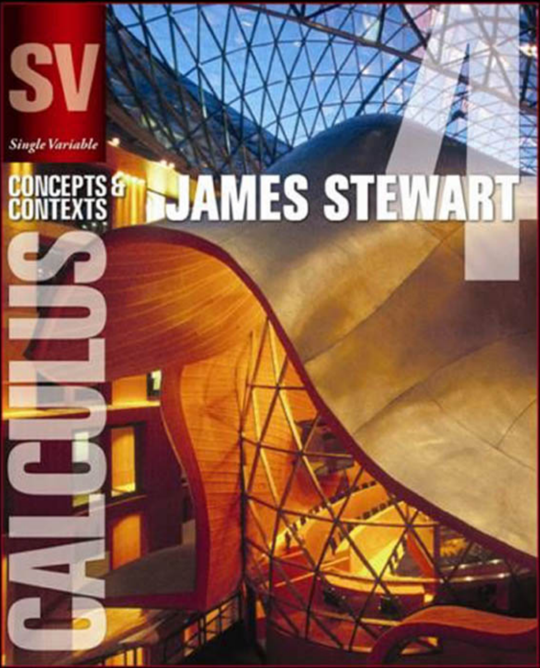
STUDY GUIDE FOR STEWART'S



ROBERT BURTON AND DENNIS GARITY

Study Guide

Single Variable Calculus
Concepts and Contexts

FOURTH EDITION

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For my family, wife, children, grandchildren, and parents: Vicki; Sarah, Maya & Kevin, Sarah, Taber & Alyssa, Harrison, Jeff & Laurie, Clare, Gray; Maurine & Robert Sr., Molcie & Simeon, with love.

RMB

For my mother and father – for their love and encouragement,

and for Marie, Diana, and Dylan – for their love, patience, and understanding

DJG

Preface

Note to Students — How to use this Study Guide

This *Study Guide* is designed to be used in a number of different ways:

- Before you attend a class that covers material from the text, read the first few paragraphs in the corresponding part of the *Study Guide* to become familiar with the key concepts and skills that you will need to master. Watch for where these concepts and skills appear during class.
- After you attend a class that covers material from the text, read the corresponding part of the *Study Guide* to get more explanation of concepts and skills in that section.
- If you get stuck while working on a problem in the text, find the corresponding SkillMaster that is needed for the problem. Read the explanation and try the problems in the *Study Guide* corresponding to that SkillMaster.
- Work through the Worked Examples to get extra practice on skills that you feel you need to spend more time on. Try each of the Worked Examples on your own before reading the solutions.
- When you are preparing for a test, use the master list of SkillMasters at the end of each chapter to find skills that you need to review. Use this list to get more practice in skills that need reinforcing.

Organization of Study Guide:

There is a section in the *Study Guide* corresponding to each section in the text. Each section contains most of the following:

- A brief introduction to the ideas in the section,
- A short list of key concepts,
- A short list of skills to master, called SkillMasters,
- A more detailed explanation of the concepts and skills, and
- Worked Examples for each of the SkillMasters.

The Worked Examples are a key feature of the *Study Guide.* Problem and hints are presented in a two column format. The problems are listed in the left column. Hints are often given in the right column. Following the problems and hints, detailed solutions are provided. You should always try to work the problems on your own before looking at the hints or solutions.

At the end of each chapter, a complete list of the SkillMasters for that chapter is provided.

Along with the material in each section of the *Study Guide* are links in the margin to earlier and later material in the text. The words that a link corresponds to are italicized, and the link is provided in the margin with the following graphic.

Acknowledgments for the First and Second Editions

The authors are grateful to Marie Franzosa for providing suggestions for improvement, for providing mathematics correction, and for reviewing the manuscript; to Jason Brown for copy edit, mathematics correction and many suggestions on the pedagogy; to Jennifer McCandless, Kathy Smith and Lisa Woodrum for proofreading. The authors also thank James Stewart for informal advice and much useful discussion. We are especially grateful for the editorial assistance, support and patience of Carol Benedict, Beth Wilbur, and Gary Ostedt. Finally, the support and encouragement of the Oregon State University Mathematics Department and of its Chair at the time, John Lee are gratefully acknowledged.

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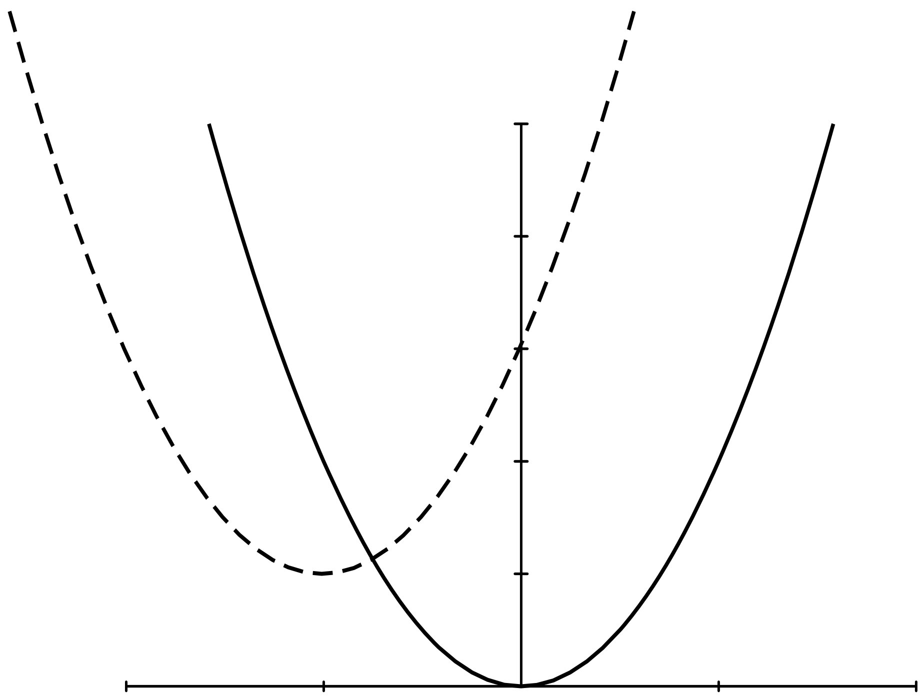
Chapter 1

Functions and Models



1.1 Four Ways to Represent a Function

Key Concepts:

- Representing functions verbally, by a table of values, by a graph, or by an explicit formula
- Functions that are defined piecewise
- Increasing or decreasing functions

Skills to Master:

- Use the different representations of functions to solve real world problems.
- Understand and be able to determine domain and range of specific functions.
- Sketch graphs of piecewise defined functions.

Discussion:

Section 1.1 defines the concept of a function. This begins with the identification of a set of inputs, called the domain, and of a set of potential outputs, called the range. A function is a rule that uniquely determines for each member of the domain a member of the range. Functions are fundamental to mathematics. They are the primary way that relationships are expressed and described. A function may have any set for a domain and any set for a range, but this text and course is most concerned with relationships between numbers. So most functions will have a numerical domain and range. The triumph of calculus is the creation of a methodology for solving problems of optimization, change, and comparison. This is the exciting world you are about to enter.

Key Concept: Representing functions verbally, by a table of values, by a graph, or by an explicit formula

In principle, any media or method of expression may be used to represent a function. For example, music is a sequence of sounds that occur as function of time. Practical ways to describe functions in mathematics fall into four categories: *verbally* by a description in words, *numerically* by a table of values, *visually* by a graph and *algebraically* by an explicit formula. Study the examples in this section that illustrate how a function can be represented in different ways. Most functions that you are familiar with such as polynomials, trigonometric functions, exponential functions and logarithmic functions are represented by formulas. However, functions that are not given by explicit formulas arise in many situations. *Section 1.2* in the text will show in more detail how to deal with such functions.

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Key Concept: Functions that are defined piecewise

A piecewise defined function $f(x)$ is a function that is given by different formulas for different values of x. For example, in some applications, a function may be given by one formula when x is positive and by a different formula when x is negative as in the case of the absolute value function. Sometimes these functions meet at the endpoints on intervals on which they are defined and are continuous. Sometimes they have jumps. Read the examples given of piecewise defined functions and make sure that you know how to graph such functions.

Key Concept: Increasing or decreasing functions

A function is *increasing* on an interval I if

$$f(x_1) < f(x_2) \text{ whenever } x_1 < x_2 \text{ in } I.$$

A function is *decreasing* on an interval I if

$$f(x_1) > f(x_2) \text{ whenever } x_1 < x_2 \text{ in } I.$$

In the graph of the function, increasing means that the graph is rising as you scan from left to right. For example, if a function represents the temperature of water as a function of time, to say the function is increasing means the temperature is going up. Notice that in this definition, a constant function (with a horizontal graph) is neither increasing nor decreasing. Study the examples and make sure that you understand what it means to say that functions are increasing or decreasing.

SkillMaster 1.1: Use the different representations of functions to solve real world problems.

In real world problems that arise in the sciences or economics, a function is often first given by a verbal description, by a table of values, or by a graph. Understanding the different ways that a function can be represented will allow you to work with functions that come up in these ways. When a function is given by a graph, it is important that you be able to interpret the graph verbally as well as analytically. Approximating such functions by other functions given by an explicit formula is often the preferred method to use because formulas may be manipulated algebraically.

SkillMaster 1.2: Understand and be able to determine domain and range of specific functions.

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The *domain* of a function $f(x)$ is the set of values x for which the function is defined. The *range* of a function $f(x)$ is the set of values taken on by the function. If a function is graphed, the domain consists of all x values for which a vertical line drawn through the point $(x,0)$ intersects the graph. The range consists of all y values for which a horizontal line drawn through $(0,y)$ intersects the graph. Make sure that you know how to find the domain and range of functions that are given verbally, by data, by a graph or by a formula.

SkillMaster 1.3: Sketch graphs of piecewise defined functions.

To sketch a graph of a piecewise defined function given by a number of different formulas, you need to separately graph the function represented by each formula for the appropriate values of x. For example, the absolute value function $f(x) = |x|$ is described as the piecewise defined function:

$$f(x) = \begin{cases} x & \text{if} \quad x \geq 0 \\ -x & \text{if} \quad x < 0 \end{cases} .$$

The graph of this piecewise defined function consists of parts of the lines $y = x$ and $y = -x$.

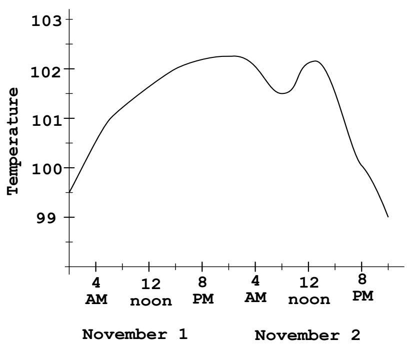
Worked Examples

For each of the following examples, first try to find the solution without looking at the hint. Cover the right column with a piece of paper. If you need a hint, uncover the column. If you need to see the worked solution, look at the solutions section that follows the examples and hints.

Example **Hint**

SkillMaster 1.1.

1. The graph below describes a person's temperature over a period of two days when he was ill. At what time and date did his temperature first reach 102°? What was his temperature at 8:00 am on November 2? When did he begin to feel healthy again (interpret this as the time his temperature first fell below 100°)?



Hint: Use a straight edge to see where the line at the correct height parallel to the x-axis first crosses the graph. Then drop a perpendicular line to the x-axis to find the time.

2. A woman is expecting a baby. She has charted her weight over a 40 week period. She weighed 140 pounds before her pregnancy and at the end of each four week period she wrote down her weight.

Week	0	4	8	12	16	20
Weight	140	140	141	143	145	150

Week	24	28	32	36	40
Weight	154	158	160	165	170

Graph the woman's weight as a function of time. Use the graph to estimate her weight at 17 weeks.

Hint: Use the time in weeks as the x-axis and the use the weight in pounds for the y-axis. Plot the points according to the information in the table.

3. A metal can is produced by rolling a rectangular sheet of metal into a cylinder and attaching circular pieces of metal for the top and the bottom. Because there is waste when the circular pieces are cut out of the sheet metal, the cost for these pieces is more per unit area than is the cost for the side. The cost for the tops and bottoms is 14 dollars per square unit and the cost for the sides is 10 dollars per square unit. Write a function that gives the cost of the can as a function of h and r.

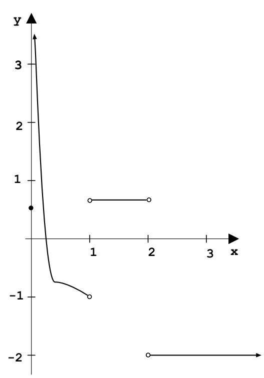
The rectangle that forms the side has height h and length $2\pi r$, the circumference of the circle. Its area is the product of these two lengths, $2\pi rh$. The area of the top πr^2. This is also the area of the bottom.

4. A man is taking a trip to Arizona. One of his tires has a slow leak and has lower than recommended pressure. He is in too much of a hurry to have it fixed so he inflates it from 24 lb/in^2 to the recommended 32 lb/in^2. As he drives the tire deflates and its pressure gradually lowers from 32 lb/in^2 to 20 lb/in^2 over the next 4 hours of driving time. At this time the tire goes over a nail and has a blow-out. Sketch a graph of the tire pressure over this time interval.

First consider what happens between time 0 hours and time 4 hours.

SkillMaster 1.2.

5. Find the domain and the range of the function shown in the graph.



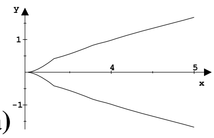
Which values of x give you a y value? These are the points on the x-axis for which a vertical line intersects the graph. These are the points in the domain. It may be easier to see certain points that are not in the domain. The range is the set of possible y values.

6. Find the domain and range of the function. Sketch a graph of the function using your calculator.

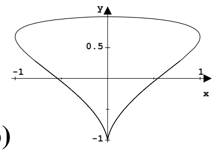
$$f(x) = \frac{1}{\sqrt{x^2 - 4}}$$

The function is NOT defined when the denominator is 0 or when the expression inside the square root is negative.

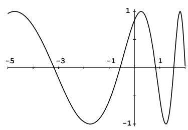
7. Which of the following graphs is a function?



(a)



(b)



(c)

The Vertical Line Test is used to determine whether a graph represents a function. It does if and only if each vertical line passes through at most one point on the graph.

SkillMaster 1.3.

8. Sketch the graph of the following piecewise defined function.

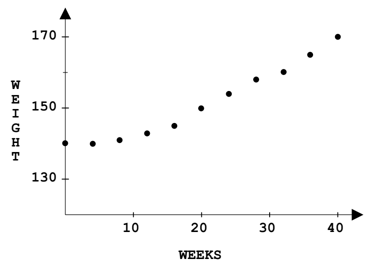
$$f(x) = \begin{cases} -2 & \text{if } x < 0 \\ x^2 & \text{if } 0 \le x \le 3 \\ x-3 & \text{if } 3 < x \end{cases}$$

Consider separately what happens for x values in each interval listed.

Solutions to worked examples

1. His temperature first reached 102 ° at approximately 4:00 P.M. on November 1. At 8:00 am on November 2 his temperature began to rise again after falling a bit. His temperature at that time was 101.5° His temperature fell to 100° for the first time at about 8:00 P.M. on November 2.

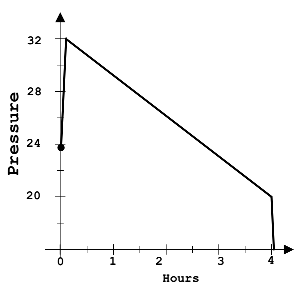
2. The woman's weight at 17 weeks was about 146.25 lb. This can be seen from the graph below.



3. The cost of the side is $14(2\pi rh)$ and the cost of the top and bottom together is $10(2\pi r^2)$. the total cost is

$$C = 28\pi rh + 20\pi r^2.$$

4.



5. The domain is $[0,1)\cup(1,2)\cup(2,\infty)$. Note that the 1 and 2 are excluded from the domain because a vertical line through these points does not intersect the graph. The range is $\{-2\}\cup(-1,\infty)$.

6. The denominator cannot be 0 so

$\sqrt{x^2-4} \neq 0$ or $x^2-4 \neq 0$

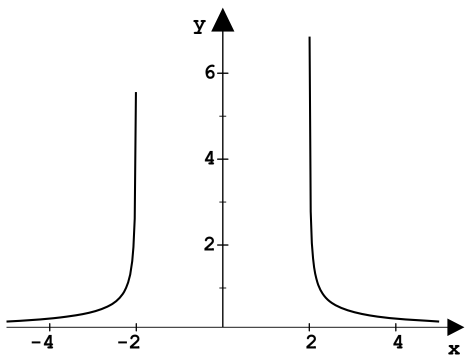
$x^2 \neq 4$ so $x \neq \pm 2$.

The expression inside the square root cannot be negative so

$x^2 - 4 > 0 \qquad (x-2)(x+2) > 0$

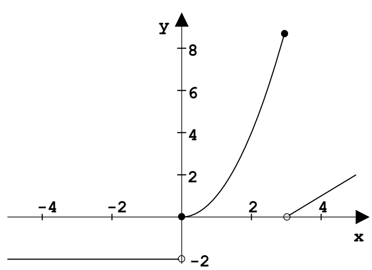
$x > 2$ or $x < -2$.

Putting these together implies the domain is all points x not in [-2,2]. The domain is $(-\infty,-2) \cup (2,\infty)$. From the graph below it is clear that the range is $(0,\infty)$.



7. The first two are not functions, while the third is a function.

8.



1.2 Mathematical Models

Key Concepts:

- Review of polynomial, rational, algebraic and trigonometric functions
- Mathematical models used to represent physical situations
- Finding a curve that best fits collected data

Skills to Master:

- Recognize different specific types of functions.
- Create scatter plots and select an appropriate model.
- Use a model to estimate and predict other values.

Discussion:

Section 1.2 introduces the process of using mathematical models to fit real-world situations and of using curve fitting to find curves that best fit certain sets of data points. These activities are used over and over again in the sciences and in other areas that use mathematics. Pay attention to the techniques in this section. They will be used later in the text. If any of the functions discussed in this section seem unfamiliar, you should spend time now reviewing the necessary material. Ask your instructor for some additional references if you need them. The material in this section is critical for understanding later sections in the text.

Key Concept: Review of polynomial, rational, algebraic and trigonometric functions

Appendix C

You should already be familiar with polynomial, root and trigonometric functions from past courses. Review the material in the text on *trigonometric functions* if you need to. An algebraic function is a function that can be constructed from polynomials by using the operations of addition, subtraction, multiplication, division, and taking roots. For example,

$$f(x) = \sqrt[5]{\frac{\pi \cdot x^3 - \sqrt{2x^2 + 4x^4}}{3x^6 + 7x^4 + x^2 + 8}}$$

is an algebraic function. Note that this function is defined for all values of x. This is because the denominator of the fraction under the 5th root is never 0, and because 5th roots are defined for all real numbers.

Key Concept: Mathematical models used to represent physical situations

A *mathematical model* is a mathematical description of a real-world situation. The process described in the text in Figure 1 is well worth learning. Given a real-world situation, variables are assigned to the quantities under consideration and a mathematical relationship among the variables is formulated. This is the model. Mathematics is then applied to the model to reach conclusions. Hopefully, this process leads to predictions about the real world situation which can be tested empirically. The results can then be used to refine the model, if necessary.

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Key Concept: Finding a curve that best fits collected data

Given a collection of data points, one way of formulating a mathematical model that fits the data is to try to find a curve that either passes through or else is close to the points. The text discusses some of the techniques used to find linear curves, or lines, that best fit the data points. Sometimes quadratic, cubic or exponential curves will give the best fit. Pay attention to the examples in this section to see situations where these various curves arise.

SkillMaster 1.4: Recognize different specific types of functions.

If you understand the different types of functions discussed in this section, you should be able to recognize these functions when they arise as a formula, a scatter plot or other type of graph, or in one of the other ways to describe a funtion. *Problems* 1*,* 2*,* 19 *and* 20 in this section should give you practice in doing this.

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SkillMaster 1.5: Create scatter plots and select an appropriate model.

Plotting data points in the plane (creating a scatter plot) is the first step in choosing an appropriate mathematical model for a real-world situation. Examining the shape suggested by the points may indicate certain relationships between the x and y-coordinates. If the points seem to lie on a line, a linear relationship is suggested. *Example 2* in the text shows a situation where a linear relationship is indicated. If the points vary up and down in a periodic manner, a trigonometric model is a possibility. If the points grow or decay quickly, then try an exponential or algebraic model.

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SkillMaster 1.6: Use a model to estimate and predict other values.

Once you have a model and a suggested relationship between variables under consideration, you can use the model to interpolate estimations of the value of the variable when the data was not collected or extend the model to predict what might happen. The examples in this section show how to do this in a number of different settings.

Worked Examples

For each of the following examples, first try to find the solution without looking at the hint. Cover the right column with a piece of paper. If you need a hint, uncover the column. If you need to see the worked solution, look at the solutions section that follows the examples and hints.

Example	Hint
SkillMaster 1.4.	
Classify the functions below as one of the types of functions that are discussed in the text.	
1. $f(x) = \cos(x)$	The function is periodic.
2. $h(x) = 6x^3 - 3x + 2$	This function is of the form $a_n x^n + \cdots + a_1 x + a_0$ where $n = 3$, $a_0 = 2$, $a_1 = -3$, $a_2 = 0$, and $a_3 = 6$.
3. $g(u) = \dfrac{1}{\sqrt{x^3+1}}$	This function can be constructed using the algebraic operations: $+, -, \times, \div, \sqrt[n]{}$

SkillMaster 1.5.

4. In different cities with similar populations, sales of a new product is tested at several different prices. The number of units sold was recorded in each case. Create a scatter plot and select the type of model that fits the data.

Price ($)	# sold (hundreds)
2.50	2
1.75	50
1.60	60
1.50	66
1.00	100

Price ($)	# sold (hundreds)
2.00	33
2.20	19
1.40	72
1.25	83

Draw the scatter plot and see what type of curve best fits these data points. Then sketch a regression line.

SkillMaster 1.6.

5. In the problem above, if the price were set at $1.35 how many units would you predict would be sold? Notice that there is not enough information to set a price because it is unknown what the cost of production is for different amounts.

Use the graph to predict the number of units when $x = 1.35$.

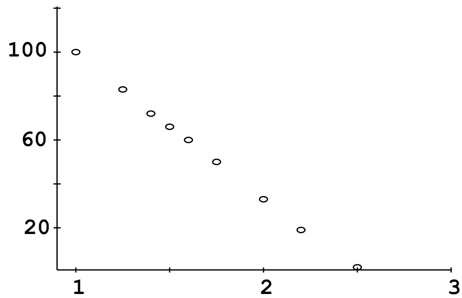
Solutions to worked examples

1. This first function is a *trigonometric function*

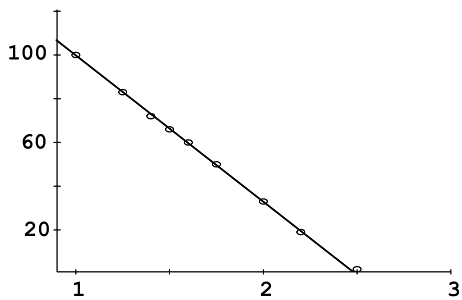
2. This function is a *polynomial function*

3. This function is a *algebraic function*

4. A linear model best fits these data points.



5. The graph is shown below.



The number of units sold would be about 7500.

1.3 New Functions from Old Functions

Key Concepts:

- Shifting, stretching and reflecting to get a new function
- Sums, products, quotients and compositions of functions

Skills to Master:

- Recognize how shifted, reflected, and stretched graphs correspond to changes in the algebraic representation and vice versa.
- Find new functions by using combinations of old functions.

Discussion:

Section 1.3 shows you how to translate, transform and combine functions discussed in the previous section to obtain new functions. Even more, this section shows how to compose different basic functions to obtain new functions. The material in this section will often be used in the text and can be thought of as part of the basic vocabulary of the course. It will be assumed that you are familiar with these terms.

Key Concept: Shifting, stretching and reflecting to get a new function

Study the descriptions in this section of *vertical and horizontal shifts* and *vertical and horizontal stretching and reflecting*. A vertical or horizontal shift keeps the shape of the graph the same, but moves it horizontally or vertically. The resulting graphs are congruent to the original graph. A stretch compresses or expands the graph. The resulting graphs are similar in shape to the original graph. A reflection flips the graph over about either the x-axis or y-axis.

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Key Concept: Sums, products, quotients and compositions of functions

The operations of addition, multiplication, or division allow you to combine existing functions to create new ones. For example, the function

$$g(x) = (x^2 + 2x - 4) \cdot \sin x$$

is the product of the two simpler functions $x^2 + 2x - 4$ and $\sin x$. A sum or product will be defined wherever both simpler functions are defined. A quotient will be defined wherever the numerator and denominator are both defined and the denominator is not 0. Thus the new domain is the set of all points that are in the domains of each of the functions used, providing those points do not force an illegal operation such as dividing by 0, etc.

The operation of taking compositions of functions consists of first applying one function and then another. For example,

$$f(x) = \cos\left(x^2 + 2x - 4\right)$$

is determined by first applying the function $x^2 + 2x - 4$ and then finding the cosine of the result. Note that this particular function is defined for all values of x since the set of all real numbers is the domain of both the cosine function and the polynomial function.

SkillMaster 1.7: Recognize how shifted, reflected, and stretched graphs correspond to changes in the algebraic representation and vice versa.

This SkillMaster asks you to recognize and understand how changes in the geometric shape of the graph of a function correspond to changes in the algebraic representation of that function. For example the graphs of $f(x)$ and $g(x) = f(x-3)$ are related in that the graph of g is just the graph of f shifted 3 units to the right. The graphs of $h(x)$ and $k(x) = h(-x)$ are related in that the graph of k is just the graph of h reflected about the y axis. Make sure that you understand the correspondence between the change in the graph and the change in the formula representing the function.

SkillMaster 1.8: Find new functions by using combinations of old functions.

You should be able to find formulas for functions that are described as sums, differences, quotients or compositions of simpler functions. You should also be able to determine the domain and range of the new functions. For example, if

$$f(x) = x^2,\ g(x) = \sin(3x), \text{ and } h(x) = \sqrt[3]{5x}$$

then the function $h \circ (f/g)$ is the function given by the formula

$$\sqrt[3]{5\frac{x^2}{\sin(3x)}}$$

and is defined except at x values where $\sin(3x) = 0$.

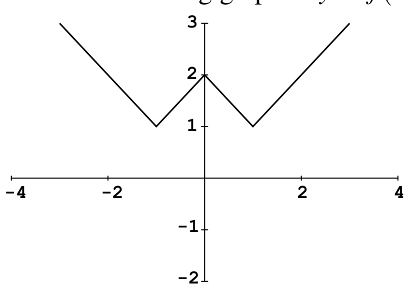
Worked Examples

For each of the following examples, first try to find the solution without looking at the hint. Cover the right column with a piece of paper. If you need a hint, uncover the column. If you need to see the worked solution, look at the solutions section that follows the examples and hints.

Example	Hint

SkillMaster 1.7.

1. Consider the following graph of $y = f(x)$.



Sketch the graphs of $f(x+1), f(2x), -f(x)+2$.

Hint: Recall $f(x+1)$ shifts the graph 1 unit to the left, $f(2x)$ compresses the graph horizontally by a factor of 2, $-f(x)$ reflects the graph about the x-axis, and adding 2 to a function shifts the function 2 units upward.

2. Graph the function

$$f(x) = x^2 + 4x + 6.$$

Hint: Complete the square and view the result as a transformation of the parabola, $y = x^2$.

SkillMaster 1.8.

3. If $f(x) = x+3$ and $g(x) = \sqrt{x-1}$, find each of the following functions and specify their domains:

$$(f+2g)(x) \quad (fg)(x) \quad (g/f)(x).$$

The domain of $f(x) = x+3$ is the set of all real numbers. The domain of $g(x) = \sqrt{x-1}$ is the set of all x for which the expression inside the radical is nonnegative.

4. If $f(x) = x+3$ and $g(x) = \sqrt{x-1}$, find each of the following functions and specify their domains.

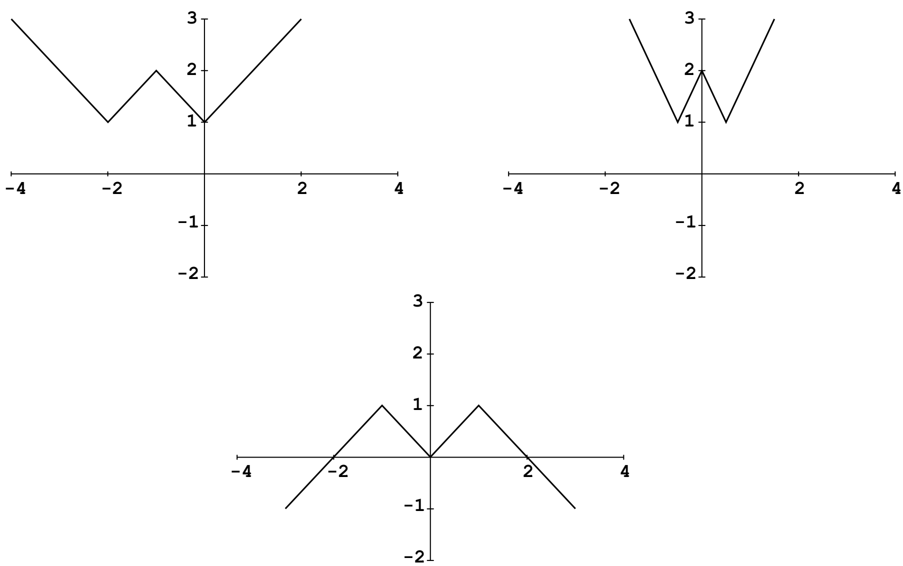
$f \circ g(x) \qquad g \circ f(x).$

Start with

$$\begin{aligned} f \circ g(x) &= f(g(x)) \\ &= g(x)+3 \end{aligned}$$

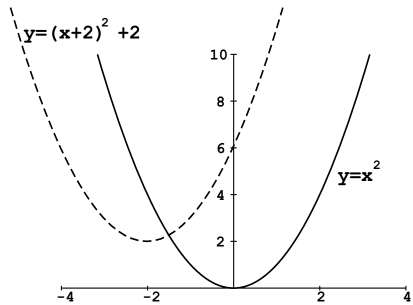
Solutions to worked examples

1. The graphs of $f(x+1), f(2x)$ and $-f(x)+2$ are shown below.



2. $y = x^2 + 4x + 6 = x^2 + 4x + 4 + 2 = (x+2)^2 + 2$

Shift the parabola $y = x^2$ 2 units to the left and 2 units upward.



3. $(f+2g)(x) = f(x) + 2g(x) = x + 3 + 2\sqrt{x-1}.$

The domain of $(f+2g)(x)$ is the set of all x values that belong to the domains of both f and g, i.e. the set of x values for which $x \geq 1$. This is the interval $[1, \infty)$.

$$(fg)(x) = f(x)g(x) = (x+3)(\sqrt{x-1})$$

The domain of fg is the set of all x values that belong to both the domain of f and the domain of g. This is the set of x for which $x \geq 1$.

$$(f/g)(x) = \frac{f(x)}{g(x)} = \frac{x+3}{\sqrt{x-1}}$$

The domain of f/g is the set of all x values that belong to the domains of both f and g provided $g(x) \neq 0$. Thus $x = 1$ is not in the domain. The domain is the set of x for which $x > 1$.

4. The domain of $f \circ g(x) = \sqrt{x-1} + 3$ is the set of all x for which the expression inside the radical is nonnegative, or the set of x for which $x \geq 1$.

$$\begin{aligned} g \circ f(x) &= g(f(x)) = \sqrt{f(x) - 1} \\ &= \sqrt{(x+3) - 1} = \sqrt{x+2} \end{aligned}$$

The domain is the set of all x for which the expression inside the radical is nonnegative, or the set of x for which $x \geq -2$.

1.4 Graphing Calculators and Computers

Key Concepts:

- Using graphing calculators for complicated functions
- Families of functions

Skills to Master:

- Become familiar with your graphing device.
- Use a graphing calculator to estimate solutions to equations and to compare functions.
- Graph families of functions.

Discussion:

Section 1.4 asks you to become more familiar with the use of your graphing calculator or graphing device. Since graphing devices show just part of the graph of a function and often just approximate the actual values of a function, you need to learn when and how to rely on these devices. The examples in this section show some of the things that you need to be wary of. No calculator will be able to show you all of the features of a function without the application of mathematical analysis.

Key Concept: Using graphing calculators for complicated functions

If you are given a formula for a complicated function, you often want to gain a better geometric understanding of what the function represents. A graphing device often can aid in gaining such understanding. Before you rely on a graphing device, make sure that you understand how to choose the viewing rectangle and how to choose reasonable x and y values to view. Study *Example 4* in the text.

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Key Concept: Families of functions

By varying the coefficients in the formula for a function you can obtain an entire family of related functions. For example, the family of functions

$$\sin(x+c)$$

are related to the function $\sin x$. The graphs of these functions are obtained from the graph of the sine function by shifting to the right or left. Review the material on *shifting, scaling and reflecting* graphs in the previous section if you need to.

pgs 38-39.

SkillMaster 1.9: Become familiar with your graphing device.

Before you can use your graphing device to help in answering questions about functions, you need to become familiar with the operation of the device. Now is the time to make sure that you know how to enter functions into the device, graph functions using the device and choose appropriate viewing rectangles for graphs. Get help on this material now if you need it. The ability to quickly and easily use your graphing device will help you in the rest of the course.

SkillMaster 1.10: Use a graphing calculator to estimate solutions to equations and to compare functions.

If you want to find a value x for which $f(x) = c$, one way to get started is to graph the function $f(x)$ and see if you can determine where the graph reaches a height c. If you are comparing two functions f and g and are trying to determine where their graphs intersect or where one function has larger values than the other, a good way to proceed is to try to graph both functions in an appropriate viewing rectangle.

SkillMaster 1.11: Graph families of functions.

At this point, the method that you have for graphing families of functions related by a varying parameter c is to choose a selection of possible values for c and graph the functions corresponding to those values. Later in the course you will see how to use Calculus to determine the the most important values of the parameter c to consider.

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Worked Examples

For each of the following examples, first try to find the solution without looking at the hint. Cover the right column with a piece of paper. If you need a hint, uncover the column. If you need to see the worked solution, look at the solutions section that follows the examples and hints.

Example	Hint
SkillMaster 1.9.	
1. Graph the function $y = \cos(113x) + .017\sin(500x)$ in the following viewing rectangles: (a) [-5,5] by [-1.2,1.2], (b) [-1,1] by [-20,20], (c) [-0.1,0.1] by [-1.1, 1.1], (d) [-0.004,0.004] by [0.8,1.02]. Describe what each graph shows about function. Which are the best viewing rectangles to understand the behavior of the function?	Carefully set each x range and y range on your graphing device.
2. Graph the function $y = x^4 - 100x^2$	Try to choose a viewing area that captures all important details.
SkillMaster 1.10.	
3. Use graphs to find the solutions to the equation $x^2 = \sin x$ correct to two decimal places.	Graph $y = x^2$ and $y = \sin x$ in the same graph and zoom in on the intersection points.

SkillMaster 1.11.

4. Graph the function

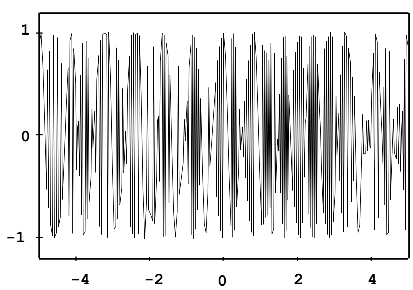
$$y = x^2 - 2cx$$

for various values of c. What is this family of functions and how does the graph change as c varies? Confirm your guess by analyzing the graph using algebra.

Try $c = -2$, $c = -1$, $c = 0$, $c = 1, c = 2$.
By completing the square, keeping c as a constant, describe the parabola you get in terms of c.

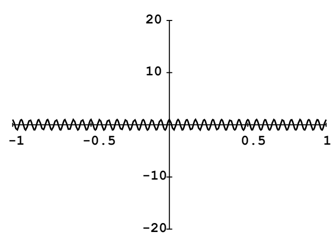
Solutions to worked examples

1. (a)



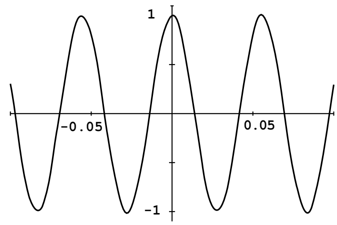
The largest the function could be is 1.017 since both $\cos(x)$ and $\sin(x)$ have values between -1 and 1. The choice of y range is good but the x range is too large to show the structure.

(b)



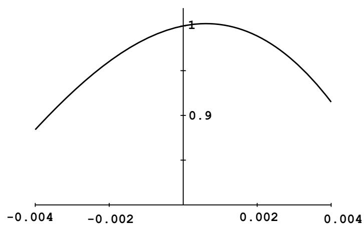
Here the choice of y range is too large and perhaps the x range should also be lowered a bit.

(c)



This is a good viewing rectangle to show the main periodic structure. The graph looks close to the graph of the $\cos(113x)$.

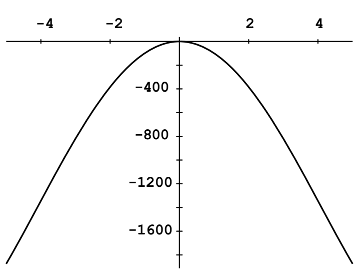
(d)

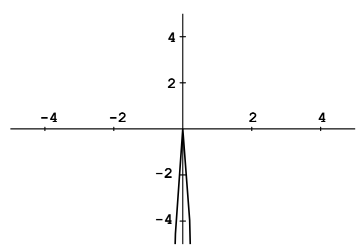


If you compare this to the graph of $cos(113x)$ on the same rectangle you can see that the $0.017\sin(500x)$ perturbs the graph of $\cos(113x)$. For example, notice the maximum just to the right of $x = 0$. These last two viewing rectangles are the most effective rectangles to observe the features of the graph.

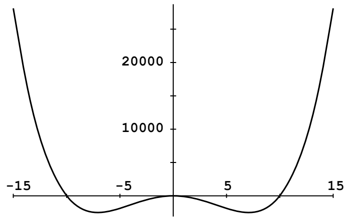
2. Graphing this function with the default scaling gives the graph on the left below. This appears to be roughly parabolic in shape.

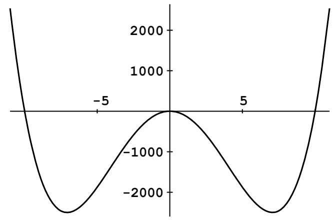
Graphing this in a [-5,5] by [-5,5] rectangle gives the graph on the right below which shows almost no detail.



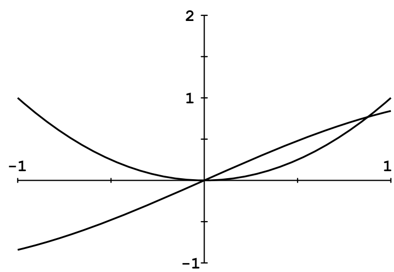


For $x > 10$, $x^4 - 100x^2$ should be positive rather than negative so try the larger rectangle shown on the left below. This shows more detail. Then zoom in closer to show the "double dip" structure shown on the right below.





3. First graph $y = x^2$ and $y = \sin x$ with viewing rectangle [-1,1] by [-1,2] to get a rough idea where the intersection points might be.

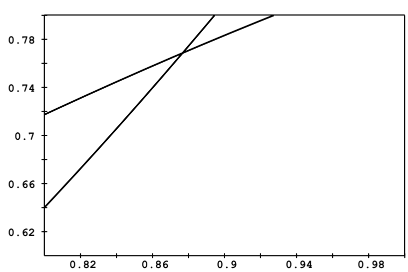


There appears to be an intersection point at $x = 0$. Check this by substituting 0 for x in the original equation to see if there is an equality.

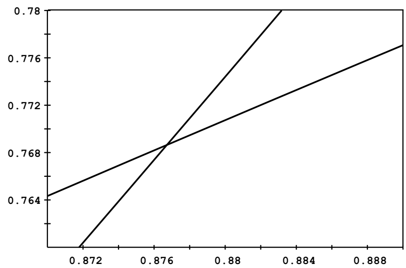
$$x^2 = \sin x \qquad 0^2 = \sin 0 \qquad 0 = 0$$

Indeed 0 is an exact solution to this equation. The other solution appears to be just a bit smaller than $x = 1$.

Choose the viewing rectangle [0.8,1.0] by [0.6,0.8] to get a closer look and obtain a better approximation.



The intersection point appears to be near (0.88,0.77). Choose a new viewing rectangle [0.87,0.89] by [0.76,0.78].



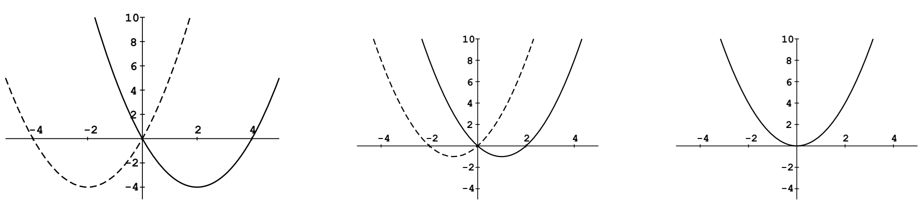
The intersection point appears to be near (0.876,0.769). Rounding to two decimal places the solution is $x = 0.88$. As a check substitute in the original equation.

$$\begin{aligned} x^2 &= \sin x \\ (0.88)^2 &= 0.7744 \\ \sin 0.88 &= 0.7707 \end{aligned}$$

This also agrees to two decimal places.

4.

$c = -2$ and $c = 2$: $c = -1$ and $c = 1$ $c = 0$



All of these graphs are parabolas that are translations of each other. The $x-$coordinate of the vertex appears to be $x = c$ and the $y-$coordinate appears to be at $y = -c^2$. Check this by completing the square.

$$\begin{aligned} y &= x^2 - 2cx, \quad y = x^2 - 2cx + c^2 - c^2 \\ y &= (x-c)^2 - c^2 \end{aligned}$$

Using the transformations of functions from Section 1.3, page 38, notice that this is the graph of $y = x^2$ shifted c units to the right and shifted c^2 units downward.

1.5 Exponential Functions

Key Concepts:

- Definition of an exponential function
- Applications of exponential functions

Skills to Master:

- Understand how a change in base affects the graph of an exponential function.
- Apply shifting, stretching, and reflection to get new functions.
- Solve word problems involving exponential functions.

Discussion:

Section 1.5 reviews properties of exponential functions. If any of this material seems unfamiliar to you, spend extra time going over the homework problems. Exponential functions are extremely important and frequently arise in applications of Calculus. Take the time to do enough exercises so that this is familiar to you.

Key Concept: Definition of an exponential function

An exponential function is a function of the form $f(x) = a^x$ where a is a positive real number called the base of the exponential. Study *Figure 3* in this section to see how the shape of the graph of an exponential function changes when different values of a are used. You should review the *Laws of Exponents* at this time:

page 54

If a and b are positive numbers and x and y are real numbers, then

$$a^{x+y} = a^x a^y, \qquad a^{x-y} = \frac{a^x}{a^y}, \qquad (a^x)^y = a^{xy}, \qquad (ab)^x = a^x b^x$$

The text also gives you a definition of the number e as the number that has the following property. The graph of $y = e^x$ has the tangent line to the graph at the point $(0, 1)$ of slope 1. It may seem strange to have such a complicated non-intuitive description. This will make much more sense when differentiation is introduced.

Key Concept: Applications of exponential functions

pages 59-61

The text points out that exponential functions often arise in situations where the amount of increase or decrease of a quantity per unit time depends on the amount of the quantity present. *Examples in the text* include population growth and radioactive decay. Note that the larger a population, the larger its ability to reproduce and create new members.

SkillMaster 1.12: Understand how a change in base affects the graph of an exponential function.

page 54

The shape of the graph of an exponential function $f(x) = a^x$ depends on the base a. If $a = 1$, $f(x)$ is a constant function and its graph is the horizontal line $y = 1$. If $a > 1$, $f(x)$ is an increasing function that is always positive and intersects the y-axis at the point $(0,1)$. If $a < 1$, then $f(x)$ is a decreasing function that is always positive and intersects the y-axis at the point $(0,1)$. Study *Figure 3* again to see these shapes.

SkillMaster 1.13: Apply shifting, stretching, and reflection to get new functions.

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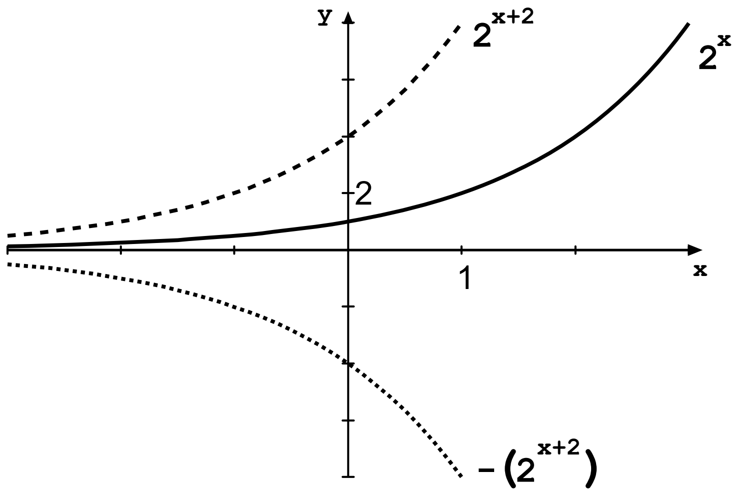
Exponential functions can be combined with the kind of transformations discussed in *Section 1.3* to get new functions. For example, the graph of the function

$$g(x) = -2^{x+2}$$

can be obtained from the graph of $f(x) = 2^x$ by shifting the graph two units the left and then reflecting the graph about the x-axis. The following graph illustrates this.



SkillMaster 1.14: Solve word problems involving exponential functions.

To solve these kinds of problems, you need to become familiar with a few ways of representing problems involving population growth or radioactive decay by exponential functions. If a population is doubling every time period, the function will look like

$$P(t) = C \cdot 2^t$$

where C is the population at time 0. If instead the population is tripling every time period, use 3 as a base instead of 2.

If you know the half life of a radioactive substance, you can write down a function that gives the amount of the substance left after t years.

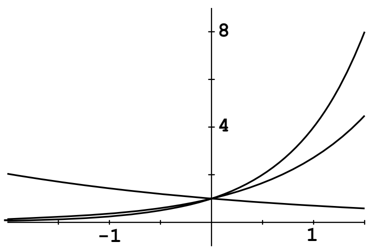
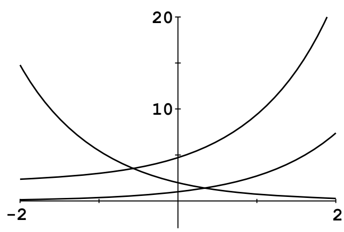
$$A(t) = C \cdot 2^{-\left(\frac{t}{h}\right)}$$

where h is the half life and C is the amount at time 0. Pay careful attention to how *these results* are derived in the text.

pages 58-60

Worked Examples

For each of the following examples, first try to find the solution without looking at the hint. Cover the right column with a piece of paper. If you need a hint, uncover the column. If you need to see the worked solution, look at the solutions section that follows the examples and hints.

Example	Hint
SkillMaster 1.12.	
1. Consider the following graphs. One is the graph of $y = 4^x$. The others are the graphs of $y = e^x$ and $y = (0.7)^x$. Identify which graph corresponds to which equation. 	Which base is largest? Which is smallest?
SkillMaster 1.13.	
2. Graph the equation $y = e^{-x+1} - 2$ by first graphing $y = f(x) = e^x$ and then applying the transformation rules.	How is $f(-x+1)$ related to $f(x)$?
3. Consider the following graphs. One is the graph of $y = e^x$. The others are the graphs of $y = e^{x+1} + 2$ and $y = 2e^{-x}$. Identify which graph corresponds to which equation. 	First find the graph of $y = e^x$. Then find the others using the transformation laws.

SkillMaster 1.14.

4. An island currently has a population of 1,000 deer. Due to an abundance of food the birthrate of the deer is 20%. This means that the equation giving the future population of deer is

$$y = 1000(1.2)^t$$

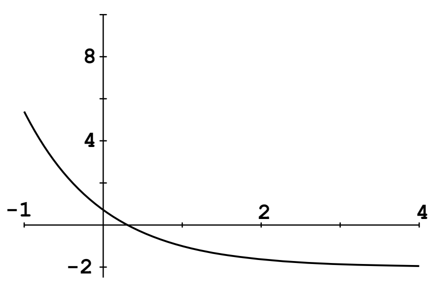
where t is the number of years in the future. Use a graph to determine when the population of deer exceeds 230,000, the number that can fit on the island.

Graph this in an appropriate viewing rectangle and observe the value of t where the graph crosses the horizontal line $y = 230,000$.

Solutions to worked examples

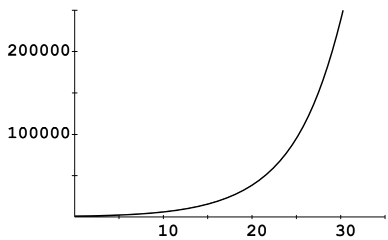
1. The graph that is higher for each value of $x > 0$ is the graph of $y = 4^x$ because the base 4 is larger than $e \approx 2.718$. The graph that is decreasing is graph of $y = (0.7)^x$. An exponential function is decreasing if and only if the base is strictly between 0 and 1. The middle graph is the graph of $y = e^x$. You could also have noted that at $x = 1$ the $y-$coordinates of these graphs should be $4 = 4^1, e = e^1$, and $0.7 = (0.7)^1$.

2. The graph of the equation $y = e^{-x}$ is the reflection of the graph of $y = e^x$ about the $y-$axis. The graph of the equation $y = e^{-x+1} - 2 = e^{-(x-1)} - 2$ is the graph of $y = e^{-x}$ shifted 2 units to the right and 1 unit downward.

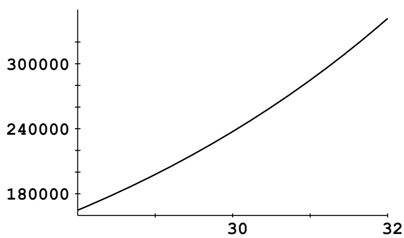


3. The lower increasing graph is the graph of $y = e^x$. It is increasing and has $y = 1$ when $x = 0$. The other increasing function is the graph of $y = e^{x+1} + 2$. It is the graph of $y = e^x$ shifted 1 unit upward and 2 units to the left. The remaining graph is decreasing so is the graph of $y = 2e^{-x}$. It is the graph of $y = e^x$ reflected about the $y-$axis and stretched by a factor of 2 in the direction.

4.



The value appears to be about $t = 30$. Zoom in near the point (30, 237,000).



This confirms the solution is approximately $t = 30$ years.

1.6 Inverse Functions and Logarithms

Key Concepts:

- One-to-one functions and inverse functions
- Logarithmic functions and the laws of logarithms
- Natural logarithms

Skills to Master:

- Determine whether or not a function is one-to-one.
- Find a formula for the inverse of a given function.
- Use laws of logarithms to solve problems involving logarithms and exponents.

Discussion:

Section 1.6 continues the review of basic functions by discussing inverse functions and logarithms. Again, as with exponential functions, if any of this material seems unfamiliar, spend extra time on this section now to avoid problems later in the course. Inverse functions are sometimes counter-intuitive and require extra study to completely understand. In this section, Logarithms are defined as the inverse functions to the exponential functions. It is well worth the extra time to become versatile with this subject matter. It will recur often throughout Calculus.

Key Concept: One-to-one functions and inverse functions

page 61

The *Horizontal Line Test* states that a function is one-to-one if and only if no horizontal line intersects the graph of the function in more then one point. This means that no value is taken on by the function more that once. If $y = f(x)$ is a function that is one-to-one, the inverse function f^{-1} is defined for f by the rule that

$$f^{-1}(a) = b \text{ if and only if } f(b) = a.$$

page 61

Pay careful attention to the *note* in the text that $f^{-1}(x)$ DOES NOT mean $1/f(x)$. If you have forgotten the other properties of inverse functions, go over them now. Note that the graph of the inverse function can be obtained from the graph of the original function by reflecting the graph about the line $y = x$.

Key Concept: Logarithmic functions and the laws of logarithms

Logarithmic functions are defined as the inverse functions of exponential functions. That is,

$$\log_a x = y \text{ if and only if } a^y = x.$$

The properties of inverse functions combined with the properties of exponential functions lead to the Laws of Logarithms:

$$\begin{aligned} \log_a(xy) &= \log_a x + \log_a y \\ \log_a \frac{x}{y} &= \log_a x - \log_a y \\ \log_a x^r &= r \log_a x \end{aligned}$$

If you have forgotten these rules, make sure that you relearn them now.

Key Concept: Natural logarithms

page 65

The natural logarithm of x is the logarithm to the base e discussed in *Section 1.5.* and is denoted $\ln x$. Some of the basic properties are:

$$\begin{aligned} \ln x &= y \text{ if and only if } e^y = x \\ \ln(e^x) &= x, \\ e^{\ln x} &= x, \text{ and} \\ \ln e &= 1. \end{aligned}$$

Logarithms to other bases can be expressed in terms of the natural logarithm by using the relationship

$$\log_a x = \frac{\ln x}{\ln a}.$$

SkillMaster 1.15: Determine whether or not a function is one-to-one.

To determine whether a function $f(x)$ is one-to-one, use the Horizontal Line Test on the graph of the function. If any horizontal line intersects the graph in more than one place, then the function is not one-to-one. The Horizontal Line Test tells whether or not there are two different domain values that are assigned the same function value. If not, the function is one-to-one.

SkillMaster 1.16: Find a formula for the inverse of a given function.

The text gives a procedure for finding the inverse of a given one-to-one function.

1. Write $y = f(x)$.
2. Solve the above equation for x in terms of y if possible.
3. To express f^{-1} as a function of x, interchange x and y. The resulting equation is $y = f^{-1}(x)$.

As an example, given $y = f(x) = 3x+2$, solve for x in terms of y yielding the equation

$$x = \frac{y-2}{3}.$$

Interchanging x and y yields $y = \frac{x-2}{3}$. This is the formula for the inverse function of f. So

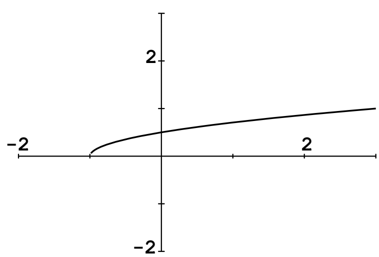
$$f^{-1}(x) = \frac{x-2}{3}.$$

SkillMaster 1.17: Use laws of logarithms to solve problems involving logarithms and exponents.

You can now use the laws of logarithms to solve problems involving both logarithmic and exponential functions. You can apply the rules for transforming functions to see what the graphs of transformations of logarithmic functions look like. For example, to find the graph of $y = \ln(2x-2)$ start with the graph of $y = \ln(x)$ and shift the graph 2 units to the right, then compress the graph by a factor of two in the horizontal direction.

Worked Examples

For each of the following examples, first try to find the solution without looking at the hint. Cover the right column with a piece of paper. If you need a hint, uncover the column. If you need to see the worked solution, look at the solutions section that follows the examples and hints.

Example	Hint
SkillMaster 1.15.	
1. Is the function $y = 0.4\tan(\pi x/4 + \pi/2)$ one-to-one?	Graph the function and apply the Horizontal Line Test.
2. Find an interval on which the function $y = 0.4\tan(\pi x/4 + \pi/2)$ is one-to-one.	Find an interval so that the graph of the function on this interval intersects each horizontal line at most once.
SkillMaster 1.16.	
3. Check that the function $y = f(x)$ shown in the graph is one-to-one and sketch the inverse function $y = f^{-1}(x)$. 	Remember what one-to-one means.
4. Find the inverse function $y = f^{-1}(x)$ where $y = f(x) = \dfrac{1+x}{3-2x}.$	Solve for x in terms of y and then interchange x and y.

SkillMaster 1.17.

5. Solve the equation

$$e^{3x-2} = 3(2^{x+1}).$$

Take natural logarithms of both sides of the equation and solve the resulting linear equation.

6. Evaluate $\log_5 3 + \log_5 10 - \log_5 6$ exactly without using a calculator.

Use the laws of logarithms to combine into a single term.

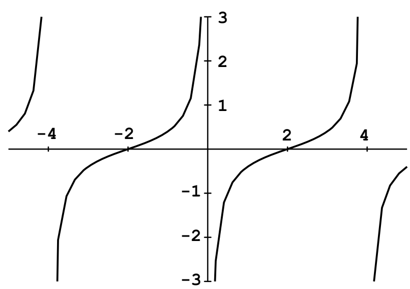
7. Solve the following equation for x.

$$\ln(1/x) + \ln(2x^3) = \ln 5$$

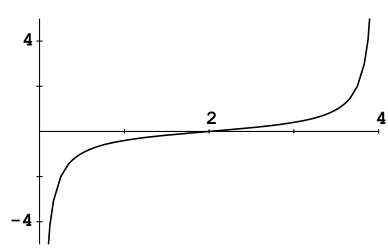
Use the laws of logarithms to simplify this equation before solving for x.

Solutions to worked examples

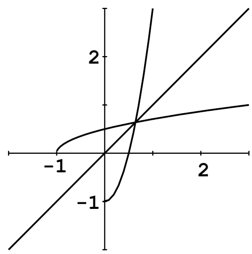
1. The function is not one-to-one because it fails the Horizontal Line Test. There are horizontal lines that cross the graph more than once.



2. Inspection of the graph reveals the function is one-to-one on the open interval (0,4). Note that the function is not defined at the endpoints $x = 0, x = 4$.



3. Each horizontal line intersects the graph at most once. The function is one-to-one by the Horizontal Line Test. To sketch the inverse function $y = f^{-1}(x)$ reflect the graph of $y = f(x)$ about the line $y = x$.



4. $$y = \frac{1+x}{3-2x} \quad\Rightarrow\quad y(3-2x) = 1+x$$
$$\Rightarrow\quad 3y - 2xy = 1+x \quad\Rightarrow\quad -2xy - x = 1-3y$$
$$\Rightarrow\quad -x(2y+1) = 1-3y \quad\Rightarrow\quad x(2y+1) = 3y-1$$
$$\Rightarrow\quad x = \frac{3y-1}{2y+1} \quad\Rightarrow\quad y = f^{-1}(x) = \frac{3x-1}{2x+1}$$

5. $$e^{3x-2} = 3(2^{x+1}) \quad\Rightarrow\quad 3x-2 = \ln 3 + (x+1)\ln 2$$
$$\Rightarrow\quad 3x-2 = (\ln 2)x + \ln 3 + \ln 2$$
$$\Rightarrow\quad 3x - (\ln 2)x = 2 + \ln 3 + \ln 2$$
$$\Rightarrow\quad (3-(\ln 2))x = 2 + \ln(3\times 2)$$
$$\Rightarrow\quad x = \frac{2+\ln 6}{3-\ln 2}$$

6. $\log_5(3) + \log_5(10) - \log_5(6) = \log_5((3\times 10)/6) = \log_5(5) = 1$

7. $$\ln(1/x) + \ln(2x^3) = \ln 5 \quad\Rightarrow\quad \ln((1/x)(2x^3)) = \ln 5$$
$$\Rightarrow\quad \ln(2x^2) = \ln 5 \quad\Rightarrow\quad 2x^2 = 5$$
$$\Rightarrow\quad x^2 = 5/2 \quad\Rightarrow\quad x = \sqrt{5/2} \approx 1.58$$

Only the positive square root is a solution because the natural logarithm is defined only for positive values. In particular $\ln(1/x)$ is only defined for $x > 0$.

1.7 Parametric Curves

Key Concepts:

- Parametric equations
- (x, y) coordinates used to represent position of a particle at time t

Skills to Master:

- Graph parametric curves by hand and by using a graphing device.
- Use parametric curves to answer questions about motion in the plane.

Discussion:

Section 1.7 show how to describe the location of a particle moving along a path in the plane. In this case, the x and y-coordinates of its location can be thought of as functions of time t. The variable t is called the *parameter* and the equations of the x and y-coordinates

$$x = f(t) \qquad y = g(t)$$

are called *parametric equations* . In this section, you learn to work with parametric equations and to use parametric equations to answer questions about motion in the plane.

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Key Concept: Parametric equations

When the variables x and y are given in terms of another variable t, as in the displayed equations above, you can think of the variable t as representing time. You can then view the equations that give x and y in terms of t as equations that tell you the x and y-coordinates of a point moving in the plane. These equations are called parametric equations. They are very useful and give more information than the graph of the function because parametric equations include motion and time information. It is useful to keep this geometric view of parametric equations in mind when working through the problems in this section.

Key Concept: (x,y) coordinates used to represent position of a particle at time t

Any path in the plane can be represented by a pair of parametric equations, one that gives the x position as a function of t and one that gives the y position as a function of t. Even a curve given in the usual x, y form such as $y = x^2$ can be given parametrically as

$$x = t \qquad y = t^2$$

These are the parametric equations that are at the point $(0, f(0))$ at time $t = 0$ and the particle moves at unit speed in the x direction constrained the graph of the function. Parametric equations are particularly useful to represent graphs that fail the *vertical line test* and are therefore not graphs of functions.

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SkillMaster 1.18: Graph parametric curves by hand and by using a graphing utility.

Most graphing devices can graph parametric curves. Make sure that you know how to use your calculator or graphing device to do this. You can also plot approximations to curves given parametrically by choosing a selection of values of t and plotting the corresponding x and y values. You can often eliminate the parameter t algebraically and find an equation that directly relates x and y.

SkillMaster 1.19: Use parametric curves to answer questions about motion in the plane.

When you have a curve in the plane that is given parametrically, you can answer questions about motion along the curve by graphing and analyzing the parametric curve. By seeing what happens for different values of t, you can decide which direction a particle moving along the curve is heading and can geometrically describe the path traced out.

Worked Examples

For each of the following examples, first try to find the solution without looking at the hint. Cover the right column with a piece of paper. If you need a hint, uncover the column. If you need to see the worked solution, look at the solutions section that follows the examples and hints.

Example	Hint
SkillMaster 1.18.	
1. Sketch the following curve by hand, first evaluating (x,y) pairs for various t values and then filling in the curve. $$\begin{aligned} x &= t^2, \\ y &= t^4, \\ 0 &\leq t \leq 1 \end{aligned}$$ Then eliminate the parameter t to find the Cartesian equation of the curve.	Try the values $t = 0$, 0.5, 0.7, 0.8, 0.9, and 1.0.
2. Graph x and y as a function of θ. $$\begin{aligned} x &= \sin\theta, \\ y &= \sin 2\theta, \\ 0 &\leq \theta \leq \pi \end{aligned}$$ Then eliminate the parameter θ to find the Cartesian equation of the curve.	To eliminate the parameter, use the double-angle formula (page A22 of Appendix C), to express y in terms of x.
3. Graph the equation by changing it to parametric form first. $$x = y^3 - 3y^2 - y$$	Make parametric equations by setting $y = t$.

SkillMaster 1.19.

4. Describe the motion of a particle with position (x, y) as t varies in the given interval.

$$\begin{aligned} x &= \sin t, \\ y &= 2\cos t, \\ 0 &\leq t \leq 2\pi \end{aligned}$$

Use the fundamental identity $\sin^2 t + \cos^2 t = 1$ to find the Cartesian equation for the graph. Then find (x, y) for a few values of t to describe the motion of the particle.

5. A baseball is thrown into the air. The $y-$axis represents the vertical direction and the $x-$axis represents the horizontal direction (both in units of feet). The baseball's position at time t (in seconds) is given parametrically below:

$$\begin{aligned} x &= 7t, \\ y &= 20t - t^2 \end{aligned}$$

Graph the trajectory of the ball. How far away does the ball land? Eliminate the parameter t to find the Cartesian equation for the graph.

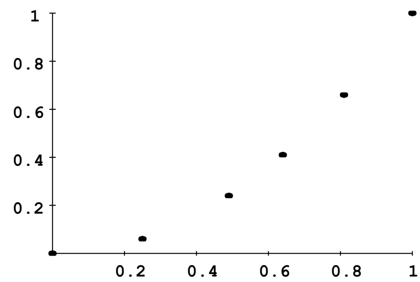
To find where the ball hits the ground, find the x value when $y = 0$.

Solutions to worked examples

1.

t	$x = t^2$	$y = t^4$
0.0	0	0
0.5	$(0.5)^2 = 0.25$	$(0.5)^4 \approx 0.06$
0.7	$(0.7)^2 = 0.49$	$(0.7)^4 \approx 0.24$
0.8	$(0.8)^2 = 0.64$	$(0.8)^4 \approx 0.41$
0.9	$(0.9)^2 = 0.81$	$(0.9)^4 \approx 0.66$
1.0	$(1.0)^2 = 1$	$(1.0)^4 = 1$

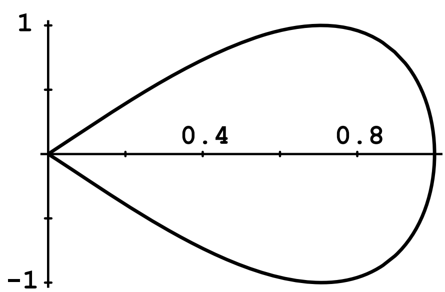
Plot the points (0,0), (0.25,0.06), (0.49,0.24), (0.64,0.41), (0.81,0.66), and (1,1) on a graph.



To eliminate t, notice that $(t^2)^2 = t^4$. So $y = t^4 = (t^2)^2 = (x)^2 = x^2$.

The graph is the part of the parabola $y = x^2$ with x between 0 and 1.

2.



The double angle formula for the sin function is

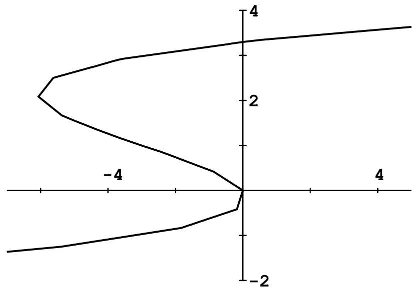
$y = \sin 2\theta = 2\sin\theta\cos\theta$

Square both x and y.

$x^2 = \sin^2\theta$ $\qquad\qquad y^2 = 4\sin^2\theta\cos^2\theta = 4x^2\cos^2\theta$

But since $x^2 = \sin^2(\theta)$ and $\sin^2(\theta) + \cos^2(\theta) = 1$ it follows that $y^2 = 4x^2 - 4x^4$ and $y = \pm 2\sqrt{x^2 - x^4}$.

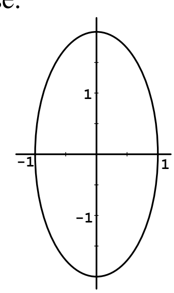
3. $x = t^3 - 3t^2 - t \qquad y = t$



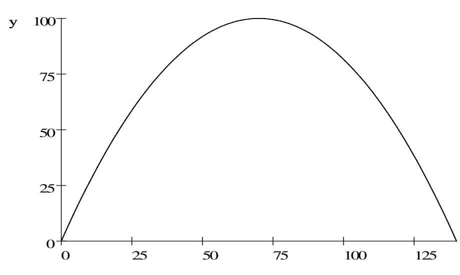
4. $y = 2\cos t$, so $\dfrac{y}{2} = \cos t$ and $x = \sin t$. From $1 = \sin^2 t + \cos^2 t$ it follows that

$1 = x^2 + \left(\dfrac{y}{2}\right)^2$, or $x^2 + \dfrac{y^2}{4} = 1$.

This is an ellipse with vertices at $(\pm 1, 0)$ and $(0, \pm 2)$. When $t = 0$ the particle is at the right vertex $(x,y) = (\sin(0), 2\cos(0)) = (1,0)$. As t increases the particle travels around the ellipse counterclockwise.



5.



The ball hits the ground when $y = 0$ for $t > 0$. Solve the following.

$$\begin{aligned} 0 &= 20t - t^2 \\ 0 &= t(20 - t) \end{aligned}$$

The roots are $t = 0$ which represents the time that the ball was thrown, and the solution to

$$\begin{aligned} 0 &= 20 - t \\ t &= 20\text{sec} \end{aligned}$$

The ball hits the ground when $t = 20$sec. At this time, $x = 7t = 7(20) = 140ft$

To eliminate the parameter t solve for x in terms of t and substitute this into the equation for y.

$$\begin{aligned} x &= 7t \\ t &= x/7 \end{aligned}$$

$$\begin{aligned} y &= 20t - t^2 \\ &= 20(x/7) - (x/7)^2 \\ &= \frac{20}{7}x - \frac{x^2}{49}. \end{aligned}$$

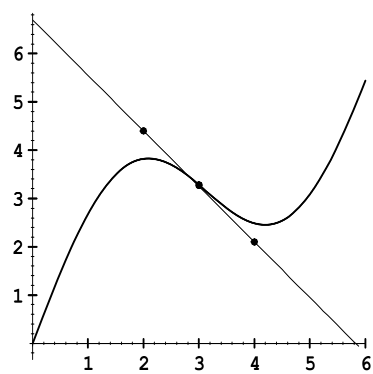
This is the equation of a parabola.

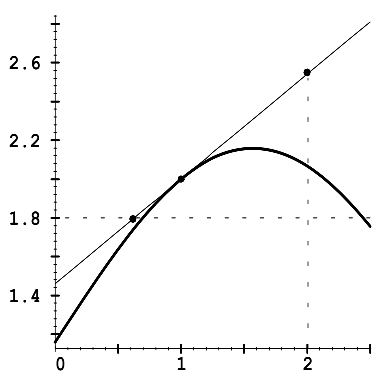
SkillMasters for Chapter 1

SkillMaster 1.1:	Use the different representations of functions to solve real world problems.
SkillMaster 1.2:	Understand and be able to determine domain and range of specific functions.
SkillMaster 1.3:	Sketch graphs of piecewise defined functions.
SkillMaster 1.4:	Recognize different specific types of functions.
SkillMaster 1.5:	Create scatter plots and select an appropriate model.
SkillMaster 1.6:	Use a model to estimate and predict other values.
SkillMaster 1.7:	Recognize how shifted, reflected, and stretched graphs correspond to changes in the algebraic representation and vice versa.
SkillMaster 1.8:	Find new functions by using combinations of old functions.
SkillMaster 1.9:	Become familiar with your graphing device.
SkillMaster 1.10:	Use a graphing calculator to estimate solutions to equations and to compare functions.
SkillMaster 1.11:	Graph families of functions.
SkillMaster 1.12:	Understand how a change in base affects the graph of an exponential function.
SkillMaster 1.13:	Apply shifting, stretching, and reflection to get new functions.
SkillMaster 1.14:	Solve word problems involving exponential functions.
SkillMaster 1.15:	Determine whether or not a function is one-to-one.
SkillMaster 1.16:	Find a formula for the inverse of a given function.
SkillMaster 1.17:	Use laws of logarithms to solve problems involving logarithms and exponents.
SkillMaster 1.18:	Graph parametric curves by hand and by using a graphing utility.
SkillMaster 1.19:	Use parametric curves to answer questions about motion in the plane.

Chapter 2

Limits and Derivatives





2.1 The Tangent and Velocity Problems

Key Concepts:

- The tangent line to the graph of a function
- Functions given by an algebraic formula, by data, or by a graph
- The relation of velocity to the tangent line

Skills to Master:

- Find the equation of a tangent line.
- Estimate the equation of a tangent line when the function is given by an algebraic formula, by a data set, or by a graph.
- Estimate instantaneous velocity.

Discussion:

Section 2.1 introduces two of the ideas that led to the development of the differential calculus: the problem of finding the tangent line to a curve at a specific point and the problem of finding the instantaneous velocity of a moving object. Both of these ideas require taking limits of certain quantities. Limits are more formally introduced and developed in Sections 2.2 and 2.3. In this section, you will use information about curves and about position functions to estimate the equations of tangent lines and to estimate instantaneous velocity. *Figures 2 and 3 and Example 1* show you the visual conception of the tangent line.

pages 90-92

Key Concept: The tangent line to the graph of a function

The tangent line to a circle at a point should be familiar to you from geometry. The fact that a tangent line to a circle is perpendicular to a radius makes it possible to find the equation of the tangent line in this setting. For more general curves, the problem of finding the tangent line at a point is more difficult. Once a point on a curve is specified, the problem of finding the tangent line to the curve at that point is reduced to finding the slope of the tangent line. For then, the point-slope form of a line can be used to get the equation of the tangent line. The text shows how to approximate the slope of the tangent line by taking slopes of secant lines that pass through two nearby points of the curve. Make sure that you study *Figures 3 and 5* in Section 2.1 to see this concept presented visually.

pages 91-93

Key Concept: Functions given by an algebraic formula, by data, or by a graph

Before you can find the tangent line to a curve at a specified point, you must understand something about the function that gives rise to the curve. Most of the functions that you are familiar with such as polynomials, trigonometric functions, exponential functions and logarithmic functions are given by a formula. The text shows you other ways that a function might be described. In many situations, a collection of data points is obtained by observation or measurements and a smooth curve is drawn through the data points to approximate the graph of the function giving rise to the points. In other situations, you may be presented with a graph without being given the algebraic formula that gives rise to it. Finally, a function may be described in words, that is, as a narrative. In each of these cases, you should be able to approximate the tangent line at a point. Notice in the case of a narrative description, you will probably not be able to find the exact value of the slope.

Key Concept: The relation of velocity to the tangent line

Pay careful attention to the discussion of velocity in the text. Note how the same process that gave rise to the tangent line seems to be at work in finding the velocity. Just as the slope of the tangent line is the limiting value of the slope of secant lines, the instantaneous velocity is the limiting value of the average velocities over shorter and shorter time intervals.

SkillMaster 2.1: Find the equation of a tangent line.

The key idea here is that to find the tangent line to a curve at a specific point, all you need to find is the slope of the tangent line. The *point-slope form* of a line then allows you to write the equation of the line. Review this form of the equation of a line if you need to. The equation of the tangent line to a curve through a point $(a, f(a))$ on the curve is

App. B

$$y - f(a) = m(x - a)$$

where m is the slope of the line.

SkillMaster 2.2: Estimate the equation of a tangent line when the function is given by an algebraic formula, by a data set, or by a graph.

For this skill, you need to estimate the slope of the tangent line. This is usually enough to complete the point-slope form of the equation of the tangent line because it will usually be clear what the coordinates of the point on the curve are. If you have a formula for the function and are trying to estimate the equation of the tangent line at $P = (a, f(a))$, find the slope of lines through P and nearby points Q. If these slopes appear to be approaching a specific value as the point Q gets closer to the point P, use this value as the estimate of the slope of the tangent line. If you have a data set for a function and are trying to estimate the equation of the tangent line at $P = (a,b)$, find the slope of the secant line $\dfrac{d-b}{c-a}$ for data points $Q = (c,\, d)$ close to $(a,\, b)$. If these slopes appear to be approaching a specific value as the point Q gets closer to the point P, use this value as the estimate of the slope of the tangent line. Finally, if you have the graph of a function, estimate the slope of the tangent line at $P = (a, f(a))$ by drawing a tangent line on the graph and measuring its slope.

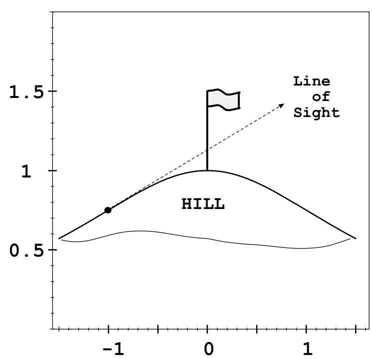
SkillMaster 2.3: Estimate instantaneous velocity.

To estimate the instantaneous velocity of a moving object at a specific time, you use the same skill as in SkillMaster 2.2. The instantaneous velocity at time $t = t_0$ is the slope of the tangent line to the curve $y = p(t)$ at the point $(t_0, p(t_0))$. Here, $y = p(t)$ is position as a function of time. So apply the process in SkillMaster 2.2 to the curve $y = p(t)$ whether this curve is given by an algebraic formula, by a data set or by a graph.

Worked Examples

For each of the following examples, first try to find the solution without looking at the hint. Cover the right column with a piece of paper. If you need a hint, uncover the column. If you need to see the worked solution, look at the solutions section that follows the examples and hints.

Example	Hint
SkillMaster 2.1.	
1. Two people are playing a laser tag game of capture the flag. The flag is on a pole at the top of a hill and is located on the y-axis. The height of the hill is given by the function $f(x) = \frac{3}{3+x^2}$. One person lies prone on the hill at $(-1, \frac{3}{4})$ facing uphill. The other sneaks up the other side of the hill and has to stand at the top to get the flag. Assume that each unit represents 80 feet. In these units, the second person is $\frac{1}{16}$ unit tall and the flag pole is $\frac{1}{2}$ unit tall. If the first person steadily shoots the laser straight ahead, will the person at the top of the hill be tagged? Does the laser hit the flag pole? Suppose as well that you know that the slope of the tangent line at $(-1, \frac{3}{4})$ is $\frac{3}{8}$.	Use the point-slope form, $y - y_1 = m(x - x_1)$ for the equation of a line with slope m and passing through the point (x_1, y_1).



2. Consider the function $y = f(x) = x^3 - 4x$. Suppose the slope of the tangent line at the point $(-1,3)$ is known to be -1. Find the equation of the tangent line, and graph both the function and this line.

Use the point-slope form, $y - y_1 = m(x - x_1)$ for the equation of a line with slope m and passing through the point (x_1, y_1).

SkillMaster 2.2.

3. Estimate the slope of the tangent line to the function

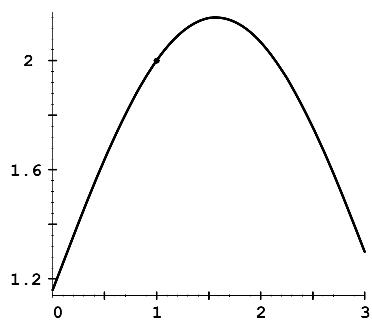
$$y = f(x) = x^3 - 4x$$

at the point $P = (0,0)$. Do this by finding the slopes of the secant lines PQ where $Q = (x, f(x))$ for $x = 1$, 0.1, 0.03, and 0.002, and for $x = -1$, -0.28, -0.05, and -0.001.

First notice that $f(0) = 0^3 - 4(0) = 0$. Form the slopes of the secant lines PQ which are given by

$$\frac{f(x) - f(0)}{x - 0} = \frac{f(x)}{x}$$

4. Estimate the equation of the tangent line at the point $(1,2)$ in the graph below.



Sketch the tangent line on the graph. Sketch in a right triangle that has part of this tangent line as the hypotenuse. Estimate the slope by dividing the length of the vertical side of the triangle by the length of the horizontal side of the triangle. Use this as the slope in the equation of the tangent line in point-slope form.

5. A function is partially defined by the data in the following table.

x	0	1	2	3	4	5	6
y	0	2.68	3.82	3.28	2.49	3.08	5.44

Estimate the slope of the tangent line to the function at the point $P = (3, 3.28)$. by averaging the slopes of the secant lines to the nearest data point on either side.

The nearest x values to 3 are $x = 2$ and $x = 4$.
Compute the slopes of the secant line through $(3,\ 3.28)$ and $(2,\ 3.82)$ and the secant line through $(3, 3.28)$ and $(4,\ 2.49)$. Then average these slopes.

6. Estimate the slope of the tangent line at the point $P = (3,\ 3.28)$ in the example above by plotting the data and then sketching a smooth curve through these points. Now sketch the tangent line and estimate its slope. Which answer do you think is closer to the true slope?

Try using the points $(4, 2.1)$ and $(2, 4.4)$ which appear to lie on the tangent line. Other choices are also possible.

SkillMaster 2.3.

7. A stone is thrown downward from a cliff toward the lake 40 ft below. It is thrown with a velocity of 10 ft/s. At time t seconds the stone is

$$p(t) = 40 - 10t - 16t^2$$

feet above the lake (at least until the stone hits the lake). Estimate the velocity of the stone at time $t = 1$. Do this by estimating the average velocity of the stone over the intervals $[1,\ 1.1]$, $[1,\ 1.01]$, $[1, 1.001]$ and then guessing the limit. As a check, graph the function and estimate the slope of the tangent line.

Note
$p(1) = 40 - 10 - 16 = 14$.
The average velocity over any interval $[1,t]$ is $\dfrac{p(t) - p(1)}{t-1} = \dfrac{p(t) - 14}{t-1}$.

Solutions to worked examples

1. The light of the laser traces out the forward ray of the tangent line to $y = f(x)$ at

$x=-1$. The equation of the tangent line is $y-\dfrac{3}{4}=\dfrac{3}{8}(x-(-1))$, or

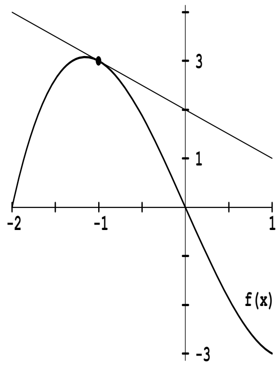
$y-\dfrac{3}{4}=\dfrac{3}{8}(x+1) \qquad 8y-6=3x+3 \qquad 8y-3x=9$

The flag pole is on the y-axis. The laser hits the y-axis at the y-intercept of the tangent line. To find this, we set $x=0$ and get

$$8y-3\cdot 0=9 \text{ or } y=9/8$$

or $1/8$ unit above the top of the hill. This overshoots the person at the top of the hill who is only $1/16$ unit tall, but easily hits the flag pole.

2. $y-3=(-1)(x-(-1))$, so $y=-(x+1)+3$ and $y=-x+2$



3. First we make a table for the positive values of x.

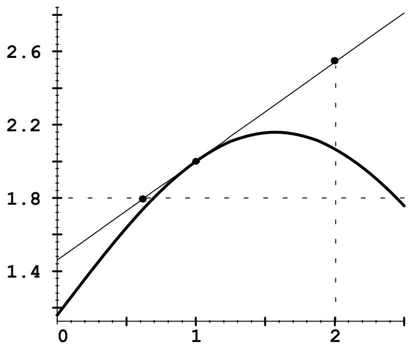
x	$f(x)$	Slope $=f(x)/x$
1	-3	-3
0.1	-0.399	-3.99
0.03	-0.119973	-3.9991
0.002	-0.007999992	-3.999996

Now we make a table for the negative values of x.

x	$f(x)$	Slope $=f(x)/x$
-1	3	-3
-0.28	1.098048	-3.9216
-0.05	0.199875	-3.9975
-0.001	0.003999999	-3.999999

The values appear to be getting closer and closer to -4. We take -4 as our estimate for the slope of the tangent line. The equation of the tangent line is then estimated by $y-0=-4(x-0)$ or $y=-4x$.

4.



The horizontal side appears to intersect the tangent line at $(0.6,\ 1.8)$ and the vertical side appears to intersect the tangent line at $(2,\ 2.55)$. So the slope is about $\dfrac{2.55-1.8}{2-.6}=0.54$. The equation of the tangent line is then $y=2+(0.54)(x-1)$, or $y=0.54x+1.46$.

5. The slope of the line through $(3,\ 3.28)$ and $(2,\ 3.82)$ is $\dfrac{3.82-3.28}{2-3}=-0.54$.

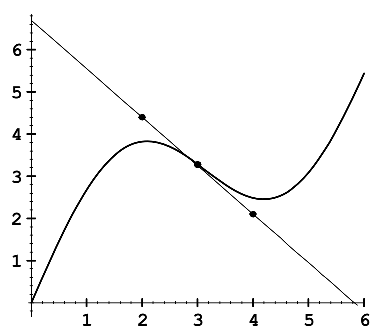
The slope of the line through $(3,\ 3.28)$ and $(4,\ 2.49)$ is $\dfrac{2.49-3.28}{4-3}=-0.79$. So $y=0.54x+1.46$

The estimate of the slope of the tangent line is the average of the slopes of these lines. $\dfrac{-0.54-0.79}{2}=-0.665$.

6. Using the points $(4,\ 2.1)$ and $(2,\ 4.4)$ which appear to lie on the tangent line, we get a slope of about

$$\frac{2.1-4.4}{4-2}=-1.15\ .$$

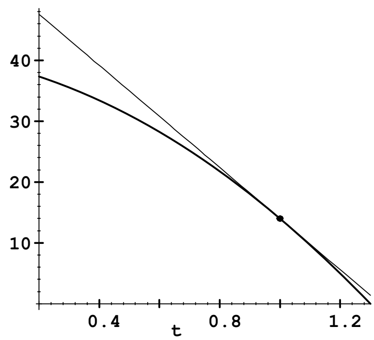
Notice that answers may vary as they depend on the sketch chosen. This is likely closer to the true value if the curve is smooth and rounded. The answer obtained by averaging the slopes of the secant line is the slope of the line through $(2,\ 3.82)$ and $(4,\ 2.49)$. The true slope is likely to be higher.



7.

Interval	Average Velocity
[1, 1.1]	$\dfrac{p(1.1)-14}{0.1} = \dfrac{9.64-14}{0.1} = -43.6$
[1, 1.01]	$\dfrac{p(1.01)-14}{0.01} = \dfrac{13.58-14}{0.01} = -42$
[1, 1.001]	$\dfrac{p(1.001)-14}{.001} = \dfrac{13.958-14}{.001} = -42$

We guess the instantaneous velocity is -42 ft/s. The negative sign reflects the fact that the stone is falling. The graph shows the tangent line decreases from a height of about 48 feet to a height of 0 feet as t increases from 0.2 to about 1.35. This is a slope of $\dfrac{48-0}{.2-1.35} \approx -41.7$ which agrees closely with the answer obtained from the intervals.



2.2 The Limit of a Function

Key Concepts:

- The limit of $f(x)$ as x approaches a
- Avoiding calculators errors

Skills to Master:

- Guess limits from the graph of a function.
- Guess limits by using a calculator.

Discussion:

Section 2.2 gives a *definition* of what it means to say that the limit as x approaches a of $f(x)$ is equal to L. In the next section, you will learn Limit Laws that allow you to compute limits involving most of the functions that you are familiar with. In this section you will approximate limits by working with graphs or with values obtained from a calculator. The concept of limit is one of the central ideas in Calculus. Think about the definition, be able to describe what it means in words, and see how it applies to the problems.

page 5

Key Concept: The limit of f(x) as x approaches a

Definition 1 in the text gives the meaning of "the limit as x approaches a of $f(x)$ is equal to L". This means that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently close to a, but not equal to a. The definitions of "the limit as x approaches a from the left of $f(x)$" and "the limit as x approaches a from the right of $f(x)$" are similar. Note that the function does not have to be defined at $x = a$ for $\lim_{x\to a} f(x)$ to exist.

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Key Concept: Avoiding calculators errors

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Example 2 in this section shows that you need to be careful when using calculators to find limits. If the function that you are taking a limit of is a quotient, and if the limit of both the numerator and denominator are 0, then the roundoff error of your calculator can give incorrect results. Roughly speaking, any computing device represents numbers and function to a certain number of decimal places. For the device to give accurate answers you must avoid calculations that depend on too many decimal places. For example, many computers cannot tell the difference between 10^{-100} and 0. Bearing this in mind, you should be able to estimate limits with assurance. The Limit Laws in the next section will help you to compute limits with precision.

SkillMaster 2.4: Guess limits from the graph of a function.

If you are given the graph of a function f, it is relatively easy to estimate $\lim_{x\to a} f(x)$, $\lim_{x\to a^+} f(x)$, and $\lim_{x\to a^-} f(x)$ where $x = a$ corresponds to a point that appears in the graph. Examine the portion of the graph that is to the left of $x = a$ if computing $\lim_{x\to a^-} f(x)$, examine the portion of the graph that is to the right of $x = a$ if computing $\lim_{x\to a^+} f(x)$, and examine the portion of the graph that is on both sides of $x = a$ if computing $\lim_{x\to a} f(x)$.

SkillMaster 2.5: Guess limits by using a calculator.

If you are given a formula for a function f you can guess limits like $\lim_{x\to a} f(x)$ by using a calculator either to graph the function and use the technique of SkillMaster 2.4, or to choose values of x closer and closer to a and see if the function values appear to approach a specific value. *Examples 3, 4, and 5* in the text illustrate this approach. Make sure that you are aware of the possibility of roundoff error as illustrated in *Example 2*.

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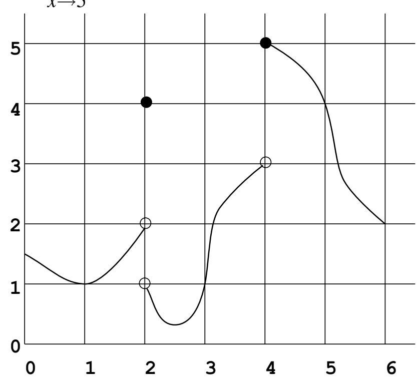
Worked Examples

For each of the following examples, first try to find the solution without looking at the hint. Cover the right column with a piece of paper. If you need a hint, uncover the column. If you need to see the worked solution, look at the solutions section that follows the examples and hints.

Example	Hint
SkillMaster 2.4.	
1. Find the following limits and function values for the graph of f shown below. $\lim_{x\to 2^-} f(x)$ $\lim_{x\to 2^+} f(x)$ $f(2)$ $f(3)$ $\lim_{x\to 4^-} f(x)$ $\lim_{x\to 4^+} f(x)$ $f(4)$ $\lim_{x\to 5^-} f(x)$ 	Remember that an open circle indicates a point not on the graph of the function. A closed circle indicates a point on the graph of the function.

2. Sketch the graph of a function with the following properties.

$$\begin{array}{lclclcl} f(-2) & = & 2 & \quad & f(-1) & = & 2 \\ \lim_{x\to -1^-} f(x) & = & 3 & & \lim_{x\to -1^+} f(x) & = & 1 \\ f(3) & = & 1 & & \lim_{x\to 3^-} f(x) & = & 3 \\ \lim_{x\to 3^+} f(x) & = & 5 & & f(4) & = & 3 \end{array}$$

There are many correct answers. First draw open circles at the one-sided limits, i.e. at $(-1, 3)$, $(-1, 1)$, $(3, 3)$, and $(3, 5)$. Then draw closed circles at the function values that are given, i.e. at $(-2, 2)$, $(-1, 2)$, $(3, 1)$, and $(4, 3)$. Then fill in the rest of the curve. This may be done in many ways, with straight line segments, with smoothly curving arcs, etc. The main thing is to be sure that the Vertical Line Test is passed (See *Page 17* in the text), that is that a vertical line intersects the graph in at most one point.

SkillMaster 2.5.

3. Estimate the following limit by (i) checking it for the values $x = 0.2, 0.1, 0.05, 0.01$ and $x = -0.2, -0.1, -0.05, -0.01$ and (ii) by graphing the function for x near 0 and estimating.

$$\lim_{x\to 0} \frac{e^x - 1}{x}$$

Review the exponential function if you need to.

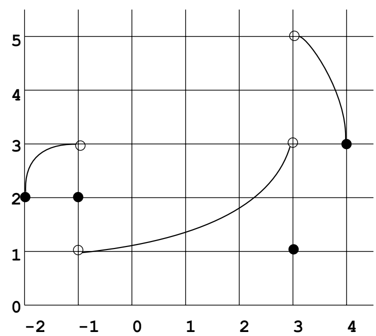
Solutions to worked examples

1. $\lim_{x\to 2^-} f(x) = 2$ $\lim_{x\to 2^+} f(x) = 1$ $f(2) = 4$

$f(3) = 1$ $\lim_{x\to 4^-} f(x) = 3$ $\lim_{x\to 4^+} f(x) = 5$

$f(4) = 5$ $\lim_{x\to 5^-} f(x) = 4$

2.

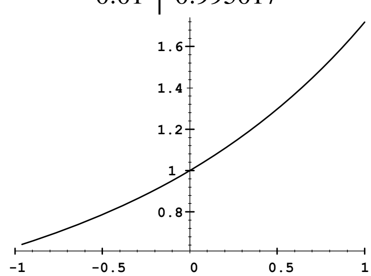


3.

x	$\dfrac{e^x - 1}{x}$
0.2	1.107014
0.1	1.051709
0.05	1.025422
0.01	1.005017

x	$\dfrac{e^x - 1}{x}$
−0.2	0.906346
−0.1	0.951626
−0.05	0.975412
−0.01	0.995017

Based on this evidence we guess the limit is 1.



The graph also appears to go through the point $(0, 1)$.

2.3 Calculating Limits Using the Limit Laws

Key Concepts:

- Limit Laws and the computations of limits
- Squeeze Theorem

Skills to Master:

- Find limits from graphs or partial information using the Limit Laws.
- Find limits using the Limit Laws and algebraic simplification.
- Use the Squeeze Theorem to find limits.

Discussion:

Section 2.3 introduces the Limit Laws that simplify many computations of limits. One of the main results in this section is the following:

If f is a polynomial or rational function and a is in the domain of f, then

$$\lim_{x \to a} f(x) = f(a)$$

This result allows you to compute limits such as $\lim_{x \to 5} \frac{3x^2-2x+7}{x^3+5x^2-6x-3}$. The result can be determined by substituting 5 for x provided that the denominator does not become 0 when this substitution is done. The resulting limit is $\frac{72}{217}$. Another important result obtained is the Squeeze Theorem. Pay careful attention to *Figures 6* and *7* in Section 2.3 and make sure that you are familiar with the Limit Laws introduced in this section.

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Key Concept: Limit Laws and the computations of limits.

The key Limit Laws introduced in this section can be summarized as follows:

$$\text{The limit of a} \left\{ \begin{array}{l} \text{sum} \\ \text{difference} \\ \text{constant multiple} \\ \text{product} \\ \text{quotient} \end{array} \right\} \text{is the} \left\{ \begin{array}{l} \text{sum} \\ \text{difference} \\ \text{constant multiple} \\ \text{product} \\ \text{quotient} \end{array} \right\} \text{of the limit(s).}$$

Of course, for these laws to hold, the individual limits must exist and for the last law to hold, the limit of the denominator must not be 0. These laws allow you to reduce the limit of an expression involving sums, differences, products and quotients to limits of simpler expressions.

Key Concept: Squeeze Theorem

The Squeeze Theorem allows you to compute limits of functions that could not be computed directly using the Limit Laws. For example, $\lim_{x\to 0}(x \cdot \sin\frac{1}{x})$ can not be computed as $\lim_{x\to 0} x \cdot \lim_{x\to 0} \sin\frac{1}{x}$ because the latter limit does not exist ($\sin\frac{1}{x}$ oscillates from $+1$ to -1 as x approaches 0). However, $x \cdot \sin\frac{1}{x}$ is always between $|x|$ and $-|x|$ and since both of these functions have limit 0 as x approaches 0, $\lim_{x\to 0} x \cdot \sin\frac{1}{x} = 0$.

SkillMaster 2.6: Find limits from graphs or partial information using the Limit Laws.

If you know the limits of the component functions that make up a more complicated function either from graphical information or from information that you are given, you can often compute the limit of the more complicated function by applying the Limit Laws. *Problems 1 and 2* in the exercises for Section 2.3 give you practice in applying this skill.

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SkillMaster 2.7: Find limits using the Limit Laws and algebraic simplification.

Even if the limit of a denominator is 0, algebraic simplification combined with the Limit Laws can allow you to compute limits of certain quotients. For example,

$$\lim_{x\to 4}\frac{x^2-2x-8}{x^2-3x-4}=\lim_{x\to 4}\frac{(x+2)(x-4)}{(x+1)(x-4)}=\lim_{x\to 4}\frac{(x+2)}{(x+1)}$$

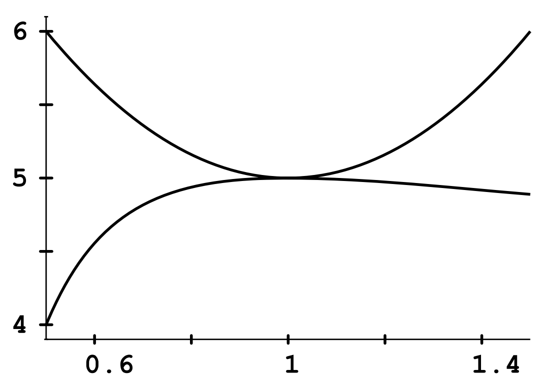
provided both of the latter limits exist. Directly substituting $x=4$ in the original limit yield $0/0$ which is undefined. But $\lim\limits_{x\to 4}\dfrac{(x+2)}{(x+1)}=6/5$ from the Limit Laws. So the desired limit is $6/5$.

SkillMaster 2.8: Use the Squeeze Law to find limits.

The Squeeze Law allows you to compute limits of certain functions based on knowledge of other functions with greater and lesser values. For example, if

$$\frac{4x^2+2x-1}{x^2}\le f(x)\le 4x^2-8x+9$$

for values of x near 1, then $\lim\limits_{x\to 1}f(x)=5$ since both $\lim\limits_{x\to 1}\dfrac{4x^2+2x-1}{x^2}=5$ and $\lim\limits_{x\to 1}4x^2-8x+9=5$. The graph below illustrates this:

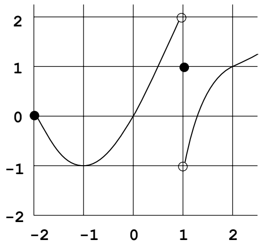


Worked Examples

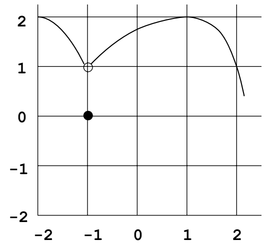
For each of the following examples, first try to find the solution without looking at the hint. Cover the right column with a piece of paper. If you need a hint, uncover the column. If you need to see the worked solution, look at the solutions section that follows the examples and hints.

Example	Hint
SkillMaster 2.6.	
Evaluate the following three limits based on the given information about the functions $f(x)$, $g(x)$, and $h(x)$. $\lim_{x\to 3} f(x) = 4 \qquad \lim_{x\to 3} g(x) = -1 \qquad \lim_{x\to 3} h(x) = 2$	
1. $\lim_{x\to 3} [f(x) + 2g(x)]$	The Limit Laws allow the limits to be evaluated term by term if they exist.
2. $\lim_{x\to 3} \frac{f(x)h(x)}{(g(x))^4}$	The Limit Laws allow the limits to be evaluated by substituting the respective limits for the functions and then evaluating if possible.
3. $\lim_{x\to 3} \frac{f(x)g(x)}{f(x) - 2h(x)}$	Check the limit of the denominator.

Use the graphs of $f(x)$ and $g(x)$ shown below to evaluate the limits in the next three problems.



$f(x)$



$g(x)$

4. $\lim\limits_{x\to 2} [f(x)+g(x)]$

The graphs indicate that each of f and g approach 1 as x approaches 2.

5. $\lim\limits_{x\to 1^+} [f(x)g(x)]$

The graphs indicate that f approaches -1 and g approaches 2 as x approaches 1 from the right.

6. $\lim\limits_{x\to -1} x^3 g(x)$

First, $\lim\limits_{x\to -1} x^3 = (-1)^3$. The graph indicates that g approaches 1 as x approaches -1 from both the left and right. So $\lim\limits_{x\to -1} g(x) = 1$

SkillMaster 2.7.

Evaluate the following limits

7. $\lim_{x\to 1} \dfrac{3x^2-4x+2}{6x-3}$

First try to substitute the value 1 for x. If the result is well-defined then this is the limit. (This will be justified by Theorem 7 on page 122 in the next section of the text.)

8. $\lim_{x\to 2} \dfrac{x^2-5x+6}{x-2}$

Here, if we substitute the value 2 for x we get $\dfrac{0}{0}$ which is undefined. Use algebra to eliminate the factor $x-2$ in the denominator by factoring the numerator and canceling.

9. $\lim_{h\to 0} \dfrac{(3+h)^2-3^2}{h}$

Again, if we substitute 0 for h we get $\dfrac{0}{0}$ which is undefined. Expand and simplify the numerator.

10. $\lim_{x\to 0^+} \dfrac{x-1}{\sqrt{x}-1}$

Rationalize the denominator, that is multiply both the numerator and the denominator by $\sqrt{x}+1$.

11. $\lim_{x\to 2^+} \dfrac{[[x]]}{\sqrt{x}}$ and $\lim_{x\to 2^-} \dfrac{[[x]]}{\sqrt{x}}$

Recall that $[[x]]$ is the greatest integer less than or equal to x. For $2 \le x < 3, [[x]] = 2$. For $1 \le x < 2, [[x]] = 1$. Thus $\lim_{x\to 2^+} [[x]] = 2$ and $\lim_{x\to 2^-} [[x]] = 1$.

SkillMaster 2.8.

12. Evaluate $\lim_{x\to 0} \tan(x)\cos(2/x)$

Use the Squeeze Theorem and notice that $-1 \le \cos(2/x) \le 1$.

Solutions to worked examples

1. $\lim_{x\to 3} [f(x)+2g(x)] = \lim_{x\to 3} f(x) + 2\lim_{x\to 3} g(x) = 4+2(-1) = 2$

2. $\lim_{x\to 3} \dfrac{f(x)h(x)}{(g(x))^4} = \dfrac{f(3)h(3)}{(g(3))^4} = \dfrac{4\cdot 2}{(-1)^4} = 8$

3. $\lim_{x\to 3} f(x)g(x) = [\lim_{x\to 3} f(x)][\lim_{x\to 3} g(x)] = (4)(-1)$ and

$\lim_{x\to 3} f(x) - 2h(x) = 4-2(2) = 0$. Since the limit of the denominator is 0 and the limit of the numerator is -4, the limit of the quotient does NOT exist as a real number. In Section 2.5, you will see how to evaluate this limit as $-\infty$.

4. $\lim_{x\to 2} [f(x)+g(x)] = \lim_{x\to 2} f(x) + \lim_{x\to 2} g(x) = f(2)+g(2) = [1+1] = 2$

5. $\lim_{x\to 1^+} [f(x)g(x)] = \lim_{x\to 1^+} f(x)\cdot \lim_{x\to 1^+} g(x) = (-1)(2) = -2$

6. $\lim_{x\to -1} x^3 g(x) = [\lim_{x\to -1} x^3][\lim_{x\to -1} g(x)] = (-1)^3(1) = -1$

7. $\lim_{x\to 1} \dfrac{3x^2-4x+2}{6x-3} = \dfrac{3(1)^2-4(1)+2}{6(1)-3} = \dfrac{3-4+2}{6-3} = \dfrac{1}{3}$

8. $\lim_{x\to 2} \dfrac{x^2-5x+6}{x-2} = \lim_{x\to 2} \dfrac{(x-2)(x-3)}{x-2} = \lim_{x\to 2} \dfrac{(x-3)}{1} = (2-3) = -1$

9. $\lim\limits_{h\to 0} \dfrac{(3+h)^2-3^2}{h} = \lim\limits_{h\to 0} \dfrac{(9+6h+h^2)-9}{h} = \lim\limits_{h\to 0} \dfrac{6h+h^2}{h} = \lim\limits_{h\to 0} \dfrac{h(6+h)}{h}$

$$= \lim_{h\to 0} (6+h) = 6$$

10. $\lim\limits_{x\to 0^+} \dfrac{x-1}{\sqrt{x}-1} = \lim\limits_{x\to 0^+} \dfrac{x-1}{\sqrt{x}-1}\dfrac{\sqrt{x}+1}{\sqrt{x}+1}$

$$= \lim_{x\to 0^+} \frac{(x-1)(\sqrt{x}+1)}{(\sqrt{x})^2-1^2} = \lim_{x\to 0^+} \frac{(x-1)(\sqrt{x}+1)}{(x-1)} = \lim_{x\to 0^+} \sqrt{x}+1 = 1$$

11. $\lim\limits_{x\to 2^+} \dfrac{[[x]]}{\sqrt{x}} = \dfrac{\lim\limits_{x\to 2^+} [[x]]}{\lim\limits_{x\to 2^+} \sqrt{x}} = \dfrac{2}{\sqrt{2}} = \sqrt{2} \lim\limits_{x\to 2^-} \dfrac{[[x]]}{\sqrt{x}}$

$$= \frac{\lim\limits_{x\to 2^-} [[x]]}{\lim\limits_{x\to 2^-} \sqrt{x}} = \frac{1}{\sqrt{2}} = \sqrt{2}/2$$

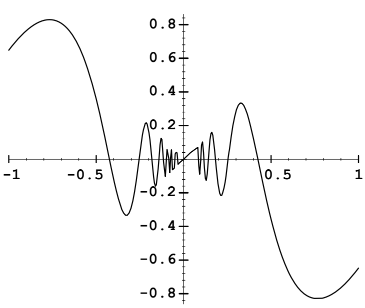
12. $-|\tan(x)| \le \tan(x)\cos(2/x) \le |\tan(x)|$

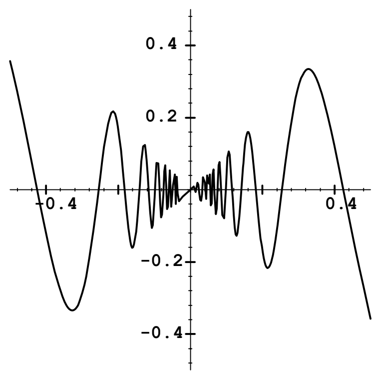
$\lim\limits_{x\to 0} -|\tan(x)| = 0$ and $\lim\limits_{x\to 0} |\tan(x)| = 0$

So the Squeeze Theorem implies

$$0 \le \lim_{x\to 0} \tan(x)\cos(2/x) \le 0 \qquad \lim_{x\to 0} \tan(x)\cos(2/x) = 0$$

Pictured below are graphs of $\tan(x)\cos(2/x)$ for two different intervals near 0.





2.4 Continuity

Key Concepts:

- The meaning of continuity
- Continuity of most familiar functions
- Intermediate Value Theorem

Skills to Master:

- Find points of discontinuity from a graph or from a function given by a formula.
- Estimate roots by using the Intermediate Value Theorem.

Discussion:

An informal definition of continuity is that a function is continuous if the graph can be drawn without lifting a pencil from the paper. The concept of continuity may seem abstract to you at this point, but it has important consequences throughout Calculus. The Intermediate Value Theorem presented at the end of Section 2.4 is one of those consequences. Later, in Chapter 4, another consequence will be the ability to compute the maximum and minimum values of continuous functions defined on closed intervals.

Key Concept: The meaning of continuity

For a function to be continuous at $x = a$, the limit as x approaches a of $f(x)$ must be equal to $f(a)$. Note that this requires three things to happen. First, f must be defined at $x = a$. Second, the limit as x approaches a of $f(x)$ must exist. And third, these two quantities must be the same. Understanding this also leads to an understanding of how functions can fail to be continuous. A function f fails to be continuous at $x = a$ if and only if any of these three conditions fail.

Key Concept: Continuity of most familiar functions

The Limit Laws immediately give the result that sums, differences, constant multiples, products and quotients of continuous functions are continuous wherever they are defined. In particular, rational functions, (quotients of polynomials) are continuous wherever they are defined. The text gives geometric and other arguments to support the fact that root functions, trigonometric functions, inverse trigonometric functions, exponential functions and logarithmic functions are continuous wherever they are defined. Another result that allows you to work with a larger class of functions is that compositions of continuous functions are continuous. These results can be combined to show that a function such as:

$$f(x) = \sqrt[3]{\sin\left(\frac{e^{\cos(x)}}{x^3 + \ln(x)}\right)}$$

is continuous wherever it is defined.

Key Concept: Intermediate Value Theorem

The Intermediate Value Theorem gives an important property of continuous functions. The proof of this result depends on a careful analysis of properties of the real numbers and is usually presented in advanced calculus courses. The result states that a continuous function f defined on an interval $[a,b]$ achieves every value between $f(a)$ and $f(b)$. This property of continuous functions can be used in approximating roots as in *Example 9* in the the text. An intuitive example of this theorem is the idea that if a person travels between two adjoining countries, at some time the person is exactly on the border.

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SkillMaster 2.9: Find points of discontinuity from a graph or from a function given by a formula.

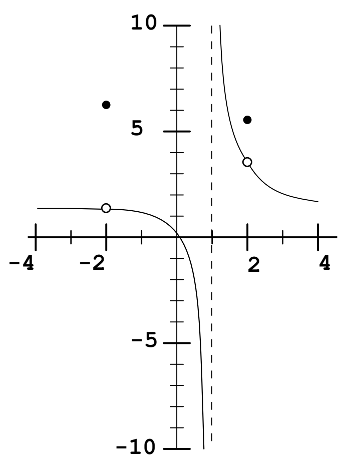
Now that you know how a function can fail to be continuous, it will be possible to find points of continuity and discontinuity from a graph. Any points where a function is not defined, where there is a jump in the graph, or where the limit of the function as you approach a particular point does not exist will be a point of discontinuity. If the function is given by a formula, evaluating the limit as you approach a point from the left and from the right and checking whether the function is defined at that point will allow you to determine whether the function is continuous at that point.

SkillMaster 2.10: Estimate roots by using the Intermediate Value Theorem.

The Intermediate Value Theorem allows you to approximate a root of a function $y = f(x)$ if you can determine whether the function is positive or negative at certain points. (Recall that a root of f is a number r such that $f(r) = 0$.) The key idea is the following: If f is continuous on an interval $[a, b]$ and if f is positive at one endpoint and negative at the other, then f must have a root at some value between a and b.

Worked Examples

For each of the following examples, first try to find the solution without looking at the hint. Cover the right column with a piece of paper. If you need a hint, uncover the column. If you need to see the worked solution, look at the solutions section that follows the examples and hints.

Example	Hint
SkillMaster 2.9.	
1. The graph of the function $y = f(x)$ is shown for x in $[-4, 4]$. What is its domain? State the intervals on which the function is continuous. 	The domain is the set of x-values for which $f(x)$ is defined.

2. Where is the following function continuous?

$$f(x) = \tan(x)\sqrt{4-x^2}.$$

The function is continuous at all points in the domain since it is composed of continuous functions.

3. Where is the following function discontinuous?

$$f(x) = \begin{cases} e^x & \text{if } x < 0 \\ 1 & \text{if } x = 0 \\ 2x+1 & \text{if } 0 < x < 2 \\ x^2 & \text{if } 2 \le x \end{cases}$$

Evaluate the one-sided limits at $x = 0$ and $x = 2$.

4. Is there a value c so that the following function is continuous? If so, what is the value?

$$f(x) = \begin{cases} 2x^2 - c & \text{if } x \le 1 \\ cx & \text{if } 1 < x \end{cases}$$

If possible find a value c so that the right and left limits of $f(x)$ at $x = 1$ agree.

SkillMaster 2.10.

5. Show that $f(x) = xe^x - 6 = 0$ has a solution on $[1,2]$ and use a graphing calculator to find it to 2 decimal places.

Use the Intermediate Value Theorem and find two x values so that f is positive on one and negative on the other.

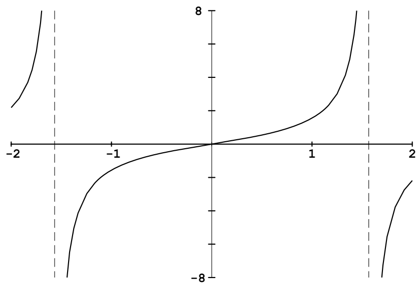
Solutions to worked examples

1. The domain is the set of all real numbers in $[-4,4]$ except $x = 1$, alternatively the domain is the intervals $[-4,1)$ and $(1,4]$. The function is continuous on the intervals $[-4,-2),(-2,1),(1,2)$, and $(2,4]$. It is not continuous at 2 or -2 because $f(2)$ and $f(-2)$ have different values than the one-sided limits.

2. The domain of $\sqrt{4-x^2}$ is the set of all x such that $4-x^2 \ge 0$, or $4 \ge x^2$, or

-2 $\leq x \leq$ 2.

The domain of $\tan(x)$ is the set of all x such that $x \neq \pi/2 + n\pi$ for some integer n. In the interval $[-2,2]$ this rules out $\pm\pi/2$. The function is continuous on the intervals $[-2,-\pi/2), (-\pi/2,\pi/2)$, and $(\pi/2,2]$.



3. The function $f(x)$ is continuous at all points except possibly $x = 0$ or $x = 2$.

$$f(0) = 1 \qquad \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} e^x = e^0 = 1$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} 2x+1 = 2(0)+1 = 1$$

Since the one-sided limits agree and equal $f(0)$ we have $\lim_{x\to 0} f(x) = f(0) = 1$. Thus f is continuous at $x = 0$.

$$f(2) = 2^2 = 4 \text{ and } \lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (2x+1) = 2(2)+1 = 5$$

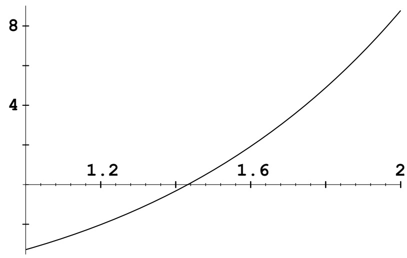
The limit from the left does not equal the function value so $f(x)$ is discontinuous at $x = 2$.

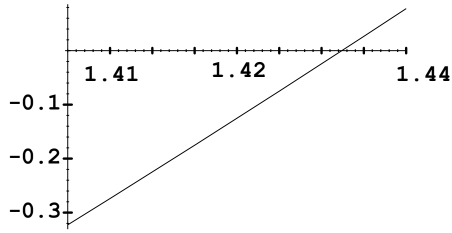
4. $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (2x^2 - c) = 2 - c = f(1).$ $\qquad \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (cx) = c$

The function is continuous if and only if both limits agree, which occurs if and only if $2 - c = c$, or $2 = 2c$, or $c = 1$

5. $f(1) = 1(e^1) - 6 \approx -3.2817 < 0$ $\qquad f(2) = 2(e^2) - 6 \approx 8.7781 > 0$

The Intermediate Value Theorem implies there is a solution.





To two decimal places the solution is 1.43.

2.5 Limits Involving Infinity

Key Concepts:

- Limits that are infinite
- Limits at infinity
- Vertical and horizontal asymptotes

Skills to Master:

- Find limits involving infinity.
- Find vertical and horizontal asymptotes.

Discussion:

The definition of $\lim_{x\to a} f(x) = L$ requires that a and L be finite real numbers. It is often useful to know what happens as x approaches $\pm\infty$ and to have some notation that indicates that $f(x)$ becomes large without bound or large and negative without bound as x approaches a. This section gives meaning to limits such as the following:

$$\begin{aligned}
\lim_{x\to\infty} f(x) &= L, \quad \lim_{x\to-\infty} f(x) = L, \quad \lim_{x\to a} f(x) = \infty, \\
\lim_{x\to a} f(x) &= -\infty, \quad \text{and} \lim_{x\to\pm\infty} f(x) = \pm\infty
\end{aligned}$$

Key Concept: Limits that are infinite

If a function f is defined on both sides of $x = a$, $\lim_{x\to a} f(x) = \infty$ means that the values of $f(x)$ can be made arbitrarily large by taking f sufficiently close to a (but not equal to a). Note the similarity to the definition of $\lim_{x\to a} f(x) = L$. One sided limits as x approaches a being infinite and limits approaching $-\infty$ are defined analogously. Note that this notation does *NOT* mean the limit exists or that we are regarding ∞ as a number. This notation is just a useful way of indicating that a function behaves in a certain way as x gets closer and closer to a.

Key Concept: Limits at infinity

It is also useful to have notation to indicate when a function appears to get larger and larger without bound, either in a positive or negative direction. The meaning of $\lim_{x\to\infty} f(x) = L$ is that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large. The notation $\lim_{x\to-\infty} f(x) = L$ is given meaning in a similar way. Finally, $\lim_{x\to\infty} f(x) = \infty$, $\lim_{x\to\infty} f(x) = -\infty$, $\lim_{x\to-\infty} f(x) = \infty$, and $\lim_{x\to-\infty} f(x) = -\infty$ are defined by combining the notion of limits that are infinite with the notion of limits at infinity.

Key Concept: Vertical and horizontal asymptotes

The line $x = a$ is a vertical asymptote of $y = f(x)$ if at least one of the limits as x approaches a from above or below is ∞ or $-\infty$. Read carefully through the examples in the text to get a better feeling for what vertical asymptotes are. The line $y = L$ is a horizontal asymptote of $y = f(x)$ if at least one of the limits as x approaches positive or negative infinity is L. Note that vertical asymptotes involve limits that are infinite and horizontal asymptotes involve limits at infinity.

SkillMaster 2.11: Find limits involving infinity.

Some facts that allow you to compute infinite limits and limits at infinity are the following:

- If $\lim_{x\to a} f(x) = 0$ and if the values of f are positive for x near a, then $\lim_{x\to a} \frac{1}{f(x)} = \infty$
- If $\lim_{x\to a} f(x) = 0$ and the values of f are negative for x near a, then $\lim_{x\to a} \frac{1}{f(x)} = -\infty$
- If $\lim_{x\to 0^+} f(x) = L$, then $\lim_{x\to\infty} f(1/x) = L$
- If $\lim_{x\to 0^-} f(x) = L$, then $\lim_{x\to-\infty} f(1/x) = L$

Rather than just memorizing facts like these, try to understand the behavior of the function as x approaches a or $\pm\infty$.

SkillMaster 2.12: Find vertical and horizontal asymptotes.

Once you understand how to find limits involving infinity, finding vertical and horizontal asymptotes should not be too difficult. To find vertical asymptotes of quotients, examine places where the denominator is 0. To find horizontal asymptotes, examine the behavior of the functions as values approach $\pm\infty$. It is always helpful to look at the graph of the function in a good viewing window.

Worked Examples

For each of the following examples, first try to find the solution without looking at the hint. Cover the right column with a piece of paper. If you need a hint, uncover the column. If you need to see the worked solution, look at the solutions section that follows the examples and hints.

Example	Hint

SkillMaster 2.11.

Find the following limits.

1. $\lim\limits_{x\to(\pi/2)^-} \sec(x)$	Recall that $\sec(x) = 1/\cos(x)$ and that $\cos(\pi/2) = 0$ and is positive for x slightly smaller than $\pi/2$.
2. $\lim\limits_{x\to(\pi/2)^-} e^{\sec(x)}$	Recall $\lim\limits_{u\to\infty} e^u = \infty$ and $\lim\limits_{x\to(\pi/2)^-} \sec(x) = \infty$
3. $\lim\limits_{x\to\infty} \dfrac{x}{x^2 - 2x - 4}$	Divide the numerator and the denominator by x^2 because 2 is the highest power that occurs in the denominator.

4. $\lim_{x\to\infty} \dfrac{3x^2}{5x^2 - 2x - 4}$

Divide the numerator and the denominator by x^2 because 2 is the highest power that occurs in the denominator.

5. $\lim_{x\to\infty} \cos(1/x)$

Think of this as a composition of limits. The inside limit $(1/x)$ is approaching 0 and $\cos(u)$ is continuous at 0.

6. $\lim_{x\to\infty} \left(\sqrt{4x+1} - 2\sqrt{x}\right)$

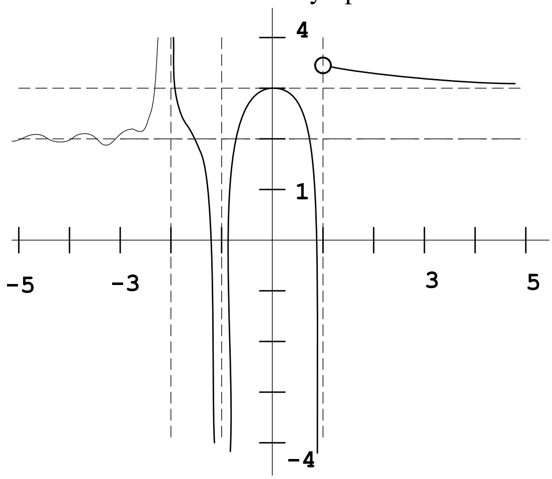
Use the trick of multiplying and dividing by $\left(\sqrt{4x+1} + 2\sqrt{x}\right)$ to get a fraction whose limit is easier to compute.

7. Suppose that $y = p(t)$ represents the population of reindeer on an island at time t, in units of years after the initial observation of the population. What does $\lim_{t\to 10} p(t) = \infty$ mean?

Interpret this as a real world situation.

SkillMaster 2.12.

8. Answer the following questions about the graph below:
Find all vertical and horizontal asymptotes. Describe the limits of the function at these asymptotes.



Find all the x-coordinates of the vertical lines where the graph appears to go to positive or negative infinity. Then find all the y-coordinates of the horizontal lines that are approached as x goes to positive or negative infinity.

9. Use a graphing calculator to guess the asymptotes of

$$f(x) = \frac{\sin(4x) + x^2}{x^2}$$

Describe the limit of the function at these asymptotes.

Where is the function undefined? What happens as x approaches positive or negative infinity?

Solutions to worked examples

1. $\lim_{x \to (\pi/2)^-} \sec(x) = \infty$

2. $\lim_{x \to (\pi/2)^-} e^{\sec(x)} = \lim_{u \to \infty} e^u = \infty$

3. $\lim\limits_{x\to\infty} \dfrac{x}{x^2-2x-4} = \lim\limits_{x\to\infty} \dfrac{x/x^2}{x^2/x^2-2x/x^2-4/x^2} = \lim\limits_{x\to\infty} \dfrac{1/x}{1-2/x-4/x^2}$
$= \lim\limits_{x\to\infty} \dfrac{0}{1-0-0} = 0$

4. $\lim\limits_{x\to\infty} \dfrac{3x^2}{5x^2-2x-4} = \lim\limits_{x\to\infty} \dfrac{3x^2/x^2}{5x^2/x^2-2x/x^2-4/x^2} = \lim\limits_{x\to\infty} \dfrac{3}{5-2/x-4/x^2}$
$= \lim\limits_{x\to\infty} \dfrac{3}{5-0-0} = \dfrac{3}{5} = 0.6$

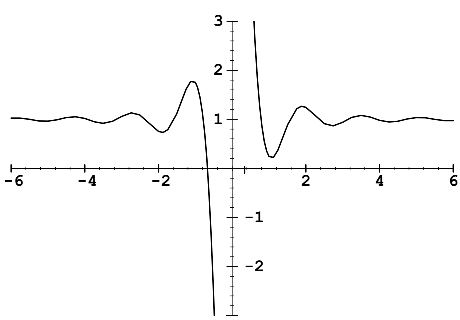
5. $\lim\limits_{x\to\infty} \cos(1/x) = \cos(\lim\limits_{x\to\infty} 1/x) = \cos(0) = 1$

6. $\lim\limits_{x\to\infty} \left(\sqrt{4x+1}-2\sqrt{x}\right) = \lim_{x\to\infty} \left(\left(\sqrt{4x+1}-2\sqrt{x}\right)\cdot\left(\dfrac{\sqrt{4x+1}+2\sqrt{x}}{\sqrt{4x+1}+2\sqrt{x}}\right)\right)$
$= \lim\limits_{x\to\infty} \dfrac{(\sqrt{4x+1})^2-(2\sqrt{x})^2}{\sqrt{4x+1}+2\sqrt{x}} = \lim\limits_{x\to\infty} \dfrac{4x+1-4x}{\sqrt{4x+1}+2\sqrt{x}} = \lim\limits_{x\to\infty} \dfrac{1}{\sqrt{4x+1}+2\sqrt{x}} = 0$

7. It means as the time approaches the 10th year the population of reindeer begins to increase without bound. Since so many reindeer will exhaust the resources of the island we can expect extinction to occur.

8. The vertical asymptotes appear to be at $x=-2, x=-1$ and $x=1$.
$\lim\limits_{x\to-2} f(x)=\infty$, $\lim\limits_{x\to-1} f(x)=-\infty$, $\lim\limits_{x\to1^-} f(x)=-\infty$, $\lim\limits_{x\to1^+} f(x)=3.5$
There appear to be horizontal asymptotes at $y=2$ and $y=3$.
$\lim\limits_{x\to-\infty} f(x)=2$, $\lim\limits_{x\to\infty} f(x)=3$

9. The vertical asymptote is at $x=0$. The horizontal asymptote is at $y=1$.
$\lim\limits_{x\to-\infty} f(x)=1$, $\lim\limits_{x\to\infty} f(x)=1$, $\lim\limits_{x\to0^-} f(x)=-\infty$, $\lim\limits_{x\to0^+} f(x)=\infty$



2.6 Derivatives and Rates of Change

Key Concepts:

- The definition of the tangent line
- Instantaneous velocity defined as a limit
- The instantaneous rate of change defined as a limit
- The definition of the derivative
- The derivative as a rate of change

Skills to Master:

- Find the slope of the tangent line using the definition.
- Find velocity and other rates of change by using the definition, by using data from a table, and by using a graph.
- Compute and estimate derivatives.
- Interpret the derivative as a rate of change.

Discussion:

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Section 2.6 gives clarity to the meaning of the tangent line to the curve $y = f(x)$ at the point $(a, f(a))$. Earlier, in Section 2.1 the *tangent line* was defined as the line with slope equal to the limiting value of the slopes of secant lines. Now that we have made the notion of limit more precise, we can make the notion of tangent line more precise. The text also makes clear what the relation of the slope of the tangent line is to instantaneous velocity and to other various instantaneous rates of change.

The derivative $f'(a)$ of a function f at a point a has the formal definition

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

provided the limit exists. In each specific context, the derivative can be interpreted as a rate of change. This section is a continues the discussion of the previous sections and shows you how to compute derivatives and how to interpret the derivative as a rate of change.

Key Concept: The definition of the tangent line

You need to learn and understand the definition of the tangent line. In the next section, you will see that the tangent line to the curve $y = f(x)$ at $x = a$ is the line with slope equal to the derivative of the function at $x = a$. The definition states that the tangent line to the curve $y = f(x)$ at the point $P = (a, f(a))$ is the line through P with slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided this limit exists.

Key Concept: Instantaneous velocity defined as a limit

If $s = f(t)$ gives the position of an object at time t, the instantaneous velocity of the object at time $t = a$, $v(a)$, is defined by

$$v(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Note that if we let $x = a + h$, this is the same as the definition of the slope of the tangent line to $s = f(t)$ at $t = a$.

Key Concept: The instantaneous rate of change defined as a limit

If $y = f(x)$ represents y as a quantity that depends on x, we may ask what the instantaneous rate of change of y with respect to x is $x = x_1$. If x_2 is a point near x_1, we can let $\Delta x = x_2 - x_1$ represent the change in x and we can let $\Delta y = f(x_2) - f(x_1)$ represent the corresponding change in y. Then

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

represents the average change in y with respect to x over the interval $[x_1, x_2]$. The instantaneous rate of change of y with respect to x at $x = x_1$ is then defined to be

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

provided this limit exists. Notice that this is the same as the slope of the tangent line to $y = f(x)$ at $x = x_1$.

Key Concept: The definition of the derivative

Make sure that you understand the definition of the derivative. An alternate definition is

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

To see that the two definitions are the same, set $h = x - a$. Notice that quantities that you take the limit of in computing the derivative are the slopes of secant lines through point $(a, f(a)$ and $(a+h, f(a+h))$. When you understand this, you will see why the derivative is the slope of the tangent to $y = f(x)$ at $x = a$.

Key Concept: The derivative as a rate of change

Examine $\lim\limits_{h\to 0} \frac{f(a+h)-f(a)}{h}$. The numerator gives the change in the values of $f(x)$ as x changes from a to $a+h$. The denominator gives the change in x as x changes from a to $a+h$. So the derivative is defined to be the limiting value of the change in values of $f(x)$ divided by the change in values of x. This is the *rate of change* of the function values with respect to the change in x values, as discussed in the previous section. Keep this concept of rate of change in mind whenever you are working with derivatives.

SkillMaster 2.13: Find the slope of the tangent line using the definition.

To find the slope of the tangent line using the definition, you need to evaluate $\lim\limits_{x\to a} \frac{f(x)-f(a)}{x-a}$ using techniques already developed. Note that both the numerator and denominator are approaching 0, so you won't just be able to substitute in $x = a$. Some algebraic simplification will be necessary before you can compute the limit.

SkillMaster 2.14: Find velocity and other rates of change by using the definition, by using data from a table, and by using a graph.

To find velocity or other rates of change from the definition, use the same techniques as in SkillMaster 2.13. To find the velocity or other rates of change from data or from a graph, use the techniques developed earlier to estimate the slope of the tangent line to the function.

SkillMaster 2.15: Compute and estimate derivatives.

This is a skill that you have already mastered. Computing the derivative from the definition is the same as computing the slope of the tangent line from the definition as described in the previous section. Estimating derivatives from data or from graphs is the same as estimating the slopes of tangent lines.

SkillMaster 2.16: Interpret the derivative as a rate of change.

Whenever you encounter a derivative, $f'(a)$, you need to be able to interpret it as an instantaneous rate of change of $y = f(x)$ with respect to x. Depending on what the function represents, this may have different physical interpretations, such as velocity, the rate of increase of temperature, etc.

Worked Examples

For each of the following examples, first try to find the solution without looking at the hint. Cover the right column with a piece of paper. If you need a hint, uncover the column. If you need to see the worked solution, look at the solutions section that follows the examples and hints.

Example	Hint
SkillMaster 2.13.	
For the next three problems, find the equation of the tangent line to the curve at the given point.	
1. $y = 3x^2 - x$ at $(1,2)$	First compute the slope of the tangent line by computing it from the definition. Next express the line in the point slope form.

2. $y = \dfrac{3}{(x-1)^2}$ at $(2,3)$

Set $f(x) = \dfrac{3}{(x-1)^2}$ and find $\lim_{h\to 0} \dfrac{f(2+h)-f(2)}{h}$.

3. $y = \sqrt{x^2+1}$ at $(1,\sqrt{2})$

Find $\lim_{h\to 0} \dfrac{f(1+h)-f(1)}{h}$ where $f(x) = \sqrt{x^2+1}$

SkillMaster 2.14.

4. A can of juice is taken from the refrigerator on a hot day. The temperature of the juice is originally 40° Fahrenheit and the room temperature is 80° Fahrenheit. The table shows the temperature of the juice as a function of how long (in hours) it has been out of the refrigerator.

Time	0	0.2	0.4	0.6	0.8	1.0
Temp	40	49.8	58.5	65.4	70.5	73.9

Time	1.2	1.4	1.6	1.8	2.0
Temp	76.2	77.7	78.6	79.1	79.5

What is the average rate of change of temperature during the first 24 minutes? What is the average rate of change of temperature in the final 24 minutes? Estimate the instantaneous rate of change at 0.9 hours by computing the average rate of change from $t = 0.8$ to $t = 1.0$. Graph the data.

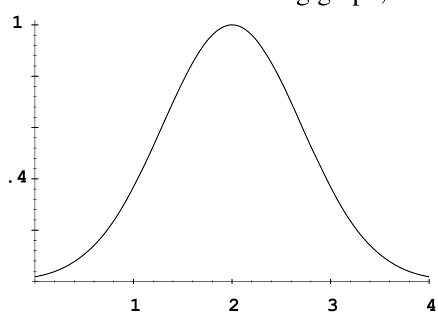
Use 24 minutes = .4 hours.

5. A hemispherical depression of depth 40 cm. is made in a horizontal rock shelf on a ledge. A gentle rain falls at a steady pace. This fills the depression in 10 minutes. Let $f(t)$ = the depth of the water in this depression at time t minutes after it begins raining. Make a possible graph of $y = f(t)$. Put in numerical order: the average rate of change from $t = 1$ to $t = 2$, the average rate of change from $t = 4$ to $t = 5$, the average rate of change from $t = 8$ to $t = 10$.

Draw a diagram. The rate at which the depth is changing depends on the amount of water already in the depression.

SkillMaster 2.15.

6. For the function in the following graph,



put the following numbers in increasing order.

$$0, f'(1), f'(2), f(2) - f(1),$$
$$f'(3), f(4) - f(3), f'(4)$$

Notice that $f(2) - f(1)$ is $\dfrac{f(2) - f(1)}{2 - 1}$ which is the slope of the line joining $(1, f(1))$ and $(2, f(2))$. Sketch tangent lines and secant lines and compare the slopes.

7. Find $f'(2)$ where

$$f(x) = x^3 - 1$$

Use the definition of derivative.

8. Sketch the graph of a function with the following properties:

$f(0) = 1 \qquad f'(0) = 1 \qquad \lim_{x\to\infty} f(x) = 3$

$\lim_{x\to-\infty} f(x) = 0 \qquad \lim_{x\to-1} f(x) = -\infty.$

First notice that there are many correct answers although they will all share properties in common. The point $(0,1)$ is on the graph. The tangent line at this point has slope 1. There is a vertical asymptote at $x = -1$. There is a horizontal asymptote of $y = 3$. The graph will have two pieces, one with derivative 1 at $x = 0$, rising toward the horizontal asymptote of $y = 3$, and falling toward the vertical asymptote at $x = -1$. The other piece can lie below the x-axis and to the left of $y = -1$ with these as asymptotes.

9. Find $f'(a)$ where

$$f(x) = \sqrt{x-4}$$

and $x \geq 4$.

Remember here a is just a constant and behaves in calculations no differently than 2 does in the previous worked out example.

SkillMaster 2.16.

10. Suppose that a company produces butterfly brakes. Suppose that $c(x)$ is equal to the cost of producing x butterfly brakes and that $p(x)$ is price that the company can get for x butterfly brakes. We assume that the quantities are sufficiently large that we may take these functions to be differentiable. Interpret $p'(x)$ and $c'(x)$. Suppose that $p'(x) > c'(x)$ for $0 < x < a$ and that $p'(a) = c'(a)$. What does this mean for the company?

Notice that the difference quotient approximation to the derivative is especially easy when the problem involves whole units. This implies that the difference in x is always 1. Here
$c'(x) \approx \dfrac{c(x+1) - c(x)}{1}$
$= c(x+1) - c(x)$ and
$p'(x) \approx \dfrac{p(x+1) - p(x)}{1}$
$= p(x+1) - p(x)$.

11. The following table gives the population $p(t)$ (in 10,000s) of a small country in recent years.

t	1994	1995	1996	1997
$p(t)$	1776	1829	1884	1942

Estimate $p'(1996)$. Interpret this as a rate of change. What is $p'(1996)/p(1996)$ and what is its interpretation?

Estimate $p'(1996)$ as a difference quotient $(p(1997) - p(1996)) \div (1997 - 1996) = p(1997) - p(1996)$.

Solutions to worked examples

1. $m = \lim_{h\to 0} \dfrac{f(1+h)-f(1)}{h} = \lim_{h\to 0} \dfrac{[3(1+h)^2-(1+h)]-[3(1)^2-1]}{h}$

$= \lim_{h\to 0} \dfrac{[3(1+2h+h^2)-1-h]-2}{h} = \lim_{h\to 0} \dfrac{3+6h+3h^2-1-h-2}{h} = \lim_{h\to 0} \dfrac{5h+3h^2}{h} = \lim_{h\to 0}$
$5+3h = 5+3(0) = 5$

The slope of the tangent line is 5 and the tangent line passes through the point $(1,2)$. Its equation is $y-2=5(x-1)$ so $y=5x-3$.

2. The slope m is found using the definition.

$$m = \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0} \frac{\dfrac{3}{(2+h-1)^2} - \dfrac{3}{(2-1)^2}}{h}$$

$$= \lim_{h\to 0} \frac{\dfrac{3}{(1+h)^2} - \dfrac{3}{(1)^2}}{h} = \lim_{h\to 0} \frac{\dfrac{3}{(1+h)^2} - 3}{h} = \lim_{h\to 0} \frac{3-3(1+h)^2}{h(1+h)^2}$$

$$= \lim_{h\to 0} \frac{3-3(1+2h+h^2)}{h(1+h)^2} = \lim_{h\to 0} \frac{3-3(1+2h+h^2)}{h(1+h)^2} = \lim_{h\to 0} \frac{3-3-6h-3h^2}{h(1+h)^2}$$

$$= \lim_{h\to 0} \frac{-6h-3h^2}{h(1+h)^2} = \lim_{h\to 0} \frac{-6-3h}{(1+h)^2} = \frac{-6-3(0)}{(1+0)^2} = -6$$

The equation of the tangent line through the point $(2,3)$ is $y-3=-6(x-2)$ or $y = -6x+15$.

3. $m = \lim_{h\to 0} \dfrac{f(1+h)-f(1)}{h} = \lim_{h\to 0} \dfrac{\sqrt{(1+h)^2+1}-\sqrt{2}}{h} = \lim_{h\to 0} \dfrac{\sqrt{(1+2h+h^2)+1}-\sqrt{2}}{h}$

$$= \lim_{h\to 0} \frac{\sqrt{2+2h+h^2}-\sqrt{2}}{h} = \lim_{h\to 0} \frac{\sqrt{2+2h+h^2}-\sqrt{2}}{h} \cdot \frac{\sqrt{2+2h+h^2}+\sqrt{2}}{\sqrt{2+2h+h^2}+\sqrt{2}}$$

$$= \lim_{h\to 0} \frac{(\sqrt{2+2h+h^2})^2-(\sqrt{2})^2}{h(\sqrt{2+2h+h^2}+\sqrt{2})} = \lim_{h\to 0} \frac{(2+2h+h^2)-2}{h(\sqrt{2+2h+h^2}+\sqrt{2})} = \lim_{h\to 0} \frac{2h+h^2}{h(\sqrt{2+2h+h^2}+\sqrt{2})}$$

$$= \lim_{h\to 0} \frac{2+h}{(\sqrt{2+2h+h^2}+\sqrt{2})} = \frac{2+0}{(\sqrt{2+2(0)+(0)^2}+\sqrt{2})} = \frac{2}{(\sqrt{2}+\sqrt{2})}$$

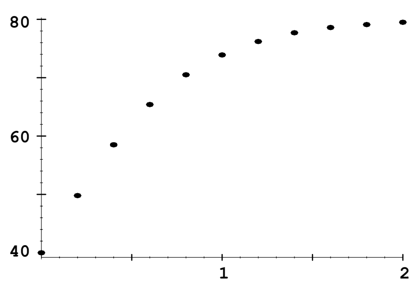
$$= \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

The equation of the tangent line through the point $(1,\sqrt{2})$ is

$y-\sqrt{2} = \dfrac{\sqrt{2}}{2}(x-1)$ or $y = \dfrac{\sqrt{2}}{2}x + \dfrac{\sqrt{2}}{2}$.

4.

Time interval	Average Rate of Change
$[0, 0.4]$	$\dfrac{58.5-40}{0.4-0} = 46.25°$ per hr
[1.6, 2.0]	$\dfrac{79.5-78.6}{2-1.6} = 2.25°$ per hr
[0.8, 1.0]	$\dfrac{73.9-70.5}{1.0-0.8} = 17°$ per hr



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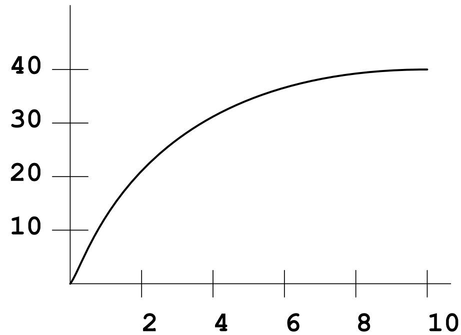


The depression gets wider as the water level gets higher so it fills at a slower rate. The average rate of change from $t=1$ to $t=2$ is greater than the average rate of change from $t=4$ to $t=5$ which is greater than the average rate of change from $t=8$ to $t=10$.

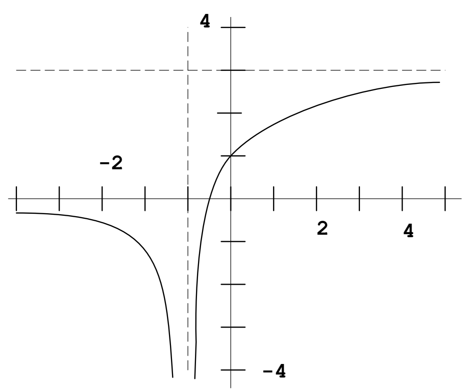
6. $f'(3) < f(4) - f(3) < f'(4) < f'(2) = 0 < f(2) - f(1) < f'(1)$

7. $f'(2) = \lim\limits_{h\to 0} \dfrac{f(2+h)-f(2)}{h} = \lim\limits_{h\to 0} \dfrac{[(2+h)^3-1]-[(2^3-1]}{h}$

$= \lim\limits_{h\to 0} \dfrac{[(2+h)^3-2^3]}{h} = \lim\limits_{h\to 0} \dfrac{[(2^3+3(2^2)h+3(2)h^2+h^3)-2^3]}{h}$

$= \lim\limits_{h\to 0} \dfrac{[12h+6h^2+h^3]}{h} = \lim\limits_{h\to 0} 12+6h+h^2 = 12$

8.



9. $f'(a) = \lim_{h\to 0} \dfrac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \dfrac{\sqrt{a+h-4}-\sqrt{a-4}}{h}$

$$= \lim_{h\to 0} \frac{\sqrt{a+h-4}-\sqrt{a-4}}{h} \cdot \frac{\sqrt{a+h-4}+\sqrt{a-4}}{\sqrt{a+h-4}+\sqrt{a-4}} = \lim_{h\to 0} \frac{(\sqrt{a+h-4})^2-(\sqrt{a-4})^2}{h(\sqrt{a+h-4}+\sqrt{a-4})}$$

$$= \lim_{h\to 0} \frac{(a+h-4)-(a-4)}{h(\sqrt{a+h-4}+\sqrt{a-4})} = \lim_{h\to 0} \frac{h}{h(\sqrt{a+h-4}+\sqrt{a-4})}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{a+h-4}+\sqrt{a-4}} = \frac{1}{\sqrt{a-4}+\sqrt{a-4}} = \frac{1}{2\sqrt{a-4}}$$

10. The cost of producing one more butterfly brake after x have been produced is approximately $c'(x)$. Similarly the price for this additional brake is approximately $p'(x)$. These are called the marginal cost and marginal price. If $c'(a) = p'(a)$ then there is no profit in producing the next brake and the company should devote its resources to something else.

11. $p'(1996) \approx p(1997) - p(1996) = 1,942 - 1,884 = 58$

This is the approximate rate of increase of the population.

$$\frac{p'(1996)}{p(1996)} \approx \frac{58}{1,884} \approx 0.031 = 3.1\%$$

This is the approximate growth rate of the population per individual.

2.7 The Derivative as a Function

Key Concepts:

- The derivative of a function as another function
- Differentiability implies continuity
- Non-differentiable functions
- The second derivative

Skills to Master:

- Sketch the derivative from the graph of a function.
- Find points where a function is differentiable.
- Interpret the second derivative.

Discussion:

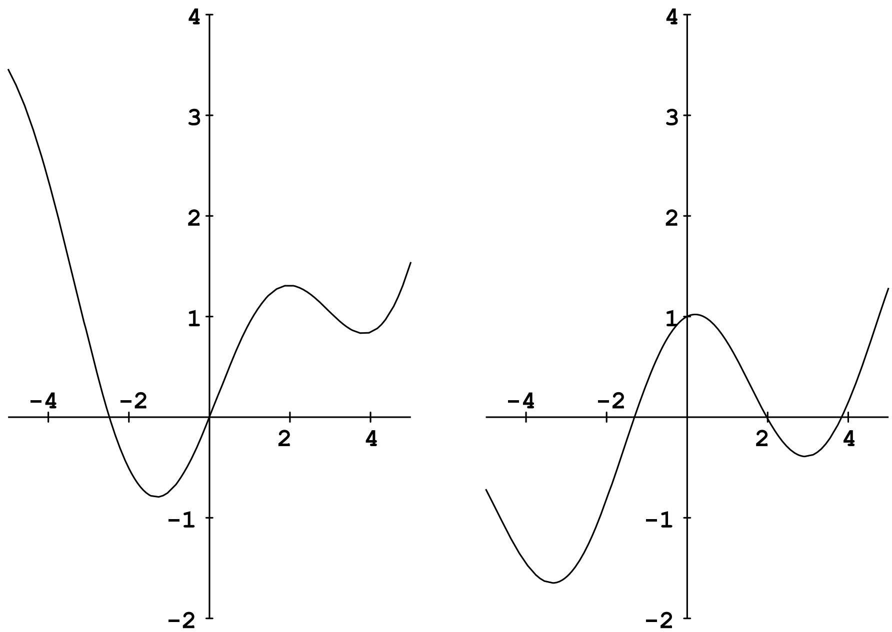
Section 2.7 explains how the derivative of a function can itself be viewed as a function. This viewpoint will be extremely useful and will allow you to use the derivative to gain information about the original function. In addition, alternative notation for derivatives is introduced as is the concept of the second derivative of a function. You will also learn how to determine places where a function fails to have a derivative.

Key Concept: The derivative of a function as another function

If $y = f(x)$ is a function, we can define another function, denoted by $f'(x)$, by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

wherever this limit exists. You should think of x as fixed and h as becoming tinier and tinier in this definition. Note that the value of $f'(x)$ is the slope of the tangent line to the curve $y = f(x)$ at the point $(x, f(x))$. Below the graphs of a function f and the graph of its derivative f' are shown.



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Examine the relation between the two graphs. Pay attention to the *alternative notation* for $f'(x)$ listed in the text.

Alternate notation for $f'(x)$:

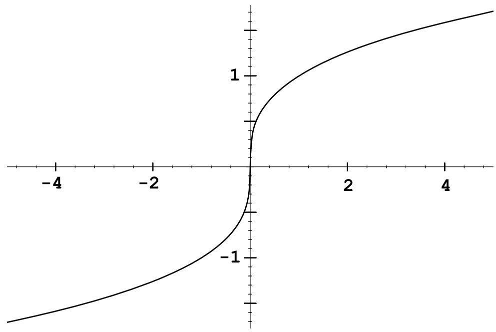
$$y', \quad \frac{dy}{dx}, \quad \frac{df}{dx}, \quad \frac{d}{dx}f(x), \quad Df(x), \quad D_x f(x)$$

Key Concept: Differentiability implies continuity

If a function f is differentiable at $x = a$, then f is continuous at $x = a$. Read through the explanation of this fact in the text. This tells you one way in which a function can fail to have a derivative at a point: if the function is not continuous at $x = a$, the function can not have a derivative at $x = a$. Note that the converse statement is not true. That is, a function can be continuous at a point $x = a$, but can fail to have a derivative at the point $x = a$. The function to keep in mind to remember this is $y = |x|$.

Key Concept: Non-differentiable functions

As just noted, a function can fail to be differentiable at a point, even if it is continuous at the point if the graph has a “corner” at the point. Two other ways for a function to fail to be differentiable at a point are for the function to not be continuous at the point, or for the function to have a vertical tangent at that point. An example to keep in mind is $y = \sqrt[3]{x}$ which has a vertical tangent at $x = 0$.



Key Concept: The second derivative

The second derivative of $y = f(x)$ at $x = a$ is the derivative of the first derivative function, $f'(x)$ at the point $x = a$. Just as the first derivative can be viewed as a function, so can the second derivative. If the original function represents position as a function of time, the first derivative represents velocity as a function of time and the second derivative represents acceleration as a function of time.

SkillMaster 2.17: Sketch the derivative from the graph of a function.

Given the graph of a function, the graph of its derivative can be sketched by noticing the following interrelationships:

f	f'
increasing	positive
decreasing	negative
level	0

SkillMaster 2.18: Find points where a function is not differentiable.

To find points where a function is not differentiable find points where the function fails to be continuous, where the graph has a "corner", or where the tangent appears to be vertical.

SkillMaster 2.19: Interpret the second derivative.

The second derivative can be interpreted as the rate of change of the first derivative. If the first derivative is increasing, then the second derivative is positive. If the first derivative is decreasing, the second derivative is negative. We will see that the second derivative determines the concavity of the graph of the original function. See *Figure 8* in Section 2.7 for examples of graphs illustrating points where a function is not differentiable.

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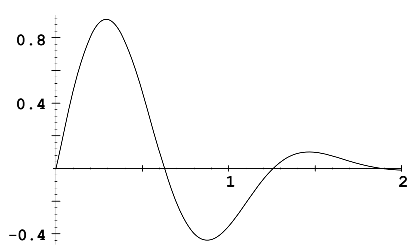
Worked Examples

For each of the following examples, first try to find the solution without looking at the hint. Cover the right column with a piece of paper. If you need a hint, uncover the column. If you need to see the worked solution, look at the solutions section that follows the examples and hints.

Example	Hint

SkillMaster 2.17.

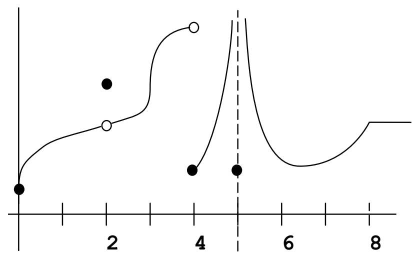
1. Sketch a graph of the derivative of the function $y = f(x)$ shown below.



Hint: First find the x-values where $f'(x) = 0$. Find the intervals where $f'(x)$ is positive and negative. Estimate the derivative by estimating the slope of the tangent line at various places.

SkillMaster 2.18.

2. What is the domain of the function graphed? Where is the function continuous? Where is the function differentiable? Where is the first derivative positive, negative and zero?



Look for anomalies to differentiability such as corners, vertical tangents, isolated points and other points of discontinuity.

3. Consider

$$y = f(x) = \sqrt{|x-2|}.$$

Find the domain of the function and explain why it is continuous. Graph $f(x)$. Where do you think it is not differentiable? Why? Make a sketch of the derivative.

Where is the function defined?

4. Compute the derivative of the function in the previous worked out example. Graph it. Does this graph resemble the sketch?

Express the function in pieces and then use the definition of the derivative function on each piece.

$$\begin{aligned} f(x) = \sqrt{x-2} \text{ if } x &\geq 2 \\ f(x) = \sqrt{2-x} \text{ if } x &< 2 \end{aligned}$$

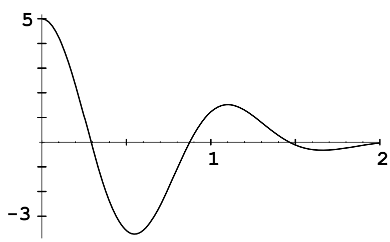
SkillMaster 2.19.

5. A student wishes to visit her friend who lives next door. Her car is parked on the street. He drives it forward, accelerating then decelerating to a stop 100 ft. ahead. Then he rings the door bell and discovers her friend is not at home. He then gets back in his car and drives the car in reverse, back to his usual parking place. Make a possible graph of the position, velocity, and acceleration as a function of time in minutes.

Remember the relationship between the sign of the derivative and whether a function is increasing or decreasing.

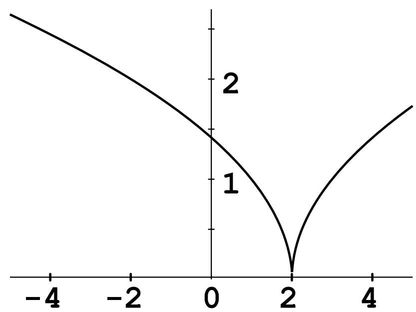
Solutions to worked examples

1. $f'(x) = 0$ at about $x = 0.3$, $x = 0.85$, and $x = 1.45$. These are points where the graph of $y = f(x)$ has horizontal tangents. The x-values where f' is positive are x-values where f is increasing. The x-values where f' is negative are x-values where f is decreasing.



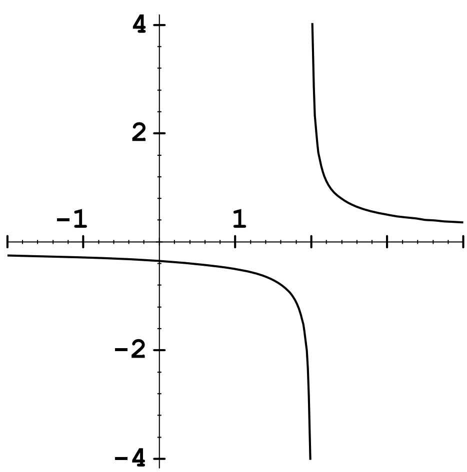
2. The domain of the function is $[0,\infty)$, the set of nonnegative real numbers. The function is continuous at all points in its domain except $x = 2, x = 4$, and $x = 5$. It is differentiable at all points except for where it is discontinuous, has a vertical tangent line, or has a corner. These points are $x = 2$, $x = 4$, $x = 5$ (discontinuities), $x = 0$, $x = 3$ (vertical tangents), $x = 8$ (corner point). The first derivative is positive where the curve is increasing, $[0,2)$, $(2,4)$, $[4,5)$ and $[6.5,8]$. The first derivative is negative on $(5,6.5)$. It is zero at points of horizontal slope, i.e. at 6.5 and on (8,9).

3. The function $y = f(x)$ has all real numbers as its domain and is continuous because it is a composition of continuous functions and for each x it is true that $|x-2| \geq 0$ so $f(x) = \sqrt{|x-2|}$ is well-defined.



The function appears to be differentiable at all points in its domain except $x = 2$ where there appears to both a corner and a vertical tangent line.

The derivative is sketched below.



4. For $x > 2$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sqrt{x+h-2} - \sqrt{x-2}}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x+h-2} - \sqrt{x-2}}{h} \cdot \frac{\sqrt{x+h-2} + \sqrt{x-2}}{\sqrt{x+h-2} + \sqrt{x-2}} = \lim_{h \to 0} \frac{(\sqrt{x+h-2})^2 - (\sqrt{x-2})^2}{h(\sqrt{x+h-2} + \sqrt{x-2})}$$

$$=\lim_{h\to 0}\frac{(x+h-2)-(x-2)}{h(\sqrt{x+h-2}+\sqrt{x-2})}=\lim_{h\to 0}\frac{h}{h(\sqrt{x+h-2}+\sqrt{x-2})}$$

$$=\lim_{h\to 0}\frac{1}{\sqrt{x+h-2}+\sqrt{x-2}}=\frac{1}{\sqrt{x+0-2}+\sqrt{x-2}}=\frac{1}{2\sqrt{x-2}}$$

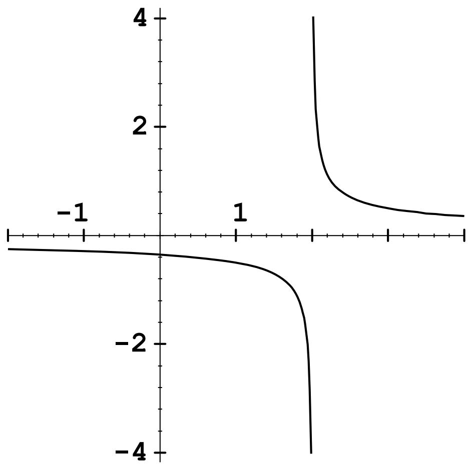
Similarly for $x<2$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\sqrt{2-(x+h)}-\sqrt{2-x}}{h}$$

$$=\lim_{h\to 0}\frac{\sqrt{2-(x+h)}-\sqrt{2-x}}{h}\times\frac{\sqrt{2-(x+h)}+\sqrt{2-x}}{\sqrt{2-(x+h)}+\sqrt{2-x}}=\lim_{h\to 0}\frac{(\sqrt{2-x-h})^2-(\sqrt{2-x})^2}{h(\sqrt{2-x-h}+\sqrt{2-x})}$$

$$=\lim_{h\to 0}\frac{(2-x-h)-(2-x)}{h(\sqrt{2-x-h}+\sqrt{2-x})}=\lim_{h\to 0}\frac{-h}{h(\sqrt{2-x-h}+\sqrt{2-x})}$$

$$=\lim_{h\to 0}\frac{-1}{\sqrt{2-x-h}+\sqrt{2-x}}=\frac{-1}{\sqrt{2-x-0}+\sqrt{2-x}}=\frac{-1}{2\sqrt{2-x}}$$



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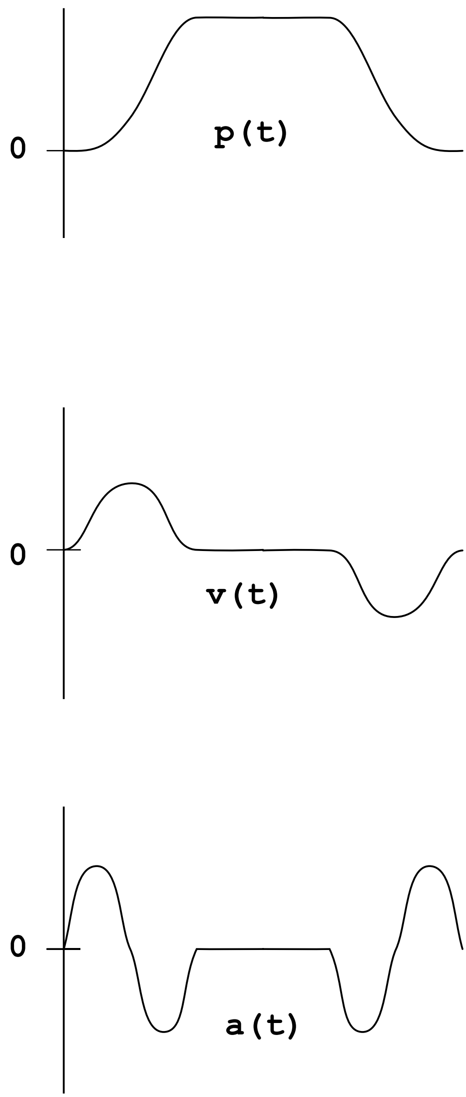


2.8 What does f' say about f ?

Key Concepts:

- The relation of the sign of f' to properties of f
- The relation of the sign of f'' to concavity
- Antiderivatives

Skills to Master:

- Use f' to determine where f is increasing, decreasing, and where f has a local maximum or minimum.
- Use f'' to determine where the graph of f is concave up or concave down.
- Sketch a graph of f from information about f' and f''.

Discussion:

Section 2.8 goes into more detail about the relation between f and f'. In Chapter 4, you will explore these ideas in even more detail. The main thing to learn is that f' tells you whether the function f is increasing or decreasing and f'' tells you whether f is concave up or concave down. The concept of antiderivatives is also introduced. Antiderivatives are a central concept in Integral Calculus.

Key Concept: The relation of the sign of f$'$ to properties of f

In the next chapter, you will see in more detail why the following facts are true. For now, learn these two facts and use them to get information about f.

$$\begin{aligned}
\text{If } f'(x) &> 0 \text{ on an interval, then } f \text{ is increasing on that interval.}\\
\text{If } f'(x) &< 0 \text{ on an interval, then } f \text{ is decreasing on that interval.}
\end{aligned}$$

As an example, consider $f(x) = 4x^2 + 3x - 5$. Then $f'(x) = 8x + 3$. This first derivative is positive when x is bigger than $3/8$ and is negative when x is less than $3/8$. Thus f is increasing on the interval $(3/8,\infty)$ and is decreasing on the interval $(-\infty,3/8)$. Check this by graphing the function.

Key Concept: The relation of the sign of f$''$ to concavity

A curve is concave upward at a point if it lies above the tangent lines to the curve near that point and is concave downward at the point if it lies below the tangent lines near that point. Keep in mind the graphs of x^2 and $-x^2$ to see the difference between these concepts. The main result to learn about second derivatives is:

$$\begin{aligned}
\text{If } f''(x) &> 0 \text{ on an interval, then } f \text{ is concave up on that interval (like a bowl).}\\
\text{If } f''(x) &< 0 \text{ on an interval, then } f \text{ is concave down on that interval (like an upside down bowl).}
\end{aligned}$$

Key Concept: Antiderivatives

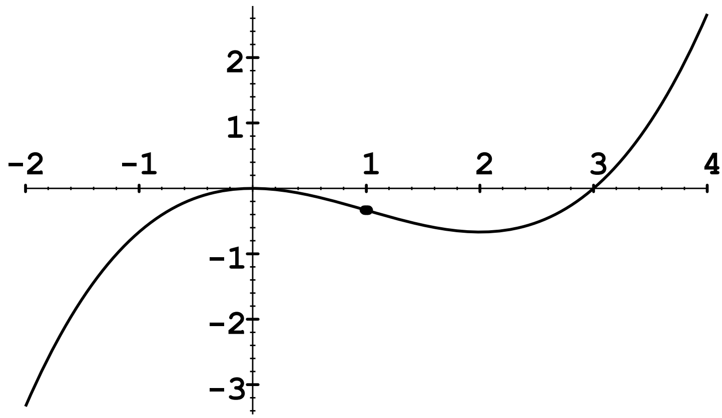
Given a function $y = f(x)$, an antiderivative for f is a function $F(x)$ such that $F' = f$. You will see much more about this concept in integral calculus. For now, make sure that you understand what an antiderivative is.

SkillMaster 2.22: Use f$'$ to determine where f is increasing, decreasing, and where f$'$ has a local maximum or minimum.

If you can determine where f' is positive and where f' is negative, that will tell you where f is increasing and decreasing. To determine where f has a local maximum, find points where f' changes from positive to negative. To determine where f has a local minimum, find points where f' changes from negative to positive.

SkillMaster 2.23: Use f$''$ to determine where the graph of f$'$ is concave up or concave down.

If you can determine where f'' is positive and where f'' is negative, that will tell you where f is concave up and concave down. For example, if $f''(x) = x - 1$, then f is concave up when x is greater than 1 and f is concave down when x is less than 1. One such function is $f(x) = \dfrac{x^3}{6} - \dfrac{x^2}{2}$. Look at the graph below and examine the concavity to see that it changes at $x = 1$.



SkillMaster 2.24: Sketch a graph of f from information about f′ and f″.

Combining the above two skills, knowledge about f' and f'' tell you where f is increasing, where f is decreasing, where f is concave up, and where f is concave down. This knowledge, together with the value of f at a few points should enable you to a reasonable graph of f.

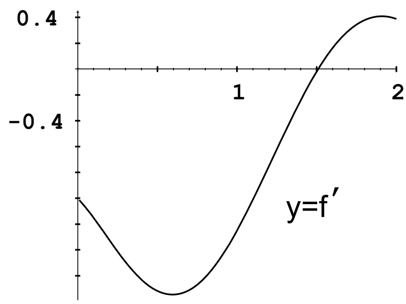
Worked Examples

For each of the following examples, first try to find the solution without looking at the hint. Cover the right column with a piece of paper. If you need a hint, uncover the column. If you need to see the worked solution, look at the solutions section that follows the examples and hints.

Example	Hint

SkillMaster 2.22.

1. The graph of a function $y = f'(x)$ is shown. For what intervals is the function $f(x)$ increasing or decreasing? For which x does f have a local maximum or local minimum?



Hint: The function f is decreasing whenever is f' negative and is increasing whenever f' is positive.

SkillMaster 2.23.

2. Where is f (from the previous problem) concave upward and downward. Where are there any inflection points?

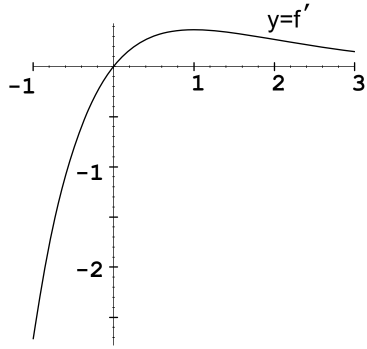
Hint: The graph of f is concave upward if f' is increasing (and if f'' is positive) and is concave downward if f' is decreasing (and if f'' is negative), so look at the graph of f' and observe where it is increasing and where it is decreasing.

SkillMaster 2.24.

3. Suppose we are given

$$f'(x) = xe^{-x} \qquad f''(x) = e^{-x}(1-x)$$

The graph of f' is shown below. Decide where $f(x)$ is increasing and where it is decreasing? Find any local maxima or minima. Where is it concave upward or downward? Find any inflection points. If $f(0) = -1$ make a rough sketch of the graph of f.



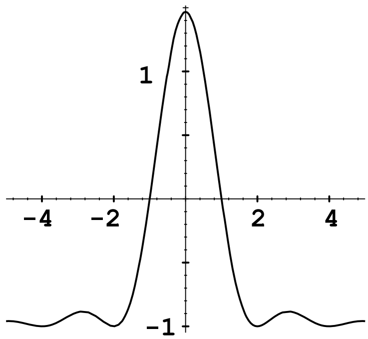
Remember what you know about the relation between f and f'.

4. Sketch a graph with the following properties:

$f(-1) = 1$, $\qquad f(0) = 2$,
$f(1) = 0$, $\qquad f'(x) > 0$ for $x < 0$,
$f'(x) < 0$ for $x > 0$, $\qquad f''(x) > 0$ for $x < -1$,
$f''(x) < 0$ for $x > -1$, $\qquad \lim_{x\to-\infty} f(x) = 0$,
$\lim_{x\to\infty} f(x) = -\infty$.

There is a local maximum at $x = 0$ and $f(0) = 2$. There is an inflection point at $x = -1$. To the left of $x = -1$ the graph is concave upward; to the right of $x = -1$ it is concave downward.

5. The graph shows an antiderivative of a function $y = f(x)$. Sketch another antiderivative $F(x)$ with the property that $F(0) = 1$.



All antiderivatives of a function are upward or downward shifts of each other.

Solutions to worked examples

1. The function f is decreasing on $(0,1.5)$ and increasing on $(1.5,2)$. There is a local minimum at $x = 1.5$.

2. The graph of f is concave upward on $(0.5,2)$ and is concave downward on $(0,.5)$. There is an inflection point at $x = 0.5$.

3. First $e^{-x} > 0$ so

$f'(x) = xe^{-x} > 0$ if $x > 0$ $\qquad$ $f'(x) = xe^{-x} < 0$ if $x < 0$

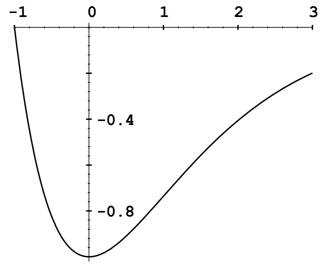
$f'(x) = 0$ if $x = 0$.

Thus f is decreasing for $x < 0$, increasing for $x > 0$, and has a local minimum at $x = 0$.

To test concavity check the second derivative.

$f''(x) = e^{-x}(1-x) > 0$ if and only if $(1-x) > 0$
if and only if $1 > x$

The graph of the function is concave upward for $x < 1$, concave downward for $x > 1$, and has an inflection point for $x = 1$. Here is a sketch.



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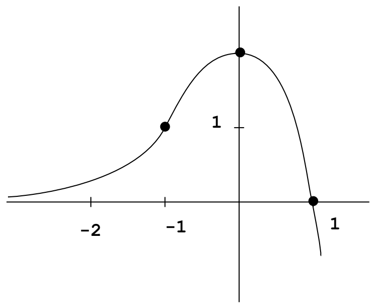

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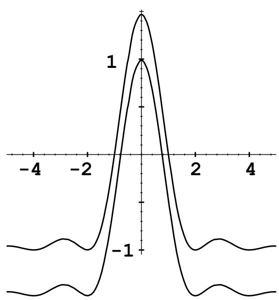

SkillMasters for Chapter 2

SkillMaster 2.1:	Find the equation of a tangent line.
SkillMaster 2.2:	Estimate the equation of a tangent line when the function is given by an algebraic formula, by a data set, or by a graph.
SkillMaster 2.3:	Estimate instantaneous velocity.
SkillMaster 2.4:	Guess limits from the graph of a function.
SkillMaster 2.5:	Guess limits by using a calculator.
SkillMaster 2.6:	Find limits from graphs or partial information using the limit laws.
SkillMaster 2.7:	Find limits using the limit laws and algebraic simplification.
SkillMaster 2.8:	Use the Squeeze Law to find limits.
SkillMaster 2.9:	Find points of continuity and discontinuity from a graph.
SkillMaster 2.10:	Estimate roots by using the Intermediate Value Theorem.
SkillMaster 2.11:	Find limits involving infinity.
SkillMaster 2.12:	Find vertical and horizontal asymptotes.
SkillMaster 2.13:	Find the slope of the tangent line using the definition.
SkillMaster 2.14:	Find velocity and other rates of change by using the definition, by using data from a table, and by using a graph.
SkillMaster 2.15:	Compute and estimate derivatives.
SkillMaster 2.16:	Interpret the derivative as a rate of change.
SkillMaster 2.17:	Sketch the derivative from the graph of a function.
SkillMaster 2.18:	Find points where a function is differentiable.
SkillMaster 2.19:	Interpret the second derivative.
SkillMaster 2.20:	Compute linear approximations.
SkillMaster 2.21:	Use linear approximations to approximate functions.
SkillMaster 2.22:	Use f' to determine where f is increasing, decreasing, and where f has a local maximum or minimum.
SkillMaster 2.23:	Use f'' to determine where the graph of f is concave up or concave down.
SkillMaster 2.24:	Sketch a graph of f from information about f' and f''.

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Chapter 3

Differentiation Rules

$$\frac{d}{dx}\left[f(x)g(x)\right] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]$$

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{\left[g(x)\right]^2}$$

3.1 Derivatives of Polynomials and Exponential Functions

Key Concepts:

- Power Rule - The derivative of x^n is nx^{n-1}
- Rules for derivatives of sums, differences and constant multiples
- The definition of the number e and the derivative of e^x

Skills to Master:

- Differentiate polynomials.
- Differentiate sums of polynomials and multiples of e^x.
- Use the differentiation rules to solve problems.

Discussion:

This section explains the basic rules for differentiating sums, differences and constant multiples of functions that you already know how to differentiate. To give you some functions to start with, the derivatives of x^n and e^x are given. You need to completely master the rules and formulas in this section. Almost everything else that you encounter about differentiation involves these basic facts.

Key Concept: Power Rule - The derivative of $\mathbf{x^n}$ is $\mathbf{nx^{n-1}}$

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The facts that $\dfrac{d}{dx}c = \dfrac{d}{dx}x^0 = 0$ and that $\dfrac{d}{dx}x = \dfrac{d}{dx}x^1 = 1$ are special cases of the Power Rule. Note how the special cases are explained first, and then the case where n is a positive integer is derived. Review the *Binomial Theorem* in Stewart in order to make sure that you understand how the more general case is derived. As you work through the examples, understand what this general rule says about the rate at which x^n is changing. For example, the rate of change of x^{100} at $x = 10$ is $100 \cdot 10^{99} = 10^{(101)}$, a number that makes the number of particles in known universe seem small. How does this compare with the rate of change of x^{10} at $x = 10$ which is merely ten billion? Clearly, for large n and x larger than 1 the function x^n has a large value and a large rate of change that gets astronomically larger as n and x get even moderately large.

The explanation of the *Power Rule* for arbitrary real numbers n is deferred until Section 3.7.

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Key Concept: Rules for derivatives of sums, differences and constant multiples

The Constant Multiple Rule, the Sum Rule, and the Difference Rule for differentiation are listed below.

$$\begin{aligned}
\frac{d}{dx}[c \cdot f(x)] &= c \cdot \frac{d}{dx} f(x) \\
\frac{d}{dx}[f(x) + g(x)] &= \frac{d}{dx} f(x) + \frac{d}{dx} g(x) \\
\frac{d}{dx}[f(x) - g(x)] &= \frac{d}{dx} f(x) - \frac{d}{dx} g(x)
\end{aligned}$$

In these rules, it is assumed that f and g are differentiable and that c is a constant. Note that in each case, the derivative of a new, more complicated, function is computed by using the derivatives of simpler "'building block"' functions. Practice computing with these rules until they become second nature to you.

Key Concept: The definition of the number e and the derivative of e^x

After a brief discussion of *exponential functions* , the number e is defined as the number such that

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$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1$$

This means that the tangent line to the curve $y = f(x) = e^x$ has slope 1 at the point $(0,1)$ or equivalently that $f'(0) = 1$. You will see an alternate definition of e in section 3.7. Since exponential functions of the form $f(x) = a^x$ have the property that $f'(x) = f'(0) \cdot a^x$, it follows that

$$\frac{d}{dx} e^x = \left[(e^x)' |_{x=0} \right] \cdot e^x = 1 \cdot e^x$$

Thus e^x is a function whose derivative is itself. In fact, constant multiples of exponential functions are the only functions that are equal to their derivative. This fact ensures that exponential functions arise in many applications in other disciplines. The properties of exponential functions will be frequently used from now on. For a review of properties of *exponential functions* see *Section 1.5*.

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SkillMaster 3.1: Differentiate polynomials.

Polynomials are sums of constant multiples of powers of x. For example,

$$5x^3 - 3x^2 + 14$$
$$5x^3 + (-3)x^2 + 14$$

is the sum of the three functions:

- 5 times the function x^3,
- -3 times the function x^2, and
- the constant function 14.

To differentiate such polynomials, apply the Sum Rule, the Constant Multiple Rule and the Power Rule.

SkillMaster 3.2: Differentiate sums of polynomials and multiples of e^x.

Functions that are sums of polynomials and multiples of e^x can also be differentiated by applying the Sum and Constant Multiple Rule. For example, the derivative of $3x^3 - 4x^2 + 12e^x$ is the sum of the derivative of $3x^3 - 4x^2$ and 12 times the derivative of e^x.

SkillMaster 3.3: Use the differentiation rules to solve problems.

Now that you know some of the basic rules of differentiation, you need to be able to use these rules to solve problems that involve computing derivatives. For example, you should be able to find *tangent* and *normal* lines to curves if the curves are given in terms of functions that you can differentiate. You should also be able to use tangent lines to determine where functions are increasing, decreasing, or have tangents of a particular slope. for such functions.

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Worked Examples

For each of the following examples, first try to find the solution without looking at the hint. Cover the right column with a piece of paper. If you need a hint, uncover the column. If you need to see the worked solution, look at the solutions section that follows the examples and hints.

Example	Hint
SkillMaster 3.1.	
Differentiate the following functions:	
1. x^{31}	Power Rule
2. $6x^{1000}$	Constant Multiple Rule and the Power Rule
3. $4x^5+10x^3-21x^2+x-2$	This is a polynomial.
SkillMaster 3.2.	
Differentiate the following functions:	
4. u^{π}	Remember, π is a constant.
5. $6x^{3.5}+4x^{2.5}$	Apply the Power Rule to each term.
6. $\sqrt{t}$	Convert to power notation: $\sqrt{t}=t^{0.5}$.
7. $t(1-\sqrt{t})$	Multiply and convert to power notation: $t(1-\sqrt{t})=t-t^{1.5}$.

8. $\dfrac{u+1}{\sqrt{u}}$

Divide, convert to power notation.

9. $5e^x$

Constant multiple of e^x

10. x^2+2e^x

Differentiate each expression.

11. ae^x+bx^6

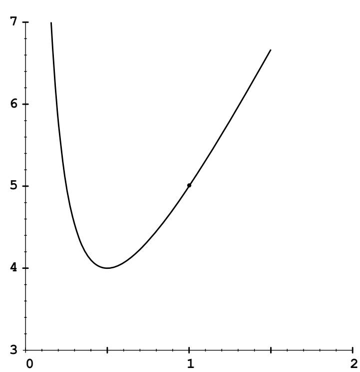
a and b are constants.

SkillMaster 3.3.

12. If $f(6)=2$, $f'(6)=1$ and $g(x)=2x^2+e^x+f(x)$, find $g(6)$ and find $g'(6)$.

Don't panic! Take a moment to think. First, find $g'(x)$ in terms of expressions involving x, $f(x)$ and $f'(x)$.
Substitute 6 for x in the expressions for g and g'.

13. Find the equations of all tangent lines with slope 3 to the graph of $y=f(x)=4x+\dfrac{1}{x}$, for $x>0$. This function is pictured below.



Rewrite $1/x$ using exponential notation.
Compute $f'(x)$. Set $f'(x)=3$ and solve for x to find the x value that on both the graph and the tangent line. Use only the positive values of x.
Express the tangent line in point-slope form.

14. Find the x-values where there are horizontal tangents to the graph of $y = f(x) = 4x - e^x$. Check that the x-values you found agree with what you observe from the graph of $y = f(x)$

Compute $f'(x)$ and set $f'(x) = 0$. Then solve the equation you get.

Solutions to worked examples

1. $31x^{30}$

2. $6000x^{999}$

3. $20x^4 + 30x^2 - 42x + 1$

4. $\pi u^{\pi-1}$

5. $21x^{2.5} + 10x^{1.5}$

6. $0.5t^{-0.5} = \dfrac{1}{2\sqrt{t}}$

7. $1 - 1.5t^{0.5} = 1 - \dfrac{3}{2}\sqrt{t}$

8. $\dfrac{d}{du}(u^{0.5} + u^{-0.5}) = 0.5u^{-0.5} - 0.5u^{-1.5}$

9. $5e^x$

10. $2x + 2e^x$

11. $ae^x + 6bx^5$

12. $g(6) = 2(6^2) + e^6 + f(6) = 72 + e^6 + 2 = 74 + e^6 \approx 477.43$

$g'(x) = 4x + e^x + f'(x)$

So $g'(6) = 4(6) + e^6 + f'(6) = 24 + e^6 + 1 = 25 + e^6 \approx 428.43$

13. $f'(x) = 4 - \dfrac{1}{x^2}$ $\quad 3 = 4 - \dfrac{1}{x^2}$ $\quad -1 = -\dfrac{1}{x^2}$

$x^2 = 1$ $\quad x = \pm 1$ $\quad x = 1$ (because $x > 0$).

When $x = 1$, $f(1) = 5$. $\quad$ So $y - 5 = 3(x-1)$, $\quad y = 3x + 2$.

14. $f'(x)$=$4 - e^x$ $\quad$ Set $f'(x) = 0$

$0 = 4 - e^x$ $\quad$ or $\quad e^x = 4$ $\quad$ So $x = \ln(4) \approx 1.386$.

3.2 The Product and Quotient Rules

Key Concepts:

- The Product Rule for derivatives
- The Quotient Rule for derivatives

Skills to Master:

- Differentiate complicated expressions involving products, sums and quotients.
- Use the Product and Quotient Rules to solve problems.

Discussion:

This section adds to the rules for differentiating functions that are combinations of functions that you already know how to differentiate. In particular, rules for differentiating products and quotients are derived. Make sure that you understand the geometric argument given to explain the Product Rule and that you understand how the Quotient Rule can be derived from the Product Rule.

Key Concept: The Product Rule for derivatives

The Product Rule for derivatives states that

$$\frac{d}{dx}[f(x)\cdot g(x)] = f(x)\cdot\left[\frac{d}{dx}g(x)\right] + g(x)\left[\frac{d}{dx}f(x)\right]$$

provided that the functions f and g are differentiable. The diagram in *Section 3.2* clarifies the geometry behind the Product Rule . Note that this formula is more complicated than the Sum and Difference Rules from the previous section. page 184

Key Concept: The Quotient Rule for derivatives

The Quotient Rule for derivatives states that

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\cdot\dfrac{d}{dx}[f(x)] - f(x)\cdot\dfrac{d}{dx}[g(x)]}{[g(x)]^{\,2}}$$

provided that both f and g are differentiable and that $g(x) \neq 0$. The Quotient Rule looks even more complicated than the Product Rule, but is easily derived from the Product Rule as shown in *Section 3.2* . This is another of the rules that you need to know perfectly and be able to apply with ease and dexterity.

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SkillMaster 3.4: Differentiate complicated expressions involving products, sums and quotients.

You should now be able to differentiate more complicated expressions involving products, sums and quotients of polynomials and multiples of e^x. Practice and work as many of the problems as possible until you are proficient at techniques of differentiation. You need to be completely comfortable using all of the differentiation rules.

SkillMaster 3.5: Use the Product and Quotient Rules to solve problems.

The Product and Quotient Rules allow you to solve more sophisticated real world problems that involve products and quotients. Such problems arise in many different ways. In the physical and biological sciences, in economics and other fields many quantities are most naturally expressed as products or quotients of simpler ones. You now have the ability to compute *rates of change* for such quantities.

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Worked Examples

For each of the following examples, first try to find the solution without looking at the hint. Cover the right column with a piece of paper. If you need a hint, uncover the column. If you need to see the worked solution, look at the solutions section that follows the examples and hints.

Example	Hint
SkillMaster 3.4.	
Differentiate the following functions:	
1. x^3e^x	Use the Product Rule with the factors $f(x) = x^3$ and $g(x) = e^x$.
2. $(x^3+4x^2+x+2)e^x$	This is a product of two functions.
3. $(x^2+x+\sqrt{x})\cdot(x^6-x^4-\sqrt{x})$	Convert square roots to power notation, then use the Product Rule.
4. $\dfrac{x^2+1}{x-1}$	Use the Quotient Rule.
5. $\dfrac{e^x}{x}$	Use the Quotient Rule.

6. $\dfrac{\sqrt{x}+x}{\sqrt{x}+x^2}$

Convert to power notation, then simplify by dividing the numerator and denominator by $x^{1/2}$. Then, begin to differentiate the expression.

7. $ae^x + be^{-x}$

Convert the e^{-x} to $\dfrac{1}{e^x}$ and use the Quotient Rule. Remember, a and b are constants.

SkillMaster 3.5.

8. Find the intervals of *concavity* and the inflection points of
$f(x) = \dfrac{1}{1+x^2}$

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Find the second derivative $f''(x)$.
Then find where $f''(x)$ is positive (concave up),
where it is negative (concave down),
and where it is zero and changing sign (inflection point).

9. Suppose that $f(3) = 1$ and $f'(3) = 2$. Find the derivative of $g(x) = f(x)e^x$ at $x = 3$.

Compute the derivative of $g(x)$ in terms of e^x, $f(x)$, and $f'(x)$ using the Product Rule. Then substitute 3 for x.

10. Use the Product Rule to find the derivative of e^{2x}. Now use the Product Rule to find the derivative of e^{3x}. Guess a formula for the derivative of e^{nx}. Prove this by induction.

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Write $e^{2x} = e^x e^x$ and use the Product Rule.
Write $e^{3x} = e^x e^{2x}$ and again use the Product Rule.
Write $e^{(n+1)x} = e^x e^{nx}$.
Again, use the Product Rule.

Solutions to worked examples

1. $x^3\dfrac{d}{dx}(e^x)+e^x\dfrac{d}{dx}(x^3)=x^3e^x+e^x 3x^2=(x^3+3x^2)e^x$

2. $(x^3+4x^2+x+2)\dfrac{d}{dx}(e^x)+e^x\dfrac{d}{dx}(x^3+4x^2+x+2)$
$=(x^3+4x^2+x+2)e^x+e^x(3x^2+8x+1)=(x^3+7x^2+9x+3)e^x$

3. $(x^2+x+x^{1/2})\dfrac{d}{dx}(x^6-x^4-x^{1/2})+(x^6-x^4-x^{1/2})\dfrac{d}{dx}(x^2+x+x^{1/2})$
$=(x^2+x+x^{1/2})(6x^5-4x^3-\dfrac{1}{2}x^{-1/2})+(x^6-x^4-x^{1/2})(2x+1+\dfrac{1}{2}x^{-1/2})$

4. $\left[(x-1)\dfrac{d}{dx}(x^2+1)-(x^2+1)\dfrac{d}{dx}(x-1)\right]\cdot\dfrac{1}{(x-1)^2}=\dfrac{(x-1)(2x)-(x^2+1)(1)}{(x-1)^2}$
$=\dfrac{2x^2-2x-x^2-1}{(x-1)^2}=\dfrac{x^2-2x-1}{(x-1)^2}$

5. $\dfrac{x\dfrac{d}{dx}(e^x)-e^x\dfrac{d}{dx}(x)}{x^2}=\dfrac{xe^x-e^x(1)}{x^2}=\dfrac{(x-1)e^x}{x^2}$

6. This simplifies to $=\dfrac{1+x^{1/2}}{1+x^{3/2}}$.

Note that this problem could have been done without the hint, but it would have been more complicated and more difficult to simplify. Always pause and think before beginning to compute so that your approach is the easiest and most straightforward possible. The derivative is

$$\left[(1+x^{3/2})\frac{d}{dx}(1+x^{1/2})-(1+x^{1/2})\frac{d}{dx}(1+x^{3/2})\right]\cdot\frac{1}{(1+x^{3/2})^2}$$
$$=\left[(1+x^{3/2})\cdot\frac{1}{2}x^{-1/2}-(1+x^{1/2})(\frac{3}{2})(x^{1/2})\right]\cdot\frac{1}{(1+x^{3/2})^2}$$
$$=\frac{x^{-1/2}+x-3x^{1/2}-3x}{2(1+x^{3/2})^2}=\frac{x^{-1/2}-2x-3x^{1/2}}{2(1+x^{3/2})^2}$$

7. $\dfrac{d}{dx}(ae^x+b\dfrac{1}{e^x})=a\dfrac{d}{dx}(e^x)+b\dfrac{d}{dx}(\dfrac{1}{e^x})=ae^x+b(\dfrac{e^x\dfrac{d}{dx}(1)-(1)\dfrac{d}{dx}(e^x)}{(e^x)^2}$
$=ae^x+b\dfrac{e^x\cdot 0-e^x}{e^{2x}}=ae^x+b(-e^{x-2x})=ae^x-be^{-x}$

8. $f'(x) = \dfrac{(1+x^2)\frac{d}{dx}(1) - (1)\frac{d}{dx}(1+x^2)}{(1+x^2)^2} = \dfrac{-2x}{(1+x^2)^2}$

$$f''(x) = \frac{(1+x^2)^2\frac{d}{dx}(-2x) - (-2x)\frac{d}{dx}(1+x^2)^2}{(1+x^2)^4}$$

$$= \frac{(1+x^2)^2\cdot(-2) + 2x\frac{d}{dx}(1+2x^2+x^4)}{(1+x^2)^4}$$

$$= \frac{(1+x^2)^2\cdot(-2) + 2x(4x+4x^3)}{(1+x^2)^4} = \frac{(1+x^2)^2\cdot(-2) + 8x^2(1+x^2)}{(1+x^2)^4}$$

$$= \frac{\left[(1+x^2)\cdot(-2)+8x^2\right]\cdot(1+x^2)}{(1+x^2)^4} = \frac{-2+6x^2}{(1+x^2)^3} = 2\cdot\frac{-1+3x^2}{(1+x^2)^3}$$

The denominator is always positive, so $f''(x) > 0 \Longleftrightarrow$ the numerator is positive.

$-1+3x^2 > 0 \Longleftrightarrow 3x^2 > 1 \Longleftrightarrow x^2 > \dfrac{1}{3} \Longleftrightarrow x > \dfrac{1}{\sqrt{3}}$ or $x < -\dfrac{1}{\sqrt{3}}$

So f is concave up on $(-\infty, -1/\sqrt{3})$ and $(1/\sqrt{3}, \infty)$, f is concave down on $(-1/\sqrt{3}, 1/\sqrt{3})$, and has inflection points at $x = \pm 1/\sqrt{3}$.

9. $g'(x) = e^x\dfrac{d}{dx}(f(x)) + f(x)\dfrac{d}{dx}(e^x) = e^x f'(x) + f(x)e^x$

$g'(3) = e^3 f'(3) + f(3)e^3 = 2e^3 + 1e^3 = 3e^3 \approx 60.26$

10. $\dfrac{d}{dx}(e^{2x}) = \dfrac{d}{dx}(e^x e^x) = e^x\dfrac{d}{dx}(e^x) + e^x\dfrac{d}{dx}(e^x) = e^x e^x + e^x e^x = 2e^{2x}$

$\dfrac{d}{dx}(e^{3x}) = \dfrac{d}{dx}(e^x e^{2x}) = e^{2x}\dfrac{d}{dx}(e^x) + e^x\dfrac{d}{dx}(e^{2x}) = e^{2x}e^x + e^x(2e^{2x})$

$= 3e^x e^{2x} = 3e^{3x}$

Recall, that to prove a formula by induction, we have to do two things. First, show that the formula is correct for $n = 1$. Then prove that the formula is correct for $n+1$ using the information that the formula is true for n.

For $n = 1$, $\dfrac{d}{dx}e^{1\cdot x} = \dfrac{d}{dx}e^x = e^x = 1\cdot e^{1\cdot x}$.

For the general step, $\dfrac{d}{dx}(e^{(n+1)x}) = \dfrac{d}{dx}(e^x e^{nx}) = e^{nx}\dfrac{d}{dx}(e^x) + e^x\dfrac{d}{dx}(e^{nx})$

$= e^{nx}e^x + e^x n e^{nx} = (n+1)e^x e^{nx} = (n+1)e^{(n+1)x}$.

3.3 Derivatives of Trigonometric Functions

Key Concepts:

- $\lim_{x \to 0} \frac{\sin x}{x} = 1$ and $\lim_{x \to 0} \frac{\cos x - 1}{x} = 0$
- $\frac{d}{dx} \sin(x) = \cos(x)$ and $\frac{d}{dx} \cos(x) = -\sin(x)$
- Derivatives of other trigonometric functions

Skills to Master:

- Differentiate expressions involving trigonometric functions and use these results to solve problems.

Discussion:

Prior to this section, you only knew how to differentiate polynomials, the function e^x, and various combinations of these of functions. This section introduces the derivatives of another important class of functions, the trigonometric functions. Review the *trigonometric functions* if you need to. Pay careful attention to the geometric derivation of the derivatives of the sine and cosine functions.

App. C

Key Concept: $\lim_{\mathbf{x} \to \mathbf{0}} \frac{\sin(\mathbf{x})}{\mathbf{x}} = \mathbf{1}$ and $\lim_{\mathbf{x} \to \mathbf{0}} \frac{\cos(\mathbf{x}) - \mathbf{1}}{\mathbf{x}} = \mathbf{0}$

Make sure that you understand the geometric reasoning that leads to these two limits. These limits are precisely the limits we need to evaluate in computing the derivative of the function $y = \sin x$ and related trigonometric functions. This is a good place to review the concept of *limit* and the definition of the derivative as a limit of *slopes of secant lines*.

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Key Concept: $\frac{\mathbf{d}}{\mathbf{dx}}\mathbf{sin(x) = cos(x)}$ and $\frac{\mathbf{d}}{\mathbf{dx}}\mathbf{cos(x) = -sin(x)}$

Once we know the two limits mentioned in the previous key concept, it is easy to compute the derivative of the function $y = \sin x$. Work through the details of *Exercise 18, page 196* to convince yourself of the formula for the derivative of the function $y = \cos x$. Section 3.3 illustrates the *geometric interpretation* of these formulas, namely that the rate of change of $\sin x$ is equal to $\cos x$ and that the rate of change of $\cos x$ is $-\sin x$.

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Key Concept: Derivatives of other trigonometric functions

Since the other trigonometric functions, $\tan x$, $\cot x$, $\sec x$, and $\csc x$ are defined as quotients of the functions $\sin x$ and $\cos x$, an application of the Quotient Rule leads to formulas for the derivatives of these other functions. We summarize the results in the following table:

$$
\begin{array}{ll}
\frac{d}{dx}[\sin(x)] = \cos(x) & \frac{d}{dx}[\cos(x)] = -\sin(x) \\
\frac{d}{dx}[\tan(x)] = \sec^2(x) & \frac{d}{dx}[\cot(x)] = -\csc^2(x) \\
\frac{d}{dx}[\sec(x)] = \sec(x)\tan x & \frac{d}{dx}[\csc(x)] = -\csc(x)\cot x
\end{array}
$$

As an aid to remembering these formulas, note how the formulas in the right column are obtained from the formulas in the left column by replacing each function by its cofunction on the right side and by adding a negative sign.

SkillMaster 3.6: Differentiate expressions involving trigonometric functions and use these results to solve problems.

Now that you know the derivatives of the trigonometric functions, you can differentiate complicated expressions involving sums, products and quotients of these functions and of polynomials and multiples of e^x. Notice how the types of functions and expressions that you can differentiate is increasing as you work through this chapter.

Many applied problems involve trigonometric functions. A number of such problems appear in the examples and exercises in this section. Make sure that you understand how to solve problems involving the derivatives of trigonometric functions as a rate of change.

Worked Examples

For each of the following examples, first try to find the solution without looking at the hint. Cover the right column with a piece of paper. If you need a hint, uncover the column. If you need to see the worked solution, look at the solutions section that follows the examples and hints.

Example	Hint
SkillMaster 3.6.	
Differentiate the following functions.	
1. $y = \cos(x) - 2\sin(x)$	Differentiate each term.
2. $y = e^x \cos(x)$	Use the Product Rule.
3. Differentiate the following function in two ways: first directly and second by reducing the expression to one involving $\sin(x)$ and $\cos(x)$ but no other trigonometric functions. $$y = \frac{x}{\tan(x) + \sec(x)}$$	First, use the Quotient Rule. For the second approach, $y = \dfrac{x}{\tan(x)+\sec(x)} = \dfrac{x}{\frac{\sin(x)}{\cos(x)} + \frac{1}{\cos(x)}} = \dfrac{x\cos(x)}{\sin(x)+1}$. Now use the Quotient Rule for the fraction and the Product Rule to differentiate the numerator of the fraction.

4. A mass on a spring vibrates. Its position at time t is

$$x(t) = \frac{2\cos(t)}{e^t}$$

Find the velocity and acceleration at time t. Graph $x(t)$ for positive t. What is the behavior of position, velocity and acceleration as $t \to \infty$?

The velocity is $\frac{dx}{dt}$.
The acceleration is $\frac{d^2x}{dt^2}$.

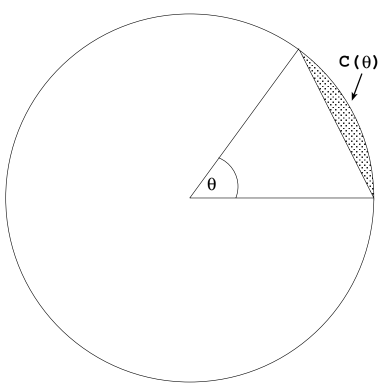
5. Evaluate the limit

$$\lim_{\theta \to 0} \frac{\tan(\theta)}{\sin(\theta) + \theta}$$

Divide both the numerator and the denominator by θ and use the limit formulas from the text. Change $\tan\theta$ to $\frac{\sin\theta}{\cos\theta}$.

6. The figure shows a unit circle with circular arc and a chord both cut off by an angle θ. Let $C(\theta)$ be the area between the chord and the arc. Compute the limit.

$$\lim_{\theta \to 0} \frac{C(\theta)}{\theta}$$



The circular sector has area $\theta/2$ and the isosceles triangle with base the chord has height $\cos(\theta/2)$ and base $2\sin(\theta/2)$. $C(\theta)$ is equal to the difference between the area of the sector cut out by θ, and the area of the isosceles triangle.

Solutions to worked examples

1. $\dfrac{dy}{dx} = -\sin(x) - 2\cos(x)$

2. $\dfrac{dy}{dx} = \cos(x)\cdot\dfrac{d}{dx}(e^x) + e^x\cdot\dfrac{d}{dx}(\cos(x))$

$= \cos(x)\cdot e^x - e^x\sin(x) = e^x\cdot(\cos(x) - \sin(x))$

3. $\dfrac{dy}{dx} = \left[\left((\tan(x)+\sec(x))\cdot\dfrac{d}{dx}(x)\right) - \left(x\cdot\dfrac{d}{dx}(\tan(x)+\sec(x))\right)\right]$

$\div(\tan(x)+\sec(x))^2$

$= \left[\tan(x)+\sec(x) - x\sec^2(x) - x\sec(x)\tan(x)\right] \div (\tan(x)+\sec(x))^2$

$= \left[(\tan(x)+\sec(x)) - x\sec(x)(\sec(x)+\tan(x))\right] \div (\tan(x)+\sec(x))^2$

$= \dfrac{(1-x\sec(x))(\tan(x)+\sec(x))}{(\tan(x)+\sec(x))^2} = \dfrac{1-x\sec(x)}{\tan(x)+\sec(x)}$

Using the second approach

$\dfrac{dy}{dx}$

$= \left[\left((\sin(x)+1)\cdot\dfrac{d}{dx}(x\cos(x))\right) - \left(x\cos(x)\cdot\dfrac{d}{dx}(\sin(x)+1)\right)\right] \div (\sin(x)+1)^2$

$= \left[(\sin(x)+1)(\cos(x)-x\sin(x)) - x\cos(x)\cos(x)\right] \div (\sin(x)+1)^2$

$= \left[\cos(x)\sin(x)+\cos(x) - x\sin^2(x) - x\sin(x) - x\cos^2(x)\right] \div (\sin(x)+1)^2$

$= \left[\cos(x)(\sin(x)+1) - x(\sin(x)+\sin^2(x)+\cos^2(x))\right] \div (\sin(x)+1)^2$

$= \dfrac{\cos(x)(\sin(x)+1) - x(\sin(x)+1)}{(\sin(x)+1)^2} = \dfrac{\cos(x)-x}{\sin(x)+1}$

Note that if we divide both the numerator and the denominator of this last answer by $\sin(x)$ and use the definitions of trig functions, we obtain the same answer as before.

4. $\dfrac{dx}{dt} = \dfrac{e^t\cdot(-2\sin(t)) - 2\cos(t)e^t}{(e^t)^2} = -2\dfrac{(\sin(t)+\cos(t))}{e^t}$

$\dfrac{d^2x}{dt^2} = (-2)\left[\left(e^t\cdot\dfrac{d}{dt}(\sin(t)+\cos(t))\right) - \left((\sin(t)+\cos(t))\cdot\dfrac{d}{dt}(e^t)\right)\right] \div (e^t)^2$

$= -2\dfrac{\cos(t)-\sin(t)-\sin(t)-\cos(t)}{e^t} = 4\dfrac{\sin(t)}{e^t} = 4\sin(t)-t$

Since the denominators of position, velocity, and acceleration go to infinity and each numerator is bounded, each of these approaches 0 as $t \to \infty$.

5. $$\lim_{\theta\to 0}\frac{\tan(\theta)}{\sin(\theta)+\theta} = \lim_{\theta\to 0}\frac{\tan(\theta)/\theta}{\sin(\theta)/\theta+\theta/\theta} = \lim_{\theta\to 0}\frac{\tan(\theta)/\theta}{\sin(\theta)/\theta+1}$$
$$= \lim_{\theta\to 0}\frac{\frac{1}{\cos\theta}\frac{\sin\theta}{\theta}}{\sin(\theta)/\theta+1} = \frac{1}{1+1} = 1/2$$

6. $$\lim_{\theta\to 0}\frac{C(\theta)}{\theta} = \lim_{\theta\to 0}\frac{\theta/2-\sin(\theta/2)\cos(\theta/2)}{\theta}$$
$$= 1/2 - \lim_{\theta\to 0}(\frac{\sin(\theta/2)}{\theta/2})(\frac{1}{2})(\cos(\theta/2)) = 1/2-(1)(\frac{1}{2})(1) = 0$$

3.4 The Chain Rule

Key Concepts:

- Chain Rule for differentiating compositions of functions
- Leibniz notation version of the Chain Rule
- $\dfrac{d}{dx}[f(x)]^n = n\cdot[f(x)]^{\,n-1}\cdot f'(x)$ and $\dfrac{d}{dx}a^x = a^x\cdot\ln(a)$

Skills to Master:

- Calculate the derivative of composed functions.
- Find tangent lines to parametric curves.

Discussion:

The Chain Rule gives you a way of differentiating much more complicated combinations of functions than the earlier Sum, Difference, Product and Quotient Rules. After mastering the Chain Rule, you will be able to differentiate complicated compositions of functions (i.e. functions inserted into other function), such as $y = \sqrt{\sin\left(e^{\cos x}\right)}$. You will also be able to find tangents to curves that are given parametrically.

Key Concept: Chain Rule for differentiating compositions of functions

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The Chain Rule allows you to differentiate *compositions* of functions that you already know how to differentiate. For example, if $g(x) = \cos x$ and if $f(u) = e^u$, then $(f\circ g)(x) = f(g(x)) = e^{\cos x}$. The inner function is $\cos x$ and the outer function is e^u. Since both f and g are differentiable, the Chain Rule tells us that

$$
\begin{aligned}
(f \circ g)'(x) &= f'(g(x)) \cdot g'(x) \\
&= e^{g(x)} \cdot (-\sin x) \\
&= e^{\cos x} \cdot (-\sin x)
\end{aligned}
$$

In particular, $(f \circ g)'(\dfrac{\pi}{2}) = e^{\cos \frac{\pi}{2}} \cdot (-\sin \dfrac{\pi}{2}) = e^0 \cdot (-1) = -1.$

Notice that it is important to tell which is the inner function and which is the outer function. In the example above, $(f \circ g) = \cos(e^x)$, a much different function, with a much different derivative.

Make sure that you understand the explanation of the special case of the Chain Rule given in the text.

Key Concept: Leibniz notation version of the Chain Rule

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If $y = f(u)$ and $u = g(x)$, and both of these functions are differentiable, another form of the Chain Rule, using *Leibniz notation,* is

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

You may find this form of the Chain Rule easier to remember. If $\Delta u \neq 0$, this formula comes from the fact that $\dfrac{\Delta y}{\Delta x} = \dfrac{\Delta y}{\Delta u} \cdot \dfrac{\Delta u}{\Delta x}$ and that the limit of difference quotients is the derivative.

Key Concept: $\mathbf{\dfrac{d}{dx}[f(x)]^n = n \cdot [f(x)]^{\,n-1} \cdot f'(x)}$ and $\mathbf{\dfrac{d}{dx}a^x = a^x \cdot \ln(a)}$

Both of these formulas follow directly from the Chain Rule. The function $[f(x)]^{\,n}$ is the composition of the functions x^n and $f(x)$. The function $a^x = e^{x \ln a}$ is the composition of the functions e^x and $x \ln a$. In the first of these, the power is a fixed number n. In the second of these, the power is a variable x.

SkillMaster 3.7: Calculate the derivative of composed functions.

You need to practice computing derivatives using the Chain Rule until you have no trouble figuring out which function to differentiate first and where to evaluate the various derivatives. As an example, the function listed in the discussion above, $y = \sqrt{\sin\left(e^{\cos x}\right)}$, is the composition of four functions: $f(x) = \sqrt{x}$ (outermost), $g(x) =$

$\sin x$, $h(x) = e^x$, and $k(x) = \cos x$ (innermost). The given function is the composition $f(g(h(k(x))))$. So the Chain Rule tells us that

$$[f \circ (g \circ (h \circ k))]'(x) = f'(g(h(k(x))) \cdot g'(h(k(x))) \cdot h'(k(x)) \cdot k'(x)$$

In this case, we get

$$[f \circ (g \circ (h \circ k))]'(x) = \frac{1}{2\sqrt{\sin(e^{\cos x})}} \cdot \cos(e^{\cos x}) \cdot e^{\cos x} \cdot (-\sin(x))$$

In many applications, there are three (or more) varying quantities where the first depends directly on the second and the second depends directly on the third. To determine the rate of change of the first quantity with respect to the third, an application of the Chain Rule is needed. Read carefully through the examples in the text so that you see how such problems arise.

SkillMaster 3.8: Find tangent lines to parametric curves.

A *parametric curve* in the plane is a curve described by giving the x and y coordinates of a point on the curve in terms of a third variable. For example,

$$x = \cos t \qquad y = \sin t$$

is a parametric description of the circle $x^2 + y^2 = 1$. If y is a function of x at some point on the circle, the Chain Rule gives us the fact that

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} \text{ or } \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

assuming that $\frac{dx}{dt} \neq 0$. This allows us to compute *tangent lines* to parametric curves.

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Worked Examples

For each of the following examples, first try to find the solution without looking at the hint. Cover the right column with a piece of paper. If you need a hint, uncover the column. If you need to see the worked solution, look at the solutions section that follows the examples and hints.

Example	Hint
SkillMaster 3.7.	
In Problems 1 - 6, use the Chain Rule to differentiate:	
1. $y=(x^3-4x^2+2)^3$	This function can be viewed as $f(g(x))$ where $y=f(u)=u^3$ and $u=g(x)=(x^3-4x^2+2)$.
2. $y=e^{(4x-1)}$	This function can be viewed as $f(g(x))$ where $y=f(u)=e^u$ and $u=g(x)=4x-1$.
3. $\dfrac{3}{\sqrt{\tan(x)+x^2}}$	This can be done using the power form of the Chain Rule. As always, convert the radical to power form.
4. $y=\sqrt[3]{\dfrac{t^3+2}{t^3-1}}$	This can be done using the power form of the Chain Rule and the Quotient Rule.
5. $y=\tan(\tan(\tan(x)))$	Think of function as $f(g(h(x)))$ and apply the Chain Rule twice.

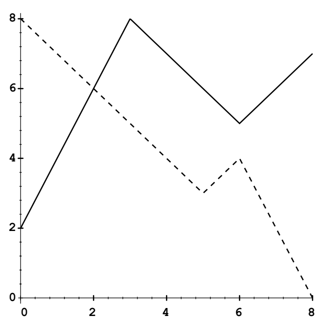
6. $y = x^x$

Write with e as the base of the exponent, $y = x^x = e^{x\ln(x)}$. Now use the the Chain Rule and then the Product Rule in differentiating $x\ln(x)$.

7. Suppose that $y = g(x)$ is a twice differentiable function with $g'(4) = 2, g''(4) = 6$. Let $f(x) = g(x^2)$. Find $f'(2)$ and $f''(2)$.

First differentiate $f(x)$ using the Chain Rule, then evaluate at $x = 2$.

8. Here is a picture of a graph that we can use to find derivatives of composed functions as in Problem 45, page 288 in Concepts and Contexts. Let f be the function with solid line graph and g be the function with dotted line graph. Find $(g \circ f)'(1)$



Read the required function values and slopes from the graph.

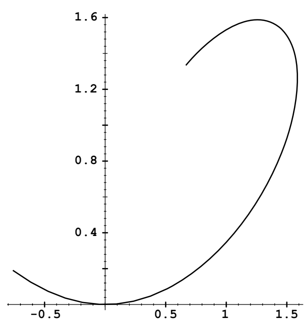
9. For what values of r does $y = e^{rt}$ satisfy the differential equation $y'' - 5y' + 6y = 0$?

Compute y' and y'' as functions of t and substitute into the differential equation.
After obtaining a product equal to 0, set each factor equal to 0 to get the results.

SkillMaster 3.8.

10. Consider the curve given by parametric equations $x = \dfrac{3t}{1+t^3} y = \dfrac{3t^2}{1+t^3}$

Pictured below is the part of the graph of the curve for t values between $-.25$ and 2.



Find the points where there are horizontal tangent lines. Find the equation of the tangent line to the curve when $t = 1$.

First find the expressions for $\dfrac{dy}{dt}$ and $\dfrac{dx}{dt}$. Then simplify the ratio of these two quantities to get the slope of the tangent line. To find horizontal tangents, find points where this slope is 0.
Note that when $t = 1$,
$x = x(1) = 3/2$ and
$y = y(1) = 3/2$

Solutions to worked examples

1. The derivatives are $\dfrac{dy}{du} = f'(u) = 3u^2$ and $\dfrac{du}{dx} = (3x^2 - 8x) = g'(x)$.

Also $f'(g(x)) = 3(g(x))^2 = 3(x^3 - 4x^2 + 2)^2$.

$\dfrac{dy}{dx} = f'(g(x))g'(x) = 3(x^3 - 4x^2 + 2)^2(3x^2 - 8x)$

Or, using Leibniz notation we arrive at the same answer.

$\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = 3u^2(3x^2 - 8x) = 3(x^3 - 4x^2 + 2)^2(3x^2 - 8x)$

2. The derivatives are $f'(u) = e^u$ and $g'(x) = 4$. Also, $f'(g(x)) = e^{g(x)} = e^{(4x-1)}$.

$\dfrac{dy}{dx} = f'(g(x)) \cdot g'(x) = e^{(4x-1)} \cdot (4) = 4e^{(4x-1)}$

In Leibniz notation $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = e^{u} \cdot (4) = 4e^{(4x-1)}$

3. $y = \dfrac{3}{\sqrt{\tan(x)+x^2}} = 3(\tan(x)+x^2)^{-1/2}$

$$\frac{dy}{dx} = 3\left(-\frac{1}{2}\right)\left(\tan(x)+x^2\right)^{-\frac{3}{2}}(\sec^2(x)+2x) = \frac{-3(\sec^2(x)+2x)}{2(\tan(x)+x^2)^{3/2}}$$

4. $y = \left(\dfrac{t^3+2}{t^3-1}\right)^{1/3}$

$$\frac{dy}{dt} = \frac{1}{3}\left(\frac{t^3+2}{t^3-1}\right)^{-2/3} \cdot \frac{d}{dt}\left(\frac{t^3+2}{t^3-1}\right)$$

$$= \frac{1}{3}\left(\frac{t^3+2}{t^3-1}\right)^{-2/3} \cdot \frac{(t^3-1)\frac{d}{dt}(t^3+2) - (t^3+2)\frac{d}{dt}(t^3-1)}{(t^3-1)^2}$$

$$= \frac{1}{3}\left(\frac{t^3+2}{t^3-1}\right)^{-2/3} \cdot \frac{(t^3-1)\cdot 3t^2 - (t^3+2)(3t^2)}{(t^3-1)^2} = \frac{1}{3}\left(\frac{t^3+2}{t^3-1}\right)^{-2/3}\frac{(-9t^2)}{(t^3-1)^2}$$

$$= \frac{-3t^2}{(t^3+2)^{2/3}(t^3-1)^{4/3}}$$

5. First apply the Chain Rule to the outer function, $\dfrac{dy}{dx} = \sec^2(\tan(\tan(x)))\dfrac{d}{dx}(\tan(\tan(x)))$. Then apply the Chain Rule again to the inner function.

$$= \sec^2(\tan(\tan(x))) \cdot \sec^2(\tan(x))\frac{d}{dx}(\tan(x))$$

$$= \sec^2(\tan(\tan(x)))\sec^2(\tan(x))\sec^2(x)$$

6. $\dfrac{dy}{dx} = e^{x\ln(x)} \cdot \dfrac{d}{dx}(x\ln(x)) = e^{x\ln(x)} \cdot (\ln(x) + x(\dfrac{1}{x}))$

$$= e^{x\ln(x)} \cdot (\ln(x)+1) = (1+\ln(x))x^x$$

7. $f'(x) = g'(x^2) \cdot \dfrac{d}{dx}(x^2) = g'(x^2) \cdot (2x) = 2x \cdot g'(x^2)$

$$f'(2) = 2 \cdot (2) \cdot g'(2^2) = 4 \cdot g'(4) = 4 \cdot (2) = 8$$

$$f''(x) = \frac{d}{dx}(f'(x)) = \frac{d}{dx}(2x \cdot g'(x^2)) = 2x \cdot \frac{d}{dx}(g'(x^2)) + g'(x^2)\frac{d}{dx}(2x)$$

$$= 2x \cdot (g''(x^2)) \cdot \frac{d}{dx}(x^2) + g'(x^2) \cdot 2 = 2x \cdot (g''(x^2)) \cdot (2x) + 2 \cdot g'(x^2)$$

$$= 4x^2 \cdot g''(x^2) + 2 \cdot g'(x^2)$$

$$f''(2) = 4 \cdot (2^2) \cdot g''(2^2) + 2 \cdot g'(2^2)$$

$= 16 \cdot g''(4) + 2 \cdot g'(4) = 16 \cdot (6) + 2 \cdot (2) = 100$

8. The Chain Rule says $(g \circ f)'(x) = g'(f(x)) \cdot f'(x)$.
Now substitute $x = 1$.
$(g \circ f)'(1) = g'(f(1)) \cdot f'(1) = g'(4) \cdot f'(1) = (-1) \cdot 2 = -2$

9. $y' = \dfrac{d}{dt}(e^{rt}) = re^{rt} \qquad y'' = \dfrac{d}{dt}(re^{rt}) = r^2 e^{rt}$
$0 = y'' - 5y' + 6y = (r^2 e^{rt}) - 5(re^{rt}) + 6e^{rt}$
$= (r^2 - 5r + 6)e^{rt} = (r-2)(r-3)e^{rt}$

Since e^{rt} is never 0, the only possible solutions are $r = 2$ and $r = 3$.

10. $$\frac{dx}{dt} = \frac{(1+t^3) \cdot \frac{d}{dx}(3t) - (3t) \cdot \frac{d}{dx}(1+t^3)}{(1+t^3)^2} = \frac{(1+t^3) \cdot 3 - (3t) \cdot (3t^2)}{(1+t^3)^2}$$
$$= \frac{3(1-2t^3)}{(1+t^3)^2}$$
$$\frac{dy}{dt} = \frac{(1+t^3) \cdot \frac{d}{dx}(3t^2) - (3t^2) \cdot \frac{d}{dx}(1+t^3)}{(1+t^3)^2} = \frac{(1+t^3) \cdot 6t - (3t^2) \cdot (3t^2)}{(1+t^3)^2}$$
$$= \frac{3t \cdot (2-t^3)}{(1+t^3)^2}$$
$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t(2-t^3)}{3(1-2t^3)} = \frac{t(2-t^3)}{1-2t^3}$$

There is a horizontal tangent line when the numerator is equal to 0. This occurs when $t = 0$ or when $(2-t^3) = 0$. That is, $t = 0$ or $t = 2^{1/3}$. Note that this corresponds to the points $(0,0)$ and $(2^{1/3}, 2^{2/3}) \approx (1.2599, 1.5874)$.

These correspond to the places where the graph appears to have horizontal tangent lines.

For the second question, the slope of the tangent line at $(3/2, 3/2)$ (i.e. when $t = 1$) is $((1)(2-1^3))/(1-2(1^3)) = 1/(-1) = -1$.

The equation of the tangent line, using the point-slope form is

$(y - 3/2) = (-1)(x - 3/2)$ or $y + x = 3$.

3.5 Implicit Differentiation

Key Concepts:

- Functions defined implicitly by an equation in x and y
- Differentiating implicit functions using the rules of derivatives

Skills to Master:

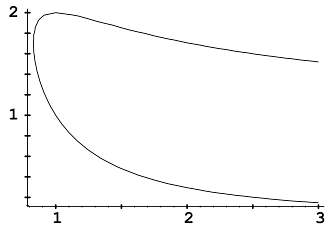
- Calculate derivatives of implicit functions.

Discussion:

Often two variables will be related by an equation in such a way that it is difficult or impossible to find a formula to solve for one of the variables in terms of the other. Nonetheless, it is reasonable to assume that if a value for one variable is given one could find a value of the other variable so that a function is defined. Such functions are called *implicit functions*. In more advanced mathematics courses, conditions under which one of the variables can be viewed as a function of the other are given. Assuming that such a functional relationship exists, one can differentiate both sides of an equation by using the Chain Rule and come up with a expression for the derivative of one variable with respect to the other.

Key Concept: Functions defined implicitly by an equation in x and y

An equation such as $x^2y+2xy=(xy)^2+2$ implicitly defines y as a function of x. There may be points x where the value of y is ambiguous. Below is a plot indicating points in the pictured region that satisfy this equation.



Note that the point $x = 1$, $y = 1$ satisfies the equation. It appears from the graph that we should be able to find the slope of the tangent line to the curve at this point. The text describes how to do this later in this section.

Key Concept: Differentiating implicit functions using the rules of derivatives

Once a function is given implicitly by an equation, you can use the various derivatives of functions that you know and the rules of differentiation such as the Product, Quotient, Sum, and Chain Rules to differentiate both sides of the equation. Then you can solve for the derivative that you want.

SkillMaster 3.9: Calculate derivatives of implicit functions.

Consider again the equation $x^2y + 2xy = (xy)^2 + 2$ discussed above. If we assume that $y = y(x)$ is an implicit function of x, by using the Product and Chain Rules, we can differentiate both sides of this equation and then solve for $\frac{dy}{dx}$:

$$\begin{aligned}
\left(2xy + x^2\frac{dy}{dx}\right) + \left(2y + 2x\frac{dy}{dx}\right) &= 2(xy)\left(y + x\frac{dy}{dx}\right), \text{ or} \\
2xy + x^2\frac{dy}{dx} + 2y + 2x\frac{dy}{dx} &= 2xy^2 + 2x^2y\frac{dy}{dx} \\
\frac{dy}{dx} \cdot (x^2 + 2x - 2x^2y) &= 2xy^2 - 2xy - 2y, \text{ or} \\
\frac{dy}{dx} &= \frac{2xy^2 - 2xy - 2y}{x^2 + 2x - 2x^2y}
\end{aligned}$$

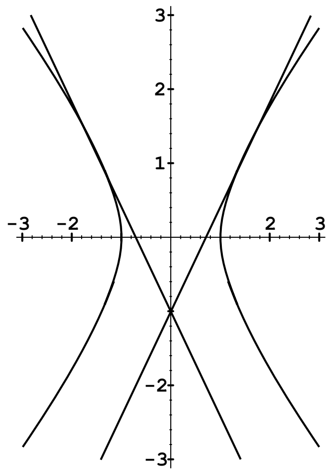
Evaluating this at $x = 1$, $y = 1$, we find that $\dfrac{dy}{dx} = -2$. That is, the slope of the tangent line at the point $(1, 1)$ is -2.

Worked Examples

For each of the following examples, first try to find the solution without looking at the hint. Cover the right column with a piece of paper. If you need a hint, uncover the column. If you need to see the worked solution, look at the solutions section that follows the examples and hints.

Example	Hint
SkillMaster 3.9.	
1. Suppose $x^4 + y^4 = 1$ for $y > 0$. Find y' in two different ways. First solve for y and using the Chain Rule. Then solve by using implicit differentiation. Check that both answers agree.	First solve for y. $y = (1 - x^4)^{1/4}$ Next differentiate implicitly. Remember that $\frac{d}{dx}(y^4)$ requires the Chain Rule.
2. Find y' in two different ways. First solve for y and using the Quotient Rule. Then solve by using implicit differentiation. Check that both answers agree. $$xy + x + y = 1$$	First solve for y and differentiate using the Quotient Rule. Then differentiate implicitly and solve for y'. Don't forget to use the Product Rule on xy.
3. Differentiate implicitly: $$\cos(x)y + xy + x^2 = 6$$	Use the Product Rule on the terms $\cos(x)y$ and xy.
4. Differentiate implicitly: $$e^{xy} = e^x + e^y$$	Use the Chain Rule and the Product Rule on e^{xy} and the Chain Rule on e^y.

5. Find the equations of the two tangent lines to $x^2 - y^2 = 1$ that pass through the point $(0,-1)$.



You don't actually know in advance which points on the curve have tangents passing through $(0,-1)$ (if any). So begin by finding the equation of the tangent line for each point (x_0, y_0) of the curve. Use implicit differentiation to find the slope. Then use the point-slope form of a line to write the equation of the tangent line through (x_0, y_0).
Next, since you want such lines passing through $(0,-1)$, substitute 0 for x and -1 for y.
Finally, (x_0, y_0) also satisfies the equation $x^2 - y^2 = 1$ so $-{x_0}^2 + {y_0}^2 = -1$. Substitute this in the previous equation.

6. Consider the unit circle

$$x^2 + y^2 = 1$$

Use implicit differentiation to find expressions in terms of x and y for the first, second, and third derivatives of y with respect to x.

Use implicit differentiation and solve for y' in terms of x and y.
Use implicit differentiation again using the Quotient Rule. Now solve for y''. Remember, $x^2 + y^2 = 1$.
Differentiate one last time to find y'''. Use the Chain Rule.

7. Show the following families of curves are orthogonal trajectories of each other.

$$\begin{aligned} y^2 &= 2(a-x) \\ y &= be^x \end{aligned}$$

Find the derivative y' for each family of curves and compare.

Solutions to worked examples

1. $y^4 = 1 - x^4$ or $y = (1-x^4)^{1/4}$ since $y > 0$.

$\dfrac{dy}{dx} = (\dfrac{1}{4})(1-x^4)^{-3/4}(-4x^3) = -x^3(1-x^4)^{-3/4}$

Now apply implicit differentiation to $x^4 + y^4 = 1$.

$\dfrac{d}{dx}(x^4) + \dfrac{d}{dx}(y^4) = \dfrac{d}{dx}(1) \qquad 4x^3 + 4y^3\dfrac{d}{dx}(y) = 0$

$4x^3 + 4y^3y' = 0$

So $y' = \dfrac{-x^3}{y^3}$

To see that these two answers agree, substitute in the last expression what y is in terms of x.

$$y' = \frac{-x^3}{((1-x^4)^{1/4})^3} = \frac{-x^3}{(1-x^4)^{3/4}} = -x^3(1-x^4)^{-3/4}$$

2. First solve $xy + x + y$ for y. $y(x+1) = 1 - x \qquad y = \dfrac{1-x}{x+1}$

Differentiate: $y' = \dfrac{-(x+1)-(1-x)}{(x+1)^2} = \dfrac{-2}{(x+1)^2}$

Implicitly differentiate the original expression $xy + x + y = 1$.

$\dfrac{d}{dx}(xy) + \dfrac{d}{dx}(x) + \dfrac{d}{dx}(y) = \dfrac{d}{dx}(1)$

$y\dfrac{d}{dx}(x) + x\dfrac{d}{dx}(y) + 1 + y' = 0 \qquad y + xy' + 1 + y' = 0$

$y'(x+1) = -(y+1) \qquad y' = \dfrac{-(y+1)}{x+1}$

To see that both answers agree, substitute in the last expression for y in terms of x.

$$y' = \frac{-(\frac{1-x}{x+1}+1)}{x+1} = \frac{-(1-x+x+1)}{(x+1)^2} = \frac{-2}{(x+1)^2}$$

3. $\dfrac{d}{dx}(\cos(x)y + xy + x^2) = \dfrac{d}{dx}(6)$

$y\dfrac{d}{dx}(\cos(x)) + \cos(x)\dfrac{dy}{dx} + y\dfrac{d}{dx}(x) + x\dfrac{dy}{dx} + 2x = 0$

$y \cdot (-\sin(x)) + \cos(x)y' + y + xy' + 2x = 0$

$y'(\cos(x) + x) = \sin(x)y - y - 2x$

$$y' = \frac{\sin(x)y - y - 2x}{\cos(x) + x}$$

4. $\frac{d}{dx}(e^{xy}) = \frac{d}{dx}(e^x) + \frac{d}{dx}(e^y) \qquad e^{xy}\frac{d}{dx}(xy) = e^x + e^y\frac{d}{dx}(y)$

$e^{xy}(y\frac{d}{dx}(x) + x\frac{d}{dx}(y)) = e^x + e^y y'$

$e^{xy}(y + xy') = e^x + e^y y'$

$y'(xe^{xy} - e^y) = e^x - ye^{xy} \qquad y' = \frac{e^x - ye^{xy}}{xe^{xy} - e^y}$

5. Implicitly Differentiate: $2x - 2yy' = 0$, or $y' = \frac{x}{y}$

If (x_0, y_0) is on the graph, then the equation of the tangent line at (x_0, y_0) is $y = \frac{x_0}{y_0}(x - x_0) + y_0$. We need $(0, -1)$to be on this line.

So $-1 = \frac{x_0}{y_0}(0 - x_0) + y_0$ or $-y_0 = -x_0{}^2 + y_0{}^2$

Since (x_0, y_0) is on the curve, $x_0{}^2 - y_0{}^2 = 1$, we have $-x_0{}^2 + y_0{}^2 = -1$, so $y_0 = 1$

$x_0{}^2 - (1)^2 = 1 \qquad x_0{}^2 = 2 \qquad x_0 = \pm\sqrt{2}$

The equations of the two tangent lines are

$y = \sqrt{2}(x - \sqrt{2}) + 1$ and $y = -\sqrt{2}(x + \sqrt{2}) + 1$

6. $x^2 + y^2 = 1 \qquad 2x + 2yy' = 0 \qquad x + yy' = 0 \qquad y' = \frac{-x}{y}$

$y'' = \frac{y\frac{d}{dx}(-x) - (-x)\frac{d}{dx}(y)}{y^2}$ or $y'' = \frac{-y + xy'}{y^2}$

Substitute for y'.

$y'' = \frac{-y + x\left(\frac{-x}{y}\right)}{y^2} = \frac{-(x^2 + y^2)}{y^3} = \frac{-1}{y^3}$ or,

$y'' = -(y^{-3}) \qquad y''' = -(-3)(y^{-4})y'$

Substitute for y'.

$y''' = -(-3)(y^{-4})(\frac{-x}{y}) = \frac{-3x}{y^5}$

7. For the first family of curves, $2yy' = -2 \qquad y' = -\frac{1}{y}$.

For the second family of curves,

$y' = be^x = y$.

Since each derivative for the first family of curves is the negative reciprocal of the derivative for the other family of curves, the two families of curves are orthogonal trajectories of each other.

8. $y' = \dfrac{1}{\sqrt{1-(\sqrt{x})^2}}\dfrac{d}{dx}(\sqrt{x}) = \dfrac{1}{\sqrt{1-x}}\dfrac{1}{2\sqrt{x}} = \dfrac{1}{2\sqrt{x-x^2}}$

9. $y' = \arctan(x^2)\dfrac{d}{dx}(1+x^4) + (1+x^4)\dfrac{d}{dx}\arctan(x^2)$

$= \arctan(x^2)4x^3 + (1+x^4)\dfrac{1}{1+(x^2)^2}(2x)$

$= \arctan(x^2)4x^3 + \dfrac{1+x^4}{1+x^4}(2x) = 4x^3\arctan(x^2) + 2x$

3.6 Inverse Trigonometric Functions

Key Concepts:

- Definitions and Properties of Inverse Trigonometric Functions

Skills to Master:

- Calculate derivatives of inverse trigonometric functions.

Discussion:

This section uses the material in the previous section to introduce the definitions of inverse trigonometric functions and their derivatives. You may remember some of this material from a precalculus trigonometry course. The important things to focus on are where the inverse trigonometric functions are defined. Make sure that you learn the new derivatives introduced in this section.

Key Concept: Definitions and Properties of Inverse Trigonometric Functions

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To define the inverse trigonometric functions, one restricts the domain of the trigonometric functions so that there is a unique y-value associated with each x-value. See *Figures 1 and 2* for an illustration of this. Make sure you review the definitions of the inverse trigonometric functions introduced in this section if you have forgotten them.

SkillMaster 3.10: Calculate derivatives of inverse trigonometric functions.

The *inverse trigonometric functions* are introduced and discussed in this section. For example,

$$\sin x = y \text{ for } -\frac{\pi}{2} \le y \le \frac{\pi}{2}$$

defines y as an implicit function of x. This implicit function is called the *inverse sine function*. One can use the technique of implicit differentiation to then calculate $\dfrac{dy}{dx}$.

We list the two main formulas that you should learn.

$$\begin{aligned} \frac{d}{dx}\left[\sin^{-1}(x)\right] &= \frac{1}{\sqrt{1-x^2}} \\ \frac{d}{dx}\left[\tan^{-1}(x)\right] &= \frac{1}{1+x^2} \end{aligned}$$

Remember there are two equivalent ways to write inverse trigonometric functions, $\sin^{-1}(x) = \arcsin(x)$ and $\tan^{-1}(x) = \arctan(x)$.

Worked Examples

For each of the following examples, first try to find the solution without looking at the hint. Cover the right column with a piece of paper. If you need a hint, uncover the column. If you need to see the worked solution, look at the solutions section that follows the examples and hints.

Example	**Hint**
SkillMaster 3.10.	
Find the derivative of each of the following functions:	
1. $y = \sin^{-1}(\sqrt{x})$	Don't forget to use the Chain Rule.
2. $y = (1+x^4)\arctan(x^2)$	Use the Product Rule and then the Chain Rule.

Solutions to worked examples

1. $y' = \dfrac{1}{\sqrt{1-(\sqrt{x})^2}}\dfrac{d}{dx}(\sqrt{x}) = \dfrac{1}{\sqrt{1-x}}\dfrac{1}{2\sqrt{x}} = \dfrac{1}{2\sqrt{x-x^2}}$

2. $$\begin{aligned} y' &= \arctan(x^2)\frac{d}{dx}(1+x^4) + (1+x^4)\frac{d}{dx}\arctan(x^2) \\ &= \arctan(x^2)4x^3 + (1+x^4)\frac{1}{1+(x^2)^2}(2x) \\ &= \arctan(x^2)4x^3 + \frac{1+x^4}{1+x^4}(2x) = 4x^3\arctan(x^2) + 2x \end{aligned}$$

3.7 Derivatives of Logarithmic Functions

Key Concepts:

- $\dfrac{d}{dx}[\ln(x)] = \dfrac{1}{x}$ and $\dfrac{d}{dx}[\log_a(x)] = \dfrac{1}{x \ln a}$
- $e = \lim\limits_{n\to\infty} (1 + \dfrac{1}{n})^n$
- Logarithmic differentiation as a tool to simplify problems

Skills to Master:

- Find the derivatives of functions involving logarithms.
- Calculate derivatives using the technique of logarithmic differentiation.

Discussion:

The technique of implicit differentiation can be used to develop the formula

$$\frac{d}{dx}[\log_a(x)] = \frac{1}{x \ln a}$$

In particular, $\dfrac{d}{dx}[\ln(x)] = \dfrac{1}{x}$. Review properties of *logarithmic functions* if you need to. The technique of logarithmic differentiation can be used to differentiate complicated functions that involve products, quotients, and powers. This technique leads to an explanation of the general Power Rule from Section 3.1:

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$$\frac{d}{dx}x^n = nx^{n-1}$$

This also leads to a new expression for e.

Key Concept: $\mathbf{\frac{d}{dx}[\ln(x)] = \frac{1}{x}}$ and $\mathbf{\frac{d}{dx}[\log_a(x)] = \frac{1}{x\ln a}}$

The fact that $y = \log_a x \Longleftrightarrow a^y = x$ together with the technique of implicit differentiation allows you to check that $\frac{d}{dx}[\log_a(x)] = \frac{1}{x\ln a}$. Since $\ln(x) = \log_e x$ and since $\ln e = 1$, the other formula follows. The text also derives the fact that

$$\frac{d}{dx}\ln(|x|) = \frac{1}{x}$$

Key Concept: $\mathbf{e = \lim_{n\to\infty}(1+\frac{1}{n})^n}$

Make sure that you understand how this formula is derived using the definition of the derivative of $\ln x$ at $x = 1$. A more general formula that is also true is

$$e^a = \lim_{n\to\infty}(1+\frac{a}{n})^n.$$

This can be derived in a similar manner.

Key Concept: Logarithmic differentiation as a tool to simplify problems

Two centuries ago logarithms were necessary for computation. For example, logarithms were critical for navigation at sea before the development of computational machines. There was no other convenient method available to calculate numbers such as π^e. To accomplish this computation via logarithms, one would use the fact that $\log(\pi^e) = e\log(\pi)$. This use of logarithms reduced an exponential problem to a multiplication problem. Also multiplication of large numbers without calculators and computers is more difficult than the method of first taking logarithms of each term in the product, adding the result and finding the inverse logarithm. This uses the fact that $\log(ab) = \log(a) + \log(b)$.

Today, in the same way, many expressions can be simplified by taking the logarithm. This is because applications of the Laws of Logarithms allow us to rewrite the logarithms of powers, products and quotients as products, sums and differences. If an expression can be simplified this way, then the technique of logarithmic differentiation may be used to take the derivative. Make sure that you understand why this technique works.

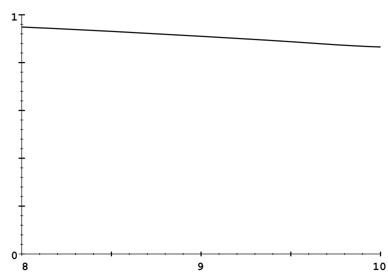
SkillMaster 3.11: Find the derivatives of functions involving logarithms.

Using the formulas developed in this section, you should now be able to differentiate products, quotients and compositions where some of the functions are logarithmic functions. For example, to find the derivative of $\sin(\log_3 x)$, use the Chain Rule after recognizing this as a composition of functions. In this case,

$$\begin{aligned}\frac{d}{dx}\sin(\log_3 x) &= \cos(\log_3 x)\cdot\frac{d}{dx}(\log_3 x)\\ &= \cos(\log_3 x)\cdot\frac{1}{x\ln 3}\ .\end{aligned}$$

Since the $\log_3 9 = 2$, the derivative of this function at $x = 9$ is $\cos(2)\cdot\dfrac{1}{9\ln 3} \approx -.042088$. Pictured below is the graph of $\sin(\log_3 x)$ near $x = 9$.

The fact that the graph of the function appears nearly straight indicates that the derivative does not change much between $x = 8$ and $x = 10$.



SkillMaster 3.12: Calculate derivatives using the technique of logarithmic differentiation.

Functions which involve products, quotients and powers can often be simplified by taking the logarithm of the function. The technique of logarithmic differentiation exploits this fact by taking the logarithm and then differentiating. For example, if

$$\begin{aligned}y &= \left(\frac{x^3}{x^5-5x}\right)^{1/2}\text{, then}\\ \ln y &= (1/2)\left(3\ln x - \ln(x^5-5x)\right)\end{aligned}$$

Differentiating this implicitly, one obtains

$$\begin{aligned}
\frac{d}{dx}(\ln y) &= \frac{y'}{y} = (1/2)\cdot\left(\frac{3}{x}-\frac{5x^4-5}{x^5-5x}\right), \text{ so} \\
y' &= y\cdot(1/2)\cdot\left(\frac{3}{x}-\frac{5x^4-5}{x^5-5x}\right) \\
&= \left(\frac{x^3}{x^5-5x}\right)^{1/2}\cdot(1/2)\cdot\left(\frac{3}{x}-\frac{5x^4-5}{x^5-5x}\right)
\end{aligned}$$

Work through as many problems from the text as possible to become familiar with this technique.

Worked Examples

For each of the following examples, first try to find the solution without looking at the hint. Cover the right column with a piece of paper. If you need a hint, uncover the column. If you need to see the worked solution, look at the solutions section that follows the examples and hints.

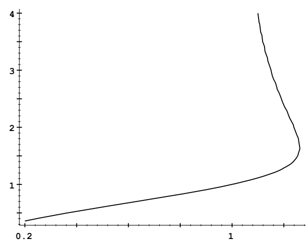
Example	Hint
SkillMaster 3.11.	
In Worked Examples 1 - 3, find derivatives of the following functions:	
1. $y=\log_{10}(e^x+1)$	Use the Chain Rule.
2. $y=x\ln(1+x^2)+2\tan^{-1}(x)$	Use the Product Rule and the Chain Rule.
3. $y=\ln\sqrt[3]{\dfrac{t^3+3t+1}{t^3-3t-2}}$	First apply the laws of logarithms to simplify. Then use the Chain Rule.
4. Find the domain of the function $f(x)=\ln(\ln(x+1)+1)$ and then differentiate the function.	Use the Chain Rule. The domain of $\ln(x)$ is the set of all $x>0$.

SkillMaster 3.12.

5. Use logarithmic differentiation to find the derivative of
$y = \dfrac{\sqrt{x^2+1}(x-3)^{10}}{(x^2-x+9)^5}$

First take natural logarithms and simplify. Then differentiate and solve for y'.

6. Find the equation of the tangent line at the point $(1,1)$ to the curve given by the equation $x^{y^2} = y^x$



Find the slope using logarithmic differentiation. Then substitute in $x=1, y=1$ and solve for y'. Then express the equation of the tangent line in point-slope form.

Solutions to worked examples

1. $y' = \dfrac{1}{\ln(10)(e^x+1)} \dfrac{d}{dx}(e^x+1) = \dfrac{e^x}{\ln(10)(e^x+1)}$

2. $y' = \ln(1+x^2)\dfrac{d}{dx}(x) + x\dfrac{d}{dx}\left(\ln(1+x^2)\right) + \dfrac{2}{1+x^2}$

$$= \ln(1+x^2) + x\frac{1}{1+x^2}\frac{d}{dx}(1+x^2) + \frac{2}{1+x^2} = \ln(1+x^2) + \frac{x(2x)}{1+x^2} + \frac{2}{1+x^2}$$

$$= \ln(1+x^2) + \frac{2(x^2+1)}{1+x^2} = \ln(1+x^2) + 2$$

3. $y = \dfrac{1}{3}\ln\left(\dfrac{t^3+3t+1}{t^3-3t-2}\right) = \dfrac{1}{3}\ln(t^3+3t+1) - \dfrac{1}{3}\ln(t^3-3t-2)$

$$y' = \frac{3t^2+3}{3(t^3+3t+1)} - \frac{3t^2-3}{3(t^3-3t-2)} = \frac{t^2+1}{t^3+3t+1} - \frac{t^2-1}{t^3-3t-2}$$

4. x is in the domain of $f \iff \ln(x+1)+1 > 0$, or when $\ln(x+1) > -1$.
This happens when $e^{\ln(x+1)} > e^{-1}$, or $x+1 > e^{-1}$, or $x > e^{-1} - 1$.

$$f'(x) = \frac{\frac{d}{dx}(\ln(x+1)+1)}{\ln(x+1)+1} = \frac{\frac{1}{x+1}}{\ln(x+1)+1} = \frac{1}{(x+1)(\ln(x+1)+1)}$$

5. $\ln(y) = \ln(\frac{\sqrt{x^2+1}(x-3)^{10}}{(x^2-x+9)^5})$

$= \ln(\sqrt{x^2+1}) + \ln((x-3)^{10}) - \ln((x^2-x+9)^5)$

$= \frac{1}{2}\ln(x^2+1) + 10\ln(x-3) - 5\ln(x^2-x+9)$

$\frac{y'}{y} = \frac{2x}{2(x^2+1)} + \frac{10}{x-3} - \frac{5(2x-1)}{x^2-x+9}$

$y' = y(\frac{x}{x^2+1} + \frac{10}{x-3} - \frac{5(2x-1)}{x^2-x+9})$

$= \left(\frac{\sqrt{x^2+1}(x-3)^{10}}{(x^2-x+9)^5}\right) \cdot \left(\frac{x}{x^2+1} + \frac{10}{x-3} - \frac{5(2x-1)}{x^2-x+9}\right)$

6. $y^2\ln(x) = x\ln(y)$ Use implicit differentiation and the product rule to get

$2yy'\ln(x) + \frac{y^2}{x} = \ln(y) + \frac{x}{y}y'$.

Now substitute the values $x = 1$ and $y = 1$.

$2(1)y'\ln(1) + \frac{1^2}{1} = \ln(1) + \frac{1}{1}y'$ or $1 = y'$

The tangent line then has equation $y = 1(x-1)+1$ $y = x-1+1$ $y = x$.

3.8 Rates of Change in the Natural and Social Sciences

Key Concepts:

- The derivative as a rate of change
- The derivative applied to the sciences

Skills to Master:

- Use the derivative to solve problems in scientific applications.

Discussion:

This section gives a number of examples showing how derivatives can be applied to solve rate of change problems in physics, chemistry, biology, economics and other sciences. In modeling within these fields, information is often given as rates of change as well as in terms of variables. Applications are the main reason that Calculus is so important. Make sure you understand the relationship between derivatives and rates of change.

Key Concept: The derivative as a rate of change

The derivative of $y = f(x)$ is the limit of the *change* in y divided by the *change* in x.

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$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

Thus, the derivative is the limit of the *average rate of change* in y with respect to x. The derivative $\dfrac{dy}{dx} = f'$ is the *instantaneous rate of change* and gives the number needed to multiply a small change in x so that the product is approximately equal to the corresponding change in y. These are the key ideas one uses to think about and apply the derivative.

Key Concept: The derivative applied to the sciences

Read carefully the various applications of the derivative in this section. The derivative can be used to compute:

- instantaneous velocity and acceleration,
- density,
- current,
- instantaneous rates of reaction,
- isothermal compressibility,
- instantaneous rates of growth, and
- marginal cost.

SkillMaster 3.13: Use the derivative to solve problems in scientific applications.

You need to be able to recognize how quantities in various scientific and economic problems can be computed by using the derivative. Any quantity that can be expressed as the instantaneous rate of change of one variable with respect to another can be computed as a derivative.

In addition to solving scientific problems involving rates of change by using the derivative, you can use your geometric understanding of the derivative as an instantaneous rate of change to get a better understanding of various scientific concepts.

Worked Examples

For each of the following examples, first try to find the solution without looking at the hint. Cover the right column with a piece of paper. If you need a hint, uncover the column. If you need to see the worked solution, look at the solutions section that follows the examples and hints.

Example	Hint

SkillMaster 3.13.

Use the following information for the next three problems. A large pine cone is thrown down from a tree house which is 15 feet above the ground. Due to strong updrafts, the equation for the height of the pine cone in feet as a function of time in seconds is

$$p(t) = 6t^3 - 16t^2 + t + 15.$$

Example	Hint
1. Use a graphing device to estimate to within one tenth of a second when the pine cone hits the ground. What is the velocity of the pine cone at the time it hits the ground?	Graph $p(t)$ to estimate when the pine cone hits the ground. Differentiate the equation for $p(t)$ and substitute in the time that the pine cone hits the ground to get the velocity of the pine cone when it hits the ground.
2. At what time does the velocity of the pine cone begin to decrease?	The velocity begins to decrease when $a(t)$, the acceleration, becomes negative.

3. At the exact same time, a second pine cone is blown from the roof of the tree house which is 20 feet above the ground. Assume the height of the second pine cone at time t is given by

$$q(t) = 6t^3 - 16t^2 + t + 20$$

Graph this equation with a graphing device. Does the second pine cone hit the ground? When does the second pine cone go past the floor of the tree house for the second time?

The equation indicates that the second pine cone is subject to the same wind updraft as the first pine cone but starts at 20 feet above the ground while the first pine cone began 15 feet above the ground. This is reflected by the fact that $p(0) = 15$.

4. The population of a colony of bacteria is given by the function

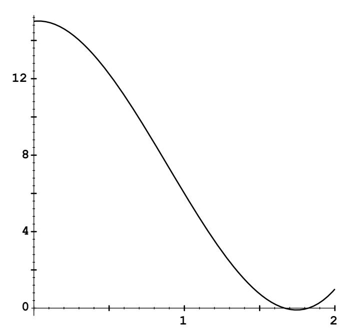
$$p(t) = \frac{10^6 e^t}{1 + e^t}$$

(a) Find the population at time 0.
(b) Find the birth rate, $p'(t)/p(t)$ and interpret it.
(c) At time $t = 2$ how many new births in the next .5 seconds would you expect to come from a sub colony of 10,000 bacteria?

Substitute 0 for t.
Then differentiate to find $p'(t)$.
To find the birth rate at $t = 2$, substitute 2 for t.
Then multiply the population by the birth rate by the change in time.

Solutions to worked examples

1. Graphing $p(t)$ shows that the first pine cone hits the ground at about $t = 1.7$ seconds.



The velocity is $v(t) = p'(t) = 18t^2 - 32t + 1$.

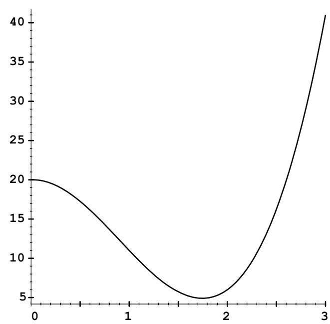
The velocity of the pine cone when it hits the ground is about $v(1.7) = -1.38$ ft/s. The negative sign before the velocity indicates that the pine cone was moving downward.

2. The acceleration is $a(t) = v'(t) = 36t - 32$.

The acceleration is positive until $a(t) = 0$ and after this time becomes negative. The velocity begins to decrease at the time t when $a(t) = 0$ or when $36t - 32 = 0$ or $t = \frac{32}{36} = \frac{8}{9} = 0.8\overline{8}$s .

3. The second pine cone has as its equation of motion $q(t) = 6t^3 - 16t^2 + t + 20$.

Graphing this equation shows that it does not cross the x-axis so the pine cone never hits the ground but is carried upward by the updraft. The graph shows that the pine cone goes up past the tree house floor (at 15 feet above the ground) at about time 2.5 s.



4. (a) At time 0 the population is $p(0) = \dfrac{10^6 e^0}{1 + e^0} = \dfrac{10^6}{2} = 5 \cdot 10^5$.

(b) For notational convenience replace 10^6 by the constant k. This will allow us to see the structure of the equations and not be distracted by the appearance of 10^6. Note: this is a good trick to use whenever there are large numbers or expressions that will not change during the calculation. You do have to remember to back substitute the number or expression for k when your calculation is complete.

$$p'(t) = k\frac{(1+e^t)\frac{d}{dt}(e^t) - e^t\frac{d}{dt}(1+e^t)}{(1+e^t)^2} = k\frac{(1+e^t)e^t - e^t(e^t)}{(1+e^t)^2} = \frac{ke^t}{(1+e^t)^2}$$

$$p'(t)/p(t) = \frac{ke^t}{(1+e^t)^2} \div (\frac{ke^t}{(1+e^t)}) = \frac{1}{1+e^t}$$

The constant k has disappeared in this calculation so there is no need to back substitute the value of k.

The birth rate is the rate at which new members of the population are born per individual bacterium.

(c) $p'(2)/p(2) = \dfrac{1}{1+e^2} \approx 0.1192$

A given subset of 10,000 bacteria would give rise to approximately
$10,000 \cdot (p'(t)/p(t)) \cdot \Delta t \approx 10,000 \cdot (0.1192) \cdot (.5) \approx 596$. So about 600 new bacteria will be produced in the next 5 seconds from each subset of $10,000$.

3.9 Linear Approximations and Differentials

Key Concepts:

- The tangent line to a function at a point as the best linear approximation near the point
- The differential dy defined as $f'(x) \cdot dx$

Skills to Master:

- Calculate the linearization of a function at a point.
- Use differentials to approximate the actual change in a function and to compute relative errors.

Discussion:

This section introduces the idea of using the *linear approximation* or tangent line approximation to approximate values of a function over a small interval containing a point where the function value is known. Since a linear function is easier to compute than a more complicated function, the linear approximation at a point is often used as an approximation to the function near that point. Remember that the linear approximation is another name for the tangent line approximation. Differentials are introduced in this section to give meaning to the symbols dx and dy. Differentials will explain the concept of the error of an estimation and relative error.

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Key Concept: The tangent line to a function at a point as the best linear approximation near the point

The tangent line to a differentiable function f at $x = a$ is the linear function: $L(x) = f(a) + f'(a)(x - a)$.

This function L is called the *linearization* of f. When x is close to a, $f'(a)$ is close to $\dfrac{f(x)-f(a)}{x-a}$ and so $L(x)$ is close to

$$f(a)+\left(\frac{f(x)-f(a)}{x-a}\right)(x-a)=f(a)+(f(x)-f(a))=f(x)$$

Thus we see that $L(x)$ is close to $f(x)$ when x is close to a and so is a good approximation. $L(x)$ is also called the linear approximation or tangent line approximation to $f(x)$ near $x=a$.

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In more advanced courses, it is shown that this is the best (i.e. most accurate) linear approximation to f near $x=a$. The *Laboratory Project* on Taylor Polynomials in the text shows how to get approximations to $f(x)$ with polynomials of higher degree.

Key Concept: The differential dy defined as $\mathbf{f'(x)\cdot dx}$

If $y=f(x)$, the differential dx is an independent variable. You should think of dx as corresponding to Δx, or the change in x.

The differential dy requires more explanation. It is defined by:

$$dy=f'(x)\cdot dx=f'(x)\cdot\ \Delta x$$

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Thus dy depends on two variables, the variable x and the variable dx. *Figure 5* in Section 3.8 in the text shows you how to interpret this. If $x=a$ and if $dx=x-a$ then the linearization of f at a, $L(x)=f(a)+f'(a)(x-a)$, can be rewritten using differentials as

$$L(x)=f(a)+dy$$

In this setting, dy represents the change in y value on the tangent line corresponding to the change in x value dx. This often is used to approximate the actual change in y value of the function, $f(a+dx)-f(a)$, corresponding to the change in x value dx.

SkillMaster 3.14: Calculate the linearization of a function at a point.

Since linear approximations at $x=a$ give close approximations to a function for values of x close to a, you can use these approximations to get approximations to function values. In a later calculus course, you will learn a more sophisticated technique for approximating functions using higher derivatives.

The linearization can be used to approximate $f(x)$ near $x=a$. For example, if $f(x)=\sqrt{x}$, the linearization at $x=100$ is

$$L(x)=\sqrt{100}+\frac{1}{2\sqrt{100}}(x-100)$$

So an approximation to $\sqrt{101}$ is $L(101) = 10 + \frac{1}{20} = 10.05$. The actual value, rounded to 5 decimal places, is 10.04988. So 10.05 is a reasonable approximation since the difference between 10.5 and 10.4988 is only about 0.0012 .

By using a graphing device, you will be able to determine intervals where the error in the linear approximation is less than some given value.

SkillMaster 3.15: Use differentials to approximate the actual change in a function and to compute relative errors.

If $y = f(x)$ and the differential dy is computed using $x = a$, the resulting value $dy = f'(a) \cdot dx$ is an approximation to the change in y for the function when x changes from a to $a + dx$. For example, using the function $f(x) = \sqrt{x}$ as above and $a = 100$, $dy = f'(100) \cdot dx = \frac{1}{20} dx$. Thus near $x = 100$, when x changes by dx, $\sqrt{x}$ changes by about $\frac{1}{20} dx$.

The *accuracy* of an approximation $L(x)$ to a function value $f(x)$ is within m over an interval if $|L(x) - f(x)| < m$ for all x in the interval. Generally, you would think of an approximation as good if the accuracy is small enough. On the other hand, if the value $v = 0.0002$ which is approximated by $c = 0.003$ the accuracy is < 0.001 which seems small although the approximation does not look very good. To avoid this kind of problem, we define the *relative error* as the proportion of the error to the value or $\frac{c - v}{v}$. In the example, the relative error equals 1 or 100%.

The relative error for $y = f(x)$ when x changes from $x = a$ to $x = a + dx$ is approximately $\frac{dy}{f(a)} = \frac{f'(a) \cdot dx}{f(a)}$. This can be (and often is) expressed as a percentage error as explained in the text.

Worked Examples

For each of the following examples, first try to find the solution without looking at the hint. Cover the right column with a piece of paper. If you need a hint, uncover the column. If you need to see the worked solution, look at the solutions section that follows the examples and hints.

Example	Hint
SkillMaster 3.14.	
1. Suppose you are given only the following information about a function $y = f(x)$: $$\begin{aligned} f(0) &= 1 \\ f'(x) &= 2e^{-x^2}. \end{aligned}$$ Find the linear approximation to $f(x)$ at the point $a = 0$ and estimate $f(0.027)$.	Use the formula for $L(x)$. Then substitute in the value 0.027 for x.
2. Find the linearization $L(x)$ of the function $f(x) = \sqrt[3]{x}$ at $a = 8$. Use this to approximate $\sqrt[3]{8.1}$ and $\sqrt[3]{7.99}$ to 6 decimal places. Estimate the error using a calculator.	First evaluate f and f' at the point $a = 8$. Then write the linearization using the given formula. Substitute in 8.1 and 7.99 for x. Then compare with what you get on a calculator.
3. Find the linear approximation of $f(x) = e^x$ at the point $a = 1$. Then, using a graphing calculator determine the values of x for which this approximation is accurate to within 0.1.	First calculate f and f' at the point $a = 1$, and find $L(x)$.

SkillMaster 3.15.

4. Calculate the relative error in the area of a circular field if the radius is measured to be 50 feet with an accuracy of $\pm.2$ feet.

The area, as a function of radius, is $A(r) = \pi r^2$. Use $dr = \pm.2$ feet at $r = 50$ feet to determine dA. Recall that $dA = A'(r) \cdot dr$.

Solutions to worked examples

1. $f'(0) = 2e^{-0^2} = 2$

$L(x) = f(0) + f'(0)(x-0) = 1 + 2(x-0) = 2x+1$

$L(0.027) = 2(0.027) + 1 = 1.054$

2. $f(x) = x^{1/3} \quad f'(x) = \frac{1}{3}x^{-2/3} f(8) = 2 \quad f'(8) = \frac{1}{3}2^{-2} = \frac{1}{12}$

$L(x) = f(8) + f'(8)(x-8) = 2 + \frac{1}{12}(x-8) = \frac{1}{12}x + \frac{4}{3}$

$L(8.1) = \frac{8.1}{12} + \frac{4}{3} = 2.0083\overline{3} \quad \sqrt[3]{8.1} \approx 2.008333$

The error is $\left|L(8.1) - \sqrt[3]{8.1}\right| < 0.00004$.

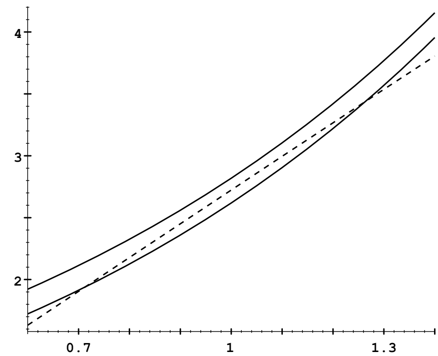
$L(7.99) = \frac{7.99}{12} + \frac{4}{3} = 1.999166\overline{6} \quad \sqrt[3]{7.99} \approx 1.999167$

The error is $\left|L(7.99) - \sqrt[3]{7.99}\right|$ which is less than 0.0000004.

3. $f(1) = e^1 = e \qquad f'(1) = e^1 = e$

$L(x) = f(1) + f'(1)(x-1) = e + e(x-1) = ex$

The graph of $L(x)$ is between the graphs of $f(x) + 0.1$ and $f(x) - 0.1$ for $0.72 < x < 1.26$.



4. $dA = A'(r) \cdot dr = 2\pi r \cdot dr = 2\pi \cdot 50 \cdot (\pm.2) = \pm 20\pi$ ft.^{2}

The accuracy is within 20π ft.^{2} = approximately 62.832 ft.^{2}. The actual area when $r = 50$ feet is 2500πft.^{2}

The relative error is thus $\dfrac{dA}{A} = \dfrac{\pm 20\pi}{2500\pi} = \pm\dfrac{1}{125} = \pm 0.008$.
Note that this is a percentage error of 0.8%.

SkillMasters for Chapter 3

SkillMaster 3.1:	Differentiate polynomials.
SkillMaster 3.2:	Differentiate sums of polynomials and multiples of e^x.
SkillMaster 3.3:	Use the differentiation rules to solve problems.
SkillMaster 3.4:	Differentiate complicated expressions involving products, sums and quotients.
SkillMaster 3.5:	Use the product and quotient rules to solve problems.
SkillMaster 3.6:	Differentiate expressions involving trigonometric functions and use these results to solve problems.
SkillMaster 3.7:	Calculate the derivative of composed functions.
SkillMaster 3.8:	Find tangent lines to parametric curves.
SkillMaster 3.9:	Calculate derivatives of implicit functions.
SkillMaster 3.10:	Calculate derivatives of inverse trigonometric functions.
SkillMaster 3.11:	Find the derivatives of functions involving logarithms.
SkillMaster 3.12:	Calculate derivatives using the technique of logarithmic differentiation.
SkillMaster 3.13:	Use the derivative to solve problems in scientific applications.
SkillMaster 3.14:	Calculate the linearization of a function at a point.
SkillMaster 3.15:	Use differentials to approximate the actual change in a function and to compute relative errors.

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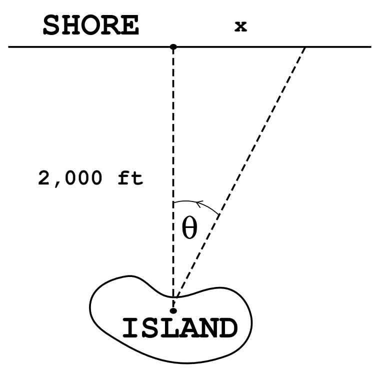
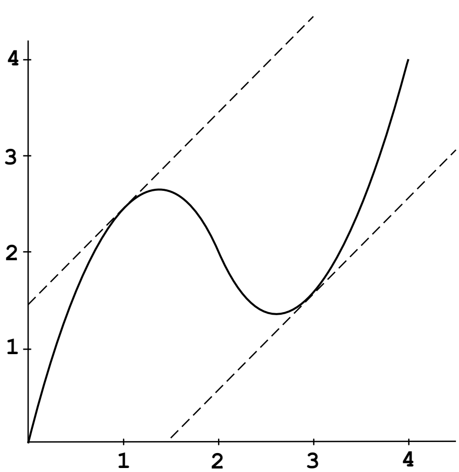
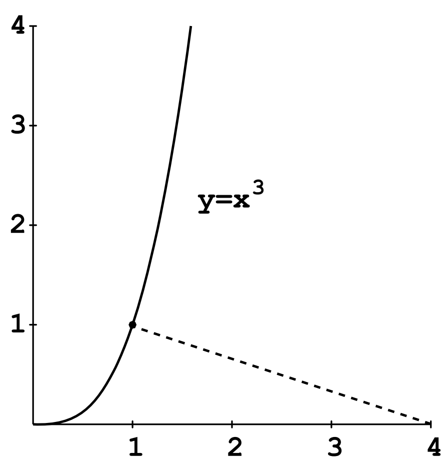
Chapter 4

Applications of Differentiation







4.1 Related Rates

Key Concepts:

- Problems in which two or more rates of change are related

Skills to Master:

- Find an equation that relates quantities that are changing.
- Use the chain rule to find the relationship between rates of change.

Discussion:

Section 4.1 shows you how to express two or more rates of change when the variables are expressed in an equation. Determining the rate of change of one variable, when the rate of change of a related variable is known, is one of the key applications of Differential Calculus. When variables are related, a change in one is accompanied by a change in the other. If an equation relating the variables can be found, the chain rule can often be used to differentiate and find another equation that relates the rates of change of the variables. Now is the time that your work on the *chain rule* and the Rules of Differentiation from Chapter 3 will pay off. Go back and make sure that you can use the Rules of Differentiation with ease.

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Key Concept: Problems in which two or more rates of change are related

The key point is the ability to recognize when two changing variables are related. First, identify the quantities in a problem and assign variables to them. Then determine what rate of change information is given. For example, in a spherical balloon, the surface area S and radius r are related. Suppose you know that the radius is increasing at a rate of 2 cm per second when the radius is 3 cm. You are asked to find the rate at which the surface area is changing when the radius is 3 cm. If the surface area changes, the radius changes, and vice versa. If air is added to the balloon, both S and r change over time and both can be thought of as functions of time.

SkillMaster 4.1: Find an equation that relates quantities that are changing.

Once you recognize that you have a problem in which two or more rates of change are related, the next step is to find an equation relating the quantities that are changing. In the above example where air is being added to a balloon, the related quantities that are changing are S and r. An equation that relates these quantities is

$$S = 4\pi r^2$$

This is the formula for surface area of a sphere in terms of radius.

SkillMaster 4.2: Use the chain rule to find the relationship between rates of change.

This SkillMaster gives the final step in solving related rates problems. First, you recognized that two rates of change were related. Next, you found an equation relating the quantities in question. In the above example, this equation was

$$S = 4\pi r^2$$

Now represent $\dfrac{dS}{dt}$ in terms of $\dfrac{dr}{dt}$ by using the chain rule and differentiating both sides of the above equation with respect to t.

$$\frac{dS}{dt} = \frac{dS}{dr}\frac{dr}{dt} = 8\pi r\frac{dr}{dt}$$

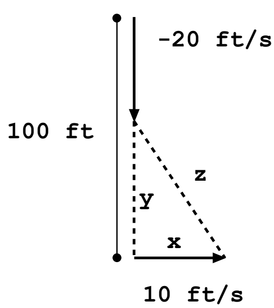
Finally, substitute in the known quantities and solve for $\dfrac{dS}{dt}$.

$$\frac{dS}{dt} = 8\pi r\frac{dr}{dt} = 8\pi \cdot 3 \text{ cm} \cdot 2 \text{ cm/s} = 48\pi \text{cm}^2/\text{s}$$

One thing to check at the end of these problems is that you have answered the original question. This seems obvious, but it is very common to give a great answer to a question different from the one posed.

Worked Examples

For each of the following examples, first try to find the solution without looking at the hint. Cover the right column with a piece of paper. If you need a hint, uncover the column. If you need to see the worked solution, look at the solutions section that follows the examples and hints.

Example	Hint
SkillMaster 4.1.	
1. A circle is expanding so that its area is increasing at a constant rate of 2 cm ^{2}/s. Suppose you need to know the rate of increase of the radius. Find an equation that relates the quantities that are changing.	The quantities to relate are the area of the circle and the radius.
2. A person is walking away from an intersection at a rate of 10 feet per second due east. 100 feet north of the intersection, another person is walking due south at a rate of 20 feet per second. We are interested in how quickly they are moving away from each other. What is the equation that relates the relevant variables? See the diagram below. 	The relevant variables are time, the distances of each person from the intersection and the distance between the two people. Once an equation relating the distance between the two people is obtained, think of the variables, for example x and y as functions of t and differentiate with respect to t. The rates $\dfrac{dx}{dt}$ and $\dfrac{dy}{dt}$ are given in the problem.

SkillMaster 4.2.

3. A circle is expanding so that its area is increasing at a constant rate of 2 cm^2/s. How fast is the radius increasing when the radius is equal to 3 cm?

This is a continuation of Exercise 1 above. You can assign the letter A to represent the area and r to represent the radius. These variables are related by the basic formula

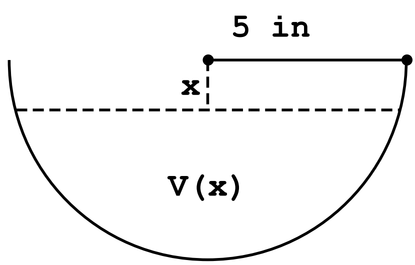
$$A = \pi r^2$$

4. A person is walking away from an intersection at a rate of 10 feet per second due east. 100 feet north of the intersection, another person is walking due south at a rate of 20 feet per second. How quickly are they receding from each other after 1 second?

Diagram this situation carefully before setting up your equation. This is similar to Worked Example 2 above. As before, we know that $\frac{dx}{dt} = 10$ ft/s $\frac{dy}{dt} = -20$ ft/s $z^2 = x^2 + y^2$. After 1 s, $x = 10$ ft and $y = 100 - 20 = 80$ ft.

5. There is a hemispherical depression in a rock ledge. It is 5 inches in radius. A gentle rain falls at the rate of 1 inch per 5 hours. How fast is the depth of water in the depression rising when it is 3 inches from the top? You may use the fact that the volume of water in the depression when it is x inches from the top is

$$V(x) = \frac{250\pi}{3} - 25\pi x + \frac{\pi}{3}x^3$$



The area of the top of the depression is

$$\pi \cdot 5^2 = 25\pi \text{ in}^2.$$

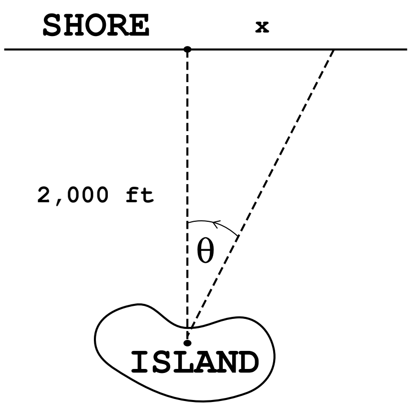
Thus water is collecting in the depression at a rate

$$25\pi \cdot (1/5) = 5\pi \text{ in/h}.$$

6. A company produces blank DVD disks. Suppose that $p(x)$ is the price that would be offered the company for each DVD it produces when x DVDs are produced. Economic theory suggests that the price per DVD will decrease as more DVDs are produced. The total revenue will be the total number of items times the price per item, $R(x) = xp(x)$. Suppose that $p(10000) = \$1.10$, $p'(10000) = -0.00005$, and the quantity being produced is being increased at a rate of 10 per day. How much is the rate of revenue increase if 10000 DVDs are being produced on a given day?

Differentiate implicitly the equation relating the revenue and the price. Substitute 10,000 for x.

7. A search light makes a complete revolution of $2\pi = 360^\circ$ every 3 minutes. It is on an island located 2000 feet from the beach. How fast does the search light go by a point on the beach 1000 feet from the closest point on the beach to the light?



Find an equation relating θ and x. Keep the actual situation in mind as you solve this Worked Example. Check that your answer makes sense.

8. A right triangle is enlarged so that its base is growing at a rate of 2cm/s and its height is growing at a rate of 4 cm/s. How fast is the area growing when the base is 5 cm and the height is 10 cm?

Let b be the base and h be the height. The the area is $A = \dfrac{bh}{2}$. Differentiate this.

Solutions to worked examples

1. If we assign the letter A to represent the area and r to represent the radius then these variables are related by the basic formula giving the area of a circle as a function of its radius.

$$A = \pi r^2$$

2. Let x be the distance the first person is from the intersection and let y be the distance the second person is from this intersection. Let z be the distance between the two people. The information in the statement of the example tells us that

$$\frac{dx}{dt} = 10 \text{ ft/sec} \qquad \frac{dy}{dt} = -20 \text{ ft/sec}$$

The Pythagorean theorem implies

$$z^2 = x^2 + y^2$$

3. You are given

$$\frac{dA}{dt} = 2 \text{ cm}^2$$

Differentiate the basic formula with respect to t.

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

Substitute $r = 3$ and $\frac{dA}{dt} = 2$ then solve for $\frac{dr}{dt}$.

$$2 = 2\pi \cdot 3 \frac{dr}{dt} \qquad \frac{dr}{dt} = \frac{1}{3\pi}$$

So the radius of the circle is increasing at a rate of $1/3\pi$ cm/s.

4. Differentiate with respect to t using the chain rule.

$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \qquad \frac{dz}{dt} = \frac{x}{z}\frac{dx}{dt} + \frac{y}{z}\frac{dy}{dt}$$

We know that $x = 10$ and $y = 80$ so $z = \sqrt{10^2 + 80^2} = \sqrt{6500} = 10\sqrt{65}$. Substitute the known quantities.

$$\frac{dz}{dt} = \frac{10}{10\sqrt{65}}10 + \frac{80}{10\sqrt{65}} \cdot (-20) = \frac{10 - 160}{\sqrt{65}} = \frac{-150}{\sqrt{65}} \approx -18.61 \text{ ft/s}$$

5. We know

$$\frac{dV}{dt} = 5\pi, \qquad \frac{dV}{dx} = -25\pi + \pi x^2, \qquad \text{and } \frac{dV}{dt} = \frac{dV}{dx}\frac{dx}{dt}$$

Substitute.

$$5\pi = (-25\pi + \pi x^2)\frac{dx}{dt}$$

We need to know dx/dt when $x = 3$.

$$5\pi = (-25\pi + \pi 3^2)\frac{dx}{dt} = (-25\pi + 9\pi)\frac{dx}{dt} \qquad \frac{dx}{dt} = -\frac{5}{16} \text{ in/h}$$

Thus the distance between the water level and the top is shrinking at a rate of 5/16 inch per hour so the water is rising at a rate of 5/16 inch per hour.

6. Differentiate $R(x) = x \cdot p(x)$ with respect to t.

$$\frac{dR}{dt} = p(x)\frac{dx}{dt} + x \cdot p'(x)\frac{dx}{dt}$$

Substitute $x = 10000$, $p(10000) = 1.10$, and $p'(10000) = -.00005$.

$$\frac{dR}{dt} = 1.10(10) + 10000(-0.00005)(10) = 11.00 - 5 = \$6.00$$

This means the company will take in an additional \$6 of revenue the next day.

7. If θ is the angle that the search light makes with the line to the closest point on the beach, and if x is the distance from where the light touches the beach to this closest point then we are given

$$\frac{d\theta}{dt} = \frac{2\pi}{3} \text{ rad/min and } \tan(\theta) = \frac{x}{2000}\ .$$

Differentiate both sides with respect to t to get

$$\sec^2(\theta)\frac{d\theta}{dt} = \frac{1}{2000}\frac{dx}{dt}$$

Substitute known values in this expression.

$$\left(\frac{1000}{1000\sqrt{5}}\right)^2 \frac{2\pi}{3} = \frac{1}{2000}\frac{dx}{dt} \qquad \text{So } \frac{dx}{dt} = 2000\frac{2\pi}{15} \approx 838 \text{ ft/min}$$

8. $\dfrac{dA}{dt} = \dfrac{h}{2}\dfrac{db}{dt} + \dfrac{b}{2}\dfrac{dh}{dt}$

Substitute the known quantities $b = 5, h = 10, db/dt = 2, dh/dt = 4$.

$$\frac{dA}{dt} = \frac{10}{2}2 + \frac{5}{2}4 = 10 + 10 = 20$$

The area is growing at a rate of 20 cm^2/s.

4.2 Maximum and Minimum Values

Key Concepts:

- Local maximum and minimum values, Fermat's theorem
- Absolute maximum and minimum, extreme values, and the Closed Interval Method

Skills to Master:

- Sketch graphs fitting specified information about maximum and minimum values.
- Find critical numbers, and absolute and local maximum and minimum values for continuous functions given by a graph.
- Find critical numbers and absolute and local maximum and minimum values for continuous functions given by an equation.

Discussion:

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Section 4.2 gives one of the most important applications of Differential Calculus. How do you optimize a quantity? That is, how to do you adjust variables you can control so that the dependent variable is maximized (if you want as much of it as possible) or so that the dependent variable is minimized (if you want as little as possible, such as cost). As Stewart points out, many applications of Calculus to other disciplines involve finding the maximum or minimum value of some physical quantity. The *Extreme Value Theorem* tells us that a continuous function on a closed interval $[a,b]$ attains an absolute maximum value and an absolute minimum value. Learn the meaning of the terms absolute minimum, absolute maximum, local minimum, local maximum and critical number.

A common mistake is failing to distinguish between the point c at which a function f attains an extreme value and the actual value of the extreme $f(c)$. Be careful not to make this mistake. Remember, the maximum or minimum occurs at c and the maximum or minimum value of f is at $f(c)$.

Key Concept: Local maximum and minimum values, Fermat's theorem

A very important theorem is *Fermat's Theorem* which states that:

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> If f has a local maximum or local minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.

Make sure that you understand the meaning of local maximum and local minimum. A function f has a *local maximum* at $x = c$ if $f(c) \geq f(x)$ for values of x near c. Note that the local maximum is $f(c)$ and the place where the maximum occurs is $x = c$. A *local minimum* is similarly defined.

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Key Concept: Absolute maximum and minimum, critical numbers, and the Closed Interval Method

Fermat's Theorem implies that if f has a local maximum or local minimum at c, then either $f'(c) = 0$ or $f'(c)$ does not exist. A number in the domain where one of these two things happens is called a critical number of f. Critical numbers are candidates for absolute maximum or absolute minimum: either they have a horizontal tangent or else something strange happens such as a discontinuity, a corner point, etc.

This leads to the following procedure, called the Closed Interval Method, for finding the absolute maximum and minimum values of a continuous function f defined on a closed interval $[a,b]$:

> 1. Find the values of f at all critical numbers of f in (a,b).
> 2. Find the values of $f(a)$ and $f(b)$.
> 3. The largest of the values from steps 1 and 2 is the absolute maximum value, and the smallest of the values from steps 1 and 2 is the absolute minimum value.

It is easy to forget to check the values of the function at the endpoints of the interval when you are trying to find the absolute maximum and minimum values. Don't forget that extreme values often occur at the endpoints of the domain.

SkillMaster 4.3: Sketch graphs fitting specified information about maximum and minimum values.

The key to this SkillMaster is understanding the meaning of the various terms. Review the meaning of the following terms now.

- Continuous function
- Differentiable
- Local and global maximum and minimum
- Critical number

Make certain that you have a clear understanding of the meaning of these four concepts because questions will asked using this terminology.

SkillMaster 4.4: Find critical numbers, and absolute and local maximum and minimum values for continuous functions given by a graph.

Again, the key to this SkillMaster is understanding the meaning of the various terms. If you are presented with a graph, it should be straightforward to find the absolute and local maximum and minimum values. You will also need to remember that a critical number is any number corresponding to a local maximum or minimum, or any number where the function is not differentiable. Places where the function is not differentiable occur where the function is not continuous, where the graph of the function has a vertical tangent line, a corner point, an end point, or some other strange behavior.

SkillMaster 4.5: Find critical numbers, and absolute and local maximum and minimum values for continuous functions given by an equation.

For this SkillMaster, use the Closed Interval Method. If you are given a continuous function on a closed interval, differentiate the given function and find each value where the derivative is 0. Then find each number where the derivative is undefined. You have now found all of the critical numbers. Each absolute maximum and minimum are either at a critical number or at an endpoint of the closed interval. You need to compute the function value at each critical number and at the endpoints to determine where the absolute maximum and minimum occur, and what the absolute maximum and minimum values are.

Worked Examples

For each of the following examples, first try to find the solution without looking at the hint. Cover the right column with a piece of paper. If you need a hint, uncover the column. If you need to see the worked solution, look at the solutions section that follows the examples and hints.

Example	Hint

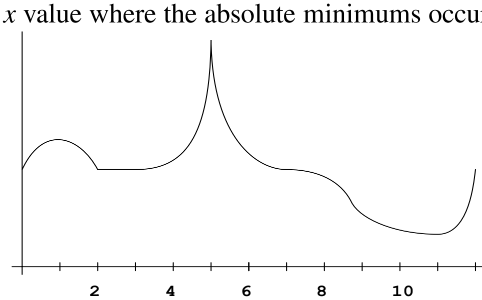
SkillMaster 4.3.

1. Sketch a graph of $y = f(x)$ with domain $[0,4]$ and the following properties:
(a) $f(0) = 5, f(1) = 0, f(2) = 2, f(4) = 1$,
(b) the absolute maximum is at $x = 0$,
(c) the function is differentiable on its domain except at $x = 2$,
(d) there is a local maximum at $x = 2$,
(e) there is an absolute minimum at $x = 1$, and
(f) there are no other critical numbers (except at $x = 1, x = 2$).

Hint: First fill in the known points on a graph. There should be a corner at $x = 2$ to ensure that the function is not differentiable there. The absolute maximum is at the endpoint $x = 0$ and an absolute minimum at $x = 1$, so the graph must stay between these points.

SkillMaster 4.4.

2. In the graph of the function find:
(a) all points where $f'(x) = 0$,
(b) all critical points,
(c) all x values where local minima occur,
(d) all x values where local maxima occur,
(e) the x value where the absolute maximum occurs,
(f) the x value where the absolute minimums occur.

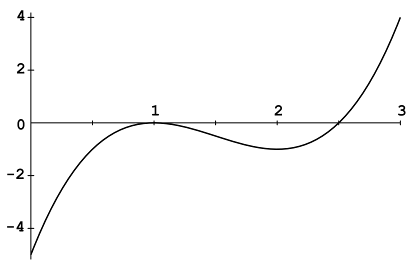


Hint: Notice that there are no tangent lines at $x = 2$ or $x = 5$. There is no derivative at these points and they are critical points. On the other hand, there is a horizontal tangent for all x values in $(2,3]$ as well as at $x = 1, x = 7$ and $x = 11$.

SkillMaster 4.5.

3. Find the critical numbers of the function $y = f(x)$ shown below with domain on the interval $[0,3]$. Find the absolute maximum and absolute minimum values.

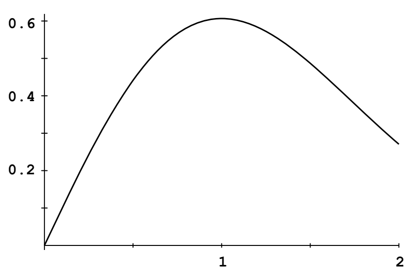
$$f(x) = 2x^3 - 9x^2 + 12x - 5$$



The critical values are the x-values for which $f'(x) = 0$ or else where $f'(x)$ is undefined. This function is a polynomial so $f'(x)$ is always defined. The critical points are only the x values where $f'(x) = 0$.

4. Find the critical numbers of the function $y = f(x)$ below with domain on the interval $[0,2]$. Find the absolute maximum and absolute minimum values.

$$f(x) = xe^{-x^2/2}$$

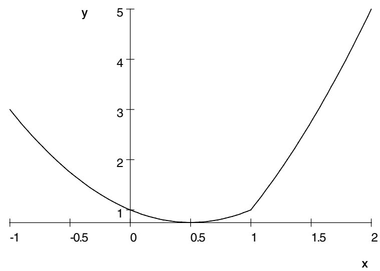


For the absolute maximum and minimum values, check the critical points and the endpoints.

5. Find all critical values and the absolute maximum and absolute minimum of

$$|x-1|+x^2$$

on the interval $[-1,2]$.



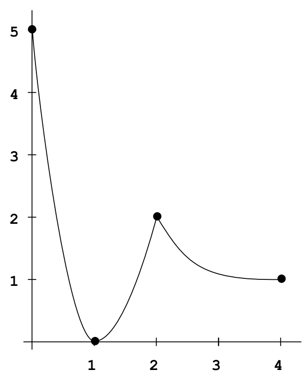
Express the function in piecewise notation:

$f(x) = 1 - x + x^2$ if $x \leq 1$,

$f(x) = x - 1 + x^2$ if $x > 1$.

Solutions to worked examples

1.



2. (a) $x = 1, 7, 11$ and $x \in (2,3]$.

(b) $x = 1, 7, 11, x \in (2,3]$, and $x = 2, 5$.

(c) $[2,3], 11$ are places where local minima occur.

(d) $1, 5$ are places where local maxima occur.

(e) The absolute maximum occurs at $x = 5$.

(f) The absolute minimum occurs at $x = 11$.

3. $f'(x) = 6x^2 - 18x + 12 = 6(x^2 - 3x + 2) = 6(x-1)(x-2)$

Thus $f'(x) = 0 \Leftrightarrow x = 1$ or $x = 2$.

The absolute maximum and minimum must occur at critical points or at endpoints of the domain.

x	0	1	2	3
$f(x)$	-5	0	-1	4

The smallest number in the $f(x)$ row is -5 at $x = 0$, so this is the absolute minimum; the largest number in the $f(x)$ row is 4 at $x = 3$, so this is the absolute maximum.

4. $f'(x) = x\dfrac{d}{dx}(e^{-x^2/2}) + \dfrac{d}{dx}(x)e^{-x^2/2} = xe^{-x^2/2}\dfrac{d}{dx}(-x^2/2) + e^{-x^2/2}$

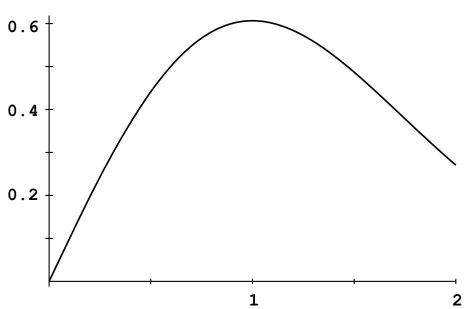
$= xe^{-x^2/2}(-x) + e^{-x^2/2} = (1-x^2)e^{-x^2/2}$

Since $e^{-x^2/2}$ is always positive the $f'(x) = 0$ if and only if

$0 = (1-x^2)$, or $x^2 = 1$, or $x \pm = 1$. The only critical value is $x = 1$ since $x = -1$ is not in the domain. Make a table of endpoints, critical values and their function values.

x	$f(x)$
0	0
1	$e^{-1/2} \approx 0.6065$
2	$2e^{-2} \approx 0.2707$

The absolute maximum is $2e^{-2}$ at $x = 1$ and the absolute minimum is 0 at $x = 0$.



5. The derivative is

$$f'(x) = \begin{cases} -1 + 2x & \text{if} \quad x < 1 \\ 1 + 2x & \text{if} \quad x > 1 \end{cases}$$

The function is not differentiable at $x = 1$ because the derivative from the left at $x = 1$ is $-1+2(1) = 1$ which is not equal to the derivative from the right at $x = 1$ which is $1+2(1) = 3$. Therefore $x = 1$ is a critical number.

Other critical numbers are those x where $f'(x) = 0$. First consider $x < 1$ and set

$$\begin{aligned} f'(x) &= -1+2x = 0 \\ 2x &= 1 \\ x &= 1/2 \end{aligned}$$

We check that $x = 1/2 < 1$ so that it lies in the part of the domain that this piece describes. So 1/2 is a critical number.

Now consider $x > 1$ and set

$$\begin{aligned} f'(x) &= 1+2x = 0 \\ 2x &= -1 \\ x &= -1/2 \end{aligned}$$

This is NOT a critical number since $-1/2 < 1$, so $-1/2$ is not in this part of the domain. The critical values are 1/2 and 1 and the endpoints are -1 and 2. To find the absolute maximum and absolute minimum make table of the critical values and the corresponding function values:

x	$\lvert x-1\rvert+x^2$
-1	3
$1/2$	$3/4$
1	1
2	5

The absolute maximum is 5 and the absolute minimum is $3/4$.

4.3 Derivatives and the Shapes of Curves

Key Concepts:

- The Mean Value Theorem
- Using the first derivative to test for increasing, decreasing, and local maxima and minima
- Using the second derivative to test concavity and for local maxima and minima

Skills to Master:

- Compute the values given by the Mean Value Theorem.
- Find information about the function f from the graphs of f, f' and f''.
- Find information about the function f by applying the first and second derivative tests.

Discussion:

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Section 4.3 develops the visual properties that derivatives tell you about the *shape of the graph.* In Section 2.9 you learned that functions with positive derivatives are increasing and functions with negative derivatives are decreasing. In this section, this result is derived from a very important theorem of Differential Calculus called the Mean Value Theorem. In *Section 2.9,* you also learned about functions that were concave upward or concave downward on certain intervals. These results are reemphasized in this section. After working through the problems in this section, you should have a much better understanding of how the first and second derivatives of a function are related to the shape of the graph of the function.

Key Concept: The Mean Value Theorem

This theorem is a central theorem in Differential Calculus and many other results can be derived from it. The theorem states that if f is differentiable on the interval $[a,b]$, then there exists a number c between a and b such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

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Study *Figures 1 and 2* in this section to get a better geometric understanding of what this theorem is saying. In the special case where $f(a) = f(b)$, this theorem states that there is a number c between a and b where $f'(c) = 0$. This special case can be derived from the Extreme Value Theorem and Fermat's Theorem as follows. If f is constant on $[a,b]$, then $f'(c) = 0$ for all numbers c between a and b; each domain point is both an absolute maximum and an absolute minimum. If f is not constant, the *Extreme Value Theorem* implies that there is a local maximum or local minimum at some point c between a and b. *Fermat's Theorem* then implies that $f'(c) = 0$. In more advanced mathematics courses, the general case of the Mean Value Theorem is derived from this special case.

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Key Concept: Using the first derivative to test for increasing, decreasing, and local maxima and minima

Make sure that you understand how the *Increasing/Decreasing Test* or the *I/D Test* is derived from the Mean Value theorem. The first derivative test follows from the Increasing/Decreasing test. Let c be a critical number of a continuous function f.

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(a) If, at c, f' changes from positive to negative then f has a local maximum.

(b) If, at c, f' changes from negative to positive then f has a local minimum.

(c) If f' does not change signs at c, f has no local maximum or minimum at c.

Study *Figure 4* in this section to gain a geometric understanding of this result.

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Key Concept: Using the second derivative to test concavity and for local maxima and minima

The second derivative test gives a way to test algebraically for local minimum and local maximum values. It states that if f'' is continuous near c, and if $f'(c) = 0$, then :

(a) If $f''(c) > 0$, then f has a local minimum at c.

(b) If $f''(c) < 0$, then f has a local maximum at c.

Note that the test gives no information if $f''(c) = 0$. One way to remember this test is to apply the test to the functions $f(x) = x^2$ and $g(x) = -x^2$. Since $f''(x) = 2$ for all x

and since $f'(0) = 0$, f has a local minimum at 0. Since $g''(x) = -2$ for all x and since $g'(0) = 0$, g has a local maximum at 0.

SkillMaster 4.6: Compute the values given by the Mean Value Theorem.

Given a differentiable function f defined on a closed interval $[a,b]$, this SkillMaster requires you to find values c between a and b where $f'(c) = \dfrac{f(b)-f(a)}{b-a}$. This can be done graphically by finding points on the graph of the function that have tangent lines with the required slope. This can also be done algebraically by finding the derivative of the function and then determining which values of c give the right value.

SkillMaster 4.7: Find information about the function f from the graphs of f, f′, and f″.

This SkillMaster will be useful in solving the problems of the next section. Make sure that you understand the correspondence between where f' is positive or negative and where f is increasing or decreasing. Also make sure that you understand the relation between where f'' is positive or negative and where f is concave upward or concave downward.

SkillMaster 4.8: Find information about the function f by applying the first and second derivative tests.

The types of information that you can be asked to find include:

- intervals where the function is increasing or decreasing,
- local and/or global maximum and minimum values of the function,
- intervals where the graph of the function is concave upward or concave downward, and
- inflection points.

In each case, if the function is given by an equation, take the first and second derivatives and apply the first derivative test or the second derivative test as needed to get the required information. Review the definition of *inflection point* if needed.

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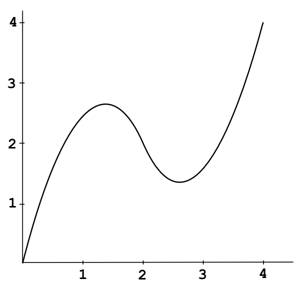
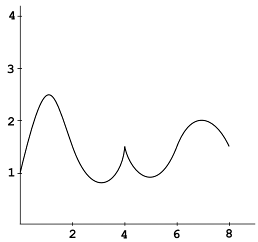
Worked Examples

For each of the following examples, first try to find the solution without looking at the hint. Cover the right column with a piece of paper. If you need a hint, uncover the column. If you need to see the worked solution, look at the solutions section that follows the examples and hints.

Example	Hint
SkillMaster 4.6.	
1. Estimate all values of c that satisfy the Mean Value Theorem for the function $y = f(x)$ on the interval $[0,4]$ pictured below. 	Draw a line between the end-points of the curve. Find all points whose tangent is parallel to this line.
2. Suppose that you get onto the state turnpike, which has a speed limit of 55 miles per hour, at 1:05 pm. You get off on an off-ramp 90 miles down the road at 2:35 pm the same day. A state trooper is at the toll booth where you get off and immediately gives you a speeding ticket. Is the trooper justified?	Let $p(t)$ = miles traveled where t = length of time (in hours) on the turnpike. So $p(0) = 0$ and $p(1.5) = 90$. Apply the Mean Value Theorem.

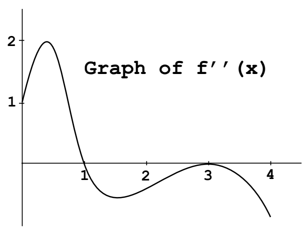
SkillMaster 4.7.

3. In the graph of $y = f(x)$ below estimate the intervals where f is concave upward, concave downward, and where there are inflection points.



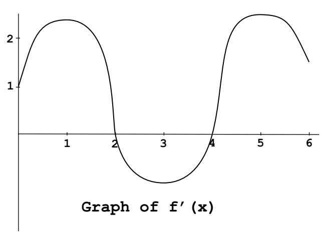
Determine the concavity of the curve by imagining that you are driving a car on the curve from left to right. The graph is concave upward if you are veering to the left and concave downward if you are veering to the right. If you are switching from right to left or from left to right then you are at an inflection point.

4. The graph of $f''(x)$, the second derivative of a function $f(x)$, is shown below. Where is the graph of the original function $y = f(x)$ concave upward, concave downward, and where is there an inflection point?



The graph of $y = f(x)$ is concave upward whenever $f''(x) > 0$ and concave downward whenever $f''(x) < 0$.

5. The graph of the derivative $f'(x)$ is shown. Find the intervals where f is increasing and decreasing. Where does f have a local maximum and local minimum? Where is f concave upward or downward? Where are there inflection points of f?



The function f is increasing when $f' > 0$ and decreasing when $f' < 0$. There is a local maximum when f' changes sign from positive to negative, and there is a local minimum when f' changes sign from negative to positive. The graph is concave upward when $f'' > 0$ or when f' is increasing.

SkillMaster 4.8.

6. For the indicated function, find the intervals where f is increasing or decreasing.

$$f(x) = 2x^3 - 9x^2 + 12x - 5$$

Take the first derivative and apply the first derivative test.

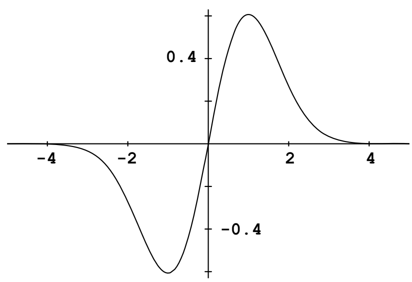
7. In the graph of $y = f(x)$ find intervals of concavity, and inflection points.

$$f(x) = 2x^3 - 9x^2 + 12x - 5$$

Take the second derivative and use the second derivative test.

8. In the graph of $y = f(x)$ find where there are local maxima and minima, intervals of concavity, and inflection points

$$f(x) = xe^{-x^2/2}$$



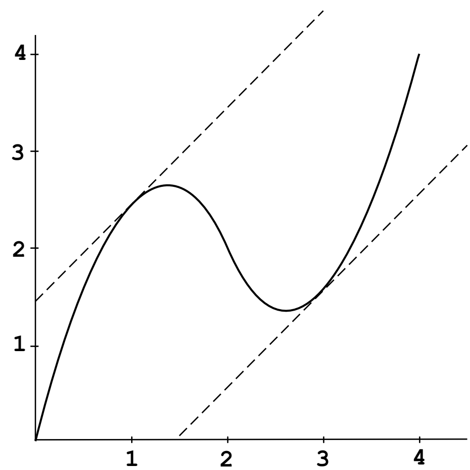
First notice that f is odd, that is, $f(-x) = -f(x)$.

9. Suppose that the functions $f(x)$ and $g(x)$ are both differentiable and positive. Suppose that the first derivative test shows that both $f(x)$ and $g(x)$ are decreasing, and the second derivative test shows that both $f(x)$ and $g(x)$ are concave upward. Let $h(x) = f(x)g(x)$. Show that h is positive, decreasing, and concave upward.

Take the first and second derivatives of $h(x) = f(x)g(x)$ using the product rule.

Solutions to worked examples

1. The values of c that work are $c = 1$ and $c = 3$.



2. The derivative $p'(t) = v(t)$ the velocity. The Mean Value Theorem says there is a time c between 0 and 90 so that

$$v(c) = \frac{p(1.5) - p(0)}{1.5 - 0} = \frac{90}{1.5} = 60 \text{ m/h}$$

At this time c you were going 60 miles per hour which is faster than the speed limit. The trooper is justified in giving you a ticket.

3. The curve is concave downward on the intervals $(0,2)$ and $(6,8)$. The curve is concave upward on the intervals $(2,4)$ and $(4,6)$. There are inflection points at x = 2 and 6.

4. The graph of $y = f(x)$ is concave upward on $(0,1)$, concave downward on $(1,4)$, and has an inflection point at $x = 1$.

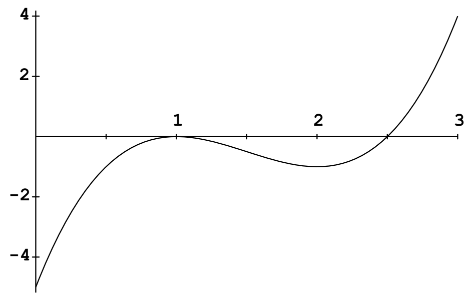
5. The function f is increasing on $(0,2)$ and $(4,6)$, decreasing on $(2,4)$, has a local maximum at $x = 2$ and a local minimum at $x = 4$. It is concave upward on $(0,1)$ and $(3,5)$. It is concave downward on $(1,3)$ and $(5,6)$. There are inflection points at $x = 1,3$, and 5.

6. $f'(x) = 6x^2 - 18x + 12 = 6(x-1)(x-2)$.

So f' is positive for $x < 1$ and $x > 2$ and is negative for $1 < x < 2$. The function is increasing for $x < 1$, has a local maximum at $x = 1$, is decreasing for $1 < x < 2$, has a local minimum at $x = 2$, and increases for $x > 2$.

7. $f''(x) = 12x - 18 = 6(2x - 13)$. So $f'(x) = 0$ when $3x - 2 = 0$ or $x = 1.5$.

So f'' is negative for x less than 1.5 where the graph of f is concave downward and positive for x greater than 1.5 where the graph of f is concave upward. There is an inflection point at $x = 1.5$.



8. $f'(x) = x\dfrac{d}{dx}(e^{-x^2/2}) + \dfrac{d}{dx}(x)e^{-x^2/2} = x(-x)e^{-x^2/2} + (1)e^{-x^2/2} = (1-x^2)e^{-x^2/2}$.

Since $e^{-x^2/2}$ is always positive, $f'(x) < 0$ if $x < 1$ or if $x > 1$, where f is decreasing, and $f'(x) > 0$ if $-1 < x < 1$, where f is increasing. There is a local maximum at $x = 1$ and a local minimum at $x = -1$.

$$\begin{aligned} f''(x) &= (1-x^2)\frac{d}{dx}(e^{-x^2/2}) + \frac{d}{dx}(1-x^2)e^{-x^2/2} = (1-x^2)(-x)e^{-x^2/2} + (-2x)e^{-x^2/2} \\ &= (-2x - x + x^3)e^{-x^2/2} = x(x^2-3)e^{-x^2/2} = x(x-\sqrt{3})(x+\sqrt{3})e^{-x^2/2} \end{aligned}$$

So f has inflection points at $x = 0, x = \pm\sqrt{3}$. It is concave downward on $(-\infty, -\sqrt{3})$ and$(0, \sqrt{3})$ and concave upward on $(-\sqrt{3}, 0)$ and $(\sqrt{3}, \infty)$.

9. The product of two positive numbers is positive, so the function $h(x) = f(x)g(x)$ is also positive because both $f(x)$ and $g(x)$ are positive.

$$h'(x) = f(x)g'(x) + f'(x)g(x)$$

Now $f'(x) < 0$ because of the first derivative test and $g(x) > 0$, so $f'(x)g(x) < 0$. Similarly $f(x)g'(x) < 0$. Thus

$$h'(x) < 0$$

so h is decreasing.

Now take the second derivative.

$$
\begin{aligned}
h''(x) &= \frac{d}{dx}(h'(x)) = \frac{d}{dx}(f(x)g'(x) + f'(x)g(x)) \\
&= [f(x)\frac{d}{dx}(g'(x)) + \frac{d}{dx}(f(x))g'(x)] + [f'(x)\frac{d}{dx}(g(x)) + \frac{d}{dx}(f'(x))g(x)] \\
&= f(x)g''(x) + f'(x)g'(x) + f'(x)g'(x) + f''(x)g(x) \\
&= f(x)g''(x) + f''(x)g(x) + 2f'(x)g'(x)
\end{aligned}
$$

Now $f''(x) > 0$ because of the second derivative test and $g(x) > 0$ is given. So $f''(x)g(x) > 0$. The first derivative test tells you that both $f'(x) < 0$ and $g'(x) < 0$, so $f'(x)g'(x) > 0$. Likewise $f(x)g''(x) > 0$. Each term in the expression of $h''(x)$ is positive, so $h''(x) > 0$. The second derivative test implies h is concave upward.

4.4 Graphing With Calculus and Calculators

Key Concepts:

- Using calculus to refine calculator generated graphs
- Investigating families of curves

Skills to Master:

- Analyze the graphs of f' and f'' to produce more accurate graphs of f.
- Observe trends in families of curves and observe transitional values for which the basic shape of the curves changes.
- Analyze graphs of functions given parametrically.

Discussion:

Section 4.4 expands on some of the ideas from *Section 1.4* and shows you how to combine Calculus and graphing devices to obtain accurate information about functions. One thing to watch out for is the fact that the initial viewing rectangle produced by your calculator or computer may not be scaled correctly to provide all the desired information about a function under consideration. You may need to re-scale, graph the derivative, and use Calculus to analyze the function before you can be sure that you have have seen all the scales necessary to reflect all the important information about a function.

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Key Concept: Using calculus to refine calculator generated graphs

The initial viewing rectangle produced by a calculator or computer may not show some important features of a function. This can happen for a number of reasons. The default choice of which x-values to use may miss some important values. If the function has a large range of values, the scaling may prevent you from seeing some important features. By using calculus and graphing the first and second derivatives, you can determine whether or not important features have been missed.

Key Concept: Investigating families of curves

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Example 5 in the text analyzes and explains how the graph of $f(x) = \dfrac{1}{x^2 + 2x + c}$ changes as c varies. Pay careful attention to this example. By taking the first and second derivative of families of curves with a varying parameter, and by correctly interpreting what the first and second derivatives tell you about the graphs, you can analyze how the graphs vary as the parameter varies. In the worked out example for SkillMaster 4.10 below, another example is presented. Make sure that you understand these examples.

SkillMaster 4.9: Analyze the graphs of f$'$ and f$''$ to produce more accurate graphs of f .

In the previous sections, you learned how f' and f'' give you information about where f is increasing, decreasing, concave upward or concave downward. By graphing f' and finding places where the graph changes from positive to negative or negative to positive, you can determine where f has local maximum and local minimum points. By graphing f'' and finding places where the graph changes from positive to negative or negative to positive, you can determine where f changes concavity.

SkillMaster 4.10: Observe trends in families of curves and observe transitional values for which the basic shape of the curve changes.

A family of related curves can be given by varying the coefficients of some of the terms of the equations giving the curves. As a simple example, consider the family of curves $f(x) = ax^2 + b$. When a is positive, the graph of $f(x)$ is a parabola opening upwards. When a is negative, the graph of $f(x)$ is a parabola opening downwards. When $a = 0$, the graph of $f(x)$ is a horizontal line with y-intercept b. The effect of varying b is to raise or lower the parabolas or horizontal lines. This is a simple example that you can analyze without using calculus because you know about parabolas and their translations and scale changes. For more complicated examples, use the first and second derivatives to gain information about the shapes of the graphs.

SkillMaster 4.11: Analyze graphs of functions given parametrically.

If the x and y values of points on a curve are given in terms of a parameter t, you can visualize the curve being traced out as t varies in the positive direction. By using the relation

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

and by differentiating the expressions $y = y(t)$ and $x = x(t)$, you can find information about the slope of tangent lines to the given curve. Any function $y = f(x)$ can be given parametrically by letting $x(t) = t$, $y(t) = f(t)$.

Worked Examples

For each of the following examples, first try to find the solution without looking at the hint. Cover the right column with a piece of paper. If you need a hint, uncover the column. If you need to see the worked solution, look at the solutions section that follows the examples and hints.

Example	Hint
SkillMaster 4.9.	
1. Draw the graph of the function $$f(x) = e^{\sin(\pi x)}$$ in a viewing rectangle that shows all the features of the graph. Estimate (to one decimal place) the minimum and maximum values and the intervals of concavity using calculus.	Note that this is a periodic function with period 2.
2. Show the graph of $$f(x) = \frac{x(x-1)}{(x+1)^2(x-3)}$$ in a viewing rectangle that shows the main features. Do not use derivative tests.	Look for asymptotes first. Then graph using the default viewing rectangle. Adjust this viewing rectangle using the information about asymptotes.

SkillMaster 4.10.

3. Consider the family of functions defined by

$$\frac{1}{x^2+cx+1}$$

for $c \geq 0$. Describe intervals of increase, local maximum and minimum values and asymptotes as c varies.

Use the first derivative test to find intervals of increase and to find local maximum and minimum values.
To find the vertical asymptotes use the quadratic formula to find the zeros of the denominator.
Graph the function for several values of c.

SkillMaster 4.11.

4. Graph the curve with parametric equations

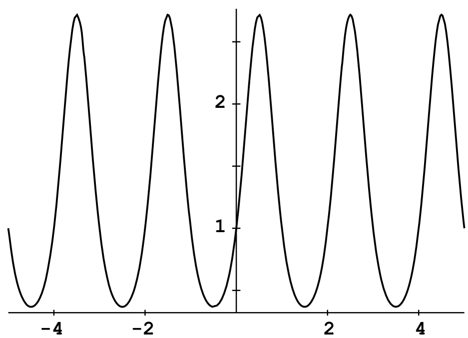
$$\begin{aligned} x &= t^2-4t \\ y &= t^4-t^2+1 \end{aligned}$$

in a viewing rectangle that displays the important features of the graph. Give the coordinates of the significant points on the curve.

First graph in the default viewing rectangle, then adjust. Find maxima and minima using the chain rule to find $dy/dx = (dy/dt)/(dx/dt)$. Find vertical asymptotes by finding where $dx/dt = 0$.

Solutions to worked examples

1. First graph this in a viewing rectangle that appears appropriate after re-sizing.



The function appears periodic and it is:

$$f(x+2) = e^{\sin(\pi(x+2))} = e^{\sin(\pi x+2\pi)} = e^{\sin(\pi x)} = f(x).$$

Now focus on a fundamental period, say for x in $[0,2]$. The first derivative is

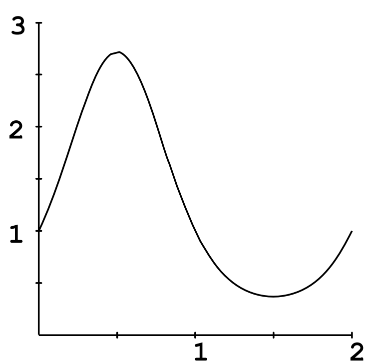
$$f'(x) = e^{\sin(\pi x)}\frac{d}{dx}(\sin(\pi x)) = \pi\cos(\pi x)e^{\sin(\pi x)}$$

Since the exponential function is always positive

$f'(x) > 0$ if and only if $\cos(\pi x) > 0$ if and only if $0 < x < 1/2$ or $3/2 < x < 2$.

Similarly, $f'(x) < 0 \Leftrightarrow 1/2 < x < 3/2$.

There is a maximum at $x = 1/2$ with a value of $e^{\sin(\pi/2)} = e^1 \approx 2.7$ and a minimum at $x = 3/2$ with a value of $e^{\sin(3\pi/2)} = e^{-1} \approx 0.4$. This is shown in the graph.



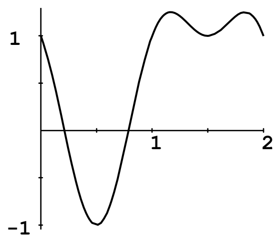
To estimate the regions of concavity compute the second derivative.

$$f''(x) = \pi\frac{d}{dx}(\cos(\pi x)e^{\sin(\pi x)}) = \pi\cos(\pi x)\frac{d}{dx}(e^{\sin(\pi x)}) + \pi e^{\sin(\pi x)}\frac{d}{dx}(\cos(\pi x))$$
$$= \pi\cos(\pi x)[\pi\cos(\pi x)e^{\sin(\pi x)}] + \pi e^{\sin(\pi x)}[(-\pi\sin(\pi x))]$$
$$= \pi^2 e^{\sin(\pi x)}[\cos^2(\pi x) - \sin(\pi x)]$$

The factors outside the expression in the square brackets are always positive, so

$f''(x) > 0$ if and only if $\cos^2(\pi x) - \sin(\pi x) > 0$

This inequality is not easy (or possible) to solve exactly, so estimate the needed points by graphing $\cos^2(\pi x) - \sin(\pi x)$.

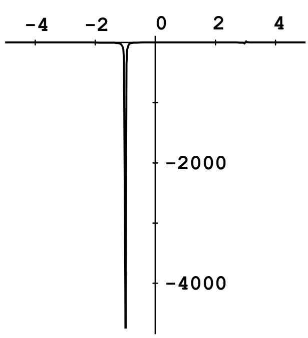


The graph is negative for x between about 0.2 and 0.8 and nonnegative elsewhere. Thus there are inflection points of the original function at about 0.2 and 0.8. The original graph is concave downward between 0.2 and 0.8; it has inflection points at $x = 0.2$ and $x = 0.8$, and is concave upward between 0 and 0.2 as well as between 0.8 and 2.

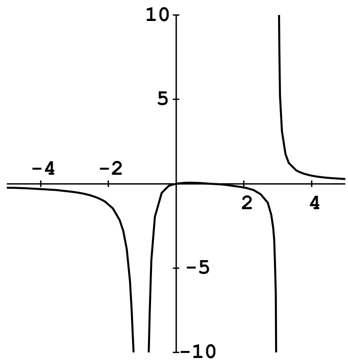
2. $\displaystyle\lim_{x\to\pm\infty} f(x) = \lim_{x\to\infty} \frac{x(x-1)}{(x+1)^2(x-3)} = 0$

because the degree of the denominator (3) is one higher than the degree of the numerator (2). The denominator has roots at $x = -1$ and $x = 3$, so there are vertical asymptotes there.

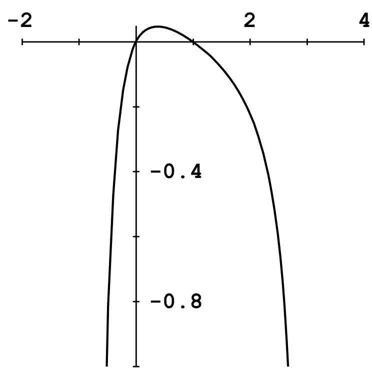
Now graph in the default viewing rectangle.



There in not much detail but you can see something unusual at $x = -1$. The vertical scale is so large because of the asymptotes. Re-scale the viewing rectangle so the vertical scale is moderate and the horizontal scale is just large enough to cover the asymptotes.



This is better and shows the main features. The behavior of the graph between $x = 0$ and $x = 2$ is still not clear. Once more adjust the vertical scale to see this part of the graph.



The last two graphs together give a complete picture of the behavior of the function.

3.

$$f'(x) = \frac{-(2x+c)}{(x^2+cx+1)^2}$$

This is positive if and only if the numerator is positive because the denominator is positive for each c.

$f'(x) > 0$ if and only if $-(2x+c) > 0$ if and only if $2x+c < 0$ if and only if $x < -c/2$

Likewise

$$f'(x) < 0 \Leftrightarrow x > -c/2$$

Thus f is increasing for each $x < -c/2$ in its domain, decreasing for each $x > -c/2$ in its domain and has a local maximum at $-c/2$.

The quadratic formula says the denominator has a 0 if and only if $c^2 - 4 \geq 0$ if and only if $c^2 \geq 4$ if and only if $c \geq 2$, since c is positive. If $0 \leq c < 2$ then there are no vertical asymptotes. If $c \geq 2$ then there are asymptotes at

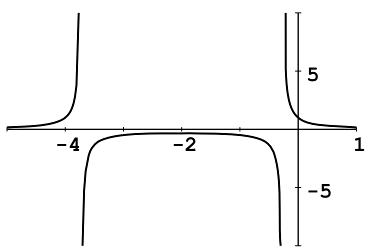
$$x = \frac{-c \pm \sqrt{c^2 - 4}}{2}$$

Note that if $c = 2$ then there is only one asymptote at $x = -1$.

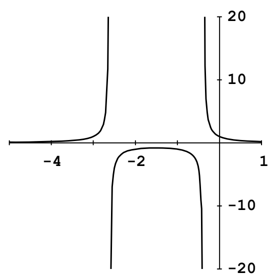
There are horizontal asymptotes at $y = 0$ since

$$\lim_{x\to\infty} \frac{1}{x^2 + cx + 1} = \lim_{x\to-\infty} \frac{1}{x^2 + cx + 1} = 0$$

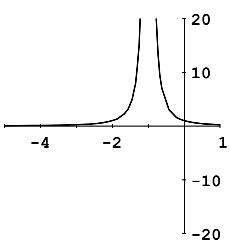
for every c.



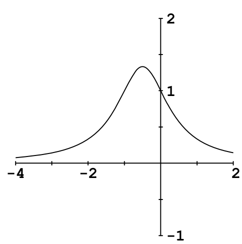
$c > 2$



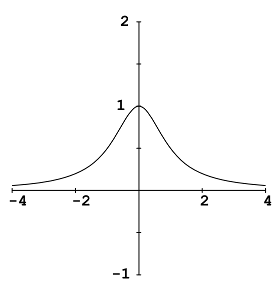
$c > 2$



$c = 2$



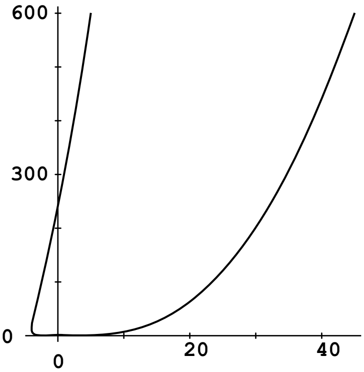
$c < 2$

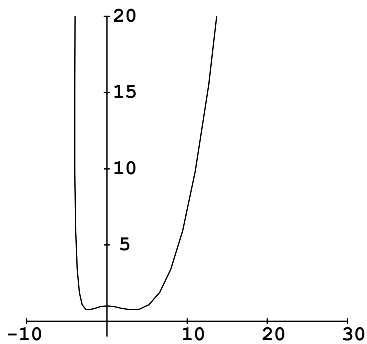


$c < 2$

For $c > 2$ there are 2 vertical asymptotes. As c approaches 2 from above, these asymptotes get closer together. At $c = 2$ there is only one vertical asymptote, and the middle piece of the function that is concave down disappears. For $c < 2$ the graph is a bell-shaped curve with no vertical asymptote. The above sequence of pictures shows this progression.

4. On the left below is a graph with the default viewing rectangle. On the right is a graph zoomed in on the region near the origin to look for finer structure.





Now take derivatives.

$$\frac{dx}{dt} = 2t - 4 \qquad \frac{dy}{dt} = 4t^3 - 2t \qquad \frac{dy}{dx} = \frac{4t^3 - 2t}{2t - 4} = \frac{t(2t^2 - 1)}{2(t-2)}$$

There is a vertical tangent in this case where $dx/dt = 0$, or when $t = 2$. The coordinates when $t = 2$ are $(2^2 - 4(2), 2^4 - 2^2 + 1) = (-4, 13)$. When horizontal tangents occur, $dy/dt = 0$. This happens at $t = 0$, and $t = \pm 1/\sqrt{2}$. The coordinates are $(0,1)$ for $t = 0$ (a maximum), $(1/2 - 2\sqrt{2}, 3/4)$ (a minimum), and $(1/2 + 2\sqrt{2}, 3/4)$ (a minimum).

Note that it is not always possible to show every interesting feature of a graph in the same viewing rectangle. Using to use more than one viewing rectangle is often necessary.

4.5 Indeterminate Forms and L'Hospital's Rule

Key Concepts:

- Indeterminate forms of types $\frac{0}{0}$ and $\frac{\infty}{\infty}$ and l'Hospital's Rule
- Indeterminate products, differences and powers

Skills to Master:

- Determine whether or not l'Hospital's Rule applies to given limits.
- Evaluate limits using appropriate methods.

Discussion:

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Section 4.5 explains l'Hopital's Rule. Often a limit of a quotient can not be computed directly because the numerator and denominator are either both approaching 0 or are both approaching infinity. L'Hospital's Rule gives a means for evaluating these limits provided the numerator and denominator are differentiable and provided that the limit of the derivative of the numerator divided by derivative of the denominator exists. Pay careful attention to the statement of *l'Hospital's Rule* in this section The examples in this section also show you how to apply the rule to limits of the indeterminate forms $0 \cdot \infty$, $\infty - \infty$, 0^{∞}, ∞^{0}, and 1^{∞}. None of these indeterminate forms have any fixed meaning. Luckily, limits involving these forms can often be evaluated by using the techniques in the section.

Key Concept: Indeterminate forms of types $\frac{0}{0}$ and $\frac{\infty}{\infty}$ and l'Hospital's Rule

An expression of the form $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)}$ where $\lim\limits_{x \to a} f(x) = 0$ and $\lim\limits_{x \to a} g(x) = 0$ is called an indeterminate form of type $\dfrac{0}{0}$. If instead, $\lim\limits_{x \to a} f(x) = \infty$ and $\lim\limits_{x \to a} g(x) = \infty$, the expression is called an indeterminate form of type $\dfrac{\infty}{\infty}$. The reason that these forms are called indeterminate is that any value is possible for such limits, and the limits may not even exist. For example, $\lim_{x\to 0}\dfrac{c\cdot x}{d\cdot x}$ where c and d are fixed constants is an indeterminate form of type $\dfrac{0}{0}$. If $d \neq 0$, the limit is c/d. Note that this is exactly what the limit of the derivative of the numerator divided by the derivative of the denominator is.

L'Hospital's Rule tells you that if $\lim_{x\to a}\dfrac{f(x)}{g(x)}$ is an indeterminate form of type $\dfrac{0}{0}$ or of type $\dfrac{\infty}{\infty}$ and if $\lim_{x\to a}\dfrac{f'(x)}{g'(x)}$ exists, then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Key Concept: Indeterminate products, differences and powers

Limits of the indeterminate forms $0\cdot\infty$, $\infty-\infty$, 0^∞, ∞^0, and 1^∞ can often be converted to limits of indeterminate form $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$ by algebraic manipulation. If this can be done, then l'Hospital's Rule often allows you to compute the limit. A product of indeterminate form $0\cdot\infty$ can be rewritten as an indeterminate quotient by rewriting an expression $f\cdot g$ as $\dfrac{f}{1/g}$ or as $\dfrac{g}{1/f}$. Indeterminate powers of forms $f(x)^{\,g(x)}$ can be rewritten as $e^{\,g(x)\cdot\ln(f(x)}$ and the exponent can often be dealt with as an indeterminate product.

SkillMaster 4.12: Determine whether or not l'Hospital's Rule applies to given limits.

A mistake that is often made in trying to evaluate limits using l'Hospital's Rule is not checking that the conditions needed to apply l'Hospital's Rule are satisfied. For example, there is no need to use l'Hospital's Rule in evaluating $\lim_{x\to\pi}\dfrac{\cos x}{\pi/2-x}$ because the numerator is approaching -1 and the denominator is approaching $-\pi/2$. Note that if you tried to apply l'Hopital's Rule then you would have yielded the answer 0 which is incorrect. However the limit $\lim_{x\to\pi/2}\dfrac{\cos x}{\pi/2-x}$ does require an application of l'Hospital's Rule since the denominator and numerator are approaching 0. In this case, the limit is equal to $\lim_{x\to\pi/2}\dfrac{-\sin x}{-1} = 1$. Make sure that you check carefully whether l'Hospital's Rule can be applied before trying to use it.

SkillMaster 4.13: Evaluate limits using appropriate methods.

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The *examples* in this section show that sometimes l'Hospital's Rule needs to be applied a number of times before an answer is obtained. These examples also show how to convert indeterminate differences and powers into indeterminate quotients so that l'Hospital's Rule can be applied. When you see a problem that is an indeterminate form, think carefully about what type of problem it is and what technique to use before trying to solve it.

Worked Examples

For each of the following examples, first try to find the solution without looking at the hint. Cover the right column with a piece of paper. If you need a hint, uncover the column. If you need to see the worked solution, look at the solutions section that follows the examples and hints.

Example	Hint
SkillMaster 4.12.	
1. Determine if l'Hospital's Rule applies to the following limit. $$\lim_{x\to 1}\frac{x^2-1}{x+1}$$ and evaluate the limit.	L'Hospital's Rule applies when the limits of the numerator and denominator are either both 0, or both $\pm\infty$.
SkillMaster 4.13.	
2. Compute $\displaystyle\lim_{x\to 2}\frac{x^3-8}{x-2}$.	Determine that l'Hospital's Rule applies and then use it.
3. Compute $\displaystyle\lim_{x\to 0}\frac{\cos(\sqrt{x})-1}{2x}$.	Determine if l'Hospital's Rule applies and if so, use it.

4. Find the limit. Does l'Hospital's Rule apply and is it the simplest way to compute the limit?

$$\lim_{x\to 2} \frac{x^2-3x+2}{x-2}$$

Take the limits of the numerator and denominator separately.

5. Find $\lim\limits_{x\to\infty} xe^{-\sqrt{x}}$.

The product is indeterminate. Express this product as a quotient (use the properties of exponents).

6. Find $\lim\limits_{x\to 0} (e^x-1)^{\sin(x)}$

This is an indeterminate of the form 0^0. Let $y=(e^x-1)^{\sin(x)}$ and compute $\lim\limits_{x\to 0} \ln(y)$ using l'Hospital's Rule.

Solutions to worked examples

1. Compute the limits of the numerator and the denominator.

$\lim\limits_{x\to 1} x^2-1 = 1^2-1=0$ $\qquad$ $\lim\limits_{x\to 1} x+1 = 1+1=2$

L'Hospital's Rule does not apply.

The limit may be computed by substitution.

$$\lim_{x\to 1} \frac{x^2-1}{x+1} = \frac{1^2-1}{1+1} = 0$$

2. Compute the limits of the numerator and the denominator.

$\lim\limits_{x\to 2} x^3-8 = 2^3-8=0$ $\qquad$ $\lim\limits_{x\to 2} x-2 = 2-2=0$

Both limits are 0, so l'Hospital's Rule applies.

$$\lim_{x\to 2} \frac{x^3-8}{x-2} = \lim_{x\to 2} \frac{\frac{d}{dx}(x^3-8)}{\frac{d}{dx}(x-2)} = \lim_{x\to 2} \frac{3x^2}{1} = 12$$

3. The limits in both the numerator and denominator are 0, so l'Hospital's Rule applies.

$$\lim_{x\to 0}\frac{\cos(\sqrt{x})-1}{2x} = \lim_{x\to 0}\frac{\frac{d}{dx}(\cos(\sqrt{x})-1)}{\frac{d}{dx}(2x)}$$

$$= \lim_{x\to 0}\frac{-\sin(\sqrt{x})\frac{1}{2\sqrt{x}}}{2} = \lim_{x\to 0}\frac{-\sin(\sqrt{x})}{4\sqrt{x}}$$

The limit still is of the form "$0/0$" so apply l'Hospital's Rule again.

$$\lim_{x\to 0}\frac{-\sin(\sqrt{x})}{4\sqrt{x}} = \lim_{x\to 0}\frac{-\frac{d}{dx}(\sin(\sqrt{x}))}{4\frac{d}{dx}(\sqrt{x})}$$

$$= \lim_{x\to 0}\frac{-\cos(\sqrt{x})(1/(2\sqrt{x}))}{4(1/(2\sqrt{x}))} = \lim_{x\to 0}\frac{-\cos(\sqrt{x})}{4} = \frac{-\cos(\sqrt{0})}{4} = -\frac{1}{4}$$

4. Direct substitution of $x = 2$ shows the denominator becomes

$2^2 - 3(2) + 2 = 4 - 6 + 2 = 0$

and the numerator becomes $2 - 2 = 0$.

Thus l'Hospital's Rule applies.

However, this problem has a more direct solution: you may factor the numerator to find the limit.

$$\lim_{x\to 2}\frac{x^2-3x+2}{x-2} = \lim_{x\to 2}\frac{(x-1)(x-2)}{x-2} = \lim_{x\to 2}(x-1) = 2-1 = 1$$

It is still that case that l'Hospital's Rule applies and gives you the same limit.

$$\lim_{x\to 2}\frac{x^2-3x+2}{x-2} = \lim_{x\to 2}\frac{\frac{d}{dx}(x^2-3x+2)}{\frac{d}{dx}(x-2)} = \lim_{x\to 2}\frac{2x-3}{1} = \frac{2(2)-3}{1} = 1$$

5. $$\lim_{x\to\infty} xe^{-\sqrt{x}} = \lim_{x\to\infty}\frac{x}{e^{\sqrt{x}}} = \lim_{x\to\infty}\frac{\frac{d}{dx}(x)}{\frac{d}{dx}(e^{\sqrt{x}})}$$

$$= \lim_{x\to\infty}\frac{1}{(e^{\sqrt{x}})\frac{d}{dx}(\sqrt{x})} = \lim_{x\to\infty}\frac{1}{e^{\sqrt{x}}(1/(2\sqrt{x}))} = \lim_{x\to\infty}\frac{2\sqrt{x}}{e^{\sqrt{x}}}$$

This is still indeterminate, so apply l'Hospital's Rule again.

$$\lim_{x\to\infty} \frac{2\sqrt{x}}{e^{\sqrt{x}}} = \lim_{x\to\infty} \frac{\frac{d}{dx}(2\sqrt{x})}{\frac{d}{dx}(e^{\sqrt{x}})} = \lim_{x\to\infty} \frac{2(1/(2\sqrt{x}))}{e^{\sqrt{x}}(1/(2\sqrt{x}))} = \lim_{x\to\infty} \frac{2}{e^{\sqrt{x}}} = 0$$

6. $\ln(y) = \ln((e^x - 1)^{\sin(x)}) = \sin(x)\ln(e^x - 1) = \dfrac{\ln(e^x - 1)}{\csc(x)}$

As $x \longrightarrow 0$ this is an indeterminate form, so use

l'Hospital's Rule to compute the limit.

$$\lim_{x\to 0} \ln(y) = \lim_{x\to 0} \frac{\ln(e^x - 1)}{\csc(x)} = \lim_{x\to 0} \frac{\frac{d}{dx}(\ln(e^x - 1))}{\frac{d}{dx}(\csc(x))}$$

$$= \lim_{x\to 0} \frac{e^x/(e^x - 1)}{-\csc(x)\cot(x)} = \lim_{x\to 0} \frac{e^x}{e^x - 1} \frac{\sin^2(x)}{\cos(x)}$$

At this point the limit is still indeterminate. The complexity can be reduced by factoring out the parts of the limit that are not approaching 0 and then applying l'Hospital's rule once more.

$$\lim_{x\to 0} \frac{e^x}{e^x - 1} \frac{\sin^2(x)}{\cos(x)} = \lim_{x\to 0} \frac{e^x}{\cos(x)} \lim_{x\to 0} \frac{\sin^2(x)}{e^x - 1} = \frac{e^0}{\cos(0)} \lim_{x\to 0} \frac{\sin^2(x)}{e^x - 1}$$

$$= \lim_{x\to 0} \frac{\frac{d}{dx}(\sin^2(x))}{\frac{d}{dx}(e^x - 1)} = \lim_{x\to 0} \frac{2\sin(x)\cos(x)}{e^x} = \frac{2\sin(0)\cos(0)}{e^0} = 0$$

Thus the limit is $e^0 = 1$.

4.6 Optimization Problems

Key Concepts:

- Steps in solving optimization problems
- The First Derivative Test for absolute maxima and minima on an interval
- Applications to Economics

Skills to Master:

- Carefully read and understand optimization problems, determine the function to be optimized and solve.
- Optimize Cost and Revenue

Discussion:

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Section 4.6 shows you how to solve optimization problems. These problems occur in many applications of Calculus to other fields. You often want to minimize cost, maximize profits, minimize amounts of material used, or maximize volume or area in various applications. The techniques that you have learned for finding maximum and minimum values of functions using Calculus are exactly the techniques that you need to solve many of these problems. Read and work through as many examples as you can to become familiar setting up the problem as a diagram with mathematical expressions and then using the techniques of differentiation to find the answer. The concept of *marginal cost* was first introduced in Chapter 3. This section also introduces the price function $p(x)$, the revenue function $R(x) = xp(x)$, the marginal revenue function $R'(x)$, the profit function $P(x) = R(x) - C(x)$, and the marginal profit function $P'(x)$. Make sure that you read this section carefully and understand what each of these terms mean.

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Key Concept: Steps in solving optimization problems

The text lists the *steps in solving optimization problems* in this section. They are:

1. Understand the problem.
2. Draw a diagram.
3. Introduce notation for the quantity to be optimized and for other relevant quantities.
4. Express the quantity to be optimized in terms of other quantities.
5. Find relationships among the quantities and express them as equations.
6. Use Calculus techniques to find the absolute maximum or minimum of the quantity to be optimized.

Not all of these will be used in every optimization problem. But you should think through these steps as you try to solve the problems in this section.

Key Concept: The First Derivative Test for absolute extrema on an interval

You know from the *Extreme Value Theorem* that a continuous function defined on a closed interval $[a,b]$ achieves a maximum and minimum value on that interval. However, you are sometimes presented with functions that are defined on unbounded intervals $[a,\infty)$ or on open intervals (a,b). The *First Derivative Test* in this section gives a method that sometimes determines absolute maximum or minimum values for functions defined on such intervals. This test states:

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Suppose that c is a critical number of a continuous function f defined on an interval.

(a) If $f'(x) > 0$ for all $x < c$ and if $f'(x) < 0$ for all $x > c$, then $f(c)$ is the absolute maximum value of f on the interval.

(b) If $f'(x) < 0$ for all $x < c$ and if $f'(x) > 0$ for all $x > c$, then $f(c)$ is the absolute minimum value of f on the interval.

Pay careful attention to how this test is used.

Key Concept: Applications to Economics

If $C(x)$ is the cost of producing x items, the average cost $c(x)$ is equal to $\dfrac{C(x)}{x}$. By taking the derivative of this quantity using the quotient rule, you see that

$$c'(x) = \frac{xC'(x) - C(x)}{x^2}$$

It follows that where the average cost is a minimum, $xC'(x) - C(x) = 0$. Another way of writing this is $C'(x) = \dfrac{C(x)}{x} = c(x)$. So to find where the average cost is a minimum, find places where the marginal cost is equal to the average cost. The profit in producing and selling x items is the revenue minus the cost of producing them, $P(x) = R(x) - C(x)$. The marginal profit is the derivative of the profit function, and the marginal revenue is the derivative of the revenue function. By taking the derivative, you see that if the profit is at a maximum, then $P'(x) = 0$, so $R'(x) = C'(x)$. Another way of saying this is that the marginal revenue is equal to the marginal cost. Given graphs of $R(x)$ and $C(x)$, look for x values where the graphs have the same slope.

SkillMaster 4.14: Carefully read and understand optimization problems, determine the function to be optimized and solve.

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This SkillMaster involves applying the *Steps In Solving Optimization Problems* described above. The most important step is reading carefully and understanding the problem. Once you clearly understand what is being asked for, you can proceed with the rest of the steps. The kinds of problems in this section often take much longer than straightforward problems about finding the maximum or minimum values of functions given by specific equations.

Don't get discouraged! These kind of problems contain several steps and are difficult for everyone in the beginning. Eventually, with enough experience, they will become routine. Work through as many problems as you can, perhaps with a study group. As a final step, make sure that you have answered the original question.

SkillMaster 4.15: Optimize Cost and Revenue

Given a specific model of an economic situation, you should be able to use the above concepts to find the number of items to produce to maximize profit or to minimize average cost. Use the relationships described above to do this. To find places where the average cost is a minimum, look for places where the marginal cost is the same as the average cost. To find places where the profit is a maximum, look for places where the marginal revenue is equal to the marginal cost. Note that the marginal cost enters into both situations, minimizing cost and maximizing profit.

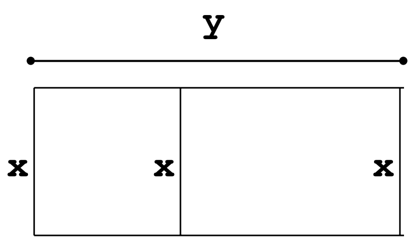
Worked Examples

For each of the following examples, first try to find the solution without looking at the hint. Cover the right column with a piece of paper. If you need a hint, uncover the column. If you need to see the worked solution, look at the solutions section that follows the examples and hints.

Example	Hint

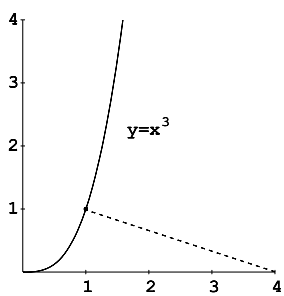
SkillMaster 4.14.

1. A farmer wishes to build fencing to contain and separate goats and pigs. The fencing will be in the form of two adjacent rectangles. If the farmer has 120 feet of fencing what is the largest area that can be contained?



Give variable names to the side lengths. Express the constraint that there is 120 feet of fencing. Use this expression to solve for one variable in terms of the other. Express the area first as a function of both variables then as a function of only one.
Notice that $A(x)$ is a parabola whose maximum is a vertex. The critical point must be this vertex, hence a maximum.

2. Find the point on the curve $y = x^3$ closest to the point $(4,0)$.

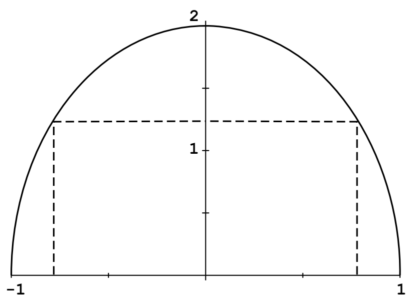


First find an expression for the distance from $(4,0)$ to a point (x,y). Turn this into an expression of one variable by substituting x^3 for y. Suggestion: you may simplify the computation by squaring this expression to eliminate the square root. These expressions have minima at the same places.

3. A half ellipse is shown. The equation of the ellipse is

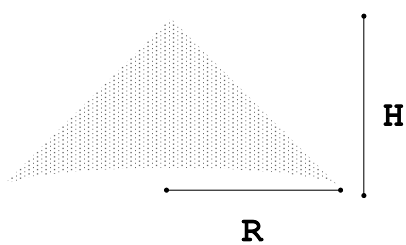
$$x^2 + \frac{y^2}{4} = 1$$

What are the dimensions of the largest rectangle with one side centered on the x-axis that can be inscribed in the half ellipse ?



Again it will be easier to work with the square of the area and maximize that instead of maximizing the area which contain square roots with complicated derivatives.

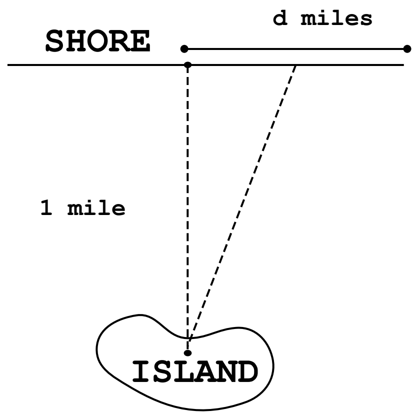
4. A right circular cone has radius R and height H. What is the radius of the inscribed cylinder with largest volume? The base of the cylinder is assumed to be on the base of cone with the same central axis.



Let r be the radius of the cylinder. Use similar triangles to find an expression for the height of the cylinder in terms of r, R and H. Remember to treat R and H as constants, not as variables.

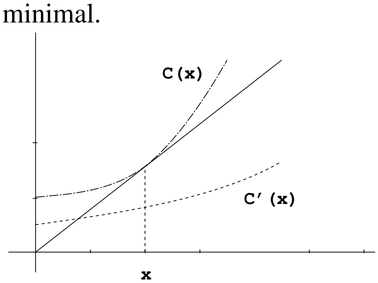
5. A lifeguard is on an island 1 mile from the shore. Someone has hurt his foot and is at a distance d up the shore from the closest point A on the shore to the lifeguard. She will row to shore and help him. She can row at a speed of 2 miles per hour and can walk at a speed of 4 miles per hour. Where should she land the boat to begin walking in the cases where $d = 2$ mi, $d = 1$ mi and $d = 1/2$ mi?

First do the calculus keeping d as a constant rather than doing the same problem 3 times. Be sure that your answers make physical sense.



SkillMaster 4.15.

6. The cost function $C(x)$ of producing x units of a product, and its derivative, the marginal cost function $C'(x)$ of producing x units are shown in the graph. What is the interpretation of $C(x)$ being concave upward with no inflection point? Graphically find the x for which the average cost $c(x)$ is minimal.



$C(x)$ concave upward means that the marginal cost function, $C'(x)$ is increasing.
The slope of the line through the points $(0,0)$ and $(x,C(x))$ is the average cost of x units, $c(x)$. This slope is smallest when the line through the points $(0,0)$ and $(x,C(x))$ is tangent to the curve $y = C(x)$.

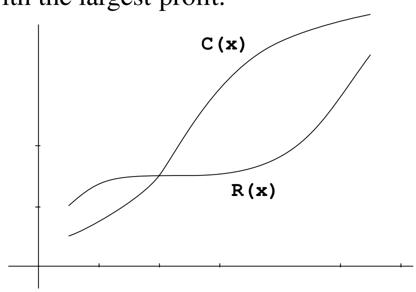
7. Sketch a possible graph of $c(x)$, the average cost function for the function $C(x)$ shown above.

The average cost function $c(x)$ intersects the marginal cost $C'(x)$ at the $x-$value where the average cost has a minimum.

8. Suppose that the cost of producing x units is
$C(x) = 402 + 30x + 0.002x^2$.
Find the expression for the average cost function $c(x)$ and the marginal cost function $C'(x)$. Find the number of units with the smallest average cost.

Remember, the minimal average cost occurs where the average cost $c(x)$ equals the marginal cost $C'(x)$

9. The graphs of the cost function $C(x)$ and the revenue function $R(x)$ are shown. Estimate graphically the number of units with the largest profit.



Profit is $P(x) = R(x) - C(x)$. This is maximal when the vertical line segment between the upper graph of $R(x)$ and the lower graph of $C(x)$ is the longest, that is when revenue minus cost is at its highest point.

10. Suppose that the cost of producing x units is

$$C(x) = 402 + 30x + 0.002x^2$$

and that the price of one unit when x are produced is

$$p(x) = 32 - 0.0015x.$$

Find the number of units that will yield the largest profit.

When is the profit largest?

11. Suppose a company makes DVD recorders and prices them at \$600. At this price, the company sells 10,000 DVD recorders per month. Research indicates that for every \$10 rebate the company offers, another 256 DVDs would be purchased by the company's clients. Suppose that the cost function is $C(x) = 12,000 + 150x$. How many dollars in rebate should be offered to maximize profit? Write down a function that gives the profit.

Solutions to worked examples

1. Give variable names x, y to the sides shown. The total perimeter is

$3x + 2y = 120y = \frac{1}{2}(120 - 3x)$. The quantity to be maximized is the total area.

$A = xy = x(1/2)(120 - 3x) = x(60 - 3/2x) = 60x - 3/2x^2$

To find the maximum set the derivative to 0.

$A'(x) = 60 - 3x = 0 \qquad 60 = 3xx = 20$

Find y by substituting $x = 20$ into the formula $y = \frac{1}{2}(120 - 3x)$.

$y = \frac{1}{2}(120 - 3x) = \frac{1}{2}(120 - 3 \cdot 20) = \frac{1}{2} \cdot 60 = 30$

The largest area that can be contained is (20)(30)=600 square feet.

2. Let D be the distance.

$D = \sqrt{(x-4)^2 + y^2} = \sqrt{(x-4)^2 + (x^3)^2} = \sqrt{(x-4)^2 + x^6}$

Instead of minimizing D let $f = D^2$ and minimize f.

(This simplifies the work. You could work with D and still

do the problem correctly.)

$f(x) = (x-4)^2 + x^6 f'(x) = 2(x-4) + 6x^5 = 6x^5 + 2x - 8\text{Set } 0 = 6x^5 + 2x - 8.$

Inspection or use of graphing device reveals that $x = 1$ is a solution. The first derivative is is negative for all $x < 1$ and positive for all $x > 1$, so f changes from decreasing to increasing across the point $x = 1$, thus f has a minimum here. As a check, the second derivative is $f''(x) = 30x^4 + 2$. At the point $x = 1$, $f''(1) = 32 > 0$ so the Second Derivative Test also show there is a minimum at this point on the curve, (x, x^3) at $x = 1$ or the point $(1, 1^3) = (1, 1)$.

3. Let (x, y) be the coordinates of the corner of the rectangle in the first quadrant. So y is the height of the rectangle and $2x$ is the width. The area is

$$A = 2xy,\ 0 \le x \le 1.$$

Eliminate the variable y by solving for it in the equation of the ellipse.

$\frac{y^2}{4} = 1 - x^2, \quad \frac{y}{2} = \sqrt{1-x^2} y = 2\sqrt{1-x^2}$

Substitute this for y in the expression for A.

$A = 2x(2\sqrt{1-x^2}) = 4x\sqrt{1-x^2}$

As before, the expression for the square of A is easier to work with than the expression for A.

Set $f = A^2, f(x) = A^2 = (4x\sqrt{1-x^2})^2 = 16x^2(1-x^2) = 16x^2 - 16x^4$

Take the derivative and set it equal to 0.

$f'(x) = 16(2x - 4x^3) \qquad 0 = 16(2x - 4x^3) = 2x - 4x^3 \qquad 0 = 2x(2 - x^2)$

The possibilities are $x = 0, 1/\sqrt{2}, -1/\sqrt{2}$.

The only positive possibility is $x = 1/\sqrt{2}$. This yields a maximum value for A because at the endpoints $x = 0$ and $x = 1$, $A = 0$, while at $x = 1/\sqrt{2}$, $A = 2$.

To find y substitute $1/\sqrt{2}$ for x in the expression for y.

$y = 2\sqrt{1-x^2} y = 2\sqrt{1-(1/\sqrt{2})^2} = 2\sqrt{1-1/2} = 2\sqrt{1/2} = 2/\sqrt{2} = \sqrt{2}$

The rectangle is $\sqrt{2}$ high and $2/\sqrt{2} = \sqrt{2}$ wide. Thus, it is a square.

4. If h is the height of the cylinder then the volume is

$$V = \pi r^2 h,\ 0 \le r \le R.$$

To eliminate the variable h use the similarity of the triangles.

$\frac{H}{R} = \frac{H-h}{r} \qquad rH = R(H-h)$

$rH = RH - Rh \qquad Rh = RH - rH \qquad h = \frac{H}{R}(R - r)$

Substitute this expression for h in the equation for the volume.

$V = \pi r^2 \frac{H}{R}(R-r) = \frac{\pi H}{R} r^2 (R-r) = \left(\frac{\pi H}{R}\right)(Rr^2 - r^3).$

To find the maximum, differentiate and set equal to 0.

$V' = \left(\frac{\pi H}{R}\right)(2Rr - 3r^2) = 0$

$0 = 2Rr - 3r^2 \qquad 0 = r(2R - 3r^2)$

The roots are $r = 0, -\sqrt{2R/3}$ and $\sqrt{2R/3}$. The only positive root is $r = \sqrt{2R/3}$

which is the radius of the inscribed cylinder of maximal volume. This is because when $r = 0$ or $r = R$, $V = 0$, and V is positive at $r = \sqrt{2R/3}$. Note that the radius does not depend on H.

5. Let x be the distance from point A to where the boat lands. The total time taken is

$$T(x) = (\sqrt{1+x^2})/2 + (d-x)/4.$$

To find the shortest time for the lifeguard to get to the injured person, differentiate and set the result equal to 0. Note that x is in the interval $[0,d]$

$$T'(x) = (1/2)(1+x^2)^{-1/2}(2x)/2 - 1/4 = \frac{x}{2\sqrt{1+x^2}} - 1/4 = 0$$

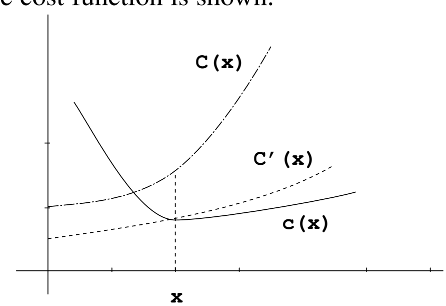
$$0 = \frac{2x}{\sqrt{1+x^2}} - 1 \qquad 1 = \frac{2x}{\sqrt{1+x^2}} \qquad \sqrt{1+x^2} = 2x$$

$$1 + x^2 = 4x^2 \qquad 1 = 3x^2 \qquad x = \pm 1/\sqrt{3}$$

Take the positive solution $x = 1/\sqrt{3} \approx 0.5774$ mi. Notice that this solution does not appear to depend on where the injured person is located. When the victim is farther away from point A than $1/\sqrt{(3)}$, this is, indeed, the correct answer. In this Worked Example, he is 1/2 mile away (less than 0.5774), so this answer must be wrong. The 0.5774 answer suggests that the lifeguard overshoot the position of the injured person and then walk back to get him. In this case, the lifeguard should row directly toward the injured person. The minimum actually occurs at $x = d$, one of the endpoints. This Worked Example illustrates that you need to keep the physical situation in mind to ensure that you arrive at answers that are both sensible and possible.

6. $C(x)$ being concave upward means that each additional unit is more expensive to produce that the one before. In such cases there is no economy of scale. A tangent line to $y = C(x)$ through (0,0) is shown.

7. A possible average cost function is shown.

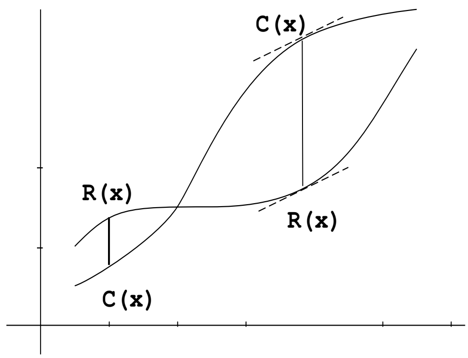


8. $c(x) = \dfrac{C(x)}{x} = 402/x + 30 + 0.002x$ $\qquad C'(x) = 30 + 0.004x$ The minimal average cost occurs if the average cost equals the marginal cost.

$c(x) = C'(x) \qquad 402/x + 30 + 0.002x = 30 + 0.004x \qquad 402/x = 0.002x$

$402/0.002 = x^2 \qquad x = \sqrt{402/0.002}\sqrt{201000} \approx$

9. The longest segment is shown in the graph. Notice the tangents to $R(x)$ and $C(x)$ are parallel at this point. Unfortunately, this is the point where costs most exceed revenue, a losing situation, rather than the other way around. $R(x) - C(x)$ is at a maximum in the first part of the graph as shown, where $R(x)$ is greater than $C(x)$.



10. First compute the revenue.

$R(x) = xp(x) = x(32 - 0.0015x) = 32x - 0.0015x^2$

Profit is largest when $R'(x) = C'(x)$.

$R'(x) = C'(x) \qquad 32 - 0.003x = 30 + 0.004x \qquad 2 = 0.007x$

$x = 2/0.007 \approx 286$

11. Let u the rebate in dollars. Then the company will sell $x = 10,000 + 256(u/10) = 10,000 + 25.6u$ units. The revenue for these units is the number of units times the price per unit.

$p = 600 - u$

$R(u) = \text{(number of units)(price)} = xp = (10,000 + 25.6u)(600 - u)$

$= 6,000,000 + 15,360u - 10,000u - 25.6u^2$

$= 6,000,000 + 5,360u - 25.6u^2$.

The cost is $C(u) = 12,000 + 150(10,000 + 25.6u) = 12,000 + 1,500,000 + 3840u$

$= 1,512,000 + 3840u$

The profit is the difference between revenue and cost.

$P(u) = R(u) - C(u) = 4,488,000 + 1520u - 25.6u^2$

Differentiate and set equal to 0 to find the rebate amount

u where $P(u)$ is a maximum.

$P'(u) = 1520 - 51.2u \qquad 0 = 1520 - 51.2u \qquad 51.2u = 1520$

$u = 1520/51.2 \approx 29.69$

To round things off the company should offer a $30 rebate.

4.7 Newton's Method

Key Concepts:

- Newton's method

Skills to Master:

- Apply Newton's method to find roots of an equation.
- Apply Newton's method to the derivatives of a function to gain more information about the function.

Discussion:

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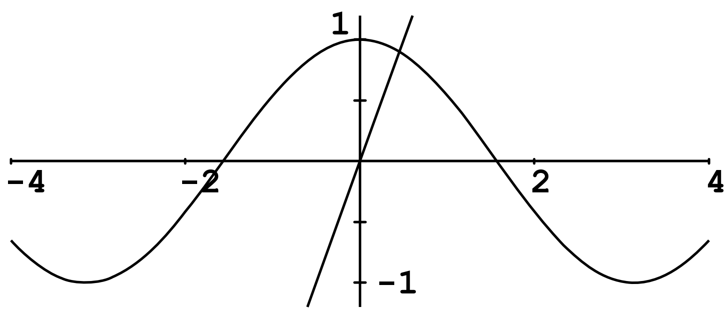
Section 4.7 shows you how to find good approximate roots for an equation using Newton's Method. For many functions the equation $f(x) = 0$ can not be algebraically solved for x. For example, there is no algebraic method for determining exactly which x satisfies $\cos(x) = 2x$. However, if you let $f(x) = \cos(x) - 2x$, since $f(0) = 1$ which is positive and $f(\frac{\pi}{2}) = -\pi$ which is negative, the *Intermediate Value Theorem* tells you that there is a value of x between 0 and $\frac{\pi}{2}$ for which $f(x) = 0$. This section explains a technique, called Newton's method, which will allow you to find accurate approximate solution to the value of x for which $f(x) = 0$. The graphs of part the functions $\cos(x)$ and $2x$ are shown below for x values between -4 and 4.



Key Concept: Newton's method

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Pay careful attention to *Figure* 2 in this section . This figure shows the geometry that makes Newton's method work. If you are trying to find a value of x for which $f(x) = 0$, and you can make a good first guess x_1 to this value, Newton's method will typically (but not always) give you a succession of better and better approximations to a value x. The formula for finding the $(n+1)^{st}$ approximation, x_{n+1}, from the, n^{th}, approximation, x_n, and the derivative of the function f is the recursive updating formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Note that if $f(x_n)$ is positive and the function f is increasing at x_n, Newton's method produces a value for $x_n + 1$ to the left of x_n. If $f(x_n)$ is positive and the function f is decreasing at x_n, Newton's method produces a value for x_{n+1} to the right of x_n. Assuming that you are close to the root and that the function has no critical points or inflection points near the root, this procedure takes you closer to the root.

SkillMaster 4.16: Apply Newton's method to find roots of an equation.

If you want to find a value x for which $f(x) = c$, Newton's method can be used by instead considering the function $g(x) = f(x) - c$ and finding a value x for which $g(x) = 0$. The key to getting started is to pick a reasonable first approximation to a root of the equation $g(x) = 0$. A graphing calculator can help you choose a first approximation. Once you have this first approximation, keep applying Newton's method to get successive approximations that agree to a desired number of decimal places. This method usually gives an approximation to the desired degree of accuracy.

SkillMaster 4.17: Apply Newton's method to the derivatives of a function to gain more information about the function.

Since maximum and minimum values of a differentiable function occur where $f'(x) = 0$, and since inflection points occur where $f''(x) = 0$, Newton's method can be applied to the equations $f'(x) = 0$ or $f''(x) = 0$ to find places where maxima, minima or inflection points occur.

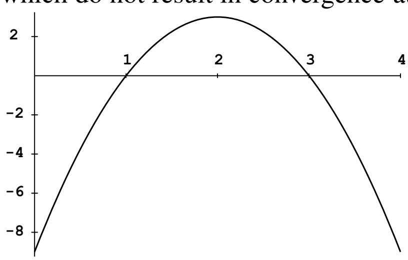
Worked Examples

For each of the following examples, first try to find the solution without looking at the hint. Cover the right column with a piece of paper. If you need a hint, uncover the column. If you need to see the worked solution, look at the solutions section that follows the examples and hints.

Example	Hint

SkillMaster 4.16.

1. In the graph shown draw several iterations of Newton's method beginning the initial point with $x_1 = 1.5$. Which initial values for x_1 result in convergence to the root at $x = 1$, which result in convergence to the root at $x = 3$ and which do not result in convergence at all?



Hint: Try values of x less than 1, between 1 and 2, between 2 and 3, and bigger than 3.

2. Suppose that some function values and derivative values are given in the following table.

x	$f(x)$	$f'(x)$
-1	-1	2
0	1/2	-1
1	2	1

Approximate a root using Newton's method starting with $x_1 = 1$. Find x_2 and x_3.

Hint: Remember the formula for Newton's method

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

Apply this formula using the values given in the table.

3. Find $\sqrt[3]{10}$ to 4 decimal places, using Newton's method.

Newton's method finds roots of equations, so you need an equation whose root is $\sqrt[3]{10}$. That is an equation satisfying $x = \sqrt[3]{10}$. Cube both sides and get $x^3 = 10$ or
$f(x) = x^3 - 10$.

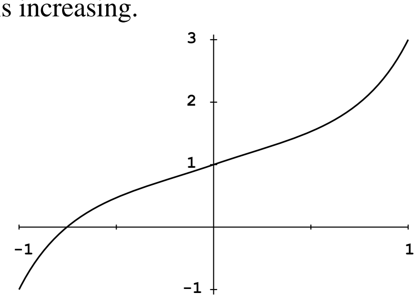
4. Find a solution of $e^x = 2 - x$ to 4 decimal places. Find a good initial guess by graphing.

First graph $y = e^x$ and $y = 2 - x$ to see if and approximately where these graphs intersect. Then apply Newton's method to the difference $f(x) = e^x - (2 - x)$.

5. Consider the equation

$$f(x) = x^5 + x + 1$$

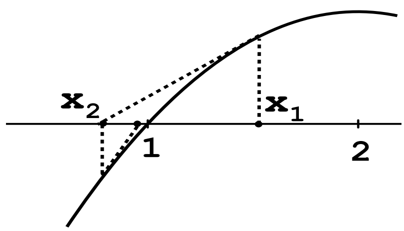
Use the Intermediate Value Theorem to show that $f(x)$ has a root between -1 and 1. Find such a root to 4 decimal places. Show that this is the only root by showing that the function is increasing.



Check if the signs of $f(1)$ and $f(-1)$ are different. To show that there is only one root, show the function is increasing by showing its derivative is always positive.

Solutions to worked examples

1.



Any initial guess that is less than 1 will converge upward toward 1. Any initial guess between 1 and 2 will first go to a number less than 1 then converge upward to 1. Thus any initial guess in $(-\infty, 2)$ will converge to the root $x = 1$. Similarly, any initial guess in $(2, \infty)$ will converge to the root $x = 3$. If 2 is the initial guess then since the tangent line is horizontal, it does not cross the $x-$axis so Newton's method will not produce a new estimate, so it fails.

2. Beginning with $x_1 = 1$ fill the values into Newton's Method.

$$x_2 = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{2}{1} = -1$$

Now find x_3 using the same formula.

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = -1 - \frac{-1}{2} = -1/2$$

3. You are asked to find a root to $f(x) = x^3 - 10$. Note that $f'(x) = 3x^2$.

The formula for Newton's method is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - 10}{3x_n^2}.$$

The integer whose cube is closest to 10 is 2, since $2^3 = 8$.

Try taking $x_1 = 2$ as the initial approximation.

$$x_2 = 2 - \frac{2^3 - 10}{3(2^2)} \approx 2.1667$$

$$x_3 = 2.1667 - \frac{(2.1667)^3 - 10}{3(2.1667^2)} \approx 2.1545$$

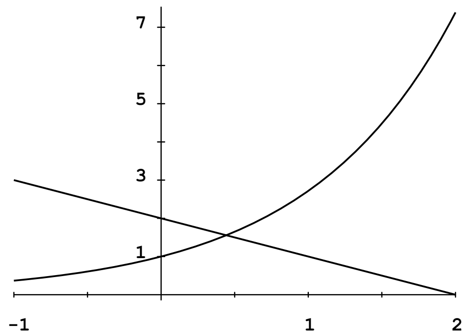
$$x_4 = 2.1545 - \frac{(2.1545)^3 - 10}{3(2.1545^2)} \approx 2.1544$$

$$x_5 = 2.1544 - \frac{(2.1544)^3 - 10}{3(2.1544^2)} \approx 2.1544$$

Conclude that $\sqrt[3]{10} \approx 2.1544$.

As a check, $(2.1544)^3 = 9.99952$ is very close to 10.

4. By graphing both functions you can see there is exactly one intersection point because e^x is increasing and $2-x$ is decreasing. This point appears to be close to 0.5, so take $x_1 = 0.5$.



$f(x) = e^x - 2 + x \qquad f'(x) = e^x + 1$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{e^{x_n} - 2 + x_n}{e^{x_n} + 1}$$

$$x_2 = 0.5 - \frac{e^{0.5} - 2 + 0.5}{e^{0.5} + 1} \approx 0.4439$$

$$x_3 = 0.4439 - \frac{e^{0.4439} - 2 + 0.4439}{e^{0.4439} + 1} \approx 0.4429$$

$$x_4 = 0.4429 - \frac{e^{0.4429} - 2 + 0.4429}{e^{0.4429} + 1} \approx 0.4429$$

Since the last two approximations agree to 4 decimal places, you can conclude that the intersection point is ≈ 0.4429.

5. $f(1) = 1^5 + 1 + 1 = 3 > 0 \qquad f(-1) = (-1)^5 + (-1) + 1 = -1 < 0$

Since the function is negative for $x = -1$ and positive for $x = 1$, the Intermediate Value Theorem implies there must be a root between -1 and 1. Take the simplest intermediate point, $x_1 = 0$.

Note that $f'(x) = 5x^4 + 1$

Since x^4 is never negative, conclude $f'(x) \geq 1$.

The First Derivative Test implies that $f(x)$ is increasing and can only cross the

$x-$axis once. This means there is only one root.

Using Newton's method

$x_{n+1} = x_n - \dfrac{x_n^5 + x_n + 1}{5x_n^4 + 1}$

$x_2 = 0 - \dfrac{0^5 + 0 + 1}{5(0^4) + 1} = -1$

$x_3 = (-1) - \dfrac{(-1)^5 - 1 + 1}{5(1^4) + 1} = -0.8333$

$x_4 = (-0.8333) - \dfrac{(-0.8333)^5 + (-0.8333) + 1}{5(-0.8333)^4 + 1} \approx -0.7644$

$x_5 = (-0.7644) - \dfrac{(-0.7644)^5 + (-0.7644) + 1}{5(-0.7644)^4 + 1} \approx -0.7550$

$x_6 = (-0.7550) - \dfrac{(-0.7550)^5 + (-0.7550) + 1}{5(-0.7550)^4 + 1} \approx -0.7549$

$x_7 = (-0.7549) - \dfrac{(-0.7549)^5 + (-0.7549) + 1}{5(-0.7549)^4 + 1} \approx -0.7549$

Since the last two approximations agree, you can conclude that the root is ≈ -0.7549.

4.8 Antiderivatives

Key Concept:

- Definition of an antiderivative
- Solving differential equations

Skills to Master:

- Find general antiderivatives and antiderivatives satisfying specific conditions.
- Solve word problems relating position, velocity and acceleration of an object moving in a straight line.

Discussion:

Section 4.8 prepares you for the material to come in future chapters. Specifically, finding *antiderivatives* is the technique of of reversing the process of differentiation. Where differentiation looks at differences and changes, integration looks at sums and page 317 accumulations. Here, the basic formulas of antidifferentiation are introduced. They are the building blocks for solving problems of reversing differentiation. A good model to pay careful attention to is the relation between position, velocity and acceleration for rectilinear motion.

Key Concept: Definition of an antiderivative

An antiderivative of a function $f(x)$ is another function $F(x)$ satisfying the condition that $F'(x) = f(x)$. The concept of antiderivative reverses the process of differentiation that you have become proficient at. For example, an antiderivative of $f(x) = x^2$ is the function $F(x) = \dfrac{x^3}{3}$ because $\dfrac{d}{dx}(\dfrac{x^3}{3}) = x^2$. Make certain that you understand *Theorem 1* which describes the relationship between two antiderivatives of the same function. It states that if you know one antiderivative $F(x)$ for a function $f(x)$, then all page 317 antiderivatives are of the form $F(x) + C$ for some constant C.

Key Concept: Solving differential equations

A differential equation is an equation that involves the derivatives of a function. For example, $f'(x) = \cos(x) + x$ is a differential equation. Solving the differential equation means finding a function $f(x)$ that makes it true. In the just mentioned example, $f(x) = \sin x + x^2/2$ is a solution as is any function of the form $\sin x + x^2/2 + C$. In fact, these are the only solutions. Solving differential equations of the type presented in this section involves finding antiderivatives.

SkillMaster 4.18: Find general antiderivatives and antiderivatives satisfying specific conditions.

Given a specific function $f(x)$, finding the general antiderivative involves first finding a function $F(x)$ that satisfies $F'(x) = f(x)$. The general antiderivative is then of the form $F(x) + C$. Finding a specific antiderivative $G(x)$ that satisfies a specific condition $G(a) = b$ involves solving the equation $F(a) + C = b$ for C. Since there are many antiderivatives of a specific function, specifying a condition that the antiderivative must satisfy focuses on a particular one of the many antiderivatives. For example, the only antiderivative $F(x)$ of $\cos x$ that satisfies $F(0) = 1$ is $F(x) = \sin x + 1$.

SkillMaster 4.19: Solve word problems relating position, velocity and acceleration of an object moving in a straight line.

For an object moving in a straight line, the derivative of the position function is the velocity function and the derivative of the velocity function is the acceleration function. By using the concept of antidifferentiation along with the concept of differentiation, you should be able to start with any one of position, velocity and acceleration and find the other two. For example, starting with the velocity, differentiating gives you the acceleration. Antidifferentiating the velocity gives you the position.

Worked Examples

For each of the following examples, first try to find the solution without looking at the hint. Cover the right column with a piece of paper. If you need a hint, uncover the column. If you need to see the worked solution, look at the solutions section that follows the examples and hints.

Example	Hint
SkillMaster 4.18.	
1. Find the most general antiderivative of the function. $$f(x) = \frac{x^2+x}{\sqrt{x}} + 3e^x + \sec^2(x)$$	First simplify the function by putting the first expression into power notation.
2. Find an antiderivative F of f that satisfies the given condition. $$f(x) = x - (1-x^2)^{-1/2},\ F(0) = 4$$	Notice that if the part of the function with an exponent is put into radical notation then its antiderivative will become clear.
3. Find $f(x)$ if $f''(x) = x - 1/x^2, \qquad f(1) = 7/6, \qquad f'(1) = 1.5.$	You can do this in two steps: first find f' then use f' to find f.
SkillMaster 4.19.	
4. A steel ball bearing at rest is accelerated in a magnetic field in a straight line with acceleration $$a(t) = 120t \text{ m/s}$$ Find the velocity when it hits a barrier 8 meters away by first finding its position, $p(t)$, and its velocity, $v(t)$, at each time t.	Try writing $v(t)$ as a function of $p(t)$ and then substitute $p = 8$.

5. Suppose that marginal revenue is

$$R'(x) = 400 - .03x$$

where x is the number of units sold. Of course, if no units are sold, there will be no revenue. Find the revenue function $R(x)$.

The revenue function is the antiderivative of the marginal revenue function, $R'(x)$. The revenue for $x = 0$ is 0.

Solutions to worked examples

1. $f(x) = x^{3/2} + x^{1/2} + 3e^x + \sec^2(x)$. Now antidifferentiate term by term, as usual.

$$F(x) = \frac{x^{5/2}}{5/2} + \frac{x^{3/2}}{3/2} + 3e^x + \tan(x) + C = \frac{2}{5}x^{5/2} + \frac{2}{3}x^{3/2} + 3e^x + \tan(x) + C$$

2. $f(x) = x - \dfrac{1}{\sqrt{1-x^2}}$

$$F(x) = \frac{x^2}{2} - \sin^{-1}(x) + C \qquad 4 = F(0) = \frac{0^2}{2} - \sin^{-1}(0) + C = 0 + C = C$$

$$F(x) = \frac{x^2}{2} - \sin^{-1}(x) + 4$$

3. Antidifferentiate to find an expression for $f'(x)$.

$f''(x) = x - x^{-2}$

$$f'(x) = \frac{x^2}{2} - \left(\frac{x^{-1}}{-1}\right) + C = \frac{x^2}{2} + \frac{1}{x} + C$$

$$1.5 = f'(1) = \frac{1^2}{2} + \frac{1}{1} + C = 1.5 + C \qquad C = 0$$

$$f'(x) = \frac{x^2}{2} + \frac{1}{x}$$

Antidifferentiate again to find an expression for $f(x)$.

$$f(x) = \left(\frac{1}{2}\right)\frac{x^3}{3} + \ln(x) + D = \frac{x^3}{6} + \ln(x) + D$$

$$7/6 = f(1) = \frac{1^3}{6} + \ln(1) + D \qquad 7/6 = 1/6 + 0 + D \qquad D = 1$$

$$f(x) = \frac{x^3}{6} + \ln(x) + 1$$

4. First antidifferentiate acceleration to get the velocity.

$v(t) = 60t^2 + C \qquad v(0) = 0 = 0^2 + C \qquad C = 0$

$v(t) = 60t^2$ Antidifferentiate again to get the position.

$$p(t) = 20t^3 + D \qquad p(0) = 0 = \frac{20(0^3)}{3} + D = D \qquad D = 0$$

$p(t) = 20t^3$ Solve for t. $t = (p/20)^{1/3}$ Substitute for t.

$$v = 60t^2 = 60((p/20)^{1/3})^2 \qquad v = \frac{60}{20^{2/3}} p^{2/3}$$

Substitute $p = 8$. $v = \dfrac{60}{20^{2/3}}(8)^{2/3} \approx 32.57$ m/s

5. First antidifferentiate and use the fact the $R(0) = 0$
(if you sell no items then you take in no money).

$$R(x) = 400x - 0.03\frac{x^2}{2} + C = 400x - 0.015x^2 + C$$

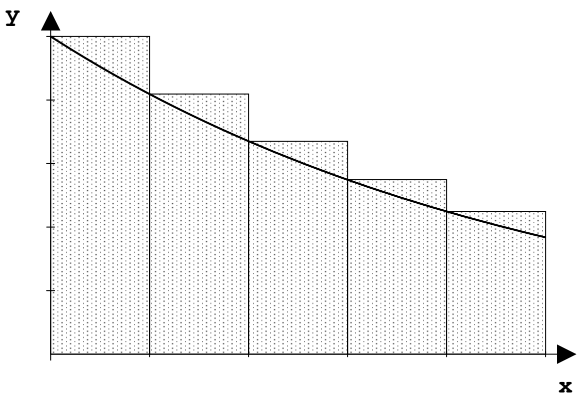
$$R(0) = 0 = 400(0) - 0.015(0^2) + C = C \qquad C = 0 \qquad R(x) = 400x - 0.015x^2$$

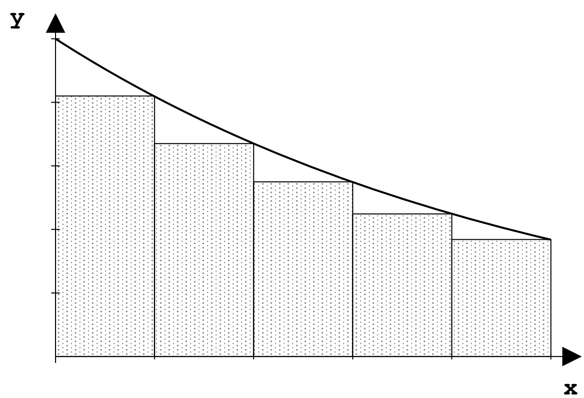
SkillMasters for Chapter 4

SkillMaster 4.1:	Find an equation that relates quantities that are changing.
SkillMaster 4.2:	Use the chain rule to find the relationship between rates of change.
SkillMaster 4.3:	Sketch graphs fitting specified information about maximum and minimum values.
SkillMaster 4.4:	Find critical numbers, and absolute and local maximum and minimum values for continuous functions given by a graph.
SkillMaster 4.5:	Find critical numbers, and absolute and local maximum and minimum values for continuous functions given by an equation.
SkillMaster 4.6:	Compute the values given by the Mean Value Theorem.
SkillMaster 4.7:	Find information about the function f from the graphs of f, f' and f''.
SkillMaster 4.8:	Find information about the function f by applying the first and second derivative tests.
SkillMaster 4.9:	Analyze the graphs of f' and f'' to produce more accurate graphs of f.
SkillMaster 4.10:	Observe trends in families of curves and observe transitional values for which the basic shape of the curve changes.
SkillMaster 4.11:	Analyze graphs of functions given parametrically.
SkillMaster 4.12:	Determine whether or not l'Hospital's rule applies to given limits.
SkillMaster 4.13:	Evaluate limits using appropriate methods.
SkillMaster 4.14:	Carefully read and understand optimization problems, determine the function to be optimized and solve.
SkillMaster 4.15:	Optimize profit, revenue and cost for specific models.
SkillMaster 4.16:	Apply Newton's method to find roots of an equation.
SkillMaster 4.17:	Apply Newton's method to the derivatives of a function to gain more information about the function.
SkillMaster 4.18:	Find general antiderivatives and antiderivatives satisfying specific conditions.
SkillMaster 4.19:	Solve word problems relating position, velocity and acceleration of an object moving in a straight line.

Chapter 5

Integrals





5.1 Areas and Distances

Key Concepts:

- Area under the graph of a continuous function
- Distance traveled by an object with variable velocity
- Relationship between the area and distance problems

Skills to Master:

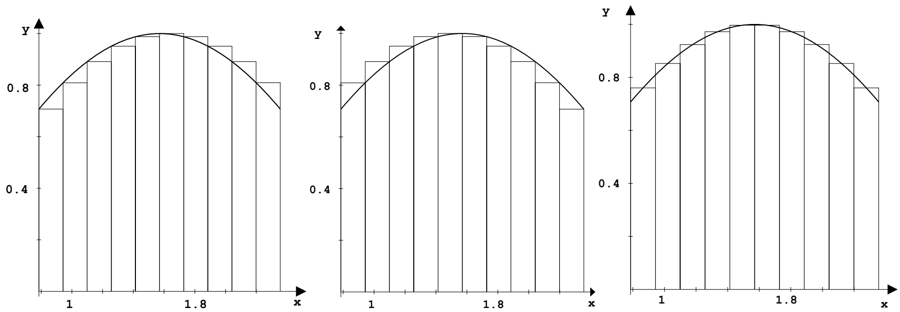
- Estimate area using left and right endpoint and midpoint rectangles.
- Estimate distances using velocity data or graphs.
- Evaluate limits to find areas.

Discussion:

Section 5.1 introduces and explains the method for finding the area under the graph of a continuous function. This is the motivating idea for Integral Calculus; Chapters 5 and 6 are devoted to the development of Integral Calculus. Make sure that you understand the method of finding areas. You are also shown that the area under the graph of the velocity function of an object, assuming that the velocity is always positive so that the object always moves forward. Therefore there are at least two interpretations for the area under a positive function; it will turn out that there are many more.

Key Concept: Area under the graph of a continuous function

The area under the graph of a positive continuous function $f(x)$ for $a \leq x \leq b$ can be approximated by the sum of the areas of rectangles. If the interval $[a,b]$ is divided into n equal subintervals of length $\dfrac{b-a}{n}$, these subintervals can be used as the base of n rectangles. The height of the rectangles can be chosen to be the value of the function at the right endpoint of each subinterval, at the left endpoint of each subinterval, at the midpoint of each subinterval or at some other point in the interval. This is illustrated below for the function $f(x) = \sin(x)$ on the interval $[\pi/4,\, 3\pi/4]$ using rectangles with heights determined by the left, right, and midpoint of the subintervals respectively.



The *area* is defined to be the limit of the sum of the areas of the approximating rectangles.

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Key Concept: Distance traveled by an object with variable velocity

The distance traveled by an object with variable, nonnegative velocity $v(t)$ over a time interval $[a,b]$ can be approximated by dividing the interval $[a,b]$ into n equal subintervals of length $\dfrac{b-a}{n}$ and using the relationship that distance =velocity $\times$ time for objects traveling at constant velocity to approximate the distance traveled during each subinterval. Read carefully *Example 4* in this section to see a specific application of this concept.

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Key Concept: Relationship between the area and distance problems

Note carefully the relationship between the previous two concepts. Note that the distance traveled by an object with variable velocity $v(t)$ over a time interval $[a,b]$ can be interpreted as having the same value as the area under the graph of $v(t)$ for $a \le x \le b$.

SkillMaster 5.1: Estimate area using left and right endpoint and midpoint rectangles.

To approximate the area under the graph of a continuous function $f(x)$ for $a \le x \le b$, where $f(x) \ge 0$, make sure that you understand how to use the left endpoint, right endpoint and midpoint rectangles as discussed in the first concept above. You will be able to use the graphs of functions (and where they are increasing or decreasing) to determine whether the approximating area is greater than or less than the actual area that you are estimating.

SkillMaster 5.2: Estimate distances using velocity data or graphs.

If you understand the relation between finding the distance traveled when you are given the velocity function and finding the area under the graph of the velocity function, you should be able to estimate distances using velocity data or velocity graphs. To estimate the distance using the velocity data, estimate the distance traveled between successive times by using the relationship that distance = velocity $\times$ time for objects traveling at constant velocity. To estimate the distance traveled by using velocity graphs, estimate the area under the graph of the velocity function.

SkillMaster 5.3: Evaluate limits to find areas.

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By evaluating limits of approximating areas, you can find the actual area under the graph of a function. Pay careful attention to *Example 2* in this section to see how this is done. To apply this technique in general, you will often need a formula giving the sum of the first n integers, the sum of the first n squares, or the sum of the first n cubes. These formulas are given in *Section 5.2*

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Worked Examples

For each of the following examples, first try to find the solution without looking at the hint. Cover the right column with a piece of paper. If you need a hint, uncover the column. If you need to see the worked solution, look at the solutions section that follows the examples and hints.

Example	Hint
SkillMaster 5.1.	
1. Estimate the area under the curve $f(x) = e^{-x}$ between $x = 0$ and $x = 1$ and above the $x-$axis using the left endpoint approximation for $n = 5$ (this is L_5). Is this an overestimate or an underestimate?	Subdivide [0,1] into 5 equal pieces. The left endpoints are 0, 0.2, 0.4, 0.6, and 0.8. The length of each subinterval is $(b-a)/n = 1/5 = 0.2$.

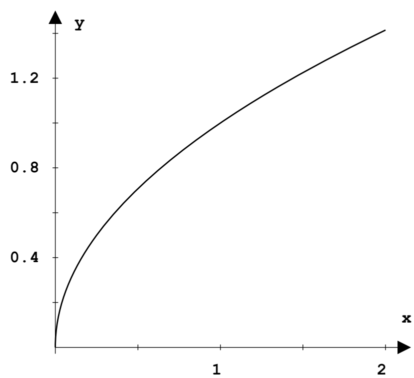
2. Repeat the worked out example above using the right endpoint approximation, R_5.

The right endpoints are 0.2, 0.4, 0.6, 0.8, and 1.0.

3. Repeat the worked out example above using the midpoint approximation, M_5.

The midpoints of the subintervals are 0.1, 0.3, 0.5, 0.7, and 0.9

4. Estimate the area under the graph using L_4 and R_4



Answers will vary a bit because the function values must be estimated from the graph.

Estimating the values $f(x_i)$ for $i = 0,1,2,3,4$ from the graph.
$f(x_0) = f(0) \approx 0.0$
$f(x_1) = f(0.5) \approx 0.7$
$f(x_2) = f(1) \approx 1.0$
$f(x_3) = f(1.5) \approx 1.2$
$f(x_4) = f(2) \approx 1.4$
$\Delta x = (2-0)/4 = 0.5.$

SkillMaster 5.2.

5. The following table gives the velocity of a slowing boulder that is rolling down a rocky ramp at 1 second intervals from time $t = 0$ to time $t = 6$. Two archaeologists hunting for rare artifacts are tied to a tree in the path of the boulder and it will take them 6 seconds to untie the ropes and flee. The tree is 300 ft away from the boulder. Can you tell if the pair escapes disaster?

Time (s)	0	1	2	3	4	5	6
Velocity (ft/s)	115	80	50	25	15	10	5

Determine which of L_6 or R_6 is an overestimate. If this number is less than 300 ft then the archaeologists escape. On the other hand, if an underestimate is more than 300 ft then the pair will get crushed.

SkillMaster 5.3.

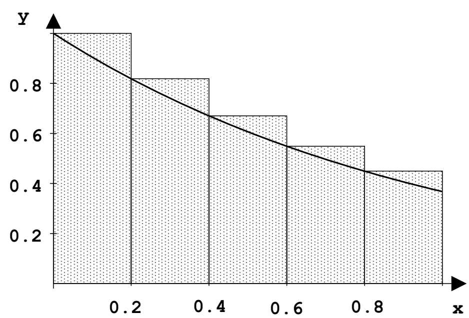
6. Find the exact area under the curve $f(x) = 2x$ from $x = 0$ to $x = 2$ in two ways: first by evaluating the limit of right endpoint approximations, and then by interpreting the area as the area of a triangle. You will need to use the formula
$\sum_{i=1}^{n} i = \dfrac{n(n+1)}{2}$

Evaluate the limit of the right endpoint approximations as n approaches infinity.

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

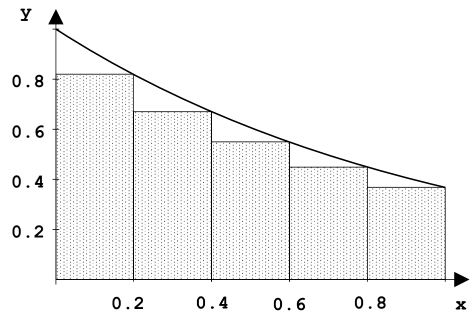
Solutions to worked examples

1. $L_5 = f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x$
$= e^{-0}(0.2) + e^{-0.2}(0.2) + e^{-0.4}(0.2) + e^{-0.6}(0.2) + e^{-0.8}(0.2)$
$\approx 1(0.2) + (0.8187)(0.2) + (0.6703)(0.2) + (0.5488)(0.2) + (0.4493)(0.2)$
≈ 0.6974



The graph is decreasing so the rectangles approximating the area under the curve in the left endpoint approximation cover the region under the graph. The approximation L_5 overestimates the true area.

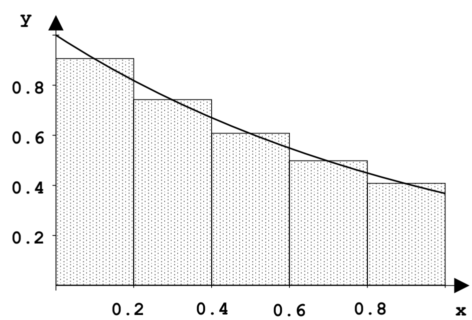
2. $R_5 = f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x + f(x_5)\Delta x$
$= e^{-0.2}(0.2) + e^{-0.4}(0.2) + e^{-0.6}(0.2) + e^{-0.8}(0.2) + e^{-1}(0.2)$
$\approx (0.8187)(0.2) + (0.6703)(0.2) + (0.5488)(0.2) + (0.4493)(0.2) + (0.3679)(0.2)$
≈ 0.5710



The graph is decreasing so the rectangles approximating the area under the curve in the right endpoint approximation is covered by the region under the graph. The approximation R_5 underestimates the true area.

3. $M_5 = f(x_1{}^*)\Delta x + f(x_2{}^*)\Delta x + f(x_3{}^*)\Delta x + f(x_4{}^*)\Delta x + f(x_5{}^*)\Delta x$

$$= e^{-0.1}(0.2) + e^{-0.3}(0.2) + e^{-0.5}(0.2) + e^{-0.7}(0.2) + e^{-0.9}(0.2)$$
$$\approx (0.9049)(0.2) + (0.7408)(0.2) + (0.6065)(0.2) + (0.4966)(0.2) + (0.4066)(0.2)$$
$$\approx 0.6311$$



The approximating rectangles are sometimes below the graph and sometimes above. It is not possible to easily tell if the approximation is larger or smaller than the true value. On the other hand it is likely to be a better approximation than either the overestimate L_5 or the underestimate R_5.

4. $L_4 = f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x$

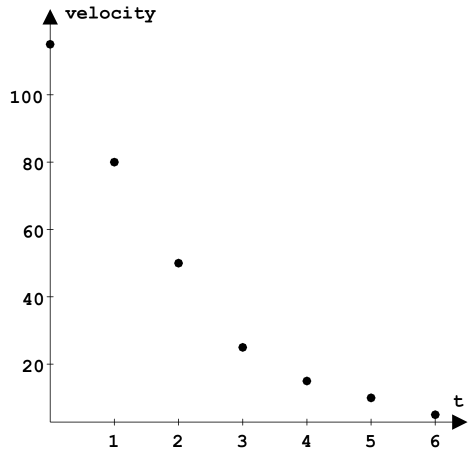
$$= f(0)\Delta x + f(0.5)\Delta x + f(1)\Delta x + f(1.5)\Delta x$$
$$\approx (0.0)(0.5) + (0.7)(0.5) + (1.0)(0.5) + (1.2)(0.5) = 1.45$$

Compute R_4 similarly.

$R_4 = f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x$

$$= f(0.5)\Delta x + f(1)\Delta x + f(1.5)\Delta x + f(2)\Delta x$$
$$\approx (0.7)(0.5) + (1.0)(0.5) + (1.2)(0.5) + (1.4)(0.5) = 2.15$$

5.



The function appears to be decreasing so the left endpoint approximation L_6 is an overestimate. Use $\Delta t = 1$. $L_6 = (115)(1) + (80)(1) + (50)(1) + (25)(1) + (15)(1) + (10)(1)$

$$= 115 + 80 + 50 + 25 + 15 + 10 = 295$$

Thus the true distance traveled by the boulder is less than or equal to 295 ft and the boulder is at least 5 feet away from the bound archaeologists at time $t = 6$ and so they barely escape certain doom!

6. First $\Delta x = (2-0)/n = 2/n$. Second $x_i = i(\Delta x) = 2i/n$.

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} (2x_i)\Delta x$$
$$= \lim_{n\to\infty} \sum_{i=1}^{n} (2x_i)(2/n) = \lim_{n\to\infty} \sum_{i=1}^{n} (4(i/n))(2/n)$$
$$= \lim_{n\to\infty} \sum_{i=1}^{n} 8i/n^2 = \lim_{n\to\infty} (8/n^2) \sum_{i=1}^{n} i = \lim_{n\to\infty} (8/n^2)\left(\frac{n(n+1)}{2}\right)$$
$$= \lim_{n\to\infty} 4\frac{n(n+1)}{n^2} = \lim_{n\to\infty} 4\frac{(n+1)}{n} = \lim_{n\to\infty} 4\left(1+\frac{1}{n}\right) = 4(1+0) = 4$$

This area may also be found by observing that it is the area of a right triangle with base 2 and height 4. The area of a triangle is (1/2)(base)(height) $= (1/2)(2)(4) = 4$.

5.2 The Definite Integral

Key Concepts:

- Riemann sums and the definition of the definite integral
- The Midpoint Rule
- Properties of the definite integral
- Comparison properties of the integral

Skills to Master:

- Approximate definite integrals by the Midpoint Rule.
- Approximate definite integrals using the area interpretation.
- Approximate definite integrals using the comparison properties.
- Evaluate definite integrals using limits and the sum formula.
- Evaluate definite integrals using the properties and previous results.

Discussion:

Section 5.2 defines the *Definite Integral* of a continuous function $f(x)$ from a to b, $\int_a^b f(x)dx$, to be the limit of certain sums called Riemann sums. These sums are similar to the sums of areas of rectangles discussed in the previous section. The only difference is that the height of the rectangle is not determined by a left endpoint, right endpoint or midpoint, but by sample points in the subintervals. If you understand the method of approximating the area under a curve discussed in *Section 5.1* , you should have little trouble with the ideas presented in this section.

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Success with Integral Calculus depends on a clear understanding of what the definite integral is and how it works. This is the time to to check that you understand the concept and the notation of the method of taking limits of Riemann Sums to define the definite integral.

Key Concept: Riemann sums and the definition of the definite integral

For a continuous function $f(x)$ defined for $a \le x \le b$, a Riemann sum is a sum of the form

$$\sum_{i=1}^{n} f(x_i{}^*)\Delta x$$

and is obtained as follows. Divide the interval into n equal subintervals of length $\Delta x = \dfrac{b-a}{n}$. For each subinterval, choose a sample point in the subinterval. Call the sample point chosen for the i^{th} subinterval $x_i{}^*$. Then form the sum indicated above. The definite integral of a continuous function $f(x)$ for $a \le x \le b$, $\int_a^b f(x)dx$, is defined to be the limiting value of these Riemann Sums as n goes to infinity:

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i{}^*)\Delta x$$

It can be shown that for continuous functions, this limit always exists and does not depend on the choice of sample points. Note that the value of the definite integral does not depend on x, indeed, it would be the same value if x were replaced throughout with u or any other variable.

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Make sure that you learn the terminology introduced in this section and are able to use the terms: *integral sign, integrand, limits of integration, lower limit* and *upper limit.*

Key Concept: The Midpoint Rule

The Midpoint Rule states that the definite integral of a continuous function $f(x)$ for $a \le x \le b$ can be approximated by a Riemann Sum using midpoints of subintervals as the sample points:

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(\overline{x_i})\Delta x = \Delta x\,(f(\overline{x_1}) + \cdots + f(\overline{x_n}))$$

where

$$\Delta x = \frac{b-a}{n} \text{ and where } \overline{x_i} = \frac{1}{2}(x_{i-1} + x_i)$$

Make sure that you understand how this rule gives an approximation to the area under a curve.

Key Concept: Properties of the definite integral

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Properties 1-4 of the integral introduced in this section are basic to evaluating definite integrals. Property 1 gives the value of the integral of a constant function; note that for positive constants, this may be interpreted as the fact that the area of a rectangle is obtained by multiplying the lengths of the sides. Properties 2, 3 and 4 correspond to the similar Rules of Differentiation, such as the Constant Multiple Rule, the Sum Rule and the Difference Rule. In other words, it is possible to integrate term by term as you do with differentiation. Later, you will see that the other Rules of Differentiation correspond to certain techniques of integration.

Key Concept: Comparison properties of the integral

Some additional properties of integrals are introduced in this section. The first two on page 361 in the text state that

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx \text{ and } \int_a^a f(x)\,dx = 0$$

This gives meaning to definite integrals where the upper limit is less than the lower limit or where the upper and lower limits are the same. *Property 5* allows you to integrate interval by interval for piecewise defined functions.

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$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

The Comparison Properties of the Integral (Properties 6 - 8) allow you to use comparison information about functions to get comparison information about integrals of those functions. The properties are listed below.

If $f(x) \geq 0$ for $a \leq x \leq b$, then $\displaystyle\int_a^b f(x)\,dx \geq 0$

If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\displaystyle\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$

If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then $\displaystyle m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$

Make sure that you understand the geometric reasoning behind the third property given in *Figure 16* in this section.

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SkillMaster 5.4: Approximate definite integrals by the Midpoint Rule.

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Make sure that you can use the Midpoint Rule to approximate definite integrals. *Example 5* in this section shows how to do this.

SkillMaster 5.5: Approximate definite integrals using the area interpretation.

For positive continuous functions, the definite integral can be interpreted as the area under the graph of the function. For continuous function that take on both positive and negative values, the definite integral can be interpreted as area above the $x-$axis *minus* the area below the $x-$axis. Using this interpretation, you can approximate definite integrals by approximating these areas. You can obtain an exact answer if you know the areas involved. For example, if the areas involve triangles or parts of circles, you can often determine the exact areas.

SkillMaster 5.6: Approximate definite integrals using the comparison properties.

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In situations where only a rough estimate of a definite integral is needed, *Property 8* from the text can be used.

SkillMaster 5.7: Evaluate definite integrals using limits and the sum formula.

Since the definite integral is defined as the limiting value of certain Riemann sums you can use these Riemann sums and take limits to get the values of the definite integrals. You may sometimes need a calculator or computer algebra system to evaluate the limits under consideration.

SkillMaster 5.8: Evaluate definite integrals using the properties and previous results.

If a function is written as a combination of other functions using the operations of addition, subtraction and multiplying by a constant, you can use the properties discussed in this section to evaluate integrals. For example, if

$$\int_2^6 f(x)\,dx = 5 \text{ and } \int_2^6 g(x)\,dx = -2$$

then

$$\begin{aligned} & \int_2^6 (2f(x) - 3g(x) + 2)\,dx \\ = \;& 2 \cdot \int_2^6 f(x)\,dx - 3 \cdot \int_2^6 g(x)\,dx + \int_2^6 2\,dx \\ = \;& 2 \cdot 5 - 3 \cdot (-2) + 2 \cdot (6-2) = 24 \end{aligned}$$

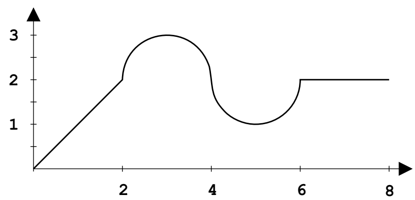
Worked Examples

For each of the following examples, first try to find the solution without looking at the hint. Cover the right column with a piece of paper. If you need a hint, uncover the column. If you need to see the worked solution, look at the solutions section that follows the examples and hints.

Example	Hint
SkillMaster 5.4.	
1. Write the following definite integral as a limit of Riemann sums. $$\int_1^2 \sqrt{1+x^3}dx$$	Recall the definition. $\lim\limits_{n\to\infty} \sum_{i=1}^n f(x_i{}^*)\Delta x$ Here $f(x)=\sqrt{1+x^3}$.
2. Approximate the following definite integral using the Midpoint Rule with $n=5$. $$\int_1^2 \sqrt{1+x^3}dx$$	Use the preceding worked out example. $\int_1^2 \sqrt{1+x^3}dx =$ $\lim\limits_{n\to\infty} \left(\sum\limits_{i=1}^{n} \sqrt{1+(x_i)^3}\Delta x\right)$ $\approx \sum\limits_{i=1}^{5} \sqrt{1+(x_i{}^*)^3}\Delta x$ Note that the midpoints are used as the sample points rather than the right endpoints.
3. Express the following limit as a definite integral. $$\lim_{n\to\infty}\sum_{i=1}^{n} x_i \tan^{-1}(x_i)\Delta x,\ [0,\pi/2]$$	The definition for the definite integral of $y=f(x)$ from $x=a$ to $x=b$ is $\int_a^b f(x)dx = \lim\limits_{n\to\infty}\sum\limits_{i=1}^{n} f(x_i^*)\Delta x$ Here $a=0, b=\pi/2$. It remains to find the expression for $f(x)$.

SkillMaster 5.5.

4. Consider the following graph of $y = f(x)$.



You may assume that the graph consists of a straight line segment for $0 \le x \le 2$, a semicircle for $2 \le x \le 4$, a semicircle for $4 \le x \le 6$ and a straight line segment for $6 \le x \le 8$. Compute the following, using the area interpretation of the definite integral.

$$\int_0^4 f(x)\,dx \qquad \int_4^8 f(x)\,dx \qquad \int_0^8 f(x)\,dx$$

The definite integral of a non-negative function is the area under the curve. Compute the areas using the formulas for the area of a triangle, circle and rectangle.

5. Evaluate the definite integral.

$$\int_0^2 x + \sqrt{4 - x^2}dx$$

Evaluate the integral as a sum of two integrals each of which represent the area of a region that may be computed using known area functions.

SkillMaster 5.6.

6. Estimate the following definite integral by finding the maximum and minimum values of e^{x^3} on the interval $[0, 0.5]$ and using one of the comparison properties.

$$\int_0^{0.5} e^{x^3}\,dx$$

The function e^{x^3} is increasing so its minimum value is at the lower endpoint, i.e. $x = 0$, and it maximum value is at the upper endpoint, $x = 0.5$.

SkillMaster 5.7.

7. Use the definition of the definite integral to evaluate

$$\int_0^1 \left(x^2 - 3x + 2\right)\, dx$$

Recall
$\int_0^1 f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$
Here $f(x) = x^2 - 3x + 2$ and
$\Delta x = (1-0)/n = 1/n$,
$x_i = i/n$.

SkillMaster 5.8.

8. Suppose that $\int_1^3 f(x)dx = 2$, $\int_1^5 f(x)dx = -5$, $\int_1^3 g(x)dx = 3$. Evaluate the following definite integrals.
$\int_1^3 (f(x) + 3g(x))\, dx \qquad \int_3^5 f(x)\, dx$

Evaluate the first by writing it as a sum of two integrals. Find the second integral by using a difference of two integrals involving $f(x)$.

Solutions to worked examples

1. $\displaystyle\int_1^2 \sqrt{1+x^3}dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i{}^*)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1+(x_i{}^*)^3}\Delta x$

2. $\Delta x = \dfrac{b-a}{n} = \dfrac{2-1}{n} = \dfrac{1}{n}$

$x_i = b + i\Delta x = 1 + i/n = 1 + i/5$

$x_0 = 1.0 \quad x_1 = 1 + 1/5 = 1.2 \quad x_2 = 1 + 2/5 = 1.4 \quad x_3 = 1 + 3/5 = 1.6$
$x_4 = 1 + 4/5 = 1.8 \quad x_5 = 1 + 5/5 = 2.0$

The midpoints are the averages of the endpoints.

$x_1{}^* = (x_0 + x_1)/2 = (1.0 + 1.2)/2 = 1.1 \quad x_2{}^* = (x_1 + x_2)/2 = (1.2 + 1.4)/2 = 1.3$
$x_3{}^* = (x_2 + x_3)/2 = (1.4 + 1.6)/2 = 1.5 \quad x_4{}^* = (x_3 + x_4)/2 = (1.6 + 1.8)/2 = 1.7$
$x_5{}^* = (x_4 + x_5)/2 = (1.8 + 2.0)/2 = 1.9$

Evaluate the Riemann sum.

$$\int_1^2 \sqrt{1+x^3}dx = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1+(x_i)^3}\Delta x \approx \sum_{i=1}^{5} \sqrt{1+(x_i{}^*)^3}\Delta x$$
$$= \sum_{i=1}^{5} \sqrt{1+(x_i{}^*)^3}\cdot(1/5) = (1/5)\sum_{i=1}^{5} \sqrt{1+(x_i{}^*)^3}$$
$$= (1/5)(\sqrt{1+(1.1)^3}+\sqrt{1+(1.3)^3}+\sqrt{1+(1.5)^3}+\sqrt{1+(1.7)^3}+\sqrt{1+(1.9)^3})$$
$$\approx (0.2)(1.5268+1.7880+2.0917+2.4317+2.8034) = (0.2)(10.6416) \approx 2.1283.$$

3. The summation formula in this example and in the definition look alike except that in the example the expression $x_i \tan^{-1}(x_i)$ appears in place of $f(x_i{}^*)$. This leads us to take $f(x) = x\tan^{-1}(x)$. The definite integral is

$$\int_0^{\pi/2} x\tan^{-1}(x)\,dx$$

4. The region under the curve between $x=0$ and $x=4$ is composed of a right triangle that has base 2 and height 2, a square of side length 2, and a half circle of radius 1. The area of the triangle is (1/2)(2)(2) = 2 square units. The area of the square is (2)(2) = 4 square units and the area of the half circle is $(1/2)\pi 1^2 = \pi/2$. The total area is

$$\int_0^4 f(x)\,dx = 1+2+\pi/2 = 3+\pi/2$$

The region under the curve between $x=4$ and $x=8$ is composed of a rectangle with a half circle removed. The rectangle has base 4 and height 2 so has area 8 while the removed circle has area $\pi/2$. The total area is

$$\int_4^8 f(x)\,dx = 8-\pi/2$$

To find the final definite integral use the properties.

$$\int_0^8 f(x)\,dx = \int_0^4 f(x)\,dx + \int_4^8 f(x)\,dx = (3+\pi/2)+(8-\pi/2) = 11$$

5. $\int_0^2 x+\sqrt{4-x^2}dx = \int_0^2 x\,dx + \int_0^2 \sqrt{4-x^2}dx = (1/2)(2)(2)+(1/2)\pi 2^2$
$= 2+2\pi$

The first part of the integral is the area of a triangle with base 2 and height 2. The second part is the area of a half circle with radius 2.

6. For x in [0,0.5] the following inequality holds.

$e^0 \leq e^{x^3} \leq e^{(0.5)^3} 1 \leq e^{x^3} \leq 1.1331$

The comparison property of the integral implies the following.

$$1(0.5-0) \leq \int_0^{0.5} e^{x^3}\,dx \leq 1.1331(0.5-0)0.5 \leq \int_0^{0.5} e^{x^3}\,dx \leq 0.5666$$

7. $$\int_0^1 x^2-3x+2dx = \lim_{n\to\infty} \sum_{i=1}^n f(x_i)\Delta x = \lim_{n\to\infty} \sum_{i=1}^n ({x_i}^2-3x_i+2)(1/n)$$

$$= \lim_{n\to\infty} (1/n) \sum_{i=1}^n ({x_i}^2-3x_i+2) = \lim_{n\to\infty} (1/n) \left[\sum_{i=1}^n {x_i}^2 - 3\sum_{i=1}^n x_i + \sum_{i=1}^n 2\right]$$

$$= \lim_{n\to\infty} (1/n) \left[\sum_{i=1}^n (i/n)^2 - 3\sum_{i=1}^n (i/n) + \sum_{i=1}^n 2\right]$$

$$= \lim_{n\to\infty} (1/n^3) \sum_{i=1}^n i^2 - 3\lim_{n\to\infty} (1/n^2) \sum_{i=1}^n i + 2\lim_{n\to\infty} (1/n) \sum_{i=1}^n 1$$

$$= \lim_{n\to\infty} \frac{n(n+1)(2n+1)}{6n^3} - 3\lim_{n\to\infty} \frac{n(n+1)}{2n^2} + 2\lim_{n\to\infty} \frac{n}{n}$$

$$= \lim_{n\to\infty} \frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) - 3\lim_{n\to\infty} \frac{1}{2}\left(1+\frac{1}{n}\right) + 2$$

$$= \frac{1}{6}(2) - 3\frac{1}{2} + 2 = \frac{1}{3} - \frac{3}{2} + 2 = \frac{5}{6}$$

8. $$\int_1^3 (f(x)+3g(x))\,dx = \int_1^3 f(x)\,dx + 3\int_1^3 g(x)\,dx = 2+3(3) = 2+9 = 11$$

$$\int_1^3 f(x)\,dx + \int_3^5 f(x)\,dx = \int_1^5 f(x)\,dx$$

So $2 + \int_3^5 f(x)\,dx = -5$ or $\int_3^5 f(x)\,dx = -5-2 = -7$

5.3 Evaluating Definite Integrals

Key Concepts:

- The Evaluation Theorem
- Indefinite Integrals
- Applications of the definite integral

Skills to Master:

- Interpret definite integrals as net change over a given interval.
- Evaluate definite integrals and find general indefinite integrals.
- Solve application problems using the definite integral.

Discussion:

Section 5.3 shows you an easier method for evaluating many definite integrals. The examples in the last two sections should convince you of the difficulty in evaluating definite integrals directly from the definition as a limit of Riemann Sums. This section and the next relate the process of integration to the process of differentiation covered in the previous chapters of the text.

Key Concept: The Evaluation Theorem

The Evaluation Theorem states that if f is continuous on the interval $[a,b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

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where F is any antiderivative of f. Recall that F is an *antiderivative* of f if $F' = f$. This is much easier than evaluating a limit of Riemann sums. If you can find an antiderivative for a function f, you can then find definite integrals.

Key Concept: Indefinite Integrals

An *indefinite integral* is a convenient notation for an antiderivative of a function.

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$$\int f(x)\,dx = F(x) \text{ means } F'(x) = f(x).$$

Recall that if $F'(x) = G'(x) = f(x)$ on an interval, then F and G differ by a constant on that interval. The indefinite integral notation is often used to represent an entire family of antiderivatives by including a constant C. For example, $\int \cos x\,dx = \sin x + C$ because $\sin x$ is one antiderivative of $\cos x$ and because any two antiderivatives for $\cos x$ differ by a constant. Study and learn the *Table of Indefinite Integrals* in this section.

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Key Concept: Applications of the definite integral

The Net Change Theorem states that the integral of a rate of change is the total change, or that

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

This follows from the Evaluation Theorem and can be used in applications. One key application is that the total change in displacement of a moving object is the definite integral of the velocity of the object. This reinforces the relationships in the Distance Problem from *Section 5.1*.

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SkillMaster 5.9: Interpret definite integrals as net change over a given interval.

The *applications* discussed in this section show you how to interpret the definite integral as the total change over a given interval in a number of different settings. Study these applications and make sure you understand how to use the Net Change Theorem.

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SkillMaster 5.10: Evaluate definite integrals and find general indefinite integrals.

The Evaluation Theorem tells you how to evaluate definite integrals if you know an antiderivative for a given function. It should be easy to find antiderivatives if you remember the *differentiation formulas* that you have learned. For example, since

Chapter 3

$$\frac{d}{dx}(\tan x) = \sec^2(x), \text{ you know that } \int \sec^2 x \; dx = \tan x\,.$$

You can use this knowledge to evaluate definite integrals of $\sec^2(x)$ over intervals where $\sec x$ is defined. In particular,

$$\int_0^{\pi/4} \sec^2 x\, dx = \tan x]_0^{\pi/4} = \tan(\pi/4) - \tan(0) = 1 - 0 = 1 .$$

To find general indefinite integrals, find a particular antiderivative and add a constant C.

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There is a brief *Table of Indefinite Integrals* in this section. Notice that this table is essentially a table of differentiation formulas written in reverse. It should be natural to find antiderivatives if you remember to simply try to undo differentiation.

SkillMaster 5.11: Solve application problems using the definite integral.

The Net Change Theorem can be combined with applications where the derivative of a function is part of the given information. Often the derivative may be given as data or as a graph and you must estimate the net change interval by interval. Study *Example 8* to see how the Net Change Theorem applies to physical situations.

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Worked Examples

For each of the following examples, first try to find the solution without looking at the hint. Cover the right column with a piece of paper. If you need a hint, uncover the column. If you need to see the worked solution, look at the solutions section that follows the examples and hints.

Example	Hint
SkillMaster 5.9.	
1. Suppose that pollutants are introduced to a lake at a rate $g(t)$ where t is in units of years and $t = 0$ at the beginning of 1965. What is the interpretation of $$\int_{25}^{30} g(t)dt?$$	When $t = 25$ the year is $1965 + 25 = 1990$. When $t = 30$ the year is 1995.

SkillMaster 5.10.

Evaluate the integrals using the Evaluation Theorem.

2. $\displaystyle\int_0^1 \left(x^2 - 3x + 2\right) dx$

Evaluate term by term and use the Evaluation Theorem applied to power functions.
$\int x^n\, dx = \dfrac{x^{n+1}}{n+1} + C.$

3. $\displaystyle\int_1^4 \frac{(x+1)^2}{\sqrt{x}}\, dx$

Simplify the expression inside the integral so that the Evaluation Theorem may be applied to power functions.

4. $\displaystyle\int_1^e 2e^x + \frac{1}{x}\, dx$

Integrate each term separately.

5. $\displaystyle\int_0^1 \left(\frac{2}{1+x^2} - 4x\right) dx$

Integrate each term separately.

6. $\displaystyle\int_0^{\pi/4} \sec(x)\tan(x)\, dx$

What function has derivative equal to $\sec(x)\tan(x)$?

7. Verify that the following antidifferentiation formula is correct.

$$\int \ln(x)\, dx = x\ln(x) - x + C, \qquad x > 0$$

Verify by differentiating the right-hand side, i.e. check that
$\dfrac{d}{dx}(x\ln(x) - x + C) = \ln(x).$

8. For the following find the general form of the antiderivative. Illustrate this by graphing several members of this family of functions on the same screen.

$$\int \cos(x) + \sin(x) + x^{1/2}\, dx$$

Antidifferentiate each term separately.

SkillMaster 5.11.

9. A rocket is accelerating at an increasing rate $a(t) = 6t+10$ ft/s^2 from $t=0$ to $t=5$. If the initial velocity is $v(0)=0$ how far did the rocket go in the first 5 seconds? What is the velocity after 5 seconds?

If $v(t)$ is the velocity, then $v(t)$ is an antiderivative of the acceleration $a(t)$. If $p(t)$ is the position then

$$p(t)-p(0)=\int_0^t v(u)du.$$

Solutions to worked examples

1. $\displaystyle\int_{25}^{30} g(t)dt$ represents the total amount of pollutants introduced between the years 1990 and 1995.

2. $$\int_0^1 \left(x^2-3x+2\right)dx=\int_0^1 x^2\,dx-3\int_0^1 x\,dx+2\int_0^1 dx=\frac{1}{3}x^3-\frac{3}{2}x^2+2x\Big]_0^1$$
$$=\left(\frac{1}{3}1^3-\frac{3}{2}1^2+2(1)\right)-\left(\frac{1}{3}0^3-\frac{3}{2}0^2+2(0)\right)=\frac{1}{3}-\frac{3}{2}+2-0=\frac{5}{6}$$

3. $$\int_1^4 \frac{(x+1)^2}{\sqrt{x}}\,dx=\int_1^4 \frac{x^2+2x+1}{\sqrt{x}}\,dx=\int_1^4 \frac{x^2+2x+1}{x^{1/2}}\,dx$$
$$=\int_1^4 \left(x^{3/2}+2x^{1/2}+x^{-1/2}\right)dx=\frac{2}{5}x^{5/2}+\frac{4}{3}x^{3/2}+2x^{1/2}\Big]_1^4$$
$$=\left(\frac{2}{5}(4)^{5/2}+\frac{4}{3}(4)^{3/2}+2(4^{1/2})\right)-\left(\frac{2}{5}(1)^{5/2}+\frac{4}{3}(1)^{3/2}+2(1)^{1/2}\right)$$
$$=\frac{64}{5}+\frac{32}{3}+4-\left(\frac{2}{5}+\frac{4}{3}+2\right)=\frac{62}{5}+\frac{28}{3}+2$$
$$=12+2/5+9+1/3+2=\frac{356}{15}\approx 23.7333$$

4. $$\int_1^e 2e^x+\frac{1}{x}\,dx=2\int_1^e e^x\,dx+\int_1^e \frac{1}{x}\,dx=2e^x+\ln|x|\Big]_1^e$$
$$=2e^e+\ln(e)-(2e+\ln(1))=2e^e-2e+1\approx 25.8720$$

5. $\int_0^1 \left(\frac{2}{1+x^2} - 4x\right)\,dx = 2\int_0^1 \frac{1}{1+x^2}\,dx - 4\int_0^1 x\,dx$

$= 2\tan^{-1}(x) - 4\left(\frac{x^2}{2}\right)\Big]_0^1 = 2\tan^{-1}(1) - 4\cdot\frac{1}{2} - \left(2\tan^{-1}(0) - 4\cdot\frac{0}{2}\right)$

$= 2\left(\frac{\pi}{4}\right) - 2 - (0-0) = \pi/2 - 2$

6. $\int_0^{\pi/4} \sec(x)\tan(x)\,dx = \sec(x)]_0^{\pi/4} = \sec(\pi/4) - \sec(0) = \sqrt{2} - 1 \approx 0.4142$

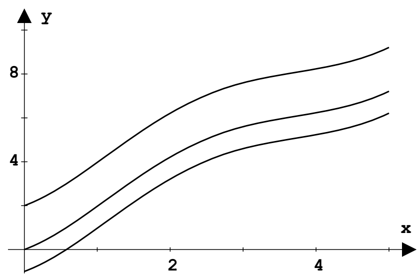
7. $\frac{d}{dx}(x\ln(x) - x + C) = \frac{d}{dx}(x\ln(x)) - \frac{d}{dx}(x) + \frac{d}{dx}(C)$

$= x\frac{d}{dx}(\ln(x)) + \ln(x)\frac{d}{dx}(x) - 1 + 0 = x\left(\frac{1}{x}\right) + \ln(x) - 1$

$= 1 + \ln(x) - 1 = \ln(x)$

8. $\int \cos(x) + \sin(x) + x^{1/2}\,dx = \int \cos(x)\,dx + \int \sin(x)\,dx + \int x^{1/2}\,dx$

$= \sin(x) - \cos(x) + \frac{2}{3}x^{3/2} + C$



9. $v(t) = v(t) - v(0) = \int_0^t a(u)\,du$

$= \int_0^t (6u+10)\,du = 3u^2 + 10u]_0^t = 3t^2 + 10t$

The velocity at $t = 5$ is $v(5) = 3(5^2) + 10(5) = 125$ ft/s.

The distance traveled is $p(5) - p(0) = \int_0^5 v(u)\,du = \int_0^5 \left(3u^2 + 10u\right)\,du$

$= u^3 + 5u^2\Big]_0^5 = 5^3 + 5(5^2) = 250$ ft

5.4 The Fundamental Theorem of Calculus

Key Concepts:

- Fundamental Theorem of Calculus

Skills to Master:

- Evaluate and analyze functions of the form $g(x) = \int_a^x f(t)\,dt$.
- Use the Fundamental theorem to find derivatives of functions of the form $g(x) = \int_a^x f(t)\,dt$.

Discussion:

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Section 5.4 explains the most important result in Calculus, the Fundamental Theorem of Calculus. The *Evaluation Theorem* from the previous section is the second part of the Fundamental Theorem of Calculus. The first part states in a precise way that the processes of differentiation and integration are inverses of each other. Spending time understanding the material in this section will help you in the rest of the sections of Chapters 5 and 6.

Key Concept: Fundamental Theorem of Calculus

There are two parts to the Fundamental Theorem of Calculus. The first part states that if $f(x)$ is continuous on $[a,b]$, and if $g(x) = \int_a^x f(t)dt$, then

$$g'(x) = f(x)$$

One thing that you need to understand about the notation $\int_a^x f(t)dt$ is that a is fixed and that x is the variable. One way of stating this part of the Fundamental Theorem of Calculus is that the derivative of the integral of a function (with respect to the upper limit of integration) is the function being integrated. Make sure that you understand the motivation given for the Fundamental Theorem of Calculus and the *proof* of this theorem given in this section.

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The second part of the Fundamental Theorem of Calculus is the Evaluation Theorem from the previous section.

SkillMaster 5.12: Evaluate and analyze functions of the form $\mathbf{g(x) = \int_a^x f(t)dt}$.

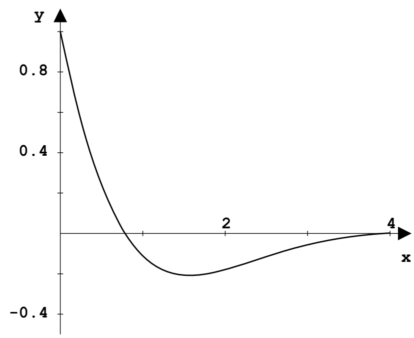
To evaluate a function of the form $g(x) = \int_a^x f(t)dt$, apply the second part of the Fundamental Theorem of Calculus (the Evaluation Theorem) by finding an antiderivative F for f. Then $g(x)$ is $F(x) - F(a)$. To analyze functions of the form $g(x) = \int_a^x f(t)dt$, use the interpretation of $\int_a^x f(t)dt$ as area.

SkillMaster 5.13: Use the Fundamental theorem to find derivatives of functions of the form $\mathbf{g(x) = \int_a^x f(t)dt}$.

To differentiate functions of the form $g(x) = \int_a^x f(t)dt$, use the first part of the Fundamental Theorem of Calculus which states that $g'(x) = f(x)$.

Worked Examples

For each of the following examples, first try to find the solution without looking at the hint. Cover the right column with a piece of paper. If you need a hint, uncover the column. If you need to see the worked solution, look at the solutions section that follows the examples and hints.

Example	Hint
SkillMaster 5.12.	
1. Let $g(x) = \int_0^x f(t)dt$ where the graph of f on the interval $0,4$ is shown. It is given that the area above the $x-$axis and under the graph is equal to the area below the $x-$axis and above the graph.  What is $g'(1)$ (to one decimal place)? What is $g(4)$? On what interval is g increasing? Where does g have a maximum? Make a rough sketch of g.	The derivative of g is f. Use what you know about how the derivative of a function and the function are related.
SkillMaster 5.13.	
2. For $g(x)$ given below find $g'(x)$ in two ways, first by evaluating the integral and differentiating, and second by using the Fundamental Theorem of Calculus. $$g(x) = \int_2^x \cos(t) - t\,dt$$	When integrating, integrate each term separately.

3. For $g(x)$ given below, find $g'(2)$.

$$g(x) = \int_1^x \frac{1}{1+u^3} du$$

Use the Fundamental Theorem of Calculus to find $g'(x)$, then evaluate at $x = 2$.

4. For $g(x)$ given below, find $g'(x)$ in two ways, first by evaluating the integral and differentiating, and second by using the Fundamental Theorem of Calculus.

$$g(x) = \int_0^{x^2} e^u du$$

When using the Fundamental Theorem of Calculus, don't forget to use the Chain Rule.

5. For $g(x)$ given below, find $g'(x)$.

$$g(x) = \int_{\ln(x)}^x e^t dt$$

You can apply the Fundamental Theorem of Calculus by first using the properties of the integral: $g(x) = \int_1^x e^t dt - \int_1^{\ln(x)} e^t dt$

Solutions to worked examples

1. $g'(1) = f(1) \approx -0.1$.

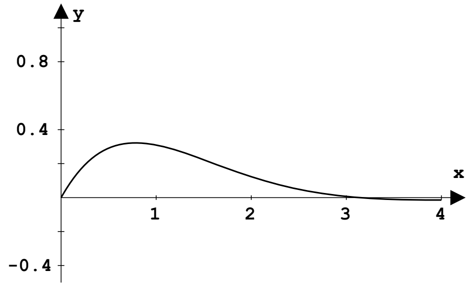
$g(4) = 0$ because it is the area above the axis minus the area below the axis and it is given that these areas are equal.

g is increasing whenever f is positive. In this case this is the interval (0,0.8).

g is decreasing when f is negative.

At $x \approx 0.8$, g goes from increasing to decreasing so has a maximum there.

A sketch of g is shown.



2. $g(x) = \int_2^x \cos(t) - t\,dt = \sin(t) - t^2/2]_2^x$

$= \sin(x) - x^2/2 - \sin(2) + 2^2/2 = \sin(x) - x^2/2 - \sin(2) + 2$

$g'(x) = \frac{d}{dx}\left(\sin(x) - x^2/2\ - \sin(2) + 2\right) = \cos(x) - 2x/2 + 0 = \cos(x) - x$

Using the Fundamental Theorem of Calculus: $g'(x) = \cos(x) - x$,

which is obtained by plugging x into the expression in the integrand.

3. $g'(x) = \frac{1}{1+x^3}$ $\qquad$ $g'(2) = \frac{1}{1+2^3} = 1/9.$

4. $g(x) = \int_0^{x^2} e^u du = e^u]_0{}^{x^2} = e^{x^2} - 1$

$g'(x) = \frac{d}{dx}(e^{x^2} - 1) = e^{x^2}(\frac{d}{dx}(x^2)) = 2xe^{x^2}$

Using the Fundamental Theorem of Calculus you get the same expression.

$g'(x) = \frac{d}{dx}\left(\int_0^{x^2} e^u\,du\right) = e^{x^2}\frac{d}{dx}(x^2) = 2xe^{x^2}$

The point is to recognize the outer function as the integral and the inner function as x^2. This may be more clear in Leibniz notation.

Let $v = x^2$.

$g'(x) = \frac{d}{dx}\left(\int_0^{x^2} e^u\,du\right) = \frac{d}{dv}\left(\int_0^{v} e^u\,du\right)\frac{dv}{dx} = e^v\frac{dv}{dx}$

$= e^{x^2}\frac{d}{dx}(x^2) = e^{x^2}(2x) = 2xe^{x^2}$

5. $g'(x) = e^x - e^{\ln(x)} \cdot \frac{d}{dx}(\ln(x)) = e^x - x\left(\frac{1}{x}\right) = e^x - 1$

5.5 The Substitution Rule

Key Concepts:

- The substitution rule for definite and indefinite integrals
- Integrals of symmetric functions

Skills to Master:

- Evaluate definite and indefinite integrals using substitution.
- Use symmetry to simplify calculations.

Discussion:

Section 5.5 gives a method that will transform integrals into other forms that may be easier to integrate. If you recognize a function $f(x)$ as the derivative of another function $F(x)$ it is not difficult to evaluate integrals of the form

$$\int_a^b f(x)\,dx$$

by using the Evaluation Theorem. The integral is then equal to $F(b) - F(a)$. The Substitution Rule for indefinite and definite integrals allows you to find integrals of many functions that you do not initially recognize as derivatives of other functions.

Key Concept: The substitution rule for definite and indefinite integrals

The Substitution Rule for indefinite integrals states:

If $u = g(x)$ is a differentiable function whose range is an interval I and if f is continuous on I, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du$$

The Substitution Rule for definite integrals states:

If $g'(x)$ is a continuous on $[a,b]$ and if f is continuous on the range of g, then

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

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Both of these are consequences of the *Chain Rule* and of course, the Fundamental Theorem of Calculus. Pay attention to the *explanation* of the Substitution Rule in this section. A common mistake is confusing the limits of integration in the Substitution Rule for definite integrals. One way to understand the Substitution Rule is to realize that dx and du behave like differentials. If $u = g(x)$ then $du = g\prime(x)dx$. Putting integral signs in front of these differentials gives the Substitution Rule for Integrals.

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Key Concept: Integrals of symmetric functions

Integrating symmetric functions over intervals centered about 0 can be simplified by using the results in this section. These results make use of the Substitution Rule. The results state that if f is continuous on $[-a,a]$, then

(a) If f is even $[f(-x) = f(x)]$, then $\displaystyle\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$ and

(b) If f is odd $[f(-x) = -f(x)]$, then $\displaystyle\int_{-a}^{a} f(x)\,dx = 0$

SkillMaster 5.14: Evaluate definite and indefinite integrals using substitution.

The key to using substitution to evaluate definite and indefinite integrals is deciding what to choose for $u = g(x)$. You then differentiate $g(x)$ and replace $g'(x)\,dx$ in the original integral by du. For example, in $\int x^2 e^{x^3}\,dx$, you can let $u = x^3$.
Then $du = 3x^2\,dx$, so $x^2 dx = \dfrac{1}{3}\,du$ and

$$\int x^2 e^{x^3}\,dx = \int \frac{1}{3}e^u\,du = \frac{1}{3}\int e^u\,du = \frac{1}{3}e^u + C = \frac{1}{3}e^{x^3} + C$$

Of course, like learning to drive a car or learning calligraphy, there is no substitute for practice. Work on as many problems in this section as possible. Piecing together the concepts introduced so far in this chapter allows you to evaluate many integrals. Algebraic manipulation combined with substitution can change many integrals to a form that can be evaluated. If the integral represents an area that you know how to compute from geometric considerations, you can use this relation to compute the integral.

SkillMaster 5.15: Use symmetry to simplify calculations.

If you can recognize a function as an even or odd function,

$$[f(-x) = f(x) \text{ or } f(-x) = -f(x)] \text{ respectively}$$

then you can use the properties of integrating symmetric functions from this section to simplify calculations.

Worked Examples

For each of the following examples, first try to find the solution without looking at the hint. Cover the right column with a piece of paper. If you need a hint, uncover the column. If you need to see the worked solution, look at the solutions section that follows the examples and hints.

Example	Hint
SkillMaster 5.14.	
Find the following integrals.	
1. $\int \sin(5x)\,dx$	This is the simplest type of substitution. The innermost function is linear. Try the substitution $u = 5x$.
2. $\int 2x(2x-1)^{10}\,dx$	Again the innermost function is linear. Try $u = 2x-1$.
3. $\int 4x\cos(x^2-1)\,dx$	The innermost expression is x^2-1 and its derivative is $2x$ which appears on the outside. Use the substitution $u = x^2 - 1$.

4. $\displaystyle\int \sin^2(x)\cos^3(x)\,dx$

Use the identity $\cos^2(x) = 1 - \sin^2(x)$ and try the substitution $u = \sin(x)$.

5. $\displaystyle\int_0^{1/\sqrt{2}} \frac{\sin^{-1}(x)}{\sqrt{1-x^2}}\,dx$

Try $u = \sin^{-1}(x)$. Don't forget to use appropriate limits of integration.

6. $\displaystyle\int_1^e e^{(\ln(x))^2}\frac{\ln(x)}{x}\,dx$

The innermost expression is $\ln(x)$ and its derivative $1/x$ appears on the outside. Try $u = \ln(x)$. This may take more than one substitution to solve.

7. $\displaystyle\int_1^e (1/x)\sqrt{1-\ln(x)^2}\,dx$

Try the substitution $u = \ln(x)$. The resulting integral may be found using the area interpretation.

8. Suppose that $\displaystyle\int_0^2 f(x)\,dx = 6$ and $\displaystyle\int_2^4 f(x)\,dx = -2$. Find $\displaystyle\int_0^2 xf(x^2)\,dx$

Formally apply the substitution $u = x^2$. Don't forget to change the limits of integration.

SkillMaster 5.15.

Find the following integrals.

9. $\displaystyle\int_{-1}^1 \frac{x}{\sqrt{1+x^4}}\,dx$

One could try many ways to find an antiderivative and still fail. If a problem is too hard there may be an easier way. Notice that the function in the integral is an odd function.

10. $\int_{-1}^{1}\left(3x^{10}-2x^6+x^2\right)dx$

Use symmetry to simplify this calculation.

Solutions to worked examples

1. $u=5x \qquad du=5\,dx \qquad \frac{1}{5}du=dx$

$$\int \sin(5x)\,dx = \int \sin(u)\frac{1}{5}\,du = \frac{1}{5}\int \sin(u)\,du$$
$$= -\frac{1}{5}\cos(u)+C = -\frac{1}{5}\cos(5x)+C$$

2. $u=2x-1 \qquad du=2\,dx \qquad \frac{1}{2}du=dx \qquad 2x=u+1$

$$\int (u+1)u^{10}\frac{1}{2}\,du = \frac{1}{2}\int u^{11}+u^{10}\,du = \frac{1}{2}\frac{u^{12}}{12}+\frac{1}{2}\frac{u^{11}}{11}+C$$
$$= \frac{(2x+1)^{12}}{24}+\frac{(2x+1)^{11}}{22}+C$$

3. $u=x^2-1 \qquad du=2x\,dx$

$$\int 4x\cos(x^2-1)\,dx = \int 2\cos(u)\,du = 2\sin(u)+C = 2\sin(x^2-1)+C$$

4. $\int \sin^2(x)\cos^3(x)\,dx = \int \sin^2(x)(1-\sin^2(x))\cos(x)\,dx$

$$\left[u=\sin(x) \qquad du=\cos(x)\,dx\right]$$
$$= \int u^2(1-u^2)\,du = \int u^2-u^4\,du = \frac{u^3}{3}-\frac{u^5}{5}+C = \frac{\sin^3(x)}{3}-\frac{\sin^5(x)}{5}+C$$

5. $u=\sin^{-1}(x) \qquad du=\frac{1}{\sqrt{1-x^2}}\,dx$

when $x=0,\ u=0$ when $x=1/\sqrt{2},\ u=\pi/4$

$$\int_0^{1/\sqrt{2}} \frac{\sin^{-1}(x)}{\sqrt{1-x^2}}\,dx = \int_0^{\pi/4} u\,du = \left.\frac{u^2}{2}\right]_0^{\pi/4} = \frac{\pi^2}{32}$$

6. $u = \ln(x) \qquad du = \frac{1}{x}\,dx$

when $x = 1,\ u = \ln(1) = 0 \qquad$ when $x = e,\ u = \ln(e) = 1$

$$\int_1^e e^{(\ln(x))^2}\frac{\ln(x)}{x}\,dx = \int_0^1 e^{u^2}u\,du$$

Now let $v = u^2 \qquad dv = 2u\,du$

$\frac{1}{2}dv = u\,du$

when $u = 0,\ v = 0$when $u = 1,\ v = 1$

$$\int_0^1 e^{u^2}u\,du = \frac{1}{2}\int_0^1 e^v\,dv = \frac{1}{2}e^v\bigg]_0^1 = (e-1)/2$$

(It is also possible to use the substitution $u = (\ln(x))^2$ and solve this in one step.)

7. $u = \ln(x) \qquad du = (1/x)\,dx$

when $x = 1,\ u = 0 \qquad$ when $x = e,\ u = 1$

$$\int_1^e (1/x)\sqrt{1-\ln(x)^2}\,dx = \int_0^1 \sqrt{1-u^2}\,du = \pi/4$$

This last equality holds because the area under the curve is the area of a quarter circle of radius 1.

8. $u = x^2 \qquad du = 2x\,dx$

when $x = 0,\ u = 0 \qquad$ when $x = 2,\ u = 4$

$$\int_0^2 xf(x^2)\,dx = \frac{1}{2}\int_0^4 f(u)\,du = \frac{1}{2}\left(\int_0^2 f(u)\,du + \int_2^4 f(u)\,du\right) = \frac{1}{2}(6-2) = 2$$

9. Because x is an odd function and x^4 is an even function, the integrand is odd. This is an integral of an odd function over a symmetric interval, so must be 0.

$$\int_{-1}^1 \frac{x}{\sqrt{1+x^4}}\,dx = 0$$

10. This may be done in the usual way. Simpler is to notice that the function is even and is equal to twice the integral from 0 to 1.

$$\int_{-1}^1 \left(3x^{10} - 2x^6 + x^2\right)\,dx = 2\int_0^1 \left(3x^{10} - 2x^6 + x^2\right)\,dx$$

$$= 2\left(\frac{3x^{11}}{11} - \frac{2x^7}{7} + \frac{x^3}{3}\right)\bigg]_0^1 = \frac{6}{11} - \frac{4}{7} + \frac{2}{3} = \frac{148}{231} \approx 0.6407$$

5.6 Integration by Parts

Key Concepts:

- Evaluation of indefinite integrals by integration by parts: $\displaystyle\int u\,dv = uv - \int v\,du$
- Evaluation of definite integrals by integration by parts
- Reduction formulas

Skills to Master:

- Use integration by parts alone or combined with previous methods to evaluate integrals.
- Evaluate integrals by repeated application of integration by parts.
- Prove and apply reduction formulas to evaluate integrals.

Discussion:

Section 5.6 explains integration by parts which is the second main technique of integration that you need to learn. The first technique was substitution introduced in the previous section. These two techniques together with algebraic manipulations and trigonometric identities are the basis for most techniques of integration. Integration by parts is another technique that will require practice on your part to master.

Key Concept: Evaluation of indefinite integrals by integration by parts: $\displaystyle\int \mathbf{u}\,\mathbf{dv} = \mathbf{uv} - \int \mathbf{v}\,\mathbf{du}$

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The Chain Rule was the basis for the substitution technique from the previous section. The *Product Rule* is the basis for Integration by Parts. Make sure that you understand the explanation for Integration by Parts given in this section. In using this technique you break up the integrand $f(x)\,dx$ into two parts, one represented as u and one represented as dv. You then replace the given integral $\int u\,dv$ by $uv - \int v\,du$. Remember to use the pattern

$$\begin{aligned} u &= \boxed{} \qquad & dv &= \boxed{} \\ du &= \boxed{} \qquad & v &= \boxed{} \end{aligned}$$

The intuition behind integration by parts lies in thinking in terms of differentials. Evaluating, using the Product Rule, $d(uv) = udv + vdu$.

$$uv = \int d(uv) = \int u\,dv + \int v\,du$$

or

$$\int u\,dv = uv - \int v\,du$$

Key Concept: Evaluation of definite integrals by integration by parts

The formula for evaluation of definite integrals using Integration by Parts is

$$\int_a^v u\,dv = uv]_a^b - \int_a^b v\,du$$

Note that this is the same as the formula for Integration by Parts for indefinite integrals with the addition of upper and lower limits of integration. This should be easier to remember than the Substitution rule since the limits of integration don't change.

Key Concept: Reduction formulas

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A reduction formula reduces an integral involving a power of a function to a simpler integral involving a lower power of a function. *Example 6* in the text shows how this works for $\int \sin^n x\,dx$. Many of the integrals formulas that you find in *integration tables* at the end of the text involve reduction formulas.

SkillMaster 5.16: Use integration by parts alone or combined with previous methods to evaluate integrals

Once you have a new technique such as Integration by Parts, you can combine it with the other techniques that you already know to evaluate new kinds of integrals. To become proficient at combining the techniques that you know, you need to work as many problems as you can. Often, you will need to try different ways to break up $f(x)\,dx$ as $u\,dv$ before you find one that works.

SkillMaster 5.17: Evaluate integrals by repeated application of integration by parts.

Repeated applications of Integration by Parts are sometimes needed before an integral can be evaluated. *Examples 3 and 4* in this section show two different kinds of problems where repeated application of Integration by Parts may be needed. In $\int x^2 e^x\,dx$, two applications of Integration by Parts leaves you with an integral of the form $\int e^x\,dx$ which you know how to evaluate. In $\int e^x \sin x\,dx$, two applications of Integration by Parts leaves you with an integral of the form $-\int e^x \sin x\,dx$ which can then be combined using algebra with the first integral to give you an answer.

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SkillMaster 5.18: Prove and apply reduction formulas to evaluate integrals.

The technique of Integration by Parts when applied to expressions involving powers of functions often yields an expression with a lower power of the function. Work as many of the problems that use this technique as possible.

Worked Examples

For each of the following examples, first try to find the solution without looking at the hint. Cover the right column with a piece of paper. If you need a hint, uncover the column. If you need to see the worked solution, look at the solutions section that follows the examples and hints.

Example	Hint

SkillMaster 5.16.

Compute the following integrals.

1. $\displaystyle\int xe^{-3x}\,dx$

This is the type of problem that is easy to recognize as requiring integration by parts. The derivative of x is 1 and the integral of e^{-3x} is $-3e^{-x}$.

2. $\displaystyle\int x\sec^2(x)\ dx$

Again x appears together with a function $\sec^2(x)$ that has a recognizable integral, namely $\tan(x)$.

3. $\displaystyle\int x\ln(\sqrt[3]{x})\,dx$

Use the laws of logarithms to simplify this.

4. $\displaystyle\int_0^{1/\sqrt{2}} \sin^{-1}(x)\,dx$

No substitution is possible here and the only possibility to integrate by parts is to integrate 1 and differentiate $\sin^{-1}(x)$.

SkillMaster 5.17.

Compute the following integrals.

5. $\displaystyle\int_1^e x(\ln(x))^2\,dx$

Differentiate the term involving $\ln(x)$. This may have to be done twice.

SkillMaster 5.18.

6. Find a reduction formula for

$$\int x^n e^{-x}\,dx$$

Use integration by parts. Differentiate x^n.

Solutions to worked examples

1. $u = x \qquad dv = e^{-3x}\,dx \qquad du = dx \qquad v = (-1/3)e^{-3x}$

$$\int xe^{-3x}\,dx = (-1/3)xe^{-3x} + (1/3)\int e^{-3x}\,dx = (-1/3)xe^{-3x} - (1/9)e^{-3x} + C$$

2. $u = x \qquad dv = \sec^2(x)\,dx \qquad du = dx \qquad v = \tan(x)$

$$\int x\sec^2(x)\,dx = x\tan(x) - \int \tan(x)\,dx = x\tan(x) + \ln|\cos(x)| + C$$

3. $\displaystyle\int x\ln(\sqrt[3]{x})\,dx = \frac{1}{3}\int x\ln(x)\,dx$

$u = \ln(x) \qquad dv = x\,dx \qquad du = (1/x)\,dx \qquad v = x^2/2$

$$\frac{1}{3}\int x\ln(x)\,dx = \frac{1}{6}x^2\ln(x) - \frac{1}{6}\int x\,dx = \frac{1}{6}x^2\ln(x) - \frac{x^2}{12} + C$$

4. $u = \sin^{-1}(x) \qquad dv = 1dx \qquad du = \dfrac{1}{\sqrt{1-x^2}}dx \qquad v = x$

$$\int_0^{1/\sqrt{2}} \sin^{-1}(x)\,dx = x\sin^{-1}(x)\Big]_0^{1/\sqrt{2}} - \int_0^{1/\sqrt{2}} \frac{x}{\sqrt{1-x^2}}\,dx$$

$$= \frac{\pi}{4\sqrt{2}} + \frac{1}{2}\int_0^{1/\sqrt{2}} \frac{-2x}{\sqrt{1-x^2}}\,dx$$

Try a substitution to evaluate the last integral.

Use the variable w since u has already been used.

let $w = 1 - x^2 \qquad dw = -2x\,dx$

when $x = 0,\ w = 1 \qquad$ when $x = 1/\sqrt{2},\ w = 1/2$

$$\frac{\pi}{4\sqrt{2}} + \frac{1}{2}\int_0^{1/\sqrt{2}} \frac{-2x}{\sqrt{1-x^2}}\,dx = \frac{\pi}{4\sqrt{2}} + \frac{1}{2}\int_1^{1/2} \frac{1}{\sqrt{w}}dw$$

$$= \frac{\pi}{4\sqrt{2}} + \frac{1}{2}\frac{w^{1/2}}{1/2}\Bigg]_1^{1/2} = \frac{\pi}{4\sqrt{2}} + (1/2)^{1/2} - 1 = \frac{\pi}{4\sqrt{2}} + \frac{1}{\sqrt{2}} - 1$$

5. $u = (\ln(x))^2 \qquad dv = x\,dx \qquad du = 2\ln(x)(1/x)\,dx \qquad v = x^2/2$

$$\int_1^e x(\ln(x))^2\,dx = \frac{x^2(\ln(x))^2}{2}\Bigg]_1^e - \int_1^e x\ln(x)\,dx = \frac{e^2}{2} - \int_1^e x\ln(x)\,dx$$

$u = \ln(x) \qquad dv = x\,dx \qquad du = (1/x)\,dx \qquad v = x^2/2$

$$\frac{e^2}{2} - \int_1^e x\ln(x)\,dx$$

$$= \frac{e^2}{2} - \frac{x^2\ln(x)}{2}\Bigg]_1^e + \int_1^e (x/2)\,dx = \frac{e^2}{2} - \left(\frac{e^2}{2} - 0\right) + \int_1^e (x/2)\,dx$$
$$= \frac{x^2}{4}\Bigg]_1^e = \frac{e^2-1}{4} \approx 1.5973$$

6. $u = x^n \qquad dv = e^{-x}\,dx \qquad du = nx^{n-1}dx \qquad v = -e^{-x}$

$$\int x^n e^{-x}\,dx = -x^n e^{-x} + n\int x^{n-1}e^{-x}\,dx$$

5.7 Additional Techniques of Integration

Key Concepts:

- Special substitutions and partial fractions

Skills to Master:

- Integrate forms like $\int \sin^n(x)\cos^m(x)dx$.
- Integrate forms containing $\sqrt{a^2 \pm r^2}$ and $\sqrt{r^2 - a^2}$.
- Integrate forms like $\displaystyle\int \frac{P(x)}{Q(x)}dx$ where $P(x)$ and $Q(x)$ are polynomials.

Discussion:

Section 5.7 provides you with some additional integration methods to add to your toolbox of techniques. Most integrals cannot be solved exactly. That is, it is not possible to find antiderivatives of most functions. Either they do not exist or are too complicated. This section gives some additional techniques that may allow you to compute integrals in special cases. All of these techniques are used in computer aided integration programs.

Key Concept: Special substitutions and Partial Fractions

Two of the integration techniques shown in this section rely on trigonometric identities like $\sin^2 x + \cos^2 x = 1$ and $\tan^2 x + 1 = \sec^2 x$. A third technique allows you to integrate quotients of polynomials. As usual, the only way to fully understand and be able to use these techniques is to read the examples and do the exercises at the end of Section 5.7. These new techniques will allow you to integrate a new group of functions.

SkillMaster 5.19: Integrate forms like $\int \sin^n(x)\cos^m(x)dx$.

If n is odd, the integral may be evaluated by replacing $\sin^n(x)$ by $(1-\cos^2 x)^{\frac{n-1}{2}}\sin x$. Then use the substitution $u=\cos x$, $du=-\sin x\,dx$. This reduces the integral to a polynomial in u.

If m is odd, the integral may be evaluated by replacing $\cos^m(x)$ by $(1-\sin^2 x)^{\frac{m-1}{2}}\cos x$. Then use the substitution $u=\sin x$, $du=\cos x\,dx$. This also reduces the integral to a polynomial in u.

If both n and m are even, then the integral must be reduced using one or both of the identities

$$\begin{aligned}\sin^2 x &= \frac{1}{2}(1-\cos(2x))\\ \cos^2 x &= \frac{1}{2}(1+\cos(2x))\end{aligned}$$

After repeating this technique if necessary, the integral will be reduced to the case where either the sine or the cosine function is raised to an odd power.

SkillMaster 5.20: Integrate forms containing $\sqrt{a^2 \pm r^2}$ and $\sqrt{r^2-a^2}$.

If the form to be integrated contains $\sqrt{a^2-r^2}$ for $a>0$, and there is no other obvious way to antidifferentiate, try the following substitution:

$$\begin{aligned}a\sin(u) &= r\\ a\cos(u)du &= dr\\ \sqrt{a^2-r^2} &= a\cos(u)\end{aligned}$$

Similarly, the substitution $a\sec u=r$ may be useful in antidifferentiating forms containing $\sqrt{r^2-a^2}$ (since $\sqrt{a^2\sec^2 u-a^2}=a\tan u$) and $a\tan u=r$ may be useful in antidifferentiating forms containing $\sqrt{a^2\pm r^2}$ (since $\sqrt{a^2\tan^2 u+a^2}=a\sec u$).

SkillMaster 5.21: Integrate forms like $\int \frac{P(x)}{Q(x)}dx$ where $P(x)$ and $Q(x)$ are polynomials.

If we wish to find an antiderivative for $\frac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomials, first do long division if necessary to get the degree of the numerator less than the degree of the denominator. If $Q(x)$ factors completely, then the antiderivative is a sum of logarithms. See the worked examples for a specific instance of this.

In general, you will need to express the quotient as a sum of partial fractions. Study carefully *Notes 1-4* in this section to see what to do in the various cases that arise.

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Worked Examples

For each of the following examples, first try to find the solution without looking at the hint. Cover the right column with a piece of paper. If you need a hint, uncover the column. If you need to see the worked solution, look at the solutions section that follows the examples and hints.

Example	Hint
SkillMaster 5.19.	
1. $\int \sin^3 x \cos^2 x \, dx$	3 is an odd number, so write $\sin^3 x$ as $\sin x(\sin^2 x) = \sin x(1-\cos^2 x)$. Use $u = \cos x$ $du = -\sin x \, dx$.
2. $\int_0^{2\pi} \sin^4 \theta \, d\theta$	4 is even, so use the identity for $\sin^2 \theta$ in terms of $\cos 2\theta$. Use the identity for $\cos^2 2\theta$ in terms of $\cos 4\theta$.
SkillMaster 5.20.	
3. $\int_0^{\sqrt{2}} r^3 \sqrt{4-r^2} dr$	Since there is no other obvious way to integrate, try $2\sin u = r$.

SkillMaster 5.21.

4. $\int \frac{x-7}{x^2-2x-3}dx$

Write x^2-2x-3 as $(x+1)(x-3)$.

Solutions to worked examples

1. $\int \sin^3 x \cos^2 x\, dx = \int \sin x(1-\cos^2 x)\cos^2 x\, dx = \int \cos^2 x(1-\cos^2 x)\sin x\, dx$

$= -\int u^2(1-u^2)du = \int -u^2+u^4 du = -\frac{u^3}{3}+\frac{u^5}{5}+C = -\frac{\cos^3 x}{3}+\frac{\cos^5 x}{5}+C$

2. $\int_0^{2\pi} \sin^4\theta\, d\theta = \int_0^{2\pi}(\sin^2\theta)^2 d\theta = \int_0^{2\pi}\left(\frac{1}{2}(1-\cos 2\theta)\right)^2 d\theta$

$= \frac{1}{4}\int_0^{2\pi} 1-2\cos 2\theta+\cos^2 2\theta d\theta = \frac{1}{4}(\theta-\sin 2\theta)\Big|_0^{2\pi}+\frac{1}{4}\int_0^{2\pi}\cos^2 2\theta d\theta$

$= \frac{\pi}{2}+\frac{1}{4}\int_0^{2\pi}\frac{1}{2}(1+\cos 4\theta)d\theta = \frac{\pi}{2}+\frac{1}{8}(\theta+\frac{\sin 4\theta}{4})\Big|_0^{2\pi} = \frac{\pi}{2}+\frac{\pi}{4} = \frac{3\pi}{4}$

3. $2\sin u = r$ $\quad$ $2\cos u du = dr$ $\quad$ The limits become 0 and $\frac{\pi}{4}$

$\int_0^{\sqrt{2}} r^3\sqrt{4-r^2}dr = \int_0^{\frac{\pi}{4}}(2\sin u)^3\sqrt{4-4\sin^2 u}\,2\cos u du$

$= \int_0^{\frac{\pi}{4}} 8\sin^3 u\sqrt{4(1-\sin^2 u)}\,2\cos u du = 16\int_0^{\frac{\pi}{4}}\sin^3 u\sqrt{4(\cos^2 u)}\cos u du$

$= 32\int_0^{\frac{\pi}{4}}\sin^3 u\cos^2 u du$

By the first problem under Skillmaster 5.19 above,

$\int \sin^3 u\cos^2 u du = -\frac{\cos^3 x}{3}+\frac{\cos^5 x}{5}+C$

So the integral we are working with becomes

$32\left(-\frac{\cos^3 x}{3}+\frac{\cos^5 x}{5}\right)\Big|_0^{\frac{\pi}{4}} = 32\left[\left(-\frac{\left(\frac{\sqrt{2}}{2}\right)^3}{3}+\frac{\left(\frac{\sqrt{2}}{2}\right)^5}{5}\right)-\left(-\frac{1}{3}+\frac{1}{5}\right)\right]$

$$= 32\left[\left(-\frac{2\sqrt{2}}{24}+\frac{4\sqrt{2}}{160}\right)-\left(-\frac{2}{15}\right)\right] = 32\left[-\frac{7}{120}\sqrt{2}+\frac{2}{15}\right] = \frac{28}{15}\sqrt{2}+\frac{64}{15}$$

4. We need to write $\frac{x-7}{x^2-2x-3}$ as a sum of partial fractions.

$$\frac{x-7}{x^2-2x-3} = \frac{x-7}{(x+1)(x-3)} = \frac{A}{x+1}+\frac{B}{x-3}$$

Now $\frac{A}{x+1}+\frac{B}{x-3} = \frac{A(x-3)+B(x+1)}{(x+1)(x-3)} = \frac{(A+B)x+(B-3A)}{(x+1)(x-3)}$

Equate coefficients and solve.

$A+B=1-3A+B=-7$

The solution is $A=2$, $B=-1$.

We can now rewrite the original problem.

$$\int\frac{x-7}{x^2-2x-3}dx = \int\frac{2}{x+1}dx+\int\frac{-1}{x-3}dx = 2\ln|x+1|-\ln|x-3|+C$$

5.8 Integration Using Tables and Computer Algebra Systems

Key Concepts:

- Tables of integrals and equivalent forms of integration problems
- Computer Algebra Systems
- Equivalent answers to integration problems

Skills to Master:

- Use algebraic manipulations to enable application of integral tables.
- Use a CAS to evaluate integrals.

Discussion:

Section 5.8 gives you strategies for effectively using integral tables and Computer Algebra Systems (CAS). Many integrals that you will encounter in applications are more difficult to evaluate than the ones you have seen so far. When you encounter such integrals, you will need to use a table of integrals or a CAS. The table in the back Endpapers (inside cover and the opposite pages) of the text give evaluation formulas for many integrals. If you have access to a CAS, now is the time to learn to how to use it find symbolic antiderivatives.

Key Concept: Tables of integrals and equivalent forms of integration problems

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There are many places to find tables of integrals. The back *Endpapers* in the text provide a brief table on *pages 6-10*. Other places to look for *tables of integrals* are listed in the *beginning of Section 5.8*. A table of integrals will provide integrals of functions given in certain standard forms. For example,

$$\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right)+C$$

as given in the text.

Given an integral such as

$$\int \frac{dx}{x^2+4x+9}$$

you need to recognize that this is close to a form given in the table and you need to be able to perform algebraic manipulation to see if you can get it in a form that matches the form in the table. You will see how to do this in SkillMaster 22 below.

Key Concept: Computer Algebra Systems (CAS)

A CAS can integrate symbolically and numerically. Almost any integration problem in an introductory calculus text that uses substitution or integration by parts can be done by a sufficiently sophisticated CAS. If you have access to such a system, try it on the integration problems in this chapter to see what kind of answers are provided and how they compare to the answers you obtain by using other techniques.

SkillMaster 5.22: Use algebraic manipulations to enable application of integral tables.

In a Key Concept above, you were given the integral

$$\int \frac{dx}{x^2+4x+9}$$

and you needed to determine whether it was of the form

$$\int \frac{du}{a^2+u^2}$$

By completing the square, the original integral can be transformed into an integral with a sum of squares in the denominator.

$$\int \frac{dx}{x^2+4x+9} = \int \frac{dx}{(x^2+4x+4)+5} = \int \frac{dx}{(x+2)^2+(\sqrt{5})^2}$$

Next, a substitution $u = x+2$, $du = dx$ again transforms the integral:

$$\int \frac{dx}{(x+2)^2+(\sqrt{5})^2} = \int \frac{du}{(u)^2+(\sqrt{5})^2}$$

Now this is in the desired form (although the terms in the denominator are reversed) with $a = \sqrt{5}$. The answer can now be obtained from the table:

$$\frac{du}{(u)^2+(\sqrt{5})^2} = \frac{1}{\sqrt{5}}\tan^{-1}\left(\frac{u}{\sqrt{5}}\right)+C = \frac{1}{\sqrt{5}}\tan^{-1}\left(\frac{x+2}{\sqrt{5}}\right)+C$$

Remember that you can check whether you have done the problem correctly by differentiating your answer to see if you get back the integrand.

SkillMaster 5.23: Use a CAS to evaluate integrals.

If you have access to a Computer Algebra System, make sure that you learn how to use it to evaluate both indefinite and definite integrals. There are two main issues to watch for when using a CAS to integrate: first, it may use an obscure identity and give you an unusual looking answer; second, it may give too much information, for example, instead of showing the answer $(2x+\cos(x))^{10}$ it might multiply this expression and give an expression with 11 terms and large coefficients.

Worked Examples

For each of the following examples, first try to find the solution without looking at the hint. Cover the right column with a piece of paper. If you need a hint, uncover the column. If you need to see the worked solution, look at the solutions section that follows the examples and hints.

Example	Hint
SkillMaster 5.22. Evaluate the following integrals using the table in the back endpapers of the text.	
1. $\int \frac{2x^2+6}{x^2+9}\,dx$	There are forms like $\frac{1}{x^2+a^2}$ in the table. Use long division or inspection to get the integral in this form.
2. $\int \frac{4x}{\sqrt{4x^2+4x+5}}\,dx$	There are forms with $\sqrt{x^2+a^2}$ in the denominator. Complete the square and use a substitution to get to one of these forms.

3. $\int x^4 e^{-x}\,dx$

The table of integrals gives a reduction formula, $\int x^n e^{-x}\,dx = -x^n e^{-x} + n\int x^{n-1} e^{-x}\,dx$. Apply this formula repeatedly beginning with $n = 4$

SkillMaster 5.23.

Make an appropriate substitution to transform the integrals so that your CAS is more effective.

4. $\int \frac{x^8}{\sqrt{1-x^6}}\,dx$

If your CAS does not do this integral, try the substitution $u = x^3$.

5. $\int 8x(1-2x)^8\,dx$

If this is put into a CAS it will multiply $(1-2x)^8$ out into 9 terms. To get a more useful answer try the substitution $u = 1-2x$.

Solutions to worked examples

1. Using long division of polynomials, you obtain $\int \frac{2x^2+6}{x^2+9}\,dx = \int 2 - \frac{12}{x^2+9}\,dx$

$$= \int 2\,dx - 12\int \frac{1}{x^2+3^2}\,dx = 2x - 12\left(\frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)\right) + C$$

$$= 2x - 4\tan^{-1}(x/3)) + C$$

2. Use parts: $u = 2x+1$, $2x = u-1$, $du = 2dx$.

$$\int \frac{4x}{\sqrt{(2x+1)^2+4}}\,dx = \int \frac{u-1}{\sqrt{u^2+2^2}}\,du$$

$$= \int \frac{u}{\sqrt{u^2+2^2}}\,du - \int \frac{1}{\sqrt{u^2+2^2}}\,du = \sqrt{u^2+2^2} - \ln(u+\sqrt{u^2+2^2}) + C$$
$$= \sqrt{4x^2+4x+5} - \ln(2x+1+\sqrt{4x^2+4x+5}) + C$$

3. $\int x^4 e^{-x}\,dx = -x^4 e^{-x} + 4\int x^3 e^{-x}\,dx$
$$= -x^4e^{-x} - 4x^3e^{-x} + 12\int x^2 e^{-x}\,dx = -x^4e^{-x} - 4x^3e^{-x} - 12x^2e^{-x} + 24\int xe^{-x}\,dx$$
$$= -x^4e^{-x} - 4x^3e^{-x} - 12x^2e^{-x} - 24xe^{-x} + 24\int e^{-x}\,dx$$
$$= -x^4e^{-x} - 4x^3e^{-x} - 12x^2e^{-x} - 24xe^{-x} - 24e^{-x} + C$$

4. $u = x^3 \qquad du = 3x^2\,dx \qquad (1/3)du = x^2\,dx$
$$\int \frac{x^8}{\sqrt{1-x^6}}\,dx = \int \frac{(1/3)x^3x^3 3x^2}{\sqrt{1-(x^3)^2}}\,dx = \int \frac{(1/3)u^2}{\sqrt{1-u^2}}\,du$$
$$= -\frac{u}{6}\sqrt{1-u^2} + \frac{1}{6}\sin^{-1}(u) + C = -\frac{x^3}{6}\sqrt{1-x^6} + \frac{1}{6}\sin^{-1}(x^3) + C$$

5. $u = 1-2x \qquad du = -2\,dx \qquad 2x = 1-u$
$$\int 8x(1-2x)^8\,dx = -\int 2(1-u)u^8\,du = -\frac{2u^9}{9} + \frac{u^{10}}{5} + C$$
$$= -\frac{2(1-2x)^9}{9} + \frac{(1-2x)^{10}}{5} + C$$

5.9 Approximate Integration

Key Concepts:

- Midpoint, Trapezoidal, and Simpson's Rules
- Estimation of errors in approximation

Skills to Master:

- Use Midpoint Rule, Trapezoidal Rule and Simpson's Rule to estimate integrals for a function given in terms of a formula, graph, or table of values.
- Estimate errors, and compare the errors made in different methods or for different values of n.
- Choose n to obtain errors within specified bounds.

Discussion:

Section 5.9 shows some methods to obtain good estimates of definite integrals using techniques of numerical integration. In the previous section you learned how to use tables of integrals and Computer Algebra Systems to aid in evaluating integrals. However, there are many integrals that are impossible to evaluate exactly with any method. When this happens there are a number of methods for approximating the actual value of an integral. In this section, you will learn three of these techniques: the Midpoint Rule, the Trapezoidal Rule and Simpson's Rule.

Key Concept: Midpoint, Trapezoidal, and Simpson's Rules

The Midpoint Rule is based on approximating the value of an integral by the sum of the areas of certain rectangles. The interval of integration is divided into n equal subintervals and the function value at the midpoint of each subinterval is used as the height of the approximating rectangle. Study *Figure 1* in this section to see a geometrical page 402

depiction of this concept. The Midpoint Rule states that

$$\begin{aligned}
\int_a^b f(x)\,dx &\approx M_n = \Delta x\,[f(\overline{x_1}) + \cdots f(\overline{x_n})] \\
\text{where } \Delta x &= \frac{b-a}{n} \text{ and} \\
\overline{x_i} &= \frac{1}{2}(x_{i-1} + x_i) = \text{ midpoint of } [x_{i-1}, x_i].
\end{aligned}$$

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The Trapezoidal Rule is based on taking the average of the left endpoint approximation and the right endpoint approximation. Study *Figure 2* in this section to see a geometrical depiction of this concept. The Trapezoidal Rule states that

$$\begin{aligned}
\int_a^b f(x)\,dx &\approx T_n = \frac{\Delta x}{2}\,[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)] \\
\text{where } \Delta x &= \frac{b-a}{n} \text{ and} \\
x_i &= a + i\Delta x.
\end{aligned}$$

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Simpson's Rule is based on using parts of parabolas to approximate a curve $f(x)$. Study *Figure 7* in this section to see a geometric depiction of this concept. Note that the interval is always divided into an even number of subintervals and that the pattern of coefficients is $1, 4, 2, 4, 2, 4, 2, \ldots, 4, 2, 4, 1$. Simpson's Rule states that

$$\begin{aligned}
\int_a^b f(x)\,dx &\approx S_n = \frac{\Delta x}{3}\,[f(x_0) + 4f(x_1) + 2f(x_2) \cdots + 4f(x_{n-1}) + f(x_n)] \\
\text{where } \Delta x &= \frac{b-a}{n} \text{ and } n \text{ is even.}
\end{aligned}$$

Key Concept: Estimation of errors in approximation

For each of the three rules discussed in this section, an estimate of the error in using the rule to approximate the integral can be determined. The second derivative of f is used in determining the error bound for the Midpoint and the Trapezoidal Rules. The fourth derivative of f is used in determining the error bound for Simpson's Rule. Suppose that $|f''(x)| \leq K$ for $a \leq x \leq b$. If E_T and E_M are the error in using the Trapezoidal Rule and Midpoint Rule respectively, then

$$|\mathrm{E}_T| \leq \frac{K(b-a)^3}{12n^2} \text{ and } |\mathrm{E}_M| \leq \frac{K(b-a)^3}{24n^2}.$$

Note how the error goes down as n increases.

Suppose that $\left|f^{(4)}(x)\right| \leq K$ for $a \leq x \leq b$. If E_S is the error in using Simpson's Rule, then

$$|\mathrm{E}_S| \leq \frac{K(b-a)^5}{180n^4}.$$

SkillMaster 5.24: Use Midpoint Rule, Trapezoidal Rule and Simpson's Rule to estimate integrals for a function given in terms of a formula, graph, or table of values.

To use any of these three rules, first determine how many subintervals that you are going to divide the interval $[a,b]$ into. The number of subintervals is the value of n in the formula given by each rule. You next need to evaluate the function at the points specified in the formulas. For the Midpoint Rule, evaluate the function at the midpoint of each subinterval. For the Trapezoidal Rule and Simpson's Rule, evaluate the function at each endpoint of each subinterval. Finally, substitute the resulting values into the formulas. If the function is given by a graph or table of values, use the graph or table to determine the function values at the appropriate points.

SkillMaster 5.25: Estimate errors, and compare the errors made in different methods or for different values of n.

To estimate the errors in using the three rules, first determine the value of K. For the Midpoint and Trapezoidal Rule, K depends on the second derivative. For Simpson's Rule, K depends on the fourth derivative. Next, determine the value of n. Again, this represents the number of subintervals that the original interval is divided into. Finally, use the error formulas and compare the error estimates if you are asked to do so in the problem.

SkillMaster 5.26: Choose n to obtain errors within specified bounds.

For each of the three error estimates, since a power of n occurs in the denominator, the error estimate decreases as n increases. To choose n to obtain an error less than a specified value, determine K, b and a and substitute those values into the error formula. Then write an inequality that specifies that the error is less that the desired amount, and solve for $\dot{n}$.

Worked Examples

For each of the following examples, first try to find the solution without looking at the hint. Cover the right column with a piece of paper. If you need a hint, uncover the column. If you need to see the worked solution, look at the solutions section that follows the examples and hints.

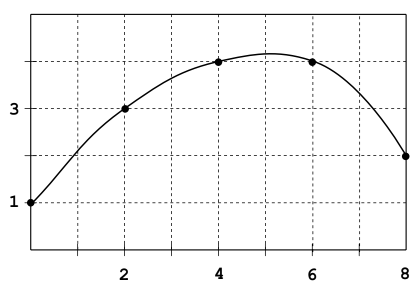
Example	Hint

SkillMaster 5.24.

1. For the function f pictured in the graph below, estimate

$$\int_0^8 f(x)dx$$

using the Trapezoidal and Simpson's Rules with $n = 4$. That is, find T_4, and S_4.



Hint: Read off the values $f(x_0) = f(0)$, $f(x_1) = f(2)$, $f(x_2) = f(4)$, $f(x_3) = f(6)$, and $f(x_4) = f(8)$ from the graph. Use the formulas for T_4 and S_4

2. Estimate

$$\int_0^1 xe^{-x}dx$$

using the Midpoint Rule and Simpson's Rule with $n = 6$ and two decimal places of accuracy when evaluating the function.

Hint: The endpoints of the subintervals when $n = 6$ are $x_0 = 0$, $x_1 = 1/6$, $x_2 = 2/6$, $x_3 = 3/6$, $x_4 = 4/6$, $x_5 = 5/6$, $x_6 = 1$. Use the formulas for the Midpoint rule and for Simpson's rule. Recall that $x_i{}^* = (x_{i-1} + x_i)/2$.

SkillMaster 5.25.

3. Estimate the errors in approximating

$$\int_0^1 xe^{-x}dx$$

using M_6. Compare this with the correct answer to 4 decimal places by exactly evaluating the integral.

Use integration by parts to evaluate the integral exactly.

4. Estimate the errors in approximating $\int_0^1 xe^{-x}dx$ using T_6,

$f^{(4)}(x) = (x-4)e^{-x}$
So $|f^{(4)}(x)| \leq 4$
on the interval $[0,1]$.

SkillMaster 5.26.

5. How large must n be for the error in using M_n to approximate

$$\int_0^1 xe^{-x}dx$$

to be less than 0.0001 in absolute value?

You need to find n so that $|E_M| \leq \dfrac{K(b-a)^3}{24n^2} \leq 0.0001$. You already know that you can take $K = 2, b = 1, a = 0$.

6. How large must n be for the error in using S_n to approximate

$$\int_0^1 xe^{-x}dx$$

to be less than 0.0001 in absolute value?

You need to find n so that $|E_S| \leq \dfrac{K(b-a)^5}{180n^4} \leq 0.0001$. You already know that you can take $K = 4, b = 1, a = 0$.

Solutions to worked examples

1. $T_4 = \left(\dfrac{8-0}{4}\right)/2[f(0)+2f(2)+2f(4)+2f(6)+f(8)]$
$= 1\cdot[1+6+8+8+2] = 25$
$S_4 = \left(\dfrac{8-0}{4}\right)/3\cdot[f(0)+4f(2)+2f(4)+4f(6)+f(8)]$
$= (2/3)[1+12+8+16+2] = (2/3)[39] = 26$

2. $M_6 = \dfrac{1}{6}[f(1/12)+f(3/12)+f(5/12)+f(7/12)+f(9/12)+f(11/12)] \approx 0.2656.$
$S_6 = \dfrac{1}{18}[f(0)+4f(1/6)+2f(2/6)+4f(3/6)+2f(4/6)+4f(5/6)+f(6/6)]$
$\approx 0.2642.$

3. $f(x) = xe^{-x}$, $f'(x) = (1-x)e^{-x}$, and $f''(x) = (x-2)e^{-x}$
On the interval $[0,1]$ the function $e^{-x} \leq 1$, so $|f''(x)| \leq |x-2| \leq 2$.
This may also be seen by graphing $|(x-2)e^{-x}|$ with domain [0,1].
The error in the Midpoint Rule is bounded by

$$|E_M| \leq \frac{K(b-a)^3}{24n^2}$$

where $|f''(x)| \leq K$, [a,b] is the interval in integration and n is the number of subdivisions in the approximation.
Here $K = 2, a = 0, b = 1$, and $n = 6$.$|E_M| \leq \dfrac{2(1-0)^3}{24(6)^2} = \dfrac{1}{432} \approx 0.0023$
Find the answer by antidifferentiating, using parts.
(You could have also used a reduction formula from the integral table in the endpapers of the text.)
$u = x \qquad dv = e^{-x}dx \qquad du = dx \qquad v = -e^{-x}$
$\displaystyle\int_0^1 xe^{-x}\,dx = -xe^{-x}\Big]_0^1 + \int_0^1 e^{-x}\,dx = -e^{-1} + 0 + \int_0^1 e^{-x}\,dx$
$= -e^{-1} + \left[-e^{-x}\right]_0^1 = -e^{-1} - e^{-1} + 1 = 1 - 2e^{-1} \approx 0.2646.$
The true error is approximately $0.2646 - 0.2653 = -0.0007$.

4. The error in Simpson's Rule is bounded by

$$|E_S| \le \frac{K(b-a)^5}{180n^4}$$

Here take $K = 4, b = 1, a = 0, n = 6. |E_S| \le \dfrac{4(1-0)^5}{180(6)^4} \approx 0.0007.$

Notice that to 4 decimal places Simpson's Rule agreed with the true value of the integral.

5. You need $\dfrac{2(1-0)^3}{24n^2} \le 0.0001 \qquad \dfrac{1}{12n^2} \le 0.0001 \qquad \dfrac{1}{12(0.0001)} \le n^2$

$833.33 \le n^2 \qquad \sqrt{\quad .33} \le n \qquad 28.9 < n$

You would need to take $n = 29$ to be sure the error is less than 0.0001.

6. $\dfrac{4(1-0)^4}{180n^4} \le 0.0001 \qquad \dfrac{1}{45(0.0001)} \le n^4 \qquad 222.22 \le n^4$

$\sqrt[4]{222.22} \le n \qquad 3.86 \le n$

Thus $n = 4$ subdivisions is already enough for such a good approximation to the integral!

5.10 Improper Integrals

Key Concepts:

- Definitions of two types of improper integrals
- Convergent and divergent improper integrals
- Comparison Test

Skills to Master:

- Determine whether improper integrals converge and if so determine their values.
- Determine whether improper integrals are convergent or divergent by examining limits and by use of the comparison test.

Discussion:

Section 5.10 introduces you to two types of improper integrals: those where the domain of integration is an infinite interval and those where the integrand is not continuous on the domain of integration. Meaning can often be given to these types of improper integrals by taking limits of integrals that are defined as earlier in this chapter.

Key Concept: Definitions of two types of improper integrals

An improper integral of Type I is an integral where the domain of integration is infinite. Examples include integrals such as

$$\int_1^\infty \frac{1}{x^4}\,dx \text{ and } \int_{-\infty}^\infty \frac{1}{x^2+1}\,dx\ .$$

The value of the first integral above is defined to be

$$\lim_{t\to\infty}\int_1^t \frac{1}{x^4}\,dx$$

provided the limit exists and the value of the second integral above is defined to be

$$\lim_{s\to-\infty}\int_s^0 \frac{1}{x^2+1}\,dx+\lim_{t\to\infty}\int_0^t \frac{1}{x^2+1}\,dx$$

provided both of these limits exist.

An improper integral of Type II is an integral where the integrand is discontinuous at one point in the domain of integration. An example of this type of integral is

$$\int_0^2 \frac{1}{\sqrt{x}}\,dx.$$

The value of this integral is defined to be

$$\lim_{t\to 0^+}\int_t^1 \frac{1}{\sqrt{x}}\,dx$$

provided that this limit exists. Such improper integrals are often associated with functions that are unbounded as they approach the point of discontinuity.

Key Concept: Convergent and divergent improper integrals

An improper integral is convergent if the limits involved in the definition of the integral exist. Otherwise, it is called divergent. Study *Examples 1-10* in this section to see both convergent and divergent improper integrals.

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Key Concept: Comparison Test

The Comparison Test gives a method for deciding the convergence or divergence of certain improper integrals of Type I. It states that if f and g are continuous functions with $f(x)\geq g(x)\geq 0$ for $x\geq a$, then

(a) If $\int_a^\infty f(x)\,dx$ is convergent, then $\int_a^\infty g(x)\,dx$ is convergent, and

(b) If $\int_a^\infty g(x)\,dx$ is divergent, then $\int_a^\infty f(x)\,dx$ is divergent.

Study *Figure 12* in this section to see a geometric interpretation of this test.

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SkillMaster 5.27: Determine whether improper integrals converge and if so determine their values.

When working with improper integrals, first determine whether the integrals are of Type I or of Type II. Then use the definition of these integrals as limits to determine whether they converge and what the values are. As an example,

$$\int_1^{\infty} \frac{1}{x^4}\,dx = \lim_{t\to\infty}\int_1^{t} \frac{1}{x^4}\,dx$$

provided this limit exists. But

$$\lim_{t\to\infty}\int_1^{t} \frac{1}{x^4}\,dx = \lim_{t\to\infty}(-3\cdot x^{-3}\big]_1^t) = \lim_{t\to\infty}\left(\frac{-3}{t^3}+3\right) = 3$$

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So the original integral has the value 3. Pay careful attention to *Example 6* and the *Warning* in this section about what can happen if you evaluate an improper integral of Type II without noticing that the integrand is discontinuous!

SkillMaster 5.28: Determine whether integrals are convergent or divergent by examining limits and by use of the comparison test.

Section 2.3 and page 291.

You may need to review some *Properties of Limits*, including *l'Hospital's Rule* when you are working with the limits involved in determining whether improper integrals converge or diverge.

When using the comparison test, try to compare with a function that is larger and that has an improper integral that converges if you think that the improper integral you are working with converges Try to compare with a smaller function that has an improper integral that diverges if you think the improper integral you are working with diverges.

Worked Examples

For each of the following examples, first try to find the solution without looking at the hint. Cover the right column with a piece of paper. If you need a hint, uncover the column. If you need to see the worked solution, look at the solutions section that follows the examples and hints.

Example	Hint
SkillMaster 5.27.	
Determine if the following improper integrals converge and if so, find their value.	
1. $\int_0^{\infty} xe^{-x}\,dx$	Use integration by parts or the reduction formula.

2. $\displaystyle\int_0^{\pi/2} \tan(x)\, dx$

There is a discontinuity at $\pi/2$.

3. $\displaystyle\int_0^1 1/\sqrt{x}\, dx$

Here the discontinuity is at 0.

SkillMaster 5.28.

4. Determine if the integral converges using the comparison test.

$$\int_1^{\infty} \frac{1}{\sqrt[3]{x^4+1}}\, dx$$

Use the fact that for $x \geq 1$, $\dfrac{1}{\sqrt[3]{x^4+1}} \leq \dfrac{1}{\sqrt[3]{x^4}} = x^{-4/3}$.

Solutions to worked examples

1. $\displaystyle\int_0^{\infty} xe^{-x}\, dx = \lim_{t\to\infty} \int_0^t xe^{-x}\, dx = \lim_{t\to\infty} (-xe^{-x} - e^{-x})\Big]_0^t$

$\displaystyle = \lim_{t\to\infty} -te^{-t} - \lim_{t\to\infty} e^{-t} + 1 = \lim_{t\to\infty} \frac{-t}{e^t} + 0 + 1$

$\displaystyle = \lim_{t\to\infty} \frac{-1}{e^t} + 1 = 0 + 1 = 1$ by l'Hospital's Rule.

2. $\displaystyle\int_0^{\pi/2} \tan(x)\, dx = \lim_{t\to\pi/2^-} \int_0^t \tan(x)\, dx = \lim_{t\to\pi/2^-} \ln|\sec(x)|\Big]_0^t$

$\displaystyle = \lim_{t\to\pi/2^-} \ln|\sec(t)| = \infty$. The integral diverges.

3. $\displaystyle\int_0^1 1/\sqrt{x}\, dx = \lim_{t\to 0^+} \int_t^1 1/\sqrt{x}\, dx = \lim_{t\to 0^+} 2\sqrt{x}\Big]_t^1 = \lim_{t\to 0^+} 2 - 2\sqrt{t} = 2$

4. $\displaystyle\int_1^{\infty} \frac{1}{\sqrt[3]{x^4+1}}\, dx < \int_1^{\infty} \frac{1}{\sqrt[3]{x^4}}\, dx = \int_1^{\infty} x^{-4/3}\, dx = \lim_{t\to\infty} -3x^{-1/3}\Big]_1^t = 0 - (-3) = 3$

Therefore by the Comparison Test, $\displaystyle\int_1^{\infty} \frac{1}{\sqrt[3]{x^4+1}}\, dx$ converges.

SkillMasters for Chapter 5

SkillMaster 5.1:	Estimate area using left and right endpoint and midpoint rectangles.
SkillMaster 5.2:	Estimate distances using velocity data or graphs.
SkillMaster 5.3:	Evaluate limits to find areas.
SkillMaster 5.4:	Approximate definite integrals by the Midpoint Rule.
SkillMaster 5.5:	Approximate definite integrals using the area interpretation.
SkillMaster 5.6:	Approximate definite integrals using the comparison properties.
SkillMaster 5.7:	Evaluate definite integrals using limits and the sum formula.
SkillMaster 5.8:	Evaluate definite integrals using the properties and previous results.
SkillMaster 5.9:	Interpret definite integrals as net change over a given interval.
SkillMaster 5.10:	Evaluate definite integrals and find general indefinite integrals.
SkillMaster 5.11:	Solve application problems using the definite integral.
SkillMaster 5.12:	Evaluate and analyze functions of the form $g(x) = \int_a^x f(t)dt$.
SkillMaster 5.13:	Use the Fundamental theorem to find derivatives of functions of the form $g(x) = \int_a^x f(t)dt$.
SkillMaster 5.14:	Evaluate definite and indefinite integrals using substitution.
SkillMaster 5.15:	Use symmetry to simplify calculations.
SkillMaster 5.16:	Use integration by parts alone or combined with previous methods to evaluate integrals.
SkillMaster 5.17:	Evaluate integrals by repeated application of integration by parts.
SkillMaster 5.18:	Prove and apply reduction formulas to evaluate integrals.
SkillMaster 5.19:	Integrate forms like $\int \sin^n(x)\cos^m(x)dx$.
SkillMaster 5.20:	Integrate forms containing $\sqrt{a^2 \pm r^2}$ and $\sqrt{r^2 - a^2}$.
SkillMaster 5.21:	Integrate forms like $\displaystyle\int \frac{P(x)}{Q(x)}dx$ where $P(x)$ and $Q(x)$ are polynomials.

SkillMaster 5.22:	Use algebraic manipulations to enable application of integral tables.
SkillMaster 5.23:	Use a CAS to evaluate integrals.
SkillMaster 5.24:	Use Midpoint Rule, Trapezoid Rule and Simpson's Rule to estimate integrals for a function given in terms of a formula, graph, or table of values.
SkillMaster 5.25:	Estimate errors, and compare the errors made in different methods or for different values of n.
SkillMaster 5.26:	Choose n to obtain errors within specified bounds.
SkillMaster 5.27:	Determine whether improper integrals converge and if so determine their values.
SkillMaster 5.28:	Determine whether integrals are convergent or divergent by examining limits and by use of the comparison test.

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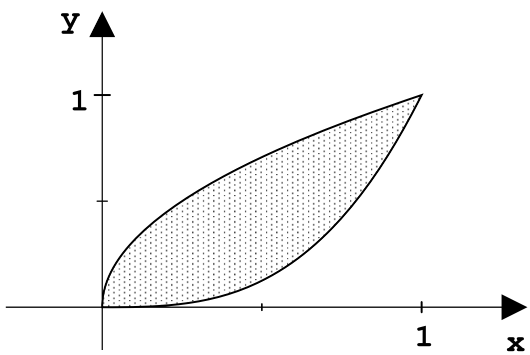
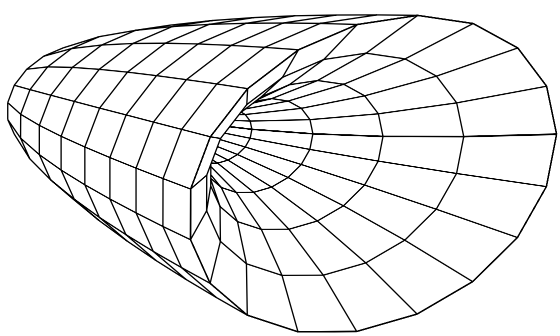
Chapter 6

Applications of Integration





6.1 More about Areas

Key Concepts:

- Area between curves
- Interpreting areas in applications
- Area bounded by parametric curves

Skills to Master:

- Sketch the region enclosed by given curves and determine the area by integration.
- Given graphs or tables representing velocities, use area to determine distances.
- Graph and determine the areas bounded by parametric curves.

Discussion:

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Section 6.1 uses the concept of the integral defined in the previous chapter to find the areas of certain regions in the plane. In the previous chapter, you learned how to find the area under a continuous curve. Make sure the concept of *area under a curve* is clear because this section extends the computation of area to include finding the area enclosed by two curves and by parametric curves.

Key Concept: Area between curves

If f and g are continuous and $f(x) \geq g(x)$ for all x in $[a,b]$, the area bounded by curves $y = f(x)$ and $y = g(x)$ for x values between a and b, is defined to be

$$A = \int_a^b [f(x) - g(x)]\, dx\ .$$

Note that you need to know that $f(x)$ is at least as large as $g(x)$ on the whole interval to apply this definition.

Key Concept: Interpreting areas in applications

In the previous chapter, you saw that the area under a velocity curve gave the distance traveled by an object. This section shows that you can interpret the area between two velocity curves as the distance between objects moving at the different velocities, provided one object has a velocity greater than the other. In the general case, the distance between the objects would have to be interpreted as the *Net Area* or the difference between the two objects' net change of position or their *Displacement*.

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Another example of this is provided in the Worked Examples below.

Key Concept: Area bounded by parametric curves

The area under a parametric curve or the area of a region bounded by a parametric curve can also be found using the techniques in this section. The area under a curve $y = h(x)$ between $x = a$ and $x = b$ is given by

$$\int_a^b h(x)\,dx = \int_a^b y\,dx\,.$$

If the x and y coordinates of a curve are given parametrically in terms of another variable t,

$$x = f(t) \qquad y = g(t),$$

then you can replace y by $g(t)$ and dx by $f'(t)\,dt$ to get the area under a parametric curve. This results in an integral of the form

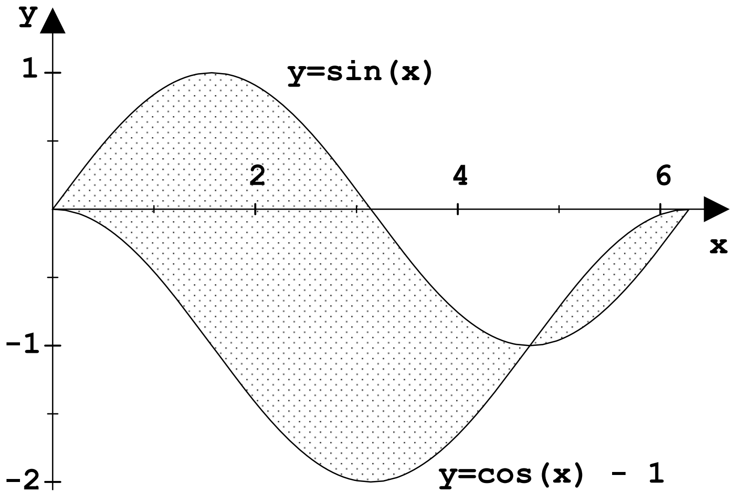
$$\int_{t_0}^{t_1} g(t)f'(t)\,dt$$

where t_0 and t_1 are the values of t corresponding to the x values a and b.

SkillMaster 6.1: Sketch the region enclosed by given curves and determine the area by integration.

If you are trying to find the area between curves $y = f(x)$ and $y = g(x)$ over some range of x values, you first need to determine where $f(x) \geq g(x)$, and where $g(x) \geq f(x)$. This can often be done by determining first where the curves intersect. To determine this, either graph the curves or equate the y values and solve for the points of intersection algebraically. After you have determined where $f(x) \geq g(x)$ and where $g(x) \geq f(x)$ you can break up the interval into separate pieces and determine the area between the two curves corresponding to each piece.

For example, to find the area between $y = \sin(x)$ and $y = \cos(x) - 1$ for x values between 0 and 2π, first graph the curves to determine points of intersection.



From the graph, it looks like the points of intersection occur at $x = 0$ and $x = 3\pi/2$. You can check algebraically that

$$\sin 0 = 0 = \cos 0 - 1 \text{ and } \sin(3\pi/2) = -1 = \cos(3\pi/2) - 1 \ .$$

The area under consideration is then

$$\begin{aligned} & \int_0^{3\pi/2} (\sin x - (\cos x - 1))\,dx + \int_{3\pi/2}^{2\pi} ((\cos x - 1) - \sin x)\,dx \\ = \quad & (-\cos x - \sin x + x)]_0^{3\pi/2} + (\sin x + \cos x - x)]_{3\pi/2}^{2\pi} \\ = \quad & \pi + 4 \end{aligned}$$

SkillMaster 6.2: Given graphs or tables representing velocities, use area to determine distances.

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In the previous chapter, you learned that the *net area under a velocity curve* represented the net distance traveled. In this section, you will see that the net area between two velocity curves for two objects starting at the same point represents the (signed) distance apart that the objects are after some time period. Study *Example 4* in the text and the Worked Example for this SkillMaster to see how this works.

SkillMaster 6.3: Graph and determine the areas bounded by parametric curves.

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Review the material from *Chapter 1* on parametric curves. If a curve is given parametrically by equations

$$x = f(t) \text{ and } y = g(t)$$

you can determine the area under the parametric curve or the area bounded by the parametric curve using the techniques in this section. Study *Example 6* in the text and the Worked Example for this SkillMaster to see how to do this.

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Worked Examples

For each of the following examples, first try to find the solution without looking at the hint. Cover the right column with a piece of paper. If you need a hint, uncover the column. If you need to see the worked solution, look at the solutions section that follows the examples and hints.

Example	Hint
SkillMaster 6.1.	
1. Find the area to the right of the y-axis enclosed by the two curves $y = 3x$ and $y = x^3 - x$.	First graph these curves to get a general picture. Then find the intersection points by equating the expressions and solving for x. Finally, integrate.
2. Find the area between the following two curves: $y = \dfrac{e - e^{-1}}{2}x + \dfrac{e + e^{-1}}{2}$ and $y = e^x$.	Make a graph to see the shape of the region. If the graph indicates possible intersection points, these may be checked directly without actually solving an equation.

3. Find the area enclosed by the two curves $x = y$ and $y^2 - 2y = x$.

These are given as functions of y. It is probably easier to compute the area as an integral in y. After graphing, this will be even more clear. If this makes you uncomfortable, you can switch the roles of x and y without affecting the calculation or the final answer. You would then get the equations $y = x$ and $y = x^2 - 2x$. This will reorient the graph but will not change the area.

4. Estimate the area between the following two curves by first finding the intersection points to two decimal places using your graphing device.

$$y = \cos(x) \text{ and } y = x^2 - 1$$

After finding the intersection points, set up the correct integral, and evaluate it.

SkillMaster 6.2.

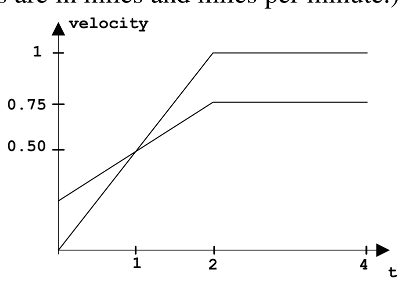
5. The graph shows the velocities of two cars that begin at the same position.

What is the significance of the intersection point?

When are the cars at the same place at the same time?

How far ahead is the faster car at $t = 4$ minutes?

(Units are in miles and miles per minute.)



These questions may be answered easiest by using the area interpretation of integration. If $v(t)$ is the velocity of the first car and if $w(t)$ is the velocity of the second car then $\int_0^t [v(u) - w(u)]\, du$ is the amount that the first car is in front of (or behind) the second car at time t.

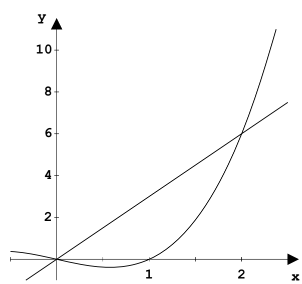
SkillMaster 6.3.

6. Set up an integral that gives the area that is enclosed by the curve given parametrically by the equations.
$x = f(t) = \dfrac{1}{1+t^2} \quad y = g(t) = \dfrac{t(t^2-1)}{1+t^2}$
(Do not attempt to solve the integral.)

First graph the curve and find the intersection points in terms of the parameter t. Since $f(t)$ is even and $g(t)$ is odd the curve is symmetric with respect to the x-axis.

Solutions to worked examples

1.



The intersection points appear to be at $x = 0$ and $x = 2$. At the intersection points the expressions for y are equal.

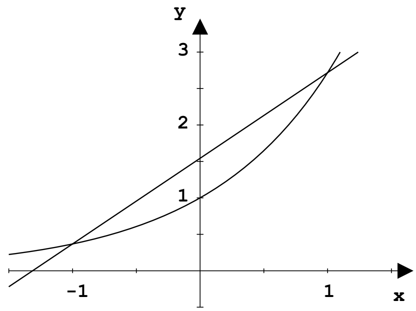
$3x = x^3 - x \qquad 0 = x^3 - 4x \qquad 0 = x(x^2-4) = x(x-2)(x+2)$

The intersection points are at $x = 0, 2, -2$.

Only the points 0 and 2 are to the right of the y-axis and $3x \geq x^3 - x$ for the entire enclosed region.

$$\int_0^2 3x - (x^3 - x)\,dx = \int_0^2 4x - x^3\,dx = 2x^2 - \frac{x^4}{4}\bigg]_0^2 = 2(2^2) - 2^2 = 8 - 4 = 4.$$

2.



There appear to be two intersection points, $x=-1$ and $x=1$. You can check this by substituting $x=-1$ in both equations and checking that the same y-values occur.

$$y=\frac{e-e^{-1}}{2}x+\frac{e+e^{-1}}{2} \qquad = \qquad e^x$$

$$\frac{e-e^{-1}}{2}(-1)+\frac{e+e^{-1}}{2}=\frac{-(e-e^{-1})+(e+e^{-1})}{2}=\frac{2e^{-1}}{2}=e^{-1}$$

Repeat this process for $x=1$:

$$y=\frac{e-e^{-1}}{2}(1)+\frac{e+e^{-1}}{2}=\frac{(e-e^{-1})+(e+e^{-1})}{2}=\frac{2e^{1}}{2}=e^{1}$$

This shows the intersection points are at $x=-1$ and $x=1$.

Now compute the integral to find the area.

Note that the line $y=\dfrac{e-e^{-1}}{2}x+\dfrac{e+e^{-1}}{2}$ lies above the curve $y=e^x$ between $x=-1$ and $x=1$.

$$\int_{-1}^{1}\left(\frac{e-e^{-1}}{2}x+\frac{e+e^{-1}}{2}\right)-e^x\,dx=\int_{-1}^{1}\frac{e-e^{-1}}{2}x\,dx+\int_{-1}^{1}\frac{e+e^{-1}}{2}\,dx-\int_{-1}^{1}e^x\,dx$$

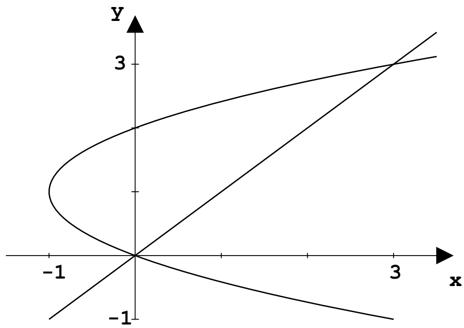
$$=0+2\int_{0}^{1}\frac{e+e^{-1}}{2}\,dx-\int_{-1}^{1}e^x\,dx$$

(because the first integral is odd and the second is even)

$$=e+e^{-1}-e^x]_{-1}^{1}=e+e^{-1}-(e^1-e^{-1})=e+e^{-1}-e^1+e^{-1}=2e^{-1}\approx 0.7357$$

or about the size of $\frac{3}{4}$ of a unit square.

3. Find the intersection points.



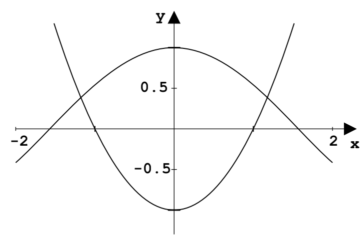
$y = y^2 - 2y \qquad 0 = y^2 - 3y \qquad 0 = y(y-3)$

The intersection points are $y = 0$ and $y = 3$. Compute the integral to find the area.

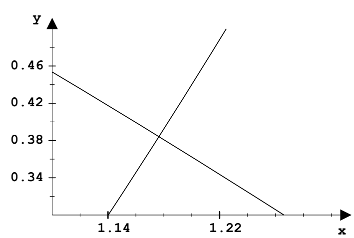
$\int_0^3 y - (y^2 - 2y)dy = \int_0^3 3y - y^2 dy = \frac{3}{2}y^2 - \frac{1}{3}y^3]_0^3$

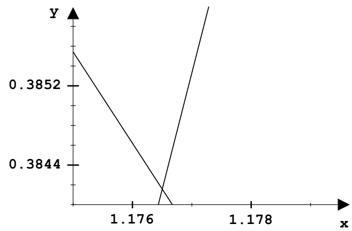
$= \frac{3}{2}3^2 - \frac{1}{3}3^3 = 27(1/2 - 1/3) = 27/6 = 9/2$

4.



Zoom in on the intersection point. You need only find the intersection point with x positive. Because both functions are even, the area enclosed is equal to twice the area the right of the x-axis.



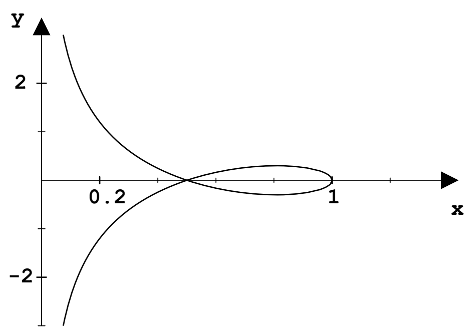


The intersection point, to two decimal places, is (1.18,0.38). The area is approximately equal to the integral below.

$$2\int_0^{1.18}(\cos(x)-(x^2-1))\,dx = 2\int_0^{1.18}(\cos(x)-x^2+1)\,dx$$
$$= 2[\sin(x)-x^3/3+x]_0{}^{1.18} = 2[\sin(1.18)-(1.18^3)/3+1.18] \approx 3.11$$

5. The velocities are equal when they intersect. It is the point where one car overtakes the other. The cars are at the same place when they start. They are also at the same place at any time t for which the net area $\int_0^t [v(u)-w(u)]\,du = 0$. This occurs at $t=2$. The areas of the two triangles bounded by the graphs between $x=0$ and $x=1$ and that between $x=1$ and $x=2$ are the same and so cancel. The integral between $t=0$ and $t=2$ is 0 as you have seen. The integral between $t=2$ and $t=4$ is the area of the rectangle between the graphs which is 2(0.25)=0.5. At time $t=4$ the second car is 2(0.25)=0.5 miles ahead of the first.

6.



The area enclosed is twice the area above the x-axis. The intersection points are the points where $y=0$. So $t=0,1$ or -1. The point corresponding to $t=0$ is (1,0) and the point corresponding to both $t=1$ and $t=-1$ is $(1/2,0)$. The area enclosed is

$$2\int_{-1}^{0} g(t)f'(t)\,dt = 2\int_{-1}^{0}\frac{t(t^2-1)}{1+t^2}\cdot\frac{(-2t)}{(1+t^2)^2}\,dt.$$

6.2 Volumes

Key Concepts:

- Definition of volume in terms of cross-sectional areas

Skills to Master:

- Compute volumes of solids obtained by rotating a bounded region about either the x-axis or y-axis.
- Compute the volume of solids by determining a function, $A(x)$, that gives areas of parallel cross-sections.

Discussion:

Section 6.2 shows how to find volumes of solids of revolution. These solids are obtained by rotating a region in the plane about (i.e. around) one of the two coordinate axes. Examples of solids of rotation include the right circular cone, the solid sphere, solid tubes, and donut shapes. If you look around your house you will see many solids of revolution: saucers for coffee cups, pipes, DVDs, some (but not all) glasses, batteries, and so on. To find the volume two kinds of approximations will be used: (i) cross-sectional areas which look like disks or annuli; and in the next section, (ii) cylindrical shells, where each piece looks like a solid tube.

Key Concept: Definition of volume in terms of cross-sectional areas

If a solid S lies between $x = a$ and $x = b$ and the cross-sectional area of S in the plane P_x through x perpendicular to the x axis is $A(x)$ where $A(x)$ is continuous, then the volume of S is

$$\int_a^b A(x)\,dx.$$

Example 1 in this section shows how to compute the volume of a sphere by using this definition. One consequence of this formula is that two solids with the same cross-sectional areas at each level have the same volume. For example, a stack of index cards fills up the same volume whether they are stacked straight or if the stack is slanted.

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SkillMaster 6.4: Compute volumes of solids obtained by rotating a bounded region about either the x or y axis.

When a bounded region is rotated about either the x axis or the y axis, the cross-sectional areas perpendicular to the x or y axis are often discs or the region between two discs of different radii. Using the formula for the area of a circle, you can determine what the cross-sectional areas are and you can integrate to find the resulting volume. *Examples 4 and 5* in this section show how to so this as do the Worked Examples for this SkillMaster.

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SkillMaster 6.5: Compute the volume of solids by determining a function, $\mathbf{A(x)}$, that gives areas of parallel cross-sections.

Since volume is given by integrating cross-sectional areas, if you can determine or are given the cross-sectional areas, you should be able to determine the volume. The Worked Example for this SkillMaster shows how to do this for a particular case where the cross-sectional areas are squares.

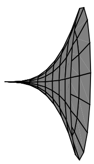
Worked Examples

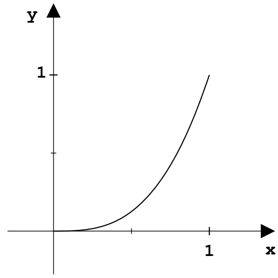
For each of the following examples, first try to find the solution without looking at the hint. Cover the right column with a piece of paper. If you need a hint, uncover the column. If you need to see the worked solution, look at the solutions section that follows the examples and hints.

Example	Hint

SkillMaster 6.4.

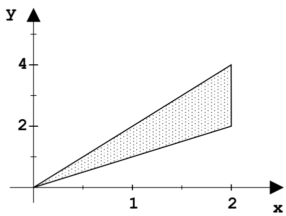
1. Find the volume of the solid obtained by rotating the region bounded by the x-axis and the curve $y = x^3$ between $x = 0$ to $x = 1$ about the $x-$axis. Pictured below is the solid that is the result of rotating the curve and the curve $y = x^3$.

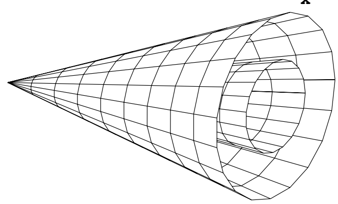




The cross-sectional areas perpendicular to the x axis are circles of radius y, and hence $A(x) = \pi y^2 = \pi(x^3)^2$.

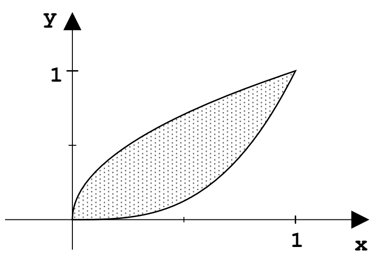
2. Find the volume of the solid obtained by rotating the region bounded by the curves $y = x$, $y = 2x$ and $x = 2$ about the x-axis in two ways: First by integration, and second by using the formula for the volume of a right circular cone. Pictured below are the region and the resulting solid. The volume that you want is that of the region between the two cones.





The cross-section is an annulus with outer radius $y = 2x$ and the inner radius $y = x$. The cross-sectional area is $A(x) = \pi((2x)^2 - x^2)$. The volume of a right circular cone is given by the formula $V = \pi r^2 h/3$. The solid may be thought of as a right circular cone which has had a right circular cone removed.

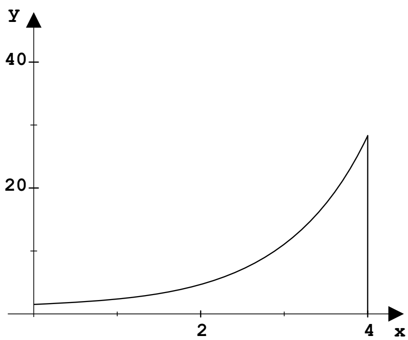
3. The region enclosed by the curves $y = \sqrt{x}$ and $y = x^3$ is rotated about the y-axis. Find the volume of the resulting solid. Pictured below are the region and the resulting solid. In the second picture, the y-axis comes out of the center of the bowl shaped region.



Since the solid is obtained by rotating about the y-axis this integral should by set up with a function of y.

SkillMaster 6.5.

4. A large loud speaker has cross-sections that are squares. The loud speaker is 4 feet long. At x feet from the small end of the speaker the cross-section is a square of side length $2f(x)$ where $f(x) = 2 + e^x$. Find the volume of the solid enclosed by the loud speaker. Pictured below is the function $f(x)$ and the speaker.



Volume is obtained by integrating cross-sectional area.

Solutions to worked examples

1. $V = \int_0^1 \pi y^2\,dx = \int_0^1 \pi(x^3)^2\,dx = \int_0^1 \pi x^6\,dx = \left.\frac{\pi x^7}{7}\right]_0^1 = \frac{\pi}{7}$

2. $\int_0^2 \pi((2x)^2 - x^2)\,dx = \int_0^2 \pi(4x^2 - x^2)\,dx = \int_0^2 3\pi x^2\,dx = \pi x^3]_0^2 = 8\pi$

The solid is a right circular cone of radius 4 and height 2 which has had a right circular cone of radius 2 and height 2 removed from it. The volume of the solid is the difference of the volumes of these two cones. $V = \pi(4^2)2/3 - \pi(2^2)2/3 = \pi 32/3 - \pi 8/3 = \pi 24/3 = 8\pi$

3. First express x as a function of y.

$y = \sqrt{x}$ $\quad y = x^3$ $\quad$ are equivalent to $\quad x = y^2$ $\quad x = y^{1/3}$.

Now compute the intersection points in order to know the limits of integration.

$y^2 = y^{1/3}$ $\quad y^6 = y$

By inspection it is clear that both $y = 0$ and $y = 1$ solve this equation and are the limits of integration. The cross-section (perpendicular to the y-axis) is an annulus with outer radius $y^{1/3}$ and inner radius y^2.

$V = \int_0^1 \pi(y^{1/3})^2 - \pi(y^2)^2 dy = \int_0^1 \pi y^{2/3} - \pi y^4 dy$

$= \pi\frac{3y^{5/3}}{5} - \pi\frac{y^5}{5}]_0^1 = \pi(\frac{3}{5} - \frac{1}{5}) = \frac{2\pi}{5}$

4. The cross-sectional area at x feet from the end of the loud speaker is

$(f(x))^2 = (2 + e^x)^2$.

So $V = \int_0^4 (2(2+e^x))^2\,dx = 4\int_0^4 4 + 4e^x + (e^x)^2\,dx = 4\int_0^4 4 + 4e^x + e^{2x}\,dx$

$= 4(4x + 4e^x + (1/2)e^{2x}]_0^4) = 4(4(4) + 4e^4 + (1/2)e^8 - 4e^0 - (1/2)e^0)$

$= 4(11.5 + 4e^4 + 0.5e^8) \approx 6880 \text{ ft}^3$

6.3 Volumes by Cylindrical Shells

Key Concepts:

- Volume computed by summing cylindrical shells

Skills to Master:

- Compute the volume of solids obtained by rotating a bounded region about either the x-axis or the y-axis by considering cylindrical shells.

Discussion:

This section continues the discussion of finding volumes by considering the method of cylindrical shells.

Key Concept: Volume computed by summing cylindrical shells

Instead of computing the volume of a solid by considering cross-sectional areas, an alternate method is to approximate the solid by thin cylindrical shells and to take the limit of the approximating volumes. If an area under a curve $y = f(x)$ is rotated about the y axis to generate a solid object, the part of the solid object obtained by rotating the area under the curve for $x_i \leq x \leq x_i + \Delta x$ can be approximated by a cylindrical shell of volume $2\pi \overline{x_i} \cdot f(\overline{x_i}) \cdot \Delta x$. See *Figures 4, 5, 8, and 9* in this section for a geometric explanation of this. An approximation to the volume is given by the Riemann sum

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$$\sum_{i=1}^{n} 2\pi \overline{x_i} \cdot f(\overline{x_i}) \cdot \Delta x.$$

SkillMaster 6.6: Compute the volume of solids obtained by rotating a bounded region about either the x or y axis by considering cylindrical shells.

When using cylindrical shells to compute volume, first determine the height and the approximate radius of the shell. If you have a shell for each value $x_i, 1 \leq i \leq n$, and if the radius is $\overline{x_i}$ and the height is $h(\overline{x_i})$, the resulting volume of the shell is

$$2\pi\cdot\overline{x_i}\cdot h(\overline{x_i})\cdot\Delta x$$

so the total volume is the limiting value of the Riemann sum

$$\sum_{i=1}^{n} 2\pi\cdot\overline{x_i}\cdot h(\overline{x_i})\cdot\Delta x \text{ , which is } \int 2\pi x\cdot h(x)dx$$

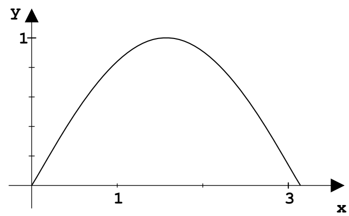
Worked Examples

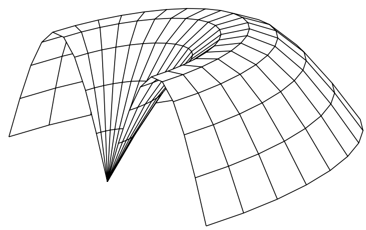
For each of the following examples, first try to find the solution without looking at the hint. Cover the right column with a piece of paper. If you need a hint, uncover the column. If you need to see the worked solution, look at the solutions section that follows the examples and hints.

Example	**Hint**

SkillMaster 6.6.

1. The region bounded by $y = \sin(x)$, the x-axis, $x = 0$ and $x = \pi$ is rotated about the y-axis. Find the volume of this solid. Pictured below are the region and part of the resulting solid.





This problem is best solved using shells rather than trying to solve for x in terms of y.

Solutions to worked examples

1. If $y = f(x) = \sin(x)$ then the radius of each shell is $\overline{x_i}$, the height of each shell is $\sin(\overline{x_i})$, and the width of each shell is Δx. The approximate volume is given by the Riemann sum:

$$\sum_{i=1}^{n} 2\pi\overline{x_i} \cdot \sin(\overline{x_i}) \cdot \Delta x$$

Taking limits expresses the volume as an integral (as it was derived in the text).

$$V = \int_0^{\pi} 2\pi x \sin(x)\, dx$$

Integrate by parts.

$$u = x \qquad dv = \sin(x)dx \qquad du = dx \qquad v = -\cos(x)$$

$$V = -2\pi x \cos(x)]_0{}^{\pi} + \int_0^{\pi} 2\pi \cos(x)\, dx = 2\pi^2 + 2\pi \sin(x)]_0{}^{\pi} = 2\pi^2$$

6.4 Arc Length

Key Concepts:

- Definitions of smooth curve and arc length
- Formulas for determining arc length

Skills to Master:

- Find the lengths of parametric curves and curves given by $y = f(x)$ or $x = f(y)$ by approximating.
- Find the exact lengths of parametric curves and curves given by $y = f(x)$ or $x = f(y)$ by integrating or by using a table of integrals or CAS.

Discussion:

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Section 6.4 explains how to express the exact length of curve with two end points as a definite integral. The integrals that arise are often difficult or impossible to evaluate exactly, but you can use the *approximation techniques* from Chapter 5 to approximate the length of the curve.

It's a good idea to check if your answer seems reasonable by approximating the curve by a few straight line segments.

Key Concept: Definitions of smooth curve and arc length

A curve C given by parametric equations

$$x = f(t) \text{ and } y = g(t) \text{ for } a \le t \le b$$

is smooth if $f'(t)$ and $g'(t)$ are continuous and are not both 0 for the same value of t between a and b. For a smooth curve that traces out a path in the plane once, the arc length is defined as the limiting value of the sum of the lengths of approximating straight line segments. The expression that you come up with when approximating by straight line segments is a *Riemann Sum* for a certain function and thus the limiting value can be expressed as an integral.

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Key Concept: Formulas for determining arc length

If the curve C is given parametrically as above, the arc length is defined to be

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt\ .$$

If the curve C is given either as $y = f(x)$ for $a \leq x \leq b$ or as $x = f(y)$ for $a \leq y \leq b$, x or y can be regarded as the parameter and you obtain the formulas

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \text{ or } L = \int_a^b \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\, dy$$

respectively.

SkillMaster 6.7: Find the lengths of parametric curves and curves given by $y = f(x)$ or $x = f(y)$ by approximating.

To find the lengths of curves using approximation techniques, set up the correct integral, and then use one of the techniques from Chapter 5 such as *Simpson's Rule* or the *Midpoint Rule* to approximate the integral. First, determine whether the curve is given parametrically or in the form $y = f(x)$ or $x = f(y)$ so you can tell which form of the arc length formula to use.

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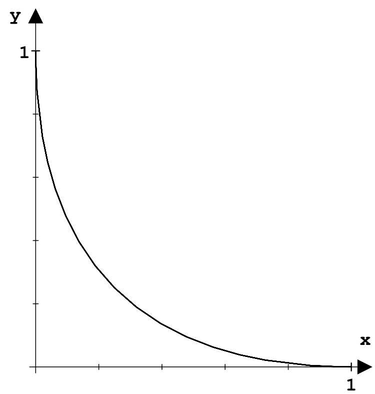
SkillMaster 6.8: Find the exact lengths of parametric curves given by $y = f(x)$ or $x = f(y)$ by integrating or by using a table of integrals or CAS.

If you are given a curve and want to determine whether or not you can find the exact length, first set up the integral as in the previous SkillMaster. Once you have the integral set up, see if any of the *integration techniques* from the previous chapter apply. If you cannot find a technique that works, use a table of integrals or a Computer Algebra System (CAS). Don't get discouraged if none of these techniques work. You may need to use approximation techniques as in the previous SkillMaster.

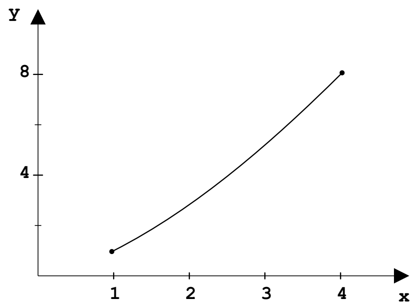
page 375, 383, 389.

Worked Examples

For each of the following examples, first try to find the solution without looking at the hint. Cover the right column with a piece of paper. If you need a hint, uncover the column. If you need to see the worked solution, look at the solutions section that follows the examples and hints.

Example	Hint
SkillMaster 6.7.	
1. Estimate the arc length of $y = f(x) = e^x$ from $(0,1)$ to $(1,e)$ using Simpson's Rule with $n = 4$. Round to 2 decimal places.	Use Simpson's Rule with $n = 4$, $\Delta x = 0.25$.
2. A curve is given parametrically by the following equations: $x = f(\theta) = \cos^4(\theta), y = g(\theta) = \sin^4(\theta), 0 \le \theta < \pi/2$.  Write an integral that equals the arc length of the curve. Do not try to evaluate this integral, but estimate it using Simpson's Rule with $n = 2$ and round the answer to 2 decimal places.	Use the formula for arc length to get the integral and then estimate its value.
SkillMaster 6.8.	
3. Show that the circumference of a circle of radius a is $2\pi a$ using the arc length formula for the circle given parametrically by $$x = f(\theta) = a\cos(\theta), y = g(\theta) = a\sin(\theta), 0 \le \theta < 2\pi.$$	Integrate $\sqrt{(f'(\theta))^2 + (g'(\theta))^2}$.

4. Find the arc length of the curve $y = f(x) = x^{3/2}$ from $(1,1)$ to $(4,8)$.



Since y is given as a function of x, use the arc length formula $\int_a^b \sqrt{1+(f'(x))^2}\,dx$. You should be able to use integration techniques to evaluate the resulting integral exactly.

5. Find the arc length of $y = f(x) = e^x$ from $(0,1)$ to $(1,e)$ by using a CAS or integral table.

First set up the integral. The solution will be easily obtained from a table or a CAS. It is actually possible to solve this integral using the substitution $u = (1+e^{2x})^{1/2}$ and applying partial fractions to the resulting integral.

Solutions to worked examples

1. The arc length is $\int_0^1 \sqrt{1+e^{2x}}\,dx$. Let $f(x) = \sqrt{1+e^{2x}}$.

Using Simpson's Rule with $n = 4, \Delta x = 0.25$.

$$L \approx \frac{1}{12}(f(0)+4f(0.25)+2f(0.5)+4f(0.75)+f(1) \approx 2.00$$

2. $f'(\theta) = -4\cos^3(\theta)\sin(\theta)$

$g'(\theta) = 4\sin^3(\theta)\cos(\theta)$

In this case, $\sqrt{(f'(\theta))^2+(g'(\theta))^2} = \sqrt{16(\cos^6(\theta)\sin^2(\theta)+\sin^6(\theta)\cos^2(\theta))}$

$= 4\sin(\theta)\cos(\theta)\sqrt{\cos^4(\theta)+\sin^4(\theta)}$

So $L = \int_0^{\pi/2} \sqrt{(f'(\theta))^2+(g'(\theta))^2}\,d\theta$

$$= \int_0^{\pi/2} 4\sin(\theta)\cos(\theta)\sqrt{\cos^4(\theta)+\sin^4(\theta)}d\theta$$

Using Simpson's Rule for $n = 2$ you obtain $\Delta\theta = \pi/4$. Notice the value of the function in the integral is 0 for both $\theta = 0$ and $\theta = \pi/2$ which are the endpoints.

$$L \approx \frac{\pi/4}{3}(0+4(4(\frac{1}{\sqrt{2}})(\frac{1}{\sqrt{2}})\sqrt{(1/\sqrt{2})^4+(1/\sqrt{2})^4}+0) = \frac{2\pi}{3\sqrt{2}} = \frac{\pi}{3}\sqrt{2} \approx 1.48$$

3. $L = \int_a^b \sqrt{(f'(\theta))^2+,(g'(\theta))^2}d\theta = \int_0^{2\pi} \sqrt{(-a\sin(\theta))^2+(a\cos(\theta))^2}d\theta$

$$= \int_0^{2\pi} \sqrt{a^2\sin^2(\theta)+a^2\cos^2(\theta)}d\theta = a\int_0^{2\pi} \sqrt{\sin^2(\theta)+\cos^2(\theta)}d\theta$$

$$= a\int_0^{2\pi} \sqrt{1}d\theta = a\theta]_0^{2\pi} = 2\pi a$$

4. $f'(x) = (3/2)x^{1/2}$, $(f'(x))^2 = (9/4)x$

$$L = \int_1^4 \sqrt{1+(9/4)x}\,dx = (1/2)\int_1^4 \sqrt{4+9x}\,dx$$

Substitute $u = 4+9x$ $\quad du = 9\,dx$ $\quad$ when $x = 1,\ u = 13$ $\quad$ when $x = 4,\ u = 40$

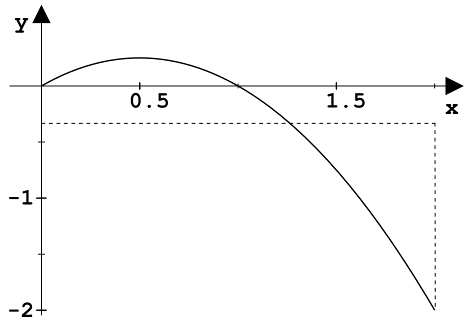
$$L = (1/18)\int_{13}^{40} \sqrt{u}\,du = \left.\frac{u^{3/2}}{27}\right]_{13}^{40} = \frac{(40)^{3/2}-(13)^{3/2}}{27} \approx 7.6337$$

5. $L = \int_0^1 \sqrt{1+(f'(x))^2}\,dx = \int_0^1 \sqrt{1+(e^x)^2}\,dx = \int_0^1 \sqrt{1+e^{2x}}\,dx.$

Using a table, a CAS, or the suggested substitution, you get

$$\left.\sqrt{1+e^{2x}}+\frac{1}{2}\ln\frac{\sqrt{(1+e^{2x})}-1}{\sqrt{(1+e^{2x})}+1}\right]_0^1$$

$$= \sqrt{1+e^2}-\sqrt{2}+1+\ln(\sqrt{1+e^2}-1)+\ln\sqrt{2}+1 \approx 2.0035$$


y
0.5
1.5
x
-1
-2

6.5 Average Value of a Function

Key Concepts:

- Definition of average value of a function
- Mean Value Theorem for Integrals

Skills to Master:

- Compute the average value of a specific function over a given interval and use the Mean Value Theorem for Integrals to find where a function value is equal to its average.

Discussion:

Section 6.5 introduces the concept of the average value of a function. If the function is positive valued, the area under the function is equal to the area of a rectangle whose base is the interval of integration and whose height is the average value. In general, the net area of a function between two end points divided by the length of the interval is the average of the function. If the function is also continuous, then there must be at least one value c where $f(c)$ is equal to the average of the function. This fact is called the Mean Value Theorem for Integrals. See *Figure 2* in this section for a geometric picture of this concept.

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Key Concept: Definition of average value of a function

The average value of a function $f(x)$ on an interval $[a,b]$ is defined to be

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Note that $\int_a^b f(x)\,dx = \int_a^b f_{ave}\,dx$ as mentioned in the discussion above.

Key Concept: Mean Value Theorem for Integrals

The Mean Value Theorem for Integrals states that for a continuous function $f(x)$ on an interval $[a,b]$, there is a number c in $[a,b]$ such that

$$\int_a^b f(x)\,dx = f(c)(b-a)$$

Another way of saying this is that the function achieves its average value at $x = c$. This makes intuitive sense. The function cannot always be less than its average or else the Comparison Properties would be violated. Similarly, the function cannot always be larger than its average. If the function is not a constant equal to its average then it must have points both less than and greater than its average. Because the function is continuous the Intermediate Value Theorem implies the function must have at least one point c where $f(c)$ is equal to its average. For example, if the average velocity of an object over some time period is 85 mph, then there was at least one time when the object was traveling exactly 85 mph. It would seem that the Mean Value Theorem for Integrals implies that a driver on a pay turnpike (that has entrance and exit times kept in its computer) who exited the turnpike too soon could be given a speeding ticket even though no one saw the driver speed.

SkillMaster 6.9: Compute the average value of a specific function over a given interval and use the Mean Value Theorem for Integrals to find where a function value is equal to its average.

To compute the average value of a specific function $f(x)$ over a given interval $[a,b]$, first evaluate $\int_a^b f(x)\,dx$ and then divide by $(b-a)$. The average is the limit of the average of $f(x_i)$ where the x_i are evenly spaced along the interval. For positive functions, the average value is the y value so that the area of a rectangle with base of length $b-a$ and height equal to this y value is exactly equal to the area under the curve.

In case the function is continuous, the Mean Value Theorem for Integrals guarantees that there is a value c such that $f(c)$ is equal to its average. Sometimes you will be asked to find a place where a continuous function achieves it's average value. To do this, set $f(x) = f_{ave}$ and solve for x.

Worked Examples

For each of the following examples, first try to find the solution without looking at the hint. Cover the right column with a piece of paper. If you need a hint, uncover the column. If you need to see the worked solution, look at the solutions section that follows the examples and hints.

Example	Hint
SkillMaster 6.9.	
1. Find the average value of the function $y = f(x) = 1 + \sin(x)$ on the interval $[0, \pi]$.	The formula for the average value of a function is $\frac{1}{b-a}\int_a^b f(x)\,dx$.
2. Find the average value of the function $y = f(x) = x - x^2$ on the interval [0,2]. Then find a point c so that $f(c) = f_{ave}$.	Solve the equation $f_{ave} = f(c)$ for c. $c - c^2 = f_{ave}$.

Solutions to worked examples

1. $f_{ave} = \frac{1}{\pi - 0}\int_0^{\pi} (1 + \sin(x))\;dx = \frac{1}{\pi}\;(x - \cos(x)]_0^{\pi} = \frac{\pi - \cos(\pi) + \cos(0)}{\pi} = \frac{\pi + 2}{\pi}$

2. $f_{ave} = \frac{1}{2}\int_0^2 x - x^2\,dx = \frac{1}{2}(\frac{x^2}{2} - \frac{x^3}{3}\Big]_0^2 = \frac{2^2}{4} - \frac{2^3}{6} = -1/3$

You need to find c so that $f(c) = c - c^2 = f_{ave} = -1/3$.

$c - c^2 = -1/3,\; 3c - 3c^2 = -1,\; 3c^2 - 3c - 1 = 0,\; c = \frac{3 \pm \sqrt{9+12}}{6} = \frac{3 \pm \sqrt{21}}{6}$

These roots are $c \approx 1.2638$ and $c \approx -0.2638$. You want to find a value in the interval $[0, 2]$ so $c \approx 1.2638$. Note in the diagram below, the area below the graph of f and above the dotted line is equal to the area above the graph of f and below the dotted line.

6.6 Applications to Physics and Engineering

Key Concepts:

- Definition of work done in moving an object by applying variable force
- Computing force due to hydrostatic pressure
- Moments and center of mass

Skills to Master:

- Compute work done by a force.
- Use integrals to determine hydrostatic force.
- Compute moments and center of mass for a flat plate in the plane.

Discussion:

Section 6.6 shows the vast applicability of the definite integral. In each of the applications in this section, you end up with a Riemann sum and then obtain an integral by taking a limit. This process is basic for problem solving throughout the physical, biological, applied, and social sciences. Recall that this process is also what leads to the computation of area, volume and arc length.

Key Concept: Definition of work done in moving an object by applying variable force

The work done in moving an object along a straight line from $x = a$ to $x = b$ by applying a variable force $f(x)$ is

$$\int_a^b f(x)\,dx$$

This formula is obtained by using the relation that work = force $\times$ distance for a constant force applied over a distance.

Key Concept: Computing force due to hydrostatic pressure

The pressure on a thin horizontal plate submerged in a fluid is defined to be the force per unit area acting on the plate. At any point in a liquid, the pressure is the same in all directions. The force on a vertical wall submerged in a stable fluid is called the hydrostatic force. The principle just mentioned about the pressure being the same in all directions can be used to compute the hydrostatic force.

Key Concept: Moments and center of mass

The moment about the $y-$axis, M_y, of a thin plate of constant density ρ that covers a region R in the plane is

$$\rho \int_a^b x f(x)\,dx$$

if the region R is the region between the x axis and the curve $f(x)$ for $a \leq x \leq b$.

The moment about the $x-$axis, M_x is

$$\rho \int_a^b \frac{1}{2}\left[f(x)\right]^2 dx$$

Read the *explanation* of how these integrals arise in the text.

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The center of mass $(\overline{x}, \overline{y})$ of the plate is given by

$$\overline{x} = \frac{1}{A}\int_a^b x f(x)\,dx \text{ and } \overline{y} = \frac{1}{A}\int_a^b \frac{1}{2}\left[f(x)\right]^2 dx$$

where A is the area. The fact that for many purposes, a mass may be replaced by a particle of the same mass at the center of mass was key in the development of integral calculus and the discovery of the laws of gravity by Isaac Newton.

SkillMaster 6.10: Compute work done by a force.

To compute the work done by a variable force acting over a distance, use the fact that this work is equal to

$$\int_a^b f(x)\,dx$$

If the force is constant but the distance varies, as is the case when lifting or pulling a rope to the top of a building, you will need to form a Riemann sum and take a limit to get an integral in order to solve the problem. This is illustrated in one of the Worked Examples for this SkillMaster.

SkillMaster 6.11: Use integrals to determine hydrostatic force.

In using an integral to determine the hydrostatic force acting against a wall or object, divide the object into thin sections at the same depth in the fluid. The pressure will be the same on all objects at this depth. Adding up the forces acting at each depth gives a Riemann sum. Taking a limit yields an integral which can be used to compute the hydrostatic force. This is illustrated in the Worked Examples for this SkillMaster.

SkillMaster 6.12: Compute moments and center of mass for a flat plate in the plane.

To compute the moments and center of mass for a flat plate in the plane, use the formulas defined above in the Key Concept: Moments and Center of Mass. Before you begin the problem, sketch the region under consideration and use symmetry if possible to determine at least one of the coordinates of the center of mass. The center of mass is the point where the plate would balance and not tend to lean in any direction. In particular the plate would balance on a straight edge if and only if the straight edge goes through the center of mass.

Worked Examples

For each of the following examples, first try to find the solution without looking at the hint. Cover the right column with a piece of paper. If you need a hint, uncover the column. If you need to see the worked solution, look at the solutions section that follows the examples and hints.

Example	Hint
SkillMaster 6.10.	
1. The force acting on a particle on the $x-$axis is $F(x) = 9800/(x+500)^2$ N when $x \geq 0$. Find the work required to move the particle from $x = 0$ to $x = 20$. By evaluating an improper integral, find the amount of work required to move the particle from $x = 0$ to $x = \infty$.	The amount of work done moving a particle from $x = a$ to $x = b$ is the definite integral $\int_a^b F(x)\,dx$.

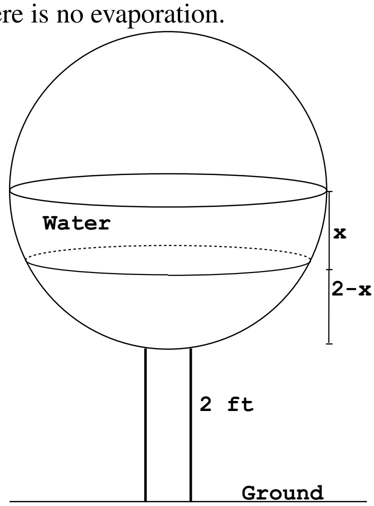
2. A spring is displaced from its natural length of 8 cm to a length of 12 cm. At this point the spring exerts a force of 20 N. How much work is done to stretch the spring further to 20 cm? How much work would be done in compressing the spring from a length of 6 cm to a length of 4 cm?

Recall that Hooke's Law says that $f(x) = kx$ where k is a constant, $f(x)$ is the force required to maintain the spring stretched x units from equilibrium. Use the first piece of information given in the problem to determine k and then evaluate an integral to solve the problem.

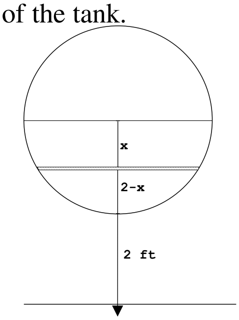
3. A rope 75 ft long hangs over the edge of a 100 ft tall building. A 30 lb weight is attached to the end of the rope. If the rope weighs 0.25 lb/ft of length, compute the work done in pulling the object to the top of the building.

Do this as two separate problems. Pulling the weight to the top requires a constant force applied over 75 ft. Pulling each length of rope to the top requires a constant force applied over a distance that will vary depending on where the length of rope is.

4. A spherical tank of radius 2 ft is suspended 2 ft above the ground. It is half full of water. There is a leak and the water is dripping out. Compute the total work that the water does on the ground if all the water leaks out. Assume there is no evaporation.

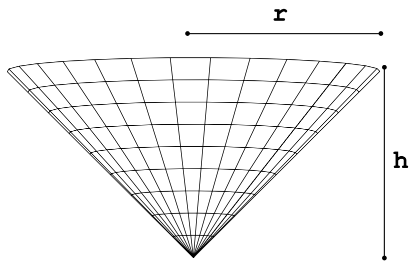


Use coordinates so that the $x-$axis is vertical, with the positive direction downward and with the origin at the center of the tank.

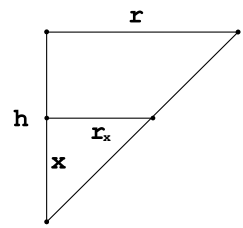


SkillMaster 6.11.

5. Find a formula for the force exerted by fluid stored in an inverted right circular cone in terms of d (the density of the fluid), g (the gravitational constant), r (the radius), and h (the height).



Use the following diagram.



SkillMaster 6.12.

6. Consider a flat plate in the x,y-plane occupying the region bounded by the x-axis and the curve $y = \cos(x)$. Find the center of mass.

Don't forget to look for symmetry. To integrate the function $\cos^2(x)$ use the double angle identity
$\cos^2(x) = \frac{1}{2}(\cos(2x)+1)$.

Solutions to worked examples

1. $W = \displaystyle\int_0^{20} \frac{9800}{(x+500)^2}\,dx = \left.\frac{-9800}{(x+500)}\right]_0^{20} = \frac{-9800}{(520)} + \frac{9800}{(500)} \approx 0.7538$ J

The work required to move the particle all the way to $x = \infty$ is given by the improper integral:

$$W = \int_0^{\infty} \frac{9800}{(x+500)^2}\,dx = \lim_{t\to\infty} \left.\frac{-9800}{(x+500)}\right]_0^t = \lim_{t\to\infty} \frac{-9800}{(t+500)} + \frac{9800}{(500)} = 19.6 \text{ J}$$

2. The displacement when the spring is stretched to 12 cm is 12-8 = 4 cm = 0.04 m.

The force at this displacement is 20 N.

$f(x) = kx20 = k(0.04)k = 500$

Stretching the spring from 12 cm to 20 cm corresponds to a displacement from 4 cm to (20-12) = 8 cm = 0.08 m.

The work done is found by evaluating the integral.

$\int_{0.04}^{0.08} 500x\,dx = 250x^2\Big]_{0.04}^{0.08} = 250(0.08^2 - 0.04^2) = 1.2$ J

Compressing the spring from 6 cm to 4 cm corresponds to a displacement from -2 to -4 cm.

The work done is found by evaluating the integral.

$\int_{-0.02}^{-0.04} 500x\,dx = 250x^2\Big]_{-0.02}^{-0.04} = 250((-0.04)^2 - (-0.02)^2) = 0.3$ J

3. The work required to pull the weight to the top is force times distance which is 30 lbs times 75 ft = $2,250$ ft-lb.

Consider a small segment of rope of length Δx at a distance x ft from the top of the building. The weight of this piece of rope is Δx ft $\times 0.25$ lb/ft $= 0.25\ \Delta x$ lb. The work required to pull this piece of rope the x ft to the top is thus $0.25x\Delta x$ ft-lb. The total work done in pulling the entire rope to the top is about

$$\sum 0.25x\Delta x \text{ ft-lb}$$

where the sum is taken over all small lengths of rope Δx. This is a Riemann sum and if you take a limit as $\Delta x \to 0$, the result is the integral:

$\int_0^{75} 0.25x\,dx \ = 0.25x^2/2\Big]_0^{75} = 0.25 \cdot 75^2/2 = 703.125$ ft-lb

Thus the total work is $2,250 + 703.125 = 2,953.125$ ft-lb.

4. Consider the work done by a thin cross-sectional slice of width Δx at a level $x_i{}^*$ below the center of the tank. Each drop in this slice has to fall approximately $(2 - x_i{}^*) + 2 = 4 - x_i{}^*$ ft. By the Pythagorean Theorem, the radius of the slice is $\sqrt{2^2 - (x_i{}^*)^2}$. Thus the volume of the slice is

$\pi(\sqrt{4 - (x_i{}^*)^2})^2\Delta x = \pi(4 - (x_i{}^*)^2)\Delta x.$

The weight of water is 625 lb/ft^3; so the work done (in ft-lb) on this slice is

$$625\pi(4 - x_i{}^*)(4 - (x_i{}^*)^2)\Delta x$$

The total work is the limit of the sum: $\lim_{n\to\infty} \sum_{i=1}^{n} 625\pi(4 - x_i)(4 - x^2)\Delta x$

$$= \int_0^2 625\pi(4-x)(4-x^2)\,dx = 625\pi \int_0^2 (4-x)(4-x^2)\,dx$$
$$= 625\pi \int_0^2 (16-4x-4x^2+x^3)\,dx = 625\pi \left[16x - 2x^2 - \frac{4}{3}x^3 + \frac{1}{4}x^4\right]_0^2$$
$$= 625\pi \left[16(2) - 2(2^2) - \frac{4}{3}(2^3) + \frac{1}{4}(2^4)\right] = 625\pi \left[32 - 8 - \frac{32}{3} + 4\right]$$
$$= 625\pi(\frac{52}{3}) \approx 34,034 \text{ ft-lb}$$

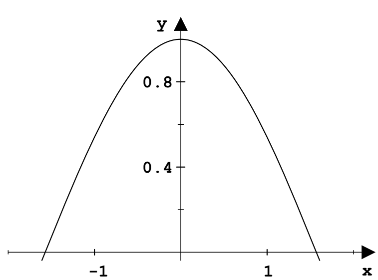
5. Use coordinates so that the x axis runs vertically through the center of the cone with the origin at the vertex of the cone. At a height x the horizontal cross section is a circle. It radius r_x can be found using similar triangles.

$$\frac{r_x}{x} = \frac{r}{h} \qquad r_x = \frac{rx}{h}$$

Now subdivide $[0,h]$ into n even subintervals. The height of the i^{th} interval is $h - x_i{}^*$. The cross-sectional volume at this height may be approximated by a disk of radius $r_{x_i{}^*} = \dfrac{rx_i{}^*}{h}$. Its volume is $2\pi\ r_{x_i{}^*}\Delta x = 2\pi\dfrac{rx_i{}^*}{h}\Delta x$. The force exerted on this slice is $dg(h - x_i{}^*)2\pi\dfrac{r}{h}x_i{}^*\Delta x$. The total force is the limit of the Riemann sums.

$$\lim_{n\to\infty} \sum_{i=1}^n dg(h-x_i{}^*)2\pi\frac{r}{h}x_i{}^*\Delta x = \int_0^h dg(h-x)2\pi\frac{r}{h}x\,dx$$
$$= \frac{2\pi dgr}{h}\int_0^h \left(hx - x^2\right)\,dx = \frac{2\pi dgr}{h}\left(\frac{h}{2}x^2 - \frac{1}{3}x^3\right)\Big]_0^h$$
$$= \frac{2\pi dgr}{h}\left(\frac{h}{2}h^2 - \frac{1}{3}h^3\right) = \frac{2\pi dgrh^3}{h}\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{\pi dgrh^2}{3}$$

6.



$$A = \int_{-\pi/2}^{\pi/2} \cos(x)\,dx = \sin(x)]_{-\pi/2}^{\pi/2} = 1 - (-1) = 2$$

The symmetry principle implies that $\overline{x} = 0$. (This may also be seen because the integrand in the numerator of $\overline{x} = \dfrac{1}{A}\displaystyle\int_{-\pi/2}^{\pi/2} x\cos(x)\,dx$ is odd.)

$$\begin{aligned}
\overline{y} &= \frac{1}{2A}\int_{-\pi/2}^{\pi/2} \cos^2(x)\,dx = \frac{1}{A}\int_0^{\pi/2} \cos^2(x)\,dx \\
&= \frac{1}{A}\int_0^{\pi/2} \frac{1}{2}(\cos(2x)+1)\,dx = \frac{1}{2}\int_0^{\pi/2} (\cos(2x)+1)\,dx \\
&= \frac{1}{2}\left(\frac{1}{2}\sin(2x)+x\right)\Bigg]_0^{\pi/2} = \pi/4
\end{aligned}$$

6.7 Applications to Economics and Biology

Key Concepts:

- Definition of consumer surplus
- Determining blood flow and cardiac output

Skills to Master:

- Use the Net Change Theorem to compute cost or revenue given marginal cost or marginal revenue.
- Compute consumer surplus given specific supply and demand.
- Use formulas given in the text to determine blood flow (flux) and cardiac output from given specifications and collected data.

Discussion:

Section 6.7 introduces the concept of consumer surplus in economics as an application of integration and also gives an application of integration to biology. This section also continues the development of the concepts of *marginal cost* and *marginal revenue* and how these relate to cost and revenue This is a good time to review the definitions and relationships between these terms because they will be used in this section. Also review the *Net Change Theorem*from Section 5.3.

page 140, Section 4.6, page 304.

Key Concept: Definition of consumer surplus

The demand function $p(x)$ is the price per unit that a company has to charge to sell x units of a commodity. The demand curve is the graph of the demand function. If X units are available, the current selling price is $P = p(X)$. The *consumer surplus* for the commodity is

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$$\int_0^X [p(x) - P]\, dx$$

and represents the amount of money saved by consumers in purchasing the commodity at price P.

Key Concept: Determining blood flow and cardiac output

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Review the *law of laminar flow* from Chapter 3.8, Example 7. Poiseuilles's Law gives the volume of blood F that passes a cross-section of a blood vessel per unit time.

$$F = \frac{\pi P R^4}{8 \eta l}$$

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Here, P is pressure difference between the ends of the blood vessel, η is the viscosity of the blood, R is the radius of the blood vessel, and l is the length of the blood vessel. Read the *explanation* of how this arises as a limit of Riemann sums in this section. Notice that one remarkable consequence of Poiseuilles' Law is that blood flow on the boundary of the blood vessel is 0. It is a general physical fact that fluid flow (liquid or gas) is 0 at the surface of a solid. You can see evidence of this by looking at a fan blade and noticing the dust particles that do not blow off even when the fan is operating at high speeds.

SkillMaster 6.13: Use the Net Change Theorem to compute cost or revenue given marginal cost or marginal revenue.

The Net Change Theorem tells you that

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

So if you know the marginal cost or marginal revenue functions (remember that these are derivatives) you can find the total cost or revenue assuming that you know the initial cost or revenue. The Worked Example for this SkillMaster gives one example of this.

SkillMaster 6.14: Compute consumer surplus given specific supply and demand.

Given a specific supply of a commodity, and given a demand function, you should be able to compute the consumer surplus using the formula

$$\int_0^X [p(x) - P]\,dx$$

If you are given the cost P you may need to set the demand function $p(X) = P$ to determine the value of X.

SkillMaster 6.15: Use formulas given in the text to determine blood flow (flux) and cardiac output from given specifications and collected data.

Poiseuille's Law to determine blood flow was discussed above. Before you apply this law, check that you know each of the quantities P, R, η, and l. The cardiac output is given by

$$F = \frac{A}{\int_0^T c(t)dt}$$

where A is the amount of dye injected into the right atrium and where $c(t)$ is the concentration of dye at time t, measured over a time interval $[0,T]$. The Worked Example for this SkillMaster shows how to approximate this from measurements by using Simpson's Rule.

Worked Examples

For each of the following examples, first try to find the solution without looking at the hint. Cover the right column with a piece of paper. If you need a hint, uncover the column. If you need to see the worked solution, look at the solutions section that follows the examples and hints.

Example	Hint
SkillMaster 6.13.	
1. The marginal revenue from selling x items is $R'(x) = 120 - 0.015x$. Find the revenue from selling 800 items.	Let $R(X)$ be the revenue. Notice that $R(0) = 0$ since there is no revenue from selling 0 items. $R(X) = R(X) - R(0)$ $= \int_0^X R'(u)du$.

SkillMaster 6.14.

2. The demand function for a product in dollars is
$p = \dfrac{8000}{x+40}$.
Find the consumer surplus when the selling price is \$100.

Recall the demand function is the price consumers are willing to pay if the quantity of the product is x. The consumer surplus is the definite integral
$\int_0^X [p(x) - P]\,dx$
where X is the amount of the quantity so that
$P = p(X)$.

SkillMaster 6.15.

3. An 8-mg injection of dye is given into the right atrium of a patient's heart. The concentration of dye $c(t)$ in milligrams per liter is measured at 1 second intervals. The results are recorded here. Use Simpson's Rule to estimate the cardiac output.

t	0	1	2	3
$c(t)$	0	3.2	6	10.4

t	4	5	6	7	8
$c(t)$	8.7	5	2.2	0.4	0

Recall that the formula for the cardiac output is $\dfrac{1}{A}\int_0^T c(t)\,dt$.

Solutions to worked examples

1. $R(800) = \int_0^{800} R'(x)\,dx = \int_0^{800} 120 - 0.015x\,dx$
$= 120x - 0.015x^2/2]_0^{800} = 120(800) - 0.015(800)^2/2$
$= 96000 - 4800 = \$91,200$

2. If the selling price is \$100 then the following equation holds.

$$100 = \frac{8000}{x+40} 1 = \frac{80}{x+40} x+40 = 80 x = 40$$

So when $P = 100$, $x = 40$.

The consumer surplus is the definite integral.

$$\int_0^{40} (p(x) - 100)\, dx = \int_0^{40} \left(\frac{8000}{x+40} - 100 \right) dx$$

$$= 8000 \ln(x+40) - 100x]_0^{40} = 8000[\ln(40+40) - \ln(40)] - 100(40)]$$

$$= 8000[\ln(80) - \ln(40)] - 4000 = 8000 \ln(80/40) - 4000$$

$$= 8000 \ln(2) - 4000 \approx \$1545$$

3. Here $T = 8, A = 8, \Delta t = 1$. Use Simpson's Rule to approximate the integral.

$$\int_0^8 c(t)\, dt \approx$$

$$\frac{1}{3}[c(0) + 4c(1) + 2c(2) + 4c(3) + 2c(4) + 4c(5) + 2c(6) + 4c(7) + c(8)]$$

$$= \frac{1}{3}[0 + 4(3.2) + 2(6.0) + 4(10.4) + 2(8.7) + 4(5.0) + 2(2.2) + 4(0.4) + 0]$$

$$= 36.6 \text{ L/min}$$

6.8 Probability

Key Concepts:

- Computing probabilities by integrating a probability density function
- Means and medians
- Normal distributions

Skills to Master:

- Use a probability density function, given as a formula or graph, to determine specified probabilities.
- Calculate mean and median for specified distributions.
- Use a calculator or CAS to determine probabilities for normal distributions.

Discussion:

Section 6.8 develops some of the ideas of probability and shows you how to use Calculus to compute the probabilities of random occurrences in various models. Probability Theory is often a course in itself but these ideas are so pervasive that it is useful to have some understanding of these methods. Of course, this section is only able to provide you with a brief introduction.

Key Concept: Computing probabilities by integrating a probability density function

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A random quantity X whose possible values occur in an interval of real numbers is called a *continuous random variable* . The probability that X is between a and b is denoted

$$P(a \leq X \leq b)$$

A probability density function f for the random variable X is a function f satisfying

$$f(x) \geq 0,$$
$$P(a \leq X \leq b) = \int_a^b f(x)\,dx, \text{ and}$$
$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

This definition reflects the facts that no probability is ever negative or ever greater than 1. In other words, nothing occurs less often than never nor occurs more often than always. The last line above indicates that the probability that X takes on some value is 1 (i.e. something occurs). To compute the probability $P(a \leq X \leq b)$, evaluate the integral $\int_a^b f(x)\,dx$.

Key Concept: Means and medians

The mean and median are two measures of the middle of the random variable. The *mean* of a probability density function f is

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$$\int_{-\infty}^{\infty} xf(x)\,dx$$

and should be interpreted as the average value of the random variable. The *median* of a probability density function is the value m such that the probability that the value of the random variable X is greater than or equal to m is $1/2$. That is, the median is the number m such that

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$$\int_m^{\infty} f(x)\,dx = 1/2$$

Key Concept: Normal distributions

A *normal distribution* is a random variable with a probability density function of the form

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$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

where μ is the mean and σ is a measure of how spread out the values of the random variable are. The value σ is called the standard deviation. Figure 5 on page 491 in the text shows the graphs of several normal distributions.

SkillMaster 6.16: Use a probability density function, given as a formula or graph, to determine specified probabilities.

The key idea here is to use the fact that $P(a \leq X \leq b) = \int_a^b f(x)\,dx$ for a random variable X with a probability density function f. You can then integrate f either using techniques of integration or by approximation techniques if you are given a formula for f. If instead, you are given a graph of f, you can use the area interpretation of integral to approximate $\int_a^b f(x)\,dx$.

SkillMaster 6.17: Calculate mean and median for specified distributions.

Given a specific density function, computing the mean is done by evaluating the integral $\int_{-\infty}^{\infty} xf(x)\,dx$. Notice that the mean μ is computed with an integral that is equivalent to finding the centroid. This would be the balance point if the probability density referred to mass. Instead the mean is the value that averages of many trials of the random variable are likely to be close to. To calculate the median, you need to find the value of m for which

$$\int_m^{\infty} f(x)\,dx = 1/2$$

To do this, evaluate the integral $\int_m^{\infty} f(x)\,dx$ and obtain an expression involving m. Then set this expression equal to $1/2$ and solve for m.

SkillMaster 6.18: Use a calculator or CAS to determine probabilities for normal distributions.

To determine probabilities for normal distributions, you will need to be able to evaluate integrals of the form

$$\int_a^b \frac{1}{\sigma\sqrt{2\pi}} e^{-\dfrac{(x-\mu)^2}{2\sigma^2}}\,dx$$

This integral has no antiderivative that may be computed or found in a table. There are tables of values of these integrals, or you can use an approximation technique such as Simpson's Rule on your CAS or calculator.

Worked Examples

For each of the following examples, first try to find the solution without looking at the hint. Cover the right column with a piece of paper. If you need a hint, uncover the column. If you need to see the worked solution, look at the solutions section that follows the examples and hints.

Example	Hint

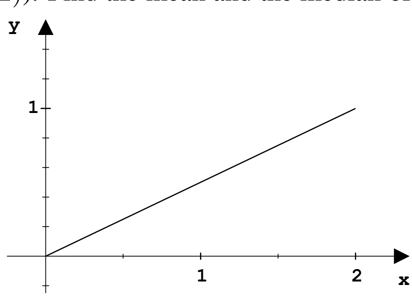
SkillMaster 6.16.

1. A new student takes a job at a donut shop. When he is not eating donuts, the student records when customers enter the shop. During the mid afternoon when business is slow he estimates that one customer enters every 8 minutes on the average. During this period of time find the probability that the first customer enters the shop sometime between the next 2 minutes and the next 8 minutes. Find the probability that no one enters the shop within the next 10 minutes. Model this using an exponential distribution.

Hint: The probability density function of an exponential distribution with mean $\mu = 8$ is $f(t) = (1/8)e^{-t/8}$.

SkillMaster 6.17.

2. Consider the graph below. Check that it is a probability density function. Let X be a random variable with this density. Find the probability that the random variable takes a value that is less than $1/2$ (that is, compute $P(X < 1/2)$). Find the mean and the median of X.



Hint: A probability density function is any function $f(x)$ that is never negative, i.e. $f(x) \geq 0$ and whose integral is 1, i.e. $\int_{-\infty}^{\infty} f(x)\,dx = 1$ (area between the graph and the $x-$axis is 1).

SkillMaster 6.18.

3. Show that 68% of a normal population is within one standard deviation from the mean. In other words if X is a normal random variable with mean μ and variance σ^2 show that $P(\mu-\sigma < X < \mu+\sigma) \approx 0.68$. Use the results of Example 5, pg. 491 to help.

In the integral that computes the probability use the substitution $u = (x-\mu)/\sigma$.

Solutions to worked examples

1. The probability that the first customer comes in between the next 2 minutes and the next 8 minutes is the definite integral shown here.

$$\int_2^8 f(t)\,dt = \int_2^8 (1/8)e^{-t/8}dt = -e^{-t/8}\Big]_2^8$$
$$= -e^{-8/8} + e^{-2/8} = -e^{-1} + e^{-1/4} \approx 0.4109$$

The probability that the first customer does not come into the donut shop within the 10 minutes is the same as the probability that the first customer enters the shop sometime after the next 10 minutes and is equal to the following definite integral.

$$\int_{10}^{\infty} f(t)\,dt = \int_{10}^{\infty} (1/8)e^{-t/8}dt = \lim_{x\to\infty} [-e^{-t/8}]_{10}^{x}$$
$$= \lim_{x\to\infty} -e^{-x/8} + e^{-10/8} = 0 + e^{-5/4} \approx 0.2865$$

2.

The graph is always positive between 0 and 1 and is 0 elsewhere so $f(x) \geq 0$.

The area is the area of a right triangle with side lengths 2 and 1, so $\int_{-\infty}^{\infty} f(x)\,dx = (1/2)(1)(2) = 1$.

(Equivalently, since $f(x)$ is a linear function for x between 0 and 2 we can see $f(x) = \frac{1}{2}x$ for x between 0 and 2 and $f(x)$ is equal to 0 outside $[0,2]$.)

Thus the graph is the graph of a probability density function.

$$P(X < 0.5) = \int_0^{\frac{1}{2}} x/2\,dx = x^2/4\Big]_0^{\frac{1}{2}} = 1/16$$

This answer may also have been found using similar triangles.

$$\mu = \int_{-\infty}^{\infty} xf(x)\,dx = \int_0^2 x(x/2)\,dx = \int_0^2 x^2/2\,dx = x^3/6\Big]_0^2 = 8/6 = 4/3 \approx 1.3333$$

The median is the value m so that $\int_{-\infty}^{m} f(x)\,dx = 1/2.$

$$1/2 = \int_{-\infty}^{m} f(x)\,dx = \int_0^m x/2\,dx = x^2/4\Big]_0^m = m^2/4$$

$$2m^2 = 4 \qquad m^2 = 2 \qquad m = \sqrt{2} \approx 1.4142$$

3. $$\frac{1}{\sigma\sqrt{2\pi}} \int_{\mu-\sigma}^{\mu+\sigma} e^{-(x-\mu)^2/(2\sigma^2)}\,dx = \frac{1}{\sigma\sqrt{2\pi}} \int_{\mu-\sigma}^{\mu+\sigma} e^{-((x-\mu)/\sigma)^2/2}\,dx$$

Let $u = (x-\mu)/\sigma$, $du = (1/\sigma)\,dx$,

$x = \mu+\sigma$, then $u = 1x = \mu-\sigma$, then $u = -1$

$$\frac{1}{\sigma\sqrt{2\pi}} \int_{\mu-\sigma} {}^{\mu+\sigma} e^{-((x-\mu)/\sigma)^2/2}\,dx = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-(x^2)/2}\,dx \approx 0.68$$

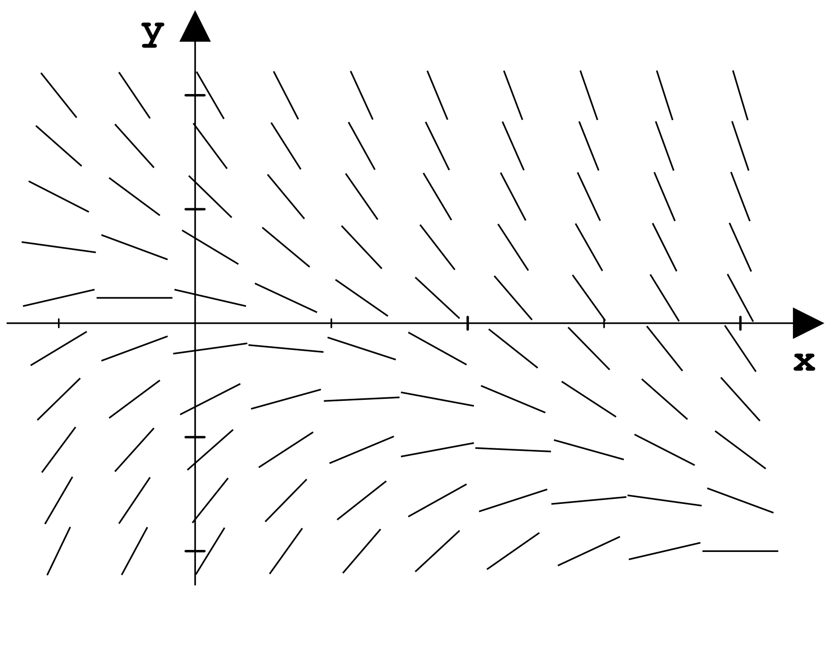
This is the correct estimate because you have already seen this integral estimated in Example 5, pg. 491 with $\mu = 100$ and $\sigma = 15$.

SkillMasters for Chapter 6

SkillMaster 6.1:	Sketch the region enclosed by given curves and determine the area by integration.
SkillMaster 6.2:	Given graphs or tables representing velocities, use area to determine distances.
SkillMaster 6.3:	Graph and determine the areas bounded by parametric curves.
SkillMaster 6.4:	Compute volumes of solids obtained by rotating a bounded region about either the $x-$axis or the $y-$axis.
SkillMaster 6.5:	Compute the volume of solids by determining a function, $A(x)$, that gives areas of parallel cross sections.
SkillMaster 6.6:	Compute the volume of solids obtained by rotating a bounded region about either the $x-$axis or the $y-$axis by considering cylindrical shells.
SkillMaster 6.7:	Find the lengths of parametric curves and curves given by $y = f(x)$ or $x = f(y)$ by approximating.
SkillMaster 6.8:	Find the exact lengths of parametric curves and curves given by $y = f(x)$ or $x = f(y)$ by integrating or by using a table of integrals or CAS.
SkillMaster 6.9:	Compute the average value of a specific function over a given interval and use the Mean Value Theorem for Integrals to find where a function value is equal to its average.
SkillMaster 6.10:	Compute work done by a force.
SkillMaster 6.11:	Use integrals to determine hydrostatic force.
SkillMaster 6.12:	Compute moments and center of mass for a flat plate in the plane.
SkillMaster 6.13:	Use the Net Change Theorem to compute cost or revenue given marginal cost or marginal revenue.
SkillMaster 6.14:	Compute consumer surplus given specific supply and demand.
SkillMaster 6.15:	Use formulas given in the text to determine blood flow (flux) and cardiac output from given specifications and collected data.
SkillMaster 6.16:	Use a probability density function, given as a formula or graph, to determine specified probabilities.
SkillMaster 6.17:	Calculate mean and median for specified distributions.
SkillMaster 6.18:	Use a calculator or CAS to determine probabilities for normal distributions.

Chapter 7

Differential Equations



7.1 Modeling with Differential Equations

Key Concepts:

- Using differential equations to model physical situations
- General solutions
- Initial value problems

Skills to Master:

- Determine whether or not specific functions provide a solution to a given differential equation.
- Given a general solution and initial value, determine a specific solution.
- Analyze properties of the solution by analyzing the differential equation.

Discussion:

Section 7.1 gives several examples of modeling physical situations with differential equations. Differential equations are equations involving variables and rates of change. In the next three sections, you will see how to solve many differential equations using geometric, numerical and symbolic methods. The material described in this section is basic to mathematical modeling and will be used later in the chapter. Differential equations are used in many diverse fields from archaeology (the study of ancient history) to zymolgy(the study of fermentation).

Key Concept: Using differential equations to model physical situations

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A *differential equation* is an equation that contains an unknown function and some of its derivatives. Suppose that you are presented with a physical situation in which the quantity changes over time. If the rate of change of the quantity depends on the quantity itself, other quantities and possibly on time, you can often write down a differential equation that models the situation. In this section the differential equations

$$\frac{dP}{dt} = kP \text{ and } \frac{dP}{dt} = kP\left(1 - \frac{P}{K}\right)$$

which model two types of population growth, and the differential equation

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

which models the motion of a spring are presented and discussed.

Key Concept: General solutions

A function $y = f(x)$ is called a solution of a differential equation if the differential equation is satisfied when f and its derivatives are substituted into the equation. When you are asked to find the general solution of a differential equation, you are being told to find all possible solutions.

Key Concept: Initial value problems

An *initial-value problem* is a problem that involves finding a solution to a differential equation that satisfies a certain initial condition. An initial condition is a condition of the form

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$$y(t_0) = y_0$$

which gives the y value at a specific time t_0. Usually you first find the general solution (really a family of solutions), and then substitute the given value of the initial condition to find a specific solution that matches the given information.

SkillMaster 7.1: Determine whether or not specific functions provide a solution to a given differential equation.

If you are given a differential equation and a specific function that you know how to differentiate, it is possible to check whether that specific function provides a solution to the differential equation. The method consists of substituting the function and its derivatives into the differential equation and checking that the differential equation is satisfied. Another example of this is provided in the Worked Example for this Skill-Master.

SkillMaster 7.2: Given a general solution and initial value, determine a specific solution.

If you are given a general solution and an initial value, substitute the values provided into the general solution and solve the resulting equation. For example, the general solution to

$$\frac{dy}{dx} = -x^2$$

is $y = -\frac{x^3}{3} + C$. If you are given the additional information that $y(0) = 4$, you can substitute 0 for x and 4 for y to obtain the equation

$$4 = 0 + C \text{ or } 4 = C$$

Thus, the solution corresponding to the initial condition $y(0) = 4$ is

$$y = -\frac{x^3}{3} + 4$$

SkillMaster 7.3: Analyze properties of the solution by analyzing the differential equation.

By analyzing a differential equation and using what you know about derivatives, you can determine where the solution function is increasing and decreasing, and you can see if the function approaches a specific value over time.

Worked Examples

For each of the following examples, first try to find the solution without looking at the hint. Cover the right column with a piece of paper. If you need a hint, uncover the column. If you need to see the worked solution, look at the solutions section that follows the examples and hints.

Example	Hint
SkillMaster 7.1.	
1. Show that for each constant C $$y = Ce^{-x^2/2}$$ is a solution for the differential equation $$y' = -xy$$	Differentiate the expression $y = Ce^{-x^2/2}$ and substitute y for $Ce^{-x^2/2}$ into the result. This will eliminate C.
SkillMaster 7.2.	
2. Consider the same differential equation as in the previous problem. Suppose $y(0) = 4$ and $y' = -xy$. Compute $y(1)$.	The general solution was found in the first Worked Example above. Use the information $y(0) = 4$ to find the specific solution. Then substitute $x = 1$ into the expression for y.
3. Find the values of r so that e^{rt} satisfies the differential equation $$y'' + 3y' - 10y = 0$$	For $y = e^{rt}$ compute y' and y''. Then substitute these expressions into the differential equation and solve for all possible values of r.

SkillMaster 7.3.

4. A population is modeled by the differential equation

$$\frac{dP}{dt} = CP(1000 - P)$$

where C is a positive constant. For what values of P is the population increasing? For what values is it decreasing? What are the equilibrium values? Since P represents population you know $P \geq 0$.

The population is increasing if $dP/dt > 0$ and the population is decreasing if $dP/dt < 0$. The equilibrium solutions are those P for which $dP/dt = 0$.

Solutions to worked examples

1. $y' = \frac{d}{dx}(Ce^{-x^2/2}) = Ce^{-x^2/2}\frac{d}{dx}(-x^2/2) = Ce^{-x^2/2}(-x) = y(-x) = -xy$

2. $y(0) = 4 = Ce^{-0^2/2}$ $\quad 4 = Ce^0$ $\quad C = 4$

$y(x) = 4e^{-x^2/2}$ $\quad y(1) = 4e^{-1^2/2}$ $\quad y(1) = 4e^{-1/2}$

3. $y = e^{rt}$ $\quad y' = re^{rt}$ $\quad y'' = r^2e^{rt}$

$y'' + 3y' - 10 = 0$ $\quad r^2e^{rt} + 3re^{rt} - 10e^{rt} = 0$ $\quad e^{rt}(r^2 + 3r - 10) = 0$

You may cancel e^{rt} because the exponential function is always positive.

$r^2 + 3r - 10 = 0$ $\quad$ To solve, factor this polynomial.

$(r-2)(r+5) = 0$ $\quad r = 2, -5$

Two solutions are $y = e^{2t}$ and$y = e^{-5t}$

4. Since $C > 0$, $dP/dt > 0$ if and only if $P(1000 - P) > 0$ P is never negative so this happens only if $P > 0$ and $1000 - P > 0$ or when $0 < P < 1000$. This is the range of P for which the population is increasing. The population is decreasing when $dP/dt < 0$. $P \geq 0$ so this occurs only for $P > 0$ and $1000 - P < 0$ or for $1000 < P$. Equilibrium solutions are those for which $dP/dt = 0 = P(1000 - P)$. These values of P are $P = 0$ and $P = 1000$. The solution $P = 0$ is an equilibrium since if the population is extinct then no new members will be born. For $0 < P < 1000$ the population is increasing and for $P > 1000$ the population is decreasing, so regardless of the initial number of members in the population (as long as it is positive) in the long run the population will converge to the equilibrium $P = 1000$.

7.2 Direction Fields and Euler's Method

Key Concepts:

- Using direction fields to represent general solutions
- Analyzing direction fields to better understand a model
- Approximating solutions to initial value problems

Skills to Master:

- Sketch direction fields and use them to find solutions satisfying initial conditions.
- Use direction fields to analyze limiting values in applications.
- Use Euler's method with specified step sizes to approximate specific y values for solutions to initial value problems.
- Compare values obtained from Euler's Method with specific step sizes and true function value, if known.

Discussion:

Section 7.2 defines a direction field for a differential equation of the form

$$y' = F(x,y)$$

consists of a collection of line segments drawn in the (x,y)-plane that have slope equal to $F(x,y)$ at the grid point (x,y). Direction fields can be used to gain a geometric understanding of the solutions to these type of differential equations.

Euler's Method gives a numerical method for finding solutions to differential equations of the form

$$y' = F(x,y)$$

The idea of this method is to approximate the solution by a curve made up of short line segments. The length of the segments is determined by the step size.

Key Concept: Using direction fields to represent general solutions

If a direction field is given to you, or if you draw a direction field, you can use the direction field to represent general solutions. To do this, draw a smooth curve or curves through the field that point in the direction of the field. This process is shown in *Figures 3, 4* and later in the Worked Examples in this section of the Study Guide.

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Key Concept: Analyzing direction fields to better understand a model

By analyzing a direction field, you can see where the solutions to a differential equation are increasing, where they are decreasing, and whether any of the solutions approach a limiting value. *Example 2* in this section provides a specific example of how to do this.

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Key Concept: Approximating solutions to initial value problems

You saw above how to sketch a direction field for a differential equation of the form

$$y' = F(x,y)$$

Euler's Method gives a way of approximating the solution to an initial value problem corresponding to this differential equation. Carefully read *Example 3* to see how this method works.

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SkillMaster 7.4: Sketch direction fields and use them to find solutions satisfying initial conditions.

To sketch a direction field, find the slope at points (x,y) for a selection of values of x and y in the region under consideration. Then draw a short line segment with the correct slope at each of the points. To find solutions satisfying initial conditions, draw a smooth curve starting at the point specified by the initial condition. Make sure that the curve you draw follows the direction field.

SkillMaster 7.5: Use direction fields to analyze limiting values in applications.

If the direction field indicates that the general solutions all approach a specific value as x increases, you can use this information to gain understanding of the physical situation that the differential equation models.

SkillMaster 7.6: Use Euler's method with specified step sizes to approximate specific y values for solutions to initial value problems.

If you are given a differential equation of the form

$$y' = F(x,y)$$

and are given a step size h and an initial condition $y(x_0) = y_0$, apply Euler's Method as follows. Let

$$x_1 = x_0 + h, \quad x_2 = x_1 + h, \quad x_3 = x_2 + h, \text{ and so on.}$$

Then let

$$\begin{aligned} y_1 &= y_0 + hF(x_0, y_0) \\ y_2 &= y_1 + hF(x_1, y_1) \end{aligned}$$

and in general let

$$y_n = y_{n-1} + hF(x_{n-1}, y_{n-1})$$

SkillMaster 7.7: Compare values obtained from Euler's Method with specific step sizes and true function value, if known.

If you know the exact solution to a differential equation, you can compare the values obtained by using Euler's Method with the exact values. The Worked Examples for this SkillMaster show another example of this.

Worked Examples

For each of the following examples, first try to find the solution without looking at the hint. Cover the right column with a piece of paper. If you need a hint, uncover the column. If you need to see the worked solution, look at the solutions section that follows the examples and hints.

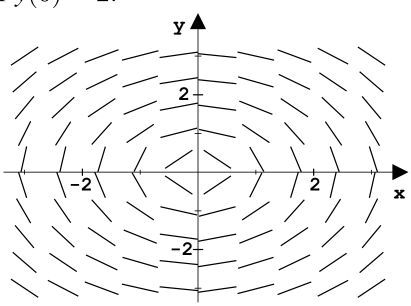
Example	Hint

SkillMaster 7.4.

1. The direction field for the differential equation

$$y' = -x/y$$

is shown below. Sketch the graphs for the solutions $y(0) = -1$ and for $y(0) = 2$.



Hint: Sketch curves starting at $(0,2)$ and at $(0,-1)$.

2. Sketch the direction field of

$$y' = y^2$$

Sketch the solution that corresponds to $y(0) = -1$ and $y(0) = 0$.

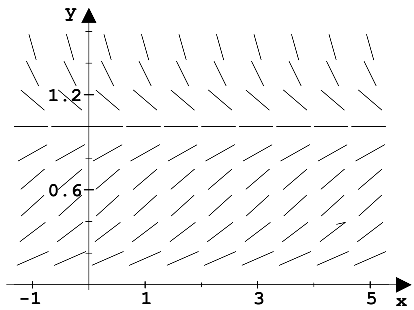
Hint: Notice that the slopes of the line segments of the direction field along any horizontal line are the same, i.e. the segments are parallel. To sketch the direction field make line segments along the $y-$axis and then make parallel translates of them along horizontal lines, as shown in Figures 10 and 11 in Stewart on page 507.

SkillMaster 7.5.

3. A population is modeled by the differential equation

$$P' = P(1 - P^{1.5}) \qquad P \geq 0$$

The direction field for this differential equation is shown.

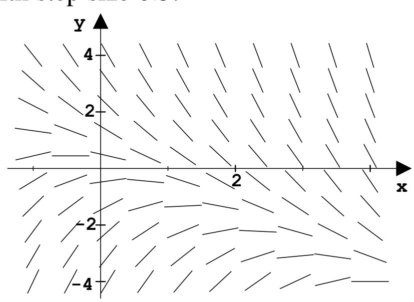


Sketch two different solutions. Identify any equilibrium points. Describe the limiting value of the population.

Equilibrium points correspond to population levels where $P' = 0$. Sometimes equilibrium points attract other solutions, that is, the limiting value may be an equilibrium point.

SkillMaster 7.6.

4. A direction field for a differential equation is shown. Use a ruler and pencil to draw an approximation to the solution that passes through the point $(-1, -2)$ using Euler's method with step size 0.5.



Start at $(-1, -2)$ and estimate the corresponding y value each time you increase x by 0.5. Each step covers a horizontal distance of 0.5.

5. Use Euler's method with $h = 0.25$ to approximate $y(1)$ where $y' = F(x,y) = \cos(xy)$, $y(0) = 1$.

Here $h = 0.25$, $x_0 = 0$, and so $x_4 = 1$. Use Euler's method to compute y_4 which is the Euler approximation to $y(1)$. To get started, $y_0 = 1$. $y_1 = y_0 + hF(x_0, y_0)$.

SkillMaster 7.7.

6. Suppose $y' = y/x^2$ and $y(1) = 1$
Use Euler's method to approximate
$y(2)$ with step size $h = 0.5$.

Here $h = 0.5$, $x_0 = 1$, $y_0 = 1$, and $F(x,y) = y/x^2$. The value $y_1 = y_0 + hF(x_0, y_0)$, and y_2 is the desired approximation.

7. Repeat the example above with step size $h = 0.2$.

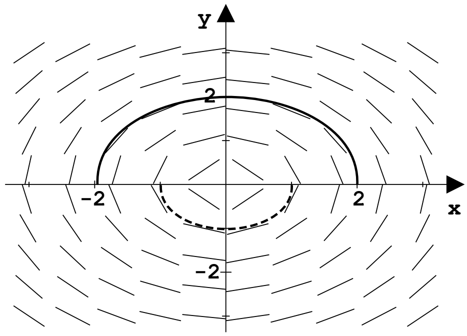
Here $h = 0.2$, $x_0 = 1$, $x_1 = 1.2$, $y_0 = 1$, and $F(x,y) = y/x^2$. The value y_5 is the desired approximation.

8. Show that $y(x) = e^{\frac{x-1}{x}}$ is a solution to the initial-value problem in problem 6 above. Compute $y(2)$. How close are the approximations computed in Worked Examples 6 and 7 above?

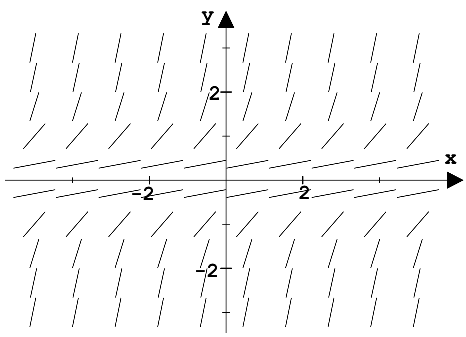
Compare $y(2)$ with the approximation from problem 6.

Solutions to worked examples

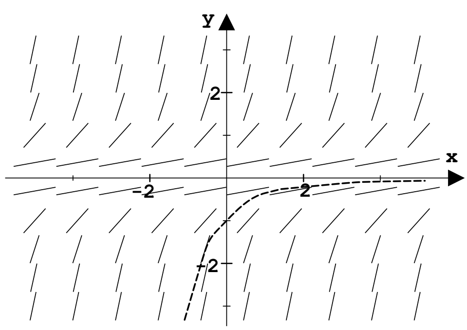
1. Below are sketched solutions through $(0,2)$ (solid line) and $(0,-1)$ (dashed) line.



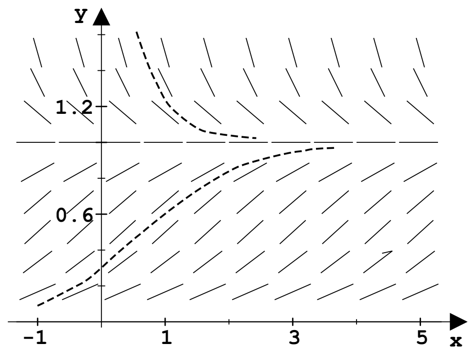
2.



The solutions for which $y(0) = -1$ is sketched below. Notice that the solution for which $y(0) = 0$ is the constant function $y = 0$. The graph of this coincides with the x axis.



3. Two solutions are sketched below with dotted lines.



Equilibrium points occur when there is no change in P, i.e. when $P' = 0$.

$0 = P(1 - P^{1.5}),\ P \geq 0$ $\qquad P = 0$ is one solution. The other is

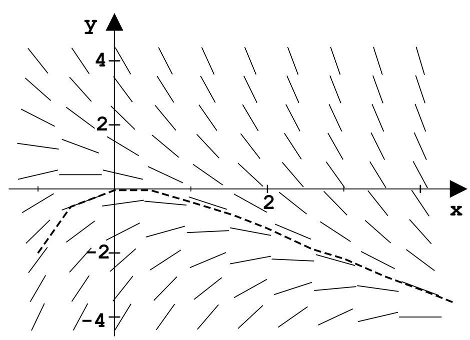
$0 = 1 - P^{1.5}$ $\qquad P^{1.5} = 1$ $\qquad P = 1.$

This is apparent from the flow lines of the direction field.

If $P = 0$ the population is extinct and cannot grow.

If $P > 0$ then all solutions converge to $P = 1$ as a limiting value. Note that $P < 0$ does not make sense physically, i.e. there is never a population of negative 10,000.

4.



5. $y_1 = 1 + 0.25\cos(0 \cdot 1) = 1.25$

$y_2 = 1.25 + 0.25\cos(0.25 \cdot 1.25) = 1.4879$

$y_3 = 1.4879 + 0.25\cos(0.5 \cdot 1.4879) = 1.6718$

$y_4 = 1.6718 + 0.25\cos(0.75 \cdot 1.6718) = 1.7498$

6. $y_1 = 1+0.5(1/1^2) = 1.5y_2 = 1.5+0.5(1.5/(1.5^2) = 1.8333$

7. $y_1 = 1+0.2(1/(1^2)) = 1.2$

$y_2 = 1.2+0.2(1.2/(1.2^2)) = 1.3667$

$y_3 = 1.3667+0.2(1.3667/(1.4^2)) = 1.5062$

$y_4 = 1.5062+0.2(1.5062/(1.6^2)) = 1.6239$

$y_5 = 1.6239+0.2(1.6239/(1.8^2)) = 1.7241$

One would expect this to be a better approximation to $y(2)$ since the step size is smaller.

8. You must show that $y(x) = e^{\frac{x-1}{x}}$ satisfies $y' = y/x^2$ and $y(1) = 1$.

$y' = \frac{d}{dx}(e^{\frac{x-1}{x}}) = e^{\frac{x-1}{x}}\dfrac{d}{dx}(\dfrac{x-1}{x}) = y\dfrac{d}{dx}(\dfrac{x-1}{x}) = y\dfrac{d}{dx}(1-\dfrac{1}{x})$

$= y(0-(\dfrac{-1}{x^2})) = y/x^2$ This agrees with the differential equation.

$y(1) = e^{\frac{1-1}{1}} = e^0 = 1$ so $y(x) = e^{\frac{x-1}{x}}$ is a solution to the initial-value problem. $y(2) = e^{\frac{2-1}{2}} = e^{1/2} = \sqrt{e} \approx 1.6487$

The approximation to $y(2)$ with step size $h = 0.2$ is closer to the true value than that with step size 0.5 but still has an error of about $1.7241 - 1.6487 = 0.0754$.

7.3 Separable Equations

Key Concepts:

- Solving separable equations
- Orthogonal trajectories of a family of curves
- Solving application problems involving separable equations

Skills to Master:

- Recognize and solve separable equations by using integration.
- Sketch orthogonal trajectories of a given family of curves.
- Solve separable equations as they arise in applications, and use the solution to answer questions about the application.

Discussion:

Section 7.3 gives you a method to solve an important class of differential equations. In the previous section you gained a geometric understanding of solutions to differential equations by using direction fields and how to approximate solutions numerically using Euler's Method. Now you will learn to solve some differential equations symbolically (and exactly).

Key Concept: Solving separable equations

A separable equation is a first order differential equation that can be written in the form

$$\frac{dy}{dx} = g(x)f(y) \text{ or } \frac{dy}{dx} = \frac{g(x)}{h(y)}$$

These types of equations can be solved by rewriting them in differential form

$$\begin{aligned} \frac{1}{f(y)}dy &= g(x)dx \\ &\text{or} \\ h(y)dy &= g(x)dx \end{aligned}$$

and then integrating. Study *Example 1* in this section to see a specific example of this.

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Key Concept: Orthogonal trajectories of a family of curves

A curve that intersects each curve in a family of curves orthogonally is an *orthogonal trajectory* of the family of curves. *Example 4* in this section shows that each curve of the form

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$$x^2 + \frac{y^2}{2} = C$$

is an orthogonal trajectory to the family of curves of the form

$$x = ky^2$$

In words, a differential equation is separable if you are able to rearrange the equation so that all of the expressions involving y are on one side and all of the expressions involving x are on the other side. Once again, you see the connection between differentials in derivatives and differentials in integrals. Always compute with the assumption that they are the same.

Key Concept: Solving application problems involving separable equations

Many application problems can be modeled by separable differential equations. If you encounter such a problem, use the techniques in this section to solve the differential equation and interpret the solution as it applies to the physical situation.

SkillMaster 7.8: Recognize and solve separable equations by using integration.

Any differential equation of the form

$$\frac{dy}{dx} = g(x)f(y) \text{ or } \frac{dy}{dx} = \frac{g(x)}{h(y)}$$

is separable and can be rewritten in the form

$$\frac{dy}{f(y)} = g(x)dx \text{ or } h(y)dy = g(x)dx$$

Then integrating both sides of the equation will lead to a solution.

SkillMaster 7.9: Sketch orthogonal trajectories of a given family of curves.

Use the techniques in this section to find orthogonal trajectories to a given family of curves. Once you have found the orthogonal trajectories, you can sketch the original family and the orthogonal trajectories to obtain a geometric understanding of the relation between them.

SkillMaster 7.10: Solve separable equations as they arise in applications, and use the solution to answer questions about the application.

Once you have modeled a physical situation by a separable differential equation, you should be able to solve it using the technique mentioned above. You then need to use your solution to answer questions about the application.

Worked Examples

For each of the following examples, first try to find the solution without looking at the hint. Cover the right column with a piece of paper. If you need a hint, uncover the column. If you need to see the worked solution, look at the solutions section that follows the examples and hints.

	Example	Hint
SkillMaster 7.8.		
1.	Solve the differential equation $$y' = y(x + \sin x).$$	This is a separable differential equation. Replace y' by dy/dx and algebraically arrange the equation so the left hand side only involves y and the right hand side only involves x.

2. Solve the initial-value problem

$$y' = \frac{e^x}{2y} \qquad y(0) = 1$$

This is also a separable equation. After separating and integrating, use the initial value to find the specific solution.

3. Find an equation of the curve that passes through the point $(2,-1/3)$ and whose slope at (x,y) is y^2x.

This is the same as finding the solution to the initial-value problem
$y' = y^2x \qquad y(2) = -1/3$

SkillMaster 7.9.

4. Consider the family of hyperbolas $xy = C$ for various constants C. Find the family of orthogonal curves to this family.

First differentiate $xy = C$ implicitly. Solve this to get an expression of the form
$y' = F(x,y)$.
The orthogonal trajectories are the solutions to
$y' = -1/F(x,y)$.

SkillMaster 7.10.

5. A pond with volume 9000 m^3 is nearly pollution free. An accident causes 1 kg of a certain pollutant to leak into the pond each day. The leak occurs continuously at a constant rate. The pond drains at a rate of 100 m^3/day and is replaced with fresh water at the same rate. How much of the pollutant is in the pond after 100 days? Does the amount reach a limiting value if the situation is not corrected?

Make a differential equation that models this. Let $y(t)$ be the amount of pollutant in the pond after t days. Find an expression for
y' = (rate pollutant is entering) - (rate pollutant is draining).

Solutions to worked examples

1. $\dfrac{dy}{dx} = y(x+\sin x) \qquad \dfrac{1}{y}\,dy = (x+\sin x)dx$

$\displaystyle\int \frac{1}{y}\,dy = \int (x+\sin x)dx \qquad \ln|y| = x^2/2 - \cos x + C\,|y| = e^{(x^2/2-\cos x+C)}$

$= e^C e^{((x^2/2)-\cos x)} y = \pm e^C e^{((x^2/2)-\cos x)} = Ae^{((x^2/2)-\cos x)}$

2. $\dfrac{dy}{dx} = \dfrac{e^x}{2y} \qquad 2y\,dy = e^x dx$

$\displaystyle\int 2y\,dy = \int e^x dx \qquad y^2 = e^x + C$

$y = \pm\sqrt{e^x + C} \qquad y(0) = -1$

$-1 = \pm\sqrt{e^0 + C} \qquad 1 = \sqrt{1+C} \qquad 1 = 1 + C$

$C = 0 \qquad y(x) = -\sqrt{e^x} = -e^{x/2}$

3. The initial-value problem is separable.

$y' = y^2 x \qquad dy/dx = y^2 x \qquad \dfrac{1}{y^2}dy = x\,dx$

$\displaystyle\int \frac{1}{y^2}dy = \int x\,dx \qquad -\frac{1}{y} = \frac{x^2}{2} + C$

$y = \dfrac{-1}{x^2/2 + C} \qquad y = \dfrac{-2}{x^2 - 2C}$

$y(2) = -1/3 \qquad -1/3 = \dfrac{-2}{(2)^2 - 2C}$

$1 = \dfrac{6}{4-2C} \qquad 4 - 2c = 6$

$-2C = 2 \qquad C = -1 \qquad y(x) = \dfrac{-2}{x^2+2}$

4. $xy = C \qquad \dfrac{d}{dx}(xy) = \dfrac{d}{dx}(C) \qquad \dfrac{d}{dx}(xy) = 0$

$xy' + y = 0 \qquad y' = -y/x$

The above is done with implicit differentiation.

With a little more work you can solve for y then differentiate.

$y = C/x \qquad y' = -C/x^2 = (-1/x)(C/x) = (-1/x)y = -y/x$

The orthogonal trajectories are the solutions to the equation obtained by setting the derivative equal to the negative reciprocal of the derivative obtained above.

$$y' = x/y \qquad \frac{dy}{dx} = x/y \qquad y\,dy = x\,dx$$

$$\int y\,dy = \int x\,dx \qquad y^2/2 = x^2/2 + C$$

$$y^2 - x^2 = 2C \qquad y^2 - x^2 = K$$

The orthogonal family is another family of hyperbolas.

5. Again, $y(t)$ is the amount of pollutant in the pond at time t. The pollutant enters the lake at rate 1 kg of pollutant per day. The concentration of pollutant in the pond after t days is $y(t)/9000$. The pond is drained at a rate of 100 m^3/day so the pollutant drains at a rate equal to $(y(t))/9000)(100) = (y(t))/90$.

So $y' = 1 - y/90y(0) = 0$. This is a separable equation.

$$90\frac{dy}{dt} = 90 - y \qquad \frac{1}{90-y}dy = \frac{1}{90}dt$$

$$-\ln|90-y| = \frac{t}{90} + C \qquad y(0) = 0,\ \text{ so } 90 - y(0) = 90 > 0$$

and obtain $-\ln(90) = C$.

Substituting this in the previous equation, you get

$$90 - y = e^{-t/90+\ln(90)} \qquad y = 90 - e^{-t/90+\ln(90)} = 90 - 90e^{-t/90}$$

After 100 days the amount of pollutant in the pond is

$$y(100) = 90 - 90e^{-100/90} \approx 60.3726 \text{ kg}$$

The limiting value of the amount of pollutant in the pond is

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty}(90 - 90e^{-t/90}) = 90 \text{ kg}$$

7.4 Exponential Growth and Decay

Key Concepts:

- The law of natural growth or decay
- Applications including relative growth rate, half-life and interest

Skills to Master:

- Solve application problems in which the model involves the law of natural growth or decay.

Discussion:

Section 7.4 returns to the discussion of growth and decay. Differential equations of the form

$$\frac{dy}{dt} = ky, \qquad y(0) = y_0$$

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were discussed earlier in *Section 7.1* . Now the general solution is obtained and applications are given to population growth, radioactive decay and continuously compounded interest.

Key Concept: The law of natural growth or decay

The differential equation

$$\frac{dy}{dt} = ky, \qquad y(0) = y_0$$

is sometimes called the *law of natural growth* if $k > 0$, or the Law of Natural Decay if $k < 0$. The solution of this initial value problem is given by

$$y(t) = y_0 e^{kt}$$

Key Concept: Applications including relative growth rate, half-life and interest

Population growth, radioactive decay and continuously compounded interest are examples of natural growth or decay. Make sure that you understand the applications in this section and that you understand how to apply the solution given above to these various problems.

SkillMaster 7.11: Solve application problems in which the model involves the law of natural growth or decay.

To solve problems modeled by the law of natural growth or decay, apply the general solution given above to the specific problem. For population growth given by

$$\frac{dP}{dt} = kP, \qquad P(0) = P_0$$

the solution is given by

$$P(t) = P_0 e^{kt}$$

If the population is known at another time, the value of k can be determined by using the given values for P and t and solving for k. Exponential growth of a population may begin at a small value and appear to change very slowly but eventually it will become large and grow quickly. For example, suppose you are given $1/1000$ of a penny on January 1, and each day in January the amount you are given doubles from the day before. It will take about 10 days to accumulate one penny but by January 31 you will be a millionaire!

For radioactive decay given by

$$\frac{dm}{dt} = km, \qquad m(0) = m_0$$

the solution is given by

$$m(t) = m_0 e^{kt}$$

If the half life of the substance is known then it can be used to determine the value of k by setting $t =$ the half life, $P = \frac{1}{2}P_0$ and solving for k.

To understand continuously compounded interest, study *Example 5* and see how the limit formula for e is needed. The bottom line is that the amount of principal in given by

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$$A(t) = A_0 e^{rt}$$

where r is the interest rate.

Worked Examples

For each of the following examples, first try to find the solution without looking at the hint. Cover the right column with a piece of paper. If you need a hint, uncover the column. If you need to see the worked solution, look at the solutions section that follows the examples and hints.

Example	Hint
SkillMaster 7.11.	
1. Suppose that the population of a country in 1960 was $1,616,213$ and its population in 2000 was $2,902,487$. If the population is governed by an exponential growth model, find its population in the year 2010 and in the year 2510.	An exponential growth model has the form $P(t) = P_0 e^{rt}$. Let t be the number of years after 1960 (so $t = 0$ corresponds to the year 1960). Substitute the values for $P(0)$ and $P(40)$ in the equation and solve for P_0 and r.
2. A population of amoebae doubles every 51 minutes. The population begins with 250 amoebae. How many amoebae will there be after one day? How long will it take for the population to reach 100,000?	Assume an exponential growth model, $P(t) = P_0 e^{rt}$. The population doubles after 51 minutes. Use this relationship to find the growth rate and thus specify the model.
3. The half-life of strontium-90 is 28 years. Suppose there is 150 grams of strontium-90 in a container. Find a formula for the amount left after t years. When is there only 10 grams remaining?	This is an exponential decay model which has the form $P(t) = P_0 e^{kt}$. The given information states: $P(0) = 150$ and $P(28) = 150/2 = 75$. Use this to solve for k and specify the model.

4. Suppose that \$10,000 is borrowed for 3 years at 9% interest. How much will be owed at the end of 3 years if the interest is compounded (i) annually, (ii) monthly, (iii) daily, (iv) continuously?

If the interest is compounded n times per year then you can solve for the total amount owed when the loan is due.

$$10,000 \cdot (1 \ +0.09/n)^{3n}$$

If the interest is compounded continuously then the amount owed is the limit as $n \to \infty$ which is

$$10,000e^{(0.09)3}$$

Solutions to worked examples

1. $P(t) = P_0 e^{rt}$ $\qquad P(0) = 1,616,213 = P_0$ $\qquad P(40) = 2,902,487$

$1,616,213e^{40r} = 2,902,487$ $\qquad e^{40r} \approx 1.7959$

$40r \approx \ln(1.7959)$ $\qquad r \approx (1/40)\ln(1.7959) \approx 0.01464$

So $P(t) \approx 1,616,213e^{0.01464t}$ $\qquad P(50) \approx 1,616,213e^{0.01464(50)} \approx 3,360,487$

$P(550) \approx 1,616,213e^{0.0464(550)} \approx 5,075,020,069$

In the year 2010 the model predicts about $3,360,500$ people. In the year 2510 the model predicts more than 5 billion people!

2. $2P_0 = P_0 e^{51r}$ $\qquad 2 = e^{51r}$ $\qquad \ln(2) = 51r$

$r = (1/51)\ln(2) \approx 0.0136$

Since the initial population is $P_0 = 250$, the model is: $P(t) = 250e^{0.0136t}$

A day has $24 \times 60 = 1440$ minutes. So after one day the population size is

$P(1440) = 80,013,476,835$. A parenthetical note about this answer is important to mention. Since the model is of a biological population which is variable, you would not expect this number to be exact. It would not be surprising if a census arrived at 79,013,470,835 members in the population (this number would not be exact either). At the present time, it is impossible to count such a large population exactly.

To find when the population first reaches the 100,000 level solve the following equation.

$250e^{0.0136t} = 100,000$ $\qquad e^{0.0136t} = 400$ $\qquad 0.0136t = \ln(400)$

$t = \dfrac{\ln(400)}{0.0136} \approx 440.55$ min or about 7.3 hours

3. $150e^{28k} = 75 \qquad e^{28k} = 0.5 \qquad 28k = \ln(0.5)$

$k = \ln(0.5)/28 \approx -0.0248 \qquad P(t) = 150e^{-0.0248t}$

To find when there are only 10 grams left solve the equation

$150e^{-0.0248t} = 10 \qquad e^{-0.0248t} = 10/150 \qquad -0.0248t = \ln(1/15)$

$t = -\ln(1/15)/0.0248 = \ln(15)/0.0248 \approx 109$ years

4. Annual compounding corresponds to $n = 1$, monthly to $n = 12$, and daily to $n = 365$. The amount owed in each case is given below.

Annual compounding: $10,000(1+0.09/1)^3 = \$12,950.29$

Monthly compounding: $10,000(1+0.09/12)^{(3)12} = \$13,086.45$

Daily compounding: $10,000(1+0.09/365)^{(3)365} = \$13,099.21$

Continuous compounding: $10,000e^{(0.09)3} = \$13,099.64$

Notice there is only 43 cents difference between daily compounding and continuous compounding.

7.5 The Logistic Equation

Key Concepts:

- The logistic differential equation
- Approximate solutions and the analytic solution
- Comparing growth models

Skills to Master:

- Use the logistic model to solve problems involving population growth and related applications.

Discussion:

Section 7.5 revisits the logistic differential equation

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{K}\right) \qquad P(0) = P_0$$

discussed earlier in *Section 7.1* . The general solution is analyzed using the geometric technique of direction fields, the numerical technique of Euler's Method and the symbolic technique of solving separable differential equations.

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Key Concept: The logistic differential equation

The logistic differential equation is the differential equation

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{K}\right)$$

This equation arises from modeling population growth.

Key Concept: Approximate solutions and the analytic solution

Approximate solutions can be obtained using direction fields or Euler's Method. The Analytic Solution, given the initial value $P(0) = P_0$, is

$$P(t) = \frac{K}{1 + Ae^{-kt}} \text{ where } A = \frac{K - P_0}{P_0}$$

Make sure that you understand how separation of variables makes the *derivation of this solution* possible.

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Key Concept: Comparing growth models

You now have two models population growth, that of exponential growth and that given by the logistic model. Exponential growth is a good approximate model as long as the population is small enough that there is no competition for resources. It makes sense that if there is only a fixed amount of food available then there will be limits forced on the population by the environment. Consider again the logistic differential equation:

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{K}\right) \qquad P(0) = P_0$$

K is the carrying capacity, the limit that the resources put on the population. If P is very small compared with K then the second factor $(1 - P/K)$ is very close to 1 and makes very little difference in the equation. The equation is thus very close to $dP/dt = kP$, which is exponential growth, representing a population size with plenty of resources. On the other hand, as the population size P grows larger, and then very close to K, the second factor becomes very close to 0. This implies that dP/dt is also close to 0, so the population, P, does not change very much and is nearly at equilibrium. This second factor is used in models to account for competition between population members for resources. These models can be compared with actual physical situations to see which model fits best. To see this done, read the details of *Example 4* .

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SkillMaster 7.12: Use the logistic model to solve problems involving population growth and related applications.

You can use the exact solution to the logistic differential equation to solve problems involving population growth. You need to determine the carrying capacity K. To find the constant k, you need to know the population at some time in addition to the value at time 0.

Worked Examples

For each of the following examples, first try to find the solution without looking at the hint. Cover the right column with a piece of paper. If you need a hint, uncover the column. If you need to see the worked solution, look at the solutions section that follows the examples and hints.

Example	Hint

SkillMaster 7.12.

1. Suppose that a logistic population model satisfies the following differential equation.

$$\frac{dP}{dt} = 0.02P(1-0.00004P)$$

What is the carrying capacity and the exponential growth rate for very small values of P?
What is the population at time $t=10$ given that $P(0) = 1,000$?

Hint: The logistic growth model is $P(t) = \dfrac{K}{1+Ae^{-kt}}$ where

$$\frac{dP}{dt} = kP(1-P/K).$$

Compare this with the given equation to determine K and k. To find A use the given initial condition

2. Suppose that the population of a country in 1960 was $1,616,213$ and its population in 2000 was $2,902,487$. Suppose that the carrying capacity of the country is 4.5 million people and that the population is described by a logistic growth model. Find its population in the year 2010 and the year 2510.

Hint: The logistic model has the form $P(t) = \dfrac{K}{1+Ae^{-kt}}$.

Use the population level in 2000 to solve for k. This will specify the parameters of the model and allow you to answer the questions in the example. You are given the carrying capacity K.
Find P_0 and use that to solve for A.

3. Suppose that a population satisfies the logistic growth model

$$P'(t) = kP(1-P/K).$$

What population level has a growth rate which is a maximum? This is an interesting question from the point of view of resource management. If the population is harvested to maintain the level of maximum growth rate then the amount harvested will also be maximized.

The growth rate of the population is $dP/dt = P' = kP(1-P/K)$. The question reduces to the easier question of finding a P for which $kP(1-P/K)$ is maximal.

Solutions to worked examples

1. The carrying capacity is $K = 25,000$ and the exponential growth rate when P is very small is $k = 0.02$.

$$P(t) = \frac{25,000}{1+Ae^{-0.02t}} \qquad P(0) = 1,000$$

$$1,000 = \frac{25,000}{1+Ae^{-0.02(0)}} \qquad 1+A = 25 \qquad A = 24$$

$$P(t) = \frac{25,000}{1+24e^{-0.02t}} \qquad P(10) = \frac{25,000}{1+24e^{-0.02(10)}} \approx \frac{25,000}{20.6495} \approx 1211$$

2. $1.616 = 4.5/(1+A) \qquad (1+A) = 4.5/1.616 \qquad 1+A \approx 2.785 A \approx 1.785$

$$P(t) = \frac{4.5}{1+1.785e^{-kt}} \qquad P(40) \approx 2.902 \qquad 2.902 = \frac{4.5}{1+1.785e^{-k(40)}}$$

$$1+1.785e^{-k(40)} = 4.5/2.902 \qquad 1+1.785e^{-40k} \approx 1.551$$

$$1.785e^{-40k} \approx 0.551 \qquad e^{-40k} \approx 0.551/1.785$$

$$e^{-40k} \approx 0.309 \qquad -40k \approx \ln(0.309) \qquad k \approx -\ln(0.309)/40$$

$$k \approx 0.0294$$

The model for this population is

$$P(t) = \frac{4.5}{1+1.785e^{-0.0295t}}.$$

In the year 2010 $t = 50$ and in the year 2510 $t = 550$. The populations for those years is

$$P(50) = \frac{4.5}{1+1.785e^{-0.0295(50)}} \approx 3.191 \text{ million}$$

$$P(550) = \frac{4.5}{1 + 1.785e^{-0.0295(550)}} \approx 4.500 \text{ million}$$

3. The growth rate is

$kP(1 - P/K) = kP - (k/K)P^2$

To find the maximum take the derivative with respect to P and set it equal to 0.

$k - 2(k/K)P = 0 \qquad (2k/K)P = k \qquad P = K/2$

This is a maximum because the second derivative is $-2k/K < 0$.

This means that to maximize yield of a renewable resource that satisfies the logistic equation, one should harvest to one half the carrying capacity.

7.6 Predator-Prey Systems

Key Concepts:

- Population models involving predators

Skills to Master:

- Use the predator-prey model to solve problems involving population growth.

Discussion:

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Section 7.6 adds a different factor to the population growth models of the previous sections. The *Logistic Equation* resulted from modeling a population with a maximum possible value, the carrying capacity. This section models two interrelated populations, that of a predator and that of a prey. Read carefully through *Example One* in this section to get a better understanding of this.

Key Concept: Population models involving predators

A predator-prey system involves two interrelated populations: the predator population W, and the prey population R. It is assumed that the populations change according to the differential equations:

$$\frac{dR}{dt} = kR - aRW \qquad\qquad \frac{dW}{dt} = -rW + bRW$$

The first equation has the population R increasing at a rate proportional to R, and decreasing at a rate proportional to the product RW. The second equation has the population W decreasing at a rate proportional to W, and increasing at a rate proportional to the product RW. It is usually impossible to solve these equations explicitly, but graphical methods can be used to get approximate solutions.

SkillMaster 7.13: Use the predator-prey model to solve problems involving population growth.

Setting both derivatives in the model equal to 0 gives the constant solution to this situation. This is the solution where the decrease in R due to predation is just balanced by the population growth of R, and the population decrease in W is just balanced by the population increase due to predation.

To graphically model other solutions, solve for $\frac{dW}{dR}$ ans in *Example One* and draw a direction field for the resulting differential equation.

Worked Examples

For each of the following examples, first try to find the solution without looking at the hint. Cover the right column with a piece of paper. If you need a hint, uncover the column. If you need to see the worked solution, look at the solutions section that follows the examples and hints.

Example	Hint
SkillMaster 7.13.	
1. The system of differential equations below models a predator prey system. Find the constant solution. $$\frac{dR}{dt} = 0.2R - 0.004RW \quad \frac{dW}{dt} = -.06W + .0002RW$$	Set the differential equations to 0 and solve.
2. In the previous problem, solve for $\dfrac{dW}{dR}$ and draw the direction field for the resulting differential equation.	Divide $\frac{dW}{dt}$ by $\frac{dR}{dt}$.

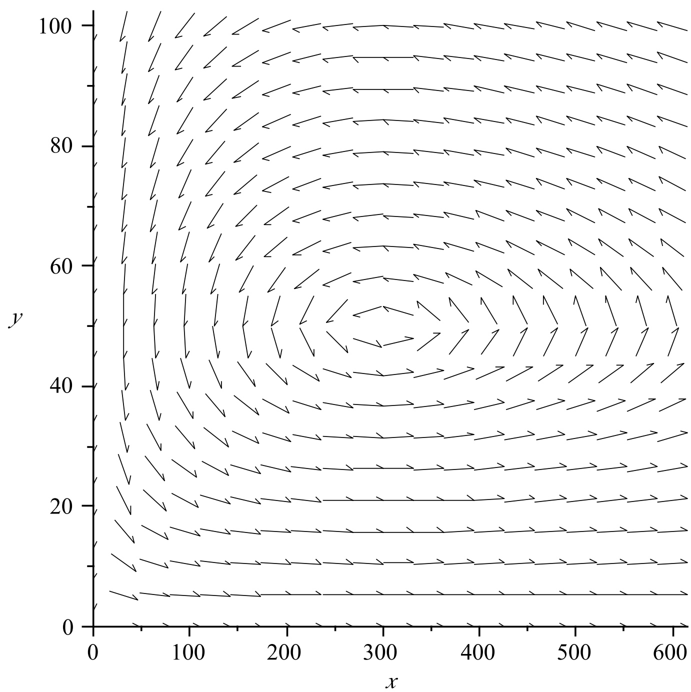
Solutions to worked examples

1. $0.2R - 0.004RW = 0 \qquad -0.06W + .0002RW = 0$

$0.2 - 0.004W = 0 \qquad -0.06 + 0.002R = 0$

$W = \dfrac{0.2}{0.004} = 50 \qquad R = \dfrac{0.06}{0.0002} = 300$

2. $\dfrac{dW}{dR} = \dfrac{-.06W + .0002RW}{0.2R - 0.004RW}$



Note that you can see the constant solution in the middle of this plot.

SkillMasters for Chapter 7

SkillMaster 7.1:	Determine whether or not specific functions provide a solution to a given differential equation.
SkillMaster 7.2:	Given a general solution and initial value, determine a specific solution.
SkillMaster 7.3:	Analyze properties of the solution by analyzing the differential equation.
SkillMaster 7.4:	Sketch direction fields and use them to find solutions satisfying initial conditions.
SkillMaster 7.5:	Use direction fields to analyze limiting values in applications.
SkillMaster 7.6:	Use Euler's method with specified step sizes to approximate specific y values for solutions to initial value problems
SkillMaster 7.7:	Compare values obtained from Euler's Method with specific step sizes and true function value, if known.
SkillMaster 7.8:	Recognize and solve separable equations by using integration.
SkillMaster 7.9:	Sketch orthogonal trajectories of a given family of curves.
SkillMaster 7.10:	Solve separable equations as they arise in applications, and use the solution to answer questions about the application.
SkillMaster 7.11:	Solve application problems in which the model involves the law of natural growth or decay.
SkillMaster 7.12:	Use the logistic model to solve problems involving population growth and related applications.
SkillMaster 7.13:	Use the predator-prey model to solve problems involving population growth.

Chapter 8

Infinite Sequences and Series

$$\sum_{n=1}^{\infty} \mathbf{a} \cdot \mathbf{r}^{\,\mathbf{n}-1} = \frac{\mathbf{a}}{1 - \mathbf{r}}$$

8.1 Sequences

Key Concepts:

- The limit of a sequence
- Limit Laws
- The Monotonic Sequence Theorem

Skills to Master:

- Find a defining equation for a sequence.
- Use the laws of limits together with known examples to determine if a sequence is divergent or convergent, and if convergent, to find the limit.
- Determine if a sequence is monotonic and use the Monotonic Convergence Theorem to show some sequences are convergent.

Discussion:

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Section 8.1 introduces the concepts of sequences and limits of sequences. You should review the material on the *limit of a function* from earlier in the text. Sequences arise in many situations. For example, if you let P_n be the number bacteria in a culture dish n hours after the start of an experiment, then the numbers $\{P_1, P_2, P_3, \ldots\}$ form a sequence; your monthly telephone bill is a sequence, as is the yearly Gross National Product and the number of microseconds between the times that water drops break off your kitchen faucet and fall. (It may seem strange but some important experiments in Chaos Theory observed exactly this sequence and found unusual patterns that had already been observed in the biological sciences.)

Key Concept: The limit of a sequence

A sequence $\{a_n\}$ has limit L , or *converges* to L, if you can make the terms a_n as close to L as you like by taking n sufficiently large. If the sequence $\{a_n\}$ has limit L you write

$$\lim_{n\to\infty} a_n = L \qquad \text{or} \qquad a_n \to L, \text{ as } n \to \infty.$$

The sequence *diverges* if it does not have a limit. The meaning of diverging to ∞ or to $-\infty$ is similar to the corresponding concepts for functions. Pay careful attention to *Theorem 2* in this section relating the limit of a sequence to the limit of a function. Note that the values of n are increasing positive integers. The point is that a sequence assigns a value a_n to each integer n and so may be thought of as a function from the positive integers to the real numbers.

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Key Concept: Limit Laws

The limit laws in this section should look familiar to you. Compare them with the limit laws for functions in Section 2.3 . Just as before,

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$$\text{The limit of a} \left\{ \begin{array}{l} \text{sum} \\ \text{difference} \\ \text{constant multiple} \\ \text{product} \\ \text{quotient} \end{array} \right\} \text{is the} \left\{ \begin{array}{l} \text{sum} \\ \text{difference} \\ \text{constant multiple} \\ \text{product} \\ \text{quotient} \end{array} \right\} \text{of the limit(s).}$$

Again, for these laws to hold, the individual limits must exist. For the last law to hold, the limit of the denominator can not be 0. There is also a Squeeze Theorem for sequences. Another important result is Theorem 4:

$$\text{If } \lim_{n\to\infty} |a_n| = 0 \text{ then } \lim_{n\to\infty} a_n = 0\ .$$

Key Concept: The Monotonic Sequence Theorem

You should make sure that you understand the meaning of the terms relating to the Monotonic Sequence Theorem. The terms increasing, decreasing, and monotonic have essentially the same meaning for sequences that they have for functions. The meaning of bounded above, bounded below and bounded as they apply to sequences should be clear. The Monotonic Sequence theorem states that every bounded monotonic (either increasing or decreasing) sequence converges.

SkillMaster 8.1: Find a defining equation for a sequence.

To find a defining equation for a sequence, examine the terms that you are given to see if you can find a pattern. If the terms involve fractions, try to find separate patterns for the numerator and for the denominator. If the terms alternately are positive and negative, include a factor of $(-1)^n$ or $(-1)^{n+1}$ in your equation.

SkillMaster 8.2: Use the laws of limits together with known examples to determine if a sequence is divergent or convergent, and if convergent, to find the limit.

If $-1 < r < 1$, $a_n = r^n$ converges to 0; if $r = 1$, r^n converges to 1; if $r > 1$ then r^n diverges to ∞; otherwise, $r \leq -1$ and r^n diverges because the sequence oscillates (that is it jumps back and forth without converging). These basic sequences, together with the laws of limits, should allow you to compute many limits.

SkillMaster 8.3: Determine if a sequence is monotone and use the Monotone Convergence Theorem to show some sequences are convergent.

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Make sure that you understand the techniques used in *Examples 9, 10* and *11* in this section. They show you how to determine if a sequence is monotonic and bounded. Review the technique of *mathematical induction* if you need to.

Worked Examples

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For each of the following examples, first try to find the solution without looking at the hint. Cover the right column with a piece of paper. If you need a hint, uncover the column. If you need to see the worked solution, look at the solutions section that follows the examples and hints.

SkillMaster 8.1.

Example	Hint
1. Find a formula for the general term a_n of the sequence below, assuming the pattern of the first few terms continues. $$\left\{\frac{1}{1}, \frac{4}{2}, \frac{9}{4}, \frac{16}{8}, \ldots\right\}$$	Look for a pattern for the numerators and a separate pattern for the denominators.
2. Write out the first few terms of the sequence $\{a_n\}$ where $$a_n = \frac{(-1)^n \ln(n^2+1)}{3^n}$$	Substitute 1 for n in the definition of a_n to find the first term, then substitute 2 for n to find the second term, and so on.

SkillMaster 8.2.

3. Determine if the sequence diverges or converges. If the sequence converges, find the limit.

$$a_n = \ln(\sqrt{n}+1)$$

Recall properties of the logarithm function.

4. Determine if the sequence diverges or converges. If the sequence converges, find the limit.

$$a_n = \frac{(-1)^n n^2}{n^3+1}$$

Divide the numerator and denominator by n^3 and apply Theorem 4.

5. Determine if the sequence diverges or converges. If the sequence converges, find the limit.

$$a_n = \frac{\ln(e^n+1)}{n}$$

The answer is not obvious. Both the numerator and denominator diverge to ∞. This suggests we consider the limit of the function below and use l'Hospital's Rule.

$$f(x) = \frac{\ln(e^x+1)}{x}$$

SkillMaster 8.3.

6. Determine whether the sequence is monotonic.

$$a_n = \frac{2n}{n+1}$$

Check the first few terms to make a guess about whether it is increasing or decreasing. Then check whether this is true or not by comparing a_{n+1} and a_n.

7. Show the following sequence is increasing and bounded by 2. Conclude that the sequence converges and find its limit.

$$a_1 = \sqrt[3]{6} \qquad a_{n+1} = \sqrt[3]{6+a_n}$$

Calculate the first four terms. Notice that they are increasing, less than 2 and appear to be converging to 2. Use induction and $a_{n+1}^3 = 6+a_n$ to show the sequence is bounded by 2. Show the sequence is increasing by induction.

Solutions to worked examples

1. $a_n = \dfrac{n^2}{2^{n-1}}$

2. $\left\{\dfrac{-\ln(2)}{3}, \dfrac{\ln(5)}{9}, \dfrac{-\ln(10)}{27}, \dfrac{\ln(17)}{81}, \ldots\right\}$

3. The sequence is unbounded and diverges to ∞.

4. $a_n = \dfrac{(-1)^n n^2}{n^3+1} \qquad |a_n| = \dfrac{n^2}{n^3+1} = \dfrac{1/n}{1+1/n^3}$

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{1/n}{1+1/n^3} = \frac{0}{1+0} = 0$$

$\lim\limits_{n\to\infty} |a_n| = 0$, so $\lim\limits_{n\to\infty} a_n = 0$.

5. $$\lim_{x\to\infty} \frac{\ln(e^x+1)}{x} = \lim_{x\to\infty} \frac{\dfrac{d}{dx}(\ln(e^x+1))}{\dfrac{d}{dx}(x)} = \lim_{x\to\infty} \frac{e^x/(e^x+1)}{1}$$

$$= \lim_{x\to\infty} \frac{e^x}{(e^x+1)(1)} = \lim_{x\to\infty} \frac{1}{1+e^{-x}} = \frac{1}{1+0} = 1$$

Thus $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \dfrac{\ln(e^n+1)}{n} = 1$

6. To show the sequence is increasing we must show

$$a_{n+1} > a_n$$

$$\frac{2(n+1)}{(n+1)+1} > \frac{2n}{n+1} \iff \quad \frac{2n+2}{n+2} > \frac{2n}{n+1}$$
$$\iff \quad (2n+2)(n+1) > (2n)(n+2) \iff \quad 2n^2+4n+2 > 2n^2+4n$$
$$\iff \quad 2 > 0.$$

The last inequality is true so the preceding inequalities must also be true and the sequence is increasing, and hence monotonic.

7. Using the hint, show that the sequence is bounded by 2 using induction. Your calculation has shown that $a_1 = \sqrt{36} \approx 1.8171 < 2$. By induction assume that $a_n < 2$. Using this fact consider $a_{n+1} = \sqrt{36} + a_n$. Cube both sides, $a_{n+1}^3 = 6 + a_n < 6 + 2 = 8$. Taking cube roots again, $a_n < \sqrt{38} = 2$. Similarly, use induction to show the sequence is increasing. Checking the first two elements of the sequence, $a_1 \approx 1.8171 < a_2 \approx 1.9846$. (You have probably noticed the terms of the sequence seem to be increasing toward the value 2 quite quickly.) Suppose $a_{n-1} < a_n$. Using this fact, show that $a_n < a_{n+1}$. Using the formula that defines the sequence, this is equivalent to showing $\sqrt{36} + a_{n-1} < \sqrt{36} + a_n$. Cubing both sides and subtracting 6 from both sides reduces things to the assumption $a_{n-1} < a_n$. The sequence is bounded and monotone so must have a limit L.

$$L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt[3]{6+a_n} = \sqrt[3]{6+L}$$

So $L = \sqrt[3]{6+L} \qquad L^3 = 6+L \qquad 0 = L^3 - L - 6.$

Inspection shows that $L = 2$ is a real root of this equation. The derivative of $L^3 - L - 6$ is positive for $L > 1$ so $L^3 - L - 6$ can't have more than one root ≥ 1. Recall the limit is trapped between 1 and 3 because the terms of the sequence are between 1 and 3.

8.2 Series

Key Concepts:

- The sum of a series
- Geometric series, telescoping series, and the harmonic series

Skills to Master:

- Determine if a geometric series is convergent, and if convergent, find the sum of the series.
- Use the laws of series together with known examples to determine if a series is divergent or convergent, and if convergent, find the sum.

Discussion:

Section 8.2 introduces infinite series. You can think of a series as an infinite sum, that is the sum of all of the terms of a sequence. To give a precise meaning to the sum of a series, partial sums are introduced. The value of a series is the limit of the partial sums if the limit exists.

Key Concept: The sum of a series

An infinite series is an expression of the form

$$a_1 + a_2 + a_3 + \cdots = \sum_{n=1}^{\infty} a_n$$

The nth partial sum of a series is the finite sum

$$s_n = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{i=1}^{n} a_i$$

If the sequence of partial sums $\{s_n\}$ of a series converges to a number s we say that the series converges and write

$$\sum_{n=1}^{\infty} a_n = s.$$

Make sure that you don't confuse the sequence of partial sums of a series with the sequence of terms in the series. For example, the sequence $a_n = 1/2^n$ converges to 0 but the sum of the series $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges to 1. Note also that the Monotone Sequence Theorem from the previous section guarantees that if the terms of the series are positive them either the series converges or else it diverges to ∞.

Key Concept: Geometric series, telescoping series, and the harmonic series

A geometric series is a series of the form

$$\sum_{n=1}^{\infty} ar^{n-1} \quad a \neq 0.$$

Note that each term in the series is obtained from the previous one by multiplying by r (this gives a test to determine if a series is geometric). Such a series is convergent if $|r| < 1$ in which case it converges to

$$\frac{a}{1-r}.$$

If $|r| \geq 1$, the geometric series diverges.

Read through *Example 1.* in the text to see how the limits of geometric series are computed. Geometric series may be the single most useful series, because they serve as a comparison and as a model for many other kinds of series. page 577

The harmonic series is given by the following simple formula:

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

The harmonic series diverges to infinity. The partial sum of 10 terms is almost 3; the first 100 terms sums to about 5.2; and the first 1,000 terms sums to about 7.4934. This series diverges to infinity but it takes a very long time to get large. When observing this on a calculator, the bigger the viewing rectangle of the plot of the partial sums, the more the series looks convergent. In this sense, it is very similar to the natural logarithm function.

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The reasoning given by *Example 7* in the text explains why the series diverges to ∞ so slowly.

This series is called harmonic because its partial sums are important to music in the western tradition (the mathematics was first described by Pythagoras, whose formula relating the sides of a right triangle is also well-known). For example, the small differences produced by these partial sums make the key of A have a different flavor from the key of C.

The third type of series, telescoping series, are series that exhibit self-cancellation, because the second part of each term is canceled by the first part of the next. These series may be expressed in the form:

$$\sum_{n=1}^{\infty}(c_n - c_{n+1})$$

The partial sums are

$$\sum_{i=1}^{n}(c_i - c_{i+1}) = c_1 - c_{n+1}$$

which converge if and only the sequence c_n converges to zero. In this case, the limit is c_1. When using telescoping series you have to be careful that c_n truly does converge to 0, otherwise you might appear to prove some very surprising facts. See Exercise 18, page 576 for an example.

Key Concept: Test for Divergence and the laws for series

The Test for Divergence (Theorem 7) states that

$$\text{if } \lim_{n\to\infty} a_n \text{ does not exist, or if } \lim_{n\to\infty} a_n \neq 0, \text{ then } \sum_{n=1}^{\infty} a_n \text{ is divergent.}$$

The laws for series are given in *Theorem 8*.

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SkillMaster 8.4: Determine if a geometric series is convergent, and if convergent, find the sum of the series.

To determine whether a given series is geometric, divide the $(n+1)^{st}$ term by the n^{th} and see if you get a constant r, no matter what n is. If you always get the same constant, the series is geometric and r is the ratio associated with the series. You may need to do some algebraic manipulation to get the series in the form

$$\sum_{n=1}^{\infty} ar^{n-1}.$$

Once you have the series in this form, the limit is

$$\frac{a}{1-r}$$

if $|r| < 1$, and the series diverges if $|r| \geq 1$.

SkillMaster 8.5: Use the laws for series together with known examples to determine if a series is divergent or convergent, and if convergent, find the sum.

Taking the ratio of successive terms allows you to determine if a series is geometric and then you can determine convergence or divergence. If the terms of a series do not approach 0, then the series diverges. If the terms of a series can be manipulated algebraically, then an underlying telescoping or other structure may be found. These facts, together with the laws for series allow you to determine convergence or divergence of other kinds of series. More techniques for determining convergence or divergence will be developed in the next few sections.

Worked Examples

For each of the following examples, first try to find the solution without looking at the hint. Cover the right column with a piece of paper. If you need a hint, uncover the column. If you need to see the worked solution, look at the solutions section that follows the examples and hints.

Example	Hint
SkillMaster 8.4.	
1. Determine if the following geometric series converges and if so find the sum. $$\sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n}}{3^n}$$	Here $a_n = (-1)^n 2^{2n}/3^n$. Rewrite 2^{2n} as 4^n.

2. Determine if the following geometric series converges and if so, find the sum.

$$\sum_{n=1}^{\infty} \frac{2^{n+2}}{\pi^{n-2}}$$

First factor the terms so that the general term appears with a power of $n-1$.

3. Express the number

$$0.135135\ldots$$

as a ratio of integers.

Express the number as a geometric series and determine its sum.
$0.135135\ldots = \sum_{n=1}^{\infty} \frac{135}{(1000)^n}$

SkillMaster 8.5.

4. Determine whether the series is convergent or divergent. If it is convergent, find its sum.

$$\sum_{n=1}^{\infty} [4(0.2)^{n+1} - (0.1)^n]$$

Use the laws for series to express this series as the difference of two geometric series.

5. Determine whether the series is convergent or divergent. If it is convergent, find its sum.

$$\sum_{n=1}^{\infty} \frac{2n}{n+1}$$

Check to see whether or not $a_n = (2n)/(n+1)$ converges to 0; if this limit is not 0 then the series diverges.

6. Determine whether the series is convergent or divergent. If it is convergent find its sum.

$$\sum_{n=1}^{\infty} [\tan(1/n) - \tan(1/(n+1))]$$

Notice that this series has a "telescoping" appearance as in Example 6 page 571 of the text. Use the definition of the sum of a series to determine if the series has a sum.

7. Determine if the series is convergent and, if so find the sum.

$$\sum_{n=1}^{\infty} \frac{1}{n^2+2n}$$

The series clearly is not geometric nor harmonic. Try to rewrite so that it appears as a telescoping series.

Solutions to worked examples

1. $a_n = (-1)^n 2^{2n}/3^n = (-4/3)^n$

Thus $r = -4/3$. Since $|r| > 1$ the series is divergent.

2. $\displaystyle\sum_{n=1}^{\infty} \frac{2^{n+2}}{\pi^{n-2}} = \frac{2^3}{\pi^{-1}} \sum_{n=1}^{\infty} \frac{2^{n-1}}{\pi^{n-1}} = 8\pi \sum_{n=1}^{\infty} \left(\frac{2}{\pi}\right)^{n-1}$

So $a = 8\pi$ and $r = 2/\pi$.

The series is convergent because $|r| < 1$ and the sum is $\dfrac{a}{1-r} = \dfrac{8\pi}{1-2/\pi} = \dfrac{8\pi^2}{\pi-2}$

3. $\displaystyle 0.135135\ldots = \frac{135}{1000} \sum_{n=1}^{\infty} \frac{1}{(1000)^{n-1}} = \frac{135}{1000}\left(\frac{1}{1-1/1000}\right)$

$$= \frac{135(1000)}{1000(999)} = \frac{135}{999} = \frac{5}{37}$$

4. $\displaystyle\sum_{n=1}^{\infty} [4(0.2)^{n+1} - (0.1)^n] = 4(0.2)^2 \left[\sum_{n=1}^{\infty} (0.2)^{n-1}\right] - (0.1)\left[\sum_{n=1}^{\infty} (0.1)^{n-1}\right]$

$$= \frac{4(0.2)^2}{1-0.2} - \frac{0.1}{1-0.1} = \frac{0.16}{0.8} - \frac{0.1}{0.9} = \frac{1}{5} - \frac{1}{9} = 4/45$$

5. $\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2n}{n+1} = \lim_{n\to\infty} \frac{2}{1+1/n} = 2 \neq 0$ The series diverges.

6. Notice that this is a telescoping series so the partial sums may be evaluated as a difference of two terms whose limits are easily found. $\displaystyle\sum_{n=1}^{\infty} [\tan(1/n) - \tan(1/(n+1))] = \lim_{N\to\infty}$ $\displaystyle\sum_{n=1}^{N} [\tan(1/n) - \tan(1/(n+1))]$

$$= \lim_{N\to\infty} [\tan(1/1) - \tan(1/N)] = \tan(1) - \lim_{N\to\infty} \tan(1/N)$$

$= \tan(1) - \tan(0) = \tan(1) \approx 1.5574$

7. This looks a bit like the telescoping series so apply partial fractions.

$\sum_{n=1}^{\infty} \frac{1}{n^2+2n} = \sum_{n=1}^{\infty} (\frac{1}{n} - \frac{1}{n+2})$.

Look at the partial sum and group the positive terms together first and then subtract the negative terms: $\sum_{n=1}^{N} (\frac{1}{n} - \frac{1}{n+2}) = (1/1 + 1/2 + 1/3 + \cdots 1/n) - (1/3 + 1/4 + \cdots + 1/(n+1) + 1/(n+2))$

$= (1 + 1/2) - (1/(n+1) + 1/(n+2))$ The last two terms converge to 0 so the series converges to 3/2.

8.3 The Integral and Comparison Tests; Estimating Sums

Key Concepts:

- Integral Test
- Comparison Test
- Series that are p-series
- Remainder Estimate for Integral Test

Skills to Master:

- Use the Integral Test to determine whether an applicable series is convergent.
- Use the Comparison Test to determine whether a series is convergent or divergent.
- Estimate the error in approximating the sum of a series.

Discussion:

Section 8.3 continues the development of series and describes the two most important methods of determining convergence or divergence: (1) comparing the series with a series whose convergence or divergence is known, and (2) comparing the series with a positive, decreasing continuous function, transforming the problem into determining the convergence of an improper integral over the interval $[1,\infty]$. This test also provides ways to estimate the number of terms needed for a partial sum to provide and adequate estimate of the sum of the series.

Key Concept: Integral Test

The Integral Test has a number of conditions that need to be satisfied by a series before it can be applied. Make sure that you check all the conditions. The Integral Test is stated as follows.

Integral Test: Suppose that f is continuous, positive and decreasing on $[1, \infty)$ and that $a_n = f(n)$.
Then the series

$$\sum_{n=1}^{\infty} a_n$$

is convergent if and only if the improper integral

$$\int_1^{\infty} f(x)\, dx$$

is convergent.

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Review the section on *improper integrals* as a warm up.

Key Concept: Comparison Test

The Comparison Test can only be applied to series with positive terms. This test is stated as follows.

Comparison Test:
Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.
(a) If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n, then $\sum a_n$ is also convergent.
(b)If $\sum b_n$ is divergent and $a_n \geq b_n$ for all n, then $\sum a_n$ is also divergent.

Note that you can only conclude convergence using this test by comparing with a series with larger terms that converges, and you can only conclude divergence with this test by comparing with a series with smaller terms that diverges.

Key Concept: Series that are p-series

A series of the form

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

is called a p-series. Such a series converges if $p > 1$ and diverges if $p \leq 1$.

Key Concept: Remainder Estimate for Integral Test

If a series

$$\sum_{n=1}^{\infty} a_n$$

converges to s by the Integral Test and if $R_n = s - s_n$, then

$$\int_{n+1}^{\infty} f(x)\,dx \le R_n \le \int_{n}^{\infty} f(x)\,dx.$$

Figures 3 and *4* in this section give the geometric reasoning behind this error estimate. Remember to check that the Integral Test applies before using this error estimate.

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SkillMaster 8.6: Use the Integral Test to determine if a series with positive decreasing terms is convergent.

Before applying the integral test to a series $\sum_{n=1}^{\infty} a_n$, make sure that you check that the terms of the series you are considering are positive and decreasing. Next, find a function $f(x)$ so that $f(n) = a_n$. Finally, determine whether the improper integral

$$\int_{1}^{\infty} f(x)\,dx$$

converges or diverges. This tells you whether the original series converges or diverges.

SkillMaster 8.7: Use the Comparison Test to determine whether a series is convergent or divergent.

Given a series $\sum_{n=1}^{\infty} a_n$, with positive terms, you should first determine whether it looks like a series that you already know either converges or diverges. If you think the series converges, try to compare it with a series with larger positive terms that converges. If you think that the series diverges, try to compare it with a series with smaller terms that diverges. You may need to multiply or divide by positive constants to get the inequalities to work out so that the Comparison Test can be correctly applied.

SkillMaster 8.8: Estimate the error in approximating the sum of a series.

When you apply the Integral Test to show that a series converges it is also possible to estimate the sum of the series. For example, if you want to estimate the sum s of the series $\sum_{n=1}^{\infty} a_n$ to within .0001, you want to find an m so that $R_m < .0001$. You know that

$$R_m \le \int_{m}^{\infty} f(x)\,dx,$$

so if you can find an m so that

$$\int_m^{\infty} f(x)\,dx < .0001$$

you will have found an m that works. To obtain the actual estimate, compute

$$\sum_{n=1}^{m} a_n.$$

Worked Examples

For each of the following examples, first try to find the solution without looking at the hint. Cover the right column with a piece of paper. If you need a hint, uncover the column. If you need to see the worked solution, look at the solutions section that follows the examples and hints.

Example	Hint
SkillMaster 8.6.	
1. Determine if the series is convergent or divergent. $$\sum_{n=1}^{\infty} ne^{-n}$$	Try the Integral Test. The terms are positive. Use the first derivative test to show that they are decreasing and then the Integral Test to determine convergence.
2. Determine if the series is convergent or divergent. $$\sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^2}$$	The terms are positive and decreasing. Try the Integral Test. It may help to use the substitution $u = ln(x)$, integration by parts would be a bit more complicated but also works.

SkillMaster 8.7.

3. Determine if the series is convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{2}{3n+\pi}$$

This series looks very much like the harmonic series $\sum(1/n)$. Use the comparison test by comparing the series with a multiple of the harmonic series. Start by observing that
$\dfrac{2}{3n+\pi} > \dfrac{2}{3n+\pi n}$.

4. Determine if the series is convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{\cos^2(n)}{n^{1.1}}$$

Use the Comparison Test. Notice that

$$0 \leq \cos^2(n) \leq 1.$$

SkillMaster 8.8.

5. We want to approximate the sum

$$\sum_{n=1}^{\infty} \frac{1}{n^5}$$

by summing the first several terms but we need to have an error less than 0.001. How many terms must we take and what is the approximation?

First note that the series is a $p-$series with $p = 5 > 1$ so the series converges. Compute the remainder term R_n and determine the value of n that will make this less than 0.001.

Solutions to worked examples

1. Let $f(x) = xe^{-x}$. Note that f is continuous and positive on $[1, \infty]$. We wish to know that f is decreasing for $x > 1$.

$f'(x) = e^{-x} - xe^{-x} = (1-x)e^{-x}$.

$e^{-x} > 0$ for all x and $(1-x) < 0$ for $x > 1$, so $f'(x) < 0$ for all $x > 1$.

Thus f is decreasing for $x > 1$, so the sequence is decreasing and the Integral Test applies. The series diverges or converges together with the divergence or convergence of the integral $\int_1^{\infty} xe^{-x}dx$. Use integration by parts.

$u = x \qquad dv = e^{-x} \qquad du = dx \qquad v = -e^{-x}$

$$\int_1^{\infty} xe^{-x}dx = \lim_{N\to\infty} -xe^{-x}\Big]_1^N + \int_1^{\infty} e^{-x}dx = 0 + e^{-1} - \lim_{N\to\infty} e^{-x}\Big]_1^N = 2e^{-1} < \infty$$

The integral converges so the series is also convergent.

2. The series diverges or converges according to the divergence or convergence of the integral $\int_2^{\infty} \frac{1}{x(\ln(x))^2}dx$

Use a substitution. $u = \ln(x) \qquad du = (1/x)dx$

$x = 2,\ u = \ln(2) \qquad x = \infty,\ u = \infty$

$$\int_2^{\infty} \frac{1}{x(\ln(x))^2}dx = \int_{\ln(2)}^{\infty} \frac{1}{u^2}du = \lim_{N\to\infty} \frac{-1}{u}\Big]_{\ln(2)}^N = 1/\ln(2)$$

Since the integral is convergent, the series is convergent also.

3. $\frac{2}{3n+\pi} > \frac{2}{3n+\pi n} = 2\frac{1}{(3+\pi)n} = \left(\frac{2}{3+\pi}\right)\frac{1}{n}$.

The comparison test shows that the series diverges because it is larger, term by term, than the divergent series

$$\sum_{n=1}^{\infty}\left(\frac{2}{3+\pi}\right)(1/n).$$

4. $\sum_{n=1}^{\infty} \frac{\cos^2(n)}{n^{1.1}} \leq \sum_{n=1}^{\infty} \frac{1}{n^{1.1}}$ The last series converges since it is $p-$series with $p > 1$.

The original series must also converge.

5. If we sum the first n terms the error (the remainder term R_n) is

$$R_n \leq \int_n^{\infty} \frac{1}{x^5}dx = \lim_{N\to\infty} (\frac{-1}{4})x^{-4}\Big]_n^N = 0 - (-1/4)n^{-4} = n^{-4}/4.$$

We need this to be less than 0.001.

$n^{-4}/4 < 0.001 \Longleftrightarrow n^{-4} < 0.004 \Longleftrightarrow 1 < 0.004n^4 \Longleftrightarrow 250 < n^4$

$\Longleftrightarrow \sqrt[4]{250} < n \Longleftrightarrow n > 3.98$

It is sufficient to take $n = 4$ terms. $\sum_{n=1}^{4} \frac{1}{n^5} = \frac{1}{1} + \frac{1}{2^5} + \frac{1}{3^5} + \frac{1}{4^5} \approx 1.0363.$

8.4 Other Convergence Tests

Key Concepts:

- Alternating Series Test
- Alternating Series Estimation Theorem
- Absolute Convergence
- Ratio Test

Skills to Master:

- Use the Alternating Series Test to determine if an alternating series is convergent.
- Use the Alternating Series Estimation Theorem to estimate the error in approximating the sum of an alternating series by a finite sum.
- Determine whether a series is absolutely convergent.
- Use the ratio test to determine whether a series is absolutely convergent.

Discussion:

Section 8.4 gives additional tests for deciding the convergence or divergence of certain series. For alternating series, an estimate is easily available for the error in approximating the sum of the infinite series by the sum of the first n terms.

Key Concept: Alternating Series Test

A series is alternating if the terms are alternately positive and negative. The Alternating Series Test is stated as follows.

Alternating Series Test:
If the alternating series

$$\sum_{n=1}^{\infty}(-1)^{n-1}b_n = b_1 - b_2 + b_3 - b_4 + \cdots$$
$$\text{where } b_n > 0$$

satisfies

(a) $b_{n+1} \leq b_n$ for all n, and
(b) $\lim_{n\to\infty} b_n = 0$,

then the series is convergent.

Note that you only need to check that the terms alternate, decrease in absolute value, and converge to 0 as a sequence. If you picture a grasshopper on the $x-$axis, beginning at the origin, that jumps forward to b_1 and then jumps backward b_2 then the grasshopper does not overtake the origin because the terms are decreasing in absolute value. Rather the grasshopper stops short of the origin at $b_1 - b_2 > 0$. Next the grasshopper turns in the positive direction again and jumps forward but because the terms are decreasing in absolute value, the grasshopper cannot overtake her earlier position at b_1 but lands short. In this way, the grasshopper is always trapped between her current position and her previous position. This is the reason that the Alternating Series Test is true and is also the reason that there is a simple estimate of the error between a finite sum and the actual sum of the infinite series. The condition that the terms go to 0 is necessary to trap the grasshopper within an interval whose length decreases to 0.

Key Concept: Alternating Series Estimation Theorem

If $s = \sum_{n=1}^{\infty}(-1)^{n-1}b_n$ is the sum of an alternating series that satisfies the conditions of the Alternating Series Test, and if $R_n = s - s_n$ is the error made in stopping after n terms, then

$$|R_n| = |s - s_n| \leq b_{n+1}.$$

Note that this says that the error is at most the magnitude of the first term not used.

Key Concept: Absolute Convergence

A series $\sum_{n=1}^{\infty} a_n$ converges absolutely if $\sum_{n=1}^{\infty} |a_n|$ converges. If a series is absolutely convergent, then it necessarily converges. Note that a series may converge without converging absolutely. For example, the series

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$$

converges by the Alternating Series Test, but the series

$$\sum_{n=1}^{\infty}\left|(-1)^{n-1}\frac{1}{n}\right| = \sum_{n=1}^{\infty}\frac{1}{n}$$

diverges by the Integral Test or the $p-$Series Test with $p = 1$.

Key Concept: Ratio Test

The Ratio Test is a strong test for determining absolute convergence. It is stated as follows.

Ratio Test:
(a) If $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum\limits_{n=1}^{\infty} a_n$ converges absolutely.
(b) If $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = L > 1$, or $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \infty$ then the series $\sum\limits_{n=1}^{\infty} a_n$ is divergent.

Make sure that you understand the Ratio Test gives no information if the limit involved is equal to 1.

SkillMaster 8.9: Use the Alternating Series Test to determine if an alternating series is convergent.

In using the Alternating Series Test, make sure that you check all the needed conditions. The terms must alternate, decrease in magnitude, and approach 0.

SkillMaster 8.10: Use the Alternating Series Estimation Theorem to estimate the error in approximating the sum of an alternating series by a finite sum.

Once you have determined that a series converges by using the Alternating Series Test, it is easy to approximate the sum and to estimate the error involved in the approximation. The error is no bigger than the magnitude of the first term not used.

SkillMaster 8.11: Determine whether a series is absolutely convergent.

To determine whether a series is absolutely convergent, form a new series by taking the absolute value of each term. Then use a test that you already know to determine whether the new series converges. If it does, the original series is absolutely convergent.

SkillMaster 8.12: Use the ratio test to determine whether a series is absolutely convergent.

The Ratio Test is a good way to determine whether or not a series is absolutely convergent. The test depends on taking the ratio of the absolute values of successive terms and seeing if these ratios converge to a number less than 1.

Worked Examples

For each of the following examples, first try to find the solution without looking at the hint. Cover the right column with a piece of paper. If you need a hint, uncover the column. If you need to see the worked solution, look at the solutions section that follows the examples and hints.

Example	Hint
SkillMaster 8.9.	
1. Determine if the series is convergent or divergent. $$\sum_{n=1}^{\infty} \frac{(-1)^n n}{\sqrt{n^3+1}}$$	The series is alternating so try the Alternating Series Test. This test will apply if the absolute values of the terms are decreasing.

SkillMaster 8.10.

2. Show that the Alternating Series Test applies and show the following series is convergent. How many terms must we take for the partial sum to approximate the sum of the series with an error less than 0.001?

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n+1)}$$

To check that the conditions of the Alternative Series Test are satisfied check that the terms alternate, that the absolute value of the terms are decreasing and that the terms converge to 0.

SkillMaster 8.11.

3. Determine whether the series is absolutely convergent.

$$\sum_{n=1}^{\infty} \frac{(-1)^n n}{(n+1)^{3/2}}$$

The series is absolutely convergent if and only if the series $\sum_{n=1}^{\infty} \left| \frac{(-1)^n n}{(n+1)^{3/2}} \right| = \sum_{n=1}^{\infty} \frac{n}{(n+1)^{3/2}}$ is convergent. This looks like a $p-$series so use the Comparison Test.

SkillMaster 8.12.

4. Use the Ratio Test to determine if the series is absolutely convergent.

$$\sum_{n=1}^{\infty} \frac{(-1)^n n^{10}}{n!}$$

Here $a_n = \frac{(-1)^n n^{10}}{n!}$. If $\lim_{n\to\infty} |a_{n+1}/a_n| < 1$ then the Ratio Test allows us to conclude that the series is absolutely convergent.

Solutions to worked examples

1. We need to check if the absolute values of the terms are decreasing.

$$\frac{n+1}{\sqrt{(n+1)^3+1}} < \frac{n}{\sqrt{n^3+1}} \Longleftrightarrow \frac{(n+1)^2}{(n+1)^3+1} < \frac{n^2}{n^3+1}$$

$$\Longleftrightarrow \frac{(n+1)^3+1}{(n+1)^2} > \frac{n^3+1}{n^2} \Longleftrightarrow n+1+\frac{1}{(n+1)^2} > n+\frac{1}{n^2}$$

$$\Longleftrightarrow 1 > \frac{1}{n^2} - \frac{1}{(n+1)^2}$$

The last inequality must be true because 1 is always larger than the difference of two unequal numbers between 0 and 1. Now we must check that the terms are converging to 0. Use the Squeeze Theorem.

$$0 \le \lim_{n\to\infty} \frac{n}{\sqrt{n^3+1}} \le \lim_{n\to\infty} \frac{n}{\sqrt{n^3}} = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$$

So the terms approach 0. The Alternating Series Test applies and the series is convergent.

2. It is easy to see that the terms alternate. The absolute value of the nth term is $1/\ln(n+1)$ which is decreasing because the reciprocals, $\ln(n+1)$, are increasing. The terms converge to 0 because the absolute values of the reciprocals, $\ln(n+1)$, are diverging toward infinity. The Alternating Series Test applies and the series is convergent.

The error in estimating the n^{th} partial sum is less than the absolute value of the $(n+1)^{st}$ term.

$$|R_n| \le \frac{1}{\ln((n+1)+1)}$$ We need this estimate to be less than 0.001.

$$\frac{1}{\ln(n+2)} < 0.001 \Longleftrightarrow 1/(0.001) < \ln(n+2) \Longleftrightarrow 1000 < \ln(n+2)$$

$$\Longleftrightarrow e^{1000} - 2 < n$$

This number is approximately 2×10^{434}. In other words, this is the number of terms that would be required to reach the desired accuracy. This number is greater than the number of elementary particles in the known universe. It is not hyperbole to observe that this is not a practical way to estimate the sum.

3. $\sum_{n=1}^{\infty}\left|\frac{(-1)^n n}{(n+1)^{3/2}}\right| = \sum_{n=1}^{\infty}\frac{n}{(n+1)^{3/2}} > \sum_{n=1}^{\infty}\frac{n}{(2n)^{3/2}}$

$= \frac{1}{2^{3/2}}\sum_{n=1}^{\infty}\frac{n}{(n)^{3/2}} = \frac{1}{2^{3/2}}\sum_{n=1}^{\infty}\frac{1}{n^{1/2}}$

which is divergent because it is a $p-$series with $p = 1/2 < 1$.

The original series is not absolutely convergent.

4. $|a_{n+1}/a_n| = \frac{(n+1)^{10}}{(n+1)!}\cdot\frac{n!}{n^{10}} = \left(\frac{n+1}{n}\right)^{10}\cdot\frac{n!}{(n+1)!} = \left(1+\frac{1}{n}\right)^{10}\frac{1}{n+1}$

$\lim_{n\to\infty}|a_{n+1}/a_n| = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{10}\frac{1}{n+1} = (1+0)^{10}\lim_{n\to\infty}\frac{1}{n+1} = 0 < 1$

The Ratio Test applies and the series is absolutely convergent.

8.5 Power Series

Key Concepts:

- Power series
- Radius and interval of convergence of a power series

Skills to Master:

- Find the radius of convergence and the interval of convergence of a power series.

Discussion:

Section 8.5 and the two following sections concern power series. A power series is an infinite series whose partial sums are polynomials. One kind of power series has the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots$$

where each c_n is a constant and x is a variable. The first question of interest is how to describe the set of x that allows the series to converge. Fortunately, there is a simple method for determining the interior points of convergence the fact that this set of points is an interval is another piece of serendipity.

Key Concept: Power series

The power series above is centered at 0. A power series centered at a is a series in $(x-a)$:

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \cdots .$$

This series may or may not converge depending on the particular value of x. Notice that the series always converges at the center $x = a$. A power series can be viewed as a function with domain the set of x values for which it converges. Substituting a specific x in the domain into the power series produces a function value. You will learn how to represent many functions as power series.

Key Concept: Radius and interval of convergence of a power series

For a given power series,

$$\sum_{n=0}^{\infty} c_n(x-a)^n,$$

there are only three possibilities:

(i)	The series converges only when $x=a$.
(ii)	The series converges for all x.
(iii)	There is a positive number R such that the series converges if $\lvert x-a\rvert < R$ and diverges if $\lvert x-a\rvert > R$.

In case (iii), the number R is called the *radius of convergence* of the series. In case (i) the radius of convergence is 0 and in case (ii) the radius of convergence is ∞. The *interval of convergence* consists of all values of x for which the series converges. In case (iii), it is one of

$$(a-R, a+R),\ (a-R, a+R],\ [a-R, a+R),\ \text{or}\ [a-R, a+R].$$

SkillMaster 8.13: Find the radius of convergence and the interval of convergence of a power series.

To find the radius of convergence of a power series

$$\sum_{n=0}^{\infty} c_n(x-a)^n,$$

apply the Ratio Test. This is effective because a power series at an x that is closer to a than R can be compared with a geometric series. The endpoints of the interval are different matter. Study *Examples 4 and 5* in this section to see how behavior at the endpoints works.

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In general, you have to test endpoint convergence for each example with no general theorem to apply in all cases. There are a few guideposts that are given in the *chart in the text*. Also, look for situations where the Alternating Series Test is necessary to prove convergence at one of the endpoints. Notice also, if the coefficients c_n form an absolutely convergent series (by themselves) and if they are algebraic then there is convergence at both endpoints so the interval of convergence is $[a-R, a+R]$. For example, the interval of convergence of $\sum_{n=1}^{\infty} \frac{x^n}{n^4}$ is $[-1, 1]$.

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Worked Examples

For each of the following examples, first try to find the solution without looking at the hint. Cover the right column with a piece of paper. If you need a hint, uncover the column. If you need to see the worked solution, look at the solutions section that follows the examples and hints.

Example	Hint
SkillMaster 8.13.	
1. Find the radius of convergence and the interval of convergence of the following power series. $$\sum_{n=0}^{\infty}(-1)^{n+1}n^6x^n$$	Use the ratio test. If a_n is the n^{th} term in the power series then x is in the interval of convergence if $\lim_{n\to\infty}\lvert a_{n+1}/a_n\rvert < 1$.
2. Find the radius of convergence and the interval of convergence of the following power series. $$\sum_{n=0}^{\infty}\frac{n^n}{n!}x^n$$ (Note: as in the text, the notation $0^0 = 1$, so the first term is $0^0/0! = 1$.)	$\lvert a_n\rvert = \dfrac{n^n}{n!}\lvert x\rvert^n$ Use the Ratio Test.
3. Find the radius of convergence and the interval of convergence of the following power series. $$\sum_{n=0}^{\infty}\frac{4n+1}{\sqrt{n+1}}(x-2)^n$$	As always to find the radius of convergence apply the Ratio Test.

Solutions to worked examples

1. $a_n = (-1)^{n+1} n^6 x^n$

$\lim\limits_{n\to\infty} |a_{n+1}/a_n| = \lim\limits_{n\to\infty} \left((n+1)^6 |x|^{n+1}\right) / \left(n^6 |x|^n\right) = \lim\limits_{n\to\infty} \left(\dfrac{n+1}{n}\right)^6 |x| = |x|$ Thus $\left|\dfrac{a_{n+1}}{a_n}\right| < 1$ if and only if $|x| < 1$.

So, $(-1,1)$ is the interval of convergence and the radius of convergence is 1. Note that the endpoints ± 1 are not in the interval of convergence because the terms of the series do not approach 0 when $x = \pm 1$.

2. $\lim\limits_{n\to\infty} |a_{n+1}/a_n| = \lim\limits_{n\to\infty} \left|\dfrac{(n+1)^{(n+1)} x^{n+1}}{(n+1)!} \dfrac{n!}{n^n x^n}\right| = \lim\limits_{n\to\infty} \left|\dfrac{(n+1)^n (n+1) n! x^{n+1}}{n^n (n+1)! x^n}\right|$

$= \lim\limits_{n\to\infty} \left|\left(\dfrac{n+1}{n}\right)^n \dfrac{(n+1)!}{(n+1)!} x\right| = \lim\limits_{n\to\infty} \left|(1+1/n)^n x\right| = e|x|$

Note that $e|x| < 1$ if and only if $|x| < 1/e$, so the radius of convergence is $1/e$ and the interval of convergence is $(-1/e, 1/e)$.

3. $\lim\limits_{n\to\infty} |a_{n+1}/a_n| = \lim\limits_{n\to\infty} \left|\dfrac{(4(n+1)+1)(x-2)^{n+1}}{\sqrt{(n+1)+1}} \cdot \dfrac{\sqrt{n+1}}{(4n+1)(x-2)^n}\right|$

$= \lim\limits_{n\to\infty} \left|\dfrac{4n+5}{4n+1}\sqrt{\dfrac{n+1}{n+2}}(x-2)\right| = \lim\limits_{n\to\infty} \left|\dfrac{4+5/n}{4+1/n}\sqrt{\dfrac{1+1/n}{1+2/n}}(x-2)\right|$

$= \left|\dfrac{4+0}{4+0}\sqrt{\dfrac{1+0}{1+0}}(x-2)\right| = |x-2|$ Note that $|x-2| < 1$ if and only if

$x \in (2-1, 2+1) = (1,)$ which gives the radius of convergence $R = 1$ and the series does not converge at the endpoints of the interval.

8.6 Representations of Functions as Power Series

Key Concepts:

- The power series for $\dfrac{1}{1-x}$
- Differentiation and integration of power series

Skills to Master:

- Represent a function as a power series using the known example $\dfrac{1}{1-x}$.
- Represent a function as a power series by differentiating and integrating known examples.

Discussion:

Section 8.6 begins representing certain functions as power series. There are many reasons this is desirable. Two of them are: (1) First, functions that are represented as power series can be differentiated and integrated term by term - the interval of convergence stays the same, although convergence at the endpoints may behave differently; (2) power series have, almost by definition, natural polynomial approximations. Polynomials are especially easy for computational devices to process. In this section, a large class of functions is shown to have a power series representation that depends only on the most basic infinite series, the geometric series.

Key Concept: The power series for $\dfrac{\mathbf{1}}{\mathbf{1-x}}$

From the formula for the sum of the geometric series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 \cdots$$

you can obtain the representation

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \text{ when } |x| < 1.$$

By replacing x by other expressions, you can obtain power series representations for many other functions, including the logarithm and inverse tangent functions.

Key Concept: Differentiation and integration of power series

Make sure that you understand the statement of *Theorem 2* in this section. This theorem tells you that if a function is represented by a power series with radius of convergence R, then the function is differentiable and can be differentiated or integrated by differentiating or integrating the power series term by term. This is an extremely important result. Note that the radius of convergence of the series that you obtain by differentiating or integrating the original series is the same as the radius of convergence of the original series.

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SkillMaster 8.14: Represent a function as a power series using the known example for $\mathbf{\frac{1}{1-x}}$.

Using the representation

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \text{ when } |x| < 1,$$

you can obtain representation for many functions that are similar in form to

$$\frac{1}{1-x}.$$

By using Theorem 2 on differentiating and integrating power series, you can obtain power series representations for still more functions.

Study *Examples 4 through 8* in this section to see how this works.

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SkillMaster 8.15: Represent a function as a power series by differentiating and integrating known examples.

Once you have a power series representation for a function, you also have a power series representation for the derivative and integral of the function. This leads to many useful power series representations.

Worked Examples

For each of the following examples, first try to find the solution without looking at the hint. Cover the right column with a piece of paper. If you need a hint, uncover the column. If you need to see the worked solution, look at the solutions section that follows the examples and hints.

Example	Hint
SkillMaster 8.14.	
1. Represent the function as a power series and give the radius of convergence. $$f(x) = \frac{x^2}{1-x}$$	We know $\frac{1}{1-x} = = \sum_{n=0}^{\infty} x^n$ and that this has a radius of convergence 1. Multiply both sides of this equation by x^2 to see the power series for the new function.
2. Represent the function as a power series and give the radius of convergence. $$f(x) = \frac{1+x^2}{1-x^2}$$	Rewrite this as $\frac{1}{1-x^2} + x^2\frac{1}{1-x^2}$. The power series for $\frac{1}{1-x^2}$ may be obtained by substituting x^2 for x in the power series for $\frac{1}{1-x}$. Start by factoring out a factor of 4 from the denominator.
3. Represent the function as a power series and give the radius of convergence. $$f(x) = \frac{1}{4+x}$$	Rewrite the function algebraically so that it looks like a function of the form $1/(1-x)$.

SkillMaster 8.15.

4. Represent the function as a power series.

$$f(x) = x\ln(1-x^2)$$

Recall $\dfrac{1}{1-x} = \sum_{n=1}^{\infty} x^n$. In Example 6 on page 602 of the text the following series was derived:
$\ln(1-x) = = -\sum_{n=1}^{\infty} \dfrac{x^n}{n}$. Substitute x^2 for x in this series and multiply by x.

5. Represent the function as a power series.

$$f(x) = \sqrt{x}\tan^{-1}(\sqrt{x})$$

In Example 7 on page 602 of the text, the series $\tan^{-1}(x) = \sum_{n=0}^{\infty} \dfrac{(-1)^n x^{2n+1}}{2n+1}$ was derived. Use this to find the required power series.

6. Evaluate the indefinite integral as a power series.

$$\int \frac{x}{1-x^4}\,dx$$

Express the integrand as a power series and integrate term by term.

Solutions to worked examples

1. $f(x) = \dfrac{x^2}{1-x} = x^2\left(\dfrac{1}{1-x}\right) = x^2\left(\displaystyle\sum_{n=0}^{\infty} x^n\right) = \displaystyle\sum_{n=0}^{\infty} x^{n+2}$

The radius of convergence is 1, the same as that of the series for $1/(1-x)$.

2. $f(x) = \dfrac{1+x^2}{1-x^2} = \dfrac{1}{1-x^2} + x^2\dfrac{1}{1-x^2} = \left(\displaystyle\sum_{n=0}^{\infty}(x^2)^n\right) + x^2\left(\displaystyle\sum_{n=0}^{\infty}(x^2)^n\right)$

$$= \left(\sum_{n=0}^{\infty} x^{2n}\right) + x^2\left(\sum_{n=0}^{\infty} x^{2n}\right) = \left(\sum_{n=0}^{\infty} x^{2n}\right) + \left(\sum_{n=0}^{\infty} x^{2n+2}\right)$$

$$= \left(\sum_{n=0}^{\infty} x^{2n}\right) + \left(\sum_{n=1}^{\infty} x^{2n}\right) = 1 + \sum_{n=1}^{\infty} 2x^{2n} = 1 + 2\sum_{n=1}^{\infty} x^{2\,n}$$

The series converges for all $x^2 < 1$ or $-1 < x < 1$ and the radius of convergence is 1.

Notice the solution shows that the original function could have been manipulated algebraically to get $\dfrac{1+x^2}{1-x^2} = -1 + 2\dfrac{1}{1-x^2}$ and then x^2 could be substituted for x in the geometric power series. It is good advice to always take a moment to first look for easier ways to solve the problem than the obvious method at hand.

3. $f(x) = \dfrac{1}{4+x} = \dfrac{1}{4}\left(\dfrac{1}{1-(-x/4)}\right)$

Substitute $(-x/4)$ for x into the power series for $\dfrac{1}{1-x}$.

$f(x) = \dfrac{1}{4}\displaystyle\sum_{n=0}^{\infty}(-x/4)^n = \sum_{n=0}^{\infty}\dfrac{(-1)^n x^n}{4^{n+1}}$

The series converges by the Ratio Test for $|-x/4| < 1$ or $|x| < 4$ or $-4 < x < 4$.

The radius of convergence is 4.

4. $f(x) = x\ln(1-x^2) = x\displaystyle\sum_{n=1}^{\infty} -\dfrac{(x^2)^n}{n} = -\sum_{n=1}^{\infty}\dfrac{x^{2n+1}}{n}$

5. There is a power series for the inverse tangent function given in the text. Substitute $x^{1/2}$ for x in this series.

$f(x) = \sqrt{x}\tan^{-1}(\sqrt{x}) = x^{1/2}\left(\displaystyle\sum_{n=0}^{\infty}\dfrac{(-1)^n (x^{1/2})^{2n+1}}{2n+1}\right)$

$$= x^{1/2} \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1/2}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{2n+1}$$

Notice that the original function is defined only for $x \geq 0$. The power series converges for all $-1 < x < 1$, so the power series represents the function f for all $0 \leq x < 1$. It is interesting to consider what the interpretation of the power series for negative x might be.

6. $\displaystyle \int \frac{x}{1-x^4}\,dx = \int x\frac{1}{1-x^4}\,dx = \int x \sum_{n=0}^{\infty} (x^4)^n\,dx$

$$= \int \sum_{n=0}^{\infty} x^{4n+1}\,dx = C + \sum_{n=0}^{\infty} \frac{x^{4n+2}}{4n+2} = C + \frac{1}{4}\ln\left|\frac{1-x^2}{1+x^2}\right|$$

8.7 Taylor and Maclaurin Series

Key Concepts:

- The Taylor and Maclaurin series of a function
- Taylor's Inequality
- Maclaurin series for familiar functions
- The Binomial Theorem and the Binomial Series

Skills to Master:

- Find and apply the Taylor and Maclaurin Series to estimate function values to a desired accuracy.
- Use the Binomial Series to find power series representations for functions.

Discussion:

Section 8.7 shows how to compute the coefficients of a power series centered at a for various functions. This is possible for a wide class of functions, including most of those you have been using in mathematics for years now. Such power series are called Taylor series, and Taylor series centered at the origin are called Maclaurin series. It is possible to get a remainder estimate, so that in principle, most functions may be approximated by polynomials to any degree of accuracy.

Key Concept: The Taylor and Maclaurin series of a function.
Suppose f has derivatives of all orders at a. The Taylor series of a function f centered at a is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots .$$

If $a = 0$, the Taylor series is called the Maclaurin series. If f has a power series representation at a , then the power series representation is the Taylor series. Be careful to note that even if f has derivatives of all orders at a, and thus has a Taylor series, f may not be equal to a power series.

Key Concept: Taylor's Inequality

A key question is whether a given function f is equal to the sum of its Taylor series at a. If

$$f(x) = T_n(x) + R_n(x)$$

where $T_n(x)$ is the nth degree Taylor polynomial of f at a and if

$$\lim_{n\to\infty} R_n(x) = 0 \text{ for } |x-a| < R,$$

then f is equal to the sum of its Taylor series for all x where $|x-a| < R$. Taylor's Inequality can help you determine if $\lim_{n\to\infty} R_n(x) = 0$ in the above setting.

Taylor's Inequality states that if

$$\left| f^{(n+1)}(x) \right| \le M \text{ for } |x-a| < R,$$

then the remainder $R_n(x)$ of the Taylor series satisfies the inequality

$$R_n(x) \le \frac{M}{(n+1)!} |x-a|^{n+1}.$$

A key fact to remember in applying this to specific situations is that

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0 \text{ for every real number } x.$$

Key Concept: Maclaurin series for familiar functions

The *table* in this section gives Maclaurin series for some familiar functions and also gives the intervals of convergence for these series. The most common functions have Maclaurin Series that are worth remembering. Although in principle, they can be computed from scratch, this can be time-consuming.

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Key Concept: The Binomial Theorem and the Binomial Series

The Binomial Theorem states that

$$(a+b)^k = \sum_{n=0}^{k} \binom{k}{n} a^{k-n} b^n \text{ where } \binom{k}{n} = \frac{k(k-1)(k-1)\cdots(k-n+1)}{n!} .$$

(If $k=0$, $\binom{n}{k}$ is defined to be 1.) You should be familiar with this in the simple cases where $k=2$ or $k=3$. In the above formula, k is a fixed positive integer.

There is a generalization of this when k is any real number. The Binomial Series for $(1+x)^k$ is

$$\sum_{n=0}^{\infty}\binom{k}{n}x^n \text{ where } \binom{k}{n}=\frac{k(k-1)(k-1)\cdots(k-n+1)}{n!}.$$

If k is not an integer, there are an infinite number of terms in this series. This series is equal to $(1+x)^k$ if $|x|<1$.

SkillMaster 8.16: Find the Taylor and Maclaurin series for a function using the definition.

To find Taylor and Maclaurin series for specific functions, use the formula

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n = f(a)+\frac{f'(a)}{1!}(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3+\cdots.$$

The coefficient of $(x-a)^n$ is $\dfrac{f^{(n)}(a)}{n!}$, so to express the series, you will compute the derivatives of f at a and find a pattern, if possible. This will allow you to determine the general term in the Taylor or Maclaurin series.

SkillMaster 8.17: Use the Binomial Series to find power series representations for functions.

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Any function that is a product of a power of x and an expression of the form $(1+x)^k$ can be written as a power series by using the Binomial Series. *Example 9* in this section show how to do this.

Worked Examples

For each of the following examples, first try to find the solution without looking at the hint. Cover the right column with a piece of paper. If you need a hint, uncover the column. If you need to see the worked solution, look at the solutions section that follows the examples and hints.

Example	Hint

SkillMaster 8.16.

1. Find the Maclaurin series for the function using the definition.

$$f(x) = \frac{1}{x+2}$$

Compute $f'(x), f''(x), \ldots$ until you see the pattern for $f^{(n)}(x)$.

2. Find the Taylor series for the function $f(x)$ at the given point a.

$$f(x) = \frac{1}{x+2}, \qquad a = 2$$

$f^{(n)}(x) = (-1)^n n!(x+2)^{-(n+1)}$. Now find $f^{(n)}(2)$.

3. Find the Taylor series for the function $f(x)$ at the given point a.

$$f(x) = \frac{1}{\sqrt{x}}, \qquad a = 1$$

Find a pattern for the n^{th} derivative of f.

4. Use the Maclaurin series for $\cos(x)$ to find the Maclaurin series for

$$f(x) = x\cos(x^3).$$

From the text, you know that $\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$.

5. Use a Maclaurin series to compute $\int_0^1 x\cos(x^3)\,dx$ to three decimal places.

Integrate the Maclaurin series for the integrand and use Taylor's Remainder Formula.

6. Find the limit using a Maclaurin series.

$$\lim_{x\to 0}\frac{e^{-x^2}-1+x^2}{x^4}$$

$e^{-x^2}-1+x^2 =$
$\left(\sum_{n=0}^{\infty}(-x^2)^n/n!\right) \quad -1+x^2 =$
$\sum_{n=2}^{\infty}(-1)^n x^{2n}/n!$

SkillMaster 8.17.

7. Use the binomial series to expand the following function as a power series. State the radius of convergence.
$f(x)=\dfrac{x^3}{\sqrt{4+x}}$

Rewrite the function as $2x^3(1+x/4)^{1/2}$ and apply the Binomial Series.

Solutions to worked examples

1. $f(x)=\dfrac{1}{x+2}=(x+2)^{-1} \qquad f'(x)=-(x+2)^{-2} \qquad f''(x)=2(x+2)^{-3}$

$f^{(3)}(x)=-2\cdot 3(x+2)^{-4} \qquad f^{(n)}(x)=(-1)^n n!(x+2)^{-(n+1)}\, f^{(n)}(0)=(-1)^n n!/2^{n+1}$

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n=\sum_{n=0}^{\infty}\frac{(-1)^n n!}{n!2^{n+1}}x^n=\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}x^n$$

2. $$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n=\sum_{n=0}^{\infty}\frac{(-1)^n n!}{n!4^{n+1}}(x-2)^n=\sum_{n=0}^{\infty}\frac{(-1)^n}{4^{n+1}}(x-2)^n$$

3. $f(x)=\dfrac{1}{\sqrt{x}}=x^{-1/2} \qquad f'(x)=(-1/2)x^{-3/2} \qquad f''(x)=\dfrac{1\cdot 3}{2\cdot 2}x^{-5/2}$

$f^{(3)}(x)=-\dfrac{1\cdot 3\cdot 5}{2\cdot 2\cdot 2}x^{-7/2} \qquad f^{(n)}(x)=(-1)^n\dfrac{1\cdot 3\cdots(2n-1)}{2^n}x^{-(2n+1)/2}$

$$f^{(n)}(1)=(-1)^n\frac{1\cdot 3\cdots(2n-1)}{2^n}=\frac{(-1)^n 1\cdot 2\cdots 2n}{2^n\cdot 2\cdot 4\cdots(2n)}=\frac{(-1)^n(2n)!}{2^n\cdot 2^n\cdot n!}$$

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n=\sum_{n=0}^{\infty}\frac{f^{(n)}(1)}{n!}(x-1)^n=\sum_{n=0}^{\infty}\frac{(-1)^n(2n)!}{4^n(n!)^2}(x-1)^n$$

4. $$f(x)=x\cos(x^3)=x\left(\sum_{n=0}^{\infty}(-1)^n\frac{(x^3)^{2n}}{(2n)!}\right)=x\left(\sum_{n=0}^{\infty}(-1)^n\frac{x^{6n}}{(2n)!}\right)=\sum_{n=0}^{\infty}(-1)^n\frac{x^{6n+1}}{(2n)!}$$

$$= x - \frac{x^7}{2!} + \frac{x^{13}}{4!} - \frac{x^{19}}{6!} + \cdots$$

5. $\displaystyle\int_0^1 x\cos(x^3)\,dx = \int_0^1 \left(\sum_{n=0}^{\infty}(-1)^n \frac{x^{6n+1}}{(2n)!}\right) dx$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\int_0^1 x^{6n+1}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\frac{x^{6n+2}}{(6n+2)}\Big]_0^1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(6n+2)(2n)!}$$

$$= \frac{1}{2} - \frac{1}{8\cdot 2} + \frac{1}{14\cdot 4!} - \frac{1}{20\cdot 6!} + \cdots = 0.5 - 0.0625 + 0.0030 - 0.00007 + \cdots$$

The Alternating Series Estimation Theorem shows that the error involved in approximating the sum of this series with the sum of the first 3 terms is less than 0.00007 (which is the value of the 4^{th} term).

$$\int_0^1 x\cos(x^3)\,dx \approx 0.5 - 0.0625 + 0.0030 \approx 0.4405$$

6. $\displaystyle\lim_{x\to 0} \frac{e^{-x^2} - 1 + x^2}{x^4} = \lim_{x\to 0} \frac{\sum_{n=2}^{\infty}(-1)^n x^{2n}/n!}{x^4} = \lim_{x\to 0} \frac{x^4\sum_{n=2}^{\infty}(-1)^n x^{2n-4}/n!}{x^4}$

$$= \lim_{x\to 0} \sum_{n=2}^{\infty}(-1)^n x^{2n-4}/n! = \lim_{x\to 0} \frac{1}{2!} - \frac{x^2}{3!} + \frac{x^4}{4!} - \cdots$$

$$= \frac{1}{2!} - \frac{0^2}{3!} + \frac{0^4}{4!} - \cdots = \frac{1}{2} = 0.5$$

7. Using the hint, $\displaystyle\frac{x^3}{\sqrt{4+x}} = \frac{x^3}{2(1+x/4)^{1/2}} = \frac{x^3}{2}(1+x/4)^{-1/2}$

The Binomial Series for $(1+x/4)^{-1/2}$ is $\displaystyle\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}\left(\frac{x}{4}\right)^n$ and converges when

$\left|\frac{x}{4}\right| < 1$, or when $|x| < 4$.

So a power series expansion for $f(x)$ is $\displaystyle\frac{x^3}{2}\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}\left(\frac{x}{4}\right)^n$

$\displaystyle= \sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}\frac{x^{n+3}}{2\cdot 4^n} = \left[\frac{1}{2}x^3 - \frac{1}{2}\cdot\frac{1}{8}x^4 + \cdots\right]$ The radius of convergence is 4.

8.8 Applications of Taylor Polynomials

Key Concepts:

- Finding the error in approximating a function by a Taylor polynomial

Skills to Master:

- Find an upper bound for the error of approximation of a function by its Taylor polynomial.
- Find how many terms in a Taylor polynomial are necessary for the error to be less than a specific amount.

Discussion:

Section 8.9 covers the case of a Taylor series that is equal to a function on an interval and determines which of the various Taylor polynomials can be used as approximations to the function. This section gives methods to determine how good these approximations are over an interval and how many terms are necessary in a Taylor polynomial to achieve a good approximation.

Key Concept: Finding the error in approximating a function by a Taylor polynomial

Suppose that a function $f(x)$ is equal to the Taylor series at a point a.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Then the nth degree Taylor polynomial of f at a is the nth partial sum of this series.

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$$

The error in using this approximation is

$$|R_n(x)| = |f(x) - T_n(x)| = \left| \sum_{i=n+1}^{\infty} \frac{f^{(i)}(a)}{i!}(x-a)^i \right|.$$

This error can be estimated by using $\left| \frac{f^{(n+1)}(a)}{(n+1)!}(x-a)^{n+1} \right|$ as an upper bound if the Taylor series is an alternating series. In general, by *Taylor's inequality* , if we can find a number M so that $\left|f^{(n+1)}(x)\right| \leq M$ for all $|x-a| \leq d$, then an upper bound for the error is $\left| \frac{M}{(n+1)!}(x-a)^{n+1} \right|$ for $|x-a| \leq d$. These error upper bounds will tell us how accurate the approximation must be. It is important to realize that the error bounds are the result of a worst-case analysis and often the approximation will be much better.

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SkillMaster 8.18: Find an upper bound for the error of approximation of a function by its Taylor polynomial.

The error estimates above provide the method to solve this kind of problem. The tricky part may be to get a good estimate of M. This estimate does not need to be exact to give good results. For example, if $f(x)$ is the sin or cos function then $M = 1$ is a good choice. If d is the interval length then the error estimate given by Taylor's Inequality is $\left| \frac{M}{(n+1)!}d^{n+1} \right|$.

SkillMaster 8.19: Find how many terms in a Taylor polynomial are necessary for the error to be less than a specific amount.

Notice that the error estimate depends on the number of terms. For large enough n the error becomes as small as we like. Use the Alternating Series Test (if it applies) or Taylor's Inequality (otherwise) to estimate the approximation error in terms of n. Choose n (use a calculator if necessary) to be as small as possible but large enough to give a good error estimate.

Worked Examples

For each of the following examples, first try to find the solution without looking at the hint. Cover the right column with a piece of paper. If you need a hint, uncover the column. If you need to see the worked solution, look at the solutions section that follows the examples and hints.

Example	Hint
SkillMaster 8.18.	
1. Use Taylor's inequality to estimate the maximum error in using Taylor approximation $T_3(x)$ about $a = 1$ to approximate the given function on the indicated interval. $f(x) = \dfrac{1}{x+2}, \qquad 0.9 \le x \le 1.1$	Taylor's inequality says that $\lvert R_3(x)\rvert \le \dfrac{M}{4!}\lvert x-a\rvert^4$ where $\left\lvert f^{(4)}(x)\right\rvert \le M$.
SkillMaster 8.19.	
2. Use a Taylor polynomial to estimate $\sqrt{1.03}$ with an error less than 0.000001.	Use the function $f(x) = \sqrt{x} = x^{1/2}$ with $a = 1$. Use the interval $[.97,\ 1.03]$. Begin, as usual, by computing derivatives.

Solutions to worked examples

1. $f(x)=\dfrac{1}{x+2}=(x+2)^{-1} \qquad f'(x)=-(x+2)^{-2} \qquad f^{(2)}(x)=2(x+2)^{-3}$

$f^{(3)}(x)=-2(3)(x+2)^{-4} \qquad f^{(4)}(x)=2(3)(4)(x+2)^{-5}=\dfrac{24}{(x+2)^5}$

We have $\left|f^{(4)}(x)\right|\leq\dfrac{24}{(2.9)^5}$ since the fraction is biggest when the denominator is the smallest. We can choose $M=\dfrac{24}{(2.9)^5}$.

Then $|R_3(x)|\leq\dfrac{24/(2.9)^5}{4!}|0.1|^4=\dfrac{1}{(2.9)^5}|0.1|^4\backsim 4.9\times 10^{-7}$

This is very good accuracy.

2. $f(x)=x^{1/2}, \qquad f'(x)=\dfrac{1}{2}x^{-1/2} \qquad f(1)=1, \qquad f'(1)=\dfrac{1}{2}$

$f^{(2)}(x)=-\dfrac{1}{2^2}x^{-3/2} \qquad f^{(2)}(1)=-\dfrac{1}{4} \qquad f^{(3)}(x)=\dfrac{3}{2^3}x^{-5/2} \qquad f^{(3)}(1)=\dfrac{3}{8}$

$f^{(4)}(x)=-\dfrac{5\cdot 3}{2^4}x^{-7/2} \qquad f^{(4)}(1)=-\dfrac{15}{16}$

We first check if $T_2(x)$ is a good enough approximation.

$\left|f^{(3)}(x)\right|=\dfrac{3}{2^3}\dfrac{1}{x^{5/2}}\leq\dfrac{3}{2^3}\dfrac{1}{(.97)^{5/2}}\backsim 0.4046708595=M$

on $[.97, 1.03]$ since the fraction is biggest when the denominator is smallest.

The error in using $T_2(x)$ to approximate $\sqrt{x}$ is then $|R_2(x)|\leq\dfrac{M}{(3)!}|.03|^3\backsim .0000018$

This is not good enough . We next check if $T_3(x)$ is.

$\left|f^{(4)}(x)\right|=\dfrac{15}{16}\dfrac{1}{x^{7/2}}\leq\dfrac{15}{16}\dfrac{1}{(.97)^{7/2}}\backsim 1.042966133=M$ on the interval $[.97, 1.03]$.

The error in using $T_3(x)$ to approximate $\sqrt{x}$ is then $|R_3(x)|\leq\dfrac{M}{(4)!}|.03|^4\backsim .000000035$

This is a good enough approximation.

$$T_3(x)=f(1)+f'(1)(x-1)+\frac{f^{(2)}(1)(x-1)^2}{2}+\frac{f^{(3)}(1)(x-1)^3}{6}$$
$$=1+\frac{1}{2}(x-1)-\frac{1}{8}(x-1)^2+\frac{3}{48}(x-1)^3$$

The approximation to $\sqrt{1.03}$ is then $T_3(1.03)=1.0148891875$

SkillMasters for Chapter 8

SkillMaster 8.1:	Find a defining equation for a sequence.
SkillMaster 8.2:	Use the laws of limits together with known examples to determine if a sequence is divergent or convergent, and if convergent, to find the limit.
SkillMaster 8.3:	Determine if a sequence is monotone and use the Monotonic Sequence Theorem to show some sequences are convergent.
SkillMaster 8.4:	Determine if a geometric series is convergent, and if convergent, find the sum of the series.
SkillMaster 8.5:	Use the laws of series together with known examples to determine if a series is divergent or convergent, and if convergent, find the sum.
SkillMaster 8.6:	Use the Integral Test to determine if a series with positive decreasing terms is convergent.
SkillMaster 8.7:	Use the Comparison Test to determine whether a series is convergent or divergent.
SkillMaster 8.8:	Estimate the error in approximating the sum of a series.
SkillMaster 8.9:	Use the Alternating Series Test to determine if an alternating series is convergent.
SkillMaster 8.10:	Use the Alternating Series Estimation Theorem to estimate the error in approximating the sum of an alternating series by a finite sum.
SkillMaster 8.11:	Determine whether a series is absolutely convergent.
SkillMaster 8.12:	Use the ratio test to determine whether a series is absolutely convergent.
SkillMaster 8.13:	Find the radius of convergence and the interval of convergence of a power series.
SkillMaster 8.14:	Represent a function as a power series using the known example $1/(1-x)$.
SkillMaster 8.15:	Represent a function as a power series by differentiating and integrating known examples.

SkillMaster 8.16:	Find the Taylor and Maclaurin series for a function using the definition.
SkillMaster 8.17:	Use the Binomial Series to find power series representations for functions.
SkillMaster 8.18:	Find an upper bound for the error of approximation of a function by its Taylor polynomial.
SkillMaster 8.19:	Find how many terms in a Taylor polynomial are necessary for the error to be less than a specific amount.