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Some basic sequences, generating functions and closed forms

Some simple sequences and their generating functions

sequence	gen. function	closed form
$\langle 1, 0, 0, 0, 0, 0, \ldots \rangle$	$\sum_{n \geqslant 0} [n = 0] z^n$	1
$\langle 0, \ldots, 0, 1, 0, 0, \ldots \rangle$	$\sum_{n \geqslant 0} [n = m] z^n$	z^m
$\langle 1, 1, 1, 1, 1, 1, \ldots \rangle$	$\sum_{n \geqslant 0} z^n$	$\frac{1}{1 - z}$
$\langle 1, -1, 1, -1, 1, -1, \ldots \rangle$	$\sum_{n \geqslant 0} (-1)^n z^n$	$\frac{1}{1 + z}$
$\langle 1, 0, 1, 0, 1, 0, \ldots \rangle$	$\sum_{n \geqslant 0} [2 \backslash n] z^n$	$\frac{1}{1 - z^2}$
$\langle 1, 0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots \rangle$	$\sum_{n \geqslant 0} [m \backslash n] z^n$	$\frac{1}{1 - z^m}$
$\langle 1, 2, 3, 4, 5, 6, \ldots \rangle$	$\sum_{n \geqslant 0} (n + 1) z^n$	$\frac{1}{(1 - z)^2}$
Triangular numbers		
$\langle 1, 3, 6, 10, 15, 21, \ldots \rangle$	$\sum_{n \geqslant 0} \binom{n+2}{2} z^n$	$\frac{1}{(1 - z)^3}$
Tetrahedral numbers		
$\langle 1, 4, 10, 20, 35, \ldots \rangle$	$\sum_{n \geqslant 0} \binom{n+3}{3} z^n$	$\frac{1}{(1 - z)^4}$
$\langle 1, 5, 15, 35, 70, \ldots \rangle$	$\sum_{n \geqslant 0} \binom{n+4}{4} z^n$	$\frac{1}{(1 - z)^5}$
$\langle 1, c, \binom{c+1}{2}, \binom{c+2}{3}, \ldots \rangle$	$\sum_{n \geqslant 0} \binom{c+n-1}{n} z^n$	$\frac{1}{(1 - z)^c}$
equivalent to	$\sum_{n \geqslant 0} \binom{n+k}{k} z^n$	$\frac{1}{(1 - z)^{k+1}}$

$\langle 0, 1, 4, 9, 16, \dots \rangle$	$\sum_{n \geqslant 0} n^2 z^n$	$\frac{z(z+1)}{(1-z)^3}$
$\langle 1, 2, 4, 8, 16, 32, \dots \rangle$	$\sum_{n \geqslant 0} 2^n z^n$	$\frac{1}{1-2z}$
$\langle 1, 4, 6, 4, 1, 0, 0, \dots \rangle$	$\sum_{n \geqslant 0} \binom{4}{n} z^n$	$(1+z)^4$
$\langle 1, c, \binom{c}{2}, \binom{c}{3}, \dots \rangle$	$\sum_{n \geqslant 0} \binom{c}{n} z^n$	$(1+z)^c$
$\langle 1, c, c^2, c^3, \dots \rangle$	$\sum_{n \geqslant 0} c^n z^n$	$\frac{1}{1-cz}$
$\langle 0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots \rangle$	$\sum_{n \geqslant 1} \frac{1}{n} z^n$	$\ln \frac{1}{1-z}$
$\langle 0, 1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \dots \rangle$	$\sum_{n \geqslant 1} \frac{-1^{n+1}}{n} z^n$	$\ln(1+z)$
$\langle 1, 1, \frac{1}{2}, \frac{1}{6}, \frac{1}{24}, \frac{1}{120}, \dots \rangle$	$\sum_{n \geqslant 0} \frac{1}{n!} z^n$	e^z
$\langle 0, 1, 0, \frac{1}{3!}, 0, \frac{1}{5!}, \dots \rangle$	$\sum_{n \geqslant 0} \frac{1}{n!} [2\backslash(n+1)] z^n$	$\sin z$
$\langle 1, 0, \frac{1}{2!}, 0, \frac{1}{4!}, 0, \dots \rangle$	$\sum_{n \geqslant 0} \frac{1}{n!} [2\backslash(n+2)] z^n$	$\cos z$
$\langle , \dots \rangle$	$\sum_{n \geqslant 0} z^n$	

Catalan numbers C_n

$\langle 1, 1, 2, 5, 14, 42, \dots \rangle$	$\sum_{n} \frac{1}{n+1} \binom{2n}{n} z^n$	$\frac{1}{2z}(1 - \sqrt{1-4z})$

sequence	gen. function	closed form
$\langle 1, 4, 12, 32, 80, 320, \ldots \rangle$	$\sum_{n\geqslant 0} 2^n(n+1)z^n$	$\frac{1}{(1-2z)^2}$
$\langle 1, 2c, 3c^2, 4c^3, 5c^4, \ldots \rangle$	$\sum_{n\geqslant 0} (n+1)c^n z^n$	$\frac{1}{(1-cz)^2}$
$\langle \overbrace{0, 0, \ldots, 0, 0}^{\text{m zeros}}, 1, 1, 1, \ldots \rangle$	$z^m \sum_{n\geqslant 0} z^n$	$\frac{z^m}{1-z}$
$\langle 1, 3, 8, 20, 48, 112, \ldots \rangle$		$\frac{1-z}{(1-2z)^2}$
$\langle 1, 2, 5, 13, 33, 81, \ldots \rangle$		
$\langle 1, 2, 5, 14, 41, 122, 365, \ldots \rangle$	$\sum_{n\geqslant 0} \frac{3^{n-1}+1}{2} z^n$	$\frac{z(2-3z)}{(1-3z)(1-z)}$
Bell (exponential) numbers		
$\langle 1, 2, 5, 15, 52, 203, \ldots \rangle$		
$\langle 1, 3, 10, 37, 151, \ldots \rangle$		

All the ways to write an integer n as a sum of 1s, 2s and 3s. Each representation $n = a \cdot 1 + b \cdot 2 + c \cdot 3$ is encoded by a monomial $x_1^a x_2^b x_3^c$ that appears in the coefficient polynomial of y^n:

$$
\begin{aligned}
\frac{1}{(1-x_1y)(1-x_2y^2)(1-x_3y^3)} &= 1 + x_1y + (x_1^2 + x_2)y^2 + (x_1^3 + x_1x_2 + \\
&+ (x_1^4 + x_1^2x_2 + x_2^2 + x_1x_3)y^4 + (x_1^5 + x_1^3x_2 + x_1x_2^2 + x_1^2x_3 + x_2x_3)y^5 \\
&+ (x_1^6 + x_1^4x_2 + x_1^2x_2^2 + x_2^3 + x_1^3x_3 + x_1x_2x_3 + x_3^2)y^6 + \ldots
\end{aligned}
$$