Vector Calculus Notes

Gabriel Ravacci

Fall - Winter 2022

Contents

Chapter 1

Line Integrals

1.1 Vector Fields

Definition 1. A vector field is defined as a function that

- In the plane, assigns each point (x, y) in a subset ω a two component vector $\mathbf{v}$.
- In space, assigns each point (x, y, z) in a subset Ω a three component vector $\mathbf{v}$.

Definition 2. A vector field is said to be **conservative** if there exists some function φ s.t.

$$\mathbf{F} = \boldsymbol{\nabla}\varphi$$

and φ is called the **potential** function.

Theorem 1. Screening test for convervative vector fields

- In $\mathbb{R}^2$ for some $\mathbf{F} = F_1(x, y)\hat{\boldsymbol{\imath}} + F_2(x, y)\hat{\boldsymbol{\jmath}}$, $\mathbf{F}$ is conservative if:

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$$

- In $\mathbb{R}^3$, for some $\mathbf{F} = F_1(x, y, z)\boldsymbol{\imath} + F_2(x, y, z)\boldsymbol{\jmath} + F_3(x, y, z)\boldsymbol{k}$,$\mathbf{F}$ is conservative if:

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x} \qquad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x} \qquad \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}$$

1.2 Line Integrals

Definition 3. Denote by C the parametrized path $\mathbf{r}(t)$ with $t_0 \leq t \leq t_1$. Then

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_C (\mathbf{F_1}\mathrm{d}x + \mathbf{F_2}\mathrm{d}y + \mathbf{F_3}\mathrm{d}z) = \int_{t_0}^{t_1} \mathbf{F}(\mathbf{r}(t)) \cdot \frac{\mathrm{d}\mathbf{r}}{\mathrm{t}}(t)\mathrm{d}t$$

Remark. Note that if C is a closed path, we may write: $\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$

Theorem 2. Let $\mathbf{F}$ be a continuous and defined vector field on all of $\mathbb{R}^2$ or $\mathbb{R}^3$. Then:

1. $\mathbf{F}$ is conservative.
2. $\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 0$
3. $\int \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ is path indepedentt. That is, for any two curves C_1,C_2 that start at P_0 and end at P_1, $\int_{C_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_{C_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$

If any of these are true, all three are true.

Chapter 2

Surface Integrals

2.1 Parametrized Surfaces

There are three common ways to specify a functio in $\mathbb{R}^3$:

1. Explicitly: $z = f(x, y)$ where $(x, y) \in \mathcal{D} \subset \mathbb{R}^2$
2. Implicitly: $G(x, y, z) = K$
3. By range of function: $\mathbf{r} : \mathcal{D} \subset \mathbb{R}^2 \to \mathbb{R}^3$ where each $(u, v) \in \mathcal{D} \mapsto \mathbf{r}(u, v) = (x(u, v), y(u, v), z(u, v))$.

Example. The unit sphere $G(x, y, z) = x^2 + y^2 + z^2 = 1$ can be parametrized as

$$\mathbf{r}(\theta, \varphi) = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$$

with $\theta \in [0, 2\pi)$ and $\varphi \in (0, \pi)$.

2.2 Tangent Planes

Theorem 3. Normal vectors to surfaces

- Let $\mathbf{r} : \mathcal{D} \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a parametrized surface and let $(x_0, y_0, z_0) = \mathrm{r}(u_0, v_0)$ be a point on the surface. Then

$$\begin{aligned} \mathrm{T}_u &= \frac{\partial \mathbf{r}}{\partial u}(u_0, v_0) \\ \mathbf{T}_v &= \frac{\partial \mathbf{r}}{\partial v}(u_0, v_0) \\ \mathbf{n} &= \mathbf{T}_u \times \mathbf{T}_v \end{aligned}$$

  is normal to the surface.

- Let $G(x, y, z) = K$ be a surface and let (x_0, y_0, z_0) be a point on the surface. Then

$$\mathbf{n} = \nabla G(x_0, y_0, z_0)$$

is normal to the surface.

2.3 Surface Integrals

Theorem 4. For a parametrized surface $\mathbf{r}(u, v)$

$$\hat{\mathbf{n}} \mathrm{d}S = \pm \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \mathrm{d}u \mathrm{d}v$$
$$\mathrm{d}S = \left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right| \mathrm{d}u \mathrm{d}v$$

Remark. Note that the $\pm$ is because there are two unit normal vectors corresponding to the inside and outside of the surface.

Corollary. For a surface $z = f(x, y)$

$$\hat{\mathbf{n}} \mathrm{d}S = \pm \left(-f_x \boldsymbol{\imath} - f_y \boldsymbol{\jmath} + \boldsymbol{k} \right) \mathrm{d}x \mathrm{d}y$$
$$\mathrm{d}S = \sqrt{1 + f_x^2 + f_y^2} \mathrm{d}x \mathrm{d}y$$

Proof. We may parametrize a surface given by $z = f(x, y)$ as

$$\mathbf{r}(x, y) = x\boldsymbol{\imath} + y\boldsymbol{\jmath} + f(x, y)\boldsymbol{k}$$

then

$$\frac{\partial \mathbf{r}}{\partial x} = \boldsymbol{\imath} + f_x \boldsymbol{k}$$
$$\frac{\partial \mathbf{r}}{\partial y} = \boldsymbol{\jmath} + f_y \boldsymbol{k}$$
$$\hat{\mathbf{n}} = \frac{\partial \mathbf{r}}{\partial x} \times \frac{\partial \mathbf{r}}{\partial y} = \det \begin{bmatrix} \boldsymbol{\imath} & \boldsymbol{\jmath} & \boldsymbol{k} \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{bmatrix} = -f_x \boldsymbol{\imath} - f_y \boldsymbol{\jmath} + \boldsymbol{k}$$

□

And from this result:

Corollary. For a surface $G(x, y, z) = K$, then

$$\hat{\mathbf{n}} \mathrm{d}S = \pm \frac{\nabla G}{\nabla G \cdot \boldsymbol{k}} \mathrm{d}x \mathrm{d}y$$
$$\mathrm{d}S = \left| \frac{\nabla G}{\nabla G \cdot \boldsymbol{k}} \right| \mathrm{d}x \mathrm{d}y$$

and holds for $\mathrm{d}x\mathrm{d}z$ and $\mathrm{d}y\mathrm{d}z$.

Definition 4. We define a surface integral to be

$$\iint_S \rho \mathrm{d}S$$

which gives the value of a function ρ across the surface. If we let $\rho = 1$ we get the surface are, that is:

$$A_S = \iint_S \mathrm{d}S$$

2.4 Flux Integrals

Definition 5. We define a flux integral to be

$$\iint_S \mathbf{F} \cdot \hat{\mathbf{n}} \mathrm{d}S$$

which describes the rate at which some vector field $\mathbf{F}$ "flows" or crosses through a surface S.

Lemma 1. Let a fluid have density described by $\rho(x, y, z, t)$ and velocity described by $\mathbf{v}(x, y, z, t)$. Then, the rate at which it is crossing through a surface S is

$$\Phi = \iint_S \rho \mathbf{v} \cdot \hat{\mathbf{n}} \mathrm{d}S$$

where $\hat{\mathbf{n}}(x, y, z)$ is a unit normal vector to S. If this is positive the fluid is crossing opposite to the normal.

Chapter 3

Grad, Div and Curl

Definition 6. We informally define the **del** ∇ operator as

$$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$$

We may apply this operator to functions in $\mathbb{R}^3$ as follows.

Definition 7. We define

a. the **gradient** of a scalar function $f(x,y,z)$ is the vector field

$$\operatorname{grad} f = \nabla f = \frac{\partial f}{\partial x}\boldsymbol{\imath} + \frac{\partial f}{\partial y}\boldsymbol{\jmath} + \frac{\partial f}{\partial z}\boldsymbol{k}$$

b. The **divergence** of a vector field $\mathbf{F}(x,y,z)$ is the scalar function

$$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

c. The **curl** of a vector field $\mathbf{F}(x,y,z)$ is the vector function

$$\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \det \begin{bmatrix} \boldsymbol{\imath} & \boldsymbol{\jmath} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{bmatrix}$$

Chapter 4

Integral Theorems

4.1 Divergence Theorem

First, some definitions:

Definition 8.
a. A surface is **smooth** if it has a parametrization $\mathbf{r}(u, v)$ with continuous partial derivative $\frac{\partial \mathbf{r}}{\partial u}$ and $\frac{\partial \mathbf{r}}{\partial \mathbf{v}}$ and their cross product is nonzeron.

b. A surface is **piecewise smooth** if it is composed of multiple smooth surfaces.

which leads us into

Theorem 5 (Divergence Theorem)**.** Let V be a bounded solid with a piecewise smooth surface ∂V and let $\mathbf{F}$ be a vector field that has continous first partial derivatives in V. Then

$$\iint_{\partial V} \mathbf{F} \cdot \hat{\mathbf{n}} \mathrm{d}S = \iiint_V \boldsymbol{\nabla} \cdot \mathbf{F} \mathrm{d}V$$

4.2 Green's Theorem

First we need some definitions:

Definition 9. A curve C with parametrization $\mathbf{r}(t), a \leq t \leq b$, is **closed** if $\mathbf{r}(a) = \mathbf{r}(b)$.

Definition 10. A curve C is **simple** if it does not cross itself.

Definition 11. A curve C is **piecewise smooth** if it has a parametrization $\mathbf{r}(t)$ which is continuous, differentiable and the derivative is also continous and nonzero.

Theorem 6 (Green's Theorem). Let R be a finite region in the xy-plane and let C bound R and consist of finite number of simple, closed and piecewise smooth curves that are oriented consitently with R. Then let F_1 and F_2 have continuous first partial derivative in R. Then

$$\oint_C [F_1(x,y)\mathrm{d}x + F_2(x,y)\mathrm{d}y] = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) \mathrm{d}x\mathrm{d}y$$

it is often more useful to define it as

$$\oint_C \langle P, Q\rangle \cdot \mathrm{d}\mathbf{r} = \iint_R \left(Q_x - P_y\right) \mathrm{d}x\mathrm{d}y$$

Corollary. Consider a region R and a curve C such that Green's Theorem applies, then

$$\mathrm{Area}(R) = \frac{1}{2}\oint_C [x\mathrm{d}y - y\mathrm{d}x]$$

Example. Let us compute the area of the circle $x^2 + y^2 \leq a^2$ using Green's Theorem. We parametrize it as

$$\mathbf{r}(t) = a\cos t\,\imath + a\sin t\,\jmath$$

with $0 \leq t \leq 2\pi$. Then

$$\begin{aligned}
A &= \iint_R \mathrm{d}x\mathrm{d}y \\
&= \frac{1}{2}\oint_C [x\mathrm{d}y - y\mathrm{d}x] \\
&= \frac{1}{2}\int_0^{2\pi} (a\cos t)(a\cos t) - (a\sin t)(-a\sin t)\mathrm{d}t \\
&= \frac{1}{2}\int_0^{2\pi} a^2\cos^2 t + a^2\sin^2 t\,\mathrm{d}t \\
&= \frac{1}{2}\int_0^{2\pi} a^2\mathrm{d}t \\
&= \frac{1}{2}a^2\int_0^{2\pi} \mathrm{d}t \\
&= \frac{1}{2}a^2(2\pi) \\
&= \pi a^2
\end{aligned}$$

4.3 Stoke's Theorem

Definition 12 (Stoke's Theorem)**.** Let S be a piecewise smooth oriented surface whose boundary (∂S) also consits of a finite number of piecewise, smooth, simple curves oriented consitently with $\hat{\mathbf{n}}$. Such that if you walk along ∂S, $\hat{\mathbf{n}}$ points upwards and S is on your left. And let $\mathbf{F}$ be a vector field that has continuous first partial derivatives in S. Then

$$\oint_{\partial S} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \iint_{S} \boldsymbol{\nabla} \times \mathbf{F} \cdot \hat{\mathbf{n}} \mathrm{d}S$$