UBINET/SI5: Performance Evaluation of Networks

Correction of homework 3

3.1 A routing problem

1. There is only one event that affects $X(t)$, the number of transmissions of a given packet at time t: it is a meeting between two nodes in the network. The inter-meeting time is an exponentially distributed rv. There could be multiple inter-meeting times that compete to change the state of the system (that is $X(t)$), so by using construction rule 2 we can say that $\{X(t), t > 0\}$ is a CTMC. It is an absorbing CTMC as states $\{1^*, 2^*, 3^*\}$ are absorbing. There are 3 transient states $\{0, 1, 2\}$. If the chain is being absorbed in state j^*, then this means that j transmissions were necessary to deliver the message to destination and the expected delivery time is the expected time until absorption in state j^* given that the chain was initially in state 0.

2. The infinitesimal generator is

$$\mathbf{Q} = \begin{bmatrix} -3\gamma & 2\gamma & 0 & \gamma & 0 & 0 \\ 0 & -3\gamma & \gamma & 0 & 2\gamma & 0 \\ 0 & 0 & -3\gamma & 0 & 0 & 3\gamma \\ & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

3. The infinitesimal generator can be seen as

$$\mathbf{Q} = \begin{bmatrix} \tilde{\mathbf{Q}} & \tilde{\mathbf{R}} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

According to Proposition 11 in the lecture notes, the matrix giving the expected sojourn times in transient states is $\tilde{\mathbf{T}} = [t_{i,j}]_{0 \le i,j \le 2} = -\tilde{\mathbf{Q}}^{-1}$ and $T(0) = t_{0,0} + t_{0,1} + t_{0,2}$. We have

$$|\tilde{\mathbf{Q}}| = \begin{vmatrix} -3\gamma & 2\gamma & 0 \\ 0 & -3\gamma & \gamma \\ 0 & 0 & -3\gamma \end{vmatrix} = -27\gamma^3 > 0 \;\Rightarrow\; \tilde{\mathbf{T}} = \frac{1}{27\gamma} \begin{bmatrix} 9 & 6 & 2 \\ 0 & 9 & 3 \\ 0 & 0 & 9 \end{bmatrix} \;\Rightarrow\; T(0) = \frac{17}{27\gamma}.$$

Sanity check: all terms in $\tilde{\mathbf{T}}$ are positive or null.

Alternative solution: According to Corollary 2 in the lecture notes, the column vector giving the expected absorption times when initially in a transient state is the solution of $\tilde{\mathbf{Q}}\mathbf{T} = -\mathbf{1}$. We can write

$$\begin{bmatrix} -3\gamma & 2\gamma & 0 \\ 0 & -3\gamma & \gamma \\ 0 & 0 & -3\gamma \end{bmatrix} \cdot \begin{bmatrix} T(0) \\ T(1) \\ T(2) \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} \;\Rightarrow\; \begin{cases} -3\gamma T(0) + 2\gamma T(1) = -1 \\ -3\gamma T(1) + \gamma T(2) = -1 \\ -3\gamma T(2) = -1 \end{cases}$$

$$\Rightarrow \quad T(2) = \frac{1}{3\gamma}, \quad T(1) = \frac{4}{9\gamma}, \quad T(0) = \frac{17}{27\gamma}.$$

Sanity check: no absorbing time is negative.

4. We obtain the embedded Markov chain by observing $\{X(t),\, t \geq 0\}$ at *jump times*. The embedded Markov chain is a homogeneous, absorbing, DTMC. The transition matrix is

$$\mathbf{P} = \begin{bmatrix} 0 & \frac{2}{3} & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ & & & & & \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

5. The transition matrix can be seen as

$$\mathbf{P} = \begin{bmatrix} \mathbf{A} & \mathbf{R} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}$$

According to Proposition 9 in the lecture notes, the matrix giving the absorption probabilities is $\mathbf{B} = [b_{i,j^*}]_{\substack{0 \leq i \leq 2 \\ 1 \leq j \leq 3}} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{R}$. We have

$$|\mathbf{I} - \mathbf{A}| = \begin{vmatrix} 1 & -\frac{2}{3} & 0 \\ 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 1 \end{vmatrix} = 1 > 0 \quad \Rightarrow \quad \mathbf{B} = \begin{bmatrix} 1 & \frac{2}{3} & \frac{2}{9} \\ 0 & 1 & \frac{1}{3} \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{2}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{4}{9} & \frac{2}{9} \\ 0 & \frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 1 \end{bmatrix}.$$

Sanity check: all terms in $\mathbf{B}$ are between 0 and 1 (they are probabilities). The sum of terms in each row is 1.

The expected number of transmissions per message delivery is

$$\mathrm{E}[X] = 1 \cdot b_{0,1^*} + 2 \cdot b_{0,2^*} + 3 \cdot b_{0,3^*} = 1 \cdot \frac{1}{3} + 2 \cdot \frac{4}{9} + 3 \cdot \frac{2}{9} = \frac{17}{9}.$$

6. The new state-space is $\mathcal{E}' = \{0, 1, 2, 3, 1^*, 2^*, 3^*, 4^*\}$.

7. The network is composed of 5 nodes so for each message there can be at most 3 relays. When the Markov chain is in state $i \in \{0, 1, 2\}$, this means that i relays have a copy of the message and $3 - i$ relays do not have a copy of it. The source can meet each of the $3-i$ relays after a time that is $\mathrm{Exp}(\gamma)$, so the first meeting between the source and any of the $3-i$ relays occurs after a time that is $\mathrm{Exp}((3-i)\gamma)$ and once this meeting occurs the Markov chain jumps to transient state $i+1$ as an additional transmission has just been made. On the other hand, when the Markov chain is in state $i \in \{0, 1, 2, 3\}$, this means that there are $i+1$ nodes (i relays and one source) that can meet the destination and deliver it the message. Each of these $i+1$ nodes can meet the destination after a time that is $\mathrm{Exp}(\gamma)$, so the first meeting between any of the $i+1$ nodes and the destination occurs after a time that is $\mathrm{Exp}((i+1)\gamma)$ and once this meeting occurs the Markov chain jumps to absorbing state $(i+1)^*$ as an additional transmission has just been made and the message has been delivered.

8. The new infinitesimal generator is

$$\mathbf{Q}' = \begin{bmatrix} -4\gamma & 3\gamma & 0 & 0 & \gamma & 0 & 0 & 0 \\ 0 & -4\gamma & 2\gamma & 0 & 0 & 2\gamma & 0 & 0 \\ 0 & 0 & -4\gamma & \gamma & 0 & 0 & 3\gamma & 0 \\ 0 & 0 & 0 & -4\gamma & 0 & 0 & 0 & 4\gamma \\ \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

9. To distinguish it from the same quantity for the case when there are 4 nodes in the network we add a prime to the notation. So we are looking for $T'(0)$. We use Corollary 2. We need to find the solution of $\tilde{\mathbf{Q}}'\mathbf{T}' = -\mathbf{1}$. We have

$$\begin{bmatrix} -4\gamma & 3\gamma & 0 & 0 \\ 0 & -4\gamma & 2\gamma & 0 \\ 0 & 0 & -4\gamma & \gamma \\ 0 & 0 & 0 & -4\gamma \end{bmatrix} \cdot \begin{bmatrix} T'(0) \\ T'(1) \\ T'(2) \\ T'(3) \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -1 \\ -1 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} -4\gamma T'(0) + 3\gamma T'(1) = -1 \\ -4\gamma T'(1) + 2\gamma T'(2) = -1 \\ -4\gamma T'(2) + \gamma T'(3) = -1 \\ -4\gamma T'(3) = -1 \end{cases}$$

$$\Rightarrow \quad T'(3) = \frac{1}{4\gamma}, \quad T'(2) = \frac{5}{16\gamma}, \quad T'(1) = \frac{13}{32\gamma}, \quad T'(0) = \frac{71}{128\gamma}\ .$$

Sanity check: no absorbing time is negative.

With 4 nodes, the expected time to deliver the message is $T(0) = \frac{17}{27\gamma} \approx \frac{0.63}{\gamma}$. With 5 nodes, $T'(0) = \frac{71}{128\gamma} \approx \frac{0.55}{\gamma}$. We have $T'(0) < T(0)$. As the number of nodes increases, the expected delivery time decreases. This result could be expected for the following reason. By allowing the use of relays, we enable the possibility to have more nodes carrying the message meet the destination and deliver the message sooner. By increasing the number of nodes, what we are increasing is the number of relays, so the time until one of them meets the destination decreases on average. As the inter-meeting time is exponentially distributed, by having more possible meetings, the time that one of them occurs is the minimum among all of the inter-meeting times, with a rate that increases with the number of possible meetings. Therefore we expect $T(0)$ to decrease as the number of relays (i.e. the number of nodes in the network) increases.

10. As the number of nodes in the network increases, the number of relays increases and so does the number of transmissions. There will be more copies of each message in the network. Therefore, the cost $\mathrm{E}[X]$ should increase with the number of nodes.