Multiplying numbers, but you are a tired seventeenth century scientist

https://tinyurl.com/LOGsigma

Math is not as simple without calculators

- Common household things are very difficult to compute without a calculator
  - Sine, cosine, inverse of secant…
  - Square root (or roots in general)
  - Multiplication (32647624.24139 x 471380478.8183)
  - Division
- In order to overcome these difficulties, people came up with several devices and algorithms.

Calculating square roots

- Useful for several formulae (Pythagoras, area of triangle, etc.)
- Heron's method

Division!
In practice, they looked up tables

$$x_{n+1} = \frac{1}{2}\left(x_n + \boxed{\frac{S}{x_n}}\right)$$

Next estimate for √S

Previous estimate for √S

Calculating trigonometric functions

- Extremely useful for basically everything, especially astronomy
- Half-angle formula:

Root!

$$\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos\theta}{2}}$$

- In practice, people looked up tables

Multiplying

- As important as… multiplication
- Prosthaphaeresis!

Sum the angles and look up the cosine

$$\cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$$

"Convert" your number to a cosine (38.9 → 0.389) and look up the corresponding angle ($\cos^{-1}0.389 = 67.108^{\circ}$)

Do the math and then "revert" to the original number (0.253 → 25.3)

Problems with prosthaphaeresis (cf. this pdf)

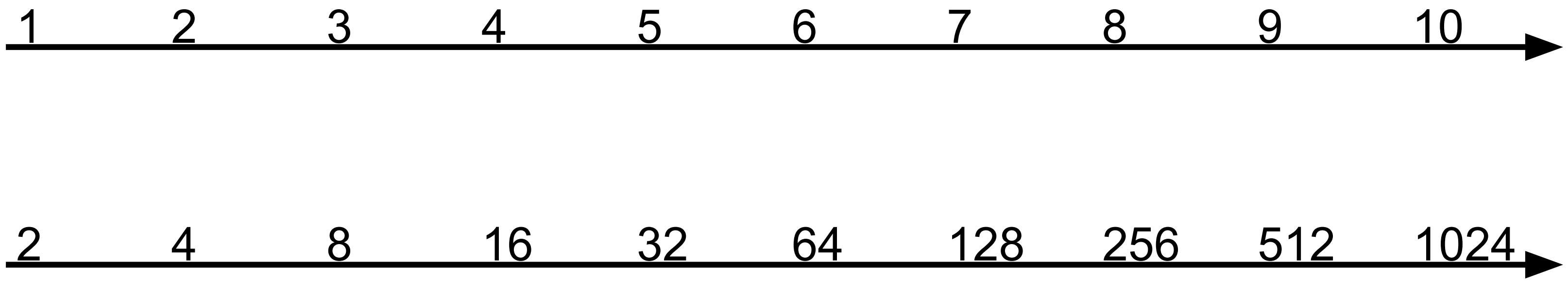
- In order to use it for division, you also need tables for secant
- If you use degree-minute-second to measure angles like early scientists, summing and subtracting angles is not that easy
- You still have to divide by 2!!!
- Only way of calculating powers of numbers is through repeated multiplication… how about fractional powers?

John Napier

- Jobs:
    - Being a noble
    - Improving mathematician’s lives
- Invented tools for computing and the logarithm

A motivation for logarithms

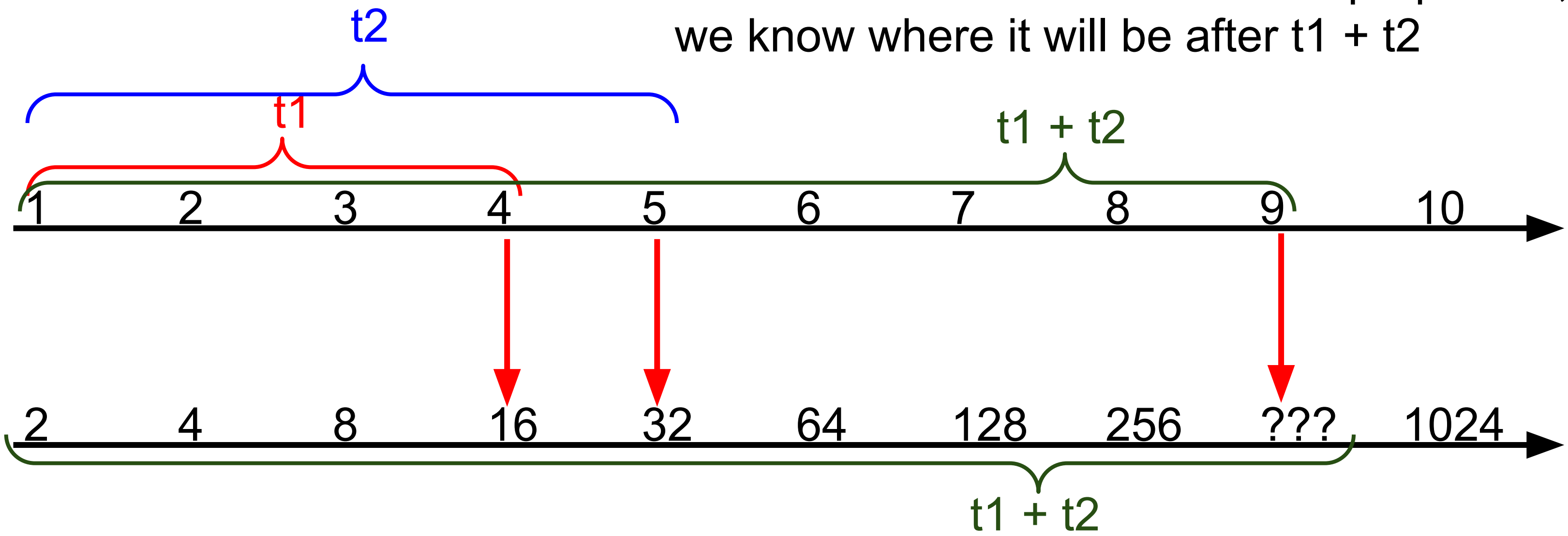
Above: constant speed
Below: speed proportional to distance from origin



Key realisation: the intervals above being in arithmetic progression corresponds to the intervals below being in geometric progression

The reason it is useful

Because this line is in arithmetic proportion, we know where it will be after t1 + t2



Because the bottom line is in geometric progression, the position at t1+t2 is the position at t1 × the position at t2

However, since the intervals above correspond to the intervals below, we can directly compute the position below from the position above!

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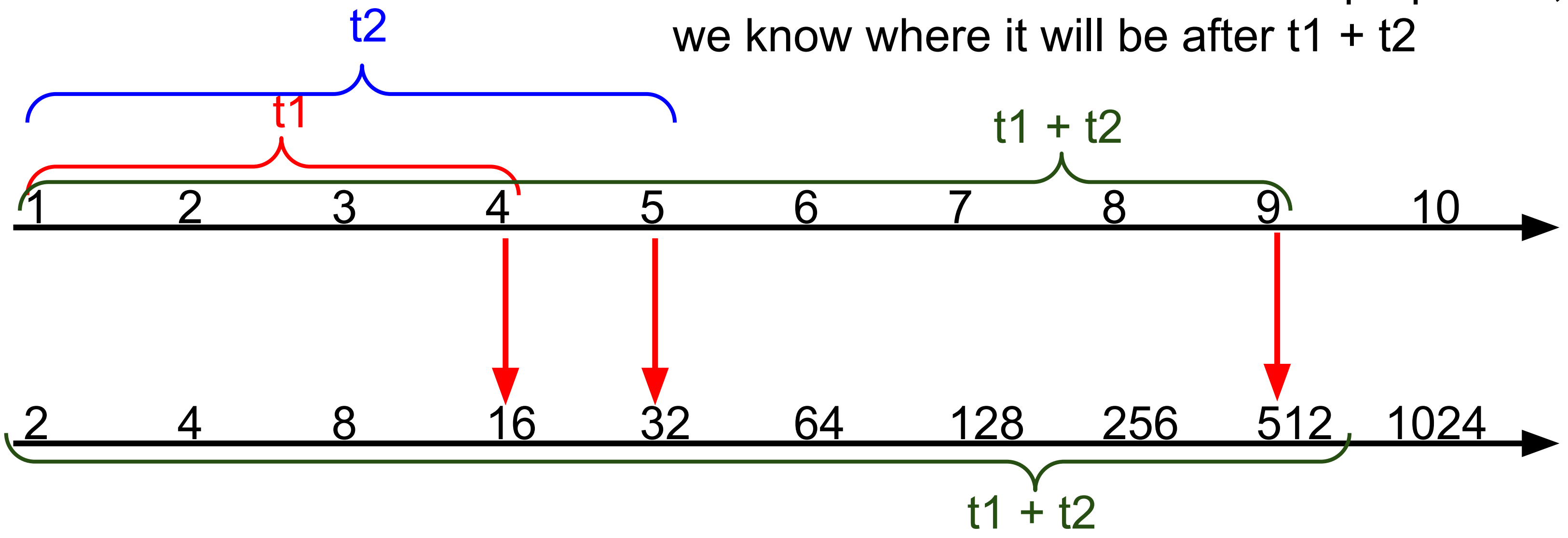

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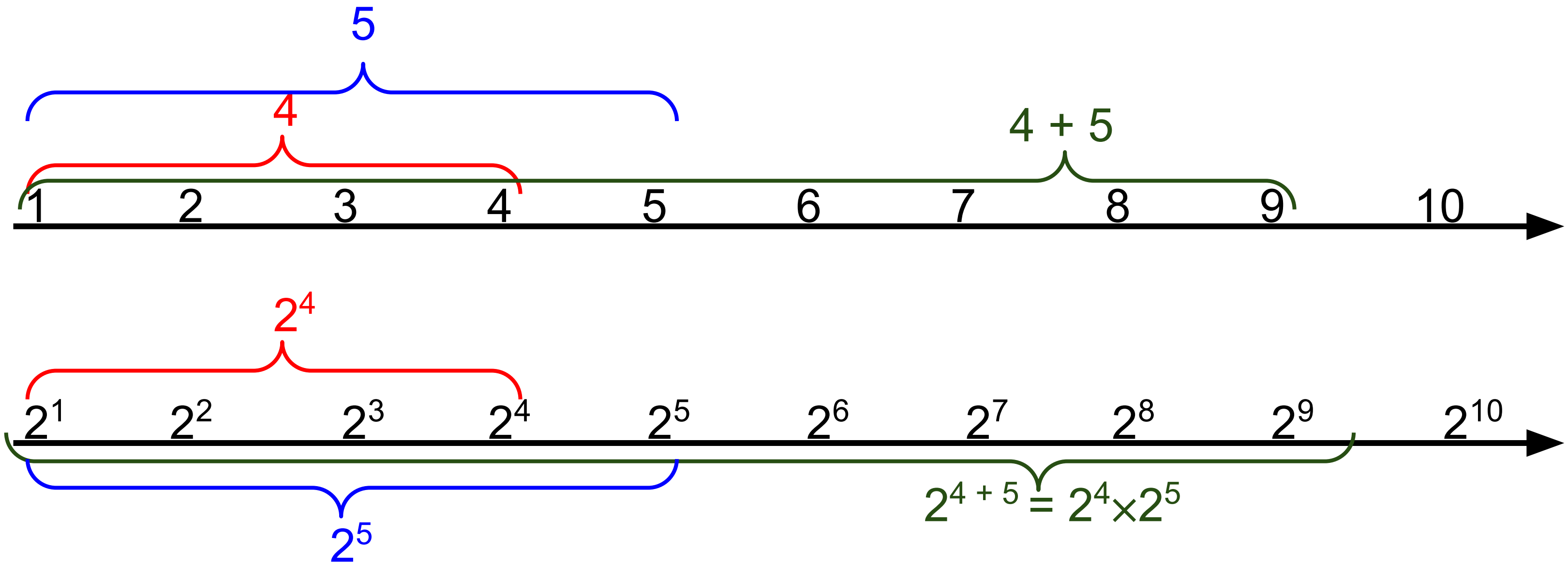
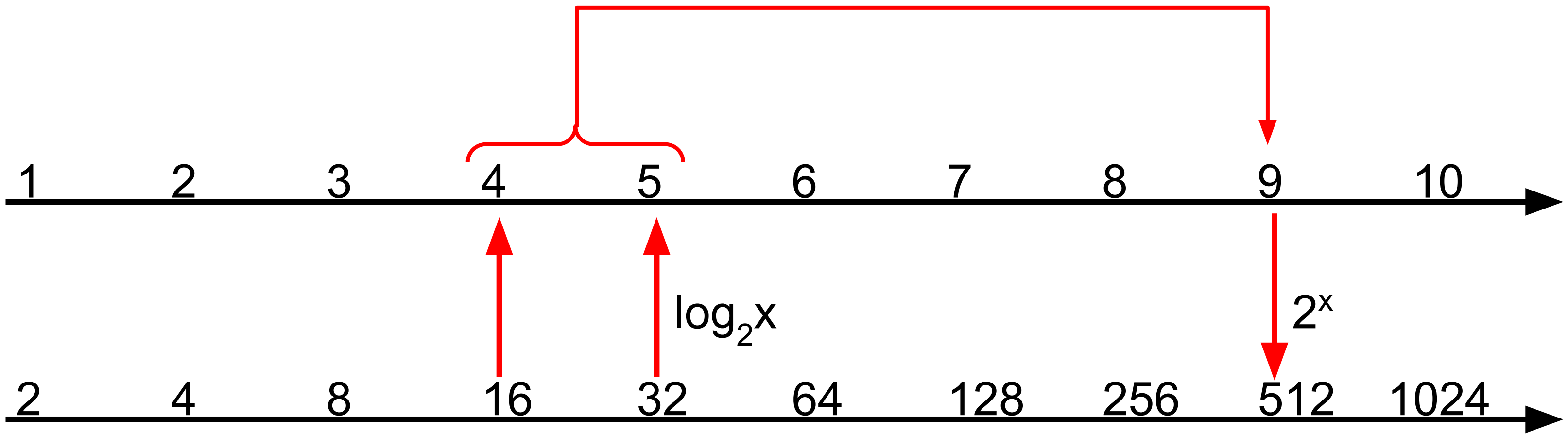
Modern notation makes everything clearer



General algorithm for multiplying two numbers



Key ideas to make it actually usable

- Instead of letting the user do the hard work of exponentiation and trying to guess logarithms, they get access to a table (as always)

- Choosing a smaller base lets the user have a finer choice of numbers to choose from (e.g. there are almost 7000 intervals between 1 and 1024 if we choose 1.001 as base instead of 2 – which has 10 intervals)

Multiplying

- As important as… multiplication
- Logarithms!

Sum the logs

$$\log A + \log B = \log AB$$

Look up the log of your number in a table

Find the log whose value is the sum. It is the log of the resulting multiplication!

Advantages of logarithms

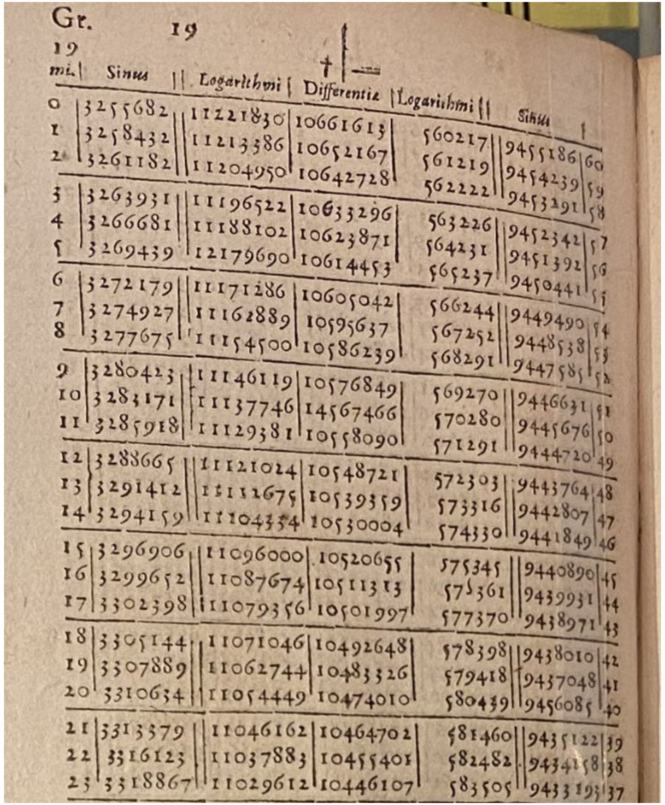
- For division, just subtract the log (no need for a second table)
- You are always summing straightforward numbers
- No division, just one very simple sum
- Can easily calculate powers, they just become multiplications

For these reasons, logarithms were ubiquitous in maths until the 70s!!!

Quirks about Napier's logarithm

- Napier used log base $1 - 10^{-7}$
  - This was because his definition made the velocity on the bottom line proportional to the distance to its end, <u>not</u> the origin
  - Having such a small base made it possible to have extremely fine-grained tables
- Also, his list computed the logs of numbers/10^7, so that the values for the log were reasonable up to big numbers, and multiplied the log by 10^7, so that he didn't have to use a decimal point
- In short: Napier's log = $10^7 \log_{1-10^{\wedge}-7}(x/10^7)$

Napier's table (edited from wikimedia)



Gr. 19

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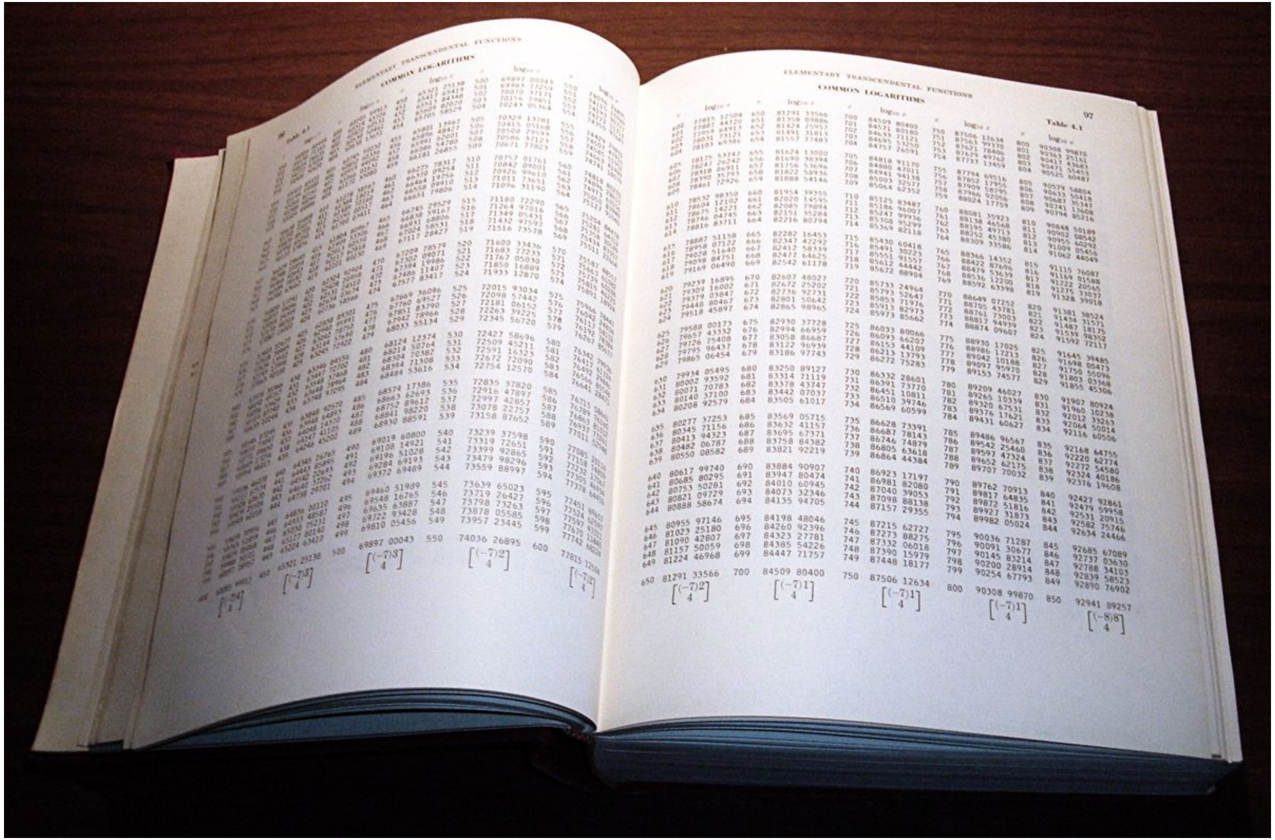
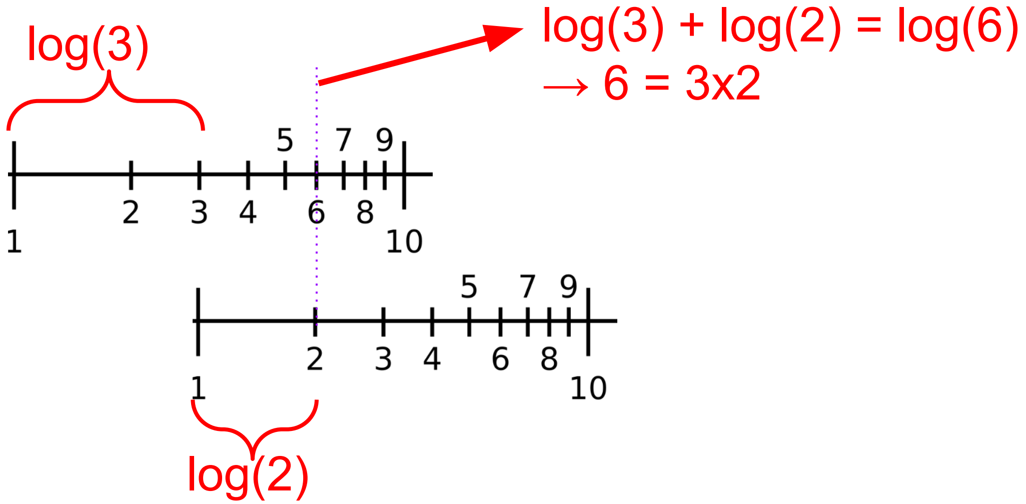
mi.	Sinus	Logarithmi	Differentiae	Logarithmi	Sinus	
0	3255682	11221830	10661613	560217	9455186	60
1	3258432	11213386	10652167	561219	9454239	59
2	3261182	11204950	10642728	562222	9453291	58
3	3263931	11196522	10633296	563226	9452342	57
4	3266681	11188102	10623871	564231	9451392	56
5	3269439	12179690	10614453	565237	9450441	55
6	3272179	11171286	10605042	566244	9449490	54
7	3274927	11162889	10595637	567252	9448538	53
8	3277675	11154500	10586239	568291	9447585	52
9	3280423	11146119	10576849	569270	9446631	51
10	3283171	11137746	14567466	570280	9445676	50
11	3285918	11129381	10558090	571291	9444720	49
12	3288665	11121024	10548721	572303	9443764	48
13	3291412	11112675	10539359	573316	9442807	47
14	3294159	11104354	10530004	574330	9441849	46
15	3296906	11096000	10520655	575345	9440890	45
16	3299652	11087674	10511313	576361	9439931	44
17	3302398	11079356	10501997	577370	9438971	43
18	3305144	11071046	10492648	578398	9438010	42
19	3307889	11062744	10483326	579418	9437048	41
20	3310634	11054449	10474010	580439	9456085	40
21	3313379	11046162	10464702	581460	9435122	39
22	3316123	11037883	10455401	582482	9434158	38
23	3318867	11029612	10446107	583505	9433193	37

Logarithm table from 1964 (cf. wikimedia)



Slide rulers

- Essentially two logarithmic scales atop each other, functionally equivalent to a table

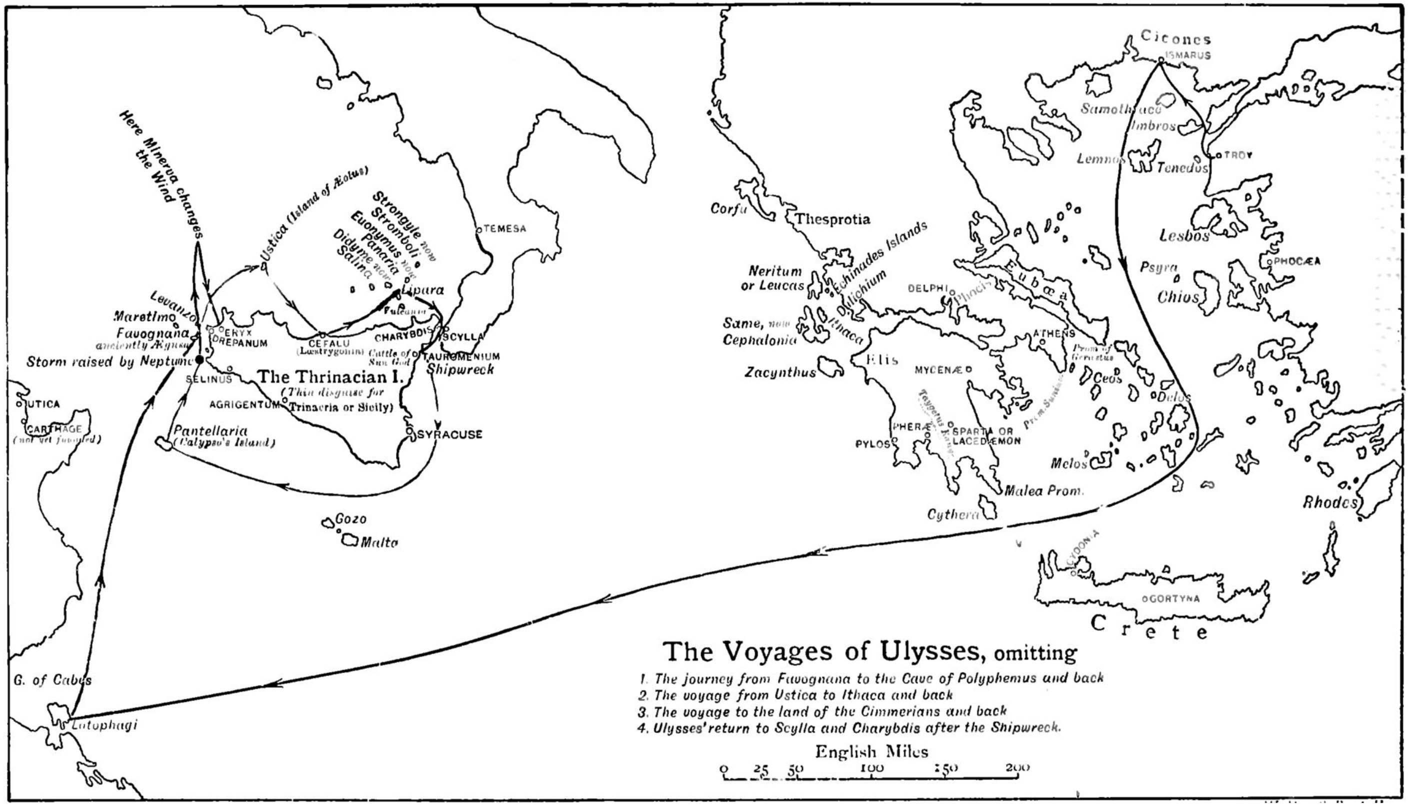


Lessons from logarithms

- Sometimes the first version of an idea is highly convoluted, like Napier's, but gradually simplifies over time
- Looking up (partial) results of calculations is extremely fast

Computers still often use look-up tables to speed computation, like the CORDIC algorithm used to calculate trigonometric functions

Your turn: Odysseus's trip



List of places Odysseus visited

Place	# Latitude	# Longitude
The Island of the Lotus-Eaters	34.7	11.3
The Island of the Cyclopes	37.75	14.0167
Aeolia	38.4833	14.95
The Laestrygonians	40	9
Aeaea	41.2522	13.1032
The Underworld	37	-8
The Sirens' Island	40.55	14.2167
Scylla and Charybdis	38.25	15.6333

Distance between two points on a sphere

Longitude

$$d = R \cdot \arccos\left[\cos\phi_1 \cos\phi_2 (1 + \cos(\lambda_2 - \lambda_1)) - \cos(\phi_1 + \phi_2)\right]$$

Latitude

Let's see who is the fastest! (earth radius: 6378 km)

Place	# Latitude	# Longitude
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$$d = R \cdot \arccos\left[\cos\phi_1 \cos\phi_2 (1 + \cos(\lambda_2 - \lambda_1)) - \cos(\phi_1 + \phi_2)\right]$$

References

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