

# Mathematical Optimization of Lightweight Bridge Structures

A comparative study of solid, hollow, and tapered beams

Gabriel Cirker

December 6, 2025

## Abstract

Designing lightweight bridge girders demands a trade-off between minimizing material usage and maintaining adequate stiffness and stability. This paper investigates how geometric design influences deflection and buckling performance for beam-like structures of equal mass. Using the Euler–Bernoulli beam theory and finite difference discretization, we compare three archetypal geometries—a uniform solid rectangular beam, a hollow rectangular box section, and a parabolically tapered beam. For each configuration, we compute static deflections under a midspan point load and estimate the critical buckling load via a generalized eigenvalue problem. These results are distilled into a nondimensional strength-to-weight ratio, which serves as a measure of efficiency. Our numerical experiments show that redistributing material away from neutral regions and toward areas of high stress markedly improves both stiffness and buckling capacity, with the tapered design offering the greatest gains.

## 1 Introduction

Modern structural engineering strives to design elements that are as light as possible while safely carrying prescribed loads. Reducing mass lowers material cost, eases transportation and erection, and can enhance dynamic performance. However, aggressive material savings risk excessive deflections or premature buckling if the underlying geometry is not carefully chosen. For slender beam-like members, the Euler–Bernoulli theory relates bending stiffness to the flexural rigidity  $EI$ , where  $E$  is the material’s Young’s modulus and  $I$  is the cross-sectional second moment of area. By choosing the cross-section wisely, designers can maintain stiffness and stability without adding material.

This paper frames lightweight bridge design as a mathematical optimization problem: given a fixed mass of material, what geometry maximizes the strength-to-weight ratio? We focus on three simplified geometries that represent idealized bridge girders: (i) a uniform solid rectangular beam, (ii) a hollow rectangular box section, and (iii) a parabolically tapered beam whose depth varies along the span. All three designs are scaled to have the same total mass for a fair comparison. We then assess their serviceability by computing static deflections under a midspan point load, and assess their ultimate limit state by estimating critical buckling loads under axial compression. From these quantities we derive a dimensionless strength-to-weight ratio (SWR), defined as the critical load divided by weight, which encapsulates performance independent of mass.

The rest of the paper is organized as follows. Section 2 summarizes the governing beam equations and defines the three geometries. Section 3 describes the finite difference discretization used for deflection and buckling analyses. Section 4 presents numerical results, comparing deflection curves, critical loads, and strength-to-weight ratios. Section 5 discusses the implications of these findings, and Section 6 offers concluding remarks.

## 2 Mathematical Model

Consider a beam of span  $L$  with vertical deflection  $y(x)$  under a transverse load. In the Euler-Bernoulli theory, the bending moment is proportional to curvature:  $M(x) = -EI(x)y''(x)$ . Equilibrium under a distributed load  $q(x)$  requires

$$EI(x)y^{(4)}(x) = q(x), \quad 0 < x < L, \quad (1)$$

subject to four boundary conditions. We model the beam as simply supported at both ends, imposing  $y(0) = y(L) = 0$  and  $y''(0) = y''(L) = 0$ , which implies zero vertical displacement and zero bending moment at the supports.

The flexural rigidity depends on the material ( $E$ ) and the cross-sectional shape through the second moment of area  $I(x)$ . We examine three geometries, all with the same material density  $\rho$  and Young's modulus  $E$ :

- (i) **Solid rectangular beam.** A uniform cross-section of width  $b$  and height  $h$  has constant second moment  $I = bh^3/12$ .
- (ii) **Hollow rectangular beam.** A thin-walled box section with outer width  $b$ , outer height  $h$ , and uniform wall thickness  $t$  has second moment  $I = \frac{bh^3 - b_i h_i^3}{12}$ , where  $b_i = b - 2t$  and  $h_i = h - 2t$  are the inner dimensions.
- (iii) **Tapered beam.** The width is constant ( $b$ ) but the height varies parabolically along the span:  $h(x) = h_0[1 - \alpha(2x/L - 1)^2]$ . The second moment at  $x$  is  $I(x) = bh(x)^3/12$ . The parameter  $h_0$  is chosen so that the total mass matches that of the baseline solid beam.

All designs are scaled to have the same mass  $M = \rho AL$ , where  $A$  is the cross-sectional area (or its integral for the tapered beam). A full derivation of the tapered mass scaling and hollow thickness selection is given in the source code accompanying this paper.

## 3 Numerical Method

To solve the deflection equation with variable  $I(x)$  and to estimate buckling loads, we discretize the span into  $N$  interior nodes and use finite difference approximations. A five-point stencil approximates the fourth derivative for deflection, while a three-point stencil approximates the second derivative for buckling. The resulting systems take the form

$$Ky = f, \quad (\text{static deflection}), \quad (2)$$

$$Ay = Py, \quad (\text{buckling eigenvalue}), \quad (3)$$

where  $K$  and  $A$  are sparse matrices incorporating the spatially varying flexural rigidity. For buckling, we solve a generalized eigenvalue problem and take the smallest positive eigenvalue  $P$  as the critical load. Further details of the finite difference scheme are available in the research code.

## 4 Results

We consider a beam of length  $L = 1$  m, density  $\rho = 7850$  kg/m<sup>3</sup>, and Young's modulus  $E = 200$  GPa, subjected to a point load of 1000 N at midspan. The baseline solid beam has width  $b = 0.02$  m and height  $h = 0.04$  m, giving a total mass of  $M = \rho b h L \approx 6.28$  kg. The hollow beam and tapered beam are scaled to have the same mass.

#### 4.1 Deflection profiles

Figure 1 plots the static deflection curves for the three geometries. The tapered beam exhibits the smallest maximum deflection, about 0.14 mm, while the solid and hollow beams both deflect approximately 0.25 mm. The similarity between the solid and hollow curves arises because the hollow walls are thin, so the second moment of area is only modestly larger than that of the solid beam for the same mass. The tapered beam, however, concentrates material where bending moments are highest, significantly increasing stiffness.

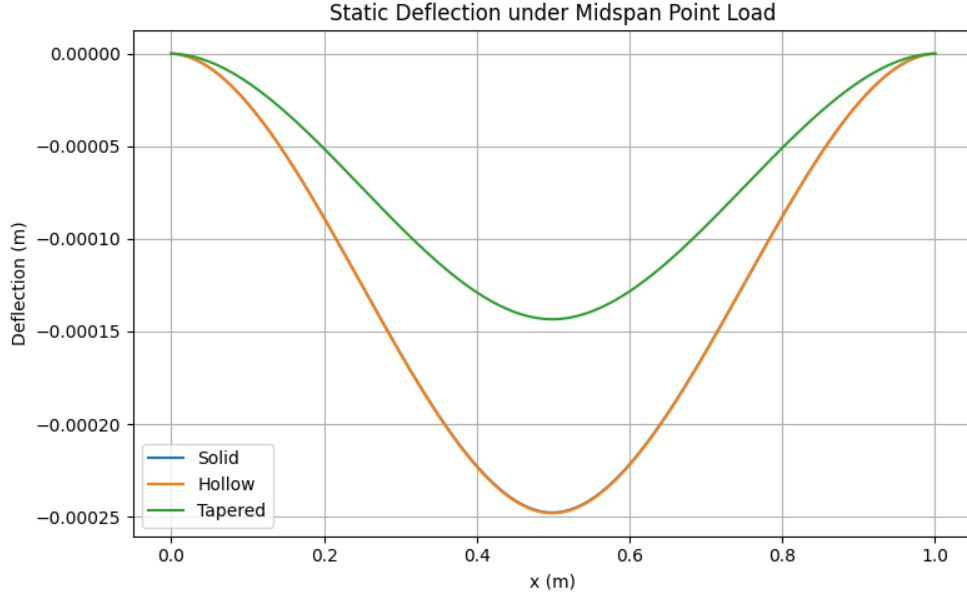


Figure 1: Static deflection under a midspan point load for solid, hollow, and tapered beams of equal mass. Deflections are plotted over the span  $x \in [0, L]$ ; the tapered beam exhibits the least deflection.

#### 4.2 Buckling loads and strength-to-weight ratios

Table 1 summarizes the key metrics for each geometry. The tapered beam has a critical buckling load of approximately  $2.77 \times 10^5$  N, substantially higher than the  $2.11 \times 10^5$  N for the solid and hollow beams. When normalized by weight, the resulting strength-to-weight ratio shows a similar trend: the tapered design achieves a 30% increase in efficiency over the solid and hollow designs.

Table 1: Comparison of maximum deflection, critical buckling load, and strength-to-weight ratio (SWR) for beams of equal mass. Deflection is measured at midspan under a 1000 N point load.

Geometry	Max deflection (mm)	$P_{cr}$ (kN)	SWR ( $\times 10^3$ )
Solid	0.248	210.5	3.42
Hollow	0.248	210.3	3.41
Tapered	0.143	276.7	4.49

## 5 Discussion

The numerical results highlight how geometry affects both serviceability and stability. Although the hollow beam redistributes material away from the neutral axis, its thin walls offer only a modest increase in second moment of area relative to the solid beam for the same mass. Consequently, its deflection and buckling performance are nearly identical to those of the solid design. The tapered beam, on the other hand, varies its depth along the span, concentrating material where bending moments are largest and thinning where they are small. This strategy yields a much larger flexural rigidity near midspan and, thus, markedly improves both stiffness and buckling resistance without adding mass.

From an optimization perspective, these findings suggest that continuous variation of section height is more effective than hollowing out a uniform section when the goal is to maximize efficiency for a given mass. In practice, fabricating a tapered girder may be more challenging than manufacturing hollow sections, and local stability of thin walls must be checked. Nonetheless, the advantages of tapering are clear in this idealized model and warrant further investigation in more realistic contexts.

## 6 Conclusion

We have presented a comparative study of three lightweight bridge beam geometries with equal mass. Using finite difference models of deflection and buckling, we showed that a parabolically tapered beam exhibits significantly higher strength-to-weight ratio and lower deflection than uniform solid or hollow beams. This improvement arises from a more efficient distribution of material, concentrating stiffness where it is most needed. Future work could explore other tapering functions, the effects of distributed loads, and potential manufacturing constraints.