# PHYS 158: Electromagnetism

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## 1 Waves and Interference

¡write some introduction here¿

In this course, our goal is to use a simplified model of two waves travelling in the SAME direction, and given their path difference, determine if they are constructive or destructive. There are many expressions for this, but the most common and easiest one to use is called the argument (arg) method, written by:

$$\boxed{\Delta\text{arg} = k\Delta r + \Delta\phi} \tag{1}$$

Where $\Delta r$ and $\phi$ are the path difference and phase difference of the two waves respectively. If:

1. $\Delta\text{arg} = 2\pi n$ where $n = 0, 1, 2, ...$, the two waves constructively interfere
2. $\Delta\text{arg} = n_{odd}\pi$ where $n = 1, 3, 5, ..$, the two waves destructively interfere

Though we won't go over the steps, the derivation of the argument method is as follows:

1. Recall the 157 definition of travelling waves $y(x,t) = A\sin(kx \pm wt + \phi)$, where the $\pm$ indicates the direction (right or left) that the wave is travelling.
2. $y_t = y_1 + y_2$, where $y_x$ is an expression for one of the two travelling waves (remember to set the same direction of the waves)
3. Use the identity $A\sin(a) \pm A\sin(b) = 2A\sin(\frac{a\pm b}{2})\cos(\frac{a\mp b}{2})$
4. Cancel terms until your cosine expression is not dependant on time - this is our amplitude modulation term of the superposition wave.
5. Observe when $\cos(\Delta\text{arg}) = 0 or 1$, build two cases for the argument within to obtain the two cases for constructive / destructive interference

We recommend working through this yourself and convincing yourself that this is true. This argument expression is the only one we'll need to use for the entirety of (sound and light) waves in PHYS 158. With this in our toolbox, let's look at a couple of applications.

## 2 Electrostatics

Every atom in the universe is composed by protons and electrons, which have charge. The attractive forces holding these particles together is the electric force. It is defined by **Coulomb's Law**:[1]

$$\boxed{\mathbf{F} = k\frac{q_1 q_2}{{r_{12}}^2}\hat{\mathbf{r}}_{12}} \tag{2}$$

Experimentally, we have determined that the electric force obeys the **superposition principle**, that is, the total force on a charge $Q$ is the vector sum of the force exerted by all other charges individually:

$$\mathbf{F}_R = \mathbf{F}_1 + \mathbf{F}_2 + \dots \tag{3}$$

## 2.1 The electric field

A simpler but more effective definition of the electric force comes from the definition of a vector field, where each point is defined by the electric force $\vec{F}$ exerted by the particle creating the electric field, thus:

$$\boxed{\mathbf{E} = \frac{\mathbf{F}}{q_0}} \tag{4}$$

Where $q_0$ is a "test charge", usually positive. As an extension of the electric force $\mathbf{F}$, it naturally also obeys the **superposition principle**:

$$\mathbf{E}_R = \mathbf{E}_1 + \mathbf{E}_2 + \ldots \tag{5}$$

Now we consider charged objects, composed of many point particles. Intuitively, the way to describe the electric field caused by such objects is by defining an infinitesimally small portion of charge $dQ$ and integrating:

$$\mathbf{E} = \int \frac{kdQ}{r^2}\hat{\mathbf{r}} \tag{6}$$

Do note that it is not unusual for $dQ$ to become something else (so you can actually evaluate the definite integral). It is common to attribute it to uniform charge distributions.

## 2.2 Electric Flux

We now define the concept of **electric flux**. Flux (represented by $\Phi$) can be interpreted as the "amount" of vector lines passing through a surface. To help visualize this notion, it is useful to imagine "flux" as the flow of a fluid, through an area. Represented mathematically by:

$$\Phi_E \equiv \mathbf{E} \cdot d\mathbf{A} \tag{7}$$

This means the "flux" of electric field going through an area is the dot product of the field vector lines and the area vector $\hat{\mathbf{n}}A$. In other words, flux is higher when the field lines are perpendicular to an area and lower at an angle (eventually 0 when the field is parallel to that area).

## 2.3 Gauss's Law

Now, we consider the flux of electric field through a **closed** surface. This is **Gauss's Law**, which states that:

$$\boxed{\Phi_{E,S} \equiv \oiint_S \mathbf{E} \cdot d\mathbf{A} = \frac{Q_{enc}}{\epsilon_0}} \tag{8}$$

This means that **flux** through a closed surface can be calculated by taking the surface integral of the electric field through that surface, which is equal to the charge enclosed that surface, divided by the permittivity of free space. This is an extremely powerful relationship, as it means that given an electric field we could calculate charge and vice-versa. This is particularly potent when the electric field caused by the charge arrangement is symmetric in some way to the Gaussian surface, because then:

$$\oiint_S \mathbf{E} \cdot d\mathbf{A} = EA = \frac{Q_{enc}}{\epsilon_0}$$

Which is beautifully simple.
Usually we look for three particular symmetries when choosing our Gaussian surface:

1. Spherical symmetry: A sphere makes sense as a Gaussian surface. In fact, with this we can derive **Coulomb's Law**.
2. Coaxial symmetry: Symmetry across an axis, which makes a cylinder a great choice.
3. Surface symmetry: The electric field is symmetrical over a surface, making a cylinder a great choice (again). A rectangular box works too. in this case.

It is sometimes useful to denote charge density for objects, mainly:

1. Linear charge density: $\lambda = \frac{Q}{\ell}$
2. Surface charge density: $\sigma = \frac{Q}{A}$
3. Volume charge density: $\rho = \frac{Q}{V}$

These are relevant when you use a Gaussian surface **inside** the desired charged object, where you'll be enclosing a fraction of its total charge.

## 2.4 Electric potential

Let's refresh our memory a bit. In classical mechanics we defined potential energy as the **work** done by a force, along a path:

$$W_{a\to b} = \int_a^b \mathbf{F} \cdot d\mathbf{l} \tag{9}$$

Which, in turn is the change in **potential** energy, $\Delta U \equiv U_b - U_a$. This also naturally applies to charged particles moving in space:

$$U_b - U_a = -\int_a^b k\frac{q_1 q_2}{{r_{12}}^2}\hat{\mathbf{r}} \cdot d\mathbf{r} \tag{10}$$

Note that we use $d\mathbf{r}$ for our path integral because the electric force is **conservative** (the path doesn't matter, only the final results), so it's simpler to take the straight path.[2] Therefore:

$$\begin{aligned} U_b - U_a &= -\Big( -k\frac{q_1 q_2}{r_{12}}\Big|_a^b \Big) \\ &= kq_1q_2\Big(\frac{1}{b} - \frac{1}{a}\Big) \end{aligned}$$

Charged particles will naturally move to points with less potential.
We can extend this definition (and it is often more useful to do so) to electric fields, by dividing by the charge. This is called **electric potential**:

$$\boxed{V_b - V_a = -\int_a^b \mathbf{E} \cdot d\mathbf{l}} \tag{11}$$

And thus electric potential is connected to potential energy by:

$$\Delta U = Q\Delta V$$

One may note that potential is a always a relationship between two points on the electric field. So when we measure potential, we always do so with a reference point. We can *usually*[3] take $V(\infty) = 0$, which sets a nice "ground" level. From Eq. 10, we can develop a nice simple equation for potential of point charges:

$$\begin{aligned} V_b - V_\infty &= -\int_\infty^b \mathbf{E} \cdot d\mathbf{l} \\ V_b - 0 &= -\int_\infty^b \frac{kQ}{r^2}\hat{\mathbf{r}} \cdot d\mathbf{r} = -\int_\infty^b k\frac{Q}{r^2}dr \\ V_b &= -\Big( -k\frac{Q}{r}\Big|_\infty^b \Big) = -\Big( -k\frac{Q}{b} + k\frac{Q}{\infty}\Big) \end{aligned}$$

$$V_b = k\frac{Q}{b} \tag{12}$$

Potential at a point a distance $b$ from the source charge.[4]. Note that this equation is very simple, and this is for our benefit. The fundamental theorem of gradients states that:

$$
\begin{aligned}
V(b) - V(a) &= \int_a^b (\nabla V) \cdot d\mathbf{l} \\
-\int_a^b \mathbf{E} \cdot d\mathbf{l} &= \int_a^b (\nabla V) \cdot d\mathbf{l} \\
\mathbf{E} &= -\nabla V
\end{aligned} \tag{13}
$$

The gradient is the **spatial derivative** of a space-dependent function:

$$
\nabla V = \frac{\partial V}{\partial x}\hat{\mathbf{i}} + \frac{\partial V}{\partial y}\hat{\mathbf{j}} + \frac{\partial V}{\partial z}\hat{\mathbf{k}}
$$

Which means the rate of change of potential defines the electric field.
In this course this is usually one dimensional, so we can take the "normal" derivative of potential to easily get the electric field.

## 2.5 Conductors

In an insulator, all electrons are attached to an atom, meaning presence or absence of charges do not affect its configuration. However, in conductors, there are free electrons, which move in the presence of a field. From this observation we can conclude that **E is 0**. The induced charges always move to the **surfaces**. As there is no **E**, it also follows from Eq. 12 that conductors are **equipotentials**.

**Charging by induction:** While charges in a conductor are free to move, conductors are still capable of being charged. When we approach a positively charged particle with charge $q$ to a conductor, it will feel an attractive force to the charge. Why? The particle *induces* a negative charge $-q$ on the surface of the conductor, creating the attractive force (by Coulomb's Law). This process is called **charging by induction**.

## 2.6 Capacitors

A specific arrangement of charged conductors with opposite (but equal) charges is what's called a **capacitor**. The opposite charged conductors create an electric field between them, which in turn means there's a potential drop between them. And all of that is proportional to the charge (by Eq. 10). This proportionality is called called **capacitance**, defined by:

$$
\boxed{C \equiv \frac{Q}{V}} \tag{14}
$$

Note that $C$ (measured in Farads) is a strictly positive quantity, so we may take $|\Delta V|$.
The common circuit capacitor is composed of two parallel metallic plates, let's derive the formula for a parallel plate capacitor. Each plate with an area $A$ and a distance $d$ separating them. By Eq. 10:

$$
\begin{aligned}
\Delta V &= \int_a^b \mathbf{E} \cdot d\mathbf{l} = \mathbf{E}d = \frac{\sigma}{\epsilon_0}d = \frac{Q}{A\epsilon_0}d \\
C &= \frac{Q}{\Delta V} = \frac{Q}{\frac{Q}{A\epsilon_0}d} \\
&= \frac{A\epsilon_0}{d}
\end{aligned}
$$

Notice that the capacitance is exclusively dependent on geometry for the parallel plate capacitor. This may not apply for different types of capacitors (cylindrical, spherical) but the idea is the same.

# 3 Magnetostatics

# 4 Maxwell's Equations (Integral Form)

$$\oiint_S \mathbf{E} \cdot d\mathbf{a} = \frac{Q_{encl}}{\epsilon_0} \tag{15}$$

$$\oiint_S \mathbf{B} \cdot d\mathbf{a} = 0 \tag{16}$$

$$\oint_P \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi_B}{dt} \tag{17}$$

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{encl} + \mu_0 \epsilon_0 \frac{d\Phi_E}{dt} \tag{18}$$

[1] Where $k$ is the "permittivity of free space" constant $k = \frac{1}{4\pi\epsilon_0}$ and is equal to about $8.85 \cdot 10^{-12}$

[2] Assuming, of course, that our path is not pre-determined. Then we have to actually evaluate the path integral.

[3] Nested charged objects usually prevent this.

[4] Be careful with using the appropriate $b$. When condensing a larger sphere onto a point charge, the minimum distance (and highest potential) is **not** 0. It is still the radius of the system. Convince yourself that this is the case.