

The Ticket Restaurant Assignment Problem

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Introduction

In this report we detail a branch-and-bound exact method based on Lagrangean relaxation to solve the ticket restaurant assignment problem (TRAP), pursuant to the project specifications set out for the Operational Research Complements course of the Università degli Studi di Milano¹.

In Section ?? we illustrate the dataset used in the experiments. In Section ?? we briefly describe the algorithm and its implementation. In Section ?? we show the results of our experiments and provide comments on them. Finally, Section 4 contains our concluding remarks.

1 Ticket Restaurant Assignment Problem

In this section we define the TRAP (Section 1.1), we formalize it as a mathematical programming problem (Section 1.2) and we examine the relevant literature (Section 1.3).

1.1 Definition

The TRAP is defined as follows. A ticket company (TC) possesses two kinds of restaurant tickets: the low-profit tickets and the high-profit tickets. TC gives a certain amount of tickets to customer companies (CC). Each CC receives only one kind of tickets. Each CC has different sets of employees and each set uses the tickets to buy meals in a specific restaurant.

For each restaurant with which TC has a deal, a given ratio between low-profit and high-profit tickets must be observed. The TC must maximize the profit while complying with this constraint. Thus, maximizing the profit amounts to minimizing the number of low-profit tickets while ensuring that a given amount of low-profit tickets is assigned for each considered restaurant.

¹ <https://homes.di.unimi.it/righini/Didattica/ComplementiRicercaOperativa/ComplementiRicercaOperativa.htm>

1.2 Formalization

Let $I = \{1, \dots, m\}$ be a set of restaurants, $J = \{1, \dots, n\}$ be a set of customer companies, $b \in \mathbb{Z}_+^m$ be the vector representing the minimum amount of low-profit tickets to be assigned for each restaurant, and $A \in \mathbb{Z}_+^{m \times n}$ be the matrix representing the amount of low-profit tickets assigned to each given customer company for each given restaurant, so that a_{ij} is the amount of low-profit tickets assigned to customer company j for restaurant i . Then, the integer linear programming model of the ticket restaurant assignment problem is the following:

$$\begin{aligned} \min \quad & z = \sum_{j \in J} \left(\sum_{i \in I} a_{ij} \right) x_j \\ \text{subject to} \quad & \sum_{j \in J} a_{ij} x_j \geq b_i \quad i = 1, \dots, m, \\ & x_j \in \{0, 1\} \quad j = 1, \dots, n, \end{aligned} \tag{1}$$

where each binary variable x_j indicates whether customer company j is assigned low-profit tickets. We define a cover a vector x such that $Ax \geq b$.

From (1), the following Lagrangean relaxation (LR) is obtained:

$$\begin{aligned} \min \quad & z_{\text{LR}} = \sum_{j \in J} \left(\sum_{i \in I} (1 - \lambda_i) a_{ij} \right) x_j + \sum_{i \in I} \lambda_i b_i \\ \text{subject to} \quad & x_j \in \{0, 1\} \quad j = 1, \dots, n, \\ & \lambda_i \geq 0 \quad i = 1, \dots, m, \end{aligned} \tag{2}$$

where λ_i are the Lagrangean multipliers.

Finally, the dual of the linear programming (LP) relaxation of (1) is:

$$\begin{aligned} \max \quad & w = \sum_{i \in I} b_i y_i \\ \text{subject to} \quad & \sum_{i \in I} a_{ij} y_i \leq c_j \quad j = 1, \dots, n, \\ & y_i \geq 0 \quad i = 1, \dots, m, \end{aligned} \tag{3}$$

where $c_j = \sum_{i \in I} a_{ij}$ for all $j \in J$. The dual of the LP relaxation may be exploited to find optimal Lagrangean multipliers (see Section 2.2).

The TRAP is a generalization of the set covering problem (SCP), as explained in the following section. The SCP is NP-hard [4], hence so is the TRAP.

1.3 Related Works

Set covering problem. The problem at hand is equivalent to the SCP when A is a binary matrix and b is an all-ones vector. The SCP has been treated extensively in literature and the ideas developed in that context can be leveraged

to solve the TRAP. For the SCP, both exact and heuristic algorithms have been devised (see the survey in [4]). We focus only on exact algorithms for our problem.

The following works in the field of SCP are of particular interest to us. Balas and Ho [2] adopted a branch-and-cut procedure, comprising a primal heuristic to find upper bounds and subgradient optimization to find lower bounds. Beasley [3] adopted a branch-and-bound procedure, computing upper bounds with a greedy heuristic, computing lower bounds with dual ascent and subgradient optimization, and devising a number of problem reduction techniques. Balas and Carrera [1] adopted a dynamic subgradient-based branch-and-bound procedure to solve the SCP, additionally employing variable fixing techniques and heuristics to obtain upper bounds.

We fashioned our branch-and-bound procedure over the algorithms described in these works, adapting their solutions to account for the peculiarities of the TRAP. In particular, these peculiarities preclude the use of techniques that rely on the fact that each given constraint can be satisfied by setting to 1 exactly one variable with a non-zero coefficient in the row corresponding to that constraint.

Multicovering problem. A problem more closely related to the TRAP is the one called multicovering problem (MCP) [7,8], defined as follows:

$$\begin{aligned} \min \quad & z = c^\top x \\ \text{subject to} \quad & Ax \geq b, \\ & x_j \in \{0, 1\} \quad \forall j = 1, \dots, n, \end{aligned}$$

where A is a binary matrix and b is a vector of positive integers. This problem differs from the one at hand in that A is binary and not simply non-negative.

Notably, Hall and Hochbaum [8] adopted a branch-and-cut procedure to solve the MCP, using a primal heuristic to find upper bounds and combining a dual heuristic with subgradient optimization to find lower bounds. From this work we derived the main primal heuristic for our branch-and-bound algorithm (see Sections 2.1 and 3.3).

Covering integer problem. Finally, a generalization of the TRAP may be identified in the so-called covering integer problem (CIP) [9,10], defined as follows:

$$\begin{aligned} \min \quad & z = c^\top x \\ \text{subject to} \quad & Ax \geq b, \\ & x_j \leq d_j \quad \forall j = 1, \dots, n, \\ & x_j \in \mathbb{Z}_+ \quad \forall j = 1, \dots, n, \end{aligned}$$

where all the entries in A , b , c and d are non-negative. This problem is equivalent to the TRAP when the multiplicity constraints $x_j \leq d_j$ force x to be binary. Unfortunately, we found only approximation algorithms for the CIP (see [9,10]).

2 Branch-and-Bound

In this section we describe the different aspects of the branch-and-bound algorithm we experimented upon to solve the TRAP: the primal heuristics used to find upper bounds (Section 2.1); the techniques used to find lower bounds (Section 2.2); the branching rules (Section 2.3); and the variable fixing strategies (Section 2.4).

2.1 Primal Heuristics

Greedy heuristic. This method selects greedily the variables of the cover, picking the row i with the largest ratio between the sum $\sum_{j \in J} a_{ij}$ and b_i and then picking the column with the largest coefficient. Each time a variable is selected, the coefficients on the left-hand side (LHS) and right-hand side (RHS) are decreased accordingly.

In this way we select the row which is easiest to cover, since a larger ratio means that more solutions might satisfy the constraint. Indeed, when the sum $\sum_{j \in J} a_{ij}$ is equal to b_i , we must select all variables with non-zero entries in A_i in order to satisfy the constraint, thus restricting the number of possible solutions. By picking the column with the largest coefficient we aim to satisfy the covering constraint in the fastest way.

Dobson heuristic. This heuristic is taken from Dobson [6] and generalizes an heuristic for the SCP from Chvatal [5]. This heuristic picks the column j that minimizes $c_j \sum_{i=1, \dots, m} a_{ij}$. Furthermore, any a_{ij} larger than b_i is lowered down to b_i .

The idea of this method is to pick the column that covers the most while costing the least. In the case of the TRAP, however, the cost of each column is equal to the sum of the coefficients of that column, which means that all columns could be equivalently selected. The only exception to this occurs when a column j is enough to cover the remaining part of b_i for any i . In this case, a_{ij} would be lowered and therefore $c_j \sum_{i=1, \dots, m} a_{ij}$ would also be lowered.

In short, this method, applied to the TRAP, prioritizes the columns whose coefficients are enough to cover the remaining part of b_i for any i .

Hall-Hochbaum heuristic. This method adapts one of the heuristics conceived by Hall and Hochbaum [8]. Specifically, this method picks the column j that maximizes $\frac{1}{c_j} \sum_{i \in L} \frac{b_i a_{ij}}{\text{space}(i)}$, where $L = \{i \in I : b_i > 0\}$ and $\text{space}(i) = \sum_{j \in J} a_{ij} - b_i$.

2.2 Lower Bounds

Lagrangean relaxation. A lower bound to the optimal value of problem (1) can be obtained by solving the LR defined in (2). When the Lagrangean multipliers λ are given, the objective function of (2) is trivially minimized by setting

x_j to 1 when $\sum_{i \in I} (1 - \lambda_i) a_{ij} < 0$ and to 0 otherwise. The only problem left is to find the optimal Lagrangean multipliers, yielding the highest lower bound.

Subgradient optimization is a popular algorithm to find the optimal Lagrangean multipliers. Our version of subgradient optimization is adapted from [1] and is described in Algorithm 1.

Algorithm 1: Subgradient optimization

```

Input:  $A, b, z_{LB}, z_{UB}; f, k, \epsilon, \omega$ 
 $t \leftarrow 1$ 
 $\lambda_i^t = 0 \ \forall i \in I$ 
 $\lambda_{\text{best}} \leftarrow \lambda^t$ 
 $z_{\text{best}} \leftarrow z_{LB}$ 
while  $z_{UB} > z_{LB}$  do
  for  $j \in J$  do
    if  $\sum_{i \in I} (1 - \lambda_i^t) a_{ij} < 0$  then
       $x_j \leftarrow 1$ 
    end
    else
       $x_j \leftarrow 0$ 
    end
  end
   $L(\lambda^t) \leftarrow \sum_{j \in J} (\sum_{i \in I} (1 - \lambda_i^t) a_{ij}) x_j + \sum_{i \in I} \lambda_i^t b_i$ 
  if  $L(\lambda^t) > z_{LB}$  then
     $z_{LB} \leftarrow L(\lambda^t)$ 
     $z_{\text{best}} \leftarrow z_{LB}$ 
     $\lambda_{\text{best}} \leftarrow \lambda^t$ 
  end
   $g(\lambda^t) \leftarrow b - A^\top x$  /* gradients */
  if  $z_{LB}$  unchanged for  $k$  iterations then
     $f \leftarrow \frac{f}{2}$ 
  end
   $\sigma^t \leftarrow \frac{f(z_{UB} - z_{LB})}{\|g(\lambda^t)\|^2}$  /* step length */
  for  $i \in I$  do
     $\lambda_i^{t+1} \leftarrow \max(0, \lambda_i^t + \sigma^t g_i^t(\lambda^t))$ 
  end
  if  $f < \epsilon \vee t > \omega$  then
    break
  end
   $t \leftarrow t + 1$ 
end
return  $z_{\text{best}}, \lambda_{\text{best}}$ 

```

When searching for the best Lagrangean multipliers in (2), we cannot assume that the Lagrangean multipliers can be confined within the range $[0, 1]$.

Lemma 1. *The optimal Lagrangean multipliers may not lie in the range $[0, 1]$.*

Proof. Consider a matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 4 \\ 2 & 2 & 2 \end{bmatrix}$ and a vector $b = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}$. If we run Algorithm 1 with parameters $f = 2$, $k = 5$, $\epsilon = 0.005$, $\omega = 150$ and we constrain each Lagrangean multiplier within the range $[0, 1]$, we obtain $z_{LB} = 8$ and $\lambda = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Running the algorithm with the same parameters, but without bounds on the Lagrangean multipliers, we obtain $z_{LB} = 10.498$ and $\lambda = \begin{bmatrix} 0 \\ 2.259 \\ 0 \end{bmatrix}$. \square

Another option to find the optimal Lagrangean multipliers is to solve the dual of the LP relaxation of 1, which we defined in Section 1.2. In the context of the SCP, this method was adopted, for instance, in [2, 3]; and it is mentioned as a viable approach in [4], due to the fact that the dual of the LP relaxation has the integrality property.

LP relaxation. Finally, one could consider solving the LP relaxation of (1) to obtain a lower bound. In fact, it is observed in [4] that “the lower bound determined by Lagrangian or alternative relaxations is much worse than the optimal solution value of the LP relaxation”. The same Authors remark that exact algorithms may behave better with LP relaxation, whereas Lagrangean relaxation may be better suited for heuristic algorithms.

2.3 Branching Rules

Reduced costs branching. For this branching rule we compute the reduced costs r of the variables with the formula $r = (1 - \lambda) \cdot A$. After that, we determine a solution by setting to 0 all the variables not fixed to 1 in the current node. Considering the computed solution, we select the variable x_j with the minimum reduced cost and a non-zero coefficient in the row with the largest violation. Finally, we generate two children nodes, fixing x_j equal to 0 in the first and equal to 1 in the second.

Cost branching for LP. This branching rule follows the same strategy as the reduced costs branching, except for the fact that we select the variable with the minimum cost instead of the variable with the minimum reduced cost. This modification allows to employ this rule with the LP relaxation, since in that case we do not have the Lagrangean multipliers (or dual variables) λ .

Beasley branching. This is the branching rule used in [3] and differs from the reduced costs branching rule only for the fact that we select the row whose corresponding Lagrangean multiplier has the largest value instead of the row with the largest violation.

2.4 Reduction

Lagrangean penalties. Following [3], we use the reduced costs to fix variables to 0 or to 1, thus reducing the size of the problem instances.

Let $r = (1 - \lambda) \cdot A$ be the vector of reduced costs, z_{LB} be the lower bound of the current node and z_{UB} be the best incumbent upper bound. Then we can fix x_j to 0 when $r_j \geq 0$ and $z_{LB} + r_j > z_{UB}$, and we can fix x_j to 1 when $r_j < 0$ and $z_{LB} - r_j > z_{UB}$.

Column inclusion. This reduction method is adapted from [3] and consists in fixing to 1 all the unassigned variables with a non-zero coefficient in an uncovered row, when said row cannot be covered otherwise.

3 Computational Results

In this section we describe the machine and the technologies used to run the experiments (Section 3.1), we illustrate how the problems were generated (Section 3.2), and then we show the results of our tests (Section 3.3).

3.1 Technologies and Hardware

We ran the branch-and-bound algorithm on a machine with 16 gigabytes of RAM and a CPU Intel(R) Core(TM) i7-9700K 3.60GHz with 8 cores. The code was implemented in Python 3.11.2.

We include in our comparisons the results obtained with a state-of-the-art commercial solver, Gurobi 10.0.2². Note that Gurobi, unlike our implementation, exploits multi-threading.

3.2 Problem Generation

We generate random TRAP instances by determining the number of constraints (m) and variables (n) and the density of the problem. The density is the percentage of non-zero coefficients in each row of the matrix A .

For each row, we pick the value of b_i uniformly at random between n and n^2 . After that, the sum of the coefficients on the LHS is picked uniformly at random between $2b_i$ and $5b_i$. Then, we determine the number of columns with a non-zero coefficient using the density parameter, picking their indices uniformly at random. Finally, the sum of the coefficients on the LHS is distributed uniformly at random among the selected column indices.

² <https://www.gurobi.com/>

3.3 Comparative Analysis

Primal heuristics. Table 1 compares the lower bounds obtained by the different primal heuristics over sets of randomly generated TRAP instances with different numbers of rows and columns. The heuristic by Hall and Hochbaum outperforms the other primal heuristics as the size of the problem increases. For this reason, only this heuristic was employed in the subsequent experiments.

Table 1. Number of times each primal heuristic provided the solution with the lowest upper bound over sets of 10 randomly generated TRAP instances with different numbers of rows and columns and with density equal to 0.5. The best result for each set of problems is highlighted in bold.

		Greedy	Dobson	Hall-Hochbaum
Rows	Cols			
5	10	1	7	2
10	20	1	3	6
20	50	0	0	10
50	100	0	0	10

Subgradient optimization parameters. We set the subgradient optimization parameters as: $f = 2$, $k = 5$, $\epsilon = 0.005$, $\omega = 150$. Setting $\omega = 150$ is enough to find the optimal Lagrangean multipliers for problems having up to 80 rows and up to 150 columns, as shown in Figure 1.

3.4 Runtime and number of nodes.

We compare different configurations for the branch-and-bound algorithm:

- S: reduced costs branching rule and subgradient optimization lower bound, without any primal heuristic and without any reduction technique;
- SP: same as S but with Hall-Hochbaum primal heuristic;
- SPR: same as SP but with both the reduction techniques described in Section 2.4;
- SPRR: same as SPR but with the primal heuristic ran only at the root node;
- SPRB: same as SPR but with Beasley branching rule instead of reduced costs branching rule;
- LP: LP branching rule, LP relaxation lower bound, Hall-Hochbaum primal heuristic and column inclusion reduction technique.

Tables 2 and 3 respectively give the runtime and number of nodes for each of these configurations over randomly generated TRAP instances. We forcibly terminated the algorithm when the runtime exceeded 5 minutes. Both tables show how SPRB outperforms the other configurations for almost every problem

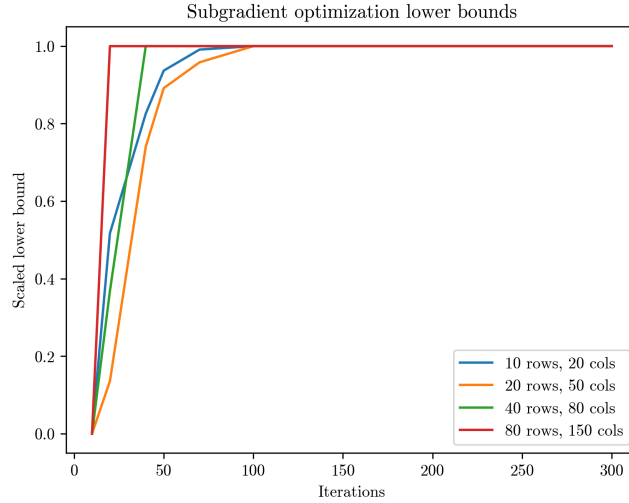


Fig. 1. Min-max scaled lower bounds obtained with subgradient optimization for different numbers of iterations.

instance, especially as the size of the instance increases. Even so, the algorithm performs poorly when compared with Gurobi. Contrary to what was observed in Section 2.2, LP yields the worst results over all problem instances.

A look at the logs of Gurobi reveals that the solver employs a branch-and-cut algorithm, performing root relaxation and finding heuristic solutions periodically. Some of the mentioned cuts are: Gomory, Cover, MIR, StrongCG, Mod-K, Zero half, RLT.

4 Conclusions

In this work we described and evaluated a branch-and-bound algorithm to solve the TRAP, employing reduction techniques, different strategies to find upper bounds and lower bounds, and different branching rules.

We were unable

More literature, especially CIP.

Further research might explore a wider set of hyperparameters, especially in the direction of lower regularization coefficient values and higher values of γ when using a Gaussian kernel.

References

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Table 2. Runtime of different configurations of the branch-and-bound algorithm over randomly generated TRAP instances with different numbers of rows and columns and different densities. Configurations whose runtime exceeded 5 minutes were assigned nan values.

Rows	Cols	Density	Gurobi	S	SP	SPR	SPRR	SPRB	LP
5	10	0.3	0.05	0.00	0.00	0.00	0.00	0.00	0.02
		0.5	0.00	0.03	0.08	0.00	0.00	0.05	0.06
		0.7	0.00	0.02	0.02	0.02	0.02	0.06	0.14
10	20	0.3	0.00	4.86	16.27	4.17	2.91	1.64	31.47
		0.5	0.02	12.47	20.56	7.38	4.22	9.66	68.22
		0.7	0.00	29.30	36.59	18.12	12.05	9.81	118.48
13	22	0.3	0.00	20.84	33.45	7.59	5.36	3.55	64.22
		0.5	0.02	29.53	38.27	17.14	14.64	10.72	288.70
		0.7	0.06	94.36	116.97	43.14	29.98	42.47	nan
15	25	0.3	0.00	62.98	93.23	24.17	17.77	9.36	nan
		0.5	0.00	nan	nan	161.31	113.97	38.22	nan
		0.7	0.06	nan	nan	291.48	248.05	127.39	nan

Table 3. Number of nodes generated by different configurations of the branch-and-bound algorithm over randomly generated TRAP instances with different numbers of rows and columns and different densities. Configurations whose runtime exceeded 5 minutes were assigned nan values.

Rows	Cols	Density	Gurobi	S	SP	SPR	SPRR	SPRB	LP
5	10	0.3	0	33	21	5	5	7	23
		0.5	1	211	203	49	51	55	183
		0.7	1	263	253	63	71	49	323
10	20	0.3	1	19973	39189	6313	6323	2231	55541
		0.5	50	54737	50515	9833	10431	15731	124795
		0.7	25	97731	80731	21107	22757	13049	209799
13	22	0.3	1	85147	82659	9889	12159	4641	117271
		0.5	1	107217	85605	25719	33459	11809	416059
		0.7	1	255533	202681	39033	44051	38831	nan
15	25	0.3	1	183249	145459	21561	27331	7761	nan
		0.5	59	nan	nan	151895	180799	35241	nan
		0.7	384	nan	nan	266983	365631	113737	nan

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