

# Software Requirements Specification for Equation-of-motion methods: A set of tools for implementing and solving EOMs

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## Revision History

Date	Version	Notes
08-10-2020	1.0	Creation of SRS

# 1 Reference Material

This section records information for easy reference.

## 1.1 Table of Units

Throughout this document SI (Système International d’Unités) is employed as the unit system. In addition to the basic units, several derived units are used as described below. For each unit, the symbol is given followed by a description of the unit and the SI name.

symbol	unit	SI
J	energy	joule

## 1.2 Table of Symbols

The table that follows summarizes the symbols used in this document along with their units. The choice of symbols was made to be consistent with the quantum chemistry literature and with existing documentation for equation-of-motion methods. The symbols are listed in alphabetical order.

symbol	unit	description
$\hat{a}_p$	unitless	annihilation operator
$\hat{a}_p^\dagger$	unitless	creation operator
$E$	J	energy
$\hat{H}$	J	Hamiltonian operator
$\gamma_{pq}$	unitless	1-electron reduced density matrix
$\Gamma_{pqrs}$	unitless	2-electron reduced density matrix
$h_{pq}$	J	1-electron integral
$v_{pqrs}$	J	2-electron integral
$N$	unitless	number of electrons in a system
$\Psi$	unitless	stationary state wavefunction
$\Psi_0^{(N)}$	unitless	N-electron ground state wavefunction
$\Psi_n^{(N\pm k)}$	unitless	n-th excited state wavefunction
$p, q, r, s$	unitless	arbitrary spinorbital index
$\hat{Q}$	unitless	Transition operator

### 1.3 Abbreviations and Acronyms

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symbol	description
<hr/>	
A	Assumption
BS	Basis set(s)
DD	Data Definition
EA	Electron affinity
EOMEE	Equation-of-motion for excited states
GD	General Definition
GS	Goal Statement
IM	Instance Model
IP	Ionization Potential
LC	Likely Change
MO	Molecular orbital
PS	Physical System Description
PC	Personal Computer
QC	Quantum chemistry
R	Requirement
SRS	Software Requirements Specification
RDM	Reduced density matrix
T	Theoretical Model
TDM	Transition density matrix

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## 2 Introduction

The spectroscopic characterization of atomic and molecular systems is one of the fundamental techniques in the fields related to Chemistry. As part of the methods developed in computational quantum chemistry, Equation-of-motion (EOM) methods offer an alternative approach to the prediction of molecular properties such as excitation energies, ionization potentials or electron affinities by directly targeting the evaluation of these magnitudes.

The present document introduces the requirements description for the development of a set of tools (EOMEE) for implementing and solving of EOM equations. The formatting of this document will follow the SRS templates introduced by [Smith \(2016\)](#) and [Smith et al. \(2007\)](#).

### 2.1 Purpose of Document

This document intends to establish the parameters under which the quality of the program EOMEE can be assessed

### 2.2 Scope of Requirements

EOMEE will evaluate the energy variations in processes that involve up to two particle transitions. In a standard PC its applicability will be limited to molecular systems with about 60 MO basis due to the demands of memory resources.

### 2.3 Characteristics of Intended Reader

The reader of this document must have a Level-1 undergraduate knowledge of mathematics (specifically of linear algebra), and quantum chemistry or physics. Additionally, familiarity with the formalism of second quantization and some programming skills are required.

### 2.4 Organization of Document

The outline of the document is as follows: In section 3 the context of the systems, its constraints and characteristics of the user are described. Section 4 deepens the description by providing information of the intended inputs and outputs, the mathematical models that characterize the physical problem and the assumptions being made. Section (4) lists the system's functional and nonfunctional requirements. The expected changes, and the invariable ones are stated in sections 5 and 6, respectively. Finally, section 7 shows the traceability matrices and graphs and section 8 lists the values of auxiliary constants.

## 3 General System Description

This section provides general information about the system. It identifies the interfaces between the system and its environment, describes the user characteristics and lists the system constraints.

### 3.1 System Context

Figure 1 describes EOMEE’s context. External interactions are indicated by circles, and the program by a rectangle area. Information pertinent to the input and output data are indicated at the left and right of the program.

The user will submit the programs arguments (1&2 electron integrals, 1&2 RDMs); then EOMEE will process that information and return to the user the resulting TDMs.

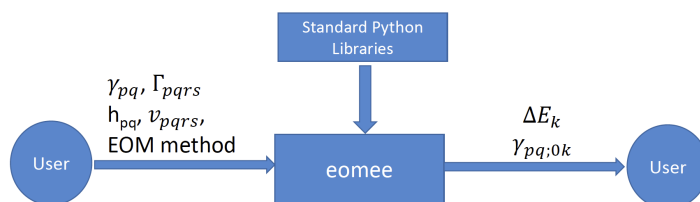


Figure 1: eomee Context

- User Responsibilities:
  - Provide the inputs for the atomic or molecular system of interest.
  - Guarantee that enough memory resources are available for the intended study system.
  - Verify that the output data has physical meaning for his intended system.
- EOMEE Responsibilities:
  - Detect data type mismatch, such as a string of characters instead of an array.
  - Detect data value mismatch, such as incorrect dimensions of input arrays.
  - Verify that the input matrices have the correct symmetries.
  - Transform the electron integrals and RDMs from the MO to the spinorbital representation.
  - Evaluate the transition energies and density matrices for the user selected EOM method.

### 3.2 User Characteristics

The user of EOMEE should have first-year undergraduate level of chemistry or physics so that she/he is familiar with the concepts of electronic excitation (or ionization) of an atomic or molecular system. To generate the program’s input data, the user needs also experience in the field of computational and theoretical chemistry or physics.

### 3.3 System Constraints

Disponibility of general Python libraries is required for the development of EOMEE.

## 4 Specific System Description

This section first presents the problem description, which gives a high-level view of the problem to be solved. This is followed by the solution characteristics specification, which presents the assumptions, theories, definitions and finally the instance models.

### 4.1 Problem Description

The development of EOMEE is motivated by the need to create a common framework for the solution of EOM methods based on the RDMs. This will allow the comparison between different wavefunction based approaches from computational quantum chemistry and serve as an intermediate point for the development of future scientific work based in the TDMs.

#### 4.1.1 Terminology and Definitions

This subsection provides a list of terms that are used in the subsequent sections and their meaning, with the purpose of reducing ambiguity and making it easier to correctly understand the requirements:

- **reference stationary state:** State onto which the excitation operator acts. Is usually taken as the lowest energy N-electron stationary state (ground state).
- **excitation or transition operator:** Operator that transforms the reference N-electron stationary state into a different stationary state.
- **excited state:** an stationary state other than the ground one. Higher in energy and potentially with a different number of electrons.
- **electron density matrix:** matrix corresponding to the electron density distribution represented in the spinorbital basis set.
- **spinorbital:** orthonormal function of the space and spin coordinates of a single electron.
- **electron integrals:** energy contributions of the different interaction between the electrons and nuclei in the system.
- **quantum chemistry methods:** methods that deal with solving the Schrödinger equation.



### 4.1.2 Physical System Description

The EOMEE's physical system includes the following elements:

PS1: The N-electron reference stationary state  $\Psi_0^{(N)}$ , characterized by the 1- and 2-RDMs.

PS2: The  $k$ th excited state  $\Psi_k^{(N\pm k)}$ .

PS3: The excitation operator  $\hat{Q}_k^{(N\pm k)}$  that transform  $\Psi_0^{(N)}$  into  $\Psi_k^{(N\pm k)}$ .

PS4: The electronic Hamiltonian operator  $\hat{H}$  which contains the information about the interactions between the particles in the system.

### 4.1.3 Goal Statements

Given the density matrices and the 1- and 2-electron integrals of a reference N-electron stationary state and an EOM method, the goal statements are:

GS1: Determine the transition energies.

GS2: Determine the transition density matrices.

## 4.2 Solution Characteristics Specification

The instance models that govern EOMEE are presented in Subsection 4.2.5. The information to understand the meaning of the instance models and their derivation is also presented, so that the instance models can be verified.

### 4.2.1 Assumptions

This section simplifies the original problem and helps in developing the theoretical model by filling in the missing information for the physical system. The numbers given in the square brackets refer to the theoretical model [T], general definition [GD], data definition [DD], instance model [IM], or likely change [LC], in which the respective assumption is used.

A1: The transition operator can be approximated by a finite expansion in second quantization operators strings (BS). [ T2],[ GD1]

A2: The EOM methods can be expressed in terms of the 1- and 2-reduced density matrices of the reference state and the 1- and 2-electron integrals. [IM1]-[IM5]

A3: The 1- and 2-electron integrals are in the MO basis representation. [DD2], [DD3]

A4: The input electron integrals and RDMs have the correct symmetries.[DD2], [DD3], [DD4], [DD5]

A5: The condition  $\hat{Q}^\dagger |\Psi_0^N\rangle = 0$  ("killer condition") is fulfilled. [GD2]

A6: The EOM method is expressed as a determined system of equations by projecting onto the transition operator’s BS. [GD2], [IM1]-[IM5]

#### 4.2.2 Theoretical Models

This section focuses on the general equations and laws that EOMEE is based on.

Number	T1
Label	<b>Time-independent Schrödinger equation</b>
Equation	$\hat{H}\Psi^{(N)} = E\Psi^{(N)}$
Description	The equation above represents the time-independent nonrelativistic Schrödinger equation. Where $\hat{H}$ is the hamiltonian operator, $\Psi^{(N)}$ , the N-electron stationary state wavefunction, and $(E)$ the corresponding total energy of the system. Finding the solutions to this equation is the main objective of quantum chemistry methods.
Source	Szabo and Ostlund (1996)
Ref. By	GD2

A more explicit representation of this equation includes its dependence on the nuclear and electronic coordinates of the particles composing the system ( $\mathbf{R}$  and  $\mathbf{r}$  respectively):

$$\hat{H}\Psi^{(N)}(\mathbf{r},\mathbf{R}) = E_{Tot}\Psi^{(N)}(\mathbf{r},\mathbf{R})$$

where  $\hat{H} = \hat{T}_{nuc} + \hat{T}_{el} + \hat{V}_{nuc-nuc} + \hat{V}_{el-el} + \hat{V}_{nuc-el}$  includes the kinetic ( $\hat{T}$ ) and potential ( $\hat{V}$ ) energies of the electrons ( $el$ ) and nuclei ( $nuc$ ).

Because quantum chemical methods are generally applied under the Born-Oppenheimer approximation (which allows to separates the nuclear and electronic motions) the actual expression used through this document is:

$$\hat{H}_{el}\Psi_{el}^{(N)}(\mathbf{r},\mathbf{R}) = E_{el}(\mathbf{R})\Psi_{el}^{(N)}(\mathbf{r},\mathbf{R})$$

in which the term  $\hat{T}_{nuc}$  is not included in the Hamiltonian operator. The coulombic interactions between the atoms ( $\hat{V}_{nuc-nuc}$ ) become a constant factor for a given set of  $\mathbf{R}$  (molecular geometry), so that the electronic energy ( $E_{el}(\mathbf{R})$ ) depends on these coordinates.

For notation simplicity we will drop the subscripts and use  $\hat{H}\Psi^{(N)} = E\Psi^{(N)}$  as the electronic Schrödinger equation for an N-electron, M-atom system.

Number	T2
Label	<b>Excited State</b>
Equation	$ \Psi_k^{(N\pm k)}\rangle = \hat{Q}_k^{(\pm k)}  \Psi_0^{(N)}\rangle$
Description	The wavefunction of the "excited" state ( $\Psi_k^{(N\pm k)}$ ) is defined by the action of a transition operator $\hat{Q}_k^{(\pm k)}$ onto a reference N-electron stationary state (ground state $\Psi_0^{(N)}$ ). As a result of the transition, the number of electrons ( $N$ ) in the initial reference system and the final one might differ ( $N \pm k$ ).
Source	—
Ref. By	GD1, GD2, T4

Number	T3
Label	<b>Transition energy</b>
Equation	$\Delta E_k = E_k - E_0$
Description	The above equation gives the energy difference between the initial ( $E_0$ ) and final states ( $E_k$ ) due to a transition process. The evaluation of this magnitude is the main goal of the present program.
Source	—
Ref. By	GD2

Number	T4
Label	<b>Transition Density Matrix</b>
Equation	$\gamma_{m;0k} = \langle \Psi_0^N   \hat{q}_m   \Psi_k^{N\pm k} \rangle$
Description	Represents the probability of finding the state defined by $\hat{q}_m^\dagger  \Psi_0^N\rangle$ in the "excited" state $ \Psi_k^{N\pm k}\rangle$ ; where $\hat{q}_m$ is some string of second quantization operators.
Source	Pernal (2018)
Ref. By	GD1, IM6

### 4.2.3 General Definitions

This section collects the laws and equations that will be used in building the instance models.

Number	GD1
Label	<b>Transition operator approximation</b>
SI Units	–
Equation	$Q_k^{(\pm k)} = \sum_n c_{n;k} \hat{q}_n$
Description	<p>The equation above represents a general notation for the definition of the transition operator (<math>Q_k^{(\pm k)}</math>) through a BS expansion (<math>\hat{q}_n</math>).</p> <p>Some common transition operators for which only the 1- and 2-RDMs would be needed are:</p> <p>electron removal: <math>\hat{Q}_k^{-1} = \sum_n c_{n;k} a_n</math></p> <p>electron attachment: <math>\hat{Q}_k^{+1} = \sum_n c_{n;k} a_n^\dagger</math></p> <p>electron excitation: <math>\hat{Q}_k^0 = \sum_{pq} c_{pq;k} a_p^\dagger a_q</math></p> <p>double electron removal: <math>\hat{Q}_k^{-2} = \sum_{pq} c_{pq;k} a_p a_q</math></p> <p>double electron attachment: <math>\hat{Q}_k^{+2} = \sum_{pq} c_{pq;k} a_p^\dagger a_q^\dagger</math></p>
Source	<a href="#">Rowe (1968)</a> ; <a href="#">Simons and Smith (1973)</a>
Ref. By	A1, A6, IM1-IM6

It is possible to define expression for  $Q_k^{(\pm k)}$  with more terms. For instance in the case of excitations, one could include higher than single excitation terms:

$$\hat{Q}_k^0 = \sum_{pq} c_{pq;k} a_p^\dagger a_q + \sum_{pqrs} c_{pqrs;k} a_p^\dagger a_q^\dagger a_s a_r + \sum_{pqrstu} c_{pqrstu;k} a_p^\dagger a_q^\dagger a_r^\dagger a_u a_t a_s + \dots$$

However the evaluation of the EOM equations would then required higher than the 1- and 2-RDMs, which is outside of the present program's scope.

Number	GD2
Label	<b>Eigenvalue Problem</b>
SI Units	–
Equation	$f(h_{pq}, v_{pqrs}, \gamma_{pq}, \Gamma_{pqrs})$ : (1) $\langle \Psi_0^N   q_m^\dagger [\hat{H}, \hat{Q}]   \Psi_0^N \rangle = \Delta E_k \langle \Psi_0^N   q_m^\dagger \hat{Q}   \Psi_0^N \rangle$ (2) $\langle \Psi_0^N   [q_m^\dagger, [\hat{H}, \hat{Q}]]_\pm   \Psi_0^N \rangle = \Delta E_k \langle \Psi_0^N   [q_m^\dagger, \hat{Q}]_\pm   \Psi_0^N \rangle$
Description	The equations above are the representation of the EOM methods as a generalized eigenvalue problem. By expanding the operator $\hat{Q}$ in some BS (see GD1 for specific expressions) and doing some algebraic manipulations of the resultant expressions, these equations can be reformulated in terms of the RDMs of the state $\Psi_0^N$ and the 1- and 2-electron integrals.
Source	Rowe (1968)
Ref. By	A2, A5, A6, T1, T2, T3, IM1-IM5

## Derivation of the eigenvalue problems

The Schrödinger equation (T1) for the excited state  $\Psi_k^{(N\pm k)}$  (T2) is:

$$\begin{aligned} \hat{H} |\Psi_k^{(N\pm k)}\rangle &= E_k |\Psi_k^{(N\pm k)}\rangle \\ \hat{H} Q_k^{(\pm k)} |\Psi_0^{(N)}\rangle &= E_k Q_k^{(\pm k)} |\Psi_0^{(N)}\rangle \end{aligned} \quad (1)$$

To directly determine the energy change due to the excitation process we apply the transition operator  $Q_k^{(\pm k)}$  to the Schrödinger equation for  $\Psi_0^{(N)}$ , subtracting it from Eq. (1):

$$Q_k^{(\pm k)} \hat{H} |\Psi_0^{(N)}\rangle = E_0 Q_k^{(\pm k)} |\Psi_0^{(N)}\rangle \quad (2)$$

$$\begin{aligned} (\hat{H} Q_k^{(\pm k)} - Q_k^{(\pm k)} \hat{H}) |\Psi_0^{(N)}\rangle &= (E_k - E_0) Q_k^{(\pm k)} |\Psi_0^{(N)}\rangle \\ [\hat{H}, Q_k^{(\pm k)}] |\Psi_0^{(N)}\rangle &= \Delta E_k Q_k^{(\pm k)} |\Psi_0^{(N)}\rangle \end{aligned} \quad (3)$$

This last expression can be solved by projecting onto a suitable set of states. Because we are interested in using the 1- and 2-RDMs, it is convenient to use the same basis set in which  $Q_k^{(\pm k)}$  is expanded [GD1] to build this set:

$$\langle \Psi_0^{(N)} | q_m^\dagger [\hat{H}, Q_k^{(\pm k)}] | \Psi_0^{(N)} \rangle = \Delta E_k \langle \Psi_0^{(N)} | q_m^\dagger Q_k^{(\pm k)} | \Psi_0^{(N)} \rangle \quad (4)$$

This last equation is suitable for the electron attachment and removal operators defined in [GD1], with which the final expressions only need the 1- and 2-RMDs.

Alternative expressions, more suitable for the cases of the excitation, double electron removal or attachment operators, can be derived by invoking the "killer condition" [A5]. This allows to add/subtract zero from (4) so that:

$$\begin{aligned} \langle \Psi_0^{(N)} | q_m^\dagger [\hat{H}, Q_k^{(\pm k)}] \pm [\hat{H}, Q_k^{(\pm k)}] q_m^\dagger | \Psi_0^{(N)} \rangle &= \Delta E_k \langle \Psi_0^{(N)} | q_m^\dagger Q_k^{(\pm k)} \pm Q_k^{(\pm k)} q_m^\dagger | \Psi_0^{(N)} \rangle \\ \langle \Psi_0^N | [q_m^\dagger, [\hat{H}, Q_k^{(\pm k)}]]_\pm | \Psi_0^N \rangle &= \Delta E_k \langle \Psi_0^N | [q_m^\dagger, Q_k^{(\pm k)}]_\pm | \Psi_0^N \rangle \end{aligned} \quad (5)$$

Expressions (4) and (5) constitute determined systems of linear equations. The equations defined by (5) can be used with any of the transition operators in [GD1]. As mentioned before they are more appropriate than (4) for the operators involving two particle transitions, because the commutation or anti-commutation relations can produce final operators with lower number of particles, so that only the 1 and 2-RDMs are required.

#### 4.2.4 Data Definitions

This section collects and defines all the data needed to build the instance models. The dimension of each quantity is also given.

Number	DD1
Label	<b>Electronic Hamiltonian operator</b>
Symbol	$\hat{H}$
SI Units	J
Equation	$\hat{H} = \sum_{pq} h_{pq} a_p^\dagger a_q + \frac{1}{2} \sum_{pqrs} v_{pqrs} a_p^\dagger a_q^\dagger a_s a_r$
Description	Second quantization notation for the electronic Hamiltonian operator under the Born-Oppenheimer approximation.  $h_{pq}$ and $v_{pqrs}$ are the one and two electron integrals in the spinorbital basis.
Sources	<a href="#">Szabo and Ostlund (1996)</a>
Ref. By	T1, DD2, DD3

Number	DD2
Label	<b>1-electron integrals</b>
Symbol	$h_{pq}$
SI Units	J
Equation	$h_{pq} = \int \phi_p^*(\mathbf{x})[\hat{t} + \hat{v}_{ext}]\phi_q(\mathbf{x})$
Description	Where: $\hat{t}$ and $\hat{v}_{ext}$ are the kinetic and external potential operators respectively. $\phi_i$ are the orthonormal spinorbitals and $\mathbf{x}$ represents the spatial and spin coordinates. The symmetry of this matrix is: $(h_{pq})^T = h_{qp}$
Sources	<a href="#">Pernal (2018)</a>
Ref. By	DD1, IM1-IM5

Number	DD3
Label	<b>2-electron integrals</b>
Symbol	$v_{pqrs}$
SI Units	J
Equation	$\langle pq rs \rangle = \int \int \phi_p^*(\mathbf{x})\phi_q^*(\mathbf{x})r_{12}^{-1}\phi_r(\mathbf{x})\phi_s(\mathbf{x})d\mathbf{x}_1d\mathbf{x}_2$
Description	These are the 2-electron integrals in the physicist notation. Where: $r_{12}$ is the distance between 2 electrons, $\phi_i$ the orthonormal spinorbitals and $\mathbf{x}$ the spatial and spin coordinates.
Sources	<a href="#">Szabo and Ostlund (1996)</a>
Ref. By	DD1, IM1-IM5

For real spinorbitals the permutational symmetries of the  $v_{pqrs}$  matrix are:

$$\begin{aligned}
\langle pq|rs \rangle &= \langle qp|sr \rangle = \langle rs|pq \rangle = \langle sr|qp \rangle \\
\langle rq|ps \rangle &= \langle qr|sp \rangle = \langle ps|rq \rangle = \langle sp|qr \rangle
\end{aligned} \tag{6}$$

The antisymmetrized 2-electron integrals are defined as:

$$\langle pq||rs \rangle = \langle pq|rs \rangle - \langle pq|sr \rangle \tag{7}$$

With the following permutational symmetries:

$$\begin{aligned} < pq|rs > = < qp|sr > = < rs|pq > = < sr|qp > \\ - < pq|sr > = - < qp|rs > = - < sr|pq > = - < rs|qp > \end{aligned} \quad (8)$$

Number	DD4
Label	<b>1-electron reduced density matrix</b>
Symbol	$\gamma_{pq}$
SI Units	–
Equation	$\gamma_{pq} = \langle \Psi_0   a_p^\dagger a_q   \Psi_0 \rangle$
Description	Where: $\Psi_0$ is an stationary state wavefunction, and $a_p^\dagger/a_q$ the creation/annihilation operator. The symmetry of this matrix is: $(\gamma_{pq})^* = \gamma_{qp}$
Sources	<a href="#">Pernal (2018)</a>
Ref. By	IM1-IM6

Number	DD5
Label	<b>2-electron reduced density matrix</b>
Symbol	$\Gamma_{pqrs}$
SI Units	–
Equation	$\Gamma_{pqrs} = \langle \Psi_0   a_p^\dagger a_q^\dagger a_s a_r   \Psi_0 \rangle$
Description	Where: $\Psi_0$ is an stationary state wavefunction, and $a_p^\dagger/a_q$ the creation/annihilation operator. The Permutational symmetries of this matrix are: $\Gamma_{pqrs} = \Gamma_{qpsr} = \Gamma_{rspq} = -\Gamma_{qprs} = -\Gamma_{pqsr}$
Sources	<a href="#">Pernal (2018)</a>
Ref. By	IM1-IM6



#### 4.2.5 Instance Models

This section transforms the problem defined in Section 4.1 into one which is expressed in mathematical terms. It uses concrete symbols defined in Section 4.2.4 to replace the abstract symbols in the models identified in Sections 4.2.2 and 4.2.3.

The goals GS1 and GS2 are solved by IM1 to IM6.

Number	IM1
Label	<b>Electron removal</b>
Input	$h_{pq}, v_{pqrs}, \gamma_{pq}, \Gamma_{pqrs}$ and IP_EOM( $h_{pq}, v_{pqrs}, \gamma_{pq}, \Gamma_{pqrs}$ )
Output	$\Delta E_k$ and $c_{n;k}$
Description	<p><math>h_{pq}</math> and <math>v_{pqrs}</math> are the 1- and 2-electron integrals in the spinorbital base</p> <p><math>\gamma_{pq}</math> and <math>\Gamma_{pqrs}</math> are the RDMs for the reference N-electron state.</p> <p>IP_EOM() is the EOM method corresponding to the electron removal transition operator.</p> <p><math>\Delta E_k</math> and <math>c_{n;k}</math> are the ionization potentials and coefficients from the BS expansion respectively.</p>
Sources	–
Ref. By	GD1, GD2, DD2, DD3, DD4, DD5

#### Derivation of the Electron removal EOM

Taking the transition operator ( $\hat{Q}_k^{-1}$ ) from GD1 and Eq. (1) from GD2 we get:

$$\begin{aligned}
 \langle \Psi_0^{(N)} | a_m^\dagger [\hat{H}, \hat{Q}_k^{(-1)}] | \Psi_0^{(N)} \rangle &= \Delta E_k \langle \Psi_0^{(N)} | a_m^\dagger \hat{Q}_k^{(-1)} | \Psi_0^{(N)} \rangle \\
 \sum_n c_{n;k} \langle \Psi_0^{(N)} | a_m^\dagger [\hat{H}, a_n] | \Psi_0^{(N)} \rangle &= \Delta E_k \sum_n c_{n;k} \langle \Psi_0^{(N)} | a_m^\dagger a_n | \Psi_0^{(N)} \rangle
 \end{aligned} \tag{9}$$

From the above equation, after some algebraic manipulations of the operators it is possible to deduct the following expression in terms of the RDMs and Hamiltonian integrals:

$$- \sum_{nq} h_{nq} \gamma_{mq} c_{n;k} + \frac{1}{2} \sum_{nqrs} v_{qnrs} \Gamma_{mqr} c_{n;k} = \Delta E_k \sum_n \gamma_{mn} c_{n;k} \tag{10}$$

Number	IM2
Label	<b>Electron attachment</b>
Input	$h_{pq}, v_{pqrs}, \gamma_{pq}, \Gamma_{pqrs}$ and EA_EOM( $h_{pq}, v_{pqrs}, \gamma_{pq}, \Gamma_{pqrs}$ )
Output	$\Delta E_k$ and $c_{n;k}$
Description	<p><math>h_{pq}</math> and <math>v_{pqrs}</math> are the 1- and 2-electron integrals in the spinorbital base</p> <p><math>\gamma_{pq}</math> and <math>\Gamma_{pqrs}</math> are the RDMs for the reference N-electron state.</p> <p>EA_EOM() is the EOM method corresponding to the electron attachment transition operator.</p> <p><math>\Delta E_k</math> and <math>c_{n;k}</math> are the electron affinities and coefficients from the BS expansion respectively.</p>
Sources	–
Ref. By	GD1, GD2, DD2, DD3, DD4, DD5

## Derivation of the Electron attachment EOM

We start with Eq. (1) from GD2 and take the transition operator ( $\hat{Q}_k^{+1}$ ) from GD1:

$$\begin{aligned}
\langle \Psi_0^{(N)} | a_m [\hat{H}, \hat{Q}_k^{(+1)}] | \Psi_0^{(N)} \rangle &= \Delta E_k \langle \Psi_0^{(N)} | a_m \hat{Q}_k^{(+1)} | \Psi_0^{(N)} \rangle \\
\sum_n c_{n;k} \langle \Psi_0^{(N)} | a_m [\hat{H}, a_n^\dagger] | \Psi_0^{(N)} \rangle &= \Delta E_k \sum_n c_{n;k} \langle \Psi_0^{(N)} | a_m a_n^\dagger | \Psi_0^{(N)} \rangle
\end{aligned} \tag{11}$$

From above's equation, after some algebraic manipulations of the operators it is possible to get the following expression in terms of the RDMs and electron integrals:

$$\left( \begin{aligned} & \sum_n h_{mn} c_{n;k} - \sum_{np} h_{pn} \gamma_{pm} c_{n;k} \\ & + \sum_{nps} v_{mpns} \gamma_{ps} c_{n;k} + \frac{1}{2} \sum_{npqs} v_{pqns} \Gamma_{pqsm} c_{n;k} \end{aligned} \right) = \Delta E_k \sum_n (\delta_{mn} - \gamma_{nm}) c_{n;k}$$

Number	IM3
Label	<b>Electron excitation</b>
Input	$h_{pq}, v_{pqrs}, \gamma_{pq}, \Gamma_{pqrs}$ and $\text{Exc\_EOM}(h_{pq}, v_{pqrs}, \gamma_{pq}, \Gamma_{pqrs})$
Output	$\Delta E_k$ and $c_{ij;k}$
Description	<p><math>h_{pq}</math> and <math>v_{pqrs}</math> are the 1- and 2-electron integrals in the spinorbital base</p> <p><math>\gamma_{pq}</math> and <math>\Gamma_{pqrs}</math> are the RDMs for the reference N-electron state.</p> <p><math>\text{Exc\_EOM}()</math> is the EOM method corresponding to the electron excitation transition operator.</p> <p><math>\Delta E_k</math> and <math>c_{ij;k}</math> are the excitation energies and coefficients from the BS expansion respectively.</p>
Sources	–
Ref. By	GD1, GD2, DD2, DD3, DD4, DD5

### Derivation of the Electron excitation EOM

Taking the transition operator ( $\hat{Q}_k^0$ ) from GD1 and Eq. (2) from GD2 we get:

$$\begin{aligned}
\langle \Psi_0^{(N)} | [a_k^\dagger a_l, [\hat{H}, \hat{Q}_k^{(0)}]] | \Psi_0^{(N)} \rangle &= \Delta E_k \langle \Psi_0^{(N)} | a_k^\dagger a_l \hat{Q}_k^{(0)} | \Psi_0^{(N)} \rangle \\
\sum_{ij} c_{ij;k} \langle \Psi_0^{(N)} | [a_k^\dagger a_l, [\hat{H}, a_i^\dagger a_j]] | \Psi_0^{(N)} \rangle &= \Delta E_k \sum_n c_{ij;k} \langle \Psi_0^{(N)} | a_k^\dagger a_l a_i^\dagger a_j | \Psi_0^{(N)} \rangle
\end{aligned} \tag{12}$$

From the above equation, after some algebraic manipulations of the operators it is possible to derive the following expression in terms of the RDMs and Hamiltonian integrals:

$$\left( \begin{aligned} & \sum_{ij} \left[ h_{li} \gamma_{kj} + h_{jk} \gamma_{il} - \sum_q (h_{jq} \delta_{li} \gamma_{kq} + h_{qi} \delta_{jk} \gamma_{ql}) \right] c_{ij;k} \\ & + \sum_{ij} \left[ \sum_{qs} (v_{lqis} \Gamma_{kqjs} + v_{jqks} \Gamma_{iqls}) \right] c_{ij;k} \\ & + \frac{1}{2} \sum_{ij} \left[ \sum_{rs} (v_{jlrs} \Gamma_{kirs} + \sum_q v_{qjrs} \delta_{li} \Gamma_{kqrs}) \right] c_{ij;k} \\ & + \frac{1}{2} \sum_{ij} \left[ \sum_{pq} (v_{pqik} \Gamma_{pqlj} + \sum_s v_{pqsi} \delta_{jk} \Gamma_{pqsl}) \right] c_{ij;k} \end{aligned} \right) = \Delta E_k \sum_{ij} (\delta_{li} \gamma_{kj} - \Gamma_{kijl}) c_{ij;k} \tag{13}$$

Number	IM4
Label	<b>Double IP</b>
Input	$h_{pq}, v_{pqrs}, \gamma_{pq}, \Gamma_{pqrs}$ and DIP_EOM( $h_{pq}, v_{pqrs}, \gamma_{pq}, \Gamma_{pqrs}$ )
Output	$\Delta E_k$ and $c_{ij;k}$
Description	<p><math>h_{pq}</math> and <math>v_{pqrs}</math> are the 1- and 2-electron integrals in the spinorbital base</p> <p><math>\gamma_{pq}</math> and <math>\Gamma_{pqrs}</math> are the RDMs for the reference N-electron state.</p> <p>DIP_EOM() is the EOM method corresponding to the double electron removal transition operator.</p> <p><math>\Delta E_k</math> and <math>c_{ij;k}</math> are the double ionization energies and coefficients from the BS expansion respectively.</p>
Sources	–
Ref. By	GD1, GD2, DD2, DD3, DD4, DD5

## Derivation of the Double IP EOM

Taking the transition operator ( $\hat{Q}_k^{-2}$ ) from GD1 and Eq. (2) from GD2 we get:

$$\begin{aligned}
\langle \Psi_0^{(N)} | [a_k^\dagger a_l^\dagger, [\hat{H}, \hat{Q}_k^{(-2)}]] | \Psi_0^{(N)} \rangle &= \Delta E_k \langle \Psi_0^{(N)} | [a_k^\dagger a_l^\dagger, \hat{Q}_k^{(-2)}] | \Psi_0^{(N)} \rangle \\
\sum_{ij} c_{ij;k} \langle \Psi_0^{(N)} | [a_k^\dagger a_l^\dagger, [\hat{H}, a_i a_j]] | \Psi_0^{(N)} \rangle &= \Delta E_k \sum_{ij} \langle \Psi_0^{(N)} | [a_k^\dagger a_l^\dagger, a_i a_j] | \Psi_0^{(N)} \rangle c_{ij;k}
\end{aligned} \tag{14}$$

From the above equation, after some algebraic manipulations of the operators it is possible to derive the following expression in terms of the RDMs and Hamiltonian integrals:

$$\left( \begin{aligned}
&2 \sum_{ij} (-h_{jl} \delta_{ik} + h_{jk} \delta_{il}) c_{ij;k} \\
&+ 2 \sum_{ij} (h_{ik} \gamma_{lj} - h_{il} \gamma_{kj}) c_{ij;k} \\
&+ 2 \sum_{ij,q} h_{jq} (\delta_{ik} \gamma_{lq} - \delta_{il} \gamma_{kq}) c_{ij;k} \\
&+ \sum_{ij} \nu_{jikl} c_{ij;k} + 2 \sum_{ij,q} \nu_{qjkl} \gamma_{qi} c_{ij;k} \\
&+ \sum_{ij,r} (\nu_{jilr} \gamma_{kr} - \nu_{jikr} \gamma_{lr}) c_{ij;k} \\
&+ 2 \sum_{ij,qr} (\nu_{iqrk} \delta_{lj} + \nu_{iqrl} \delta_{kj}) \gamma_{qr} c_{ij;k} \\
&+ 2 \sum_{ij,qr} (\nu_{jqrk} \Gamma_{qlri} + \nu_{jqlr} \Gamma_{qkri}) c_{ij;k} \\
&+ \sum_{ij,qrs} \nu_{qjrs} (\delta_{ki} \Gamma_{qlrs} - \delta_{li} \Gamma_{qkrs}) c_{ij;k}
\end{aligned} \right) = \Delta E_k \sum_{ij} \begin{pmatrix} 2\delta_{jk} \gamma_{li} + 2\delta_{il} \gamma_{kj} \\ -2\delta_{jk} \delta_{il} \end{pmatrix} c_{ij;k} \tag{15}$$

Number	IM5
Label	<b>Double EA</b>
Input	$h_{pq}, v_{pqrs}, \gamma_{pq}, \Gamma_{pqrs}$ and DEA_EOM( $h_{pq}, v_{pqrs}, \gamma_{pq}, \Gamma_{pqrs}$ )
Output	$\Delta E_k$ and $c_{ij;k}$
Description	<p><math>h_{pq}</math> and <math>v_{pqrs}</math> are the 1- and 2-electron integrals in the spinorbital base</p> <p><math>\gamma_{pq}</math> and <math>\Gamma_{pqrs}</math> are the RDMs for the reference N-electron state.</p> <p>DEA_EOM() is the EOM method corresponding to the double electron attachment transition operator.</p> <p><math>\Delta E_k</math> and <math>c_{ij;k}</math> are the double electron attachment energies and coefficients from the BS expansion respectively.</p>
Sources	–
Ref. By	GD1, GD2, DD2, DD3, DD4, DD5

## Derivation of the Double EA EOM

We start with Eq. (2) from GD2 and take the transition operator ( $\hat{Q}_k^{+2}$ ) from GD1:

$$\begin{aligned}
\langle \Psi_0^{(N)} | [a_k a_l, [\hat{H}, \hat{Q}_k^{(+2)}]] | \Psi_0^{(N)} \rangle &= \Delta E_k \langle \Psi_0^{(N)} | [a_k a_l, \hat{Q}_k^{(+2)}] | \Psi_0^{(N)} \rangle \\
\sum_{ij} c_{ij;k} \langle \Psi_0^{(N)} | [a_k a_l, [\hat{H}, a_i^\dagger a_j^\dagger]] | \Psi_0^{(N)} \rangle &= \Delta E_k \sum_{ij} \langle \Psi_0^{(N)} | [a_k a_l, a_i^\dagger a_j^\dagger] | \Psi_0^{(N)} \rangle c_{ij;k}
\end{aligned} \tag{16}$$

From the above equation, after some algebraic manipulations of the operators it is possible to derive the following expression in terms of the RDMs and Hamiltonian integrals:

$$\left( \begin{aligned}
&2 \sum_{ij} (h_{li} \delta_{kj} - h_{ki} \delta_{lj}) c_{ij;k} \\
&+ 2 \sum_{ij} (h_{ki} \gamma_{jl} - h_{li} \gamma_{jk}) c_{ij;k} \\
&+ 2 \sum_{ijp} (h_{pi} \delta_{lj} \gamma_{pk} + h_{pj} \delta_{ki} \gamma_{pl}) c_{ij;k} \\
&+ \sum_{ij} \nu_{lkij} c_{ij;k} + 2 \sum_{ijr} \nu_{lkjr} \gamma_{ir} c_{ij;k} \\
&+ \sum_{ijq} (\nu_{qlij} \gamma_{qk} - \nu_{qkij} \gamma_{ql}) c_{ij;k} \\
&+ 2 \sum_{ij,qr} (\nu_{qljr} \delta_{ki} - \nu_{qkjr} \delta_{li}) \gamma_{qr} c_{ij;k} \\
&+ 2 \sum_{ij,qr} (\nu_{qlir} \Gamma_{qjrk} - \nu_{qkir} \Gamma_{qjrl}) c_{ij;k} \\
&+ \sum_{ij,pqr} \nu_{pqjr} (\delta_{li} \Gamma_{pqrk} - \delta_{ki} \Gamma_{pqrl}) c_{ij;k}
\end{aligned} \right) = \Delta E_k \sum_{ij} \begin{pmatrix} 2\delta_{li} \delta_{kj} - 2\delta_{li} \gamma_{jk} \\ -2\delta_{kj} \gamma_{il} \end{pmatrix} c_{ij;k} \tag{17}$$

Number	IM6
Label	<b>Transition Density Matrix</b>
Input	$\gamma_{pq}$ , $\Gamma_{pqrs}$ and $c_{n;k}$
Output	$\gamma_{n;0k}$
Description	$\gamma_{pq}$ and $\Gamma_{pqrs}$ are the RDMs for the reference N-electron state. $c_{n;k}$ are the amplitudes of the BS expansion of the transition operator.
Sources	—
Ref. By	T4, GD1, DD4, DD5

### Derivation of the Transition Density Matrices

Starting from the definitions given in T2, T4 and GD1 for the TDM, excited state and transition operator the following expressions can be derived:

$$\hat{Q}_k^{(-1)} : \quad \gamma_{m;0k} = \sum_n c_{n;k} \langle \Psi_0^{(N)} | a_m^\dagger a_n | \Psi_0^{(N)} \rangle = \sum_n \gamma_{mn} c_{n;k} \quad (18)$$

$$\hat{Q}_k^{(+1)} : \quad \gamma_{m;0k} = \sum_n c_{n;k} \langle \Psi_0^{(N)} | a_m a_n^\dagger | \Psi_0^{(N)} \rangle = \sum_n (\delta_{mn} - \gamma_{mn}) c_{n;k} \quad (19)$$

$$\hat{Q}_k^{(0)} : \quad \gamma_{kl;0k} = \sum_{ij} c_{ij;k} \langle \Psi_0^{(N)} | a_k^\dagger a_l a_i^\dagger a_j | \Psi_0^{(N)} \rangle = \sum_{ij} (\delta_{li} \gamma_{kj} - \Gamma_{kijl}) c_{ij;k} \quad (20)$$

$$\hat{Q}_k^{(-2)} : \quad \gamma_{kl;0k} = \sum_{ij} c_{ij;k} \langle \Psi_0^{(N)} | a_k^\dagger a_l^\dagger a_i a_j | \Psi_0^{(N)} \rangle = \sum_{ij} \Gamma_{klji} c_{ij;k} \quad (21)$$

$$\hat{Q}_k^{(+2)} : \quad \gamma_{kl;0k} = \sum_{ij} c_{ij;k} \langle \Psi_0^{(N)} | a_k a_l a_i^\dagger a_j^\dagger | \Psi_0^{(N)} \rangle = \sum_{ij} (2\delta_{li} \delta_{kj} + 2\delta_{lj} \gamma_{ik} + 2\delta_{ki} \gamma_{jl} + \Gamma_{ijlk}) c_{ij;k} \quad (22)$$

#### 4.2.6 Input Data Constraints

Table 1 shows the data constraints on the input output variables. The column for physical constraints gives the physical limitations on the range of values that can be taken by the variable. The column for software constraints restricts the range of inputs to reasonable values. The software constraints will be helpful in the design stage for picking suitable

algorithms. The constraints are conservative, to give the user of the model the flexibility to experiment with unusual situations. The column of typical values is intended to provide a feel for a common scenario. The uncertainty column provides an estimate of the confidence with which the physical quantities can be measured. This information would be part of the input if one were performing an uncertainty quantification exercise.

The specification parameters in Table 1 are listed in Table 2.

Table 1: Input Variables

Var	Physical Constraints	Software Constraints	Typical Value	Uncertainty
$\gamma_{pq}$	$Tr(\gamma_{pq}) = N$	$\gamma_{pq} \geq \epsilon$	0 or 1	—
$\Gamma_{pqrs}$	$Tr(\Gamma_{pq}) = 0.5 * N(N - 1)$	$\Gamma_{pq} \geq \epsilon$	0 or 1	—
$h_{pq}$	$h_{pq} \geq 0$	$h_{pq} \geq \epsilon$	0.5	—
$v_{pqrs}$	$v_{pqrs} \geq 0$	$v_{pqrs} \geq \epsilon$	0.5	—

(\*) The value of  $\epsilon$  is to prevent numerical instabilities during the solution of the EOM equations (IM1-IM5)

Table 2: Specification Parameter Values

Var	Value
$N$	Number of electrons
$\epsilon$	$10^{-10}$

#### 4.2.7 Properties of a Correct Solution

EOMEE will provide and estimate of the single excitation, ionization potentials or electron affinity energies for an atomic or molecular system as selected by the user. These are expected to be positive energy values. For the transition density matrices the values should be between 0 and 1 for the calculation to be correct. These constraints are summarized in Table 3

## 5 Requirements

This section provides the functional requirements, the business tasks that the software is expected to complete, and the nonfunctional requirements, the qualities that the software is expected to exhibit.

Table 3: Output Variables

Var	Physical Constraints
$\Delta E_k$	$\Delta E_k > 0$
$\gamma_{n;0k}$	$0 \leq \gamma_{n;0k} \leq 1$

## 5.1 Functional Requirements

- R1: Input the 1- and 2-electron integrals in the MO basis representation, the 1- and 2-RDMs, and a selection of the EOM method to solve.
- R2: Verify that the input RDMs satisfy the physical constraints listed in Table 1.
- R3: Verify that the input electron integrals satisfy the symmetry properties in DD3.
- R4: Calculate the energies and TDMs using the instance models corresponding to the method selected by the user (IM1 - IM6)
- R5: Output  $\Delta E_0$  (from IM1 - IM5 as applicable) and  $\gamma_{n;0k}$  (from IM6).
- R6: The user should have the knowledge to assess whether the returned parameters have physical meaning for their system of interest.

## 5.2 Nonfunctional Requirements

- The code should be modular so that future improvements (like the implementation of new EOMs) don't imply major structural changes.
- It should be tested using as reference of correctness available literature data for atomic IP and electronic excitation energies.

## 6 Likely Changes

- LC1: Alternative expressions for the evaluation of IP, EA or excitation energies can be derived based on the application of the "killer condition" (A5)
- LC2: A different projection space than the one generated by the transition operator's BS could be used (A6), in which case TDMs might be required as inputs instead of the RDMs



## 7 Unlikely Changes

LC3: The matrix representation of the EOM equations.

## 8 Traceability Matrices and Graphs

The purpose of the traceability matrices is to provide easy references on what has to be additionally modified if a certain component is changed. Every time a component is changed, the items in the column of that component that are marked with an “X” may have to be modified as well. Table 4 shows the dependencies of theoretical models, general definitions, data definitions, and instance models with each other. Table 5 shows the dependencies of instance models, requirements, and data constraints on each other. Table 6 shows the dependencies of theoretical models, general definitions, data definitions, instance models, and likely changes on the assumptions.

	T1	T2	T3	T4	GD??	GD2	DD1	DD2	DD3	DD4	DD5	IM1	IM2	IM3	IM4	IM5	IM6
T1						X	X										
T2						X											
T3						X											
T4		X															
GD??		X		X								X	X	X	X	X	X
GD2	X	X	X									X	X	X	X	X	
DD1								X	X								
DD2							X					X	X	X	X	X	
DD3							X					X	X	X	X	X	
DD4												X	X	X	X	X	X
DD5												X	X	X	X	X	X
IM1					X	X		X	X	X	X						
IM2					X	X		X	X	X	X						
IM3					X	X		X	X	X	X						
IM4					X	X		x	X	X	X						
IM5					X	X		X	X	X	X						
IM6					X					X	X						

Table 4: Traceability Matrix Showing the Connections Between Items of Different Sections

The purpose of the traceability graphs is also to provide easy references on what has to be additionally modified if a certain component is changed. The arrows in the graphs represent

	IM1	IM2	IM3	IM4	IM5	IM6	R1	R2	R3	R4	R5	R6
IM1										X	X	
IM2										X	X	
IM3										X	X	
IM4										X	X	
IM5										X	X	
IM6										X	X	
R1												
R2												
R3												
R4												
R5												
R6												

Table 5: Traceability Matrix Showing the Connections Between Requirements and Instance Models

	A1	A2	A3	A4	A5	A6
T1						
T2	X					
T3						
T4						
GD??	X					
GD2					X	X
DD1						
DD2			X	X		
DD3			X	X		
DD4				X		
DD5				X		
IM1		X				X
IM2		X				X
IM3		X				X
IM4		X				X
IM5		X				X
IM6						

Table 6: Traceability Matrix Showing the Connections Between Assumptions and Other Items

dependencies. The component at the tail of an arrow is depended on by the component at the head of that arrow. Therefore, if a component is changed, the components that it points to should also be changed. Figure ?? shows the dependencies of theoretical models, general definitions, data definitions, instance models, likely changes, and assumptions on each other. Figure ?? shows the dependencies of instance models, requirements, and data constraints on each other.

## 9 Values of Auxiliary Constants

There are not auxiliary constants defined.

## References

- Katarzyna Pernal. Correlation energy from random phase approximations: A reduced density matrices perspective. *International Journal of Quantum Chemistry*, 118(1):e25462, 2018.
- D. J. Rowe. Equations-of-motion method and the extended shell model. *Reviews of Modern Physics*, 40(1):153–166, 1968. ISSN 00346861. doi: 10.1103/RevModPhys.40.153.
- Jack Simons and Wesley D Smith. Theory of electron affinities of small molecules. *The Journal of Chemical Physics*, 58(11):4899–4907, jun 1973. ISSN 0021-9606. doi: 10.1063/1.1679074.
- W. Spencer Smith. A rational document driven design process for scientific computing software. In Jeffrey C. Carver, Neil Chue Hong, and George Thiruvathukal, editors, *Software Engineering for Science*, chapter Section I – Examples of the Application of Traditional Software Engineering Practices to Science, pages 33–63. Taylor & Francis, 2016.
- W. Spencer Smith, Lei Lai, and Ridha Khedri. Requirements analysis for engineering computation: A systematic approach for improving software reliability. *Reliable Computing, Special Issue on Reliable Engineering Computation*, 13(1):83–107, February 2007.
- Attila Szabo and Neil S. Ostlund. *Modern Quantum Chemistry: Introduction to Advanced Electronic Structure Theory*. Dover Publications, Inc., 1996. ISBN 0-486-69186-1.