Equational Reasoning - Presentation 1

Gabriel Silva

https://github.com/gabriel951/my_work

UnB

Professor: Daniele Nantes

November 9, 2020

Overview

Goal

To present Lemma 10.3.25 of Baader's book ([1]): "Term Rewriting and All That" and its proof. We follow the approach in [1].

I decided to use a **structured proof** as described by Leslie Lamport in [2], [3].

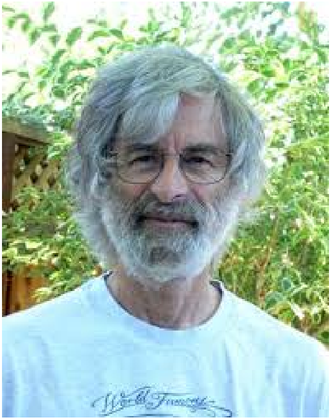

Figure 1: Leslie Lamport.

Why Use a Structured Proof?

A structured proof allows us to present a proof in details without compromising readability, due to its **hierarchical organization**.

Definition

For a vector $y\downarrow \in \mathbb{R}^n$, we denote the unit cube with lower left corner $y\downarrow$ by $C_{y\downarrow}$, i.e.

$$C_{y\downarrow} = \{z\downarrow \in \mathbb{R}^n \mid z\downarrow = y\downarrow + \sum_{1 \leq i \leq n} e\downarrow^{(i)} \cdot r_i, \text{ for some } r_i \in [0,1]\}.$$

Definition

We denote the line between $0{\downarrow}$ and $y{\downarrow}$ by $[0{\downarrow}, y{\downarrow}]$, i.e.

$$[0{\downarrow}, y{\downarrow}] := \{y{\downarrow} \cdot r \mid r \in [0, 1]\}.$$

The Lemma

Lemma (10.3.25 of Baader's book)

Let $x{\downarrow} \in \mathbb{N}^n$ *and* $p := ||x{\downarrow}||$. *There exist sequences of vectors* $y{\downarrow}^{(0)}, \ldots, y{\downarrow}^{(p)} \in \mathbb{N}^n$ *and* $z{\downarrow}^{(0)}, \ldots, z{\downarrow}^{(p)} \in \mathbb{R}^n$ *such that:*

1. $y{\downarrow}^{(0)} = 0{\downarrow} < y{\downarrow}^{(1)} < \ldots < y{\downarrow}^{(p)} = x{\downarrow}$.
2. $y{\downarrow}^{(i+1)} = y{\downarrow}^{(i)} + e{\downarrow}^{(j_i)}$ *for some* $j_i, 1 \leq j_i \leq n$.
3. $z{\downarrow}^{(i)} \in C_{y{\downarrow}^{(i)}} \cap [0{\downarrow}, x{\downarrow}]$.

Quick Comment

1. Item (1) just says that we can obtain a sequence $(y\downarrow)_n$ starting at $0\downarrow$ and going to $x\downarrow$.
2. Item (2) says that this sequence is obtained by adding unit vectors.
3. Item (3) says that $y\downarrow^{(i)}$ doesn't get too far from the straight route from $0\downarrow$ to $x\downarrow$.

Picture Proof

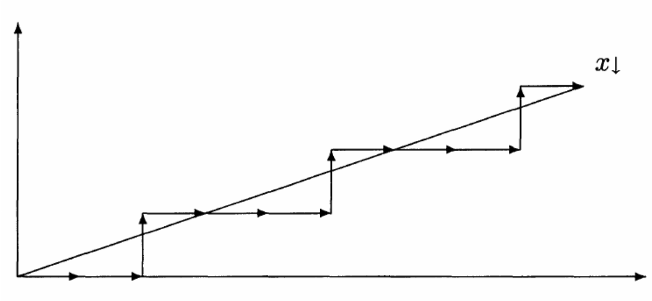

Figure 2: Illustration of The Idea of The Proof. Figure from [1].

Proof

Proof: We define the required sequence by induction.

$\langle 1 \rangle 1.$ Base Case.

$\langle 2 \rangle 1.$ **Define:** $y{\downarrow}^{(0)} := 0{\downarrow}$ and $z{\downarrow}^{(0)} := 0{\downarrow}$.

$\langle 2 \rangle 2.$ Item 1 is satisfied: $y{\downarrow}^{(0)} = 0{\downarrow}$. Additionally, if $p = 0$ then $y{\downarrow}^{(0)} = y{\downarrow}^{(p)} = 0{\downarrow} = x{\downarrow}$.

$\langle 2 \rangle 3.$ Item 2 is vacuously satisfied.

$\langle 2 \rangle 4.$ Item 3 is satisfied: $0{\downarrow} \in C_{0\downarrow} \cap [0{\downarrow}, x{\downarrow}]$.

$\langle 1 \rangle 2.$ Inductive Step.

$\langle 2 \rangle 1.$ **Assume:** $0 \leq l < p$.

$\langle 2 \rangle 2.$ **Let:** $y{\downarrow}^{(0)}, \ldots, y{\downarrow}^{(l)}$ and $z{\downarrow}^{(0)}, \ldots, z{\downarrow}^{(l)}$ be the sequences satisfying:

$$y{\downarrow}^{(0)} = 0{\downarrow} < y{\downarrow}^{(1)} < \ldots < y{\downarrow}^{(l)} < x{\downarrow},$$

and the parts (2) (for $0 \leq i < l$) and (3) (for $0 \leq i \leq l$).

These sequences exist by induction.

Proof (cont.)

⟨2⟩3. The elements of $C_{y\downarrow^{(l)}} \cap [0\downarrow, x\downarrow]$ are the vectors $z\downarrow$ such that:

$$z\downarrow = x\downarrow \cdot r = \sum_{1 \leq i \leq n} e\downarrow^{(i)} \cdot x_i \cdot r \qquad (1)$$

for some $r \in [0, 1]$ and

$$y_i^{(l)} \leq z_i = x_i \cdot r \leq y_i^{(l)} + 1, \text{ for } i = 1, \ldots, n. \qquad (2)$$

The vectors $z\downarrow$ satisfy (1) since $z\downarrow \in [0\downarrow, x\downarrow]$. They satisfy (2) since $z\downarrow \in C_{y\downarrow^{(l)}}$.

Proof (cont.)

$\langle 2 \rangle 4.$ **Let:** r_l be the largest number in $[0, 1]$ that satisfies Equation (2).
Such r_l exists since $C_{y\downarrow^{(l)}} \cap [0\downarrow, x\downarrow]$ is not empty (it contains $z\downarrow^{(l)}$ by item 3 of induction hypothesis).

$\langle 2 \rangle 5.$ **Let:** j_l, $1 \leq j_l \leq n$, be the index such that $x_{j_l} \cdot r_l = y_{j_l}^{(l)} + 1$
In Step $\langle 2 \rangle 4$ we picked r_l to be the largest number in $[0, 1]$ that satisfies Equation (2). Therefore, by the maximality of r_l there will be an index j_l such that $x_{j_l} \cdot r_l \leq y_{j_l}^{(l)} + 1$ is actually $x_{j_l} \cdot r_l = y_{j_l}^{(l)} + 1$.

$\langle 2 \rangle 6$. **Define:** $y{\downarrow}^{(l+1)} := y{\downarrow}^{(l)} + e{\downarrow}^{(j_l)}$ and $z{\downarrow}^{(l+1)} := x{\downarrow} \cdot r_l$.

Quick Comment: Intuitively, the appropriate vector $z{\downarrow}^{(l+1)}$ can be found by considering the intersection of the faces of the unit cube $C_{y{\downarrow}^{(l)}}$ with the line $[0{\downarrow}, x{\downarrow}]$, and then taking the point that is nearest to $x{\downarrow}$.

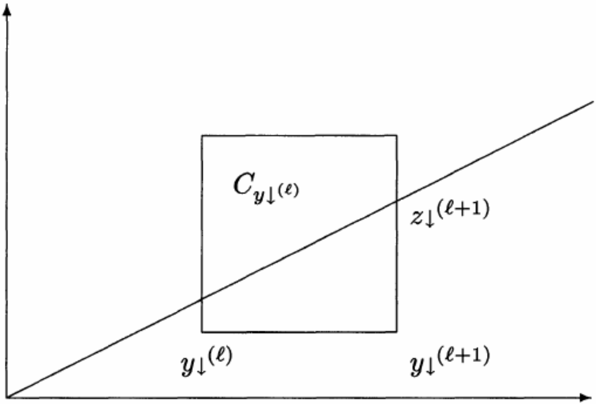

Fig. 10.3. How to find $z\downarrow^{(\ell+1)}$.

Figure 3: Figure from [1].

Proof (cont.)

$\langle 2 \rangle 7.$ Item 1 is satisfied.

$\langle 3 \rangle 1.$ $y{\downarrow}^{(0)} = 0{\downarrow} < y{\downarrow}^{(1)} < \ldots < y{\downarrow}^{(l)}$.
By item 1 of the induction hypothesis.

$\langle 3 \rangle 2.$ $y{\downarrow}^{(l)} < y{\downarrow}^{(l)} + e{\downarrow}^{(j_l)} = y{\downarrow}^{(l+1)}$.

$\langle 3 \rangle 3.$ $y{\downarrow}^{(l+1)} \leq x{\downarrow}$, and the equality holds iff $l + 1 = p$.

$\langle 4 \rangle 1.$ $y{\downarrow}^{(l+1)} \leq x{\downarrow}$ since $y_i^{(l+1)} \leq x_i$, for $i = 1, \ldots, n$.

$\langle 5 \rangle 1.$ **Case:** $i = j_l$. Then $y_{j_l}^{(l+1)} = y_{j_l}^{(l)} + 1 = x_{j_l} \cdot r_l \leq x_{j_l}$.

$\langle 5 \rangle 2.$ **Case:** $i \neq j_l$. Then $y_i^{(l+1)} = y_i^{(l)} \leq x_i$ since $y{\downarrow}^{(l)} < x{\downarrow}$ by item 1 of the induction hypothesis.

$\langle 4 \rangle 2.$ Since $||x{\downarrow}|| = p$ and $||y{\downarrow}^{(l+1)}|| = l + 1$ and by Step $\langle 4 \rangle 1$ we have that $y{\downarrow}^{(l+1)}$ and $x{\downarrow}$ are comparable, the equality holds iff $l + 1 = p$.

$\langle 2 \rangle 8.$ Item 2 is satisfied: $y{\downarrow}^{(i+1)} = y{\downarrow}^{(i)} + e{\downarrow}^{(j_i)}$ for some $j_i, 1 \leq j_i \leq n$

- $\langle 3 \rangle 1.$ **Case:** $i = l$. It holds by the definition of $y{\downarrow}^{(i+1)}$ in Step $\langle 2 \rangle 6$.
- $\langle 3 \rangle 2.$ **Case:** $i < l$. It holds by item 2 of the induction hypothesis.

$\langle 2\rangle 9.$ Item 3 is satisfied: $z{\downarrow}^{(l+1)} \in C_{y{\downarrow}^{(l+1)}} \cap [0{\downarrow}, x{\downarrow}]$.

$\langle 3\rangle 1.$ $z{\downarrow}^{(l+1)} \in C_{y{\downarrow}^{(l+1)}}$.

$\langle 4\rangle 1.$ **Case:** $i = j_l$. Then, $z_{j_l}^{(l+1)} = y_{j_l}^{(l+1)}$.
$z_{j_l}^{(l+1)} = x_{j_l} \cdot r_l$ by definition of $z{\downarrow}$ in Step $\langle 2\rangle 6$.
$x_{j_l} \cdot r_l = y_{j_l}^{(l)} + 1$ by Step $\langle 2\rangle 5$. Finally, $y_{j_l}^{(l)} + 1 = y_{j_l}^{(l+1)}$ by Step $\langle 2\rangle 6$.

$\langle 4\rangle 2.$ **Case:** $i \neq j_l$. Then, $y_i^{(l+1)} \leq z_i^{(l+1)} \leq y_i^{(l+1)} + 1$.
Then by Equation (2) and the fact that $y_i^{(l+1)} = y_i^{(l)}$ for $i \neq j_l$ we obtain $y_i^{(l+1)} \leq z_i^{(l+1)} = x_i \cdot r_l \leq y_i^{(l+1)} + 1$.

$\langle 4\rangle 3.$ By Steps $\langle 4\rangle 1$ and $\langle 4\rangle 2$ we have:
$y_i^{(l+1)} \leq z_i^{(l+1)} \leq y_i^{(l+1)} + 1$ for all i. We conclude that $z{\downarrow}^{(l+1)} \in C_{y{\downarrow}^{(l+1)}}$.

$\langle 3\rangle 2.$ $z{\downarrow}^{(l+1)} = x{\downarrow} \cdot r_l \in [0{\downarrow}, x{\downarrow}]$, as $r_l \in [0, 1]$.

Thank You

Thank you! Any doubts? [1]

[1] I now have a webpage, where I make my texts and my slides available: https://github.com/gabriel951/my_work

Bibliography

[1] F. Baader and T. Nipkow, *Term Rewriting and All That*. 1998.

[2] L. Lamport, "How to write a 21st century proof," *J. of Fixed Point Theory and Applications*, vol. 11, no. 1, pp. 43–63, 2012.

[3] L. Lamport, "How to write a proof," *The American math. monthly*, vol. 102, no. 7, pp. 600–608, 1995.