1. (5 points) A sample space consists of five events: E_1, E_2, E_3, E_4 and E_5. If $P(E_1) = 3P(E_2) = 0.3$. Find the probability of the remaining events if you know that the remaining events are equally probable (i.e., the remaining events have the same probability of occurrence).

Let S be the sample space. It is clear that

$$S = \{E_1, E_2, E_3, E_4, E_5\}$$

We are given that

$$P(E_1) = 3P(E_2) = 0.3 \quad \Rightarrow \quad P(E_1) = 0.3 \quad \text{and} \quad P(E_2) = 0.1$$

We also know that $P(E_3) = P(E_4) = P(E_5)$. Moreover,

$$P(E_1) + P(E_2) + P(E_3) + P(E_4) + P(E_5) = 1 \quad \Rightarrow \quad P(E_3) + P(E_4) + P(E_5) = 0.6$$

Therefore

$$P(E_3) = P(E_4) = P(E_5) = 0.2$$

2. Suppose two balanced coins (i.e., $P(\text{head}) = P(\text{tail}) = 0.5$) are tossed and the upper faces are observed.

   (a) (3 points) List the sample points for this experiment.

   Let S be the sample space. Let H and T denote heads and tails, respectively. Then

   $$S = \{HH, HT, TH, TT\}$$

   (b) (3 points) Assign a reasonable probability to each sample point. (Are the sample points equally likely?)

   Since we are working with balanced coins, it seems reasonable to assign the following probabilities to each of the events of the sample space:

   $$P(HH) = P(HT) = P(TH) = P(TT) = \frac{1}{4}$$

   All the sample points are equally likely.

   (c) (3 points) Let A denote the event that *exactly* one head is observed and B the event that *at least one* head is observed. List the sample points in both A and B.

   The events A and B are given by

   $$A = \{HT, TH\} \quad \text{and} \quad B = \{HT, TH, HH\}$$

(d) (5 points) From your answers to part (c), find $P(A), P(B), P(A \cap B)$, and $P(A \cup B)$.

From part (c), we know that

$$A = \{HT, TH\} \quad \Rightarrow \quad P(A) = P(HT) + P(TH) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

Also, we know that

$$B = \{HT, TH, HH\} \quad \Rightarrow \quad P(A) = P(HT) + P(TH) + P(HH) = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}$$

It is clear that

$$A \cap B = \{HT, TH\} \quad \Rightarrow \quad P(A \cap B) = P(HT) + P(TH) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

Finally

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{1}{2} + \frac{3}{4} - \frac{1}{2} = \frac{3}{4}$$

The End.