{1}	6.	$\forall x\ [\mathrm{P} \to Fx]$	5 UI

7. $\forall x\ [Fx] \text{ v } \forall x\ [Gx] : \forall x\ [Fx \text{ v } Gx]$

{1}	1.	$\forall x\ [Fx] \text{ v } \forall x\ [Gx]$	Premise
{2}	2.	$\forall x\ [Fx]$	Assumption
{2}	3.	Fa	2 UE
{2}	4.	$Fa \text{ v } Ga$	3 vI
{5}	5.	$\forall x\ [Gx]$	Assumption
{5}	6.	Ga	5 UE
{5}	7.	$Fa \text{ v } Ga$	6 vI
{1}	8.	$Fa \text{ v } Ga$	1, 2, 4, 5, 7 vE
{1}	9.	$\forall x\ [Fx \text{ v } Gx]$	8 UI

8. $\forall x\ [Fx \text{ v } Gx], \forall x\ [Fx \rightarrow Gx] : \forall x\ [Gx]$

{1}	1.	$\forall x\ [Fx \text{ v } Gx]$	Premise
{2}	2.	$\forall x\ [Fx \rightarrow Gx]$	Premise
{1}	3.	$Fa \text{ v } Ga$	1 UE
{2}	4.	$Fa \rightarrow Ga$	2 UE
{5}	5.	Fa	Assumption
{2, 5}	6.	Ga	4, 5 MP
{7}	7.	Ga	Assumption
{1, 2}	8.	Ga	3, 4, 6, 7, 7 vE
{1, 2}	9.	$\forall x\ [Gx]$	8 UI

9. $: \forall x\ [Fx \rightarrow Fx]$

{1}	1.	Fa	Assumption
–	2.	$Fa \rightarrow Fa$	1, 1 CP
–	3.	$\forall x\ [Fx \rightarrow Fx]$	2 UI

10. $: \forall x\ [Fx \text{ v } \sim Fx]$

{1}	1.	$\sim (Fa \text{ v } \sim Fa)$	Assumption
{2}	2.	Fa	Assumption
{2}	3.	$Fa \text{ v } \sim Fa$	2 vI
{1, 2}	4.	$(Fa \text{ v } \sim Fa)\ \&\ \sim (Fa \text{ v } \sim Fa)$	1, 3 &I
{1}	5.	$\sim Fa$	2, 4 RAA
{1}	6.	$Fa \text{ v } \sim Fa$	4 vI
{1}	7.	$(Fa \text{ v } \sim Fa)\ \&\ \sim (Fa \text{ v } \sim Fa)$	1, 5 &I
–	8.	$\sim\sim (Fa \text{ v } \sim Fa)$	1, 6 RAA
–	9.	$Fa \text{ v } \sim Fa$	8 DNE
–	10.	$\forall x\ [Fx \text{ v } \sim Fx]$	9 UI

EXERCISE 6.3

1 The proofs are the followings.

1. $(Fa \ \& \ Ga) : \exists x \ [Fx \ \& \ Gx]$

{1}	1.	$Fa \ \& \ Ga$	Premise
{1}	2.	$\exists x \ [Fx \ \& \ Gx]$	1 EI

2. $(Fa \ \& \ Ga) : \exists x \ [Fx] \ \& \ \exists x \ [Gx]$

{1}	1.	$Fa \ \& \ Ga$	Premise
{1}	2.	Fa	1 &E
{1}	3.	$\exists x \ [Fx]$	2 EI
{1}	4.	Ga	1 &E
{1}	5.	$\exists x \ [Gx]$	4 EI
{1}	6.	$\exists x \ [Fx] \ \& \ \exists x \ [Gx]$	3, 5 &I

3. $\forall x \ [Fx] : \exists x \ [Fx]$

{1}	1.	$\forall x \ [Fx]$	Premise
{1}	2.	Fa	1 UE
{1}	3.	$\exists x \ [Fx]$	2 EI

4. $\forall x \ [Fx \ \& \ Gx] : \exists x \ [Fx] \ \& \ \exists x[Gx]$

{1}	1.	$\forall x \ [Fx \ \& \ Gx]$	Premise
{1}	2.	$Fa \ \& \ Ga$	1 UE
{1}	3.	Fa	2 &E
{1}	4.	$\exists x \ [Fx]$	3 EI
{1}	5.	Ga	2 &E
{1}	6.	$\exists x \ [Gx]$	5 EI
{1}	7.	$\exists x \ [Fx] \ \& \ \exists x \ [Gx]$	4, 6 &I

5. $\forall x \ [Fx] : \exists x \ [Fx \text{ v } Gx]$

{1}	1.	$\forall x \ [Fx]$	Premise
{1}	2.	Fa	1 UE
{1}	3.	$Fa \text{ v } Ga$	2 vI
{1}	4.	$\exists x \ [Fx \text{ v } Gx]$	3 EI

6. $\forall x\ [Fx \rightarrow (Gx \rightarrow Hx)], (Fa \ \&\ Ga) : \exists x\ [Hx]$

{1}	1.	$\forall x\ [Fx \rightarrow (Gx \rightarrow Hx)]$	Premise
{2}	2.	$Fa \ \&\ Ga$	Premise
{1}	3.	$Fa \rightarrow (Ga \rightarrow Ha)$	1 UE
{2}	4.	Fa	2 &E
{2}	5.	Ga	2 &E
{1, 2}	6.	$Ga \rightarrow Ha$	3, 4 MP
{1, 2}	7.	Ha	5, 6 MP
{1, 2}	8.	$\exists x\ [Hx]$	7 EI

7. $\forall x\ [(Fx \ \&\ \sim Gx) \rightarrow Hx)], (Fa \ \&\ \sim Ga) : \exists x\ [Fx \ \&\ Hx]$

{1}	1.	$\forall x\ [(Fx \ \&\ \sim Gx) \rightarrow Hx)]$	Premise
{2}	2.	$Fa \ \&\ \sim Ga$	Premise
{1}	3.	$(Fa \ \&\ \sim Ga) \rightarrow Ha$	1 UE
{1, 2}	4.	Ha	2, 4 MP
{2}	5.	Fa	2 &E
{1, 2}	6.	$Fa \ \&\ Ha$	4, 5 &I
{1, 2}	7.	$\exists x\ [Fx \ \&\ Hx]$	6 EI

8. $(Fa \ \&\ Ga), \forall x\ [Hx \rightarrow \sim Gx] : \exists x\ [Fx \ \&\ \sim Hx]$

{1}	1.	$Fa \ \&\ Ga$	Premise
{2}	2.	$\forall x\ [Hx \rightarrow \sim Gx]$	Premise
{2}	3.	$Ha \rightarrow \sim Ga$	2 UE
{1}	4.	Fa	1 &E
{1}	5.	Ga	1 &E
{1}	6.	$\sim\sim Ga$	5 DNI
{1, 2}	7.	$\sim Ha$	3, 6 MT
{1, 2}	8.	$Fa \ \&\ \sim Ha$	4, 7 &I
{1, 2}	9.	$\exists x\ [Fx \ \&\ \sim Hx]$	8 EI

9. $\exists x\ [Fx] \rightarrow \mathrm{P} : \forall x\ [Fx \rightarrow \mathrm{P}]$

{1}	1.	$\exists x\ [Fx] \rightarrow \mathrm{P}$	Premise
{2}	2.	Fa	Assumption
{2}	3.	$\exists x\ [Fx]$	2 EI
{1, 2}	4.	P	1, 3 MP
{1}	5.	$Fa \rightarrow \mathrm{P}$	2, 4 CP
{1}	6.	$\forall x\ [Fx \rightarrow \mathrm{P}]$	5 UI

10. $\forall y\ [Gy \rightarrow Hy] : \exists x\ [Gx] \rightarrow \exists y\ [Hy]$
Note: I believe you need existential elimination rule to prove this sequent, and I'm not sure whether this can be done in 7 steps.

{1}	1.	$\forall y\ [Gy \rightarrow Hy]$	Premise
{1}	2.	$Ga \rightarrow Ha$	1 UE
{3}	3.	$\exists x\ [Gy]$	Assumption
{4}	4.	Ga	Assumption
{1, 4}	5.	Ha	2, 4 MP
{1, 4}	6.	$\exists x\ [Hx]$	5 EI
{1, 3}	7.	$\exists x\ [Hx]$	3, 4, 6 EE
{1}	8.	$\exists x\ [Gx] \rightarrow \exists x\ [Hx]$	3, 7 CP

EXERCISE 6.4

1 The proofs are the followings.

1. $: \forall x\ [Fx] \rightarrow \exists x\ [Fx]$

{1}	1.	$\forall x\ [Fx]$	Assumption
{1}	2.	Fa	1 UE
{1}	3.	$\exists x\ [Fx]$	2 EI
–	4.	$\forall x\ [Fx] \rightarrow \exists x\ [Fx]$	1, 3 CP

2. $\forall x\ [\mathrm{P}\ \&\ Fx] : \mathrm{P}\ \&\ \exists x\ [Fx]$

{1}	1.	$\forall x\ [\mathrm{P}\ \&\ Fx]$	Premise
{1}	2.	$\mathrm{P}\ \&\ Fa$	1 UE
{1}	3.	P	2 &E
{1}	4.	Fa	2 &E
{1}	5.	$\exists x\ [Fx]$	4 EI
{1}	6.	$\mathrm{P}\ \&\ \exists x\ [Fx]$	3, 5 &I

3. $\sim \exists x\ [Fx] :\sim Fa$

{1}	1.	$\sim \exists x\ [Fx]$	Premise
{2}	2.	Fa	Assumption
{2}	3.	$\exists x\ [Fx]$	2 EI
{1, 2}	4.	$\exists x\ [Fx]\ \&\ \sim \exists x\ [Fx]$	1, 3 &I
{1}	5.	$\sim Fa$	2, 4 RAA

4. $\sim \exists\ [Fx] : \forall x\ [\sim Fx]$

{1}	1.	$\sim \exists\ [Fx]$	Premise
{2}	2.	Fa	Assumption
{2}	3.	$\exists x\ [Fx]$	2 EI
{1, 2}	4.	$\exists x\ [Fx]\ \&\ \sim \exists x\ [Fx]$	1, 3 &I
{1}	5.	$\sim Fa$	2, 4 RAA
{1}	6.	$\forall x\ [\sim Fx]$	5 UI

5. $\sim \forall x\ [Fx] : \exists x\ [\sim Fx]$

{1}	1.	$\sim \forall x\ [Fx]$	Premise
{2}	2.	$\sim \exists x\ [\sim Fx]$	Assumption
{3}	3.	$\sim Fa$	Assumption
{3}	4.	$\exists x\ [\sim Fx]$	3 EI
{2, 3}	5.	$\exists x\ [\sim Fx]\ \&\ \sim \exists x\ [\sim Fx]$	2, 4 &I
{2}	6.	$\sim\sim Fa$	3, 5 RAA
{2}	7.	Fa	6 DNE
{2}	8.	$\forall x\ [Fx]$	7 UI
{1, 2}	9.	$\forall x\ [Fx]\ \&\ \sim \forall x\ [Fx]$	1, 8 &I
{1}	10.	$\sim\sim \exists x\ [\sim Fx]$	2, 9 RAA
{1}	11.	$\exists x\ [\sim Fx]$	10 DNE

6. $\sim \exists x\ [\sim Fx] : \forall x\ [Fx]$

{1}	1.	$\sim \exists x\ [\sim Fx]$	Premise
{2}	2.	$\sim Fa$	Assumption
{2}	3.	$\exists x\ [\sim Fx]$	2 EI
{1, 2}	4.	$\exists x\ [\sim Fx]\ \&\ \sim \exists x\ [\sim Fx]$	1, 3 &I
{1}	5.	$\sim\sim Fa$	2, 4 RAA
{1}	6.	Fa	5 DNE
{1}	7.	$\forall x\ [Fx]$	6 UI

7. $\sim \forall x\ [\sim Fx] : \exists x\ [Fx]$

{1}	1.	$\sim \forall x\ [\sim Fx]$	Premise
{2}	2.	$\sim \exists x\ [Fx]$	Assumption
{3}	3.	Fa	Assumption
{3}	4.	$\exists x\ [Fx]$	3 EI
{2, 3}	5.	$\exists x\ [Fx]\ \&\ \sim \exists x\ [Fx]$	2, 4 &I
{2}	6.	$\sim Fa$	3, 5 RAA
{2}	7.	$\forall x\ [\sim Fx]$	6 UI
{1, 2}	8.	$\forall x\ [\sim Fx]\ \&\ \sim \forall x\ [\sim Fx]$	1, 7 &I
{1}	9.	$\sim\sim \exists x\ [Fx]$	2, 8 RAA
{1}	10.	$\exists x\ [Fx]$	9 DNE

8. $: \exists x\ [Fx \text{ v } \sim Fx]$

{1}	1.	$\sim (Fa \text{ v } \sim Fa)$	Assumption
{2}	2.	Fa	Assumption
{2}	3.	$Fa \text{ v } \sim Fa$	2 vI
{1, 2}	4.	$(Fa \text{ v } \sim Fa)\ \&\ \sim (Fa \text{ v } \sim Fa)$	1, 3 &I
{1}	5.	$\sim Fa$	2, 4 RAA
{1}	6.	$Fa \text{ v } \sim Fa$	5 vI
{1}	7.	$(Fa \text{ v } \sim Fa)\ \&\ \sim (Fa \text{ v } \sim Fa)$	1, 6 &I
–	8.	$\sim\sim (Fa \text{ v } \sim Fa)$	1, 7 RAA
–	9.	$Fa \text{ v } \sim Fa$	8 DNE
–	10.	$\exists x\ [Fx \text{ v } \sim Fx]$	9 EI

9. $\exists x\ [Fx] \rightarrow \forall x\ [Gx]$,
$\forall x\ [\sim Hx \rightarrow \sim Gx] : \exists x\ [Fx] \rightarrow \exists x\ [Hx]$

{1}	1.	$\exists x\ [Fx] \rightarrow \forall x\ [Gx]$	Premise
{2}	2.	$\forall x\ [\sim Hx \rightarrow \sim Gx]$	Premise
{3}	3.	$\exists x\ [Fx]$	Assumption
{1, 3}	4.	$\forall x\ [Gx]$	1, 3 MP
{1, 3}	5.	Ga	4 UE
{1, 3}	6.	$\sim\sim Ga$	5 DNI
{2}	7.	$\sim Ha \rightarrow \sim Ga$	2 UE
{1, 2, 3}	8.	$\sim\sim Ha$	6, 7 MT
{1, 2, 3}	9.	Ha	8 DNE
{1, 2, 3}	10.	$\exists x\ [Hx]$	9 EI
{1, 2}	11.	$\exists x\ [Fx] \rightarrow \exists x\ [Hx]$	3, 10 CP

10. $\forall x\ [Fx \leftrightarrow Gx] : \exists x\ [Fx] \leftrightarrow \exists x\ [Gx]$
Note: I believe you need existential elimination rule to prove this sequent, and I'm not sure whether this can be done in 16 steps.

{1}	1.	$\forall x\ [Fx \leftrightarrow Gx]$	Premise
{1}	2.	$Fa \leftrightarrow Ga$	1 UE
{1}	3.	$(Fa \to Ga)\ \&\ (Ga \to Fa)$	2 $\leftrightarrow$ E
{1}	4.	$Fa \to Ga$	3 &E
{1}	5.	$Ga \to Fa$	3 &E
{6}	6.	$\exists x\ [Fx]$	Assumption
{7}	7.	Fa	Assumption
{1, 7}	8.	Ga	4, 7 MP
{1, 7}	9.	$\exists x\ [Gx]$	8 EI
{1, 6}	10.	$\exists x\ [Gx]$	6, 7, 9 EE
{1}	11.	$\exists x\ [Fx] \to \exists x\ [Gx]$	6, 10 CP
{12}	12.	$\exists x\ [Gx]$	Assumption
{13}	13.	Ga	Assumption
{1, 13}	14.	Fa	5, 13 MP
{1, 13}	15.	$\exists x\ [Fx]$	14 EI
{1, 12}	16.	$\exists x\ [Fx]$	12, 13, 15 EE
{1}	17.	$\exists x\ [Gx] \to \exists x\ [Fx]$	12, 16 CP
{1}	18.	$\exists x\ [Fx] \leftrightarrow \exists x\ [Gx]$	11, 17 $\leftrightarrow$ I

EXERCISE 6.5

1 The proofs are the followings.

1. $\exists x\ [Fx] : \exists y\ [Fy]$

{1}	1.	$\exists x\ [Fx]$	Premise
{2}	2.	Fa	Assumption
{2}	3.	$\exists y\ [Fy]$	2 EI
{1}	4.	$\exists y\ [Fy]$	1, 2, 3 EE

2. $\exists x\ [Fx\ \&\ Gx]$

{1}	1.	$\exists x\ [Fx\ \&\ Gx]$	Premise
{2}	2.	$Fa\ \&\ Ga$	Assumption
{2}	3.	Fa	2 &E
{2}	4.	$\exists x\ [Fx]$	3 EI
{2}	5.	Ga	2 &E
{2}	6.	$\exists x\ [Gx]$	5 EI
{2}	7.	$\exists x\ [Fx]\ \&\ \exists x\ [Fx]$	4, 6 &I
{1}	8.	$\exists x\ [Fx]\ \&\ \exists x\ [Fx]$	1, 2, 7 EE

3. $\forall x\ [Gx \to Hx], \exists x\ [Fx \ \&\ Gx] : \exists x\ [Fx \ \&\ Hx]$

{1}	1.	$\forall x\ [Gx \to Hx]$	Premise
{2}	2.	$\exists x\ [Fx \ \&\ Gx]$	Premise
{3}	3.	$Fa \ \&\ Ga$	Assumption
{3}	4.	Fa	3 &E
{3}	5.	Ga	3 &E
{1}	6.	$Ga \to Ha$	1 UE
{1, 3}	7.	Ha	5, 6 MP
{1, 3}	8.	$Fa \ \&\ Ha$	4, 7 &I
{1, 3}	9.	$\exists x\ [Fx \ \&\ Hx]$	8 EI
{1, 2}	10.	$\exists x\ [Fx \ \&\ Hx]$	2, 3, 9 EE

4. $\forall x\ [Fx \to Gx], \exists y\ [Fy \ \&\ Hy] : \exists z\ [Gz \ \&\ Hz]$

{1}	1.	$\forall x\ [Fx \to Gx]$	Premise
{2}	2.	$\exists y\ [Fy \ \&\ Hy]$	Premise
{3}	3.	$Fa \ \&\ Ha$	Assumption
{3}	4.	Fa	3 &E
{3}	5.	Ha	3 &E
{1}	6.	$Fa \to Ga$	1 UE
{1, 3}	7.	Ga	6, 4 MP
{1, 3}	8.	$Ga \ \&\ Ha$	5, 7 &I
{1, 3}	9.	$\exists z\ [Gz \ \&\ Hz]$	8 EI
{1, 2}	10.	$\exists z\ [Gz \ \&\ Hz]$	2, 3, 9 EE

5. $\exists x\ [Fx \ \&\ Gx], \forall x\ [Hx \to\sim Gx] : \exists x\ [Fx \ \&\ \sim Hx]$

{1}	1.	$\exists x\ [Fx \ \&\ Gx]$	Premise
{2}	2.	$\forall x\ [Hx \to\sim Gx]$	Premise
{3}	3.	$Fa \ \&\ Ga$	Assumption
{3}	4.	Fa	3 &E
{3}	5.	Ga	3 &E
{2}	6.	$Ha \to\sim Ga$	2 UE
{3}	7.	$\sim\sim Ga$	5 DNI
{2, 3}	8.	$\sim Ha$	6, 7 MT
{2, 3}	9.	$Fa \ \&\ \sim Ha$	4, 8 &I
{2, 3}	10.	$\exists x\ [Fx \ \&\ \sim Hx]$	9 EI
{1, 2}	11.	$\exists x\ [Fx \ \&\ \sim Hx]$	1, 3, 10 EE

6. $\forall x\ [Gx \to Hx] : \exists x\ [Gx] \to \exists y\ [Hy]$

$\{1\}$	1.	$\forall x\ [Gx \to Hx]$	Premise
$\{2\}$	2.	$\exists x\ [Gx]$	Assumption
$\{3\}$	3.	Ga	Assumption
$\{1\}$	4.	$Ga \to Ha$	1 UE
$\{1, 3\}$	5.	Ha	3, 4 MP
$\{1, 3\}$	6.	$\exists x\ [Hx]$	5 EI
$\{1, 2\}$	7.	$\exists x\ [Hx]$	2, 3, 6 EE
$\{1\}$	8.	$\forall x\ [Gx \to Hx] \to \exists x\ [Hx]$	2, 7 CP

7. $\exists x\ [Fx \text{ v } Gx] : \exists x\ [Fx] \text{ v } \exists x\ [Gx]$

$\{1\}$	1.	$\exists x\ [Fx \text{ v } Gx]$	Premise
$\{2\}$	2.	$Fa \text{ v } Ga$	Assumption
$\{3\}$	3.	Fa	Assumption
$\{3\}$	4.	$\exists x\ [Fx]$	3 EI
$\{3\}$	5.	$\exists x\ [Fx] \text{ v } \exists x\ [Gx]$	4 vI
$\{6\}$	6.	Ga	Assumption
$\{6\}$	7.	$\exists x\ [Gx]$	6 EI
$\{6\}$	8.	$\exists x\ [Fx] \text{ v } \exists x\ [Gx]$	7 vI
$\{2\}$	9.	$\exists x\ [Fx] \text{ v } \exists x\ [Gx]$	2, 3, 5, 6, 8 vE
$\{1\}$	10.	$\exists x\ [Fx] \text{ v } \exists x\ [Gx]$	1, 2, 9 EE

8. $\exists x\ [Fx] \text{ v } \exists x\ [Gx] : \exists x\ [Fx \text{ v } Gx]$

$\{1\}$	1.	$\exists x\ [Fx] \text{ v } \exists x\ [Gx]$	Premise
$\{2\}$	2.	$\exists x\ [Fx]$	Assumption
$\{3\}$	3.	Fa	Assumption
$\{3\}$	4.	$Fa \text{ v } Ga$	3 vI
$\{3\}$	5.	$\exists x\ [Fx \text{ v } Gx]$	4 EI
$\{2\}$	6.	$\exists x\ [Fx \text{ v } Gx]$	2, 3, 6 EE
$\{7\}$	7.	$\exists x\ [Gx]$	Assumption
$\{8\}$	8.	Ga	Assumption
$\{8\}$	9.	$Fa \text{ v } Ga$	8 vI
$\{8\}$	10.	$\exists x\ [Fx \text{ v } Gx]$	9 EI
$\{7\}$	11.	$\exists x\ [Fx \text{ v } Gx]$	7, 8, 10 EE
$\{1\}$	12.	$\exists x\ [Fx \text{ v } Gx]$	1, 2, 6, 7, 11 vE

9. $\exists x\ [Fx] :\sim \forall x\ [\sim Fx]$

$\{1\}$	1.	$\exists x\ [Fx]$	Premise
$\{2\}$	2.	Fa	Assumption
$\{3\}$	3.	$\forall x\ [\sim Fx]$	Assumption
$\{3\}$	4.	$\sim Fa$	2 UE
$\{3, 4\}$	5.	$Fa\ \&\ \sim Fa$	2, 4 &I
$\{2\}$	6.	$\sim \forall x\ [\sim Fx]$	3, 5 RAA
$\{1\}$	7.	$\sim \forall x\ [\sim Fx]$	1, 2, 6 EE

10. $\exists x\ [Fx\ \&\ \sim Gx] :\sim \forall x\ [Fx \to Gx]$

$\{1\}$	1.	$\exists x\ [Fx\ \&\ \sim Gx]$	Premise
$\{2\}$	2.	$Fa\ \&\ \sim Ga$	Assumption
$\{2\}$	3.	Fa	2 &E
$\{2\}$	4.	$\sim Ga$	2 &E
$\{5\}$	5.	$\forall x\ [Fx \to Gx]$	Assumption
$\{5\}$	6.	$Fa \to Ga$	2 UE
$\{2, 5\}$	7.	Ga	3, 6 MP
$\{2, 5\}$	8.	$Ga\ \&\ \sim Ga$	4, 7 &I
$\{2\}$	9.	$\sim \forall x\ [Fx \to Gx]$	5, 8 RAA
$\{1\}$	10.	$\sim \forall x\ [Fx \to Gx]$	1, 2, 9 EE

EXERCISE 6.6

1 The proofs are the followings.

1. $\forall y\ [\forall x\ [Rxy]] : \forall x\ [\forall y\ [Rxy]]$

$\{1\}$	1.	$\forall y\ [\forall x\ [Rxy]]$	Premise
$\{1\}$	2.	$\forall x\ [Rxb]$	1 UE
$\{1\}$	3.	Rab	2 UE
$\{1\}$	4.	$\forall y\ [Ray]$	3 UI
$\{1\}$	5.	$\forall x\ [\forall y\ [Rxy]]$	4 UI

2. $\forall x\ [\forall y\ [\forall z\ [Rxyz]]] : \forall z\ [\forall y\ [\forall x\ [Rxyz]]]$

$\{1\}$	1.	$\forall x\ [\forall y\ [\forall z\ [Rxyz]]]$	Premise
$\{1\}$	2.	$\forall y\ [\forall z\ [Rxyc]]$	1 UE
$\{1\}$	3.	$\forall z\ [Rxbc]$	2 UE
$\{1\}$	4.	$Rabc$	3 UE
$\{1\}$	5.	$\forall x\ [Rxbc]$	4 UI
$\{1\}$	6.	$\forall y\ [\forall x\ [Rxyc]]$	5 UI
$\{1\}$	7.	$\forall z\ [\forall y\ [\forall x\ [Rxyz]]]$	6 UI

3. $\exists x\ [\forall y\ [Rxy]] : \exists x\ [Rxa]$

{1}	1.	$\exists x\ [\forall y\ [Rxy]]$	Premise
{2}	2.	$\forall y\ [Rby]$	Assumption
{2}	3.	Rba	2 UE
{2}	4.	$\exists x\ [Rxa]$	3 EI
{1}	5.	$\exists x\ [Rxa]$	1, 2, 5 EE

4. $\exists x\ [\exists y\ [Rxy]] : \exists y\ [\exists x\ [Rxy]]$

{1}	1.	$\exists x\ [\exists y\ [Rxy]]$	Premise
{2}	2.	$\exists y\ [Ray]$	Assumption
{3}	3.	Rab	Assumption
{3}	4.	$\exists x\ [Rxb]$	3 EI
{3}	5.	$\exists y\ [\exists x\ [Rxy]]$	4 EI
{2}	6.	$\exists y\ [\exists x\ [Rxy]]$	2, 3, 5 EE
{1}	7.	$\exists y\ [\exists x\ [Rxy]]$	1, 2, 6 EE

5. $\forall x\ [\forall y\ [Rxy \to Ryx]] : \forall x\ [\exists y\ [Rxy] \to \exists y\ [Ryx]]$

{1}	1.	$\forall x\ [\forall y\ [Rxy \to Ryx]]$	Premise
{1}	2.	$\forall y\ [Ray \to Rya]$	1 UE
{1}	3.	$Rab \to Rba$	2 UE
{4}	4.	$\exists y\ [Ray]$	Assumption
{5}	5.	Rab	Assumption
{1, 5}	6.	Rba	3, 5 MP
{1, 5}	7.	$\exists y\ [Rya]$	6 EI
{1, 4}	8.	$\exists y\ [Rya]$	4, 5, 7 EE
{1}	9.	$\exists y\ [Ray] \to \exists y\ [Rya]$	4, 8 CP
{1}	10.	$\forall x\ [\exists y\ [Rxy] \to \exists y\ [Ryx]]$	9 UI

6. $\forall x\ [Fx \to Gx] : \forall x\ [\exists y\ [Fy\ \&\ Rxy] \to \exists y\ [Gy\ \&\ Rxy]]$

{1}	1.	$\forall x\ [Fx \to Gx]$	Premise
{1}	2.	$Fb \to Gb$	1 UE
{3}	3.	$\exists y\ [Fy\ \&\ Ray]$	Assumption
{4}	4.	$Fb\ \&\ Rab$	Assumption
{4}	5.	Fb	4 &E
{4}	6.	Rab	4 &E
{1, 4}	7.	Gb	2, 5 MP
{1, 4}	8.	$Gb\ \&\ Rab$	6, 7 &I
{1, 4}	9.	$\exists y\ [Gy\ \&\ Ray]$	8 EI
{1, 3}	10.	$\exists y\ [Gy\ \&\ Ray]$	3, 4, 9 EE
{1}	11.	$\exists y\ [Fy\ \&\ Ray] \to \exists y\ [Gy\ \&\ Ray]$	3, 10 CP
{1}	12.	$\forall x\ [\exists y\ [Fy\ \&\ Rxy] \to \exists y\ [Gy\ \&\ Rxy]]$	11 UI

7. $\forall x\ [\exists y\ [\forall z\ [Rxyz]]] : \forall x\ [\forall z\ [\exists y\ [Rxyz]]]$

{1}	1.	$\forall x\ [\exists y\ [\forall z\ [Rxyz]]]$	Premise
{1}	2.	$\exists y\ [\forall z\ [Rayz]]$	1 UE
{3}	3.	$\forall z\ [Rabz]$	Assumption
{3}	4.	$Rabc$	3 UE
{3}	5.	$\exists y\ [Rayc]$	4 EI
{2}	6.	$\exists y\ [Rayc]$	2, 3, 5 EE
{1}	7.	$\forall z\ [\exists y\ [Rayz]]$	6 UI
{1}	8.	$\forall x\ [\forall z\ [\exists y\ [Rxyz]]]$	7 UI

8. $\exists x\ [Fx\ \&\ \forall y\ [Gy \rightarrow Rxy]]$,
$\forall x\ [Fx \rightarrow \forall y\ [Hy \rightarrow\sim Rxy]] : \forall x\ [Gx \rightarrow\sim Hx]$
Note: I believe the premises have typos in the book.
The believed correct premises are written here.

{1}	1.	$\exists x\ [Fx\ \&\ \forall y\ [Gy \rightarrow Rxy]]$	Premise
{2}	2.	$\forall x\ [Fx \rightarrow \forall y\ [Hy \rightarrow\sim Rxy]]$	Premise
{3}	3.	Gb	Assumption
{4}	4.	$Fa\ \&\ \forall y\ [Gy \rightarrow Ray]$	Assumption
{4}	5.	Fa	4 &E
{4}	6.	$\forall y\ [Gy \rightarrow Ray]$	4 &E
{4}	7.	$Gb \rightarrow Rab$	6 UE
{3, 4}	8.	Rab	3, 7 MP
{3, 4}	9.	$\sim\sim Rab$	8 DNI
{2}	10.	$Fa \rightarrow \forall y\ [Hy \rightarrow\sim Ray]$	2 UE
{2, 4}	11.	$\forall y\ [Hy \rightarrow\sim Ray]$	5, 10 MP
{2, 4}	12.	$Hb \rightarrow\sim Rab$	11 UE
{2, 3, 4}	13.	$\sim Hb$	9, 12 MT
{2, 4}	14.	$Gb \rightarrow\sim Hb$	3, 13 CP
{2, 4}	15.	$\forall x\ [Gx \rightarrow\sim Hx]$	14 UI
{1, 2}	16.	$\forall x\ [Gx \rightarrow\sim Hx]$	1, 4, 15 EE

2 The proofs are the followings.

(i) $\forall x\ [\forall y\ [Rxy \rightarrow\sim Ryx]] : \forall x\ [\sim Rxx]$

{1}	1.	$\forall x\ [\forall y\ [Rxy \rightarrow\sim Ryx]]$	Premise
{2}	2.	Raa	Assumption
{1}	3.	$\forall y\ [Ray \rightarrow\sim Rya]$	1 UE
{1}	4.	$Raa \rightarrow\sim Raa$	3 UE
{1}	5.	$\sim Raa$	2, 4 MP
{1, 2}	6.	$Raa\ \&\ \sim Raa$	2, 5 &I
{1}	7.	$\sim Raa$	2, 6 RAA
{1}	8.	$\forall x\ [\sim Rxx]$	7 UI

(ii) $\forall x\ [\forall y\ [\forall z\ [(Rxy\ \&\ Ryz) \rightarrow \sim Rxz]]] : \forall x\ [\sim Rxx]$

{1}	1.	$\forall x\ [\forall y\ [\forall z\ [(Rxy\ \&\ Ryz) \rightarrow \sim Rxz]]]$	Premise
{2}	2.	Raa	Assumption
{1}	3.	$\forall y\ [\forall z\ [(Ray\ \&\ Ryz) \rightarrow \sim Raz]]$	1 UE
{1}	4.	$\forall z\ [(Raa\ \&\ Raz) \rightarrow \sim Raz]$	3 UE
{1}	5.	$(Raa\ \&\ Raa) \rightarrow \sim Raa$	4 UE
{2}	6.	$Raa\ \&\ Raa$	2, 2 &I
{1, 2}	7.	$\sim Raa$	5, 6 MP
{1, 2}	8.	$Raa\ \&\ \sim Raa$	2, 7 &I
{1}	9.	$\sim Raa$	2, 8 RAA
{1}	10.	$\forall x\ [\sim Rxx]$	8 UI

3 $\forall y\ [\exists x\ [Rxy]] : \exists x\ [\forall y\ [Rxy]]$. One might try to solve it the following way.

{1}	1.	$\forall y\ [\exists x\ [Rxy]]$	Premise
{1}	2.	$\exists x\ [Rxb]$	1 UE
{3}	3.	Rab	Assumption
?	4.	$\forall y\ [Ray]$	? UI

It is not possible to introduce the last line, because the term b in the previous line exists in it's assumptions (namely in the same line). This is not a definitive proof.

One definitive argument for this, is that: if it would be true, then this is true for all relations, even for the equality relation. If that would be the case, we can show, that all terms are equal to each other. And this is contradictory, because there are interpretations where not all terms are the same. The proof for this is the following (R is the $=$ relation).

{1}	1.	$\forall y\ [\exists x\ [x = y]] \rightarrow \exists x\ [\forall y\ [x = y]]$	Premise
–	2.	$a = a$	=I
–	3.	$\exists x\ [x = a]$	2 EI
–	4.	$\forall y\ [\exists x\ [x = y]]$	3 UI
{1}	5.	$\exists x\ [\forall y\ [x = y]]$	1, 4 MP
{6}	6.	$\forall y\ [a = y]$	Assumption
{6}	7.	$a = b$	6 UE
{6}	8.	$a = c$	6 UE
{6}	9.	$b = c$	7, 8 =E
{6}	10.	$\forall y\ [b = y]$	9 UI
{6}	11.	$\forall x\ [\forall y\ [x = y]]$	10 UI
{1}	12.	$\forall x\ [\forall y\ [x = y]]$	5, 6, 11 EE

The conclusion is not true in all interpretations: not all terms are the same in all interpretations. So if the premise cannot be true in all interpretations, therefore, the original argument is invalid.

EXERCISE 6.7

1 The proofs are the followings.

1. $: \exists x\ [x = x]$

–	1.	$a = a$	=I
–	2.	$\exists x\ [x = x]$	1 EI

2. $(Fa\ \&\ Ga), (a = b) : (Fb\ \&\ Gb)$

{1}	1.	$(Fa\ \&\ Ga)$	Premise
{2}	2.	$(a = b)$	Premise
{1, 2}	3.	$Fb\ \&\ Gb$	1, 2 =E

3. $\forall x\ [Fx \rightarrow (x = a)] : \exists x\ [Fx \rightarrow Fa]$

{1}	1.	$\forall x\ [Fx \rightarrow (x = a)]$	Premise
{2}	2.	Fa	Assumption
–	3.	$Fa \rightarrow Fa$	2, 2 CP
–	4.	$\exists x\ [Fx \rightarrow Fa]$	3 EI
{1}	5.	$(\exists x\ [Fx \rightarrow Fa])\ \&\ \forall x\ [Fx \rightarrow (x = a)]$	1, 4 &I
{1}	6.	$\exists x\ [Fx \rightarrow Fa]$	5 &E

4. $: ((a = b)\ \&\ (b = c)) \rightarrow (a = c)$

{1}	1.	$(a = b)\ \&\ (b = c)$	Premise
{1}	2.	$a = b$	1 &E
{1}	3.	$b = c$	1 &E
{1}	4.	$a = c$	2, 3 =E
–	5.	$((a = b)\ \&\ (b = c)) \rightarrow (a = c)$	1, 4 CP

5. $: \forall x\ [\exists y\ [x = y]]$

–	1.	$a = a$	=I
–	2.	$\exists y\ [a = y]$	1 EI
–	3.	$\forall x\ [\exists y\ [x = y]]$	2 UI

6. $Fa : \exists x\ [(x = a)\ \&\ Fx]$

{1}	1.	Fa	Premise
–	2.	$a = a$	=I
{1}	3.	$(a = a)\ \&\ Fa$	1, 2 &I
{1}	4.	$\exists x\ [(x = a)\ \&\ Fx]$	3 EI

7. $\exists x\ [(x = a)\ \&\ Fx] : Fa$

{1}	1.	$\exists x\ [(x = a)\ \&\ Fx]$	Premise
{2}	2.	$(b = a)\ \&\ Fb$	Assumption
{2}	3.	$b = a$	2 &E
{2}	4.	Fb	2 &E
{2}	5.	Fa	3, 4 =E
{1}	6.	Fa	1, 2, 5 EE

8. $\forall x\ [Fx \rightarrow (Gx \text{ v } (x = a))] : Ga \rightarrow \forall x\ [Fx \rightarrow Gx]$

{1}	1.	$\forall x\ [Fx \rightarrow (Gx \text{ v } (x = a))]$	Premise
{2}	2.	Ga	Assumption
{3}	3.	Fb	Assumption
{1}	4.	$Fb \rightarrow (Gb \text{ v } (b = a))$	1 UE
{1, 3}	5.	$Gb \text{ v } (b = a)$	3, 4 MP
{6}	6.	Gb	Assumption
{7}	7.	$b = a$	Assumption
{2, 7}	8.	Gb	2, 7 =E
{1, 2, 3}	9.	Gb	5, 6, 6, 7, 8 vE
{1, 2}	10.	$Fb \rightarrow Gb$	3, 9 CP
{1, 2}	11.	$\forall x\ [Fx \rightarrow Gx]$	10 UI
{1}	12.	$Ga \rightarrow \forall x\ [Fx \rightarrow Gx]$	2, 11 CP

9. $\forall x\ [Fx \rightarrow Gx] : \forall x\ [\forall y\ [(Fx\ \&\ \sim Gy) \rightarrow \sim (x = y)]]$

{1}	1.	$\forall x\ [Fx \rightarrow Gx]$	Premise
{2}	2.	$Fa\ \&\ \sim Gb$	Assumption
{3}	3.	$a = b$	Assumption
{1}	4.	$Fa \rightarrow Ga$	1 UE
{2}	5.	Fa	2 &E
{1, 2}	6.	Ga	4, 5 MP
{2}	7.	$\sim Gb$	2 &E
{2, 3}	8.	$\sim Ga$	3, 7 =E
{1, 2, 3}	9.	$Ga\ \&\ \sim Ga$	6, 8 &I
{1, 2}	10.	$\sim (a = b)$	3, 9 RAA
{1}	11.	$(Fa\ \&\ \sim Gb) \rightarrow \sim (a = b)$	2, 10 CP
{1}	12.	$\forall y\ [(Fa\ \&\ \sim Gy) \rightarrow \sim (a = y)]$	11 UI
{1}	13.	$\forall x\ [\forall y\ [(Fx\ \&\ \sim Gy) \rightarrow \sim (x = y)]]$	12 UI

10. $\exists x\ [\forall y\ [Fy \rightarrow (x = y)]] : \forall x\ [\forall y\ [(Fx\ \&\ Fy) \rightarrow (x = y)]]$

{1}	1.	$\exists x\ [\forall y\ [Fy \rightarrow (x = y)]]$	Premise
{2}	2.	$\forall y\ [Fy \rightarrow (a = y)]$	Assumption
{3}	3.	$Fb\ \&\ Fc$	Assumption
{3}	4.	Fb	3 &E
{3}	5.	Fc	3 &E
{2}	6.	$Fb \rightarrow (a = b)$	2 UE
{2}	7.	$Fc \rightarrow (a = c)$	2 UE
{2, 3}	8.	$a = b$	4, 6 MP
{2, 3}	9.	$a = c$	5, 7 MP
{2, 3}	10.	$b = c$	8, 9 =E
{2}	11.	$(Fb\ \&\ Fc) \rightarrow (b = c)$	3, 10 CP
{1}	12.	$(Fb\ \&\ Fc) \rightarrow (b = c)$	1, 2, 11 EE
{1}	13.	$\forall y\ [(Fb\ \&\ Fy) \rightarrow (b = y)]$	12 UI
{1}	14.	$\forall x\ [\forall y\ [(Fx\ \&\ Fy) \rightarrow (x = y)]]$	13 UI

REVISION EXERCISE I

1 See previous exercise solutions for the answers.

REVISION EXERCISE II

1 See previous exercise solutions for the answers.

REVISION EXERCISE III

1 The proofs are the followings.

1. $: \exists x\ [x = a]$

–	1.	$a = a$	=I
–	2.	$\exists x\ [x = a]$	1 EI

2. $\forall x\ [Fx] :\sim \exists x\ [\sim Fx]$

{1}	1.	$\forall x\ [Fx]$	Premise
{2}	2.	$\forall x\ [Fx]$	Assumption
{3}	3.	$\exists x\ [\sim Fx]$	Assumption
{4}	4.	$\sim Fa$	Assumption
{2}	5.	Fa	2 UE
{2, 4}	6.	$Fa\ \&\ \sim Fa$	4, 5 &I
{4}	7.	$\sim \forall x\ [Fx]$	2, 6 RAA
{3}	8.	$\sim \forall x\ [Fx]$	3, 4, 7 EE
{1, 3}	9.	$\forall x\ [Fx]\ \&\ \sim \forall x\ [Fx]$	1, 8 &I
{1}	10.	$\sim \exists x\ [\sim Fx]$	3, 9 RAA

3. $\exists x\ [\exists y\ [\forall z\ [Rxyz]]] : \forall z\ [\exists y\ [\exists x\ [Rxyz]]]$

{1}	1.	$\exists x\ [\exists y\ [\forall z\ [Rxyz]]]$	Premise
{2}	2.	$\exists y\ [\forall z\ [Rayz]]$	Assumption
{3}	3.	$\forall z\ [Rabz]$	Assumption
{3}	4.	$Rabc$	3 UE
{3}	5.	$\exists x\ [Rxbc]$	4 EI
{3}	6.	$\exists y\ [\exists x\ [Rxyc]]$	5 EI
{3}	7.	$\forall z\ [\exists y\ [\exists x\ [Rxyc]]]$	6 UI
{2}	8.	$\forall z\ [\exists y\ [\exists x\ [Rxyc]]]$	2, 3 EE
{1}	9.	$\forall z\ [\exists y\ [\exists x\ [Rxyc]]]$	1, 2 EE

4. $\exists x\ [\sim Fx] : \exists x\ [Fx \to \mathrm{P}]$

{1}	1.	$\exists x\ [\sim Fx]$	Premise
{2}	2.	$\sim Fa$	Assumption
{3}	3.	Fa	Assumption
{2, 3}	4.	$Fa\ \&\ \sim Fa$	2, 3 &I
{5}	5.	$\sim \mathrm{P}$	Assumption
{2, 3, 5}	6.	$(Fa\ \&\ \sim Fa)\ \&\ \sim \mathrm{P}$	4, 5 &I
{2, 3, 5}	7.	$Fa\ \&\ \sim Fa$	6 &E
{2, 3}	8.	$\sim\sim \mathrm{P}$	5, 7 RAA
{2, 3}	9.	P	8 DNE
{2}	10.	$Fa \to \mathrm{P}$	3, 9 CP
{2}	11.	$\exists x\ [Fx \to \mathrm{P}]$	10 EI
{1}	12.	$\exists x\ [Fx \to \mathrm{P}]$	1, 2, 11 EE

5. $: \exists x\ [Fx] \text{ v } \forall y\ [\sim Fy]$

{1}	1.	$\sim (\exists x\ [Fx] \text{ v } \forall y\ [\sim Fy])$	Assumption
{2}	2.	Fa	Assumption
{2}	3.	$\exists x\ [Fx]$	2 EI
{2}	4.	$\exists x\ [Fx] \text{ v } \forall y\ [\sim Fy]$	3 vI
{1, 2}	5.	$(\exists x\ [Fx] \text{ v } \forall y\ [\sim Fy])\ \&\ \sim (\exists x\ [Fx] \text{ v } \forall y\ [\sim Fy])$	1, 4 &I
{1}	6.	$\sim Fa$	2, 5 RAA
{1}	7.	$\forall y\ [\sim Fy]$	6 UI
{1}	8.	$\exists x\ [Fx] \text{ v } \forall y\ [\sim Fy]$	7 vI
{1}	9.	$(\exists x\ [Fx] \text{ v } \forall y\ [\sim Fy])\ \&\ \sim (\exists x\ [Fx] \text{ v } \forall y\ [\sim Fy])$	1, 8 &I
–	10.	$\sim\sim (\exists x\ [Fx] \text{ v } \forall y\ [\sim Fy])$	1, 9 RAA
–	11.	$\exists x\ [Fx] \text{ v } \forall y\ [\sim Fy]$	10 DNE

6. $: \forall\ [Fx] \text{ v } \exists\ [\sim Fx]$

{1}	1.	$\sim (\forall\ [Fx] \text{ v } \exists\ [\sim Fx])$	Assumption
{2}	2.	$\sim Fa$	Assumption
{2}	3.	$\exists x\ [\sim Fx]$	1 EI
{2}	4.	$\forall\ [Fx] \text{ v } \exists x\ [\sim Fx]$	3 vI
{1, 2}	5.	$(\forall\ [Fx] \text{ v } \exists x\ [\sim Fx])\ \&\ \sim (\forall\ [Fx] \text{ v } \exists x\ [\sim Fx])$	1, 4 &I
{1}	6.	$\sim\sim Fa$	2, 5 RAA
{1}	7.	Fa	6 DNE
{1}	8.	$\forall x\ [Fx]$	7 UI
{1}	9.	$\forall\ [Fx] \text{ v } \exists x\ [\sim Fx]$	8 vI
{1}	10.	$(\forall\ [Fx] \text{ v } \exists x\ [\sim Fx])\ \&\ \sim (\forall\ [Fx] \text{ v } \exists x\ [\sim Fx])$	1, 10 &I
–	11.	$\sim\sim (\forall\ [Fx] \text{ v } \exists\ [\sim Fx])$	1, 10 RAA
–	12.	$\forall\ [Fx] \text{ v } \exists\ [\sim Fx]$	11 DNE

7. $\forall x\ [Fx \rightarrow ((x = a) \text{ v } (x = b))], \exists x\ [Fx\ \&\ Gx] : Ga \text{ v } Gb$

{1}	1.	$\forall x\ [Fx \rightarrow ((x = a) \text{ v } (x = b))]$	Premise
{2}	2.	$\exists x\ [Fx\ \&\ Gx]$	Premise
{3}	3.	$Fc\ \&\ Gc$	Assumption
{3}	4.	Fc	3 &E
{3}	5.	Gc	3 &E
{1}	6.	$Fc \rightarrow ((x = a) \text{ v } (x = b))$	1 UE
{1, 3}	7.	$(x = a) \text{ v } (x = b)$	4, 6 MP
{8}	8.	$x = a$	Assumption
{3, 8}	9.	Ga	5, 8 =E
{3, 8}	10.	$Ga \text{ v } Gb$	9 vI
{11}	11.	$x = b$	Assumption
{3, 11}	12.	Gb	5, 11 =E
{3, 11}	13.	$Ga \text{ v } Gb$	12 vI
{1, 3}	14.	$Ga \text{ v } Gb$	7, 8, 10, 11, 13 vE
{1, 2}	15.	$Ga \text{ v } Gb$	2, 3, 14 EE

8. $\exists x\ [Fx\ \&\ Gx], \exists x\ [Fx\ \&\ \sim Gx] : \sim \forall x\ [\forall y\ [(Fx\ \&\ Fy) \to (x = y)]]$

{1}	1.	$\exists x\ [Fx\ \&\ Gx]$	Premise
{2}	2.	$\exists x\ [Fx\ \&\ \sim Gx]$	Premise
{3}	3.	$\forall x\ [\forall y\ [(Fx\ \&\ Fy) \to (x = y)]]$	Assumption
{3}	4.	$\forall y\ [(Fa\ \&\ Fy) \to (a = y)]$	3 UE
{3}	5.	$(Fa\ \&\ Fb) \to (a = b)$	4 UE
{6}	6.	$Fa\ \&\ Ga$	Assumption
{6}	7.	Fa	6 &E
{6}	8.	Ga	6 &E
{9}	9.	$Fb\ \&\ \sim Gb$	Assumption
{9}	10.	Fb	9 &E
{9}	11.	$\sim Gb$	9 &E
{6, 9}	12.	$Fa\ \&\ Fb$	7, 10 &I
{3, 6, 9}	13.	$a = b$	5, 12 MP
{3, 6, 9}	14.	$\sim Ga$	11, 13 =E
{3, 6, 9}	15.	$Ga\ \&\ \sim Ga$	8, 14 &I
{6, 9}	16.	$\sim \forall x\ [\forall y\ [(Fx\ \&\ Fy) \to (x = y)]]$	3, 15 RAA
{6, 2}	17.	$\sim \forall x\ [\forall y\ [(Fx\ \&\ Fy) \to (x = y)]]$	2, 9 EE
{1, 2}	18.	$\sim \forall x\ [\forall y\ [(Fx\ \&\ Fy) \to (x = y)]]$	1, 6 EE

9. $\exists x\ [Fx\ \&\ Gx], \exists x\ [Fx\ \&\ \forall y\ [Gy \to \sim Rxy]] : \exists x\ [Fx\ \&\ \sim \forall y\ [Fy \to Ryx]]$

{1}	1.	$\exists x\ [Fx\ \&\ Gx]$	Premise
{2}	2.	$\exists x\ [Fx\ \&\ \forall y\ [Gy \to \sim Rxy]]$	Premise
{3}	3.	$Fa\ \&\ Ga$	Assumption
{3}	4.	Fa	3 &E
{3}	5.	Ga	3 &E
{6}	6.	$Fb\ \&\ \forall y\ [Gy \to \sim Rby]$	Assumption
{6}	7.	Fb	6 &E
{6}	8.	$\forall y\ [Gy \to \sim Rby]$	6 &E
{9}	9.	$\forall y\ [Fy \to \sim Rya]$	Assumption
{6}	10.	$Ga \to \sim Rba$	8 UE
{3, 6}	11.	$\sim Rba$	5, 9 MP
{9}	12.	$Fb \to \sim Rba$	9 UE
{6, 9}	13.	Rba	7, 12 MP
{3, 6, 9}	14.	$Rba\ \&\ \sim Rba$	11, 13 &I
{3, 6}	15.	$\sim \forall y\ [Fy \to \sim Rya]$	9, 14 RAA
{3, 6}	16.	$Fa\ \&\ \sim \forall y\ [Fy \to \sim Rya]$	3, 15 &I
{3, 6}	17.	$\exists x\ [Fx\ \&\ \sim \forall y\ [Fy \to \sim Ryx]$	16 EI
{3, 2}	18.	$\exists x\ [Fx\ \&\ \sim \forall y\ [Fy \to \sim Ryx]$	2, 6, 17 EE
{1, 2}	19.	$\exists x\ [Fx\ \&\ \sim \forall y\ [Fy \to \sim Ryx]$	1, 3, 18 EE

10. $\forall x\ [\forall y\ [(Fx\ \&\ Fy) \rightarrow (x = y)]] : \exists x\ [\forall y\ [Fy \rightarrow (x = y)]]$
The proof intuitively is the following.
Let's assume that the consequent is not true, and that Fa.
Then, we get a contradiction, because a can be this element
which equals to any other element for which Fy is true (using the premise).
Therefore, $\sim Fa$. Then if Fa would be true, then anything is true
(principle of explosion), therefore, it's also true that $b = a$.
Because a was choosen arbitrarily, it's true for every term y,
we can say that there exists an element b such that if Fy, then $b = y$.
This contradicts our first assumption. Therefore, it must be false.

{1}	1.	$\forall x\ [\forall y\ [(Fx\ \&\ Fy) \rightarrow (x = y)]]$	Premise
{2}	2.	$\sim \exists x\ [\forall y\ [Fy \rightarrow x = y]]$	Assumption
{3}	3.	Fa	Assumption
{1}	4.	$\forall y\ [(Fa\ \&\ Fy) \rightarrow (a = y)]$	1 UE
{1}	5.	$(Fa\ \&\ Fb) \rightarrow (a = b)$	4 UE
{6}	6.	Fb	Assumption
{3, 6}	7.	$Fa\ \&\ Fb$	3, 6 &I
{1, 3, 6}	8.	$a = b$	5, 7 MP
{1, 3}	9.	$Fb \rightarrow a = b$	6, 8 CP
{1, 3}	10.	$\forall y\ [Fy \rightarrow a = y]$	9 UI
{1, 3}	11.	$\exists x\ [\forall y\ [Fy \rightarrow x = y]]$	10 EI
{1, 2, 3}	12.	$\exists x\ [\forall y\ [Fy \rightarrow x = y]]\ \&\ \sim \exists x\ [\forall y\ [Fy \rightarrow x = y]]$	11, 2 &I
{1, 2}	13.	$\sim Fa$	3, 12 RAA
{1, 2, 3}	14.	$Fa\ \&\ \sim Fa$	3, 13 &I
{15}	15.	$\sim b = a$	Assumption
{1, 2, 3, 15}	16.	$(Fa\ \&\ \sim Fa)\ \&\ \sim b = a$	14, 15 &I
{1, 2, 3, 15}	17.	$Fa\ \&\ \sim Fa$	16 &E
{1, 2, 3}	18.	$\sim\sim b = a$	15, 17 RAA
{1, 2, 3}	19.	$b = a$	18 DNE
{1, 2}	20.	$Fa \rightarrow b = a$	3, 19 CP
{1, 2}	21.	$\forall y\ [Fy \rightarrow b = y]$	20 UI
{1, 2}	22.	$\exists x\ [\forall y\ [Fy \rightarrow x = y]]$	21 EI
{1, 2}	23.	$\exists x\ [\forall y\ [Fy \rightarrow x = y]]\ \&\ \sim \exists x\ [\forall y\ [Fy \rightarrow x = y]]$	2, 22 &I
{1}	24.	$\sim\sim \exists x\ [\forall y\ [Fy \rightarrow x = y]]$	2, 23 RAA
{1}	25.	$\exists x\ [\forall y\ [Fy \rightarrow x = y]]$	24 DNE

Examination 4 in Formal Logic

1 The proofs are the followings.

(i) : $\forall x\ [Fx \to Gx] \to (\forall x\ [Fx] \to \forall x\ [Gx])$

Note: I believe the premises have typos in the book.
The believed correct premises are written here.

{1}	1.	$\forall x\ [Fx \to Gx]$	Assumption
{2}	2.	$\forall x\ [Fx]$	Assumption
{2}	3.	Fa	2 UE
{1}	4.	$Fa \to Ga$	1 UE
{1, 2}	5.	Ga	3, 4 MP
{1, 2}	6.	$\forall x\ [Gx]$	5 UI
{1}	7.	$\forall x\ [Fx] \to \forall x\ [Gx]$	2, 6 CP
–	8.	$\forall x\ [Fx \to Gx] \to (\forall x\ [Fx] \to \forall x\ [Gx])$	1, 7 CP

(ii) : $\exists x\ [Fx\ \&\ Gx] \to (\exists x\ [Fx]\ \&\ \exists x\ [Gx])$

{1}	1.	$\exists x\ [Fx\ \&\ Gx]$	Assumption
{2}	2.	$Fa\ \&\ Ga$	Assumption
{2}	3.	Fa	2 &E
{2}	4.	$\exists x\ [Fx]$	3 EI
{2}	5.	Ga	2 &E
{2}	6.	$\exists x\ [Gx]$	5 EI
{2}	7.	$\exists x\ [Fx]\ \&\ \exists x\ [Gx]$	4, 6 &I
{1}	8.	$\exists x\ [Fx]\ \&\ \exists x\ [Gx]$	1, 2, 7 EE
–	9.	$\exists x\ [Fx\ \&\ Gx] \to (\exists x\ [Fx]\ \&\ \exists x\ [Gx])$	1, 8 CP

(iii) : $\forall x\ [Fx\ \&\ Gx] \leftrightarrow (\forall x\ [Fx]\ \&\ \forall x\ [Ga])$

{1}	1.	$\forall x\ [Fx\ \&\ Gx]$	Assumption
{1}	2.	$Fa\ \&\ Ga$	1 UE
{1}	3.	Fa	2 &E
{1}	4.	$\forall x\ [Fx]$	3 UI
{1}	5.	Ga	2 &E
{1}	6.	$\forall x\ [Gx]$	5 UI
{1}	7.	$\forall x\ [Fx]\ \&\ \forall x\ [Gx]$	4, 6 &I
–	8.	$\forall x\ [Fx\ \&\ Gx] \to (\forall x\ [Fx]\ \&\ \forall x\ [Gx])$	1, 7 CP
{9}	9.	$\forall x\ [Fx]\ \&\ \forall x\ [Gx]$	Assumption
{9}	10.	$\forall x\ [Fx]$	9 &E
{9}	11.	Fa	10 UE
{9}	12.	$\forall x\ [Gx]$	9 &E
{9}	13.	Ga	12 UE
{9}	14.	$Fa\ \&\ Ga$	11, 13 &I
{9}	15.	$\forall x\ [Fx\ \&\ Gx]$	14 UI
–	16.	$(\forall x\ [Fx]\ \&\ \forall x\ [Gx]) \to \forall x\ [Fx\ \&\ Gx]$	9, 15 CP
–	17.	$\forall x\ [Fx\ \&\ Gx] \leftrightarrow (\forall x\ [Fx]\ \&\ \forall x\ [Gx])$	8, 16 $\leftrightarrow$ I

(iv) : $\exists x\ [\exists y\ [Rxy]] \leftrightarrow \exists y\ [\exists x\ [Rxy]]$

{1}	1.	$\exists x\ [\exists y\ [Rxy]]$	Assumption
{2}	2.	$\exists y\ [Ray]$	Assumption
{3}	3.	Rab	Assumption
{3}	4.	$\exists x\ [Rxb]$	3 EI
{3}	5.	$\exists y\ [\exists x\ [Rxy]]$	4 EI
{2}	6.	$\exists y\ [\exists x\ [Rxy]]$	2, 3, 5 EE
{1}	7.	$\exists y\ [\exists x\ [Rxy]]$	1, 2, 6 EE
–	8.	$\exists x\ [\exists y\ [Rxy]] \to \exists y\ [\exists x\ [Rxy]]$	1, 7 CP
{9}	9.	$\exists y\ [\exists x\ [Rxy]]$	Assumption
{10}	10.	$\exists x\ [Rxb]$	Assumption
{11}	11.	Rab	Assumption
{11}	12.	$\exists y\ [Ryb]$	11 EI
{11}	13.	$\exists x\ [\exists y\ [Rxy]]$	12 EI
{10}	14.	$\exists x\ [\exists y\ [Rxy]]$	10, 11, 13 EE
{9}	15.	$\exists x\ [\exists y\ [Rxy]]$	9, 10, 14 EE
–	16.	$\exists y\ [\exists x\ [Rxy]] \to \exists x\ [\exists y\ [Rxy]]$	9, 15 CP
–	17.	$\exists x\ [\exists y\ [Rxy]] \leftrightarrow \exists y\ [\exists x\ [Rxy]]$	8, 16 $\leftrightarrow$ I

2 The proof is the following.

{1}	1.	$\exists x\ [Fx \to (a = x)]$	Premise
{2}	2.	$\boldsymbol{Fb \to (a = b)}$	Assumption TD
{3}	3.	$\boldsymbol{\sim (b = a)}$	Assumption
{4}	4.	Fb	Assumption
{2, 4}	5.	$\boldsymbol{a = b}$	2, 4 MP
–	6.	$\boldsymbol{a = a}$	=I
{2, 4}	7.	$(b = a)$	5, 6 =E
{2, 3, 4}	8.	$(b = a)\ \&\ \sim (b = a)$	3, 7 &I
{2, 3}	9.	$\sim Fb$	**4, 8 RAA**
{2}	10.	$\boldsymbol{\sim (b = a) \to \sim Fb}$	3, 9 CP
{2}	11.	$\boldsymbol{\exists x\ [\sim (x = a) \to \sim Fx]}$	10 EI
{1}	12.	$\boldsymbol{\exists x\ [\sim (x = a) \to \sim Fx]}$	1, 2, 11 EE

3

(i) The domain is all animals and all body parts. F: ... is a horse. G: ... is an animal. R: ... is the head of ....

(ii) $\forall x\ [Fx \to Gx] : \forall x\ [\exists y\ [Fy\ \&\ Rxy] \to \exists y\ [Gy\ \&\ Rxy]]$

{1}	1.	$\forall x\ [Fx \to Gx]$	Premise
{2}	2.	$\exists y\ [Fy\ \&\ Ray]$	Assumption
{3}	3.	$Fb\ \&\ Rab$	Assumption
{3}	4.	Fb	3 &E
{3}	5.	Rab	3 &E
{1}	6.	$Fb \to Gb$	1 UE
{1, 3}	7.	Gb	4, 6 MP
{1, 3}	8.	$Gb\ \&\ Rab$	5, 7 &I
{1, 3}	9.	$\exists y\ [Gy\ \&\ Ray]$	8 EI
{1, 2}	10.	$\exists y\ [Gy\ \&\ Ray]$	2, 3, 9 EE
{1}	11.	$\exists y\ [Fy\ \&\ Ray] \to \exists y\ [Gy\ \&\ Ray]$	2, 10 CP
{1}	12.	$\forall x\ [\exists y\ [Fy\ \&\ Rxy] \to \exists y\ [Gy\ \&\ Rxy]]$	11 UI

4 The proof is the following.
$\exists x\ [Fx\ \&\ \forall y\ [Gy \to Rxy]]$,
$\forall x\ [Fx \to \forall y\ [Hy \to \sim Rxy]] : \forall x\ [Gx \to \sim Hx]$
Note: I believe the conclusion have a typo in the book.
The believed correct conclusion is written here.
(See also exercise 6.6, 8, which is believed to be the same.)

{1}	1.	$\exists x\ [Fx\ \&\ \forall y\ [Gy \to Rxy]]$	Premise
{2}	2.	$\forall x\ [Fx \to \forall y\ [Hy \to \sim Rxy]]$	Premise
{3}	3.	Gb	Assumption
{4}	4.	$Fa\ \&\ \forall y\ [Gy \to Ray]$	Assumption
{4}	5.	Fa	4 &E
{4}	6.	$\forall y\ [Gy \to Ray]$	4 &E
{4}	7.	$Gb \to Rab$	6 UE
{3, 4}	8.	Rab	3, 7 MP
{3, 4}	9.	$\sim\sim Rab$	8 DNI
{2}	10.	$Fa \to \forall y\ [Hy \to \sim Ray]$	2 UE
{2, 4}	11.	$\forall y\ [Hy \to \sim Ray]$	5, 10 MP
{2, 4}	12.	$Hb \to \sim Rab$	11 UE
{2, 3, 4}	13.	$\sim Hb$	9, 12 MT
{2, 4}	14.	$Gb \to \sim Hb$	3, 13 CP
{2, 4}	15.	$\forall x\ [Gx \to \sim Hx]$	14 UI
{1, 2}	16.	$\forall x\ [Gx \to \sim Hx]$	1, 4, 15 EE

Chapter Seven: Formal Logic and Formal Semantics #2

EXERCISE 7.1

1 The consistency-trees are the followings.

1. $\sim \exists x\ [Fx \text{ v } \sim Fx]$
   Valid.

1.	$\sim \exists x\ [Fx \text{ v } \sim Fx]$	Negated conclusion.
2.	$\forall x\ [\sim (Fx \text{ v } \sim Fx)]$	1 quantifier-equivalence.
3.	$\sim (Fa \text{ v } \sim Fa)$	2 UIN.
4.	$\sim Fa$	From line 3.
5.	$\sim\sim Fa$ ✗ 4, 6	From line 3.

2. $\forall x\ [Fx \rightarrow Gx], \forall x\ [Fx] : \forall x\ [Gx]$
   Valid.

1.	$\forall x\ [Fx \rightarrow Gx]$		Premise.
2.	$\forall x\ [Fx]$		Premise.
3.	$\sim \forall x\ [Gx]$		Negated conclusion.
4.	$\exists x\ [\sim Gx]$		3 quantifier-equivalence.
5.	$\sim Ga$		4 EIN.
6.	$Fa \rightarrow Ga$		1 UIN.
7.	Fa		2 UIN.
8.	$\sim Fa$ ✗ 7, 8	Ga ✗ 5, 8	From line 6.

3. $\forall x\ [Fx \to Gx], \exists x\ [Fx] : \exists x\ [Gx]$
Valid.

1.	$\forall x\ [Fx \to Gx]$		Premise.
2.	$\exists x\ [Fx]$		Premise.
3.	$\sim \exists x\ [Gx]$		Negated conclusion.
4.	$\forall x\ [\sim Gx]$		3 quantifier-equivalence.
5.	Fa		2 EIN.
6.	$Fa \to Ga$		1 UIN.
7.	$\sim Ga$		4 UIN.
8.	$\sim Fa$ ✗ 5, 8	Ga ✗ 7, 8	From line 6.

4. $\forall x\ [Fx \to Gx], \forall x\ [\sim Gx] : \forall x\ [\sim Fx]$
Valid.

1.	$\forall x\ [Fx \to Gx]$		Premise.
2.	$\forall x\ [\sim Gx]$		Premise.
3.	$\sim \forall x\ [\sim Gx]$		Negated conclusion.
4.	$\exists x\ [\sim\sim Fx]$		3 quantifier-equivalence.
5.	$\sim\sim Fa$		4 EIN.
6.	$Fa \to Ga$		1 UIN.
7.	$\sim Ga$		2 UIN.
8.	$\sim Fa$ ✗ 5, 8	Ga ✗ 7, 8	From line 6.

5. $: \forall\ [Fx] \to \exists x\ [Fx]$
Valid.

1.	$\sim (\forall\ [Fx] \to \exists x\ [Fx])$	Negated conclusion.
2.	$\forall x\ [Fx]$	From line 1.
3.	$\sim \exists x\ [Fx]$	From line 1.
4.	$\forall x\ [\sim Fx]$	From line 3.
5.	Fa	2 UIN.
6.	$\sim Fa$ ✗ 5, 6	4 UIN.

6. $\forall x\ [Fx]\ \&\ \forall x\ [Gx] : \forall x\ [Fx\ \&\ Gx]$
Valid.

1.	$\forall x\ [Fx]\ \&\ \forall x\ [Gx]$		Premise.
2.	$\sim \forall x\ [Fx\ \&\ Gx]$		Negated conclusion.
3.	$\exists x\ [\sim (Fx\ \&\ Gx)]$		2 quantifier-equivalence.
4.	$\sim (Fa\ \&\ Ga)$		3 EIN.
5.	$\forall x\ [Fx]$		From line 1.
6.	$\forall x\ [Gx]$		From line 1.
7.	Fa		5 UIN.
8.	Ga		6 UIN.
9.	$\sim Fa$ ✗ 7, 9	$\sim Ga$ ✗ 8, 9	From line 4.

7. : $\forall x\ [Fx \rightarrow Gx] \rightarrow (\forall x\ [Fx] \rightarrow \forall x\ [Gx])$
Valid.

1.	$\sim (\forall x\ [Fx \rightarrow Gx] \rightarrow (\forall x\ [Fx] \rightarrow \forall x\ [Gx]))$	Negated conclusion.
2.	$\forall x\ [Fx \rightarrow Gx]$	From line 1.
3.	$\sim (\forall x\ [Fx] \rightarrow \forall x\ [Gx])$	From line 1.
4.	$\forall x\ [Fx]$	From line 3.
5.	$\sim \forall x\ [Gx]$	From line 3.
6.	$\exists x\ [\sim Gx]$	5 quantifier-equivalence.
7.	$\sim Ga$	6 EIN.
8.	Fa	4 UIN.
9.	$Fa \rightarrow Ga$	2 UIN.
10.	$\sim Fa$ ✗ 8, 10 Ga ✗ 7, 10	From line 9.

8. : $\forall x\ [Fx\ \&\ Gx] \rightarrow (\forall x\ [Fx]\ \&\ \forall x\ [Gx])$
Valid.

1.	$\sim (\forall x\ [Fx\ \&\ Gx] \rightarrow (\forall x\ [Fx]\ \&\ \forall x\ [Gx]))$	Negated conclusion.
2.	$\forall x\ [Fx\ \&\ Gx]$	From line 1.
3.	$\sim (\forall x\ [Fx]\ \&\ \forall x\ [Gx])$	From line 1.
4.	$\sim \forall x\ [Fx]$ $\sim \forall x\ [Gx]$	From line 3.
5.	$\exists x\ [\sim Fx]$ $\exists x\ [\sim Gx]$	4 quantifier-equivalence.
6.	$\sim Fa$ $\sim Ga$	5 EIN.
7.	$Fa\ \&\ Ga$ $Fa\ \&\ Ga$	2 UIN.
8.	Fa ✗ 6, 8 Ga ✗ 6, 8	From line 7.

9. $\exists x\,[(Fx \;\&\; Gx) \to Hx] : \forall x\,[(Fx \;\&\; Gx) \to Ha]$
Invalid.

1.	$\exists x\,[(Fx \;\&\; Gx) \to Hx]$	Premise.
2.	$\sim \forall x\,[(Fx \;\&\; Gx) \to Ha]$	Negated conclusion.
3.	$\exists x\,[\sim ((Fx \;\&\; Gx) \to Ha)]$	2 quantifier-equivalence.
4.	$\sim ((Fb \;\&\; Gb) \to Ha)$	3 EIN.
5.	$Fb \;\&\; Gb$	From line 4.
6.	$\sim Ha$	From line 5.
7.	Fb	From line 5.
8.	Gb	From line 5.
9.	$(Fc \;\&\; Gc) \to Hc$	1 EIN.
10.	$\sim (Fc \;\&\; Gc)$ | Hc ✓	From line 9.
11.	$\sim Fc$ ✓ | $\sim Gc$ ✓	From line 10.

10. $: \exists x\,[Fx \;\&\; Gx] \to (\exists x\,[Fx] \;\&\; \exists x\,[Gx])$
Valid.

1.	$\sim (\exists x\,[Fx \;\&\; Gx] \to (\exists x\,[Fx] \;\&\; \exists x\,[Gx]))$	Negated conclusion.
2.	$\exists x\,[Fx \;\&\; Gx]$	From line 1.
3.	$\sim (\exists x\,[Fx] \;\&\; \exists x\,[Gx])$	From line 1.
4.	$Fa \;\&\; Ga$	2 EIN.
5.	Fa	From line 4.
6.	Ga	From line 4.
7.	$\sim \exists x\,[Fx]$ | $\sim \exists x\,[Gx]$	From line 3.
8.	$\forall x\,[\sim Fx]$ | $\forall x\,[\sim Gx]$	7 quantifier-equivalence.
9.	$\sim Fa$ ✗5, 9 | $\sim Ga$ ✗5, 6	8 UIN.

EXERCISE 7.2

1 The consistency-trees are the followings.
When an IQLI is specified, the domain is the whole numbers.

1. $\exists x\ [Fx] : \forall x\ [Fx]$
   Invalid. IQLI: F: $\{1\}$, a: 1, b: 2.

1.	$\exists x\ [Fx]$	Premise.
2.	$\sim \forall x\ [Fx]$	Negated conclusion.
3.	$\exists x\ [\sim Fx]$	2 quantifier-equivalence.
4.	Fa	1 EIN.
5.	$\sim Fb$ ✓	3 EIN.

2. $\forall x\ [\sim Fx] :\sim \exists x\ [Fx]$
   Valid.

1.	$\forall x\ [\sim Fx]$	Premise.
2.	$\sim\sim \exists x\ [Fx]$	Negated conclusion.
3.	$\exists x\ [Fx]$	From line 2.
4.	Fa	3 EIN.
5.	$\sim Fa$ ✗ 4, 5	1 UIN.

3. $\exists x\ [Fx \text{ v } Gx], \exists x\ [\sim Fx] : \exists x\ [Gx]$
Invalid. IQLI: F: $\{1\}$, G: $\{\}$, a: 1, b: 2.

1.	$\exists x\ [Fx \text{ v } Gx]$	Premise.
2.	$\exists x\ [\sim Fx]$	Premise.
3.	$\sim \exists x\ [Gx]$	Negated conclusion.
4.	$\forall x\ [\sim Gx]$	From line 3.
5.	$Fa \text{ v } Ga$	1 EIN.
6.	$\sim Fb$	2 EIN.
7.	$\sim Ga$	4 UIN.
8.	$\sim Gb$	4 UIN.
9.	Fa ✓ ; Ga ✗ 7, 9	From line 5.

4. $\forall x\ [Fx \to Gx], \forall x\ [Gx \to Hx] : \forall x\ [Fx \to Hx]$
Valid.

1.	$\forall x\ [Fx \to Gx]$	Premise.
2.	$\forall x\ [Gx \to Hx]$	Premise.
3.	$\sim \forall x\ [Fx \to Hx]$	Negated conclusion.
4.	$\exists x\ [\sim (Fx \to Hx)]$	3 quantifier-equivalence.
5.	$\sim (Fa \to Ha)$	4 EIN.
6.	Fa	From line 5.
7.	$\sim Ha$	From line 5.
8.	$Fa \to Ga$	1 UIN.
9.	$Ga \to Ha$	2 UIN.
10.	$\sim Fa$ ✗ 7, 10 ; Ga	From line 8.
11.	$\sim Ga$ ✗ 10, 11 ; Ha ✗ 7, 11	From line 9.

5. $\forall x\ [Fx \to Gx], \exists x\ [\sim Fx] : \exists x\ [\sim Gx]$
Invalid. IQLI: F: $\{\}$, G: all numbers.

1.	$\forall x\ [Fx \to Gx]$	Premise.
2.	$\exists x\ [\sim Fx]$	Premise.
3.	$\sim \exists x\ [\sim Gx]$	Negated conclusion.
4.	$\forall x\ [\sim\sim Gx]$	3 quantifier-equivalence.
5.	$\sim Fa$	2 EIN.
6.	$Fa \to Ga$	1 UIN.
7.	$\sim\sim Ga$	4 UIN.
8.	Ga	From line 7.
9.	$\sim Fa$ ✓ $\quad$ Ga ✓	From line 6.

6. $\exists x\ [Fx \to Gx] : \exists x\ [Fx] \to \exists x\ [Gx]$
Invalid. IQLI: F: $\{2\}$, G: $\{\}$, a: 1, b: 2.

1.	$\exists x\ [Fx \to Gx]$	Premise.
2.	$\sim (\exists x\ [Fx] \to \exists x\ [Gx])$	Negated conclusion.
3.	$\exists x\ [Fx]$	From line 2.
4.	$\sim \exists x\ [Gx]$	From line 2.
5.	$\forall x\ [\sim Gx]$	4 quantifier-equivalence.
6.	$Fa \to Ga$	1 EIN.
7.	Fb	3 EIN.
8.	$\sim Ga$	5 UIN.
9.	$\sim Gb$	5 UIN.
10.	$\sim Fa$ ✓ $\quad$ Ga ✗ 8, 10	From line 6.

7. : $(\exists x\ [Fx]\ \&\ \exists x\ [Gx]) \rightarrow (\exists x\ [Fx\ \&\ Gx])$
Invalid. IQLI: F $\{1\}$:, G: $\{2\}$, a: 1, b: 2.

1.	$\sim ((\exists x\ [Fx]\ \&\ \exists x\ [Gx]) \rightarrow (\exists x\ [Fx\ \&\ Gx]))$	Negated conclusion.
2.	$\exists x\ [Fx]\ \&\ \exists x\ [Gx]$	From line 1.
3.	$\sim (\exists x\ [Fx\ \&\ Gx])$	From line 1.
4.	$\exists x\ [Fx]$	From line 2.
5.	$\exists x\ [Gx]$	From line 2.
6.	$\forall x\ [\sim (Fx\ \&\ Gx)]$	3 quantifier-equivalence.
7.	Fa	4 EIN.
8.	Gb	5 EIN.
9.	$\sim (Fa\ \&\ Ga)$	6 UIN.
10.	$\sim (Fb\ \&\ Gb)$	6 UIN.
11.	$\sim Fa$ ✗ 7, 11 / $\sim Ga$	From line 9.
12.	(under $\sim Ga$:) $\sim Fb$ ✓ / $\sim Gb$ ✗ 8, 12	From line 10.