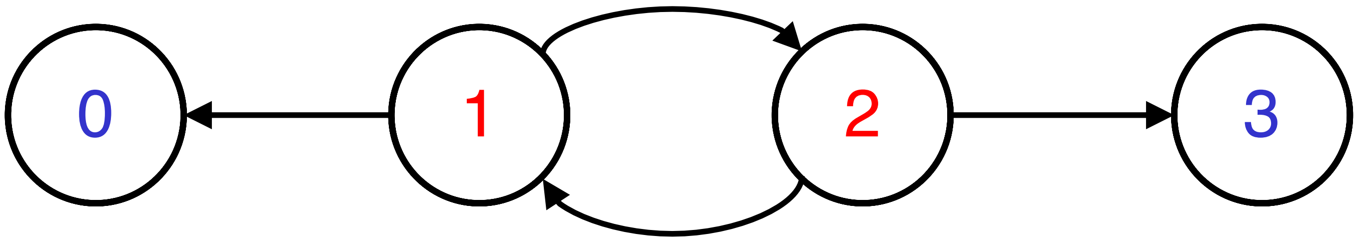
Performance Evaluation of Networks

Sara Alouf

Ch 3 – Absorbing Markov Chains

- Chapters 1 and 2: Irreducible Markov chains
  - Transient distribution
  - Steady-state / limiting distribution
- Absorbing Markov Chains

  ➜ some states are a dead end



0, 3: absorbent

1, 2: transient

- Transient distribution
- Time until absorption
- Probability to be absorbed in a given absorbing state

Discrete-Time Absorbing Markov Chain

- Homogeneous DTMC $\{X(n), n \geq 0\}$
- State space $\mathcal{E} := \{\boxed{1, 2, \ldots, N}, \boxed{1^*, 2^*, \ldots, M^*}\}$

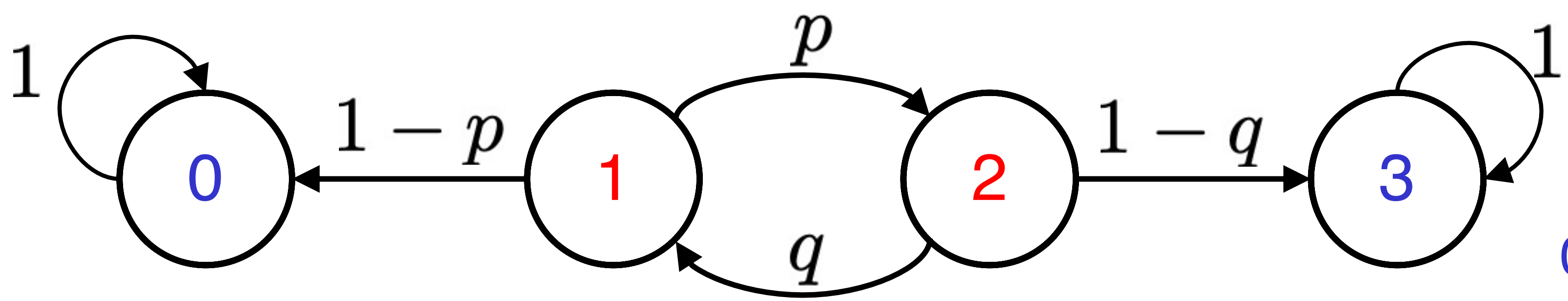
  N transient states M absorbing states
- Transition matrix $\mathbf{P} = [p_{i,j}]_{i,j \in \mathcal{E}}$

$$\mathbf{P} = \begin{pmatrix} a_{1,1} & \cdots & a_{1,N} & r_{1,1*} & \cdots & r_{1,M*} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{N,1} & \cdots & a_{N,N} & r_{N,1*} & \cdots & r_{N,M*} \\ 0 & \cdots & 0 & 1 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 1 \end{pmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{R} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}$$

$\mathbf{A} = [a_{i,j}]_{1 \leq i,j \leq N}$ $\mathbf{R} = [r_{i,j}]_{1 \leq i \leq N, 1^* \leq j \leq M^*}$

Example

- DTMC with transition diagram



1, 2: transient

0, 3: absorbent

- State-space $\mathcal{E} = \{1, 2, 0, 3\}$ ➜ order is important!

- Transition matrix (follow order)

$$\mathbf{A} = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} = \begin{bmatrix} 0 & p \\ q & 0 \end{bmatrix}$$

$$\mathbf{P} = \begin{array}{c} \\ 1 \\ 2 \\ 0 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 0 & 3 \end{array} \\ \left[\begin{array}{cc|cc} 0 & p & 1-p & 0 \\ q & 0 & 0 & 1-q \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \end{array} = \begin{bmatrix} \mathbf{A} & \mathbf{R} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}$$

$$\mathbf{R} = \begin{bmatrix} r_{1,0} & r_{1,3} \\ r_{2,0} & r_{2,3} \end{bmatrix} = \begin{bmatrix} 1-p & 0 \\ 0 & 1-q \end{bmatrix}$$

n-step Transition Matrix

- Transition matrix $\mathbf{P} = \begin{bmatrix} \mathbf{A} & \mathbf{R} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}$

- From Chapman-Kolmogorov equation $\mathbf{P}^{(n)} = \mathbf{P}^n$

$$\mathbf{P}^2 = \begin{bmatrix} \mathbf{A} & \mathbf{R} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{A} & \mathbf{R} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} = \begin{bmatrix} \mathbf{A}^2 & \mathbf{A}\mathbf{R} + \mathbf{R} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}$$

$$\mathbf{P}^3 = \begin{bmatrix} \mathbf{A}^2 & \mathbf{A}\mathbf{R} + \mathbf{R} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{A} & \mathbf{R} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} = \begin{bmatrix} \mathbf{A}^3 & \mathbf{A}^2\mathbf{R} + \mathbf{A}\mathbf{R} + \mathbf{R} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}$$

$$\boxed{\mathbf{P}^n = \begin{bmatrix} \mathbf{A}^n & \sum_{k=0}^{n-1} \mathbf{A}^k\mathbf{R} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}}$$

- Transient distribution $\pi(n) = \pi(0)\,\mathbf{P}^n$

- Limiting distribution $\lim_{n\to\infty} \pi(n) = \pi(0) \boxed{\lim_{n\to\infty} \mathbf{P}^n}$

Fundamental Matrix

- Define $N \times N$ matrix $\mathbf{N} = [n_{i,j}]_{1 \le i,j \le N}$

  $n_{i,j}$ expected number of visits to state j if initially in i

- Define $X_j^{(n)} = \mathbb{1}(X(n) = j) = \begin{cases} 1 & X(n) = j \\ 0 & \text{otherwise} \end{cases}$

- We have $n_{i,j} = E\left[\sum_{n \ge 0} X_j^{(n)} \,|\, X(0) = i\right]$

- Proposition 8: the fundamental matrix is

$$\boxed{\mathbf{N} = (\mathbf{I} - \mathbf{A})^{-1}}$$

Nota bene:
elements in $\mathbf{N}$ are
positive or null

Proof of $\mathbf{N} = (\mathbf{I} - \mathbf{A})^{-1}$

$$E\left[\sum_{k=0}^{n} X_j^{(k)} \,\middle|\, X(0) = i\right] = \sum_{k=0}^{n} E\left[X_j^{(k)} \,\middle|\, X(0) = i\right]$$

$$= \sum_{k=0}^{n} E\left[\mathbb{1}(X(k) = j) \,|\, X(0) = i\right]$$

expected number of visits

in $n+1$ steps

$$= \sum_{k=0}^{n} P\big(X(k) = j \,|\, X(0) = i\big)$$

$$= \sum_{k=0}^{n} \boxed{a_{i,j}^{(k)}}$$

(i, j) element in $\mathbf{A}^k$

$$E\left[\lim_{n\to\infty} \sum_{k=0}^{n} X_j^{(k)} \,\middle|\, X(0) = i\right] = \lim_{n\to\infty} E\left[\sum_{k=0}^{n} X_j^{(k)} \,\middle|\, X(0) = i\right]$$

$$n_{i,j} = \sum_{k\geq 0} a_{i,j}^{(k)}$$

Proof of $\mathbf{N} = (\mathbf{I} - \mathbf{A})^{-1}$

- For any $i, j = 1, \ldots N$ $\quad n_{i,j} = \sum_{k \geq 0} a_{i,j}^{(k)}$

$$\Rightarrow \quad \boxed{\mathbf{N} = \sum_{k \geq 0} \mathbf{A}^k}$$

- Consider finite sum

$$(\mathbf{I} - \mathbf{A}) \sum_{k=0}^{n} \mathbf{A}^k = \sum_{k=0}^{n} \mathbf{A}^k - \sum_{k=1}^{n+1} \mathbf{A}^k = \mathbf{I} - \mathbf{A}^{n+1}$$

- Let $n \to \infty$ $\quad a_{i,j}^{(n+1)} \to 0 \quad \Rightarrow \quad \mathbf{A}^{n+1} \to 0$

$$\rightarrow \quad (\mathbf{I} - \mathbf{A})\mathbf{N} = \mathbf{I}$$

- If inverse exists ➜ $\boxed{\mathbf{N} = (\mathbf{I} - \mathbf{A})^{-1}}$

Absorption Probabilities

- Define $N \times M^*$ stochastic matrix $\mathbf{B} = \left[b_{i,j} \right]_{\substack{i \in \{1,\ldots,N\} \\ j \in \{1^*,\ldots,M^*\}}}$

  $b_{i,j}$ probability to be absorbed in j if initially in i

- Proposition 9: $\boxed{\mathbf{B} = \mathbf{N}\mathbf{R}}$ recall $\mathbf{P} = \begin{bmatrix} \mathbf{A} & \mathbf{R} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}$

- For $i \in \{1,\ldots,N\},\ j \in \{1^*,\ldots,M^*\}$

$$b_{i,j} = \sum_{n \geq 0} P(X(n+1) = j \,|\, X(0) = i)$$

use law of tot. prob.

k last transient state

$$= \sum_{n \geq 0} \sum_{k=1}^{N} \overbrace{\boxed{P(X(n+1) = j \,|\, X(n) = k,}}^{r_{k,j}} \cancel{X(0) = i})$$

$$\times \underbrace{\boxed{P(X(n) = k \,|\, X(0) = i)}}_{a_{i,k}^{(n)}}$$

use Markov property

Absorption Probabilities

- For $i \in \{1, \ldots, N\},\ j \in \{1^*, \ldots, M^*\}$

$$\begin{aligned} b_{i,j} &= \sum_{n \geq 0} \sum_{k=1}^{N} r_{k,j} a_{i,k}^{(n)} \\ &= \sum_{k=1}^{N} \left(\sum_{n \geq 0} a_{i,k}^{(n)} \right) r_{k,j} \\ &= \sum_{k=1}^{N} n_{i,k} r_{k,j} \end{aligned}$$

- In matrix notation

$$\mathbf{B} = \mathbf{NR} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{R}$$

Nota bene:
elements in $\mathbf{B}$ are probabilities
$\mathbf{B}$ is stochastic matrix

Limit of $\mathbf{P}^n$

- *n*-step transition matrix

$$\mathbf{P}^n = \begin{bmatrix} \mathbf{A}^n & \sum_{k=0}^{n-1} \mathbf{A}^k \mathbf{R} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}$$

- As $n \to \infty$ $\qquad \mathbf{A}^n \to 0$

$$\sum_{k=0}^{n-1} \mathbf{A}^k \mathbf{R} \to \mathbf{N}\mathbf{R}$$

$$\mathbf{P}^n \to \begin{bmatrix} \mathbf{0} & \mathbf{B} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}$$

stochastic matrix

Expected Absorption Time

- Define column vector $\mathbf{T} = (T(i), i = 1, \ldots, N)^T$

  $T(i)$ expected time until absorption if initially in state i

  time = number of steps

- For any absorbing state $j \in \{1^*, \ldots, M^*\}, \quad T(j) = 0$

- Proposition 10: $\boxed{\mathbf{T = N.1}}$

- $n_{i,j}$ expected number of visits to state j if initially in i

  ➜ $T(i) = \sum_{j=1}^{N} n_{i,j}$ for $i \in \{1, 2, \ldots, N\}$

Expected Absorption Time

- Corollary 1: Column vector $\mathbf{T} = (T(i), i = 1, \dots, N)^T$ is solution of $\boxed{\mathbf{T} = \mathbf{1} + \mathbf{A}\mathbf{T}}$
- Fundamental matrix $\mathbf{N} = (\mathbf{I} - \mathbf{A})^{-1}$
- $\mathbf{T} = \mathbf{N}.\mathbf{1} \quad \Leftrightarrow \quad \mathbf{T} = (\mathbf{I} - \mathbf{A})^{-1}.\mathbf{1}$

$$\Leftrightarrow \quad (\mathbf{I} - \mathbf{A})\mathbf{T} = \mathbf{1}$$

$$\Leftrightarrow \quad \mathbf{T} - \mathbf{A}\mathbf{T} = \mathbf{1}$$

$$\Leftrightarrow \quad \mathbf{T} = \mathbf{1} + \mathbf{A}\mathbf{T}$$

Nota bene:
elements in $\mathbf{T}$ are
strictly positive

Example

- Document in P2P system is replicated over K peers
- Retrieve requests occur at beginning of every minute
- $X(n)$ number of replicas available just before minute n
- Peers connect/disconnect from system ➜ $X(n)$ stochastic
- If at n no copy is available ➜ request fails $X(n) = F$
- State-space $\mathcal{E} = \{1, \ldots, K, F\}$
- Transition matrix

$$\mathbf{P} = \left[\begin{array}{cccc|c} \frac{1}{K+1} & \frac{1}{K+1} & \cdots & \frac{1}{K+1} & \frac{1}{K+1} \\ \vdots & \vdots & & \vdots & \vdots \\ \frac{1}{K+1} & \frac{1}{K+1} & \cdots & \frac{1}{K+1} & \frac{1}{K+1} \\ \hline 0 & 0 & \cdots & 0 & 1 \end{array}\right]$$

Example

- Describe $\{X(n), n \geq 0\}$
  - Process observed every minute when requests arrive

    ➜ discrete-time process
  - Transition probabilities independent of step

    ➜ homogeneous
  - States 1 to K are transient
  - State F is absorbent

  $\{X(n), n \geq 0\}$ is absorbing homogeneous DTMC
- Limiting probability of a failure?
  - Regardless of initial distribution the chain will be absorbed in its unique absorbing state

    ➜ limiting probability of failure is 1

Example

- Compute expected absorption times when $K = 3$

$$\mathbf{P} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{R} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \quad \Rightarrow \quad \mathbf{A} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

- $T(F) = 0$
- $\mathbf{T} = \mathbf{1} + \mathbf{AT}$ $\Rightarrow \begin{bmatrix} T(1) \\ T(2) \\ T(3) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} T(1) \\ T(2) \\ T(3) \end{bmatrix}$

$$\Rightarrow T(1) = T(2) = T(3) = 1 + \frac{1}{4}\Big(T(1) + T(2) + T(3)\Big)$$

$$\Rightarrow 4T(1) = 4 + 3T(1) \qquad \boxed{\Rightarrow T(1) = T(2) = T(3) = 4}$$

- $\mathbf{N} = (\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} \boxed{2 \quad 1 \quad 1} \\ \boxed{1 \quad 2 \quad 1} \\ \boxed{1 \quad 1 \quad 2} \end{bmatrix}$ $\quad \sum = T(1) = 4$, $\sum = T(2) = 4$, $\sum = T(3) = 4$ $\qquad \mathbf{T} = \mathbf{N}.\mathbf{1}$

15 minutes break

Continuous-Time Absorbing Markov Chain

- Homogeneous CTMC $\{X(t), t \geq 0\}$
- State space $\mathcal{E} := \{\boxed{1, 2, \dots, N}, \boxed{1^*, 2^*, \dots, M^*}\}$

N transient states M absorbing states

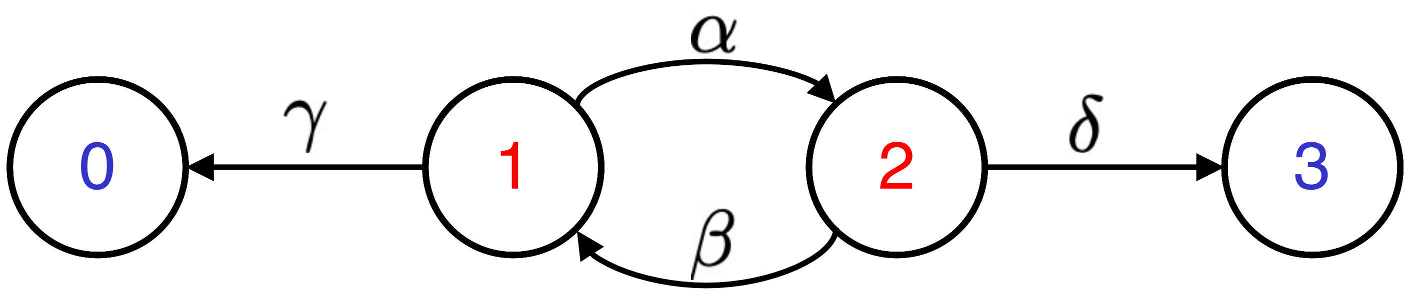
- Infinitesimal generator $\mathbf{Q} = \left[q_{i,j}\right]_{i,j \in \mathcal{E}}$

$$\mathbf{Q} = \begin{pmatrix} q_{1,1} & \cdots & q_{1,N} & \tilde{r}_{1,1*} & \cdots & \tilde{r}_{1,M*} \\ \vdots & & \vdots & \vdots & & \vdots \\ q_{N,1} & \cdots & q_{N,N} & \tilde{r}_{N,1*} & \cdots & \tilde{r}_{N,M*} \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix} = \begin{bmatrix} \tilde{\mathbf{Q}} & \tilde{\mathbf{R}} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

$\tilde{\mathbf{Q}} = [q_{i,j}]_{1 \leq i,j \leq N}$ $\tilde{\mathbf{R}} = [\tilde{r}_{i,j}]_{1 \leq i \leq N, 1^* \leq j \leq M^*}$

Example

- CTMC with transition rate diagram



1, 2: transient

0, 3: absorbent

- State-space $\mathcal{E} = \{1, 2, 0, 3\}$ ➜ order is important!
- Infinitesimal generator (follow order)

$$\mathbf{Q} = \begin{array}{c} \\ 1 \\ 2 \\ 0 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 0 & 3 \end{array} \\ \begin{bmatrix} -(\alpha+\gamma) & \alpha & \gamma & 0 \\ \beta & -(\beta+\delta) & 0 & \delta \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$= \begin{bmatrix} \tilde{\mathbf{Q}} & \tilde{\mathbf{R}} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

$$\tilde{\mathbf{Q}} = \begin{bmatrix} -(\alpha+\gamma) & \alpha \\ \beta & -(\beta+\delta) \end{bmatrix}$$

$$\tilde{\mathbf{R}} = \begin{bmatrix} \tilde{r}_{1,0} & \tilde{r}_{1,3} \\ \tilde{r}_{2,0} & \tilde{r}_{2,3} \end{bmatrix} = \begin{bmatrix} \gamma & 0 \\ 0 & \delta \end{bmatrix}$$

Embedded Markov Chain at Jump Times

- Construct DTMC from CTMC
- Observe $\{X(t), t \geq 0\}$ at jump times ➜ $\{X(n), n \geq 0\}$
- Transition probabilities at jump times

$$p(i,j) = \begin{cases} \frac{q_{i,j}}{-q_{i,i}} & i \in \{1, \ldots, N\}, j \in \mathcal{E}, j \neq i \\ 1 & j = i \in \{1^*, \ldots, M^*\} \text{ (by convention)} \\ 0 & \text{otherwise} \end{cases}$$

- Transition matrix (stochastic) $\mathbf{P} = \Big[p(i,j)\Big]_{i,j,\in\mathcal{E}} = \begin{bmatrix} \mathbf{A} & \mathbf{R} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}$

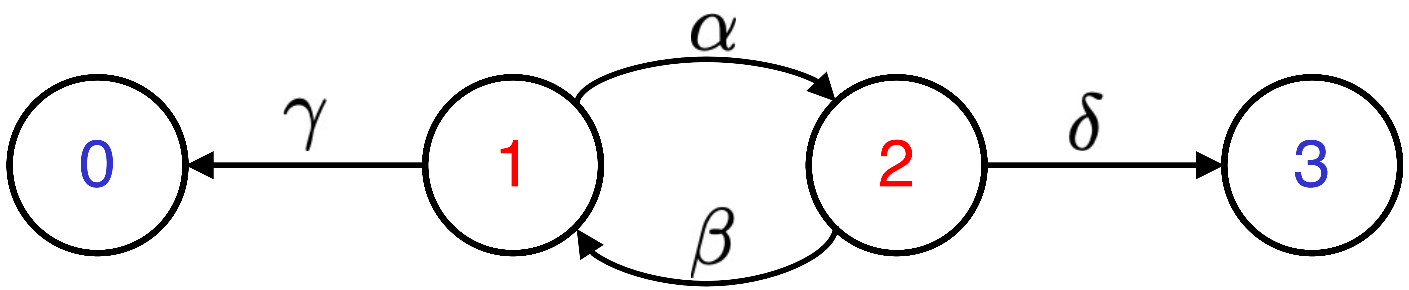
$$\mathbf{A} = [a_{i,j}]_{1 \leq i,j \leq N}$$

$$a_{i,j} = \begin{cases} \frac{q_{i,j}}{-q_{i,i}} & j \neq i \\ 0 & j = i \end{cases}$$

$$\mathbf{R} = [r_{i,j}]_{1 \leq i \leq N, 1^* \leq j \leq M^*}$$

$$r_{i,j} = \frac{q_{i,j}}{-q_{i,i}}$$

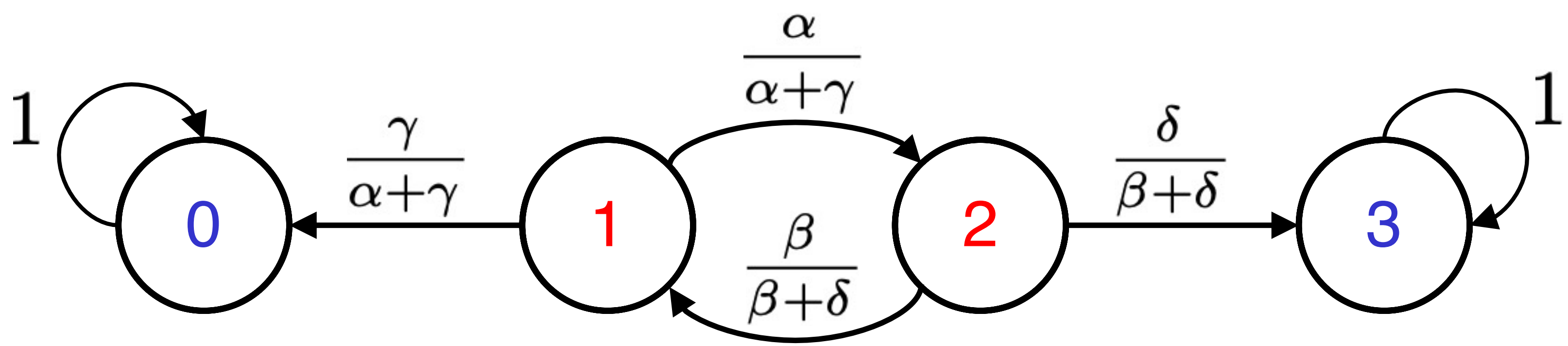
Example

- 

1, 2: transient

0, 3: absorbent

- Embedded Markov chain at jump times



$$\mathbf{P} = \begin{array}{c} \\ 1 \\ 2 \\ 0 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 0 & 3 \end{array} \\ \left[\begin{array}{cc|cc} 0 & \frac{\alpha}{\alpha+\gamma} & \frac{\gamma}{\alpha+\gamma} & 0 \\ \frac{\beta}{\beta+\delta} & 0 & 0 & \frac{\delta}{\beta+\delta} \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \end{array}$$

$$\mathbf{A} = \begin{bmatrix} 0 & \frac{\alpha}{\alpha+\gamma} \\ \frac{\beta}{\beta+\delta} & 0 \end{bmatrix}$$

$$\mathbf{R} = \begin{bmatrix} \frac{\gamma}{\alpha+\gamma} & 0 \\ 0 & \frac{\delta}{\beta+\delta} \end{bmatrix}$$

Mean Number of Visits

- Use embedded Markov chain and results of absorbing DTMC

$$\mathbf{N} = [n_{i,j}]_{1 \le i,j \le N} \qquad \boxed{\mathbf{N} = (\mathbf{I} - \mathbf{A})^{-1}}$$

$n_{i,j}$ expected number of visits to state j if initially in i

- Visits durations are different!

- In absorbing DTMC

➜ a visit lasts for a constant step time

- In Markov chain embedded at jump times of absorbing CTMC

➜ a visit in transient i lasts for a random time that is Exp($-q_{i,i}$)

Absorption Probabilities

- Use embedded Markov chain and results of absorbing DTMC

$$\mathbf{B} = \left[b_{i,j} \right]_{\substack{i \in \{1,\ldots,N\} \\ j \in \{1^*,\ldots,M^*\}}} \qquad \mathbf{B} = \mathbf{N}\mathbf{R} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{R}$$

$b_{i,j}$ probability to be absorbed in j if initially in i

- Matrix is stochastic

Expected Absorption Time

- In absorbing CTMC

  number of visits to transient state $\neq$ expected time spent in that state

- Define column vector $\mathbf{T} = (T(i), i = 1, \ldots, N)^T$

  $T(i)$ expected time until absorption if initially in state i

  time $\neq$ number of steps

- Define $N \times N$ matrix $\tilde{\mathbf{T}} = \left[t_{i,j}\right]_{1 \le i,j \le N}$

  $t_{i,j}$ expected time spent in transient j if initially in state i

- $\boxed{T(i) = \sum_{j=1}^{N} t_{i,j}}$ $\quad t_{i,j} = n_{i,j} \times \frac{1}{-q_{j,j}}, \quad i, j = 1, \ldots, N$

Expected Absorption Time

- Proposition 11: $\tilde{\mathbf{T}} = -\tilde{\mathbf{Q}}^{-1}$ $\boxed{\mathbf{T} = -\tilde{\mathbf{Q}}^{-1}.\mathbf{1}}$
- Define $\mathbf{D} = \mathbf{diag}\left(\frac{1}{q_{1,1}}, \dots, \frac{1}{q_{N,N}}\right)$

$$\mathbf{ND} = \begin{bmatrix} n_{1,1} & \cdots & n_{1,N} \\ \vdots & & \vdots \\ n_{N,1} & \cdots & n_{N,N} \end{bmatrix} \begin{bmatrix} \frac{1}{q_{1,1}} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{q_{N,N}} \end{bmatrix} = \begin{bmatrix} \frac{n_{1,1}}{q_{1,1}} & \cdots & \frac{n_{1,N}}{q_{N,N}} \\ \vdots & & \vdots \\ \frac{n_{N,1}}{q_{1,1}} & \cdots & \frac{n_{N,N}}{q_{N,N}} \end{bmatrix}$$

$$\Rightarrow \tilde{\mathbf{T}} = -\mathbf{N}\,\mathbf{D} = -(\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{D} = -\left(\mathbf{D}^{-1}\,(\mathbf{I} - \mathbf{A})\right)^{-1}$$

$$\mathbf{D}^{-1}\,(\mathbf{I} - \mathbf{A}) = \begin{bmatrix} q_{1,1} & & 0 \\ & \ddots & \\ 0 & & q_{N,N} \end{bmatrix} \begin{bmatrix} 1 & \cdots & \frac{q_{1,N}}{q_{1,1}} \\ \vdots & & \vdots \\ \frac{q_{N,1}}{q_{N,N}} & \cdots & 1 \end{bmatrix}$$

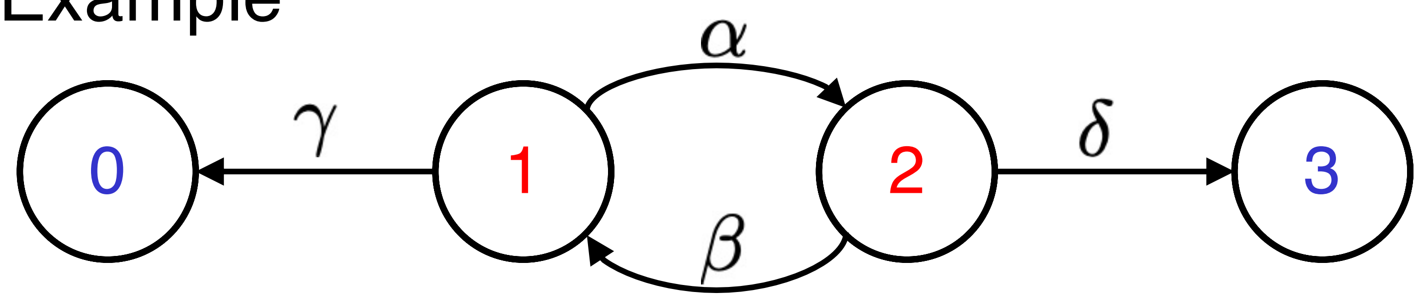
Nota bene:
elements in $\mathbf{T}$ are
strictly positive

$$= \begin{bmatrix} q_{1,1} & \cdots & q_{1,N} \\ \vdots & & \vdots \\ q_{N,1} & \cdots & q_{N,N} \end{bmatrix} = \tilde{\mathbf{Q}} \qquad \Rightarrow \boxed{\tilde{\mathbf{T}} = -\tilde{\mathbf{Q}}^{-1}}$$

Expected Absorption Time

- Corollary 2: Column vector $\mathbf{T} = (T(i), i = 1, \ldots, N)^T$ is solution of $\boxed{\tilde{\mathbf{Q}}\mathbf{T} = -\mathbf{1}}$

- Example



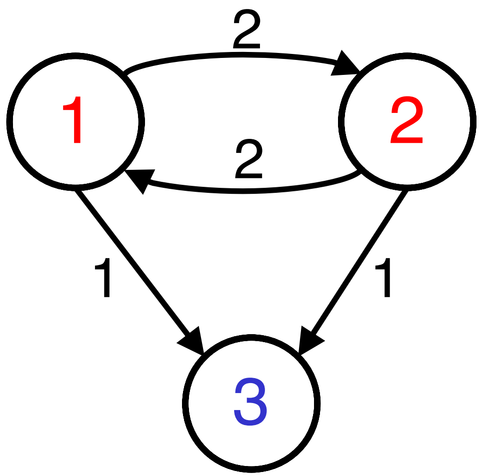
$$\tilde{\mathbf{Q}}\mathbf{T} = \begin{bmatrix} -(\alpha+\gamma) & \alpha \\ \beta & -(\beta+\delta) \end{bmatrix} \begin{bmatrix} T(1) \\ T(2) \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

$$\Rightarrow \begin{cases} -(\alpha+\gamma)\,T(1) + \alpha\,T(2) & = & -1 \\ \beta\,T(1) - (\beta+\delta)\,T(2) & = & -1 \end{cases} \Rightarrow \begin{cases} T(1) = \frac{\alpha+\beta+\delta}{\beta\gamma+\delta\alpha+\delta\gamma} \\ T(2) = \frac{\alpha+\beta+\gamma}{\beta\gamma+\delta\alpha+\delta\gamma} \end{cases}$$

Example 4 page 30

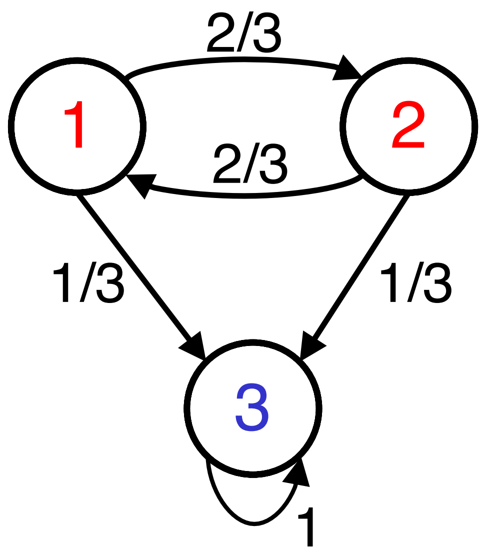
- Absorbing homogeneous CTMC

$$\mathbf{Q} = \left[\begin{array}{cc|c} -3 & 2 & 1 \\ 2 & -3 & 1 \\ \hline 0 & 0 & 0 \end{array}\right]$$



- Expected absorption time

$$\begin{aligned} -3\,T(1) + 2\,T(2) &= -1 \\ 2\,T(1) - 3\,T(2) &= -1 \end{aligned} \quad \Rightarrow T(1) = T(2) = 1$$



- Embedded Markov chain

$$\mathbf{P} = \left[\begin{array}{cc|c} 0 & \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & 0 & \frac{1}{3} \\ \hline 0 & 0 & 1 \end{array}\right] \Rightarrow \mathbf{A} = \begin{bmatrix} 0 & \frac{2}{3} \\ \frac{2}{3} & 0 \end{bmatrix} \quad \mathbf{R} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}$$

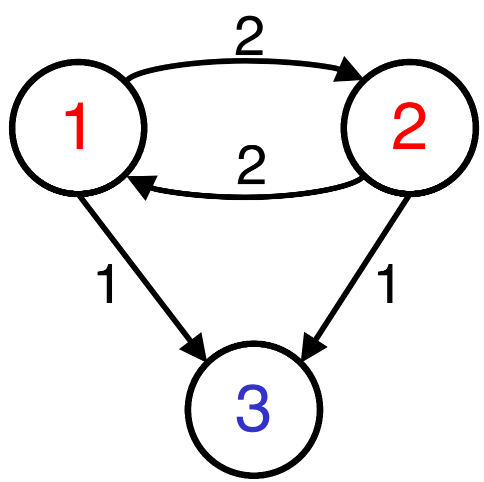
Example 4 page 30

- Mean number of visits in transient states

$$\mathbf{I}-\mathbf{A}=\begin{bmatrix}1 & -\frac{2}{3}\\ -\frac{2}{3} & 1\end{bmatrix}=\frac{1}{3}\begin{bmatrix}3 & -2\\ -2 & 3\end{bmatrix}$$

$$(\mathbf{I}-\mathbf{A})^{-1}=3\begin{bmatrix}3 & -2\\ -2 & 3\end{bmatrix}^{-1}=\frac{3}{5}\begin{bmatrix}3 & 2\\ 2 & 3\end{bmatrix}=\mathbf{N}$$

- Absorbing probabilities



$$\mathbf{B}=\mathbf{NR}=\frac{3}{5}\begin{bmatrix}3 & 2\\ 2 & 3\end{bmatrix}\frac{1}{3}\begin{bmatrix}1\\ 1\end{bmatrix}=\begin{bmatrix}1\\ 1\end{bmatrix}=\begin{bmatrix}b_{1,3}\\ b_{2.3}\end{bmatrix}$$

For next week

- Lesson 3 to revise
- Homework 3 to return on Tuesday 1 October before 9 am
- Lesson 4 to read before Lecture 4