1 Differential Forms

In $\mathbb{R}^3$ differential forms can be defined as:

- A 0-form is a function $f(x, y, z)$.
- A 1-form is of the form
$$F_1 \mathrm{d}x + F_2 \mathrm{d}y + F_3 \mathrm{d}z$$
- A 2-form is of the form
$$F_1 \mathrm{d}y \wedge \mathrm{d}z + F_2 \mathrm{d}z \wedge \mathrm{d}x + F_3 \mathrm{d}x \wedge \mathrm{d}y$$
- A 3-form is of the form
$$f(x, y, z) \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$$

Differential forms behave somewhat as expected with addition.

1.1 Multiplication of Differential Forms

Multiplication of differential forms is **noncommutative** (but it is associative). Multiplication by 0-forms (i.e. functions) behaves as you would expect and is usually written without the $\wedge$.
Let ω be a k-form and ω' be a k'-form. Their product is a $(k + k')$-form and is denoted by $\omega \wedge \omega'$. It behaves as follows:

a. $\omega \wedge \omega'$ is linear. That is, $\omega = f_1\omega_1 + f_2\omega_2$ then
$$(f_1\omega_1 + f_2\omega_2) \wedge \omega' = f_1(\omega_1 \wedge \omega') + f_2(\omega_2 \wedge \omega')$$

b. $\omega \wedge \omega' = (-1)^{kk'} \omega' \wedge \omega$

c. $\omega \wedge \omega = 0$

1.1.1 Addendum: The Hodge Star operator

We define the Hodge Star $(*)$ operation on a form ω of $k \geq 2$in $\mathbb{R}^n$ as follows
$$\omega \wedge *\omega = \Omega_n$$
where Ω_n is the complete $n = k$-form. An example makes this clearer. In $\mathbb{R}^3$:
$$*\mathrm{d}x = \mathrm{d}y \wedge \mathrm{d}z$$

1.2 Differentiation of Differential Forms

One of the main reasons differential forms are so useful is that they very nicely with differentiation. Let ω be a k-form, then
$$\mathrm{d} := \omega_k \to \omega_{k+1}$$
Still working within the scope of $\mathbb{R}^3$, consider a 0-form f. Then
$$\mathrm{d}f = \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y + \frac{\partial f}{\partial z}\mathrm{d}z$$
Is the equivalent $(k+1)$-form after differentation. Note that this is an edge case, for all other k-forms, the differential operator behaves as follows:

1. Apply d to coefficients;
2. Expand with the wedge product.

1.2.1 Example:

Let ω be a 1-form. Then
$$\begin{aligned}
\omega &= P\mathrm{d}x + Q\mathrm{d}y + R\mathrm{d}z \\
\mathrm{d}\omega &= \mathrm{d}P \wedge \mathrm{d}x + \mathrm{d}Q \wedge \mathrm{d}y + \mathrm{d}R \wedge \mathrm{d}z \\
&= (P_x\mathrm{d}x + P_y\mathrm{d}y + P_z\mathrm{d}z) \wedge \mathrm{d}x + \\
&+ (Q_x\mathrm{d}x + Q_y\mathrm{d}y + Q_z\mathrm{d}z) \wedge \mathrm{y} + \\
&+ (R_x\mathrm{d}x + R_y\mathrm{d}y + R_z\mathrm{d}z) \wedge \mathrm{d}z = \\
&= P_y\mathrm{d}y \wedge \mathrm{d}x + P_z\mathrm{d}z \wedge \mathrm{d}x + \\
&+ Q_x\mathrm{d}x \wedge \mathrm{d}y + Q_z\mathrm{d}z \wedge \mathrm{d}y + \\
&+ R_x\mathrm{d}x \wedge \mathrm{d}z + R_y\mathrm{d}y \wedge \mathrm{d}z = \\
&= (R_y - Q_z)\mathrm{d}y \wedge \mathrm{d}z + (P_z - R_x)\mathrm{d}z \wedge \mathrm{d}x + (Q_x - P_y)\mathrm{d}x \wedge \mathrm{d}y
\end{aligned}$$
Which corresponds to the **curl** operation.

1.3 Integration of Differential Forms

A k-form may be integrated over a k-dimensional manifold. Or, in simpler terms: We may integrate k times a k-form. For example, we integrate a 1-form over a 1-manifold C (curve):
$$\int_C P\mathrm{d}x + Q\mathrm{d}y + R\mathrm{d}z$$
Similarly, the integral of a 3-form over a 3-manifold (solid) is just:
$$\iiint_{\mathcal{V}} f\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$$
Where the $\wedge$'s are usually omitted.

1.3.1 Parametrization

Consider a parametrization of a surface $\mathcal{S}$:
$$\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle$$
Then we can change the basis of our differential forms using the definition of the differential operator:
$$\mathrm{d}x = \frac{\partial x}{\partial u}\mathrm{d}u + \frac{\partial x}{\partial \mathbf{v}}\mathrm{d}v = x_u\mathrm{d}u + x_v\mathrm{d}v$$
and similarly for y, z. Then for 2-form ω defined as
$$\omega = P\mathrm{d}y \wedge \mathrm{d}z + Q\mathrm{d}z \wedge \mathrm{d}x + R\mathrm{d}x \wedge \mathrm{d}y$$
notice that
$$\begin{aligned}
\mathrm{d}x \wedge \mathrm{d}y &= (x_u\mathrm{d}u + x_v\mathrm{d}v) \wedge (y_u\mathrm{d}u + y_v\mathrm{d}v) \\
&= (x_u y_v - x_v y_u)\mathrm{d}u \wedge \mathrm{d}v
\end{aligned}$$

1.4 Generalized Stoke's Theorem

Let ω be a k-form and Ω be a $(k + 1)$ manifold. Then
$$\int_\Omega d\omega = \int_{\partial\Omega} \omega$$