Performance Evaluation of Networks

Sara Alouf

Objectives of Course

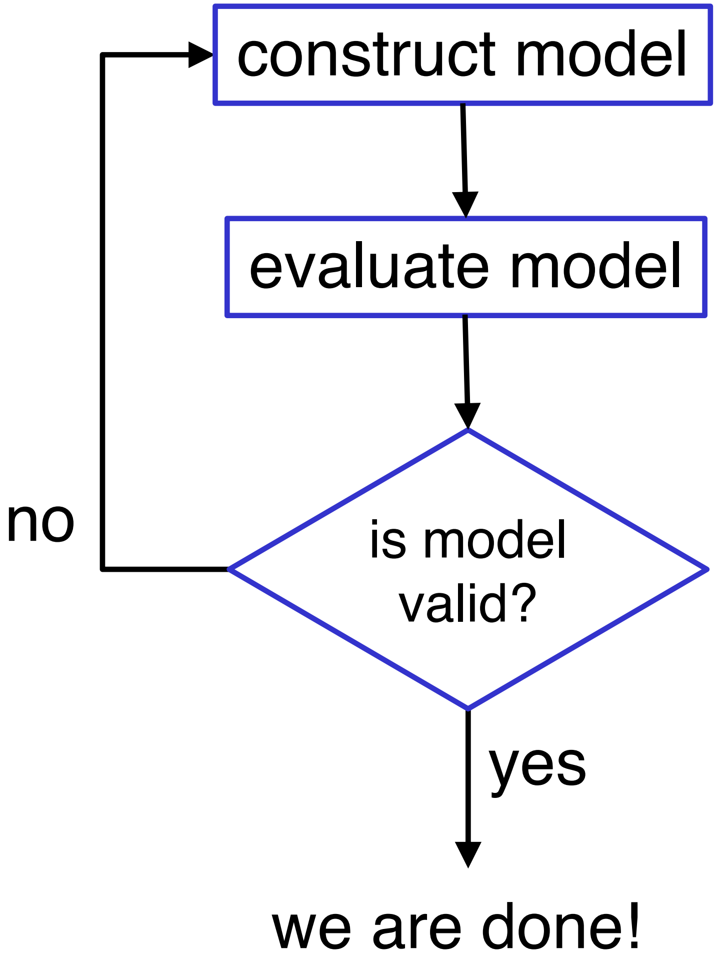
- Introduce analytical tools

- Answer questions like
  - Throughput of WiFi
  - If arrival/service rates double will response time stay same?
  - « 1 machine speed *s* » or « *n* machines speed *s/n* » ?
  - How many staff in call center to keep call rejection low?
  - and many others …

Performance Evaluation

	Measurement	**Simulation**	**Analysis**
When	Prototyping Monitoring Tuning	Anytime	Anytime
Cost	High	Moderate	Low
Accuracy	Varies	Moderate	Low
Scalability	Low	Medium	High
Salability	High	Medium	Low
Tools	Instrumentation	Languages	Mathematics

Modeling Cycle



construct model

evaluate model

is model valid?

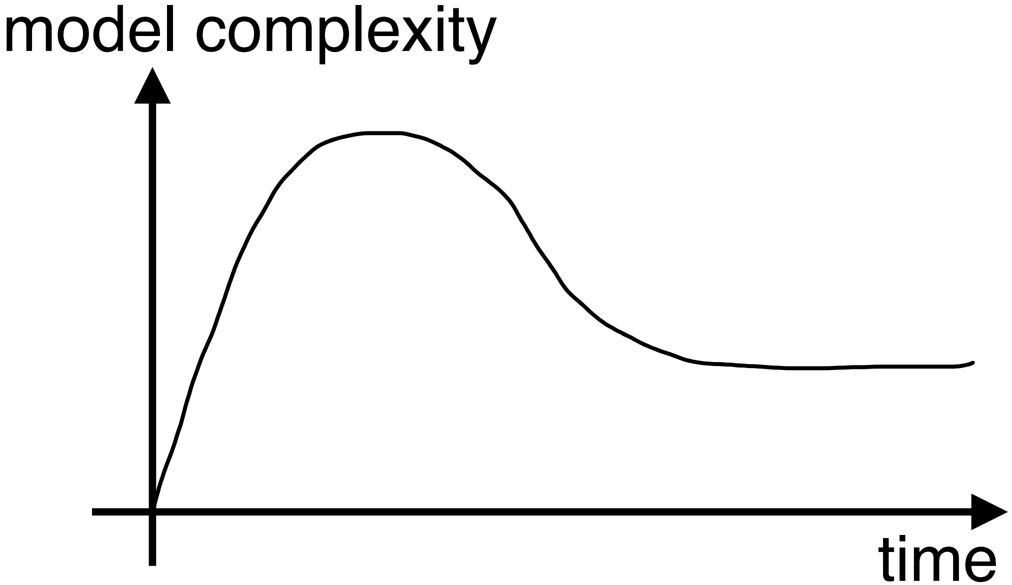
no

yes

we are done!

- abstract essential features
- ignore non-essential ones

- measurement
- simulations
- analysis



About the Course

- Website
  - https://lms.univ-cotedazur.fr/course/view.php?id=14278

    EIIN925 - ECUE Perform.Evaluation of Networks

    8 participants

    Lecture notes, slides, homeworks
  - http://www-sop.inria.fr/members/Sara.Alouf/PEN/

    Homeworks of past years
- Schedule: every Tuesday for 8 weeks
  - Markov Chains (3 lectures), Queues (3 lectures), Use cases (1 lecture), Exam (last session)
- Grade
  - 6 homeworks (60%), 1 exam (40%)

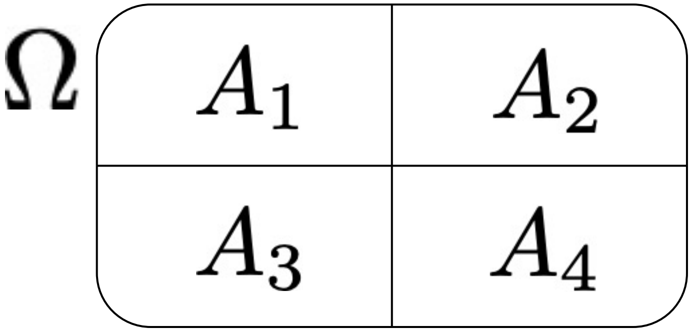
Brief Refresher

- Bayes formula

$$\boxed{P(A\,|\,B) = \frac{P(A \cap B)}{P(B)}}$$

- Law of total probability (use a partition)

$$\boxed{P(A) = \sum_{i=1}^{n} P(A \cap A_i) = \sum_{i=1}^{n} P(A\,|\,A_i)\,P(A_i)}$$



$$A_i \cap A_j = \emptyset$$

$$A_1 \cup A_2 \cup \cdots \cup A_n = \Omega$$

Brief Refresher

- Stochastic process : collection of random variables (rvs)

$$\mathbf{X} = \{X(t),\, t \in T\}$$

$X(t)$ is a rv mapping from Ω into some set $\mathcal{E} \subset \mathbb{R}$

- Poisson process : counting process rate λ

$$\{N(t), t \in T\} \qquad \boxed{\mathrm{E}[N(t)] = \lambda t}$$

  - start at 0
  - independent increments
  - count of events in $t-$long interval is Poisson variable

$$\boxed{P(N(t+s) - N(s) = k) = \frac{e^{-\lambda t}(\lambda t)^k}{k!} \qquad k = 0, 1, \ldots}$$

Part 1 Markov Chains

- Definition:

  A Markov process is a stochastic process that verifies the Markov property

$$\begin{aligned} P(X(t) \leq x \,|\, X(t_1) = x_1, \ldots, X(t_n) = x_n) \\ = P(X(t) \leq x \,|\, X(t_n) = x_n) \end{aligned}$$

$$x_1, \ldots, x_n, x \in \mathcal{E}$$

$$t_1, \ldots, t_n, t \in T$$

$$t_1 < t_2 < \cdots < t_n < t$$

- Discrete space ➜ Markov chain

Ch 1 - Discrete-Time Markov Chain (DTMC)

- Discrete-time version of Markov property

$$P(X(n+1) = j \,|\, X(0) = i_0, \ldots, X(n) = i) \\ = P(X(n+1) = j \,|\, X(n) = i)$$

$$i_0, i_1, \ldots, i_{n-1}, i, j \in \mathcal{E}$$

- DTMC finite : state-space is finite

- DTMC is homogeneous : transition independent of step

$$p_{i,j} = P(X(n+1) = j \,|\, X(n) = i) \quad \forall i, j \in \mathcal{E}$$

One-step transition probability from state i to state j

Transition Matrix

- Square matrix containing all one-step transition prob.

$$\mathbf{P} = \begin{pmatrix} \boxed{p_{0,0} & p_{0,1} & \cdots & p_{0,j} & \cdots} \\ p_{1,0} & p_{1,1} & \cdots & p_{1,j} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ p_{i,0} & p_{i,1} & \cdots & p_{i,j} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \quad \sum = 1$$

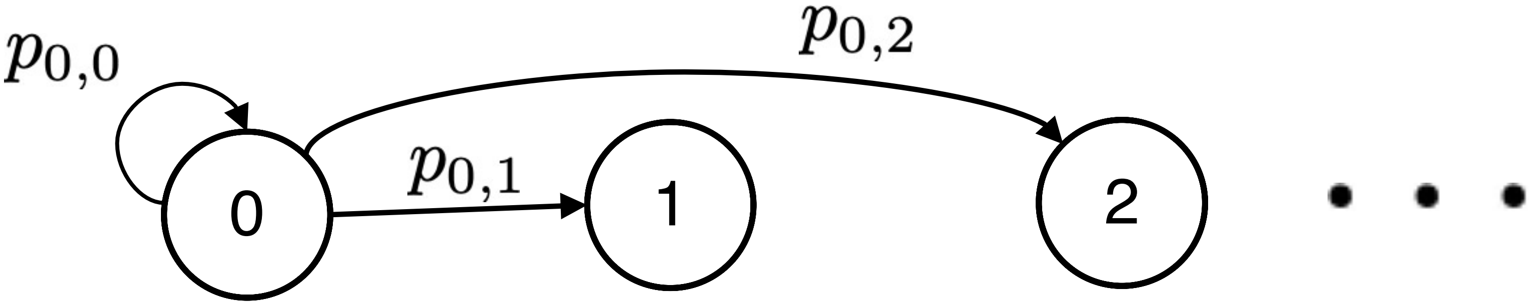
$$p_{i,j} \geq 0 \quad \forall i, j \in \mathcal{E}$$

$$\boxed{\sum_{j \in \mathcal{E}} p_{i,j} = 1 \quad \forall i \in \mathcal{E}}$$

➜ stochastic matrix

normalizing equation

Transition Diagram



Sum of arrows out of a state is 1

n-Step Transition Probability / Matrix

- n-step transition probability

$$\boxed{p_{i,j}^{(n)} = P(X(n) = j \,|\, X(0) = i)}$$

$$= P(X(n+1) = j \,|\, X(1) = i)$$

what counts is the difference between the two time steps

- n-step transition matrix

$$\mathbf{P}^{(n)} := \left[p_{i,j}^{(n)}\right] = \begin{pmatrix} p_{0,0}^{(n)} & p_{0,1}^{(n)} & \cdots \\ p_{1,0}^{(n)} & p_{1,1}^{(n)} & \cdots \\ \vdots & \vdots & \vdots \end{pmatrix} \qquad \sum = 1$$

Chapman-Kolmogorov Equation

- Proposition 1: For all $n \geq 0, m \geq 0, i, j \in \mathcal{E}$

$$p_{i,j}^{(n+m)} = \sum_{k \in \mathcal{E}} p_{i,k}^{(n)} \, p_{k,j}^{(m)}$$

$$\mathbf{P}^{(n+m)} = \mathbf{P}^{(n)} \, \mathbf{P}^{(m)}$$

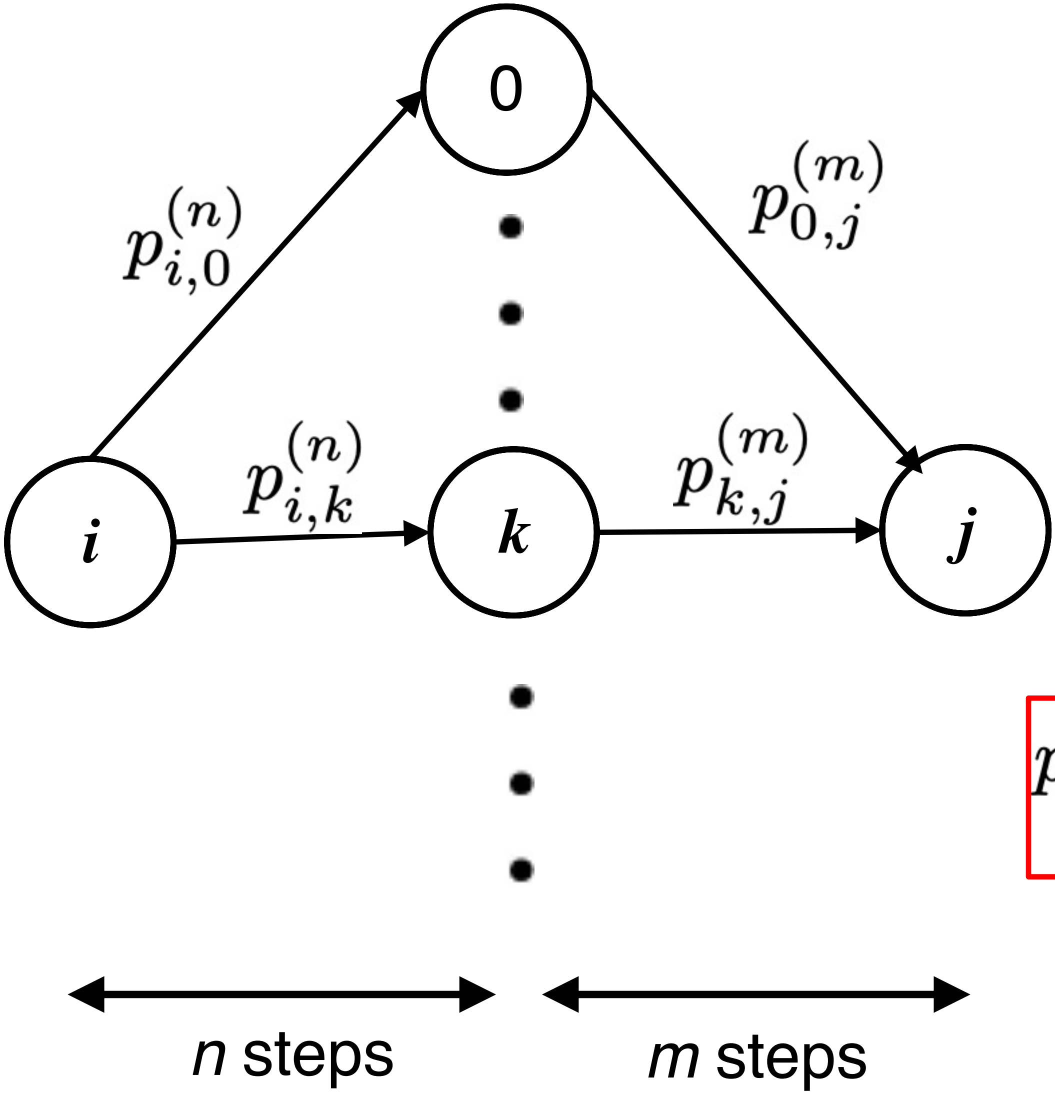
- Therefore $\mathbf{P}^{(n)} = \mathbf{P}^{n}$

  n-step transition matrix = n-th power of transition matrix

- Proof: use law of total probability and Markov property

  (see lecture notes page 8)

Chapman-Kolmogorov Equation

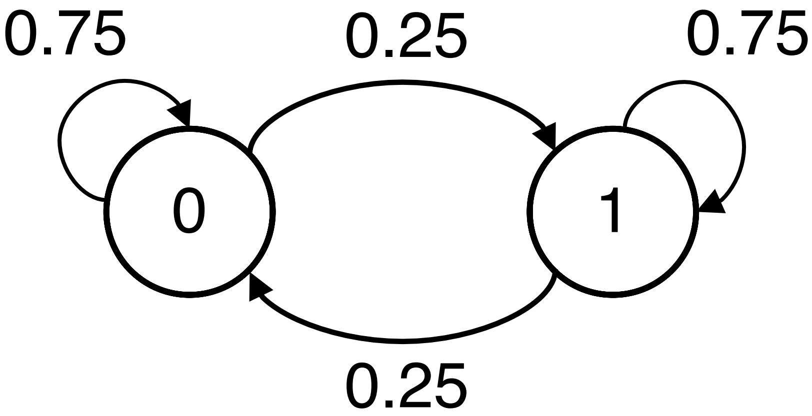


$$\boxed{p_{i,j}^{(n+m)} = \sum_{k \in \mathcal{E}} p_{i,k}^{(n)} \, p_{k,j}^{(m)}}$$

Example 2 page 8

- Communication channel transmits 0/1 through several stages
- From one stage to another, digit is unchanged with probability 0.75
- Question: Giving a 0 to stage 1, what is the probability that it is received as a 0 after stage 5?
- $X(n)$ state of system at step *n* is digit value after stage *n*
- $X(0)$ is value entered to stage 1
- State space = {0, 1}
- Markov property is verified ➜ $\{X(n), n \geq 0\}$ is a DTMC
- We are looking for $p_{0,0}^{(5)}$ the first element of matrix **P**^{5}

Example 2 page 8



$$\mathbf{P} = \begin{bmatrix} 0.75 & 0.25 \\ 0.25 & 0.75 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

We need to compute the 5th power of the transition matrix

$$\mathbf{P}^2 = \frac{1}{8}\begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$$

$$\mathbf{P}^5 = \frac{1}{64}\begin{bmatrix} 33 & 31 \\ 31 & 33 \end{bmatrix}$$

$$\mathbf{P}^4 = \frac{1}{32}\begin{bmatrix} 17 & 15 \\ 15 & 17 \end{bmatrix}$$

$$\boxed{p_{0,0}^{(5)} = 33/64 = 0.515625}$$

15 minutes break

Transient State Distribution

- We want probability that the system is in state *i* at time *n*

$$\pi_i(n) := P(X(n) = i)$$

- Assume initial distribution is known

$$\pi_i(0) = P(X(0) = i), \quad \forall i \in \mathcal{E}$$

normalization $\displaystyle\sum_{i \in \mathcal{E}} \pi_i(0) = 1$

- Law of total probability

$$P(X(n) = j) = \pi_j(n) = \sum_{i \in \mathcal{E}} \pi_i(0)\, p_{i,j}^{(n)}$$

- In matrix notation $\boxed{\pi(n) = \pi(0)\, \mathbf{P}^n}$

Limiting State Distribution

- From equation giving transient distribution

$$\pi(n) = \pi(n-1)\,\mathbf{P}$$

- If limit exists $\quad \pi = \lim_{n \to \infty} \pi(n) = (\pi_i, i \in \mathcal{E})$

$$\boxed{\pi = \pi\,\mathbf{P}}$$

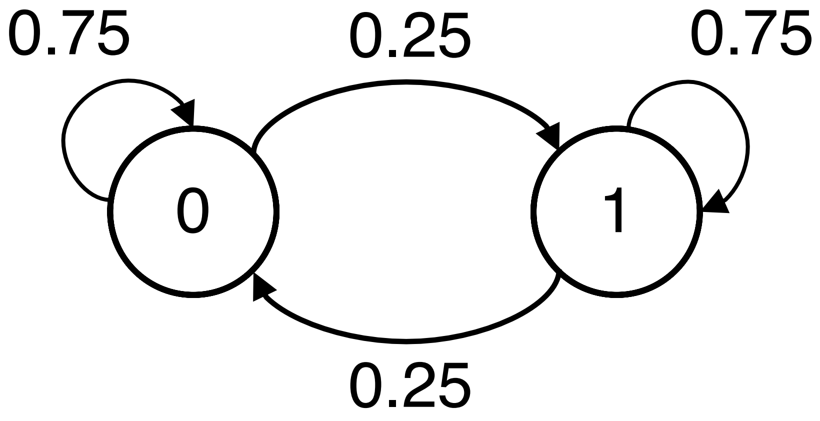
➜ $$\pi_i = \sum_{j \in \mathcal{E}} \pi_j p_{j,i} \ , \quad i \in \mathcal{E}$$

- Normalization $\quad \sum_{i \in \mathcal{E}} \pi_i = 1$

$$\boxed{\pi\,\mathbf{1} = 1}$$

DTMC Property 1: Aperiodicity?

- For state *i* define $d(i) = \gcd\{n | p_{i,i}^{(n)} > 0\}$
- If 1 then state is aperiodic, otherwise state is periodic
- If all states are aperiodic ➜ DTMC is aperiodic



$$d(0) = \gcd\{1, 2, 3, 4, \ldots\} = 1$$
$$d(1) = \gcd\{1, 2, 3, 4, \ldots\} = 1$$

this DTMC is aperiodic

DTMC Property 2: Irreducibility?

- State *j* is reachable from state *i* if for some *n* $p_{i,j}^{(n)} > 0$

- Two states communicate if each can reach the other

- DTMC is irreducible if any pair of states communicate

- Checking irreducibility on transition diagram
  - If there is a path going through all states then DTMC is irreducible

DTMC Property 3: Positive Recurrence?

- Does the DTMC return to a given state?

Let f_i be probability to return to state *i* if starting there

- $f_i < 1$ ➜ state *i* is transient
- $f_i = 1$ ➜ state *i* is recurrent
  - Mean time between visits is finite

    ➜ state *i* is positive recurrent
  - Mean time between visits is infinite

    ➜ state *i* is null recurrent

- If DTMC irreducible, all states are the same
- DTMC is positive recurrent if all its states are

DTMC Property 4: Ergodicity?

- A DTMC is ergodic if it is aperiodic, irreducible and positive recurrent

- Limiting distribution $\lim_{n \to \infty} \pi(n) = \pi(0) \lim_{n \to \infty} \mathbf{P}^n$

- Invariant measure is solution of system (if it exists)

$$\pi = \pi \mathbf{P}$$

$$\pi \mathbf{1} = 1$$

- Long-run distribution $\lim_{n \to \infty} \frac{S_j(n)}{n}$

DTMC Property 4: Ergodicity?

- If DTMC ergodic

Long-run distribution = invariant measure
= limiting distribution

Existence of Limiting Distribution

- If homogeneous DTMC is aperiodic and irreducible
- If system of equations

$$\pi = \pi \mathbf{P}$$

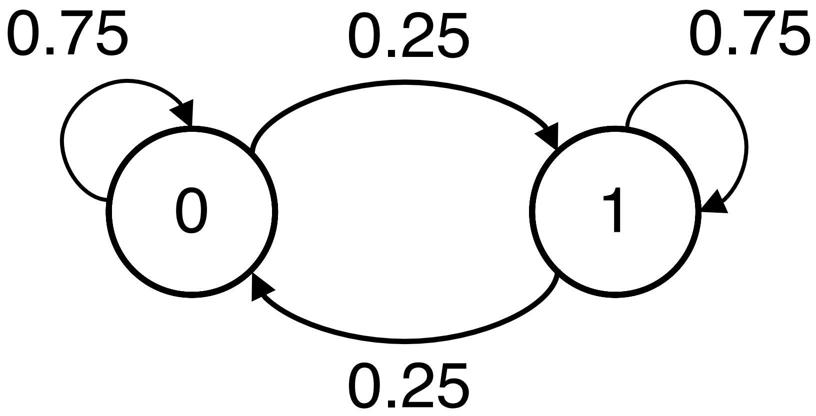
$$\pi \mathbf{1} = 1$$

has unique strictly positive solution

➜ $$\lim_{n \to \infty} \Big(P(X(n) = i), i \in \mathcal{E} \Big) = \pi$$

The limiting distribution exists and it is the invariant measure

Example 2 page 8



$$\mathbf{P} = \begin{bmatrix} 0.75 & 0.25 \\ 0.25 & 0.75 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

- DTMC is aperiodic and irreducible
- $\pi = \pi\mathbf{P} \Leftrightarrow \pi_0 = 0.75\pi_0 + 0.25\pi_1 \qquad \Rightarrow \pi_0 = \pi_1$

  $\pi\mathbf{1} = 1 \Leftrightarrow \pi_0 + \pi_1 = 1$

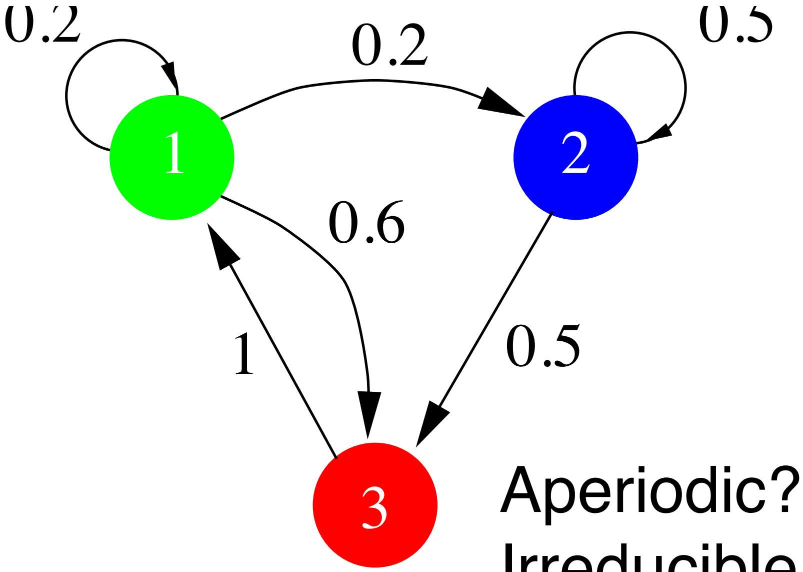
- The solution is unique and strictly positive $\pi = \frac{1}{2}(1,1)$

➜ this is the limiting distribution

Example on page 12

- DTMC with $\mathcal{E} = \{1, 2, 3\}$

$$\mathbf{P} = \begin{pmatrix} 0.2 & 0.2 & 0.6 \\ 0 & 0.5 & 0.5 \\ 1 & 0 & 0 \end{pmatrix}$$



Aperiodic? ✔
Irreducible? ✔

- From transition diagram

$$\begin{cases} \pi_1 &= 0.2\pi_1 + \pi_3 \\ \pi_2 &= 0.2\pi_1 + 0.5\pi_2 \\ \pi_3 &= 0.6\pi_1 + 0.5\pi_2 \\ 1 &= \pi_1 + \pi_2 + \pi_3 \end{cases} \Rightarrow \begin{cases} \pi_3 &= \frac{4}{5}\pi_1 \\ \pi_2 &= \frac{2}{5}\pi_1 \\ & \\ 1 &= \pi_1\left(1 + \frac{2}{5} + \frac{4}{5}\right) \end{cases}$$

$$\Rightarrow \boxed{\pi_1 = \frac{5}{11}, \pi_2 = \frac{2}{11}, \pi_3 = \frac{4}{11}}$$

For next week

- Lesson 1 to revise
- Homework 1 to return on Tuesday 11 January before 9.30 am
- Lesson 2 to read before Lecture 2