Nominal AC-Matching

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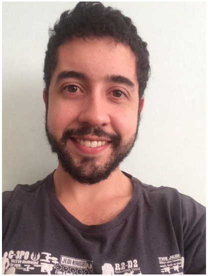

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Outline

Systems with Bindings

Systems with bindings frequently appear in mathematics and computer science, but are not captured adequately in first-order syntax.

For instance, the formulas

$$\forall x_1, x_2 : x_1 + 1 + x_2 > 0 \quad \text{and} \quad \forall y_1, y_2 : 1 + y_2 + y_1 > 0$$

are not syntactically equal, but should be considered equivalent in a system with binding and AC operators.

Nominal

The nominal setting extends first-order syntax, replacing the concept of syntactical equality by α-equivalence, which let us represent smoothly those systems.

Atoms and Variables

Consider a set of variables $\mathbb{X} = \{X, Y, Z, \ldots\}$ and a set of atoms $\mathbb{A} = \{a, b, c, \ldots\}$.

Nominal Terms

Definition 1 (Nominal Terms)

Nominal terms are inductively generated according to the grammar:

$$s, t \ ::= \ a \ | \ \pi \cdot X \ | \ \langle\rangle \ | \ [a]t \ | \ \langle s, t\rangle \ | \ f\ t \ | \ f^{AC}\ t$$

where π is a permutation that exchanges a finite number of atoms.

Freshness predicate

$a\#t$ means that if a occurs in t then it does so under an abstractor $[a]$.

A context is a set of constraints of the form $a\#X$. Contexts are denoted as Δ, Γ or ∇.

Alpha-equality

Our equality constraints $s \approx t$ take into account renaming of bound names:

$$[a]s \approx [b]t \implies s \approx (a\ b) \cdot t \wedge a \# t$$

Example 2 (In λ-calculus.)

$\lambda a.M \approx \lambda b.M[a := b]$ provided that b does not occur in M

Unification

Unification consists of “finding a way” to equal two terms by instantiating variables.

$$\boxed{s \approx_? t \quad \rightsquigarrow \quad \sigma s \approx \sigma t}$$

Matching

Matching can be seen as a simpler version of unification, where the terms on the right-hand side do not contain variables that can be instantiated.

$$\boxed{s \approx_? t \quad \rightsquigarrow \quad \sigma s \approx t}$$

Matching has applications in rewriting, functional programming, and metaprogramming.

Our Work in First-Order AC-Unification in a Nutshell

We **formalised** Stickel's seminal AC-unification algorithm in the PVS proof assistant. We proved the algorithm's termination, soundness, and completeness [AFSS22].

What is Tricky About AC? An Example

Let f be an AC function symbol. The solutions that come to mind when unifying:

$$f(X, Y) \approx_? f(a, W)$$

are:

$$\{X \to a, Y \to W\} \text{ and } \{X \to W, Y \to a\}$$

Are there other solutions?

What is Tricky About AC? An Example

Yes!

For instance,

- $\sigma_1 = \{X \to f(a, Z_1),\ Y \to Z_2,\ W \to f(Z_1, Z_2)\}$ and
- $\sigma_2 = \{X \to Z_1,\ Y \to f(a, Z_2),\ W \to f(Z_1, Z_2)\}$.

What is Tricky About AC? An Example

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$$f(X, Y) \approx_? f(a, W)$$

$$f(f(a, Z_1), Z_2) \approx_? f(a, f(Z_1, Z_2))$$

Nominal AC-matching

Nominal AC-matching is matching in the nominal setting in the presence of associative-commutative function symbols.

We proposed (to the best of our knowledge) the first nominal AC-matching algorithm, and formalised it in the PVS proof assistant ([AFFKS23]).

From unification to matching via $\mathcal{X}$

Given an algorithm of unification, one can adapt it by adding as a parameter a set of *protected variables* $\mathcal{X}$, which cannot be instantiated.

The adapted algorithm can then be used for:

- **Unification** - By putting $\mathcal{X} = \emptyset$.
- **Matching** - By putting $\mathcal{X}$ as the set of variables in the right-hand side.
- α-**Equivalence** - By putting $\mathcal{X}$ as the set of variables that appear in the problem.

From First-Order AC-Unification to Nominal AC-Matching

We modify our first-order AC-unification formalisation to obtain a formalised algorithm for nominal AC-matching.

Input

The algorithm is recursive and needs to keep track of

- the current context Γ,
- the equational constraints we must unify P,
- the substitution σ computed so far,
- the set of variables V that are/were in the problem, and
- the set of protected variables $\mathcal{X}$.

Hence, it's input is a quintuple $(\Gamma, P, \sigma, V, \mathcal{X})$.

$\mathit{Vars}(\mathit{rhs}(P)) \subseteq \mathcal{X}$

We assume the input satisfies $\mathit{Vars}(\mathit{rhs}(P)) \subseteq \mathcal{X}$.

Output

The output is a list of solutions, each of the form (Γ_1, σ_1).

applyACStep

The AC part of the algorithm (`ACMatch`) is handled by function `applyACStep`, which relies on two functions: `solveAC` and `instantiateStep`.

- `solveAC` builds the linear Diophantine equational system associated with the AC-matching equational constraint, generates the basis of solutions, and uses these solutions to generate the new AC-matching equational constraints.
- `instantiateStep` instantiates the moderated variables that it can.

Termination

Idea: for the particular case of matching (unlike unification) all the new moderated variables introduced by `solveAC` are instantiated by `instantiateStep`.

Termination is Easier

Hence, termination is much easier in nominal AC-matching than in first-order AC-unification.

Notation

$\nabla' \vdash \sigma\nabla$ denotes that $\nabla' \vdash a\#\sigma X$ holds for each $(a\#X) \in \nabla$.

$\nabla \vdash \sigma \approx_V \sigma'$ denotes that $\nabla \vdash \sigma X \approx_\alpha \sigma' X$ for all X in V. When V is the set of all variables $\mathbb{X}$, we write $\nabla \vdash \sigma \approx \sigma'$.

Solution to a Quintuple I

Our algorithm receives as input quintuples. Hence, to state the theorems of soundness and completeness, we need the definition of a solution (Δ, δ) to a quintuple $(\Gamma, P, \sigma, V, \mathcal{X})$.

Solution to a Quintuple II

Definition 3 (Solution for a Quintuple)

A solution to a quintuple $(\Gamma, P, \sigma, V, \mathcal{X})$ is a pair (Δ, δ), where the following conditions are satisfied:

1. $\Delta \vdash \delta\Gamma$.
2. if $a\#_{?}t \in P$ then $\Delta \vdash a\#\delta t$.
3. if $t \approx_{?} s \in P$ then $\Delta \vdash \delta t \approx_{\alpha} \delta s$.
4. there exists λ such that $\Delta \vdash \lambda\sigma \approx_{V} \delta$.
5. $dom(\delta) \cap \mathcal{X} = \emptyset$.

Solution to a Quintuple III

Note that if (Δ, δ) is a solution of $(\Gamma, \emptyset, \sigma, \mathbb{X}, \mathcal{X})$ this corresponds to the notion of (Δ, δ) being an instance of (Γ, σ) that does not instantiate variables in $\mathcal{X}$.

Soundness

Theorem 4 (Soundness for AC-Matching)

Let the pair (Γ_1, σ_1) *be an output of* $\mathtt{ACMatch}(\emptyset, \{t \approx_? s\}, id, Vars(t, s), Vars(s))$.

If (Δ, δ) *is an instance of* (Γ_1, σ_1) *that does not instantiate the variables in* s*, then*

$$(\Delta, \delta) \text{ is a solution to } (\emptyset, \{t \approx_? s\}, id, \mathbb{X}, Vars(s)).$$

Completeness

Theorem 5 (Completeness for AC-Matching)

Suppose that (Δ, δ) *is a solution to* $(\emptyset, \{t \approx_? s\}, id, \mathbb{X}, Vars(s))$*, that* $\delta \subseteq V$ *and that* $Vars(\Delta) \subseteq V$*.*

Then, there exists

$$(\Gamma, \sigma) \in \mathtt{ACMatch}(\emptyset, \{t \approx_? s\}, id, V, Vars(s))$$

such that (Δ, δ) *is an instance (restricted to the variables of* V*) of* (Γ, σ) *that does not instantiate the variables of* s*.*

Formalisation Size

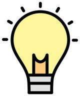
Theory	**Theorems**	**TCCs**	**Size (.pvs)**	**Size (.prf)**	**Size (%)**
AC Match Algorithm	22	35	12 kB	2.6 MB	10%
Auxiliary Lemmas AC Part	297	85	91 kB	12.2 MB	47%
Auxiliary Lemmas Nominal	592	140	124 kB	8.1 MB	31%
Diophantine & Data Structures	340	153	84 kB	3.3 MB	12%
Total	1251	413	311 kB	26.2 MB	100%

The Loop in Nominal AC-Unification

If we apply `ACMatch` to $f(X, W) \approx_? f(\pi \cdot X, \pi \cdot Y)$, where $X \notin \mathcal{X}$, we obtain a loop (more details in Appendix).

The problem happens when the same variable occurs as an argument of an AC operator multiple times with **different** suspended permutations.

A Different Approach to Nominal AC-Unification I



Idea: Explore the connection between nominal unification and higher-order pattern unification and the work of Boudet and Contejean in AC higher-order pattern unification [BC97].

Future Work

1. Consider the alternative approach to AC-unification proposed by Boudet, Contejean and Devie [BCD90, Bou93], which was used to define AC higher-order pattern unification.
2. Explore the connection between nominal and higher order patterns to obtain a nominal AC-unification algorithm.

Thank You

Thank you! Any comments/suggestions/doubts?

References I

- Mauricio Ayala-Rincón, Maribel Fernández, Gabriel Ferreira Silva, and Daniele Nantes Sobrinho, *A Certified Algorithm for AC-Unification*, Formal Structures for Computation and Deduction, FSCD 2022 (2022).
- Alexandre Boudet and Evelyne Contejean, *AC-Unification of Higher-Order Patterns*, Third International Conference on Principles and Practice of Constraint Programming CP97, 1997.
- Alexandre Boudet, Evelyne Contejean, and Hervé Devie, *A New AC Unification Algorithm with an Algorithm for Solving Systems of Diophantine Equations*, Proceedings of the Fifth Annual Symposium on Logic in Computer Science, LICS, 1990.
- Alexandre Boudet, *Competing for the AC-Unification Race*, J. of Autom. Reasoning (1993).

The loop in $f(X, W) \approx_? f(\pi \cdot X, \pi \cdot Y)$

We found a loop while solving $f(X, W) \approx_? f(\pi \cdot X, \pi \cdot Y)$.

Table of Solutions

The Diophantine equation associated[1] is $U_1 + U_2 = V_1 + V_2$.

The table with the solutions of the Diophantine equations is shown below. The name of the new variables was chosen to make clearer the loop we will fall into.

Table: Solutions for the Equation $U_1 + U_2 = V_1 + V_2$

U_1	U_2	V_1	V_2	$U_1 + U_2$	$V_1 + V_2$	New Variables
0	1	0	1	1	1	Z_1
0	1	1	0	1	1	W_1
1	0	0	1	1	1	Y_1
1	0	1	0	1	1	X_1

[1] variable U_1 is associated with argument X, variable U_2 is associated with argument W, variable V_1 is associated with argument $\pi \cdot X$ and variable V_2 is associated with argument $\pi \cdot Y$.

After `solveAC`

$$\{X \approx_? X_1, W \approx_? Z_1, \pi \cdot X \approx_? X_1, \pi \cdot Y \approx_? Z_1\}$$
$$\{X \approx_? Y_1, W \approx_? W_1, \pi \cdot X \approx_? W_1, \pi \cdot Y \approx_? Y_1\}$$
$$\{X \approx_? Y_1 + X_1, W \approx_? W_1, \pi \cdot X \approx_? W_1 + X_1, \pi \cdot Y \approx_? Y_1\}$$
$$\{X \approx_? Y_1 + X_1, W \approx_? Z_1, \pi \cdot X \approx_? X_1, \pi \cdot Y \approx_? Z_1 + Y_1\}$$
$$\{X \approx_? X_1, W \approx_? Z_1 + W_1, \pi \cdot X \approx_? W_1 + X_1, \pi \cdot Y \approx_? Z_1\}$$
$$\{X \approx_? Y_1, W \approx_? Z_1 + W_1, \pi \cdot X \approx_? W_1, \pi \cdot Y \approx_? Z_1 + Y_1\}$$
$$\{X \approx_? Y_1 + X_1, W \approx_? Z_1 + W_1, \pi \cdot X \approx_? W_1 + X_1, \pi \cdot Y \approx_? Z_1 + Y_1\}$$

After `instantiateStep`

7 branches are generated:

$B1 - \{\pi \cdot X \approx_? X\}, \sigma = \{W \mapsto \pi \cdot Y\}$

$B2 - \sigma = \{W \mapsto \pi^2 \cdot Y, X \mapsto \pi \cdot Y\}$

$B3 - \{f(\pi^2 \cdot Y, \pi \cdot X_1) \approx_? f(W, X_1)\}, \sigma = \{X \mapsto f(\pi \cdot Y, X_1)\}$

$B4 -$ *No solution*

$B5 -$ *No solution*

$B6 - \sigma = \{W \mapsto f(Z_1, \pi \cdot X), Y \mapsto f(\pi^{-1} \cdot Z_1, \pi^{-1} \cdot X)\}$

$B7 - \{f(\pi \cdot Y_1, \pi \cdot X_1) \approx_? f(W_1, X_1)\},$

$\quad \sigma = \{X \mapsto f(Y_1, X_1),\ W \mapsto f(Z_1, W_1), Y \mapsto f(\pi^{-1} \cdot Z_1, \pi^{-1} \cdot Y_1)\}$

The Loop

Focusing on *Branch7*, notice that the problem before the AC Step and the problem after the AC Step and instantiating the variables are:

$$P = \{f(X, W) \approx_? f(\pi \cdot X, \pi \cdot Y)\}$$
$$P_1 = \{f(X_1, W_1) \approx_? f(\pi \cdot X_1, \pi \cdot Y_1)\}$$

Formalisation Size in More Details

Theory	Theorems	TCCs	Size (.pvs)	Size (.prf)	Size (%)
ac_match_alg	22	35	12 kB	2.6 MB	10%
variant_inputs	22	5	8 kB	1.4 MB	5%
ac_step	48	11	13 kB	1.6 MB	6%
inst_step	75	17	21 kB	2.1 MB	8%
aux_unification	152	52	49 kB	7.1 MB	27%
Diophantine	77	44	24 kB	1.1 MB	4%
unification	120	13	28 kB	1.8 MB	7%
fresh_subs	38	5	12 kB	0.6 MB	2%
substitution	175	36	30 kB	2.6 MB	10%
equality	83	20	15 kB	1.7 MB	6%
freshness	15	10	5 kB	0.1 MB	$<$ 1 %
terms	147	53	30 kB	1.2 MB	5 %
atoms	14	3	4 kB	0.1 MB	$<$ 1 %
list	263	109	60 kB	2.2 MB	8 %
Total	1251	413	311 kB	26.2 MB	100%

The hypotheses $\delta \subseteq V$ and $Vars(\Delta) \subseteq V$

The hypotheses $\delta \subseteq V$ and $Vars(\Delta) \subseteq V$ are just a technicality that was put to guarantee that the new variables introduced by the algorithm in the AC-part do not clash with the variables in $dom(\delta)$ or in the terms in $im(\delta)$ or in $Vars(\Delta)$.

New: Removing Hypotheses $\delta \subseteq V$ and $Vars(\Delta) \subseteq V$ From the Proof of Completeness

Theorem 6 (Completeness for AC-Matching II)

Suppose that (Δ, δ) *is a solution to* $(\emptyset, \{t \approx_? s\}, id, \mathbb{X}, Vars(s))$.

Then, there exists

$$(\Gamma, \sigma) \in \mathtt{ACMatch}(\emptyset, \{t \approx_? s\}, id, Vars(t, s), Vars(s))$$

such that (Δ, δ) *is an instance (restricted to* $Vars(t, s)$*) of* (Γ, σ) *that does not instantiate the variables of* s.