

Voronoi Diagrams and Nearest-Neighbor Search

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Abstract—The abstract goes here.

I. INTRODUCTION

Robotic motion planning is the problem of finding a path from the starting configuration of a robot to some goal configuration that does not collide with any obstacles. In the case where the positions of the obstacles are known *a priori* but the start and goal configurations are not, it can be useful to create a *roadmap* of the configuration space. Roadmaps demonstrate connected edges through the free space such that a robot can easily move to some edge from a starting configuration, and easily move to the goal configuration from some edge. If there is a path between the two edges in free space, the roadmap should have a path between them.

A. Voronoi Diagrams

Voronoi diagrams are a class of roadmap. The basic definition of a Voronoi diagram is that its edges represent the points that are equidistant from the closest obstacles to that point. This means that each edge represents the path that maintains the maximal separation from two obstacles while traveling between them. Following the Voronoi edge between two obstacles minimizes the chance of a collision with an obstacle, which is a desirable property when the transition dynamics for the environment are non-deterministic and the cost of a collision is high. Visually, a Voronoi diagram divides the free space into *Voronoi regions* of points that share a common closest obstacle.

Voronoi diagrams are theoretically defined in spaces of arbitrary dimension and for obstacles of any shape. However, the complexity of creating the diagram grows quickly with the dimensionality of the configuration space. Additionally, more complex obstacle shapes bring more complex edges on the diagram. In this paper, we focus on two-dimensional environments with obstacles that are points.

B. Nearest-Neighbors Classification

Voronoi diagrams have other uses beyond robot motion planning. One application, which we explore in this paper, is to use a Voronoi diagram to build a *k-Nearest-Neighbors (KNN) classifier*. KNN classifiers come from the field of machine

learning. The classification problem is as follows: given a *training set* of feature vectors with labels of either 0 or 1, produce a machine that, given a new feature vector T , will identify the k members of the training set with the closest feature vectors to T . The output label for T will be the class that the majority of the k nearest neighbors belong to. The metric used to define the “closest” feature vectors can vary based on the characterization of the available features, but the simplest way to measure it is to take the distance between the two points plotted in Euclidean space.

A complete Voronoi diagram can be used trivially as a KNN classifier with $k = 1$; finding the Voronoi region that a test sample belongs to is sufficient, as the nearest neighbor is the training sample associated with that region. For higher values of k , the classification problem is non-trivial. We explore how to implement such a system in this paper.

II. METHODS

A. Creating Voronoi Diagrams

The problem of an efficient algorithm for creating Voronoi diagrams has been thoroughly explored in the literature. One of the most prominent is the Sweepline algorithm introduced by Fortune [1]. Conceptually, the Sweepline algorithm moves a line (the eponymous Sweepline) through the plane containing the obstacles. In this paper, we will use the convention that the sweepline begins as a horizontal line at the obstacle with the highest y-value and moves downwards through the rest of the obstacles.

1) *Incremental Construction*: As we sweep down the plane, some obstacles will be above the sweepline and others will be below. The half-plane above the sweepline is then further separated into a region of points whose nearest obstacle we can be sure of, and a region whose points may be closer to an obstacle below the sweepline than to any obstacle above the sweepline. The finalized and unknown regions are separated by a *beach line*.

The beach line consists of a set of parabolic arcs joined together at the intersections of the arcs. An example beach line is shown in Figure 1. Each arc of the beach line has a corresponding obstacle that is the closest point to it. Geometrically, the arcs take the form of parabolas because each obstacle is the focus and the sweepline is the directrix of a parabola.

The points on the parabola are equidistant from the focus and the directrix.

The intersections of the beach line arcs are points that will be on Voronoi edges, equidistant from the obstacles corresponding to the two arcs meeting at that intersection. As the sweepline moves down, the arcs and the intersection do as well. A Voronoi edge is created by connecting the point where the intersection first occurred to the point where it disappears.

Computationally, it is impossible to move the sweepline down the plane in a continuous manner. We have to choose discrete points where the updates to the beach line will occur. Fortunately, there are a limited number of points that can change the structure of the beach line. We refer to these points as *obstacle events* and *circle events*.

2) *Obstacle Events*: Obstacle events occur when the sweepline encounters a new obstacle. That obstacle then must be inserted into the beachline. We locate the arc that the new obstacle is directly under and split it into two parts, one on either side of the new obstacle. At the moment when the obstacle is first encountered, it resides exactly on the beach line. This means that its “arc” in the beach line is actually given by a vertical line segment up to the existing arc that it splits. As the sweepline moves further down the plane, the arc widens and takes the usual parabolic curve.

In general, two concave-up parabolas given by $y = Ax^2 + Bx + C$ can have a varying number of points of intersection. If the two have equal first derivatives and some vertical offset, there are no points of intersection. If the two have equal first derivatives and some horizontal offset, there is one point of intersection. If the first derivatives are unequal, there are two points of intersection. If the first derivatives are equal and there is no offset, there are infinite points of intersection as the parabolas are identical.

Fortunately, we only need to consider the cases with one and two points of intersection. The case with only a vertical offset is impossible during the sweepline algorithm, as a vertical offset means that the first derivatives will be different. The case with infinite points of intersection can be averted by eliminating repeated obstacles before diagram construction. The problem that remains is finding the one or two intersection points.

The equation for parabolas intersections is given in [2] as

$$x = \frac{a_y b_x - \sqrt{a_y b_y ((a_y - b_y)^2 + b_x^2)}}{a_y - b_y} \quad (1)$$

However, this only gives one intersection.

$$x = \frac{a_y b_x \pm \sqrt{a_y b_y ((a_y - b_y)^2 + b_x^2)}}{a_y - b_y} \quad (2)$$

III. CONCLUSION

The conclusion goes here.

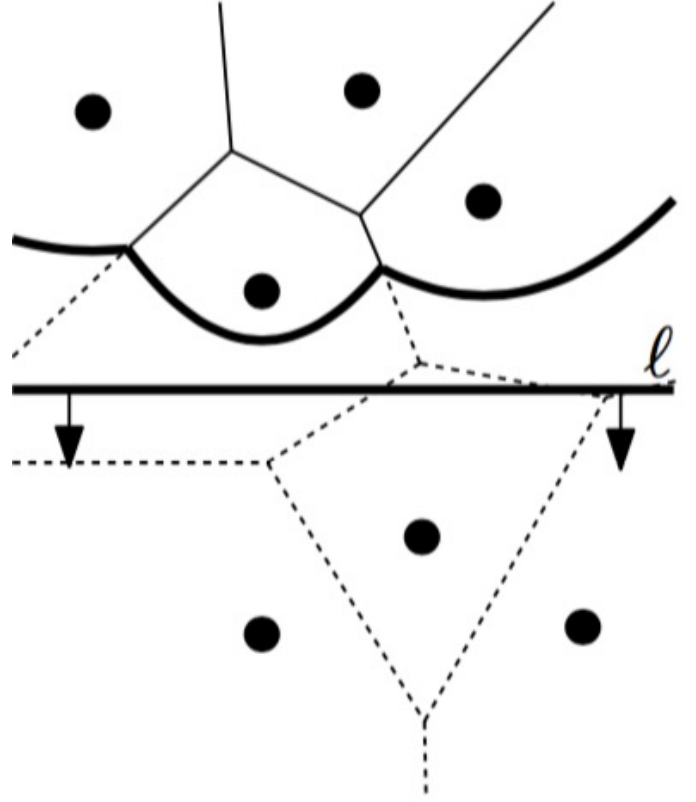


Fig. 1: Example of a beach line. The parabolic arcs are the beach line. The line segments above the beach line are finalized. The dashed line segments are the edges of the true Voronoi diagram that are not yet finalized. Taken from [3].

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