Topics on Hindley's Simple Type Theory*: 8E and 8F

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* - Presentation adapted from the work of Thiago Mendonça and Washington Ribeiro
Type Theory Class - 1/2020 - Universidade de Brasília - UnB

June 22, 2020

Outline of Presentation

Section 1

8E - The Structure of a nf-scheme

Subsection 1

Overview and Motivation

8E - The Structure of a nf-scheme

This section is about analysing the structure of an arbitrary long typed nf-scheme.

Motivation

Earlier sections left the stretching and shrinking lemmas unproved (8D2 and 8D3), as well as the completeness part of the searching theorem for $\mathtt{Long}(\tau)$ (8C5(iii)).

- **8E** - We lay the foundation by analysing the structure of an arbitrary long typed nf-scheme.
- **8F** - Having the necessary foundation, we fill the mentioned gaps.

Subsection 2

Key Role, Remarks and Notations

Structured Proofs à la Leslie Lamport

In this presentation, the mathematical proofs are presented as structured proofs à la Leslie Lamport [3, 2].

Notation for Components

We will underline a term Y when we want to indicate that it is being analysed as a component of a term X.

Organization of the Present Section

The early parts of this section will apply to both typed and untyped nf-schemes. Therefore, when we write nf-schemes, types will be omitted.

The final parts of this section will only apply to typed nf-schemes, so types will be displayed.

Key Role

A key role in our analysis will be played by a slightly strengthened form of the subformula property. This property says that the types of all the components of a closed β-nf M^τ are subtypes of τ.

This way, as the algorithm searches deeper and deeper, the types of the components we are working with remain drawn from the same fixed finite set.

nf-scheme

Remark 1

A nf-scheme is essentially a β-nf that may contain meta-variables under suitable restrictions.

nf-scheme

Remark 2

According to 8A5, every non-atomic nf-scheme X can be expressed uniquely in the form:

$$X \equiv \lambda x_1 \ldots x_m.vY_1 \ldots Y_n$$

with $m + n \geq 1$ and where v is one of the x_i if X is closed.

The head and arguments of X are v and $Y_1, \ldots, Y_n$. If X is an atom its head is X and it has no arguments.

The construction-tree of such an X is shown in Fig 1. Note that the position of Y_i is: $0^m 1^{n-i} 2$, for $1 \leq i \leq n$.

An Example of a Construction Tree

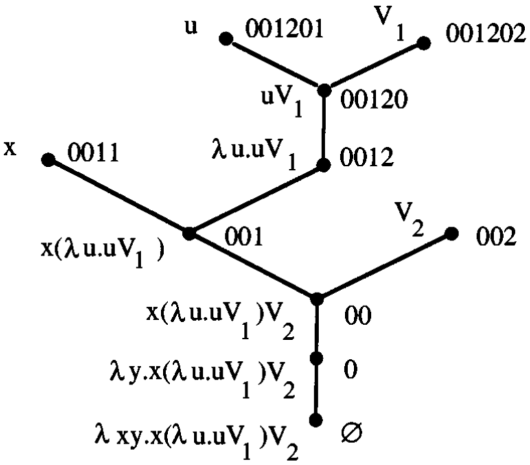

Figure 1: Construction Tree for $\lambda xy.x(\lambda u.uV_1)V_2$. Figure obtained from [1].

Subsection 3

The Foundation for 8F

Subarguments

Definition 1 (8E2 in Hindley's)

A subargument of a typed or untyped nf-scheme X is a component that is an argument of X or an argument of a proper component of X.

Lemma 8E2.1

Lemma 2 (8E2.1 in Hindley's)

A component $\underline{Y}$ of a typed or untyped nf-scheme X is a subargument iff its position is not $\emptyset$ and the last symbol in its position is 2.

Proof sketch: By induction on $|X|$. One can, for instance, consider the subarguments in the tree of Figure 1.

Remarks about Subarguments

Remark 3 (8E2.2(i) in Hindley's)

All occurrences of meta-variables in a composite nf-scheme are subarguments.

PROOF: *By restriction 8C1(iii) in the definition of nf-scheme.*

Remark 4 (8E2.2(ii) in Hindley's)

A subargument of a subargument of X is a subargument of X.

PROOF: *By the definition of subargument.*

2-length and depth

Definition 1 (8E3 in Hindley's)

The 2-length of a position string p is the number of 2's in p.

Definition 2 (8E3 in Hindley's)

The depth in X of a subargument $\underline{Z}$ of X is the 2-lenght of its position.

Remark 5 (8E3 in Hindley's)

The depth in X of a subargument $\underline{Z}$ is the number of right-hand choices made when "travelling up" the tree of X from the bottom node to $\underline{Z}$.

Lemma 3 (8E3.1 in Hindley's)

Let X be a typed or untyped nf-scheme with $Depth(X) \geq 1$, *where the Depth of an nf-scheme is defined as in 8A6. Then:*

1. $Depth(X)$ *is the maximum of the depths in X of all subarguments in X,*
2. *X has at least one subargument whose depth in X is the same as* $Depth(X)$, *and each such subargument is an atom or abstracted atom.*

PROOF SKETCH: By induction on $|X|$, using 8A6.

Argument-Branch

Definition 3 (8E4 in Hindley's)

If Z is a subargument of a typed or untyped nf-scheme X, the **argument-branch** *from X to Z is the sequence:*

$$\langle \underline{Z}_0, \underline{Z}_1, \ldots, \underline{Z}_k \rangle$$

such that $\underline{Z}_0 \equiv \underline{X}$, $\underline{Z}_k \equiv \underline{Z}$ and for each $i = 1, \ldots, k$, we have $\underline{Z}_i$ is an argument of $\underline{Z}_{i-1}$.

It is **unextendable** *iff $\underline{Z}$ is an atom or abstracted atom.*

Its **length** *is k (not $k+1$).*

Lemma 8E4.1

Lemma 4 (8E4.1 in Hindley's)

For any typed or untyped nf-scheme X:

1. *The depth in X of a subargument $\underline{Z}$ is the same as the length of the argument-branch from $\underline{X}$ to $\underline{Z}$,*
2. *Depth(X) is the maximum of the lengths of all argument-branches in X.*

PROOF SKETCH: For (1) use induction on $|X|$, for (2) use 8E3.1.

IA, CA

Definition 4 (8E5 in Hindley's)

Let $\underline{Z}$ be a subargument of a typed or untyped nf-scheme X, for instance:

$$Z \equiv \lambda x_1 \ldots x_m.yZ_1 \ldots Z_n \qquad (m \geq 0, n \geq 0)$$

The **Initial Abstractors sequence IA(Z)** *is the (possibly empty) sequence:*

$$IA(Z) = \langle x_1, \ldots, x_m \rangle$$

The **Covering Abstractors sequence CA($\underline{Z}$, X)** *is defined as:*

$$CA(\underline{Z}, X) = \langle z_1, \ldots, z_q \rangle,$$

where $\underline{\lambda z}_1$, ..., $\underline{\lambda z}_q$ are the abstractors in X whose scopes contain $\underline{Z}$, written in the order they occur in X from left to right. Also, define:

$$Length(IA(Z)) = m,$$
$$Length(CA(\underline{Z}, X)) = q.$$

IA, CA

Remark 6 (8E5.1 in Hindley's)

1. *If X has no bound-variable clashes, the members of $IA(Z)$ are distinct and so are those of $CA(\underline{Z}, X)$.*
2. *$IA(Z)$ and $CA(\underline{Z}, X)$ are sequences of variables, not components.*
3. *For typed nf-schemes each variable in $IA(Z)$ or $CA(\underline{Z}, X)$ is typed.*
4. *If the argument-branch from $\underline{X}$ to $\underline{Z}$ is $\langle \underline{Z}_0, \ldots, \underline{Z}_k \rangle$ $(k \geq 1)$, then:*

$$CA(\underline{Z}, X) = IA(Z_0) * \ldots * IA(Z_{k-1})$$

where "" denotes concatenation of sequences.*

Warning

The remaining part of this section applies only for typed nf-schemes.

IAT

Definition 5 (IAT)

Let $\underline{Z^\sigma}$ be a subargument of a typed nf-scheme X^τ, say:

$$Z^\sigma \equiv \lambda x_1^{\sigma_1} \dots x_m^{\sigma_m}.yZ_1 \dots Z_n \qquad (m \geq 0, n \geq 0)$$

The **Initial Abstractors' Types Sequence (**$IAT(Z^\sigma)$**)** *is defined as:*

$$IAT(Z^\sigma) = \langle \sigma_1, \dots, \sigma_m \rangle;$$

And we also define:

$$Length(IAT(Z^\sigma)) = m$$

Premises

If $\rho \equiv \rho_1 \to \ldots \to \rho_m \to a$, we call $\rho_1, \ldots, \rho_m$ the premises of ρ and we call a the tail of ρ.

Positions in a Term

Consider $\tau \equiv (a \to b \to c) \to (a \to b) \to a \to c$.

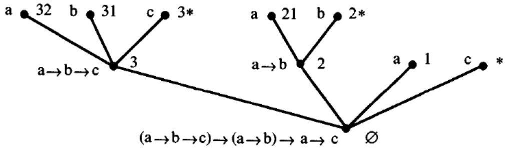

Fig. 9E2.1a.

Figure 2: Positions in the term $\tau \equiv (a \to b \to c) \to (a \to b) \to a \to c$. Figure obtained from [1].

Subpremises

A subpremise of τ is a premise of some component of τ (possibly of τ itself).

Subpremises

Example 1

Let $\tau \equiv (a \to b \to c) \to (a \to b) \to a \to c$ *(see Figure 2). Let's represent subpremisses as triples "*$\langle term, position, \tau \rangle$*". The six subpremisses of* τ *are:*

1. $\langle a, 1, \tau \rangle$,
2. $\langle a \to b, 2, \tau \rangle$,
3. $\langle a \to b \to c, 3, \tau \rangle$,
4. $\langle a, 21, \tau \rangle$,
5. $\langle b, 31, \tau \rangle$,
6. $\langle a, 32, \tau \rangle$.

Positive Subpremises

A subpremise of τ is positive if and only if its position has even length (the symbol $*$ does not count when computing the length).

In Example 1 the positive subpremises are:

1. $\langle a, 21, \tau \rangle$,
2. $\langle b, 31, \tau \rangle$,
3. $\langle a, 32, \tau \rangle$.

NSS

Definition 6

$NSS(\tau)$ is the set of all finite sequences $\langle \sigma_1, \ldots, \sigma_n \rangle$ $(n \geq 1)$ such that:

$$\sigma_1 \to \ldots \to \sigma_n \to a \tag{1}$$

is positive in τ.

Remark 7

Each member of $NSS(\tau)$ is called a negative subpremise-sequence, because it is a sequence of terms that have occurrences as negative subpremises in τ.

NSS

Example 2

Let $\tau \equiv (a \to (b \to d \to c) \to d) \to (a \to b \to c) \to d \to d$.
We have $NSS(\tau) = \{\langle a \to (b \to d \to c) \to d, a \to b \to c, d\rangle, \langle b, d\rangle\}$.

PROOF:

$\langle 1\rangle 1.$ $NSS(\tau) \supseteq \{\langle a \to (b \to d \to c) \to d, a \to b \to c, d\rangle, \langle b, d\rangle\}$

$\langle 2\rangle 1.$ $\langle a \to (b \to d \to c) \to d, a \to b \to c, d\rangle \in NSS(\tau)$

By definition, since:

$$(a \to (b \to d \to c) \to d) \to (a \to b \to c) \to d \to d$$

is positive in τ, as it has position $\emptyset$, of even length.

$\langle 2\rangle 2.$ $\langle b, d\rangle \in NSS(\tau)$

By definition, since:

$$b \to d \to c$$

is positive in τ as it has position 31, of even length.

$\langle 1\rangle 2.$ $NSS(\tau) \subseteq \{\langle a \to (b \to d \to c) \to d, a \to b \to c, d\rangle, \langle b, d\rangle\}$

By checking that no other finite sequence $\langle \sigma_1, \ldots, \sigma_n\rangle$ is such that $\sigma_1 \to \ldots \to \sigma_n \to a$ is positive in τ.

Definition 7

The set of all the members of the sequences in $NSS(\tau)$ will be called $\cup NSS(\tau)$

Example 3

In Example 2, we had:

$$NSS(\tau) = \{\langle a \to (b \to d \to c) \to d, a \to b \to c, d\rangle, \langle b, d\rangle\}$$

Therefore,

$$\cup NSS(\tau) = \{a \to (b \to d \to c) \to d, a \to b \to c, d, b\}$$

Lemma 8E7

Lemma 5 (8E7 in Hindley's)

If $\underline{Z^\sigma}$ is a subargument of a closed long typed nf-scheme X^τ, then

1. *σ occurs as a positive subpremise in τ (as defined in 9E6-8),*
2. *If σ is an atom, $IAT(Z^\sigma) = \emptyset$,*
3. *If σ is composite, $IAT(Z^\sigma) \in NSS(\tau)$ (defined in 9E9),*
4. *$NSS(\sigma) \subseteq NSS(\tau)$.*

PROOF:

$\langle 1 \rangle 1.$ σ occurs as a positive subpremise in τ (as defined in 9E6-8).

$\langle 2 \rangle 1.$ If X^τ is a long member of $\mathbb{TNS}(\Gamma)$ and $\Gamma = \{u_1 : \theta_1, \ldots, u_p : \theta_p, V_1 : \phi_1, \ldots, V_q : \phi_q\}$ and $\underline{Z^\sigma}$ is a subargument of X^τ, then σ occurs as a positive subpremise of $\theta_1 \to \ldots \to \theta_p \to \tau$.

The proof is by induction on $|X^\tau|$.

$\langle 3 \rangle 1.$ **Basis** If X^τ is an atom there is no $\underline{Z^\sigma}$ subargument of X^τ, and the conclusion holds vacuously.

Proof of Lemma 8E7 II

$\langle 3\rangle 2.$ **Induction Step**

$\langle 4\rangle 1.$ X has the form:

$$(\lambda x_1^{\tau_1} \ldots x_m^{\tau_m}.(y^{(\rho_1 \to \ldots \to \rho_n \to e)} X_1^{\rho_1} \ldots X_n^{\rho_n})^e)^{\tau_1 \to \ldots \to \tau_m \to e}$$

where $m, n \geq 0$ and $\tau \equiv \tau_1 \to \ldots \to \tau_m \to e$.

$\langle 4\rangle 2.$ CASE: $Z^\sigma \equiv X_j^{\rho_j}$.

$\langle 5\rangle 1.$ Since $Z^\sigma \equiv X_j^{\rho_j}$, we have $\sigma \equiv \rho_j$.

$\langle 5\rangle 2.$ Each of $\rho_1, \ldots, \rho_n$ occurs as a positive subpremise of $\theta_1 \to \ldots \to \theta_p \to \tau$.

$\langle 6\rangle 1.$ Using the notation of $\langle 4\rangle 1$ and $\langle 2\rangle 1$, we have that either $y \equiv x_i$ for some $i \leq m$ or $y \equiv u_i$ for some $i \leq p$.

$\langle 6\rangle 2.$ CASE: $y \equiv x_i$. We have that $\rho_1 \to \ldots \to \rho_n \to e \equiv \tau_i$. Then, the position of each ρ_j in $\theta_1 \to \ldots \to \theta_p \to \tau$ has length 2, making it a positive subpremise.

Proof of Lemma 8E7 III

$\langle 6 \rangle 3.$ CASE: $y \equiv u_i$. Then $\rho_1 \to \ldots \to \rho_n \to e \equiv \theta_i$. Then, the position of each ρ_j has length 2, making it a positive subpremise.

$\langle 4 \rangle 3.$ CASE: Z^σ is a subargument of $X_j^{\rho_j}$.

$\langle 5 \rangle 1.$ $X_j^{\rho_j}$ is a long member of $\mathbb{TNS}(\{x_1 : \tau_1, \ldots, x_m : \tau_m\} \cup \Gamma)$
$\langle 5 \rangle 2.$ By IH, σ occurs as a positive subpremise of

$$\tau_1 \to \ldots \to \tau_m \to \theta_1 \to \ldots \to \theta_p \to \rho_j$$

.

$\langle 5 \rangle 3.$ CASE: σ is a positive subpremise of ρ_j. Then, by using the result $\langle 5 \rangle 2$ of branch $\langle 4 \rangle 2$ (notice the result holds because we can repeat the argument), we conclude.
$\langle 5 \rangle 4.$ CASE: σ is a negative subpremise of one of $\tau_1, \ldots, \tau_m, \theta_1, \ldots, \theta_p$. Then σ will be a positive subpremise of $\theta_1 \to \ldots \to \theta_p \to \tau$.

Proof of Lemma 8E7 IV

$\langle 2 \rangle 2$. Since X^τ is closed, we can apply $\langle 2 \rangle 1$ with $\Gamma = \emptyset$ and conclude.

Proof of Lemma 8E7 V

$\langle 1 \rangle 2.$ If σ is an atom, $IAT(Z^\sigma) = \emptyset$.

$\langle 2 \rangle 1.$ $IAT(Z^\sigma)$ coincides with the sequences of all premises of σ.

$\langle 3 \rangle 1.$ Z^σ is long.

$\langle 3 \rangle 2.$ LET: $IAT(Z^\sigma) = \langle \sigma_1, \ldots, \sigma_m \rangle$. Since Z^σ is long, $\sigma \equiv \sigma_1 \to \ldots \to \sigma_m \to e$.

$\langle 2 \rangle 2.$ Since σ is an atom, there are no premises. By $\langle 2 \rangle 1$, $IAT(Z^\sigma) = \emptyset$.

Proof of Lemma 8E7 VI

$\langle 1\rangle 3$. If σ is composite, $IAT(Z^\sigma) \in NSS(\tau)$ (defined in 9E9)

$\langle 2\rangle 1$. $IAT(Z^\sigma) \in NSS(\sigma)$.

$\langle 3\rangle 1$. Z^σ is long.

$\langle 3\rangle 2$. LET: $IAT(Z^\sigma) = \langle \sigma_1, \ldots, \sigma_m \rangle$. Then, since Z^σ is long, $\sigma \equiv \sigma_1 \to \ldots \to \sigma_m \to e$.

$\langle 3\rangle 3$. By $\langle 3\rangle 1$, and the definition of $NSS(\sigma)$ (which is only defined for σ composite) we conclude.

$\langle 2\rangle 2$. $NSS(\sigma) \subseteq NSS(\tau)$.

It will be proved in $\langle 1\rangle 4$.

Proof of Lemma 8E7 VII

$\langle 1 \rangle 4.$ $\mathit{NSS}(\sigma) \subseteq \mathit{NSS}(\tau)$.

$\langle 2 \rangle 1.$ By $\langle 1 \rangle 1$, σ occurs as a positive subpremise in τ.

$\langle 2 \rangle 2.$ By the technical lemma 9E9.2(iii), since σ occurs as a positive subpremise of τ, we have $\mathit{NSS}(\sigma) \subseteq \mathit{NSS}(\tau)$.

Lemma 8E7

Remark 8

Notice that Lemma 8E7 connects $IAT(Z^{\sigma})$, which in general depends on the structure of Z^{σ} and hence implicitly on that of X^{τ}, with $NSS(\tau)$, which depends on τ and nothing else.

Corollary 8E7.1

Corollary 6 (8E7.1 in Hindley's)

If X^τ is a closed long typed nf-scheme, the type of each meta-variable in X^τ either occurs as a positive subpremise of τ or is τ itself.

$\langle 1 \rangle$1. CASE: X^τ is a composite nf-scheme.

- $\langle 2 \rangle$1. LET: Z^σ be an arbitrary metavariable in X^τ.
- $\langle 2 \rangle$2. An occurrence of Z^σ in X^τ is a subargument.
- $\langle 2 \rangle$3. Then, by Lemma 8E7, σ occurs as a positive subpremise of τ.

$\langle 1 \rangle$2. CASE: X^τ is an atomic nf-scheme.

In this case, X^τ is a meta-variable.

Corollary 7 (8E7.2 in Hindley's)

If X^τ is a closed long typed nf-scheme and $\underline{Z^\sigma}$ is a subargument of X^τ or $\underline{Z^\sigma} \equiv \underline{X^\tau}$, then:

1. *Length(IA(Z^σ)) = Length(IAT(Z^σ)) $\leq |\tau| - 1$,*
2. *Length(CA($\underline{Z^\sigma}$, X^τ)) $\leq (|\tau| - 1)\times$ Depth(X^τ),*
3. *If $\underline{\lambda v_1^{\rho_1}}, \ldots, \underline{\lambda v_r^{\rho_r}}$ are all abstractors in X^τ (not just the initial ones), then $\{\rho_1, \ldots, \rho_r\}$ has $\leq |\tau| - 1$ distinct members.*

Corollary 8E7.2 I

$\langle 1\rangle 1.$ Length($IA(Z^\sigma)$) = Length($IAT(Z^\sigma)$) $\leq |\tau| - 1$

$\langle 2\rangle 1.$ LET: $Z^\sigma \equiv \lambda x_1^{\sigma_1} \ldots x_m^{\sigma_m}.yZ_1 \ldots Z_n$

$\langle 2\rangle 2.$ Length($IA(Z^\sigma)$) = Length($IAT(Z^\sigma)$) $= m$.
By definition, $IA(Z^\sigma) = \langle x_1, \ldots, x_m\rangle$ and $IAT(Z^\sigma) = \langle \sigma_1, \ldots \sigma_m\rangle$.

$\langle 2\rangle 3.$ Length($IAT(Z^\sigma)$) $\leq |\tau| - 1$

$\langle 3\rangle 1.$ CASE: σ is atomic. Then, $IAT(Z^\sigma) = \emptyset$ by Lemma 8E7(ii). Therefore, Length($IAT(Z^\sigma)$) $= 0$ and since $|\tau| \geq 1$ the inequality holds.

$\langle 3\rangle 2.$ CASE: σ is composite.

$\langle 4\rangle 1.$ $IAT(Z^\sigma) \in NSS(\tau)$
By Lemma 8E7(iii).

$\langle 4\rangle 2.$ $IAT(Z^\sigma) = \langle \sigma_1, \ldots, \sigma_m\rangle$.
By definition.

$\langle 4\rangle 3.$ If $\langle \sigma_1, \ldots, \sigma_m\rangle \in NSS(\tau)$ then $m \leq |\tau| - 1$.
By Lemma 9E9.3(iv)

Corollary 8E7.2 II

$\langle 1 \rangle 2.$ Length($CA(\underline{Z^\sigma}, X^\tau)$) $\leq (|\tau| - 1) \times$ Depth(X^τ)

$\langle 2 \rangle 1.$ Case: $\underline{\mathsf{Z}} \equiv \underline{\mathsf{X}}$.
Since no abstractor in X has scope containing $\underline{\mathsf{Z}} \equiv \underline{\mathsf{X}}$,
Length($CA(\underline{Z^\sigma}, X^\tau)$) $= 0$.

$\langle 2 \rangle 2.$ Case: $\underline{\mathsf{Z}} \not\equiv \underline{\mathsf{X}}$.

$\langle 3 \rangle 1.$ Let: $\langle\ \underline{Z_0}, \ldots, \underline{Z_k}\ \rangle$, with $k \geq 1$ be the argument-branch from $\underline{\mathsf{X}}$ to $\underline{\mathsf{Z}}$.

$\langle 3 \rangle 2.$ Length(CA($\underline{\mathsf{Z}}$, X)) $=$ Length(IA(Z_0))$+ \ldots +$ Length(IA(Z_{k-1})).
By 8E5.1, remembering that in an nf-scheme there are no bound-variable clashes.

$\langle 3 \rangle 3.$ Length(IA(Z_0))$+ \ldots +$ Length(IA(Z_{k-1})) $\leq k(|\tau| - 1)$
By Step $\langle 1 \rangle 1$ we have Length(IA(Z_i)) $\leq (|\tau| - 1)$.

$\langle 3 \rangle 4.$ $k(|\tau| - 1) \leq (|\tau| - 1) \times \mathit{Depth}(X)$
By 8E4.1(ii), $\mathit{Depth}(X)$ is greater or equal than the length of the argument-branch from X to Z, which is k.

Corollary 8E7.2 III

$\langle 1 \rangle 3$. If $\underline{\lambda v}_1^{\rho_1}$, ..., $\underline{\lambda v}_r^{\rho_r}$ are all abstractors in X^τ (not just the initial ones), then $\{\rho_1, \ldots, \rho_r\}$ has $\leq |\tau| - 1$ distinct members.

$\langle 2 \rangle 1$. $\rho_i \in \cup NSS(\tau)$.

$\langle 3 \rangle 1$. Each ρ_i is in $IAT(X^\tau)$ or in $IAT(Y^\theta)$ for some subargument Y^θ of X^τ.

$\langle 3 \rangle 2$. Case: $\rho_i \in IAT(X^\tau)$. By the definition of $IAT(X^\tau)$ and of $\cup NSS(\tau)$ we get that $\rho_i \in \cup NSS(\tau)$.

$\langle 3 \rangle 3$. Case: $\rho_i \in IAT(Y^\theta)$. By Lemma 8E7(iii) we get that $\rho_i \in \cup NSS(\tau)$.

$\langle 2 \rangle 2$. $| \cup NSS(\tau)| \leq |\tau| - 1$
By Lemma 9E9.3

Section 2

8F - Stretching, Shrinking and Completeness

Subsection 1

Search Completeness Lemma

Search Completeness Lemma

Lemma 8 (8F1 in Hindley's)

Part (iii) of the search theorem 8C5 holds; i.e. if τ is composite and $d \geq 0$, then:

$$Long(\tau, d) \subseteq \mathcal{A}(\tau, \leq d + 1)$$

The way to prove the lemma would be by induction on d, however to make the induction hypothesis work, we need to strength it a bit...

An Auxiliary Lemma For Completeness

Lemma 9

Let $\mathbb{L}^*(\tau, d)$ *be the set of all long typed closed nf-schemes* X^τ *such that* $Depth(X^\tau) = d$ *and*

1. X^τ *is proper and all its meta-variables have depth* d *in* X^τ*.*
2. *all subarguments with depth* d *in* X^τ *are meta-variables.*

Then,

$$\mathbb{L}^*(\tau, d) \subseteq \mathcal{A}(\tau, \leq d) \tag{1}$$

and

$$Long(\tau, d) \subseteq \mathcal{A}(\tau, \leq d + 1) \tag{2}$$

where (1) is understood modulo renaming of meta-variables.

An Auxiliary Lemma For Completeness I

PROOF: The proof is by induction on d:

$\langle 1\rangle 1$. **Basis:** d $= 0$.

$\langle 2\rangle 1$. $\mathbb{L}^*(\tau, 0) \subseteq \mathcal{A}(\tau, 0)$

$\langle 3\rangle 1$. LET: $X^\tau \in \mathbb{L}^*(\tau, d)$, with $d = 0$.

$\langle 3\rangle 2$. X^τ is a meta-variable, as the only proper nf-schemes with depth 0 are meta-variables.

$\langle 3\rangle 3$. $\mathcal{A}(\tau, 0) = \{V^\tau\}$, by step 0 of the search algorithm (8C6).

$\langle 3\rangle 4$. Renaming the meta-variable X^τ to V^τ we see the result holds.

$\langle 2\rangle 2$. $Long(\tau, 0) \subseteq \mathcal{A}(\tau, \leq 1)$

$\langle 3\rangle 1$. LET: $M^\tau \in Long(\tau, 0)$, with $\tau \equiv \tau_1 \to \ldots \to \tau_m \to e \quad (m \geq 1)$

$\langle 3\rangle 2$. M^τ has form $\lambda y_1^{\tau_1} \ldots y_m^{\tau_m}.y_i^{\tau_i}$ with $1 \leq i \leq m, \tau_i \equiv e$.

$\langle 3\rangle 3$. $\mathcal{A}(\tau, 0) = \{V^\tau\}$, by step 0 of the search algorithm (8C6).

$\langle 3\rangle 4$. The search algorithm 8C6 Step 1:Part IIa1 adds M^τ (it may be necessary a renaming of bound variables) to $\mathcal{A}(\tau, 1)$.

Notice that the condition that the tail of τ_i (which is $\tau_i \equiv e$ itself) is isomorphic to the tail of τ (which is e) is indeed satisfied.

An Auxiliary Lemma For Completeness II

$\langle 1 \rangle 2$. **Induction Step:** d to $d+1$.

$\langle 2 \rangle 1$. $\mathbb{L}^*(\tau, d+1) \subseteq \mathcal{A}(\tau, \leq d+1)$

$\langle 3 \rangle 1$. LET: $X \in \mathbb{L}^*(\tau, d+1)$.

$\langle 3 \rangle 2$. LET: $\underline{W_1}, \ldots, \underline{W_r}$ with $r \geq 1$ the subarguments of X of depth d and let X' be the result of replacing each $\underline{W_i}$ in X by a distinct new meta-variable $\underline{V_i}$ of the same type as $\underline{W_i}$.

$\langle 4 \rangle 1$. Since $X \in \mathbb{L}^*(\tau, d+1)$, $Depth(X) = d+1$.

$\langle 4 \rangle 2$. By 8E3.1(ii), X has a subargument whose depth in X is $d+1$.

$\langle 4 \rangle 3$. By 8E4.1, X has a subargument whose depth in X is d.

$\langle 4 \rangle 4$. Therefore, if $\underline{W_1}, \ldots, \underline{W_r}$ are the subarguments of X of depth d, we must have $r \geq 1$.

An Auxiliary Lemma For Completeness III

$\langle 3 \rangle 3.$ $X' \in \mathbb{L}^*(\tau, d)$.

$\langle 4 \rangle 1.$ X' is a nf-scheme.
By definition. Notice that each new meta-variable $\underline{V_i}$ will occur in an argument position because each $\underline{W_i}$ is a subargument.

$\langle 4 \rangle 2.$ X' is long, closed and has depth d.
X' is long since X is long and each replacement of $\underline{W_i}$ by $\underline{V_i}$ preserves type. It is closed since X was closed and each replacement of $\underline{W_i}$ by $\underline{V_i}$ adds no free variable. It has depth d as every subargument of depth d is a meta-variable.

$\langle 4 \rangle 3.$ X' is proper and all its meta-variables have depth d in X'.
The proof is by contradiction. If X' contained a meta-variable occurrence $\underline{V}$ at a depth $< d$, such a $\underline{V}$ could not be a $\underline{V_i}$ and hence would also occur in X at a depth $< d$. This contradicts the fact that X is proper and all its meta-variables have depth d in X.

$\langle 4 \rangle 4.$ All subarguments with depth d in X' are meta-variables.
By the construction of X'.

An Auxiliary Lemma For Completeness IV

$\langle 3 \rangle$4. There is a $X'' \in \mathcal{A}(\tau, \leq d)$ that is identical to X' except perhaps for alphabetic variations of meta-variables.
By the induction hypothesis, since $X' \in \mathbb{L}^*(\tau, d)$.

$\langle 3 \rangle$5. Apply Step d+1 of Algorithm 8C6 to each V_i in X''. The algorithm will give X as an extension of X''.

$\langle 4 \rangle$1. Each W_i has form $W_i \equiv \lambda x_{i,1} \ldots x_{i,m_i}.y_i V_{i,1} \ldots V_{i,n_i}$
Since $Depth(X) = d + 1$, we have $Depth(W_i) \leq 1$. Since X satisfies the conditions (1) of $\mathbb{L}^*(\tau, d+1)$, W_i is not a meta-variable. Since X satisfies the condition (2) of $\mathbb{L}^*(\tau, d+1)$, the result holds.

$\langle 4 \rangle$2. By the form of W_i (see Step $\langle 4 \rangle$1) and the algorithm 8C6, each $\underline{W_i}$ will be a suitable replacement for $\underline{V_i}$.

$\langle 4 \rangle$3. X is an extension of X''.

$\langle 3 \rangle$6. $X \in \mathcal{A}(\tau, \leq d+1)$.

An Auxiliary Lemma For Completeness V

$\langle 2\rangle 2.$ $Long(\tau, d+1) \subseteq \mathcal{A}(\tau, \leq d+2)$

$\langle 3\rangle 1.$ LET: $M \in Long(\tau, d+1)$.

$\langle 3\rangle 2.$ LET: $\underline{U_1}, \ldots, \underline{U_r}$ with $r \geq 1$ be the subarguments of M, without repetition, whose depth in M is $d+1$.

By 8E3.1, M has a subargument whose depth in M is $d+1$. Therefore, $r \geq 1$.

$\langle 3\rangle 3.$ Each U_i is of the form $U_i \equiv \lambda x_{i,1} \ldots x_{i,m_i}.y_i$

Since $Depth(M) = d+1$, each U_i must have depth 0 and we conclude.

$\langle 3\rangle 4.$ LET: M' be the result of replacing each $\underline{U_i}$ in M by a distinct new meta-variable $\underline{V_i}$ with the same type as $\underline{U_i}$.

$\langle 3\rangle 5.$ $M' \in \mathbb{L}^*(\tau, d+1)$.

$\langle 4\rangle 1.$ M' is a nf-scheme.

Because M is a nf-scheme and the replacement of $\underline{U_i}$ by $\underline{V_i}$ preserves the restrictions necessary for a nf-scheme.

$\langle 4\rangle 2.$ M' is long and closed.

An Auxiliary Lemma For Completeness VI

Since M is long and closed and each replacement of $\underline{U_i}$ by $\underline{V_i}$ preserves type and adds no free variables, we conclude that M' is long and closed respectively.

$\langle 4 \rangle 3.$ M' has depth $d+1$.
When going from M to M' all subarguments whose depth in M was $d+1$ had depth 0 (when viewed as terms, instead of subarguments of M) and were replaced by a meta-variable, of depth 0. Therefore, $Depth(M') = Depth(M) = d+1$.

$\langle 4 \rangle 4.$ M' is proper and all it's meta-variables have depth $d+1$ in M'.
Because this result holds for M and all meta-variables introduced replace subarguments whose depth in M was $d+1$.

$\langle 4 \rangle 5.$ All subarguments with depth $d+1$ in M' are meta-variables.
Because all the subarguments of depth $d+1$ in M were replaced by meta-variables to obtain M'.

An Auxiliary Lemma For Completeness VII

$\langle 3 \rangle 6$. There is a M'', differing from M' only by renaming meta-variables, such that $M'' \in \mathcal{A}(\tau, \leq d+1)$.

Because $\mathbb{L}^*(\tau, d+1) \subseteq \mathcal{A}(\tau, \leq d+1)$ (see Step $\langle 2 \rangle 1$)

$\langle 3 \rangle 7$. Applying Step $d+2$ of Algorithm 8C6 to M'' will give us that M is an extension of M''.

By the Algorithm 8C6, since each $\underline{U_i}$ is a suitable replacement for $\underline{V_i}$ in M''.

$\langle 3 \rangle 8$. $M \in \mathcal{A}(\tau, \leq d+2)$.

Lemma 10 (8F1 in Hindley's)

Part (iii) of the search theorem 8C5 holds; i.e. if τ is composite and $d \geq 0$, then:

$$Long(\tau, d) \subseteq \mathcal{A}(\tau, \leq d+1)$$

PROOF: By Result (2) of Lemma 9.

Subsection 2

Stretching Lemma

Detailed Stretching Lemma

Lemma 11 (8F2 in Hindley's)

If $Long(\tau)$ has a member M^{τ} with depth $\geq ||\tau||$ then:

1. *there exists $(M^{*})^{\tau} \in Long(\tau)$ with $Depth((M^{*})^{\tau}) \geq Depth(M^{\tau}) + 1$,*
2. *$Long(\tau)$ is infinite.*

Proof of Detailed Stretching Lemma I

PROOF:

$\langle 1 \rangle 1.$ There exists $(M^*)^\tau \in \mathit{Long}(\tau)$ with $\mathit{Depth}((M^*)^\tau) \geq \mathit{Depth}(M^\tau) + 1$.

$\langle 2 \rangle 1.$ LET: M be a typed closed long β-nf with type τ and without bound-variable clashes. LET: $d = \mathit{Depth}(M) \geq ||\tau|| \geq 1$.

$\langle 2 \rangle 2.$ LET: $\langle\ \underline{N_0}, \ldots, \underline{N_d}\ \rangle$ be an argument-branch of length d. Here $\underline{N}_0 \equiv \underline{M}$ and $\underline{N}_{i+1}$ is an argument of N_i.

$\langle 2 \rangle 3.$ Each N_i has form:

$$\lambda x_{i,1} \ldots x_{i,m_i}.y_i P_{i,1} \ldots P_{i,n_i} \qquad (m_i, n_i \geq 0.)$$

$\langle 2 \rangle 4.$ LET: $\underline{B_i}$ be the body of N_i for $i = 0, \ldots, d$. That is:

$$\underline{B_i} \equiv y_i P_{i,1} \ldots P_{i,n_i}$$

Proof of Detailed Stretching Lemma II

$\langle 2 \rangle$5. At least two of these $\underline{B_i}$ have the same type.

$\langle 3 \rangle$1. The type of each B_i is an atom, since N_i is long.

$\langle 3 \rangle$2. Each one of this atoms occur in τ, by 2B3(i).

$\langle 3 \rangle$3. The number of type-variables in τ is $||\tau|| \leq d$ (by hypothesis).

$\langle 3 \rangle$4. Since there are $d + 1$ components $\underline{B_0}$, ..., $\underline{B_d}$ at least two of these must have the same type.

$\langle 2 \rangle$6. LET: $\underline{\mathsf{B}}_p$ and $\underline{\mathsf{B}}_{p+r}$, with $r \geq 1$ be a pair with the same type.
LET: M^* be the result of replacing $\underline{\mathsf{B}}_{p+r}$ in M by a copy of $\underline{\mathsf{B}}_p$ (after changing bound variables in this copy to avoid clashes).

Proof of Detailed Stretching Lemma III

$\langle 2\rangle 7.$ *Depth*$(M^*) \geq d + 1$.

$\langle 3\rangle 1.$ *Depth*$(B_p) \geq r +$ *Depth*(B_{p+r}).
Since $\underline{\mathsf{B}}_p$ properly contains $\underline{\mathsf{B}}_{p+r}$ and B_{p+r}, when seeing as a subargument of B_p, has depth r in B_p.

$\langle 3\rangle 2.$ M^* has an argument-branch with length $d + r$.
The members of the argument-branch are:

$$\underline{\mathsf{N}}^*_0, \ldots, \underline{\mathsf{N}}^*_{p+r}, \underline{\mathsf{N}}^o_{p+1}, \ldots \underline{\mathsf{N}}^o_d$$

where for $0 \leq i \leq p + r$ each $\underline{\mathsf{N}}^*_i$ has the same position in M^* as $\underline{\mathsf{N}}_i$ had in M and for $p + 1 \leq j \leq d$ we have $N^o_j \equiv N_j$.

$\langle 3\rangle 3.$ *Depth*$(M^*) \geq d + r \geq d + 1$.
Depth$(M^*) \geq d + r$ by Step $\langle 3\rangle 2$ and 8E4.1 and $d + r \geq d + 1$ since $r \geq 1$ (Step $\langle 2\rangle 6$).

Proof of Detailed Stretching Lemma IV

$\langle 2 \rangle 8.$ M^* is indeed a long typed term.

$\langle 3 \rangle 1.$ LET: Γ_i be the context that assigns to the initial abstractors of N_i the types they have in M.

$\langle 3 \rangle 2.$ The set $Con(B_p) \cup Con(M) \cup \Gamma_0 \cup \ldots \cup \Gamma_{p+r}$ is consistent.

$\langle 4 \rangle 1.$ $\Gamma_0 \cup \ldots \cup \Gamma_d$ is consistent.

Since M has no bound variable clashes, the variables in $\Gamma_0, \ldots, \Gamma_d$ are all distinct.

$\langle 4 \rangle 2.$ $Con(B_p) \subseteq \Gamma_0 \cup \ldots \cup \Gamma_d$.

$\langle 5 \rangle 1.$ Every variable free in B_p is bound in one of $N_0, \ldots, N_p$ because M is closed and $\underline{B_p}$ is in $\underline{N_p}$.

$\langle 5 \rangle 2.$ Therefore, by the definition of typed term (5A1) we get $B_p \in \mathbb{TT}(\Gamma_0, \ldots, \Gamma_p)$.

$\langle 5 \rangle 3.$ By the definition of $Con()$ we obtain $Con(B_p) \subseteq \Gamma_0 \cup \ldots \cup \Gamma_p$.

$\langle 5 \rangle 4.$ $\Gamma_0 \cup \ldots \cup \Gamma_p \subseteq \Gamma_0 \cup \ldots \cup \Gamma_d$.

Proof of Detailed Stretching Lemma V

$\langle 4 \rangle 3$. $Con(M) \subseteq \Gamma_0 \cup \ldots \cup \Gamma_d$.
Since M is closed, $Con(M) = \emptyset$.

$\langle 4 \rangle 4$. $\Gamma_0 \cup \ldots \cup \Gamma_{p+r} \subseteq \Gamma_0 \cup \ldots \cup \Gamma_d$.

$\langle 3 \rangle 3$. Since M is a genuine typed term and Step $\langle 3 \rangle 2$ holds and the abstractors in M whose scope contain $\underline{\mathrm{B}}_{p+r}$, are exactly the initial abstractors of $N_0, \ldots, N_{p+r}$ we can apply Lemma 5B2.1(ii) and conclude that M^* is a genuine typed term.

$\langle 3 \rangle 4$. M^* is long since M is long and in the substitution of $\underline{B}_{p+r}$ by $\underline{B}_p$ the types of $\underline{B}_{p+r}$ and $\underline{B}_p$ are the same.

$\langle 3 \rangle 5$. M^* is closed since M is closed and the substitution of $\underline{B}_{p+r}$ by $\underline{B}_p$ has not removed any abstractor.

Proof of Detailed Stretching Lemma VI

$\langle 1 \rangle 2$. *Long*(τ) is infinite.
By repetition of Step $\langle 1 \rangle 1$.

Subsection 3

Shrinking Lemma

Detailed Shrinking Lemma

Lemma 12 (8F3 in Hindley's)

If $Long(\tau)$ has a member M^{τ} with depth $\geq \mathbb{D}(\tau)$ then

1. *it has a member $M^{*\tau}$ with*

$$Depth(M^{\tau}) - ||\tau|| \leq Depth(M^{*\tau}) < Depth(M^{\tau})$$

2. *it has a member N^{τ} with*

$$\mathbb{D}(\tau) - ||\tau|| \leq Depth(N^{\tau}) < \mathbb{D}(\tau)$$

Proof of Detailed Shrinking Lemma I

$\langle 1 \rangle 1.$ If $Long(\tau)$ has a member M^{τ} with depth $\geq \mathbb{D}(\tau)$ then it has a member $M^{*\tau}$ with:

$$Depth(M^{\tau}) - ||\tau|| \leq Depth(M^{*\tau}) < Depth(M^{\tau})$$

$\langle 2 \rangle 1.$ LET: M be a member of $Long(\tau)$ without bound-variable clashes.

$\langle 2 \rangle 2.$ LET: $d = Depth(M)$. $d \geq \mathbb{D}(\tau) \geq 2$.

$\langle 3 \rangle 1.$ $d = Depth(M) > \mathbb{D}(\tau)$ by hypothesis.

$\langle 3 \rangle 2.$ By Definition, $\mathbb{D}(\tau) = |\tau| \times ||\tau||$.

$\langle 3 \rangle 3.$ $|\tau| \geq 2$ since τ is composite. Notice that τ must be composite since atomic types have no inhabitants.

$\langle 3 \rangle 4.$ $\mathbb{D}(\tau) \geq 2$.

Proof of Detailed Shrinking Lemma II

$\langle 2 \rangle 3.$ Consider any argument-branch of M with lenght d. It has form

$$\langle N_0, \dots, N_d \rangle$$

where $\underline{\mathsf{N}}_0 \equiv \underline{\mathsf{M}}$ and $\underline{\mathsf{N}}_{i+1}$ is an argument of $\underline{\mathsf{N}}_i$ for $i = 0, \dots, d-1$. We will shrink this branch.

By 8E4.1, since $Depth(M) = d$, M has at least one argument-branch with length d.

$\langle 2 \rangle 4.$ Each N_i has form

$$N_i \equiv \lambda x_{i,1} \dots x_{i,m_i} . y_i P_{i,1} \dots P_{i,n_i} \qquad (m_i, n_i \geq 0)$$

$\langle 2 \rangle 5.$ LET: $\rho_i \equiv \rho_{i,1} \to \dots \to \rho_{i,m_i} \to a_i$ be the type of N_i.

$\langle 2 \rangle 6.$ $IAT(N_i) = \langle \rho_{i,1}, \dots, \rho_{i,m_i} \rangle$.

Since $\underline{\mathsf{N}}_i$ is long, the types of $x_{i,1}, x_{i,2}, \dots$ are exactly $\rho_{i,1}, \rho_{i,2}, \dots$. By the definition of IAT we obtain $IAT(N_i) = \langle \rho_{i,1}, \dots, \rho_{i,m_i} \rangle$.

$\langle 2 \rangle 7.$ LET: $\underline{\mathsf{B}}_i$ be the body of N_i, just as in the proof of Lemma 8F2 (the previous lemma). The type of $\underline{\mathsf{B}}_i$ is a_i.

Since the type of $\underline{\mathsf{B}}_i$ is the tail of the type of $\underline{\mathsf{N}}_i$.

Proof of Detailed Shrinking Lemma III

$\langle 2 \rangle 8$. LET: the sequence $d_0, d_1, \ldots$ be defined as follows. $d_0 = 0$. d_{j+1} is the least index greater than d_j such that $IAT(N_{d_{j+1}})$ differs from all of:

$$IAT(N_{d_0}), \ldots, IAT(N_{d_j})$$

.

$\langle 2 \rangle 9$. LET: n be the greatest integer such that d_n is defined.

$\langle 2 \rangle 10$. $d_0, \ldots, d_n$ partition the set $\{0, 1, \ldots, d\}$ into the following $n+1$, non empty sets, which will be called **IAT-intervals**:

$$\mathbb{I}_j = \{d_j, d_j + 1, \ldots, d_{j+1} - 1\} \qquad (0 \leq j \leq n-1)$$
$$\mathbb{I}_n = \{d_n, d_n + 1, \ldots, d\}$$

$\langle 2 \rangle 11$. If $\mathbb{I}_j$ contains two numbers p and $p+r$, with $r \geq 1$ and B_p and B_{p+r} have the same type we shal call $\langle p, p+r \rangle$ a **tail-repetition**. It will be called **minimal** iff there is no other tail-repetition $\langle p', q' \rangle$ with $p \leq p' < q' \leq p + r$.

Proof of Detailed Shrinking Lemma IV

$\langle 2 \rangle$12. At least one *IAT*-interval contains a tail-repetition.

$\langle 3 \rangle$1. Suppose, by contradiction, that no interval contained a tail-repetition.

$\langle 3 \rangle$2. An $\mathbb{I}_j$ that contains no tail-repetition must have $\leq ||\tau||$ members.

$\langle 4 \rangle$1. For such an $\mathbb{I}_j$, the atoms:

$$a_{d_j}, \ldots, a_{d_{j+1}-1}$$

must all be distinct.

$\langle 4 \rangle$2. By Step $\langle 2 \rangle$5, each a_i occurs in ρ_i.

$\langle 4 \rangle$3. By 8E7, ρ_i occurs in τ. So, a_i occurs in τ.

$\langle 4 \rangle$4. By definition, there are only $||\tau||$ distinct atoms in τ.

$\langle 4 \rangle$5. Hence, $\mathbb{I}_j$ has $\leq ||\tau||$ members.

Proof of Detailed Shrinking Lemma V

$\langle 3 \rangle$3. Since there are $n+1$ *IAT* intervals in the given branch, the branch would have $\leq (n+1) \times ||\tau||$ members.

$\langle 3 \rangle$4. $n+1 \leq |\tau|$. So, the branch would have $\leq |\tau| \times ||\tau||$ members.

$\langle 4 \rangle$1. Since our argument-branch has d members after $\underline{\mathsf{N}}_0$, we have $n \leq d$ and $d_n \leq d$.

$\langle 4 \rangle$2. $0 = d_0 < d_1 < \ldots < d_n \leq d$.

$\langle 4 \rangle$3. For each i, $IAT(N_i)$ is identical to one of:

$$IAT(N_{d_0}), IAT(N_{d_1}), \ldots, IAT(N_{d_n})$$

where each one of the *IAT*'s in the equation above are distinct.

$\langle 4 \rangle$4. $n+1 \leq \#(NSS(\tau)) + 1$
By 8E7, each one of the $n+1$ $IAT's$ are empty or members of $NSS(\tau)$. Since they are distinct, at most one of them is empty.

$\langle 4 \rangle$5. $\#(NSS(\tau)) \leq |\tau| - 1$
By 9E9.3(ii)

$\langle 3 \rangle 5$. However the branch has $d+1$ members and using Step $\langle 2 \rangle 2$ we obtain

$$d + 1 = Depth(M) + 1 \geq \mathbb{D}(\tau) + 1 > |\tau| \times ||\tau||$$

which contradicts Step $\langle 3 \rangle 4$.

$\langle 2 \rangle 13$. We start to build M^* as follows. In the given branch take the last $\mathbb{I}_j$ containing a tail-repetition, choose a minimal tail-repetition $\langle p, p+r \rangle$ in it and change M to a new term M' by replacing B_p by B_{p+r}.

Proof of Detailed Shrinking Lemma VII

$\langle 2\rangle$14. M' is a genuine typed term. M' is a long β-nf with the same type as M. Also $|M'| < |M|$.

$\langle 3\rangle$1. M' is a genuine typed term, with the same type as M.
We repeat the argument used in the proof of the Stretching Lemma (8F2):

$\langle 4\rangle$1. LET: Γ_i be the context that assigns to the initial abstractors of N_i the types they have in M.

$\langle 4\rangle$2. The set $Con(B_{p+r}) \cup Con(M) \cup \Gamma_0 \cup \ldots \cup \Gamma_p$ is consistent.

$\langle 5\rangle$1. $\Gamma_0 \cup \ldots \cup \Gamma_d$ is consistent.
Since M has no bound variable clashes, the variables in $\Gamma_0, \ldots, \Gamma_d$ are all distinct.

$\langle 5\rangle$2. $Con(B_{p+r}) \subseteq \Gamma_0 \cup \ldots \cup \Gamma_d$.

$\langle 6\rangle$1. Every variable free in B_{p+r} is bound in one of $N_0, \ldots, N_{p+r}$ because M is closed and $\underline{\mathsf{B}}_{p+r}$ is in $\underline{\mathsf{N}}_{p+r}$.

Proof of Detailed Shrinking Lemma VIII

$\langle 6 \rangle$2. Therefore, by the definition of typed term (5A1) we get $B_{p+r} \in \mathbb{TT}(\Gamma_0 \cup \ldots \cup \Gamma_{p+r})$.

$\langle 6 \rangle$3. By the definition of *Con*() we obtain $Con(B_{p+r}) \subseteq \Gamma_0 \cup \ldots \cup \Gamma_{p+r}$.

$\langle 6 \rangle$4. $\Gamma_0 \cup \ldots \cup \Gamma_{p+r} \subseteq \Gamma_0 \cup \ldots \cup \Gamma_d$.

$\langle 5 \rangle$3. $Con(M) \subseteq \Gamma_0 \cup \ldots \cup \Gamma_d$.
Since M is closed, $Con(M) = \emptyset$.

$\langle 5 \rangle$4. $\Gamma_0 \cup \ldots \cup \Gamma_{p+r} \subseteq \Gamma_0 \cup \ldots \cup \Gamma_d$.

$\langle 4 \rangle$3. Since M is a genuine typed term and Step $\langle 4 \rangle$2 holds and the abstractors in M whose scope contain $\underline{B}_p$, are exactly the initial abstractors of $N_0, \ldots, N_p$ we can apply Lemma 5B2.1(ii) and conclude that M' is a genuine typed term with the same type as M.

Proof of Detailed Shrinking Lemma IX

$\langle 3 \rangle 2.$ M' is a long β-nf.
Since M is a long β-nf and B_p and B_{p+r} have the same type.

$\langle 3 \rangle 3.$ $|M'| < |M|$.
Since B_p properly contains B_{p+r} we have $|B_{p+r}| < |B_p|$ and hence $|M'| < |M|$.

Proof of Detailed Shrinking Lemma X

$\langle 2 \rangle 15.$ Although M' might not be closed, there is a procedure in which, from M', we can obtain a long β-nf M'' with the same type and depth as M' which is closed. Notice that we are not claiming that M' and M'' are related by α-conversion or any other way.

$\langle 3 \rangle 1.$ First, notice that M' might not be closed.
M' might not be closed because the change from M to M' has removed the initial abstractors of $\underline{\mathsf{N}}_{p+1}, \ldots, \underline{\mathsf{N}}_{p+r}$ from M, and so some free variables occurrences in $\underline{\mathsf{B}}_{p+r}$ that were bound in M might now be free in M'.

$\langle 3 \rangle 2.$ LET: $\underline{\mathsf{v}}$ be free in the occurrence of $\underline{\mathsf{B}}_{p+r}$ in M' that has replaced $\underline{\mathsf{B}}_p$ in M. LET: $\underline{\mathsf{v}}$ be also free in M'.

Proof of Detailed Shrinking Lemma XI

$\langle 3 \rangle 3.$ There is a variable in $x_{d_q,k} \in IA(N_{d_q})$, with $d_q \leq p$ that has the same type as v.

$\langle 4 \rangle 1.$ v occurs in $IA(\underline{\mathsf{N}}_h)$ for some h with $p + 1 \leq h \leq p + r$.
Since $\underline{\mathsf{v}}$ is free in M', v does not occur in a covering abstractor of this occurrence of B_{p+r} in M'. This covering abstractors are exactly the initial abstractors of $\underline{\mathsf{N}}_0, \ldots, \underline{\mathsf{N}}_p$ in M so:

$$v \notin IA(\underline{\mathsf{N}}_0) \cup \ldots \cup IA(\underline{\mathsf{N}}_p)$$

However, M is closed and therefore our $\underline{\mathsf{v}}$, in M, must be in the scope of a $\underline{\lambda v}$ in one of $IA(\underline{\mathsf{N}}_0), \ldots, IA(\underline{\mathsf{N}}_{p+r})$. Hence, v occurs in $IA(\underline{\mathsf{N}}_h)$ for some h with $p + 1 \leq h \leq p + r$.

$\langle 4 \rangle 2.$ In our notation, we have $v \equiv x_{h,k}$ for some $k \leq m_h$. Also, the type of v is $\rho_{h,k} \in IAT(\underline{\mathsf{N}}_h)$.

Proof of Detailed Shrinking Lemma XII

$\langle 4 \rangle 3.$ $IAT(\underline{\mathrm{N}}_h) = IAT(\underline{\mathrm{N}}_{d_q})$ for some $q \leq j$.
Since the tail-repetition $\langle p, p + r \rangle$ is in the interval $\mathbb{I}_j$, by our definition of $d_0, \ldots, d_n$, we get that $IAT(\underline{\mathrm{N}}_h)$ coincides with:

$$IAT(\underline{\mathrm{N}}_{d_0}), \ldots, IAT(\underline{\mathrm{N}}_{d_j})$$

$\langle 4 \rangle 4.$ Hence, there is a variable $x_{d_q,k} \in IA(N_{d_q})$ with the same type as v.

$\langle 4 \rangle 5.$ $d_q \leq p$.
From Step $\langle 4 \rangle 3$, we have $q \leq j$, which implies $d_q \leq d_j$. Since the tail-repetition $\langle p, p + r \rangle$ occurs in $\mathbb{I}_j$ we have $p \geq d_j$.

Proof of Detailed Shrinking Lemma XIII

$\langle 3 \rangle 4$. Replace v by this variable. The result will be a long β-nf with the same type and depth as M' and containing one less free variable.

From $\langle 3 \rangle 3$, we see that this variable is bound by an abstractor in N_{d_q}, where $d_q \leq p$. Since the change from M to M' has only removed the initial abstractors of $\underline{\mathsf{N}}_{p+1}, \ldots, \underline{\mathsf{N}}_{p+r}$, this variable is still a bound variable in M'. Therefore, the result has one less free variable than M'. The result has the same type and depth because we substituted a variable v by another variable that has the same type as v.

Proof of Detailed Shrinking Lemma XIV

$\langle 3 \rangle 5.$ By similarly replacing every variable of $\underline{\mathsf{B}}_{p+r}$ that is free in M' by a new one which has the same type but is bound in M' we obtain a long β-nf M'' with the same type and depth as M' and which is closed.

Proof of Detailed Shrinking Lemma XV

$\langle 2 \rangle$16. $d - ||\tau|| \leq Depth(M'') \leq d$.

$\langle 3 \rangle$1. The number of arguments removed from the argument-branch is r, so our argument-branch now contains $d - r$ arguments.

$\langle 3 \rangle$2. Hence, $d - r \leq Depth(M'') \leq d$.

$\langle 3 \rangle$3. $r \leq ||\tau||$.
By definition, there are only $||\tau||$ distinct atoms in τ. Since the tail repetition $\langle p, p + r \rangle$ we took is minimal, we have $r \leq ||\tau||$.

$\langle 3 \rangle$4. $d - ||\tau|| \leq Depth(M'') \leq d$.

Proof of Detailed Shrinking Lemma XVI

$\langle 2 \rangle$17. If $\mathit{Depth}(M'') < d$ define $M^* \equiv M''$. If not, select a branch in M'' with length d and apply the removal procedure to it (the removal procedure is the one that from M produced M''). Keep doing this to shorten the branches with length d until there are none left. Define M^* to be the first term produced by this procedure whose depth is less than d.

$\langle 2 \rangle$18. Then:

$$d - ||\tau|| \leq \mathit{Depth}(M^*) < d$$

as required.

Proof of Detailed Shrinking Lemma XVII

$\langle 1 \rangle$2. If $Long(\tau)$ has a member M^τ with depth $\geq \mathbb{D}(\tau)$ then it has a member N^τ with:

$$\mathbb{D}(\tau) - ||\tau|| \leq Depth(N^\tau) < \mathbb{D}(\tau)$$

By repeating the whole procedure described in Step $\langle 1 \rangle$1 until you obtain an output with depth $< \mathbb{D}(\tau)$.

Example 8F3.1 I

Let $\tau \equiv (a \to a) \to a \to a$, and let M^τ be a typed version of the Church numeral for the number four, i.e:

$$M^\tau \equiv (\lambda u^{a\to a} v^a.(u(u(u(uv)))))^\tau$$

Then: $||\tau|| = 1$, $|\tau| = 4$, $\mathbb{D}(\tau) = |\tau| \times ||\tau|| = 4$.

Since $Depth(M) = 4$, the above shrinking procedure can be applied to M. There is only one argument-branch in M containing four subarguments, and its members are:

$$\underline{\lambda uv.u^4 v}, \quad \underline{u^3 v}, \quad \underline{u^2 v}, \quad \underline{uv}, \quad \underline{v}$$

Example 8F3.1 II

Let's call them $N_0, \ldots, N_4$ respectively. We have:

$$IAT(N_0) = \langle a \to a, a \rangle$$
$$IAT(N_1) = IAT(N_2) = IAT(N_3) = IAT(N_4) = \emptyset$$

Since the only change in $IAT(N_i)$ comes at $i = 1$, using the notation of the proof of 8F3, we have:

$$n = 1, \quad d_0 = 0, \quad d_1 = 1, \quad \mathbb{I}_0 = \{0\}, \quad \mathbb{I}_1 = \{1, 2, 3, 4\}$$

There are 3 minimal repetitions in $\mathbb{I}_1$ ($\langle 1, 2 \rangle$, $\langle 2, 3 \rangle$, $\langle 3, 4 \rangle$).

Example 8F3.1 III

According to our procedure, we pick the last one. We replace $\underline{uv}$ by $\underline{v}$ and this changes M to:

$$M^* \equiv \lambda uv.u^3 v$$

And now, notice that $Depth(M^*) = 3 < \mathbb{D}(\tau)$.

Warning: A Limitation of the Shrinking Lemma

As mentioned in 8D10(iii) the proof of the shrinking lemma does not necessarily apply to restricted systems of λ-terms, for example the λI-calculus. In fact, there is no guarantee that if we shrink a λI-term the result will still be a λI-term, since shrinking may cut out some variables.

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