UBINET/SI5: Performance Evaluation of Networks

Correction of homework 5

5.1 $M/G/1$ queue with priorities

1. The curve $R(t)$ is similar to the one in the $M/G/1$ FIFO queue, composed of half-square triangles separated by idle periods. The heights of the triangles come from two different distributions, according to the type of the customer being served.

2. The computation is similar to the one done for the $M/G/1$ FIFO queue without priority and without vacations. We need to account for the number of customers of each class that are served in an interval $(0, C)$, given that the system is empty at instants 0 and C.

$$
\begin{aligned}
\overline{R} &= \lim_{C\to\infty} \frac{1}{C} \left(\sum_{i=1}^{Y_1(C)} \frac{\sigma_{1,i}^2}{2} + \sum_{j=1}^{Y_2(C)} \frac{\sigma_{2,j}^2}{2} \right) \\
&= \lim_{C\to\infty} \left(\frac{Y_1(C)}{C} \right) \lim_{C\to\infty} \left(\frac{1}{Y_1(C)} \sum_{i=1}^{Y_1(C)} \frac{\sigma_{1,i}^2}{2} \right) + \lim_{C\to\infty} \left(\frac{Y_2(C)}{C} \right) \lim_{C\to\infty} \left(\frac{1}{Y_2(C)} \sum_{j=1}^{Y_2(C)} \frac{\sigma_{2,j}^2}{2} \right) \\
&= \lambda_1 \frac{E[\sigma_1^2]}{2} + \lambda_2 \frac{E[\sigma_2^2]}{2} \ .
\end{aligned}
$$

   As the system is at steady-state and stable, the output rate of each class is equal to its input rate.

3. The computation is similar to the one done for the $M/G/1$ FIFO queue without priority.

$$\overline{W}_1 = \overline{R} + \frac{\overline{X}_1}{\mu_1}$$

   and $\overline{X}_1 = \lambda_1 \overline{W}_1$ by applying Little's formula to type 1 customers in the waiting room. By combining the two equalities we find the result.

4. Let $\overline{X}_2$ denote the expected number of type 2 customers waiting in the queue, then $\overline{X}_1 = \lambda_1 \overline{W}_1$ by applying Little's formula to type 1 customers in the waiting room. Let $\overline{Z}_1$ denote the expected number of type 1 customers that arrive during $\overline{W}_2$. Since type 1 customers arrive according to a Poisson process with rate λ_1, by conditioning on the waiting room W_2 we get $\overline{Z}_1 = \lambda_1 \overline{W}_2$. We have

$$\overline{W}_2 = \overline{R} + \frac{\overline{X}_1}{\mu_1} + \frac{\overline{X}_2}{\mu_2} + \frac{\overline{Z}_1}{\mu_1} = \overline{R} + \rho_1 \overline{W}_1 + (\rho_1 + \rho_2) \overline{W}_2 \ .$$

5. By combining the results found in the previous two questions we derive

$$\overline{W}_2 = \frac{1}{1 - \rho_1 - \rho_2} \left(\overline{R} + \frac{\rho_1 \overline{R}}{1 - \rho_1} \right) = \frac{1}{1 - \rho_1 - \rho_2} \frac{\overline{R}}{1 - \rho_1} \ .$$

6. The stability condition is $\rho_1 + \rho_2 < 1$, otherwise the expression found for $\overline{W}_2$ does not make sense.

5.2 M/G/1 with different job types

1. The arrival process to the M/G/1 queue is the aggregation of two independent Poisson processes, it is a Poisson process whose rate is $\lambda = \lambda_B + \lambda_B = \frac{3}{4}$.

2. We first observe that a job picked at random is red with probability $\frac{\lambda_R}{\lambda} = \frac{1}{3}$ and is blue with probability $\frac{\lambda_B}{\lambda} = \frac{2}{3}$. The expected service time of a job picked at random is

$$\mathrm{E}\left[\sigma\right] = \frac{1}{3} \cdot 1 + \frac{2}{3} \cdot \frac{1}{2} = \frac{2}{3}.$$

3. The second moment of a variable X is $\mathrm{E}\left[X^2\right] = \mathrm{Var}(X) + (\mathrm{E}\left[X\right])^2$. Therefore, the second moment of red job sizes is $1 + 1^2 = 2$ and that of blue job sizes is $1 + 0.5^2 = \frac{5}{4}$. The second moment of a job picked at random is

$$\mathrm{E}\left[\sigma^2\right] = \frac{1}{3} \cdot 2 + \frac{2}{3} \cdot \frac{5}{4} = \frac{3}{2}.$$

4. The load of this M/G/1 queue is

$$\rho = \lambda \mathrm{E}\left[\sigma\right] = \frac{3}{4} \cdot \frac{2}{3} = \frac{1}{2}.$$

5. All jobs, red or blue, experience the same waiting time. We use the Pollaczek-Khinchin formula to write the expected waiting time

$$\overline{W} = \frac{\lambda \mathrm{E}\left[\sigma^2\right]}{2(1-\rho)} = \frac{\frac{3}{4} \cdot \frac{3}{2}}{2\left(1 - \frac{1}{2}\right)} = \frac{9}{8}.$$

6. To get the mean response time of a particular type of jobs we add its expected service time (job size) to its expected waiting time. We obtain

$$\overline{T}_R = 1 + \frac{9}{8} = \frac{17}{8}, \qquad \overline{T}_B = \frac{1}{2} + \frac{9}{8} = \frac{13}{8}.$$