model specifications

$$\underline{y_i} = Z\underline{\alpha_i} + \underline{\theta_i} + \underline{\varepsilon_i}$$

$\theta_{it} = \rho_i \theta_{i,t-1} + v_{it}$ $\quad v_{it} \sim N(0, \sigma_i^2)$

$i = 1, \dots, n$ where $\underline{y_i} = [y_{i1}, \dots, y_{iT}]$

$\varepsilon_i \sim N_T(\underline{0}, \sigma^2_{\varepsilon_i} \mathbb{1}_T)$

$\alpha_i \overset{ind\,id}{\sim} N_p(0, \Sigma_\alpha)$ with $\Sigma_\alpha = \begin{bmatrix} \sigma_{\alpha_1} & & \\ & \ddots & \\ & & \sigma_{\alpha_p} \end{bmatrix}$

$\gamma_i = (\sigma_i^2, \rho_i)$

$\theta_i \sim N_T(\underline{0}, R(\rho_i))$ $\quad R_{ls} = \sigma_i^2 \rho_i^{|l-s|}$

$\gamma_i | P \overset{iid}{\sim} P$

$P \sim Dir(\alpha^P, P_0)$

$\sigma_{\varepsilon_i}^2 \sim IGa(c_0^\varepsilon, c_1^\varepsilon)$ $\quad \sigma_{\alpha_k}^2 \sim IGa(c_0^\alpha, c_1^\alpha)$

$$P_0(\gamma_i) = P_0(\sigma_i^2) \times P_0(\rho_i) \propto IGa(a, b) \times \sqrt{\frac{1+\rho_i^2}{1-\rho_i^2}}$$

$\alpha_i \sim N_p(\underline{0}, \Sigma_\alpha)$ $\quad \Sigma_\alpha = diag(\sigma_{\alpha_1}, \dots, \sigma_{\alpha_p})$

$$f(\underline{y} | \underline{\alpha}, \underline{\theta}, \sigma_\varepsilon) = \prod_{i=1}^{n} N_T(\underline{y_i} | Z\alpha_i + \theta_i, \sigma^2_{\varepsilon_i} \mathbb{1}_T)$$

(1) We'll make use of some well-known properties :

Marginal and Conditional Gaussians

Given a marginal Gaussian distribution for $\mathbf{x}$ and a conditional Gaussian distribution for $\mathbf{y}$ given $\mathbf{x}$ in the form

$$p(\mathbf{x}) = \mathcal{N}(\mathbf{x}|\boldsymbol{\mu}, \boldsymbol{\Lambda}^{-1}) \quad (2.113)$$
$$p(\mathbf{y}|\mathbf{x}) = \mathcal{N}(\mathbf{y}|\mathbf{A}\mathbf{x}+\mathbf{b}, \mathbf{L}^{-1}) \quad (2.114)$$

the marginal distribution of $\mathbf{y}$ and the conditional distribution of $\mathbf{x}$ given $\mathbf{y}$ are given by

$$p(\mathbf{y}) = \mathcal{N}(\mathbf{y}|\mathbf{A}\boldsymbol{\mu}+\mathbf{b}, \mathbf{L}^{-1}+\mathbf{A}\boldsymbol{\Lambda}^{-1}\mathbf{A}^{\mathrm{T}}) \quad (2.115)$$
$$p(\mathbf{x}|\mathbf{y}) = \mathcal{N}(\mathbf{x}|\boldsymbol{\Sigma}\{\mathbf{A}^{\mathrm{T}}\mathbf{L}(\mathbf{y}-\mathbf{b})+\boldsymbol{\Lambda}\boldsymbol{\mu}\}, \boldsymbol{\Sigma}) \quad (2.116)$$

where

$$\boldsymbol{\Sigma} = (\boldsymbol{\Lambda}+\mathbf{A}^{\mathrm{T}}\mathbf{L}\mathbf{A})^{-1}. \quad (2.117)$$

$$\theta_i \sim N_T(\underline{0}, R(\rho_i))$$

$$f(\underline{y}_i | \underline{\alpha}, \underline{\theta}_i, \sigma^2_{\varepsilon_i}) = N_T(\underline{y}_i | Z\underline{\alpha}_i + \mathbb{1}\,\underline{\theta}_i,\ \sigma^2_{\varepsilon_i}\mathbb{1}_T)$$

using 2.114

$$f(\underline{y}_i | \underline{\alpha}_i, \sigma^2_{\varepsilon_i}) = N_T(\underline{y}_i | \mathbb{1}_T \underline{0}_T + Z\underline{\alpha}_i,\ \mathbb{1}_T R(\rho_i)\mathbb{1}_T^T + \sigma_{\varepsilon_i}^2 \mathbb{1}_T)$$

$$f(\underline{y}_i | \underline{\alpha}, \sigma^2_{\varepsilon}) = N_T(\underline{y}_i | Z\alpha_i,\ \overbrace{R(\rho_i)+\sigma_{\varepsilon_i}^2\mathbb{1}_T}^{W_i})$$

$$\alpha_i | \Sigma_\alpha \sim N_p(\underline{0}, \Sigma_\alpha)$$

$$f(\underline{y}_i | \underline{\alpha}, \sigma^2_{\varepsilon}) = N_T(\underline{y}_i | \overset{A}{Z}\alpha_i, W_i)$$

Using 2.116

$$V_\alpha = (Z^T W_i^{-1} Z + \Sigma_\alpha^{-1})$$

$$f(\underline{\alpha}_i | y_i, \sigma_{\varepsilon_i}^2, \Sigma_\alpha) \sim N(\underline{\alpha}_i | V_\alpha Z^T W_i^{-1} y_i + \Sigma_\alpha^{-1}\underline{0})$$

$$f(\underline{\alpha}_i | y_i, \sigma_{\varepsilon_i}^2, \Sigma_\alpha) \sim N(\underline{\alpha}_i | V_\alpha Z^T W_i^{-1} y_i) \quad (1)$$

(2) $\sigma_{\varepsilon_i}^2 \sim IGa(c_0^{\varepsilon}, c_1^{\varepsilon})$

$\underline{y}_i = Z\underline{\alpha}_i + \underline{\theta}_i + \underline{\varepsilon}_i$ where $\underline{\varepsilon}_i \sim N_T(\underline{0}, \sigma^2_{\varepsilon_i}\mathbb{1}_T)$

$y_i \mid Z\alpha_i, \theta_i, \sigma^2_{\varepsilon_i} \sim N(Z\alpha_i + \theta_i, \sigma^2_{\varepsilon_i}\mathbb{1}_T)$

Inv. Gamma is conjugate prior for the Gaussian likelihood

$$\sigma^2_{\varepsilon_i} \mid y, \text{rest} \sim IGa\Big(\sigma^2_{\alpha_j} \mid c_0^{\varepsilon} + \frac{n}{2},\; c_1^{\varepsilon} + \frac{1}{2}(\underbrace{\underline{y}_i - Z\underline{\alpha}_i + \underline{\theta}_i}_{:=r_i})^T(\underline{y}_i - Z\underline{\alpha}_i + \underline{\theta}_i)\Big)$$

$$\sigma^2_{\varepsilon_i} \mid y, \text{rest} \sim IGa\Big(\sigma^2_{\alpha_j} \mid c_0^{\varepsilon} + \frac{n}{2},\; c_1^{\varepsilon} + r_i^T r_i\Big) \qquad (2)$$

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(3)

$\sigma_{\alpha_j}^2 \sim IGa(c_0^{\alpha}, c_1^{\alpha})$

$\underline{\alpha}_i \sim N_p(\underline{0}, \Sigma_\alpha)$ where $\Sigma_\alpha := \mathrm{diag}(\sigma^2_{\alpha_1}, \dots, \sigma^2_{\alpha_p})$

Inv. Gamma is conjugate prior for the Gaussian likelihood

$$\sigma^2_{\alpha_j} \mid y, \text{rest} \sim IGa\Big(c_0^{\alpha} + \frac{n}{2},\; c_1^{\alpha} + \frac{1}{2}\sum_{i=1}^{n}(\alpha_{ij} - 0)\Big) \qquad j = 1, \dots, p \qquad (3)$$

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(4) $\theta_i \sim N_T(\overset{\mu}{\underline{0}}, \overset{\Lambda^{-1}}{R(\rho_i)})$

$f(\underline{y}_i \mid \underline{\alpha}_i, \underline{\theta}_i, \sigma^2_{\varepsilon_i}) = N_T(\underline{y}_i \mid \overset{b}{Z\underline{\alpha}_i} + \overset{A}{\mathbb{1}_T}\underline{\theta}_i, \overset{L^{-1}}{\sigma^2_{\varepsilon_i}\mathbb{1}_T})$

Applying 2.116

$S_\theta = (R(\rho_i)^{-1} + (\sigma^2_{\varepsilon_i}\mathbb{1}_T)^{-1})^{-1}$

$$\underline{\theta}_i \mid y, \sigma^2_{\varepsilon_i}, R(\rho_i), \alpha_i \sim N_T\big(\theta_i; S_\theta(\sigma^2_{\varepsilon_i}\mathbb{1}_T)^{-1}(\underline{y}_i - Z\underline{\alpha}_i) + \cancel{R^{-1}(\rho_i)\underline{0}}, S_\theta\big) \quad (4)$$

let $c = (c_1, \dots, c_n)$ labels

Riparametrizzando $y_i \mid c_i, \gamma \sim F(\gamma_{c_i})$

$c_i \mid p \sim \text{Discrete}(p_1, \dots, p_k)$

$\gamma_c \sim p_0$

$p \sim \text{Dirichlet}\left(\frac{\partial^p}{D}, \dots, \frac{\partial^p}{D}\right)$

$p(y_i \mid \gamma_i, \rho_{\varepsilon_i}^2, \alpha_i) = N(y \mid Z\alpha_i, \sigma_{\varepsilon_i}^2 \mathbb{1}_T + R(\rho_i))$

<u>Algoritmo</u>

$$P(c_i = c \mid c_1, \dots, c_{-i}) = \frac{n_{ic} + \frac{\alpha}{k}}{i - 1 + \alpha}$$

For $i = 1, \dots, n$

Let m_i the number of distinct c_j for $j \neq i$

$h = m_i + n_{aux}$

Draw a new value for c_i from $\{1, \dots, h\}$ using the following probabilities:

$$P(c_i = c \mid c_{-i}, y_i, \gamma_1, \dots, \gamma_h) = \begin{cases} b \dfrac{n^*_{i,c}}{n - 1 + \partial^p} F(y_i \mid \gamma^*_c, \alpha_i, \sigma^2_{\varepsilon_i}) \\ b \dfrac{\alpha / n_{aux}}{n - 1 + \partial^p} F(y_i \mid \gamma^*_c, \alpha_i, \sigma^2_{\varepsilon_i}) \end{cases}$$

where n^*_{ic} is the number of c_j for $j \neq i$ are equal to c and b the appropiate normalizing constant.

the values of γ^*_c where $m_i < c \leq h$ are drawn independently from p^0

DATI SINTETICI

1. Fix Z
2. sample σ_{ε_i} and $\sigma_{\alpha_i}^2$ having fixed $c_0^{\varepsilon}, c_1^{\varepsilon}, c_0^{\alpha}, c_1^{\alpha}$

   $\sigma_{\varepsilon_i}^2 \sim IGa(c_0^{\varepsilon}, c_1^{\varepsilon})$

   $\sigma_{\alpha_i}^2 \sim IGa(c_0^{\alpha}, c_1^{\alpha})$
3. sample $(\rho, \sigma^2) \sim DP$ using stick-
4. $p(y_i | \rho_i, \sigma_{\varepsilon_i}^2) = N(y | Z\alpha_i, \sigma_{\varepsilon_i}^2 \mathbb{1}_T + R(\rho_i))$