

Multiplying numbers, but you are a
tired seventeenth century scientist

<https://tinyurl.com/LOGsigma>

Math is not as simple without calculators

- Common household things are very difficult to compute without a calculator
 - Sine, cosine, inverse of secant...
 - Square root (or roots in general)
 - Multiplication ($32647624.24139 \times 471380478.8183$)
 - Division
- In order to overcome these difficulties, people came up with several devices and algorithms.

Calculating square roots

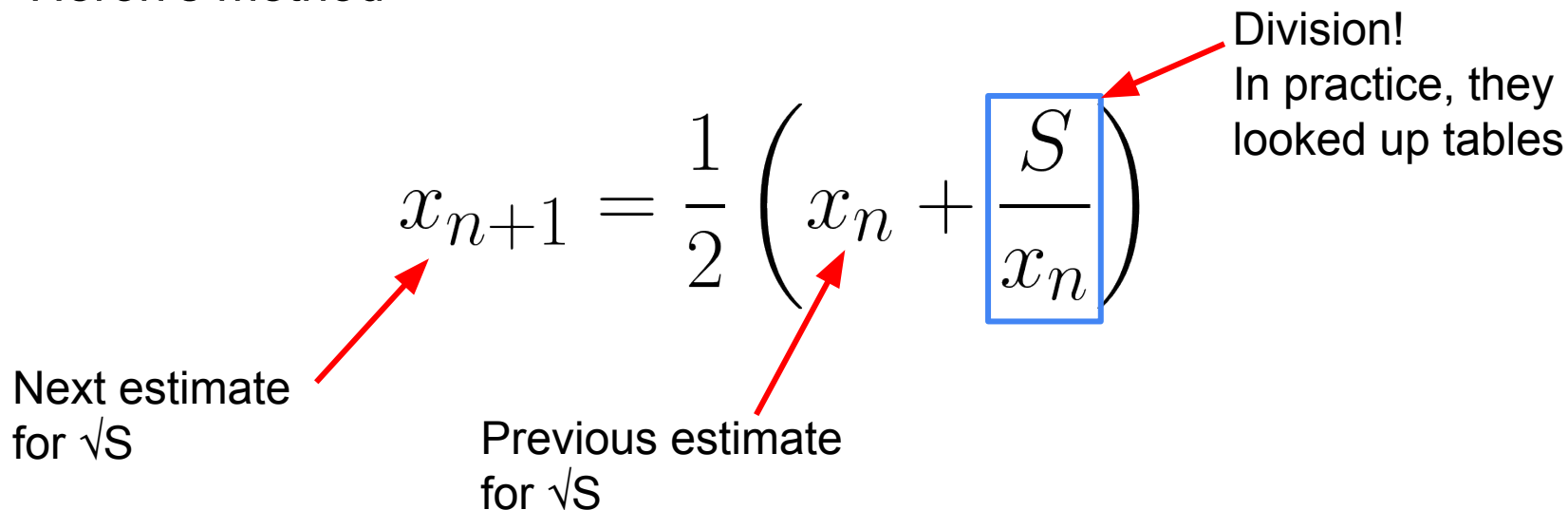
- Useful for several formulae (Pythagoras, area of triangle, etc.)
- Heron's method

$$x_{n+1} = \frac{1}{2} \left(x_n + \frac{S}{x_n} \right)$$

Next estimate for \sqrt{S}

Previous estimate for \sqrt{S}

Division!
In practice, they looked up tables


The diagram shows the formula for Heron's method: $x_{n+1} = \frac{1}{2} \left(x_n + \frac{S}{x_n} \right)$. Three red arrows point from text labels to parts of the formula. The first arrow points from 'Next estimate for \sqrt{S} ' to x_{n+1} . The second arrow points from 'Previous estimate for \sqrt{S} ' to x_n . The third arrow points from 'Division! In practice, they looked up tables' to the fraction $\frac{S}{x_n}$, which is enclosed in a blue rectangular box.

Calculating trigonometric functions

- Extremely useful for basically everything, especially astronomy
- Half-angle formula:

$$\sin \left(\frac{\theta}{2} \right) = \pm \sqrt{\frac{1 - \cos \theta}{2}}$$

Root!



- In practice, people looked up tables

Multiplying

- As important as... multiplication
- Prosthaphaeresis!

$$\cos A \cos B = \frac{1}{2} [\cos(A + B) + \cos(A - B)]$$

Sum the angles
and look up the
cosine

“Convert” your number to a
cosine ($38.9 \rightarrow 0.389$) and
look up the corresponding
angle ($\cos^{-1}0.389 = 67.108^\circ$)

Do the math and then
“revert” to the original
number ($0.253 \rightarrow 25.3$)

Problems with prosthaphaeresis (cf. [this](#) pdf)

- In order to use it for division, you also need tables for secant
- If you use degree-minute-second to measure angles like early scientists, summing and subtracting angles is not that easy
- You still have to divide by 2!!!
- Only way of calculating powers of numbers is through repeated multiplication... how about fractional powers?

John Napier

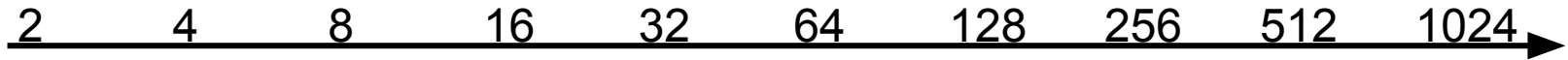
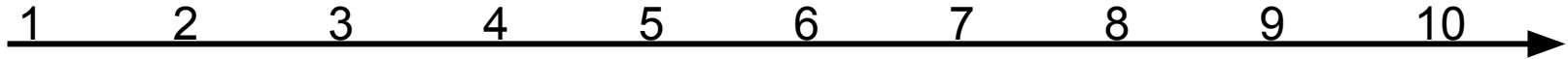


- Jobs:
 - Being a noble
 - Improving mathematician's lives
- Invented tools for computing and the logarithm

A motivation for logarithms

Above: constant speed

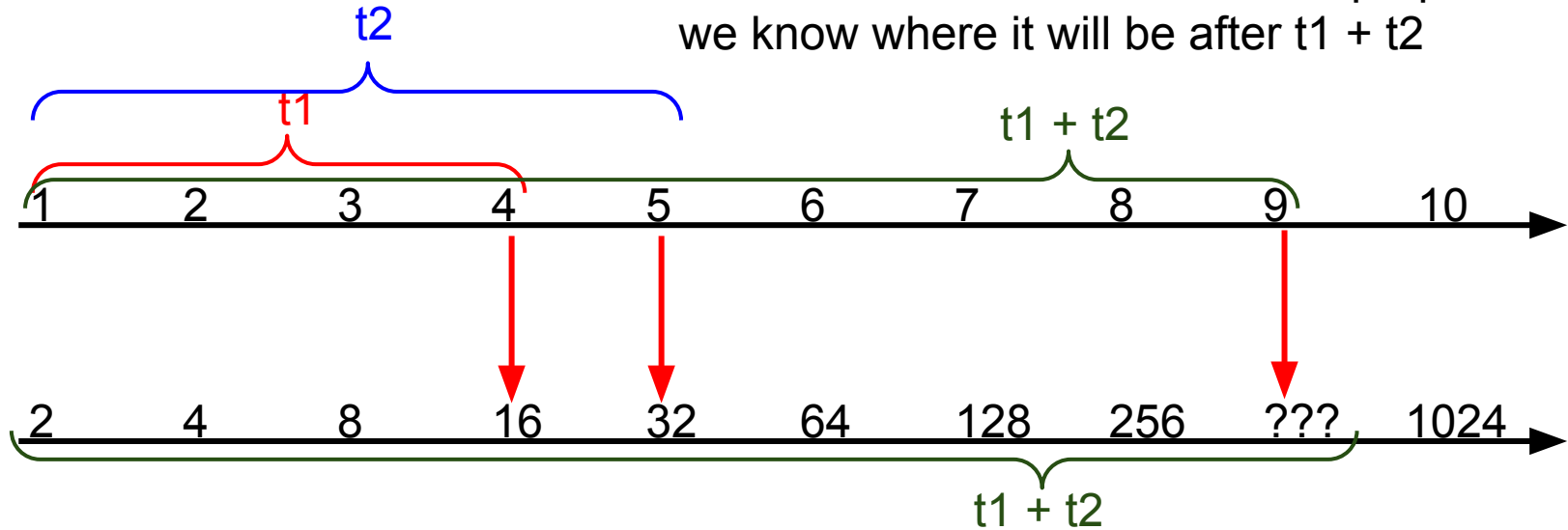
Below: speed proportional to distance from origin



Key realisation: the intervals above being in arithmetic progression corresponds to the intervals below being in geometric progression

The reason it is useful

Because this line is in arithmetic proportion, we know where it will be after $t_1 + t_2$

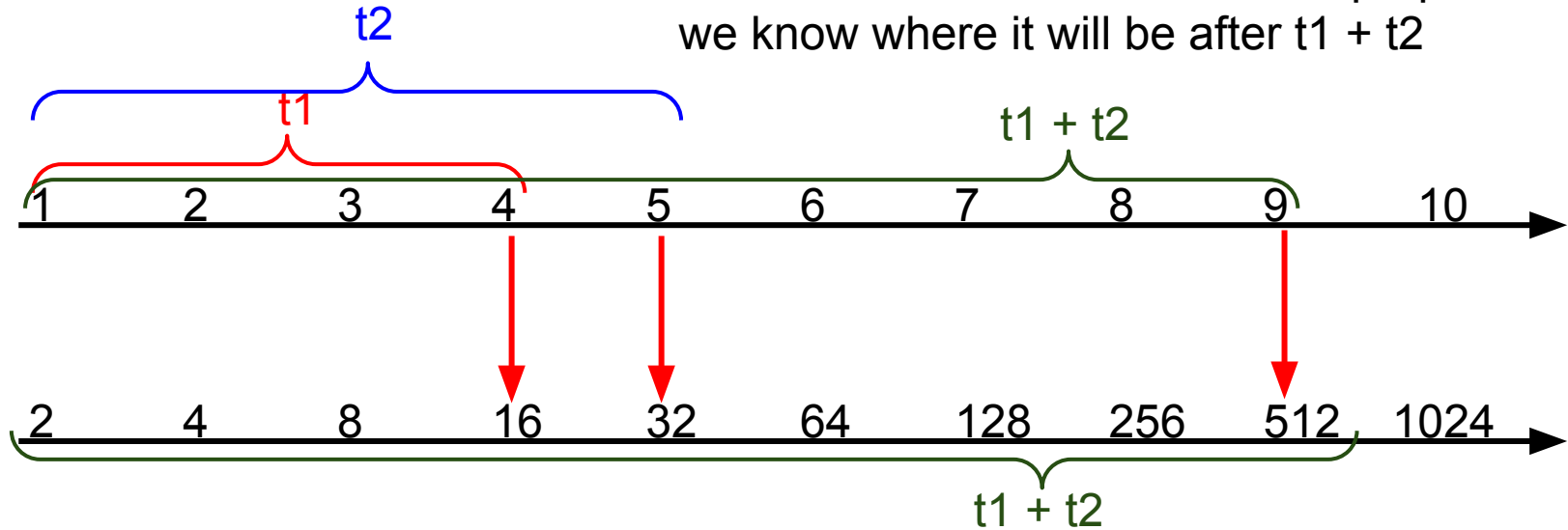


Because the bottom line is in geometric progression, the position at $t_1 + t_2$ is the position at $t_1 \times$ the position at t_2

However, since the intervals above correspond to the intervals below, we can directly compute the position below from the position above!

The reason it is useful

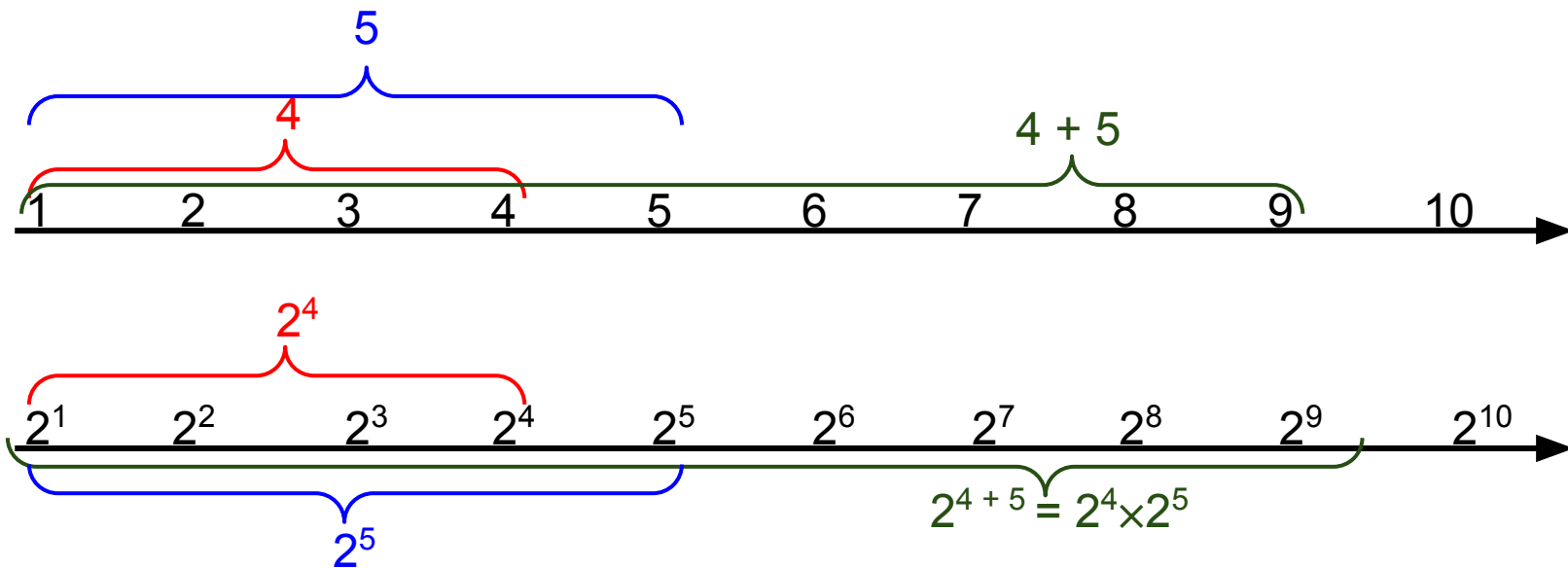
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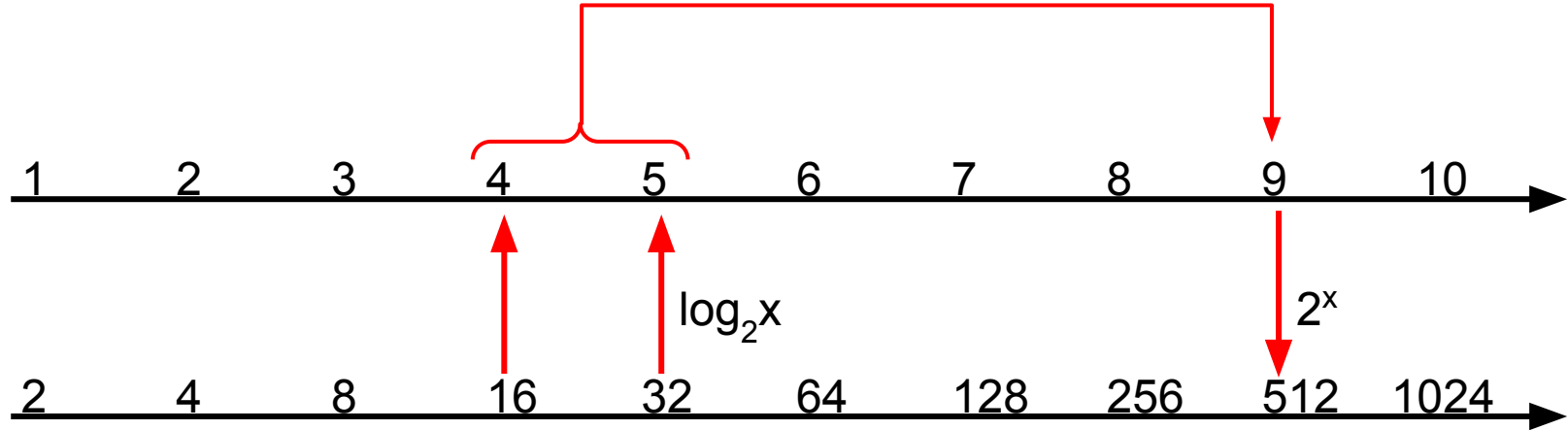
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However, since the intervals above correspond to the intervals below, we can directly compute the position below from the position above!

Modern notation makes everything clearer



General algorithm for multiplying two numbers

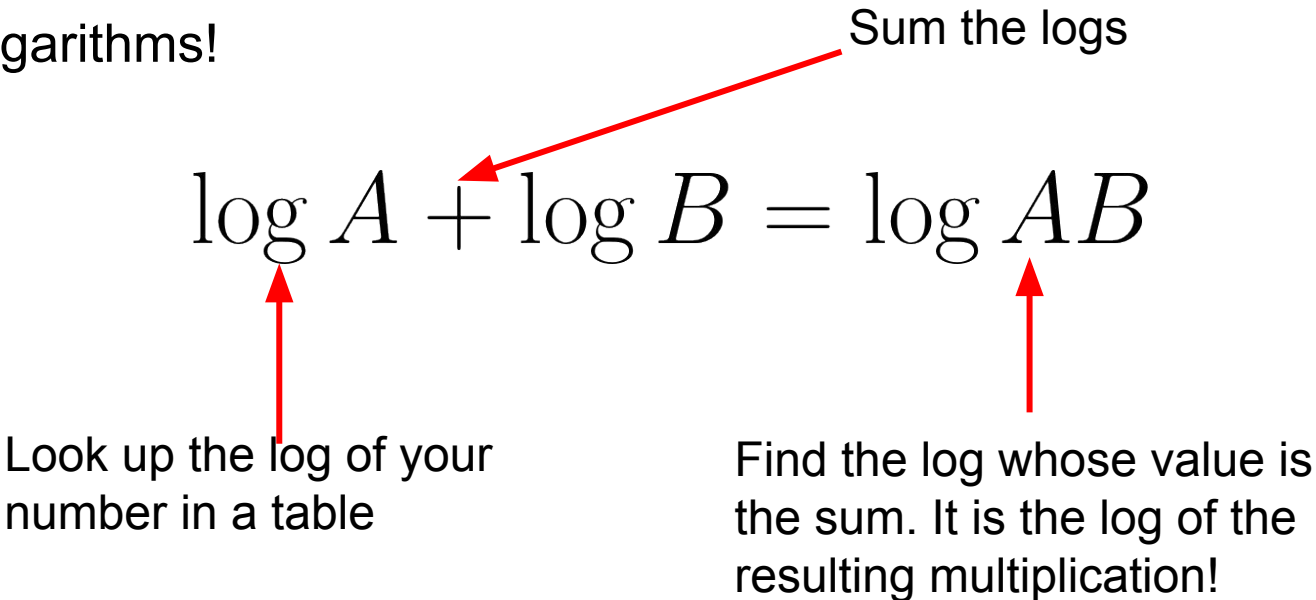


Key ideas to make it actually usable

- Instead of letting the user do the hard work of exponentiation and trying to guess logarithms, they get access to a table (as always)
- Choosing a smaller base lets the user have a finer choice of numbers to choose from (e.g. there are almost 7000 intervals between 1 and 1024 if we choose 1.001 as base instead of 2 – which has 10 intervals)

Multiplying

- As important as... multiplication
- Logarithms!



Sum the logs

$$\log A + \log B = \log AB$$

Look up the log of your
number in a table

Find the log whose value is
the sum. It is the log of the
resulting multiplication!

Advantages of logarithms

- For division, just subtract the log (no need for a second table)
- You are always summing straightforward numbers
- No division, just one very simple sum
- Can easily calculate powers, they just become multiplications

For these reasons, logarithms were ubiquitous in maths until the 70s!!!

Quirks about Napier's logarithm

- Napier used log base $1 - 10^{-7}$
 - This was because his definition made the velocity on the bottom line proportional to the distance to its end, not the origin
 - Having such a small base made it possible to have extremely fine-grained tables
- Also, his list computed the logs of numbers/ 10^7 , so that the values for the log were reasonable up to big numbers, and multiplied the log by 10^7 , so that he didn't have to use a decimal point
- In short: Napier's log = $10^7 \log_{1-10^{-7}}(x/10^7)$

Napier's table (edited from wikimedia)

Gr.	19				
mi.	Sinus	Logarithmi	Differentia	Logarithmi	Sinus
0	3255682	11221830	10661613	560217	9455186
1	3258432	11213386	10652167	561219	9454239
2	3261182	11204950	10642728	562222	9453291
3	3263931	11196522	10633296	563226	9452342
4	3266681	11188102	10623871	564231	9451392
5	3269439	11179690	10614453	565237	9450441
6	3272179	11171286	10605042	566244	9449490
7	3274927	11162889	10595637	567252	9448538
8	3277675	11154500	10586239	568291	9447585
9	3280423	11146119	10576849	569270	9446631
10	3283171	11137746	10567466	570280	9445676
11	3285918	11129381	10558090	571291	9444720
12	3288665	11121024	10548721	572303	9443764
13	3291412	11112675	10539359	573316	9442807
14	3294159	11104334	10530004	574330	9441849
15	3296906	11096000	10520655	575345	9440890
16	3299652	11087674	10511313	576361	9439931
17	3302398	11079356	10501997	577370	9438971
18	3305144	11071046	10492648	578398	9438010
19	3307889	11062744	10483326	579418	9437048
20	3310634	11054449	10474010	580439	9436085
21	3313379	11046162	10464702	581460	9435122
22	3316123	11037883	10455401	582482	9434158
23	3318867	11029612	10446107	583505	9433193

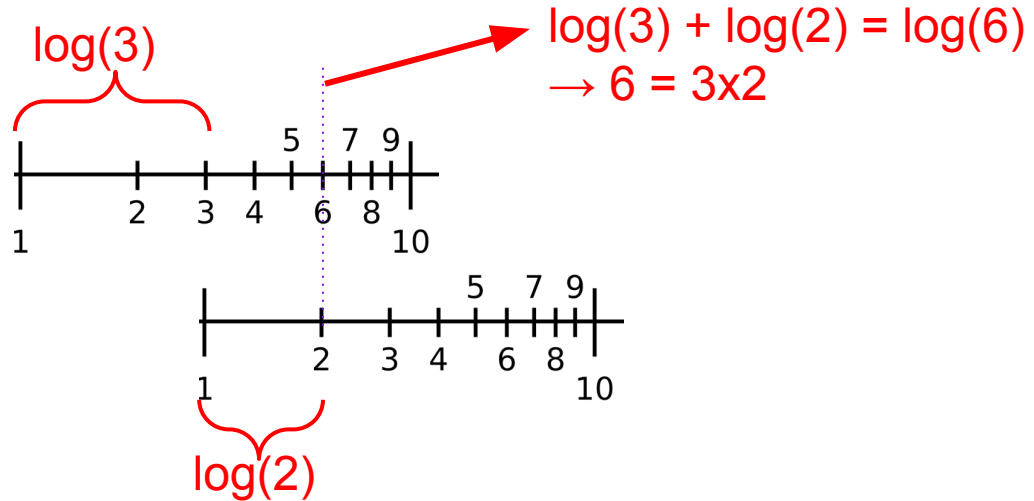
Logarithm table from 1964 (cf. wikipedia)

The image shows an open logarithm table from 1964. The left page is titled "ELEMENTARY TRANSCENDENTAL FUNCTIONS" and "COMMON LOGARITHMS". It contains a large table of logarithmic values for various numbers, including a section for "Table A.1" at the bottom. The right page is titled "ELEMENTARY TRANSCENDENTAL FUNCTIONS" and "COMMON LOGARITHMS". It contains a large table of logarithmic values for various numbers, including a section for "Table A.1" at the bottom. The tables are organized into columns and rows, with numbers and their corresponding logarithmic values listed. The pages are numbered 96 and 97.

Table A.1	
x	log ₁₀ x
1.00	0.0000
1.01	0.0043
1.02	0.0086
1.03	0.0128
1.04	0.0170
1.05	0.0212
1.06	0.0253
1.07	0.0294
1.08	0.0335
1.09	0.0376
1.10	0.0418
1.11	0.0459
1.12	0.0500
1.13	0.0541
1.14	0.0582
1.15	0.0623
1.16	0.0664
1.17	0.0705
1.18	0.0746
1.19	0.0787
1.20	0.0828
1.21	0.0869
1.22	0.0910
1.23	0.0951
1.24	0.0992
1.25	0.1033
1.26	0.1074
1.27	0.1115
1.28	0.1156
1.29	0.1197
1.30	0.1238
1.31	0.1279
1.32	0.1320
1.33	0.1361
1.34	0.1402
1.35	0.1443
1.36	0.1484
1.37	0.1525
1.38	0.1566
1.39	0.1607
1.40	0.1648
1.41	0.1689
1.42	0.1730
1.43	0.1771
1.44	0.1812
1.45	0.1853
1.46	0.1894
1.47	0.1935
1.48	0.1976
1.49	0.2017
1.50	0.2058
1.51	0.2099
1.52	0.2140
1.53	0.2181
1.54	0.2222
1.55	0.2263
1.56	0.2304
1.57	0.2345
1.58	0.2386
1.59	0.2427
1.60	0.2468
1.61	0.2509
1.62	0.2550
1.63	0.2591
1.64	0.2632
1.65	0.2673
1.66	0.2714
1.67	0.2755
1.68	0.2796
1.69	0.2837
1.70	0.2878
1.71	0.2919
1.72	0.2960
1.73	0.3001
1.74	0.3042
1.75	0.3083
1.76	0.3124
1.77	0.3165
1.78	0.3206
1.79	0.3247
1.80	0.3288
1.81	0.3329
1.82	0.3370
1.83	0.3411
1.84	0.3452
1.85	0.3493
1.86	0.3534
1.87	0.3575
1.88	0.3616
1.89	0.3657
1.90	0.3698
1.91	0.3739
1.92	0.3780
1.93	0.3821
1.94	0.3862
1.95	0.3903
1.96	0.3944
1.97	0.3985
1.98	0.4026
1.99	0.4067
2.00	0.4108

Slide rulers

- Essentially two logarithmic scales atop each other, functionally equivalent to a table

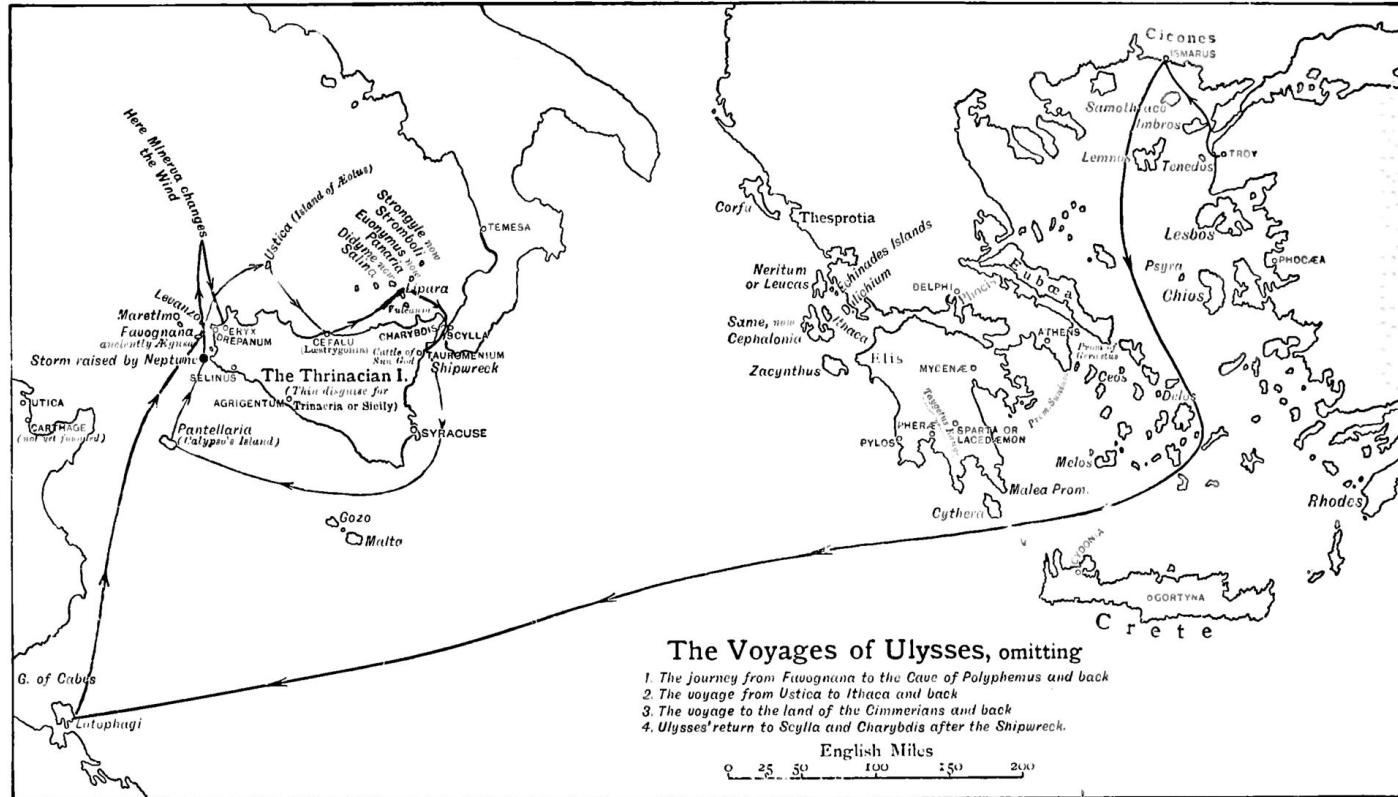


Lessons from logarithms

- Sometimes the first version of an idea is highly convoluted, like Napier's, but gradually simplifies over time
- Looking up (partial) results of calculations is extremely fast

Computers still often use look-up tables to speed computation, like the CORDIC algorithm used to calculate trigonometric functions

Your turn: Odysseus's trip



List of places Odysseus visited

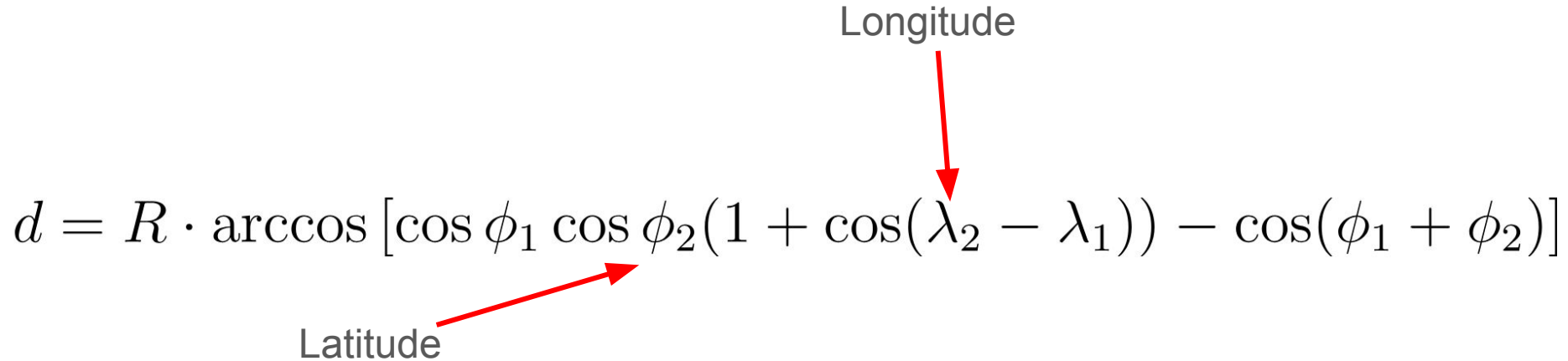
Place	▼	#	Latitude	▼	#	Longitude	▼
The Island of the Lotus-Eaters			34.7			11.3	
The Island of the Cyclopes			37.75			14.0167	
Aeolia			38.4833			14.95	
The Laestrygonians			40			9	
Aeaea			41.2522			13.1032	
The Underworld			37			-8	
The Sirens' Island			40.55			14.2167	
Scylla and Charybdis			38.25			15.6333	

Distance between two points on a sphere

$$d = R \cdot \arccos [\cos \phi_1 \cos \phi_2 (1 + \cos(\lambda_2 - \lambda_1)) - \cos(\phi_1 + \phi_2)]$$

Longitude

Latitude

The diagram shows the spherical distance formula with two red arrows pointing to specific terms. One arrow, labeled 'Longitude', points to the term $\cos(\lambda_2 - \lambda_1)$ within the formula. Another arrow, labeled 'Latitude', points to the term $\cos \phi_2$ within the formula.

Let's see who is the fastest! (earth radius: 6378 km)

Place	▼	#	Latitude	▼	#	Longitude	▼
The Island of the Lotus-Eaters				34.7		11.3	
The Island of the Cyclopes				37.75		14.0167	
Aeolia				38.4833		14.95	
The Laestrygonians				40		9	
Aeaea				41.2522		13.1032	
The Underworld				37		-8	
The Sirens' Island				40.55		14.2167	
Scylla and Charybdis				38.25		15.6333	

$$d = R \cdot \arccos [\cos \phi_1 \cos \phi_2 (1 + \cos(\lambda_2 - \lambda_1)) - \cos(\phi_1 + \phi_2)]$$

References

<https://math.stackexchange.com/questions/47927/motivation-for-napiers-logarithms>

<https://math.stackexchange.com/questions/469074/how-was-e-first-calculated/469088#469088>

https://en.wikipedia.org/wiki/Slide_rule

<https://en.wikipedia.org/wiki/Logarithm>

https://en.wikipedia.org/wiki/History_of_logarithms