UBINET: Performance Evaluation of Networks

Correction of homework 1

1.1 A dysfunctional laptop

1. The laptop's state is A when running CPU-intensive jobs, B when running memory-intensive jobs, C when swapping heavily and D when rebooting. The state-space is $\mathcal{E} = \{A, B, C, D\}$. It is enough to know the previous state of the laptop to know the transition probabilities to the future state. The Markov property is verified and we have a homogeneous discrete-time Markov chain. The transition matrix is

$$\mathbf{P} = \begin{bmatrix} \frac{19}{20} & \frac{1}{20} & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{4} & 0 & 0 & \frac{3}{4} \end{bmatrix} .$$

Sanity check: the elements in each row of $\mathbf{P}$ sum to 1.

2. The state at minute 1 is A then $\pi(1) = (1, 0, 0, 0)$. We want to know $\pi_A(3)$ and $\pi_A(5)$. We know that

$$\pi(3) = \pi(1)\mathbf{P}^2 \ , \quad \text{also} \quad \pi(5) = \pi(3)\mathbf{P}^2 \ .$$

Let us compute $\mathbf{P}^2$, we get

$$\mathbf{P}^2 = \begin{bmatrix} \frac{361}{400} & \frac{29}{400} & \frac{10}{400} & 0 \\ 0 & \frac{1}{4} & \frac{7}{12} & \frac{1}{6} \\ \frac{1}{12} & 0 & \frac{4}{9} & \frac{17}{36} \\ \frac{17}{40} & \frac{1}{80} & 0 & \frac{9}{16} \end{bmatrix} .$$

Sanity check: the elements in each row of $\mathbf{P}^2$ sum to 1.

Therefore

$$\pi(3) = \left(\frac{361}{400}, \frac{29}{400}, \frac{10}{400}, 0\right) \qquad \Rightarrow \quad \pi_A(3) = \frac{361}{400} = 0.9025$$

$$\pi_A(5) = \left(\frac{361}{400}, \frac{29}{400}, \frac{10}{400}, 0\right) \cdot \begin{bmatrix} \frac{361}{400} \\ 0 \\ \frac{1}{12} \\ \frac{17}{40} \end{bmatrix} \qquad \Rightarrow \quad \pi_A(5) = \frac{391963}{480000} \approx 0.81659 \ .$$

3. The diagonal terms of $\mathbf{P}$ are all strictly positive, which means that each state in $\mathcal{E}$ has a loop. Therefore each state is aperiodic and so is the DTMC. There is a path that cycles through all states, namely $A \to B \to C \to D \to A$, which means that all states communicate with each other and so the DTMC is irreducible. As the state-space is finite, we can conclude that the DTMC is positive recurrent. The limiting distribution exists and is equal to the stationary distribution, which we compute next.

The stationary equations are

$$\begin{cases} \pi_A = \frac{19}{20}\pi_A + \frac{1}{4}\pi_D & \Rightarrow \quad \pi_D = \frac{1}{5}\pi_A \\ \pi_B = \frac{1}{20}\pi_A + \frac{1}{2}\pi_B & \Rightarrow \quad \pi_B = \frac{1}{10}\pi_A \\ \pi_C = \frac{1}{2}\pi_B + \frac{2}{3}\pi_C & \Rightarrow \quad \pi_C = \frac{3}{2}\pi_B = \frac{3}{20}\pi_A \\ \pi_D = \frac{1}{3}\pi_C + \frac{3}{4}\pi_D \ . \end{cases}$$

The normalizing equation allows to compute π_A. We can write

$$1 = \pi_A + \pi_B + \pi_C + \pi_D = \pi_A \left(1 + \frac{1}{10} + \frac{3}{20} + \frac{1}{5}\right) \quad \Rightarrow \quad \pi_A = \frac{20}{29} \ .$$

The stationary solution is

$$\pi = \left(\frac{20}{29}, \frac{2}{29}, \frac{3}{29}, \frac{4}{29}\right) \ .$$

Sanity check: each probability is in the interval $(0, 1)$ and the sum of the probabilities is equal to 1.

4. The expected power consumption in steady-state is

$$P = 15 \cdot \pi_A + 20 \cdot \pi_B + 25 \cdot \pi_C + 10 \cdot \pi_D = \frac{455}{29} \approx 15.69 \text{ W}.$$

1.2 Exponential variables and Poisson processes

1. We have

$$Z \le t \quad \Leftrightarrow \quad \min\{X, Y\} \le t \quad \Leftrightarrow \quad X \le t \text{ or } Y \le t \ .$$

Therefore $C = A \cup B$. We observe that $\overline{C} = \overline{A} \cap \overline{B}$. Also

$$V \le t \quad \Leftrightarrow \quad \max\{X, Y\} \le t \quad \Leftrightarrow \quad X \le t \text{ and } Y \le t \ .$$

Consequently $D = A \cap B$.

2. We observe first that as X and Y are independent rvs the events A and B are also independent. We also observe that $P(A) = F_X(t)$ and $P(B) = F_Y(t)$. The cumulative distribution functions of Z and V are

$$\begin{aligned} F_Z(t) &= P(Z \le t) = P(C) = 1 - P(\overline{C}) = 1 - P(\overline{A} \cap \overline{B}) \overset{\text{indep.}}{=} 1 - P(\overline{A})P(\overline{B}) \\ &= 1 - (1 - F_X(t))(1 - F_Y(t)) \\ F_V(t) &= P(V \le t) = P(D) = P(A \cap B) \overset{\text{indep.}}{=} P(A)P(B) = F_X(t)F_Y(t) \end{aligned}$$

3. We can compute

$$F_X(t) = \int_0^t f_X(x)dx = \int_0^t \lambda e^{-\lambda x} dx = \lambda \left[\frac{e^{-\lambda x}}{-\lambda}\right]_0^t = \lambda \left[\frac{e^{-\lambda t} - 1}{-\lambda}\right] = 1 - e^{-\lambda t}.$$

Similarly $F_Y(t) = 1 - e^{-\mu t}$.

4. The computation can be done as in Example 23 page 64 of the lecture notes. A faster computation is possible as X and Y are positive rvs taking values in the interval $[0, +\infty)$. We can compute the expectations $\mathrm{E}[X]$ and $\mathrm{E}[Y]$ as follows.

$$\mathrm{E}[X] = \int_0^\infty P(X > t)dt = \int_0^\infty \mathrm{e}^{-\lambda t} = \left[\frac{\mathrm{e}^{-\lambda t}}{-\lambda}\right]_0^\infty = \frac{0-1}{-\lambda} = \frac{1}{\lambda}.$$

Similarly $\mathrm{E}[Y] = \frac{1}{\mu}$.

5. We use the expression found for $F_Z(t)$ in 2 and the expressions for $F_X(t)$ and $F_Y(t)$ found in 3. We obtain

$$F_Z(t) = 1 - \mathrm{e}^{-\lambda t}\mathrm{e}^{-\mu t} = 1 - \mathrm{e}^{-(\lambda+\mu)t}.$$

Similarly to what was done in 4, we can derive $\mathrm{E}[Z] = \frac{1}{\lambda+\mu}$. According to the cumulative distribution function found for Z, we can say that Z follows an exponential distribution with parameters $\lambda + \mu$. We can simply write $Z \sim \mathrm{Exp}(\lambda + \mu)$.

We have shown that the minimum among exponentially distributed rvs is an exponentially distributed rv whose parameter is the sum of the parameters.

6. Let X be the rv representing the time without SLA violations in the less reliable cluster, and Y be the same for the more reliable cluster. According to the statement, $\mathrm{E}[X] = 1$ month and $\mathrm{E}[Y] = 2$ months. Also $X \sim \mathrm{Exp}(\lambda)$ and $Y \sim \mathrm{Exp}(\mu)$. We then have $\mathrm{E}[X] = 1/\lambda$ and $\mathrm{E}[Y] = 1/\mu$, therefore $\lambda = 1$ violation/month and $\mu = 0.5$ violation/month.

The number of occurrences of SLA violations in a given cluster is a counting stochastic process. The time between two increments of this process is the time between two consecutive violations. This time is exponentially distributed as observed by the administrator. By assuming that all inter-violations durations are independent, then the counting process is a Poisson process with rate equal to the parameter of the exponential distribution of the inter-event time.

Therefore, the SLA violations in the less reliable cluster form a Poisson process with rate $\lambda = 1$ violation/month, and the SLA violations in the more reliable cluster form a Poisson process with rate $\mu = 0.5$ violation/month.

7. The number of occurrences of SLA violations of the data center is the aggregation of the violation processes in the two clusters. Assuming the SLA violations in the two cluster are independent, then we can use Proposition 32 page 73 in the lecture notes to conclude that the SLA violations in the datacenter form a Poisson process with rate $\lambda + \mu = 1.5$ violations/month.

Alternatively, one can consider the time without SLA violation in the datacenter. It is nothing but $\min\{X, Y\} = Z$. We know that $Z \sim \mathrm{Exp}(\lambda + \mu)$, therefore the SLA violations in the datacenter form a Poisson process with rate $\lambda + \mu = 1.5$ violations/month.

8. As there are 3 months between December 2020 and March 2021, we are looking for the probability that Z is less than 3. We have

$$P(Z \leq 3) = F_Z(3) = 1 - \mathrm{e}^{-(\lambda+\mu)3} = 1 - \mathrm{e}^{-4.5} \approx 0.988891\ .$$

9. We know that Z is already larger than 5 months and we are looking for the (conditional) probability that Z is larger than 8 ($= 5 + 3$) months. We use the memoryless property of the exponential distribution (Example 21 in page 63) to write

$$P(Z \geq 8 | Z \geq 5) = P(Z \geq 3) = 1 - F_Z(3) = \mathrm{e}^{-(\lambda+\mu)3} = \mathrm{e}^{-4.5} \approx 0.0111 \ .$$