Performance Evaluation of Networks

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Ch 6 – Queueing Networks

- Until now only one queue (one service)
  - Chapter 4: $M/M/1$, $M/M/1/K$, $M/M/c$, $M/M/c/c$
  - Chapter 5: $M/G/1$ FIFO, $M/G/1$ FIFO with vacations
- Multiple queues ➜ network of queues
  - Open versus closed network

    Open ➜ customers enter and leave network of queues

    Closed ➜ fixed number of customers in network of queues
  - Single class versus multi-class

    Single ➜ all customers are identical

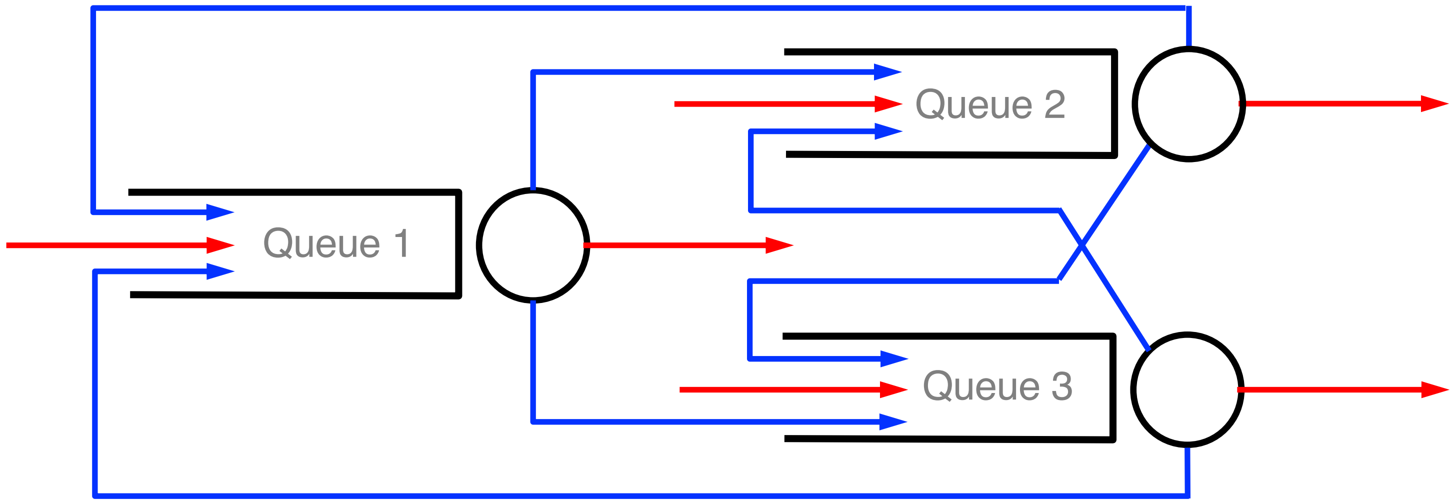
    ➜ same routing rules among queues for all

    Multi-class ➜ customers are identical within each class

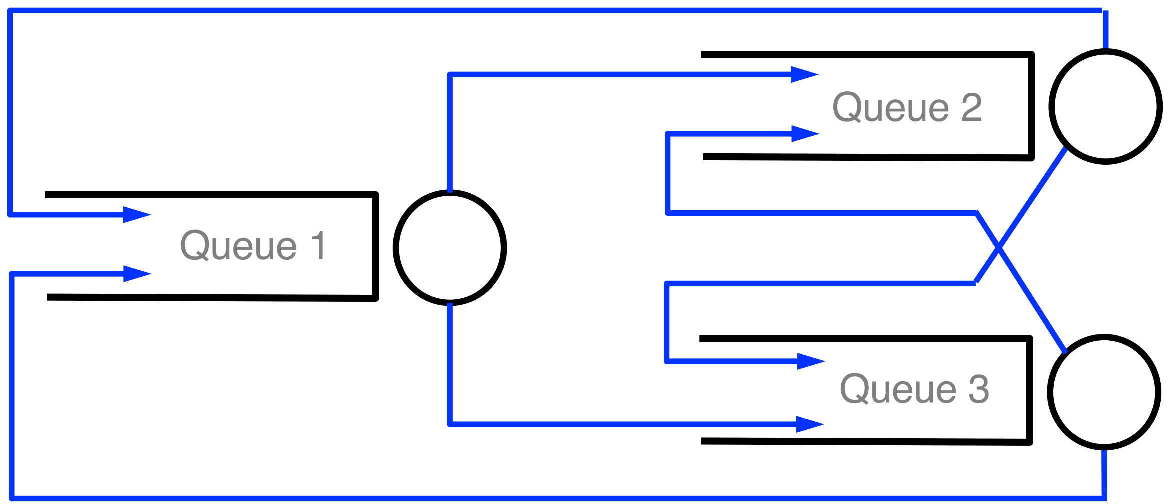
    ➜ routing rules among queues depend on class

Open Versus Closed Network

- Open network

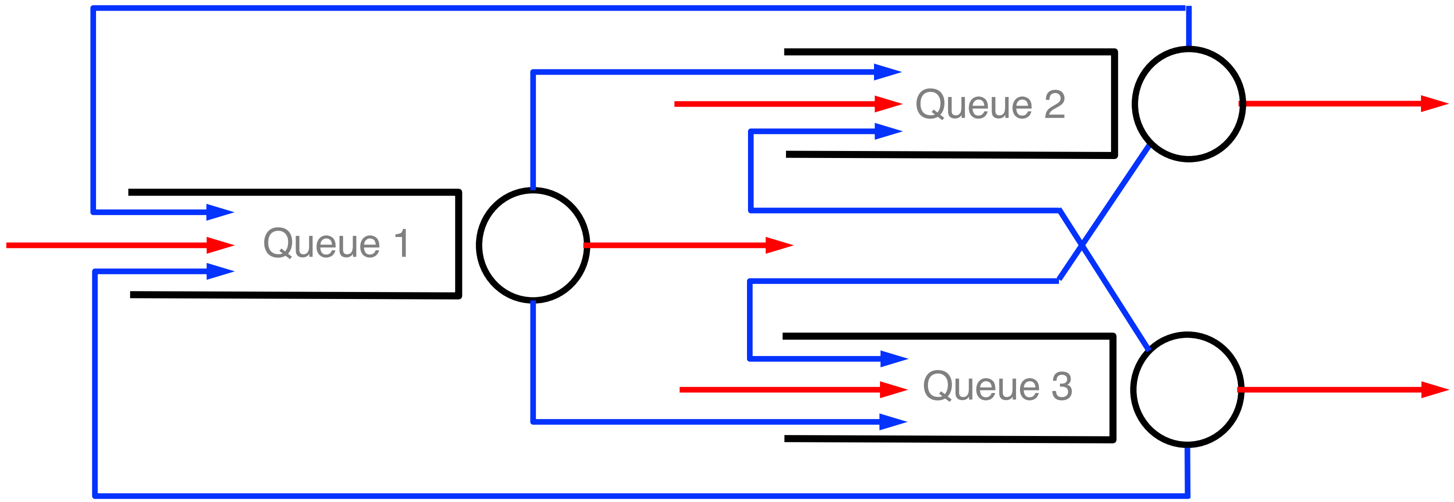


- Closed network

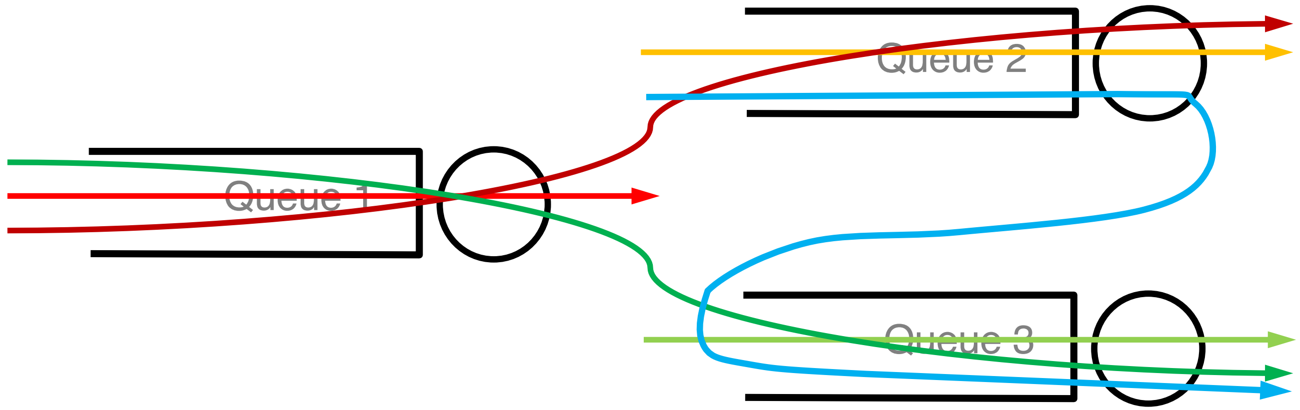


Single Class Versus Multi-Class

- Single class

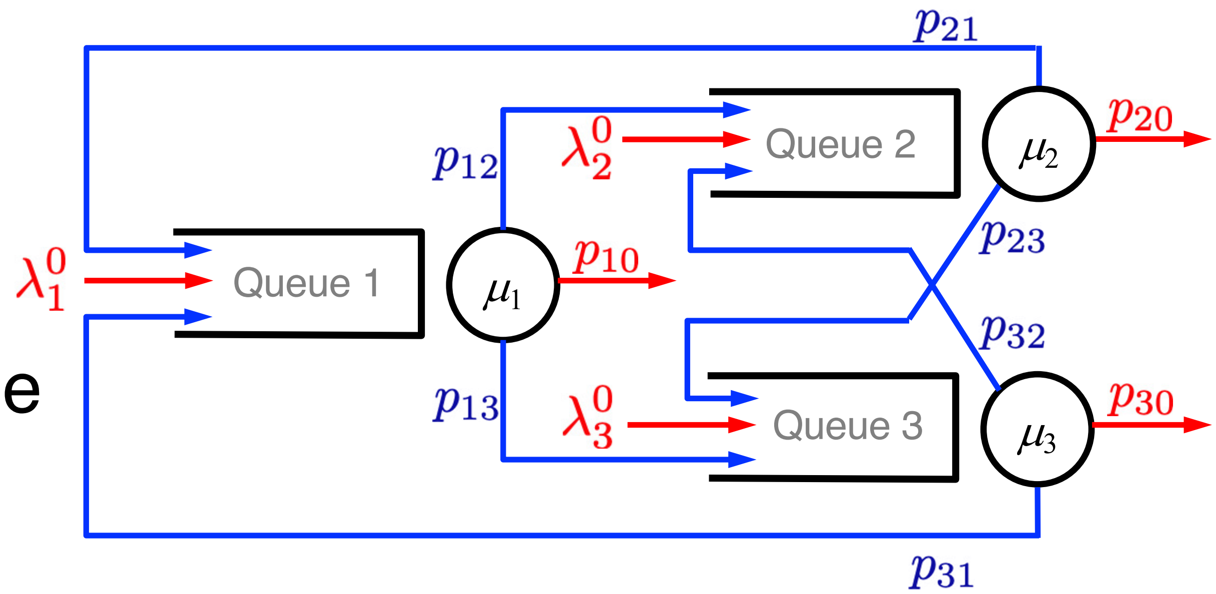


- Multi-class ➜ specific routes



Open Jackson Network

- Open network
- Single class
- K queues
- One server per queue
- Infinite capacity
- In queue i
  - External customers Poisson process rate λ_i^0
  - Service time is Exp(μ_i)
  - Served customer
    - leaves system with probability p_{i0}
    - goes to queue j with probability p_{ij}
  - Arrivals and service independent



$$\Rightarrow \sum_{j=0}^{K} \boxed{p_{ij}} = 1$$

routing probabilities

Open Jackson Network

- How to study this system?
- Can we study each queue separately?

No!

inter-arrivals in one queue depend on events in other queues

- State of system = queue size in each queue
- $X_i(t)$ queue size at time t in queue i
- Define vector $\mathbf{X}(t) = \left(X_1(t), \ldots, X_K(t)\right) \quad \forall t \geq 0$
- State space $\mathcal{E} = \mathbb{N}^K$
- Is $\{\mathbf{X}(t),\, t \geq 0\}$ a Markov process?

Yes!

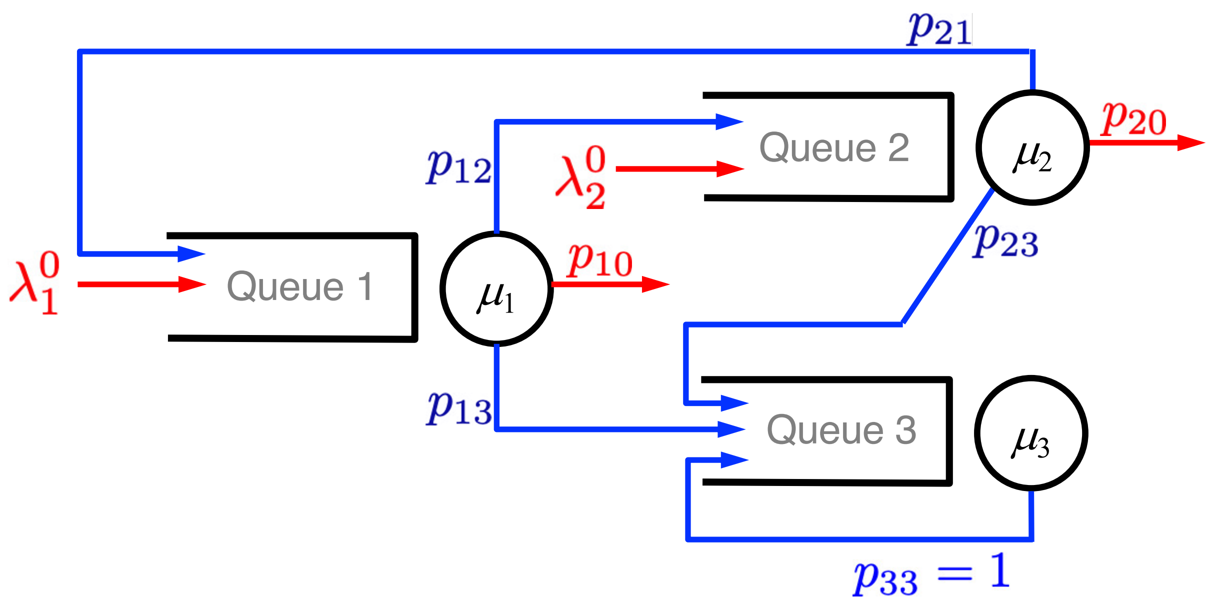
use construction rule #2

Using Construction Rule #2

- Consider state $\mathbf{n} = (n_1, \ldots, n_K) \in \mathbb{N}^K$
- Define $\mathbf{e}_i = (0, \ldots, 0, \underset{\substack{\uparrow \\ \text{position } i}}{1}, 0, \ldots, 0) \in \mathbb{N}^K$
- Possible transitions?
  - $\mathbf{n} \to \mathbf{n} + \mathbf{e}_i$ external arrival in queue i, time Exp(λ_i^0)
  - $\mathbf{n} \to \mathbf{n} - \mathbf{e}_i$ service end in queue i + leave system
    $n_i > 0$ time Exp($\mu_i p_{i0}$)
  - $\mathbf{n} \to \mathbf{n} - \mathbf{e}_i + \mathbf{e}_j$ service end in queue i + move to queue j
    $n_i > 0$ $j \neq i$ time Exp($\mu_i p_{ij}$)
- $\{\mathbf{X}(t),\, t \geq 0\}$ is a homogeneous CTMC ✔

Irreducibility?

- Need to check that any two states communicate



customers in queue 3 need a way out $p_{33} < 1$

- Can state $\mathbf{n} = (n_1, n_2, n_3)$ with $n_3 > 0$ reach $\mathbf{0} = (0, 0, 0)$?

No!

- Markov chain of shown network is not irreducible

Routing Matrix

- Define matrix with internal routing probabilities

$$\mathbf{P} = \begin{pmatrix} \boxed{p_{11} & p_{12} & \cdots & p_{1K}} \\ p_{21} & p_{22} & \cdots & p_{2K} \\ \vdots & \vdots & \ddots & \vdots \\ p_{K1} & p_{K2} & \cdots & p_{KK} \end{pmatrix} \qquad \sum = 1 - p_{10}$$

- Definitions
  - Queue is open if its customers are certain to leave system
  - If all queues open ➜ network of queues is completely open

sufficient condition

one queue with $p_{i0} > 0$ and paths from all other queues to it

Irreducibility Condition

- Matrix $\mathbf{I} - \mathbf{P}$ is invertible

$$\mathbf{I}-\mathbf{P}=\begin{pmatrix} \boxed{1-p_{11} & -p_{12} & \cdots & -p_{1K}} \\ -p_{21} & 1-p_{22} & \cdots & -p_{2K} \\ \vdots & \vdots & \ddots & \vdots \\ -p_{K1} & -p_{K2} & \cdots & 1-p_{KK} \end{pmatrix} \qquad \sum = p_{10}$$

- Jackson queueing network is completely open

- Above statements are equivalent

Stationary/Limiting Distribution

- We want to compute vector $\pi = \left(\pi_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^K\right)$

$$\pi_{\mathbf{n}} = \lim_{t \to \infty} P\left(\mathbf{X}(t) = \mathbf{n}\right)$$
$$= \lim_{t \to \infty} P\left(X_1(t) = n_1, X_2(t) = n_2, \ldots, X_K(t) = n_K\right)$$

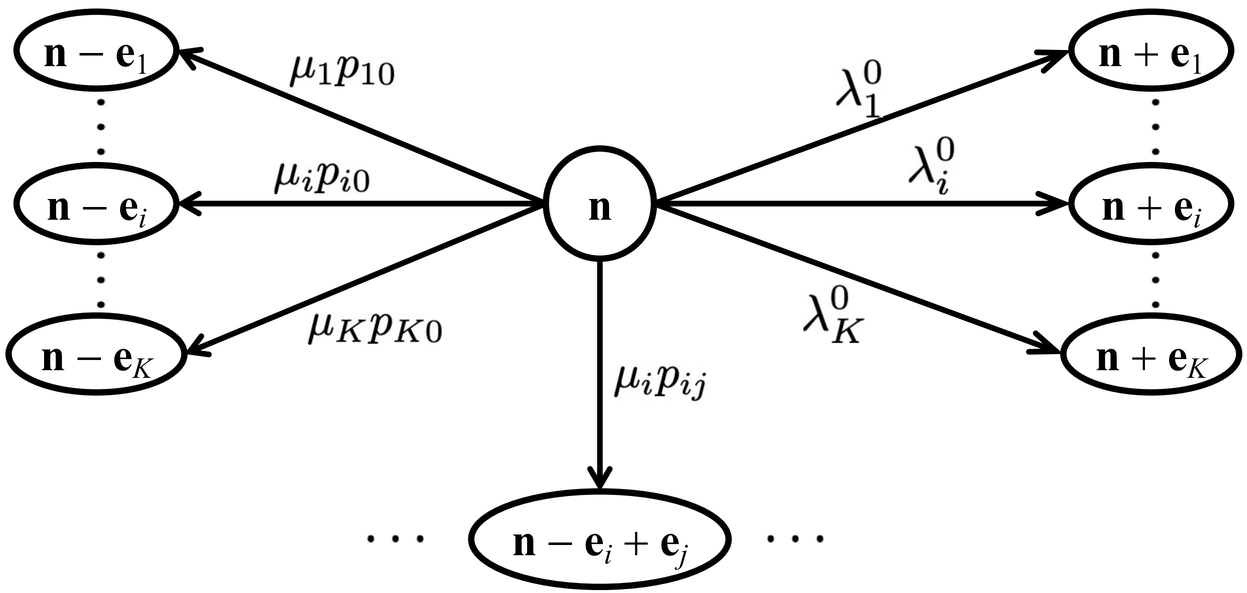
- Existence of Limiting Distribution
  - If homogeneous CTMC is irreducible ✔
  - If system of equations $\pi \mathbf{Q} = 0$
    $\pi \mathbf{1} = 1$?

    has unique strictly positive solution

    ➜ limiting distribution exists and it is the solution found

Balance Equations: Flow Out of State **n**

- Showing transitions out of state $\mathbf{n}$



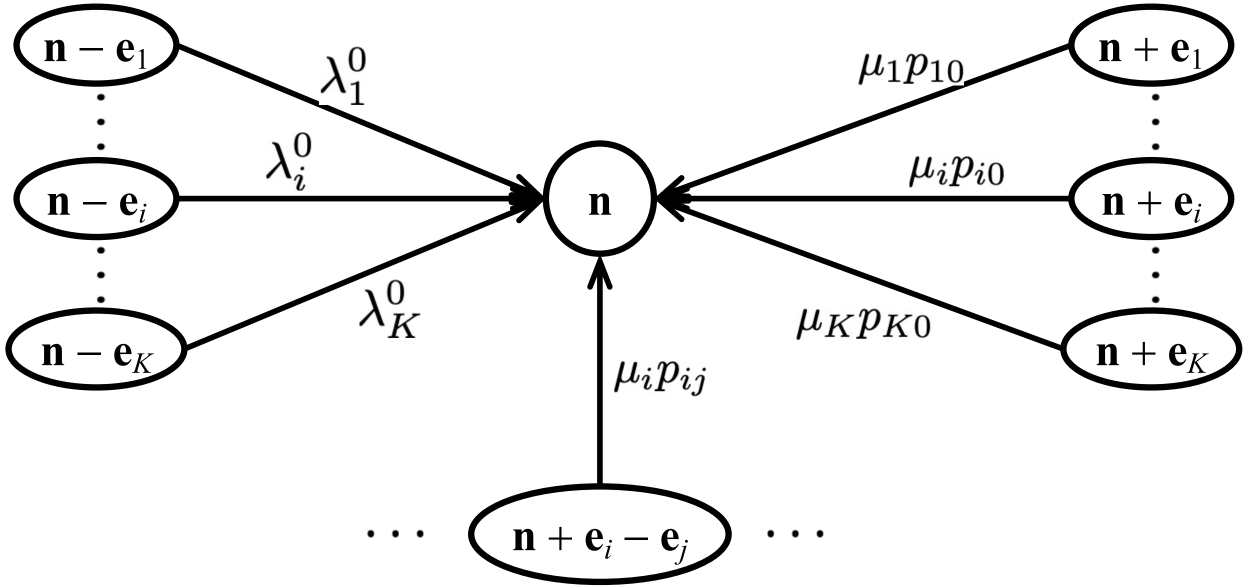
- Flow out $\pi_{\mathbf{n}}\left(\sum_{i=1}^{K}\lambda_i^0+\sum_{i=1}^{K}\boxed{\mathbb{1}_{n_i>0}}\boxed{(1-p_{ii})}\mu_i\right)$

prob to leave queue

$= 1$ if $n_i > 0$

Balance Equations: Flow In State $\mathbf{n}$

- Showing transitions into state $\mathbf{n}$



- Flow in

$$\underbrace{\boxed{\sum_{i=1}^{K} \mathbb{1}_{n_i>0}\lambda_i^0 \pi_{\mathbf{n}-\mathbf{e}_i}}}_{\text{exogenous arrivals}} + \underbrace{\boxed{\sum_{i=1}^{K} \mu_i p_{i0} \pi_{\mathbf{n}+\mathbf{e}_i}}}_{\text{departure from queue}} + \underbrace{\boxed{\sum_{i=1}^{K} \sum_{\substack{j=1 \\ j\neq i}}^{K} \mathbb{1}_{n_j>0}\, \mu_i p_{ij} \pi_{\mathbf{n}+\mathbf{e}_i-\mathbf{e}_j}}}_{\text{movements between queues}}$$

Proposition 18 (Jackson, 1957)

- Find unique nonnegative solution of system

traffic equations

$$\lambda_i = \lambda_i^0 + \sum_{j=1}^{K} p_{ji}\,\lambda_j, \qquad i = 1, 2, \ldots, K$$

- Define $\rho_i = \lambda_i / \mu_i$ for all i
- If $\lambda_i < \mu_i$ for all i

$$\boxed{\pi_{\mathbf{n}} = \prod_{i=1}^{K} (1-\rho_i)\rho_i^{n_i}, \quad \forall \mathbf{n} \in \mathbb{N}^K}$$

- Traffic equations using routing matrix $\mathbf{P}$

$$\left.\begin{aligned} \boldsymbol{\lambda} &= (\lambda_1, \ldots \lambda_K) \\ \boldsymbol{\lambda}^0 &= (\lambda_1^0, \ldots \lambda_K^0) \end{aligned}\right\} \qquad \begin{aligned} &\boldsymbol{\lambda} = \boldsymbol{\lambda}^0 + \boldsymbol{\lambda}\mathbf{P} \\ \Leftrightarrow \quad &\boldsymbol{\lambda} = \boldsymbol{\lambda}^0 \underbrace{\boxed{(\mathbf{I} - \mathbf{P})^{-1}}}_{\text{irreducibility}} \end{aligned}$$

Proposition 19 (Jackson, 1957)

- If matrix $\mathbf{I} - \mathbf{P}$ invertible
- If $\boxed{\lambda_i < \mu_i \text{ for all } i}$ stability condition
- Then

$$\pi_{\mathbf{n}} = \prod_{i=1}^{K} (1 - \rho_i)\rho_i^{n_i}, \quad \forall \mathbf{n} \in \mathbb{N}^K$$

where $\boldsymbol{\lambda} = \boldsymbol{\lambda}^0 (\mathbf{I} - \mathbf{P})^{-1}$

- What is the meaning of λ_i?

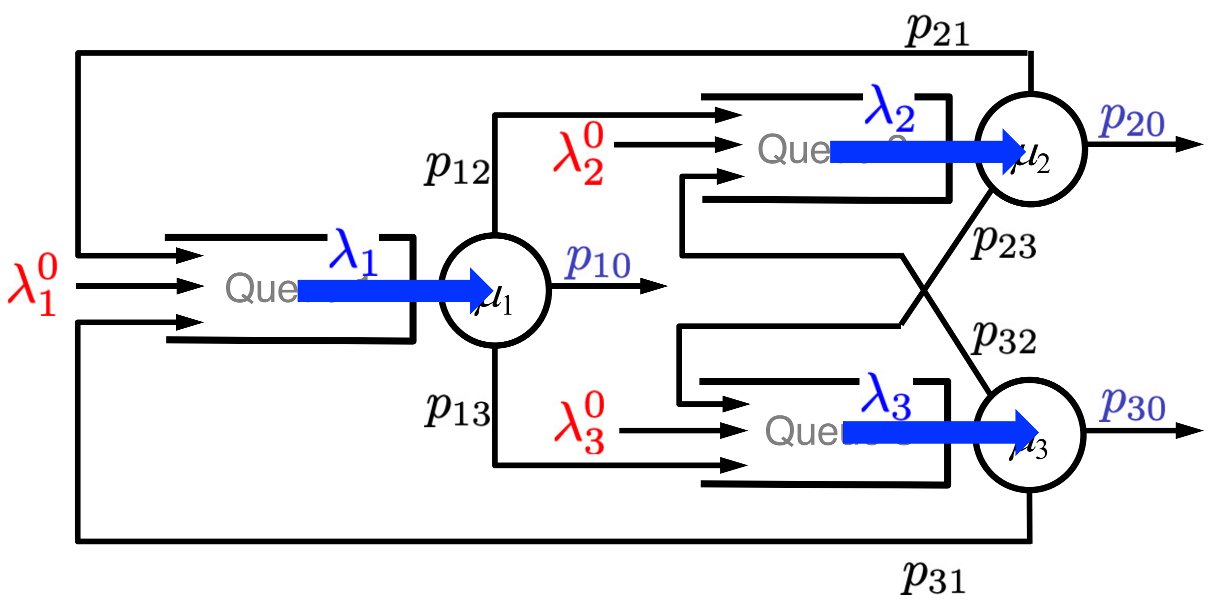
➜ total arrival rate to queue i

also throughput of queue i

Example

- In steady-state

$$\lambda_1 = \lambda_1^0 + p_{21}\lambda_2 + p_{31}\lambda_3$$
$$\lambda_2 = \lambda_2^0 + p_{12}\lambda_1 + p_{32}\lambda_3$$
$$\lambda_3 = \lambda_3^0 + p_{13}\lambda_1 + p_{23}\lambda_2$$



- By summing traffic equations

$$p_{10}\lambda_1 + p_{20}\lambda_2 + p_{30}\lambda_3 = \lambda_1^0 + \lambda_2^0 + \lambda_3^0$$

system output rate = system input rate

- In general

$$\sum_{i=1}^{K} p_{i0}\,\lambda_i = \sum_{i=1}^{K} \lambda_i^0$$

Stationary/Limiting Distribution

- Let us check that

$$\pi_{\mathbf{n}} = \prod_{i=1}^{K} (1 - \rho_i)\rho_i^{n_i}, \quad \forall \mathbf{n} \in \mathbb{N}^K$$

is solution of the system of equation

$$\pi \mathbf{Q} = 0$$

$$\pi \mathbf{1} = 1$$

Checking $\pi \mathbf{1} = 1$

- We have
$$\sum_{\mathbf{n}\in\mathbb{N}^K} \pi_{\mathbf{n}} = \sum_{n_1\in\mathbb{N},\ldots,n_K\in\mathbb{N}} \prod_{i=1}^{K}(1-\rho_i)\rho_i^{n_i}$$
$$= \sum_{n_1\in\mathbb{N}} \cdots \sum_{n_K\in\mathbb{N}} \prod_{i=1}^{K}(1-\rho_i)\prod_{i=1}^{K}\rho_i^{n_i}$$
$$= \prod_{i=1}^{K}(1-\rho_i)\sum_{n_1\in\mathbb{N}}\rho_1^{n_1}\cdot\ldots\cdot\sum_{n_K\in\mathbb{N}}\rho_K^{n_K}$$

If $\rho_i < 1$ for all i
$$= \prod_{i=1}^{K}(1-\rho_i)\prod_{i=1}^{K}\frac{1}{1-\rho_i}$$
$$= 1 \checkmark$$

Checking $\pi\,\mathbf{Q} = 0$

- From $\quad \pi_{\mathbf{n}} = \prod_{i=1}^{K} (1-\rho_i)\rho_i^{n_i}, \quad \forall \mathbf{n} \in \mathbb{N}^K$

  we find following relations

$$\pi_{\mathbf{n}-\mathbf{e}_i} = \frac{\pi_{\mathbf{n}}}{\rho_i} \qquad \pi_{\mathbf{n}+\mathbf{e}_i} = \rho_i \pi_{\mathbf{n}} \qquad \pi_{\mathbf{n}+\mathbf{e}_i-\mathbf{e}_j} = \frac{\rho_i \pi_{\mathbf{n}}}{\rho_j}$$

- Balance equation for state $\mathbf{n}$

$$\cancel{\pi_{\mathbf{n}}}\left(\sum_{i=1}^{K} \lambda_i^0 + \sum_{i=1}^{K} \mathbb{1}_{n_i>0}\,(1-p_{ii})\mu_i\right)$$

$$= \sum_{i=1}^{K} \mathbb{1}_{n_i>0}\lambda_i^0 \boxed{\pi_{\mathbf{n}-\mathbf{e}_i}} + \sum_{i=1}^{K} \mu_i p_{i0} \boxed{\pi_{\mathbf{n}+\mathbf{e}_i}} + \sum_{i=1}^{K} \sum_{\substack{j=1 \\ j\neq i}}^{K} \mathbb{1}_{n_j>0}\,\mu_i p_{ij} \boxed{\pi_{\mathbf{n}+\mathbf{e}_i-\mathbf{e}_j}}$$

$\rho_i = \lambda_i / \mu_i$

$$= \sum_{i=1}^{K} \mathbb{1}_{n_i>0}\lambda_i^0 \frac{\cancel{\pi_{\mathbf{n}}}}{\rho_i} + \sum_{i=1}^{K} \mu_i p_{i0} \rho_i \cancel{\pi_{\mathbf{n}}} + \sum_{i=1}^{K} \sum_{\substack{j=1 \\ j\neq i}}^{K} \mathbb{1}_{n_j>0}\,\mu_i p_{ij} \frac{\rho_i \cancel{\pi_{\mathbf{n}}}}{\rho_j}$$

$$\sum_{i=1}^{K} \lambda_i^0 + \sum_{i=1}^{K} \mathbb{1}_{n_i>0}\,(1-p_{ii})\mu_i = \sum_{i=1}^{K} \mathbb{1}_{n_i>0} \frac{\lambda_i^0 \mu_i}{\lambda_i} + \sum_{i=1}^{K} p_{i0}\,\lambda_i + \sum_{i=1}^{K} \sum_{\substack{j=1 \\ j\neq i}}^{K} \mathbb{1}_{n_j>0} p_{ij} \frac{\lambda_i \mu_j}{\lambda_j}$$

Checking $\pi \mathbf{Q} = 0$

$$\sum_{i=1}^K \lambda_i^0 + \boxed{\sum_{i=1}^K \mathbb{1}_{n_i>0}\,(1-\cancel{p_{ii}})\mu_i} = \sum_{i=1}^K \mathbb{1}_{n_i>0}\,\frac{\lambda_i^0\mu_i}{\lambda_i} + \boxed{\sum_{i=1}^K p_{i0}\,\lambda_i} + \sum_{i=1}^K \sum_{\substack{j=1\\ \cancel{j\neq i}}}^K \mathbb{1}_{n_j>0}\,p_{ij}\,\frac{\lambda_i\mu_j}{\lambda_j}$$

$$\sum_{i=1}^K \lambda_i^0 - \sum_{i=1}^K p_{i0}\,\lambda_i = -\sum_{i=1}^K \mathbb{1}_{n_i>0}\,\mu_i + \sum_{i=1}^K \mathbb{1}_{n_i>0}\,\frac{\lambda_i^0\mu_i}{\lambda_i} + \sum_{i=1}^K \sum_{j=1}^K \mathbb{1}_{n_j>0}\,p_{ij}\,\frac{\lambda_i\mu_j}{\lambda_j}$$

$$= -\sum_{i=1}^K \mathbb{1}_{n_i>0}\,\mu_i + \sum_{i=1}^K \mathbb{1}_{n_i>0}\,\frac{\lambda_i^0\mu_i}{\lambda_i} + \sum_{j=1}^K \sum_{i=1}^K \mathbb{1}_{n_i>0}\,p_{ji}\,\frac{\lambda_j\mu_i}{\lambda_i}$$

$$= -\sum_{i=1}^K \mathbb{1}_{n_i>0}\,\frac{\mu_i}{\lambda_i}\left(\boxed{\lambda_i - \lambda_i^0 - \sum_{j=1}^K p_{ji}\lambda_j}\right)$$

traffic equations

$$0 = 0 \quad ✔$$

Conclusion: limiting distribution exists and is

$$\pi_{\mathbf{n}} = \prod_{i=1}^K (1-\rho_i)\rho_i^{n_i}, \quad \forall \mathbf{n} \in \mathbb{N}^K$$

Product-Form Solution $\pi_{\mathbf{n}} = \prod_{i=1}^{K} (1-\rho_i)\rho_i^{n_i}$

- Queue size of $M/M/1$ with arrival rate λ_i and service rate μ_i has limiting distribution

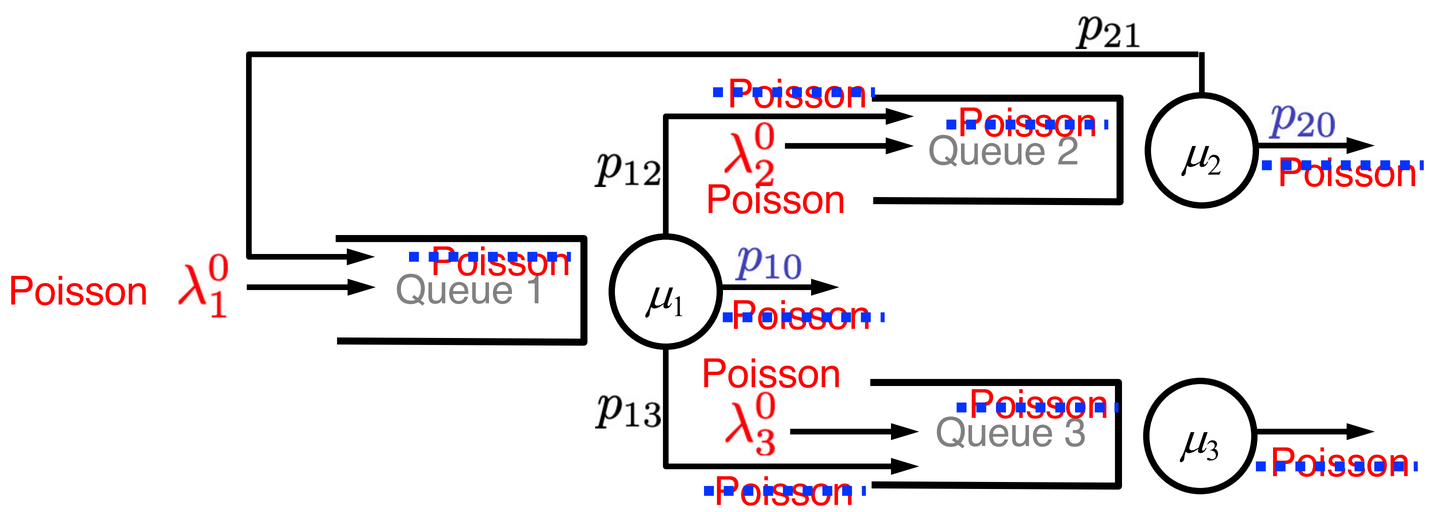
$$\pi_{n_i} = \lim_{t\to\infty} P(X_i(t) = n_i) = (1-\rho_i)\rho_i^{n_i}, \quad \forall n_i \in \mathbb{N}$$

- Solution is product of solution of each queue taken alone
- But
  - Queue sizes are correlated between queues
  - Arrivals are not Poisson in each queue in general
  - Each queue is not $M/M/1$
- Yet
  - Queue size of each queue same distribution as $M/M/1$

Arrivals are Not Poisson in Each Queue

- Burke's theorem
  - Departure process of stationary $M / M / 1$ is Poisson
- Thinning of Poisson process ➜ Poisson process
- Aggregation of independent Poisson process

  ➜ Poisson process



15 minutes break

Jackson Network of Multi-Servers Queues

- Open Jackson network where queue i has c_i servers
- Find unique nonnegative solution of system

traffic equations

$$\lambda_i = \lambda_i^0 + \sum_{j=1}^{K} p_{ji}\,\lambda_j, \qquad i = 1, 2, \ldots, K$$

- Define $\mu_i(r) = \mu_i \min(r, c_i)$ for any positive r and for all i
- Define $\rho_i = \lambda_i / c_i \mu_i$ for all i
- If $\lambda_i < c_i \mu_i$ for all i

$$\pi_{\mathbf{n}} = \prod_{i=1}^{K} C_i \left(\frac{\lambda_i^{n_i}}{\prod_{r=1}^{n_i} \mu_i(r)} \right), \quad \forall \mathbf{n} \in \mathbb{N}^K$$

$$C_i = \left[\sum_{r=0}^{c_i - 1} \left(\frac{\lambda_i}{\mu_i} \right)^r \frac{1}{r!} + \left(\frac{\lambda_i}{\mu_i} \right)^{c_i} \frac{1}{c_i!} \left(\frac{1}{1 - \rho_i} \right) \right]^{-1}$$

Jackson Network of Multi-Servers Queues

- Product-form solution

- Each term in product relates to distribution of specific queue

- Queue i is not $M/M/c_i$ but distribution of queue size is same

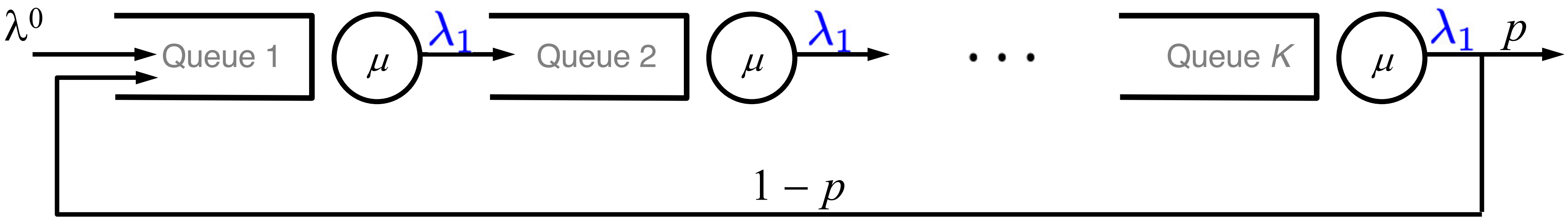
Example 10 Page 52

- Packets travel through K identical nodes to reach destination
- Destination cannot decode a ratio of $1-p$ packets due to transmission errors along the path

  ➜ a negative acknowledgement (NACK) is sent to source
- Source resends packet upon receiving a NACK
- Source generates packets according to Poisson rate λ^0
- NACK travel time is negligible
- Each node has service time Exp(μ)
- Questions:
  - Mean number of packets in network?
  - Expected sojourn time in network?

Example 10 Page 52

- We have a Jackson network



- Routing probabilities

$$p_{i,i+1} = 1 \quad i = 1, \ldots, K-1 \qquad p_{K,0} = p \qquad p_{K,1} = 1 - p$$

- All nodes are open ➜ network completely open

$p > 0$

- Traffic equations

$$\left.\begin{aligned} \lambda_1 &= \lambda^0 + (1-p)\lambda_K \\ \lambda_i &= \lambda_{i-1} \quad i = 2, \ldots, K \end{aligned}\right\} \Rightarrow \quad \lambda_i = \frac{\lambda^0}{p} \quad i = 1, \ldots, K$$

Example 10 Page 52

- Irreducibility condition: $p > 0$
- Stability condition: $\lambda^0 < p\mu$
- Joint distribution of number of customers in system

$$\pi_{\mathbf{n}} = \prod_{i=1}^{K} (1-\rho)\rho^{n_i}, \quad \forall \mathbf{n} \in \mathbb{N}^K$$

$$= \left(\frac{p\mu - \lambda^0}{p\mu}\right)^K \left(\frac{\lambda^0}{p\mu}\right)^{n_1+\cdots+n_K}$$

- Queue size in queue i similar to $M / M / 1$ with arrival rate λ^0/p and service rate μ

$$\overline{X_i} = \frac{\lambda^0}{p\mu - \lambda^0}$$

Example 10 Page 52

- Expected number of packets in system

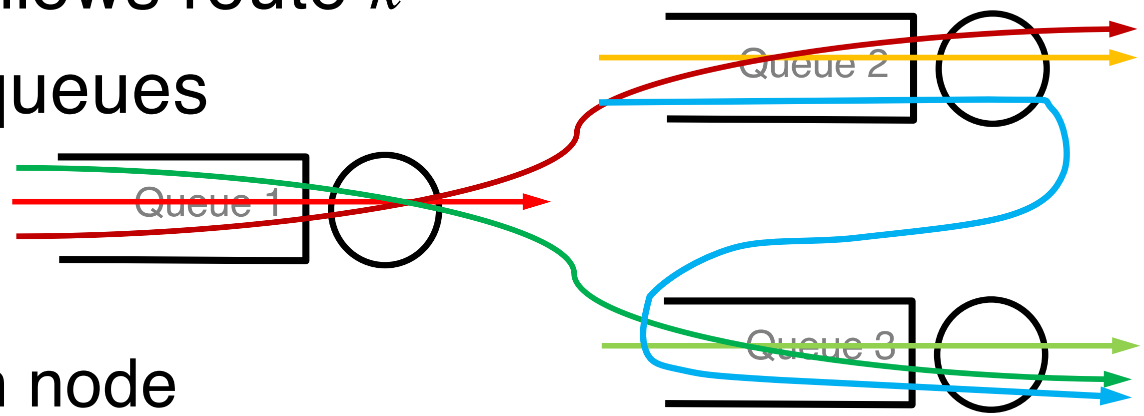
$$\overline{N} = \sum_{i=1}^{K} \overline{X_i} = \frac{K\lambda^0}{p\mu - \lambda^0}$$

- By Little: expected sojourn time in system is

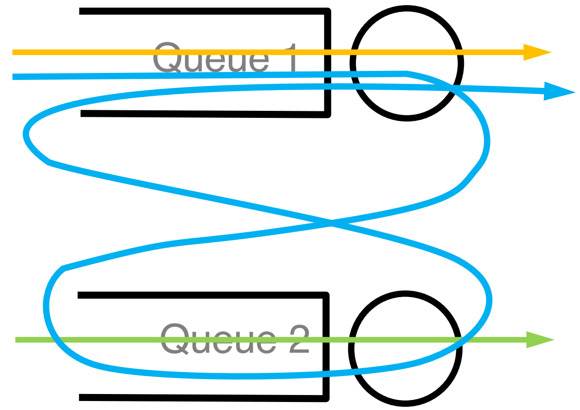
$$\overline{T} = \frac{\overline{N}}{\lambda^0} = \frac{K}{p\mu - \lambda^0}$$

Multiclass Kelly Network

- Objective: study systems where paths through network are deterministic
- Class k ➜ route $r_k = (r_k^1, \ldots, r_k^{n_k})$
  - repetitions are possible within a route
- Customer in class k follows route k
- System = network of queues
  - K nodes/queues
  - R routes
  - Single server at each node
  - Infinite waiting room
  - Poisson arrivals in class k rate λ_k
  - Service time Exp(μ_i) at node i



Multiclass Kelly Network



- How to study this system?
- Can we study each queue separately?

No!

arrivals in one queue depend on events in other queues

- State = queue size in each queue and each class
- $X_{ik}(t)$ number of class k customers at time t in queue i
- Define matrix $\mathbf{X}(t) = [X_{ik}(t)]_{\substack{1 \le i \le K \\ 1 \le k \le R}} \quad \forall t \geq 0$
- State space $\mathcal{E}_{KR}$: set of K–by–R matrices w/ entries in $\mathbb{N}$
- Is $\{\mathbf{X}(t),\, t \geq 0\}$ a Markov process?

No!

need to track number of visits to same node

Stationary/Limiting distribution

- For $\mathbf{N} = \left[n_{ik}\right]_{\substack{1\le i\le K\\1\le k\le R}} \in \mathcal{E}_{KR}$ define

$$\pi_{\mathbf{N}} = \lim_{t\to\infty} P(\mathbf{X}(t) = \mathbf{N}) = \lim_{t\to\infty} P\big(X_{ik}(t) = n_{ik}; 1 \le i \le K, 1 \le k \le R\big)$$

- Kelly (1975)
  - Compute global arrival rate of class k customers in node i

$$\hat{\lambda}_{ik} = \lambda_k \sum_{j=1}^{n_k} \mathbb{1}_{r_k^j = i} = \begin{cases} 0 & \text{if node } i \text{ not in route } r_k \\ \ell\lambda_k & \text{if node } i \text{ appears } \ell \text{ times in route } r_k \end{cases}$$

  - Compute global arrival rate in node i $\quad \hat{\lambda}_i = \sum_{k=1}^{R} \hat{\lambda}_{ik}$
  - If $\hat{\lambda}_i < \mu_i$ (stability condition) for each node

$$\pi_{\mathbf{N}} = \prod_{i=1}^{K} \left(1 - \frac{\hat{\lambda}_i}{\mu_i}\right) \boxed{\binom{\sum_{k=1}^{R} n_{ik}}{n_{i1}, n_{i2}, \ldots, n_{iR}}} \prod_{k=1}^{R} \left(\frac{\hat{\lambda}_{ik}}{\mu_i}\right)^{n_{ik}}$$

multinomial coefficient

$$\frac{(\sum_{k=1}^{R} n_{ik})!}{\prod_{k=1}^{R} n_{ik}!}$$

Stationary/Limiting distribution

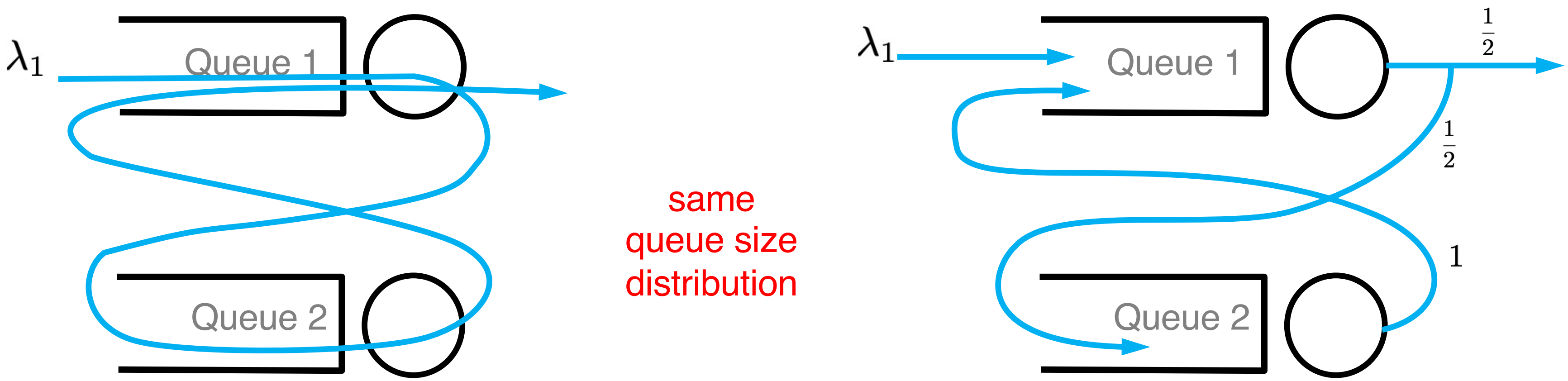
- Product-form solution

$$\pi_{\mathbf{N}} = \prod_{i=1}^{K} \boxed{\left(1 - \frac{\hat{\lambda}_i}{\mu_i}\right) \binom{\sum_{k=1}^{R} n_{ik}}{n_{i1}, n_{i2}, \ldots, n_{iR}} \prod_{k=1}^{R} \left(\frac{\hat{\lambda}_{ik}}{\mu_i}\right)^{n_{ik}}}$$

distribution of queue size in queue i

- If single class ($R = 1$) ➜ same as in Jackson network

$$\pi_{\mathbf{N}} = \prod_{i=1}^{K} \left(1 - \frac{\hat{\lambda}_i}{\mu_i}\right) \left(\frac{\hat{\lambda}_i}{\mu_i}\right)^{n_i}$$



Expected Number of Customers

- Expected number of class k customers in queue i

$$\overline{N}_{ik} = E[X_{ik}] = \frac{\hat{\lambda}_{ik}}{\mu_i - \hat{\lambda}_i}$$

- Expected number of customers in queue i

$$\overline{N}_i = \sum_{k=1}^{R} \overline{N}_{ik} = \frac{\sum_{k=1}^{R} \hat{\lambda}_{ik}}{\mu_i - \hat{\lambda}_i} = \frac{\hat{\lambda}_i}{\mu_i - \hat{\lambda}_i}$$

- Expected number of class k customers in network

$$\overline{N}^{(k)} = \sum_{i=1}^{K} \overline{N}_{ik}$$

Expected Sojourn Time

- Use Little's formula
- Expected sojourn time of class k customers

$$\overline{T}_k = \frac{\overline{N}^{(k)}}{\lambda_k} = \frac{1}{\lambda_k} \sum_{i=1}^{K} \frac{\hat{\lambda}_{ik}}{\mu_i - \hat{\lambda}_i}$$

$$\hat{\lambda}_{ik} = \begin{cases} 0 \\ \ell \lambda_k \end{cases}$$

- Expected sojourn time of arbitrary customer

$$\overline{T} = \frac{\sum_{i=1}^{K} \overline{N}_i}{\sum_{k=1}^{R} \lambda_k} = \frac{1}{\sum_{k=1}^{R} \lambda_k} \sum_{i=1}^{K} \frac{\hat{\lambda}_i}{\mu_i - \hat{\lambda}_i}$$

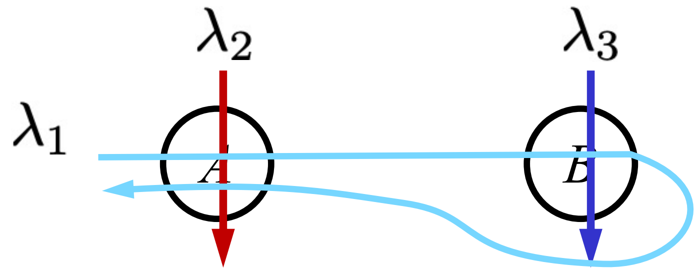
- We have $$\overline{T} = \sum_{k=1}^{R} \boxed{\frac{\lambda_k}{\sum_{l=1}^{R} \lambda_l}} \overline{T}_k$$

probability customer is from class k

Example

- Kelly network 2 nodes 3 routes

  $r_1 = (A, B, A) \quad r_2 = (A) \quad r_3 = (B)$



- Service rates μ_A and μ_B
- We can compute

$$\hat{\lambda}_{A1} = 2\lambda_1 \qquad \hat{\lambda}_{A2} = \lambda_2 \qquad \hat{\lambda}_{A3} = 0 \qquad \hat{\lambda}_A = 2\lambda_1 + \lambda_2$$

$$\hat{\lambda}_{B1} = \lambda_1 \qquad \hat{\lambda}_{B2} = 0 \qquad \hat{\lambda}_{B3} = \lambda_3 \qquad \hat{\lambda}_B = \lambda_1 + \lambda_3$$

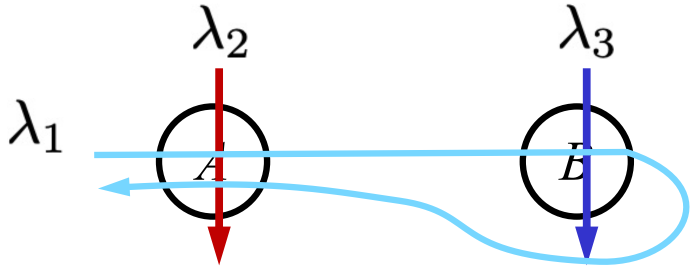
- Expected number of customers

$$\boxed{\overline{N}_{A1} = \frac{2\lambda_1}{\mu_A - 2\lambda_1 - \lambda_2}} \quad \boxed{\overline{N}_{A2} = \frac{\lambda_2}{\mu_A - 2\lambda_1 - \lambda_2}} \quad \boxed{\overline{N}_{A3} = 0} \quad \boxed{\overline{N}_A = \frac{2\lambda_1 + \lambda_2}{\mu_A - 2\lambda_1 - \lambda_2}}$$

$$\boxed{\overline{N}_{B1} = \frac{\lambda_1}{\mu_B - \lambda_1 - \lambda_3}} \quad \boxed{\overline{N}_{B2} = 0} \quad \boxed{\overline{N}_{B3} = \frac{\lambda_3}{\mu_B - \lambda_1 - \lambda_3}} \quad \boxed{\overline{N}_B = \frac{\lambda_1 + \lambda_3}{\mu_B - \lambda_1 - \lambda_3}}$$

$$\boxed{\overline{N}^{(1)} = \overline{N}_{A1} + \overline{N}_{B1}} \quad \boxed{\overline{N}^{(2)} = \overline{N}_{A2} + \overline{N}_{B2}} \quad \boxed{\overline{N}^{(3)} = \overline{N}_{A3} + \overline{N}_{B3}} \quad \boxed{\overline{N}_A + \overline{N}_B}$$

Example



- Expected number of customers per class / in network

$$\boxed{\overline{N}_{A1} = \frac{2\lambda_1}{\mu_A - 2\lambda_1 - \lambda_2}} \quad \boxed{\overline{N}_{A2} = \frac{\lambda_2}{\mu_A - 2\lambda_1 - \lambda_2}} \quad \boxed{\overline{N}_{A3} = 0} \quad \boxed{\overline{N}_{A} = \frac{2\lambda_1 + \lambda_2}{\mu_A - 2\lambda_1 - \lambda_2}}$$

$$\boxed{\overline{N}_{B1} = \frac{\lambda_1}{\mu_B - \lambda_1 - \lambda_3}} \quad \boxed{\overline{N}_{B2} = 0} \quad \boxed{\overline{N}_{B3} = \frac{\lambda_3}{\mu_B - \lambda_1 - \lambda_3}} \quad \boxed{\overline{N}_{B} = \frac{\lambda_1 + \lambda_3}{\mu_B - \lambda_1 - \lambda_3}}$$

$$\boxed{\overline{N}^{(1)} = \overline{N}_{A1} + \overline{N}_{B1}} \quad \boxed{\overline{N}^{(2)} = \overline{N}_{A2} + \overline{N}_{B2}} \quad \boxed{\overline{N}^{(3)} = \overline{N}_{A3} + \overline{N}_{B3}} \quad \boxed{\overline{N}_A + \overline{N}_B}$$

- Expected sojourn time per class / in network

$$\overline{T}_1 = \frac{\overline{N}^{(1)}}{\lambda_1} \boxed{= \frac{2}{\mu_A - 2\lambda_1 - \lambda_2} + \frac{1}{\mu_B - \lambda_1 - \lambda_3}}$$

$$\overline{T}_2 = \frac{\overline{N}^{(2)}}{\lambda_2} \boxed{= \frac{1}{\mu_A - 2\lambda_1 - \lambda_2}}$$

$$\overline{T}_3 = \frac{\overline{N}^{(3)}}{\lambda_3} \boxed{= \frac{1}{\mu_B - \lambda_1 - \lambda_3}}$$

$$\boxed{\overline{T} = \frac{\overline{N}_A + \overline{N}_B}{\lambda_1 + \lambda_2 + \lambda_3}}$$

$$= \frac{\lambda_1 \overline{T}_1 + \lambda_2 \overline{T}_2 + \lambda_3 \overline{T}_3}{\lambda_1 + \lambda_2 + \lambda_3}$$

$$= \frac{\overline{N}^{(1)} + \overline{N}^{(2)} + \overline{N}^{(3)}}{\lambda_1 + \lambda_2 + \lambda_3}$$

For next time (in two weeks)

- Lesson 6 to revise
- Homework 6 to return on Tuesday 5 November before 9 am
- Next lecture given by Alain Jean-Marie
  - Use case
  - pyMarmote tool
- Instructions to install Marmote before next lesson
  - https://marmote.gitlabpages.inria.fr/marmote/