Performance Evaluation of Networks

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Ch 1 - Discrete-Time Markov Chain (DTMC)

- Properties of DTMC:
  - aperiodicity, irreducibility, positive recurrence, ergodicity
- *n*-step transition matrix $\mathbf{P}^n$
- Transient distribution

$$\pi(n) = \pi(0)\,\mathbf{P}^n$$

- Limiting distribution

$$\lim_{n\to\infty} \pi(n) = \pi(0)\ \lim_{n\to\infty} \mathbf{P}^n$$

- Stationary distribution

$$\pi = \pi\mathbf{P}$$

$$\pi\,\mathbf{1} = 1$$

Ch 2 - Continuous-Time Markov Chain (CTMC)

- Stochastic process $\{X(t), t \geq 0\}$ with discrete state-space and verifying Markov property

$$P(X(t) = j \,|\, X(s_1) = i_1, \ldots, X(s_{n-1}) = i_{n-1}, X(s) = i)$$
$$= P(X(t) = j \,|\, X(s) = i)$$
$$i_1, \ldots, i_{n-1}, i, j \in \mathcal{E}$$
$$0 \leq s_1 < \ldots < s_{n-1} < s < t$$

is a continuous-time Markov chain

- CTMC is homogeneous if

$$P(X(t) = j \,|\, X(s) = i) = p_{i,j}(t-s)$$
$$\forall i, j \in \mathcal{E}, 0 \leq s < t$$

Chapman-Kolmogorov Equation

- Proposition 4: For all $t > 0, s > 0, i, j \in \mathcal{E}$

$$\boxed{p_{i,j}(t+s) = \sum_{k \in \mathcal{E}} p_{i,k}(t)\, p_{k,j}(s)} \qquad \boxed{\mathbf{P}(t+s) = \mathbf{P}(t) \cdot \mathbf{P}(s)}$$

DTMC:

$$p_{i,j}^{(n+m)} = \sum_{k \in \mathcal{E}} p_{i,k}^{(n)}\, p_{k,j}^{(m)} \qquad \mathbf{P}^{(n+m)} = \mathbf{P}^{(n)}\, \mathbf{P}^{(m)}$$

$$\mathbf{P}^{(n)} = \mathbf{P}^{n}$$

- Proof: use law of total probability and Markov property
- Matrix notation

$$\mathbf{P}(t) = \left[p_{i,j}(t) \right]_{i,j \in \mathcal{E}}$$

- CTMC: real steps vs. DTMC: integer steps

Infinitesimal Generator

- Define

$$q_{i,i} := \lim_{h\to 0} \frac{p_{i,i}(h) - 1}{h} \leq 0$$

$$q_{i,j} := \lim_{h\to 0} \frac{p_{i,j}(h)}{h} \geq 0$$

$$q_{i,i} = -\sum_{j\neq i} q_{i,j}$$

- Matrix $\boxed{\mathbf{Q} = [q_{i,j}]} = \lim_{h\to 0} \frac{\mathbf{P}(h) - \mathbf{I}}{h}$

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & \cdots \\ 0 & 1 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

- If CTMC can take any integer value

$$\mathbf{Q} = \begin{pmatrix} -\sum_{j\neq 0} q_{0,j} & q_{0,1} & q_{0,2} & \cdots \\ q_{1,0} & -\sum_{j\neq 1} q_{1,j} & q_{1,2} & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} \qquad \sum = 0$$

Interpretation

- $q_{i,j}$ is the transition rate to state j when in state i
- $-q_{i,i}$ is transition rate out of state i

- Sojourn time in state i is Exp($-q_{i,i}$) $P(S(i) > x) = \mathrm{e}^{q_{i,i}x}$

Proof: assume $X(0) = i$ and sojourn time is $S(i)$

$$P(S(i) > x + h) = P\big(S(i) > x \textbf{ and } X(t) = i, x < t \leq x + h\big)$$
$$= P(S(i) > x)\, P\big(X(t) = i, x < t \leq x + h\big)$$

When $h \to 0, \quad P\big(X(t) = i, x < t \leq x + h\big) \approx p_{i,i}(h)$

$$q_{i,i} = \lim_{h \to 0} \frac{p_{i,i}(h) - 1}{h} \quad \Rightarrow \quad p_{i,i}(h) = 1 + hq_{i,i} + o(h)$$

Proof of Exp Sojourn Time (continued)

$$P(S(i) > x + h) - P(S(i) > x) = P(S_i > x)(hq_{i,i} + o(h))$$

Divide by h and take limit as $h \to 0$

$$\frac{dP(S(i) > x)}{dx} = q_{i,i}\, P(S(i) > x)$$

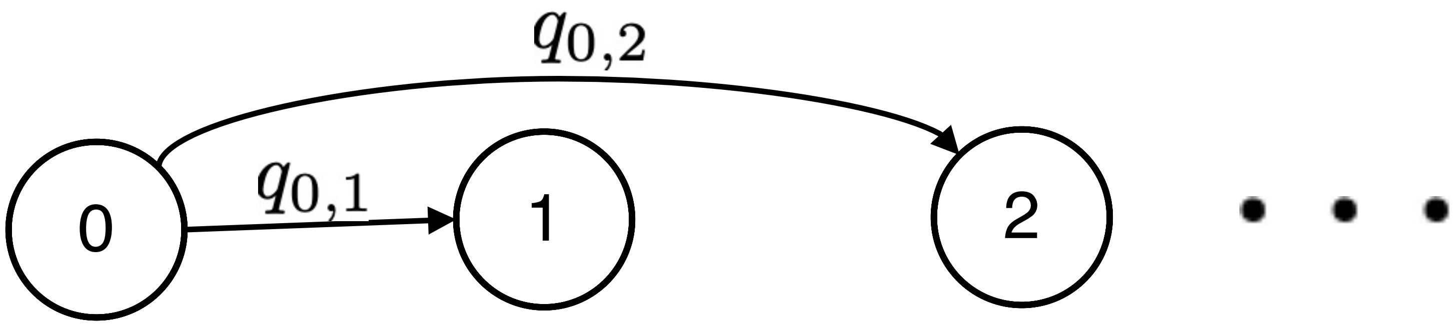
Initial condition is $P(S(i) > 0) = 1$

Solution of differential equation is

$$P(S(i) > x) = \exp(q_{i,i} x)$$

➜ Sojourn time is exponentially distributed

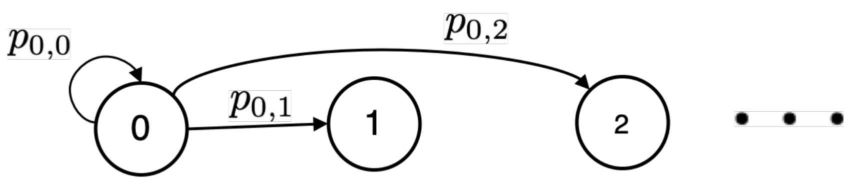
Transition Rate Diagram



- Sum of arrows out of a state i is $-q_{i,i}$

- For DTMC

Transition Diagram



Sum of arrows out of a state is 1

- Probability to go to state j when in i is

$$p(i,j) = \frac{q_{i,j}}{\sum_{j \neq i} q_{i,j}} = \frac{q_{i,j}}{-q_{i,i}} \qquad \boxed{\Rightarrow \ q_{i,j} = -q_{i,i}p(i,j)}$$

Transient State Distribution

- We want probability that the system is in state *i* at time *t*

$$\pi_i(t) := P(X(t) = i)$$

- Law of total probabilities: for any j

$$\pi_j(t+h) = \sum_{i \in \mathcal{E}} p_{i,j}(h)\,\pi_i(t) = \sum_{i \neq j} p_{i,j}(h)\pi_i(t) + p_{j,j}(h)\pi_j(t)$$

- Subtract $\pi_j(t)$ from both sides then divide by h

$$\boxed{\frac{\pi_j(t+h) - \pi_j(t)}{h}} = \sum_{\substack{i \in \mathcal{E} \\ i \neq j}} \boxed{\frac{p_{i,j}(h)}{h}} \pi_i(t) + \boxed{\frac{p_{j,j}(h) - 1}{h}} \pi_j(t)$$

- $h \to 0:$ $$\frac{d\pi_j(t)}{dt} = \sum_{\substack{i \in \mathcal{E} \\ i \neq j}} q_{i,j}\pi_i(t) + q_{j,j}\pi_j(t) = \sum_{i \in \mathcal{E}} q_{i,j}\pi_i(t)$$

Transient State Distribution

- For any j, $$\frac{d\pi_j(t)}{dt} = \sum_{i \in \mathcal{E}} q_{i,j} \pi_i(t)$$
- Row vector $$\pi(t) = (\pi_i(t), i \in \mathcal{E})$$
- In matrix notation $$\frac{d}{dt}\pi(t) = \pi(t)\mathbf{Q}, \qquad t \geq 0$$
- Solution $$\boxed{\pi(t) = \pi(0)\, e^{\mathbf{Q}t}}, \qquad t \geq 0$$
- By definition $$e^{\mathbf{Q}t} = \sum_{k=0}^{\infty} \frac{(\mathbf{Q}t)^k}{k!}$$
- For DTMC $$\boxed{\pi(n) = \pi(0)\, \mathbf{P}^n}$$

Limiting State Distribution

- From equation giving transient distribution
$$\pi(t) = \pi(0)\, e^{\mathbf{Q}(t-h)} e^{\mathbf{Q}h} = \pi(t-h) e^{\mathbf{Q}h}$$
- For DTMC $\quad \pi(n) = \pi(n-m)\mathbf{P}^m$
- If limit exists $\quad \pi = \lim_{t\to\infty} \pi(t) = (\pi_i, i \in \mathcal{E})$
- $h \to 0: \quad \pi(t) = \pi(t)(\mathbf{I} + \mathbf{Q}h + o(h))$
- $t \to \infty: \quad \pi = \pi + \pi\mathbf{Q}h + \pi o(h)$
- Divide by h and take limit as $h \to 0$ $\quad \boxed{\pi\mathbf{Q} = 0}$
- Normalization $\boxed{\pi\mathbf{1} = 1}$ $\quad$ For DTMC $\pi\mathbf{P} = \pi$

CTMC Properties

- A CTMC is irreducible if all pairs of states communicate
- A CTMC is positive recurrent if all states are positive recurrent
- A CTMC is ergodic if irreducible and positive recurrent
- Irreducible + finite state-space ➜ positive recurrent
  ➜ ergodic

➜ long-run distribution = invariant measure
= limiting distribution

Existence of Limiting Distribution

- If homogeneous CTMC is irreducible
- If system of equations

$$\pi \mathbf{Q} = 0$$

$$\pi \mathbf{1} = 1$$

has unique strictly positive solution

➜ $$\lim_{t \to \infty} \Big(P\big(X(t) = i\big), i \in \mathcal{E} \Big) = \pi$$

The limiting distribution exists and it is the stationary distribution

15 minutes break

Balance Equations

- Develop $\pi \mathbf{Q} = 0$

- For all $i \in \mathcal{E}$ $\quad \sum_{j \in \mathcal{E}} \pi_j \, q_{j,i} = 0$

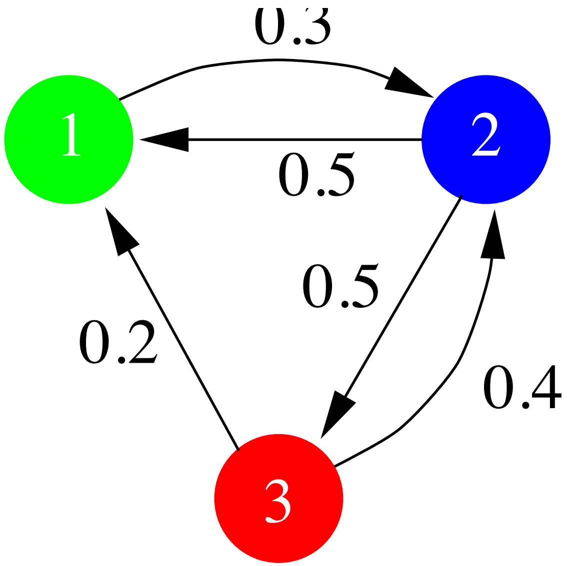
$$-\pi_i \, q_{i,i} = \sum_{j \neq i} \pi_j \, q_{j,i}$$

$$\boxed{\left(\sum_{j \neq i} q_{i,j} \right) \pi_i = \sum_{j \neq i} q_{j,i} \, \pi_j}$$

probability flow rate out of state i = probability flow rate into state i

- $q_{j,i}$: transition rate to state i when in state j
- $q_{j,i}\pi_j$: probability flow rate from state j to state i

Example



$$\mathcal{E} = \{1, 2, 3\}$$

CTMC irreducible? ✔

$$\mathbf{Q} = \begin{pmatrix} -0.3 & 0.3 & 0 \\ 0.5 & -1 & 0.5 \\ 0.2 & 0.4 & -0.6 \end{pmatrix}$$

flow out = flow in

$$\left\{ \begin{array}{rcl} 0.3\pi_1 & = & 0.5\pi_2 + 0.2\pi_3 \\ (0.5+0.5)\pi_2 & = & 0.3\pi_1 + 0.4\pi_3 \\ (0.2+0.4)\pi_3 & = & 0.5\pi_2 \\ \pi_1 + \pi_2 + \pi_3 & = & 1 \end{array} \right. \Rightarrow \left\{ \begin{array}{l} \pi_2 = 0.3\pi_1 + 0.4\pi_3 \\ 1.2\pi_3 = \pi_2 \\ \pi_1 + \pi_2 + \pi_3 = 1 \end{array} \right.$$

Example

$$\begin{cases} \pi_2 = 0.3\pi_1 + 0.4\pi_3 \\ 1.2\pi_3 = \pi_2 \quad ✔ \\ \pi_1 + \pi_2 + \pi_3 = 1 \end{cases} \qquad \Rightarrow 1.2\pi_3 = 0.3\pi_1 + 0.4\pi_3$$

$$\Rightarrow \pi_1 = \frac{8}{3}\pi_3 \quad ✔$$

- Normalization

$$\pi_3\left(\frac{8}{3}+\frac{6}{5}+1\right) = 1 \quad \Leftrightarrow \quad \pi_3\left(\frac{40+18+15}{15}\right) = 1 \quad \Leftrightarrow \quad \pi_3 = \frac{15}{73}$$

$$\boxed{\Rightarrow \quad \pi = (\pi_1, \pi_2, \pi_3) = \frac{1}{73}(40, 18, 15)}$$

sanity check
$\Sigma = 1$

- Solution is unique and strictly positive

  ➜ this is the limiting distribution

Recap

	DTMC	CTMC
Chapman-Kolmogorov	$\mathbf{P}^{(n+m)} = \mathbf{P}^{(n)}\,\mathbf{P}^{(m)}$	$\mathbf{P}(t+s) = \mathbf{P}(t)\cdot\mathbf{P}(s)$
Main matrix	Transition matrix P	Infinitesimal generator Q
with characteristics	Row sum is 1	Row sum is 0
Diagram	Transition probabilities	Transition rates
with characteristics	Arrows out sum to 1	No loops
Transient distribution	$\pi(n) = \pi(0)\,\mathbf{P}^n$	$\pi(t) = \pi(0)\,e^{\mathbf{Q}t}$
Stationary distribution	$\pi\mathbf{P} = \pi$ $\quad \pi\mathbf{1} = 1$	$\pi\mathbf{Q} = 0$ $\quad \pi\mathbf{1} = 1$
Properties to check	Aperiodicity, irreducibility	Irreducibility
Sojourn time	Geometric	$\boxed{\text{Exp}(-q_{i,i})}$
Markov property	Easy to check	????

$$P(X(t) = j \,|\, X(s_1) = i_1, \ldots, X(s_{n-1}) = i_{n-1}, X(s) = i)$$

$$= P(X(t) = j \,|\, X(s) = i) = p_{i,j}(t-s)$$

$$i_1, \ldots, i_{n-1}, i, j \in \mathcal{E}$$

$$0 \leq s_1 < \ldots < s_{n-1} < s < t$$

Impossible to check for CTMC

Construction Rule #1

- Continuous-time stochastic process $\{X(t), t \geq 0\}$
- If for each state i
  - Process stays in i for a time that is Exp(τ_i)
  - Once sojourn time is over, process jumps to state j with prob a_{ij} (a_{ii} = 0, their sum for all j is 1)

➜ $\{X(t), t \geq 0\}$ is a CTMC

- Infinitesimal generator

$$\mathbf{Q} = \left[q_{i,j}\right]_{i,j \in \mathcal{E}}, \quad q_{i,j} = \begin{cases} \tau_i a_{ij} & i \neq j \\ -\tau_i & i = j \end{cases}$$

Construction Rule #2

- Continuous-time stochastic process $\{X(t), t \geq 0\}$
- For each state i, as soon as process enters state i
  - For each other state j generate sample for $Y_{i,j}$ from Exp($\mu_{i,j}$)
    (if no transition from i to j, then $\mu_{i,j}$ is 0 and sample is ∞)
  - The first sample to expire makes process jump to corresponding state

➜ $\{X(t), t \geq 0\}$ is a CTMC

- Infinitesimal generator

$$\mathbf{Q} = \left[q_{i,j}\right]_{i,j\in\mathcal{E}}, \quad q_{i,j} = \begin{cases} \mu_{i,j} & j \neq i \\ -\sum_{k\neq i} \mu_{i,k} & j = i \end{cases}$$

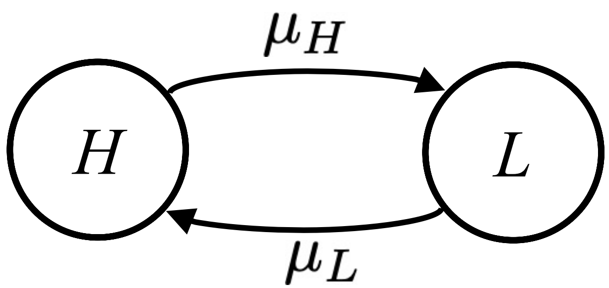
Construction Rule #2

- In practice we use this construction when at each state i
  - Several processes or events can cause a state change
  - Each lasts for a time Exp()
- Example:
  - Jobs are submitted to a server according to Poisson process rate λ ➜ time to submit job is Exp(λ)
  - A job has 1 task with prob ½ and 2 tasks with prob ½
  - Server processes tasks one at a time, service time Exp(μ)
  - State is number of tasks, state-space is set of integers

$$\left.\begin{array}{lll} i \to i+1 & \mathrm{Exp}(\lambda/2) & \\ i \to i+2 & \mathrm{Exp}(\lambda/2) & \\ i \to i-1 & \mathrm{Exp}(\mu) & i > 0 \end{array}\right\} \text{ we have a CTMC}$$

Example 2.1 Page 19

- Stochastic process $\mathbf{Y} = \{Y(t), t \geq 0\}$ alternates between states *H* and *L*
- Sojourn time in *H* is Exp(μ_H) with mean $1/\mu_H$
- Sojourn time in *L* is Exp(μ_L) with mean $1/\mu_L$
- $\mathcal{E} = \{H, L\}$ prob 1 to switch state
- Construction rule #1 ➜ $\mathbf{Y}$ is a CTMC
- Infinitesimal generator $\mathbf{Q} = \begin{pmatrix} -\mu_H & \mu_H \\ \mu_L & -\mu_L \end{pmatrix}$
- Transition diagram



Global Balance Equations

- Balance equations: for any state i

$$\left(\sum_{j\neq i} q_{i,j}\right)\pi_i = \sum_{j\neq i} q_{j,i}\,\pi_j$$

probability flow rate out of state i

= probability flow rate into state i

- Global balance equations: S subset of $\mathcal{E}$

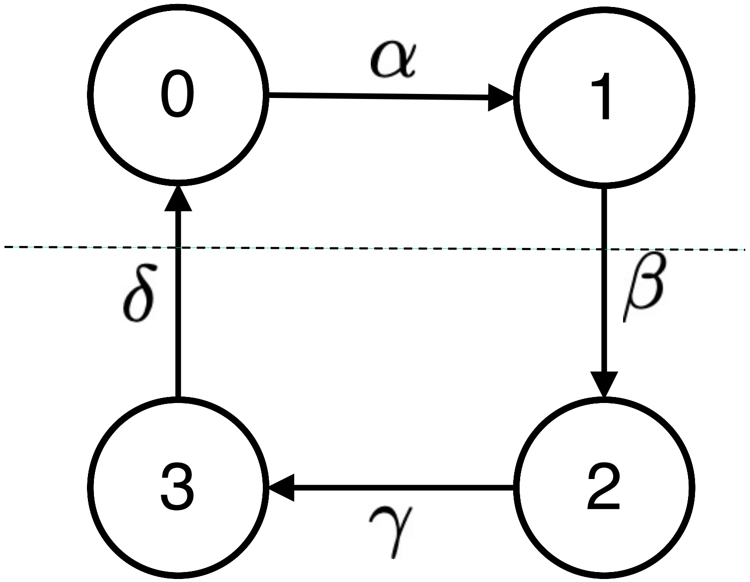
probability flow rate out of S = probability flow rate into S

$$\boxed{\sum_{i\in\mathcal{S}}\sum_{j\in\overline{\mathcal{S}}}\pi_i q_{i,j} = \sum_{i\in\overline{\mathcal{S}}}\sum_{j\in\mathcal{S}}\pi_i q_{i,j}}$$

▶ If $\mathcal{S}=\{i\}$ ➜ balance equation for state i

Global Balance Equations

- Proof: use $\pi\mathbf{Q} = 0$ and $\mathbf{Q1} = 0$ and find that diff is 0
- Example $\mathcal{S} = \{0, 1\}$



global balance equation

$$\beta\pi_1 = \delta\pi_3$$

balance equations

$$\boxed{\begin{aligned} \alpha\pi_0 &= \delta\pi_3 \\ \beta\pi_1 &= \alpha\pi_0 \end{aligned}}$$

$$\gamma\pi_2 = \beta\pi_1$$

$$\delta\pi_3 = \gamma\pi_2$$

- By summing first two expressions we find the result

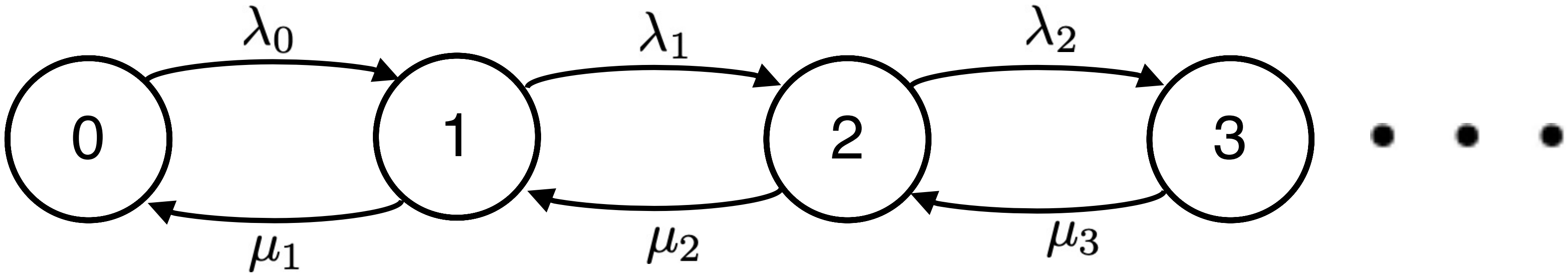
Birth and Death Process

- Particular CTMC over set of integers (or a subset)
  - New state is a neighbor of the previous state
- Birth and Death process seen as population size
- Birth rate when state is $i \geq 0, \quad \lambda_i = q_{i,i+1}$
- Death rate when state is $i \geq 1, \quad \mu_i = q_{i,i-1}$
- Infinitesimal generator is tri-diagonal matrix

$$\mathbf{Q} = \begin{pmatrix} -\lambda_0 & \lambda_0 & 0 & \cdots & & \\ \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 & 0 & \cdots & \\ 0 & \mu_2 & -(\lambda_2+\mu_2) & \lambda_2 & 0 & \cdots \\ \vdots & 0 & \mu_3 & -(\lambda_3+\mu_3) & \lambda_3 & \ddots \\ & \vdots & 0 & \mu_4 & \ddots & \ddots \\ & & \vdots & \ddots & \ddots & \end{pmatrix}$$

Birth and Death Process

- Transition rate diagram $\mathcal{E} = \mathbb{N}$



- Balance equations

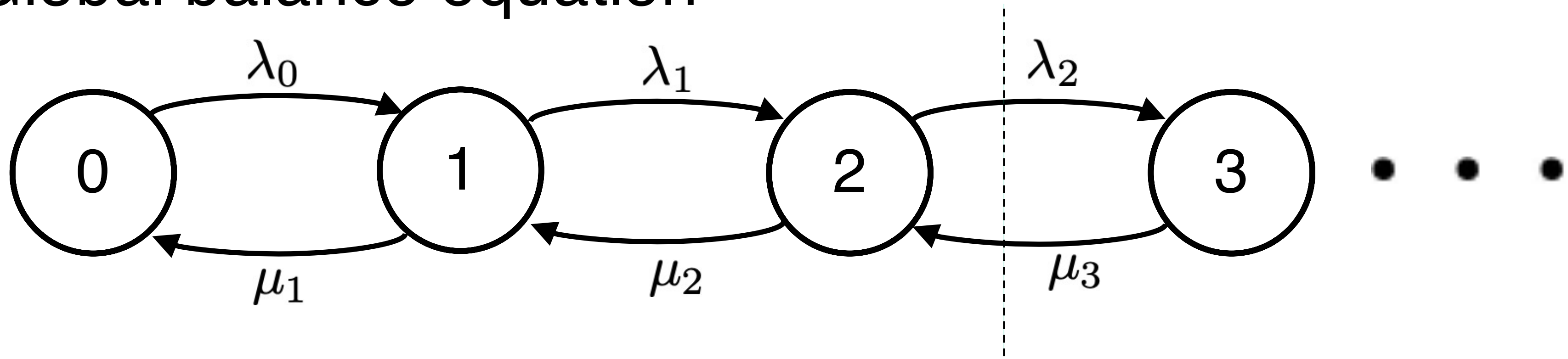
$$
\begin{aligned}
\cancel{\lambda_0\pi_0} &= \cancel{\mu_1\pi_1} \\
(\cancel{\lambda_1} + \cancel{\mu_1})\pi_1 &= \cancel{\lambda_0\pi_0} + \cancel{\mu_2\pi_2} \\
&\vdots \\
(\cancel{\lambda_{i-1}} + \cancel{\mu_{i-1}})\pi_{i-1} &= \cancel{\lambda_{i-2}\pi_{i-2}} + \cancel{\mu_i\pi_i} \\
(\lambda_i + \cancel{\mu_i})\pi_i &= \cancel{\lambda_{i-1}\pi_{i-1}} + \mu_{i+1}\pi_{i+1}
\end{aligned}
$$

$$\lambda_i\pi_i = \mu_{i+1}\pi_{i+1}$$

$$\boxed{\Rightarrow \quad \pi_{i+1} = \frac{\lambda_i}{\mu_{i+1}}\pi_i, \quad i = 0, 1, \ldots}$$

Birth and Death Process

- Global balance equation



$$\lambda_2 \pi_2 = \mu_3 \pi_3$$

- For any state $i > 0$ $\quad \lambda_{i-1}\pi_{i-1} = \mu_i \pi_i$

Birth and Death Process

- By recurrence $$\boxed{\pi_i = \frac{\lambda_0 \lambda_1 \cdots \lambda_{i-1}}{\mu_1 \mu_2 \cdots \mu_i} \pi_0, \quad i = 1, 2, \ldots}$$
- Normalization $$\sum_{i=0}^{\infty} \pi_i = 1$$

$$\pi_0 \left(1 + \frac{\lambda_0}{\mu_1} + \frac{\lambda_0 \lambda_1}{\mu_1 \mu_2} + \cdots + \frac{\lambda_0 \lambda_1 \cdots \lambda_{i-1}}{\mu_1 \mu_2 \cdots \mu_i} + \cdots \right) = 1$$

- Define $$C := \sum_{i \geq 0} \prod_{j=1}^{i} \frac{\lambda_{j-1}}{\mu_j}$$
- If $C < \infty$ (stability condition) then $\boxed{\pi_0 = \frac{1}{C} > 0}$
- Irreducible ➜ limiting distribution exists ($C < \infty$)

Time-Reversibility

- A CTMC is time-reversible if for any two states $i, j \in \mathcal{E}$

$$\pi_i q_{i,j} = \pi_j q_{j,i}$$

- A birth and death process is time-reversible
- Consequence:

  forwards chain and reverse chain are statistically identical and described by same transition diagram

- Example

  Forward trajectory …012123210123210101 0…

  Backward trajectory …010101232101232121 0…

  Probability flow rates between any two states are same in either direction (e.g. #23 = #32)

For next week

- Lesson 2 to revise
- Homework 2 to return on Tuesday 24 September before 9.00 am
- Lesson 3 to read before Lecture 3