2 Continuous-Time Markov Chain

A continuous-time Markov chain (abbreviated as CTMC) is a continuous-time (with index set $[0,\infty)$), discrete-space (with state-space $\mathcal{E}$) stochastic process $(X(t),\, t \geq 0)$ such that

$$P(X(t) = j \,|\, X(s_1) = i_1, \ldots, X(s_{n-1}) = i_{n-1}, X(s) = i) = P(X(t) = j \,|\, X(s) = i) \tag{16}$$

for all $i_1, \ldots, i_{n-1}, i, j \in \mathcal{E}$, $0 \leq s_1 < \ldots < s_{n-1} < s < t$. This is the Markov property.

A CTMC is homogenous if for all $i, j \in \mathcal{E}$, $0 \leq s < t$, $u \geq 0$

$$P(X(t+u) = j \,|\, X(s+u) = i) = P(X(t) = j \,|\, X(s) = i) := p_{i,j}(t-s).$$

From now on we shall only consider homogeneous CTMC's even if we simply speak of a CTMC. The following proposition is analog to Proposition 1 and its proof is similar.

Proposition 4 (Chapman-Kolmogorov equation for CTMC's)**.** *For all $t > 0$, $s > 0$, $i, j \in \mathcal{E}$,*

$$p_{i,j}(t+s) = \sum_{k \in \mathcal{E}} p_{i,k}(t)\, p_{k,j}(s), \tag{17}$$

$$\mathbf{P}(t+s) = \mathbf{P}(t) \cdot \mathbf{P}(s). \tag{18}$$

■

Infinitesimal generator

Define

$$q_{i,i} := \lim_{h \to 0} \frac{p_{i,i}(h) - 1}{h} \leq 0 \tag{19}$$

$$q_{i,j} := \lim_{h \to 0} \frac{p_{i,j}(h)}{h} \geq 0 \tag{20}$$

and let $\mathbf{Q}$ be the matrix $\mathbf{Q} = [q_{i,j}] = \lim_{h \to 0} \frac{\mathbf{P}(h) - \mathbf{I}}{h}$ (we will assume that the limits in (19)-(20) always exist. They will exist in all cases to be considered in this course). The matrix $\mathbf{Q}$ is called the *infinitesimal generator* of the CTMC. If $\mathcal{E} = \mathbb{N}$, then

$$\mathbf{Q} = \begin{pmatrix} -\sum_{j \neq 0} q_{0,j} & q_{0,1} & q_{0,2} & \cdots & \cdots & \cdots & \cdots \\ q_{1,0} & -\sum_{j \neq 1} q_{1,j} & q_{1,2} & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ q_{i,0} & \cdots & \cdots & q_{i,i-1} & -\sum_{j \neq i} q_{i,j} & q_{i,i+1} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

In contrast with (5) note that

$$\sum_{j \in \mathcal{E}} q_{i,j} = 0, \quad \forall i \in \mathcal{E}. \tag{21}$$

The quantity $q_{i,j}$ has the following interpretation: when the system is in state i then the rate at which it departs state i is $-q_{i,i}$, and the rate at which it moves from state i to state j, $j \neq i$, is $q_{i,j}$. In the literature $q_{i,j}$ is referred to as the *transition rate* from state i to state $j \neq i$ and $-q_{i,i}$ is the transition rate out of state i.

What is the probability $p(i,j)$ that the CTMC goes to state j after leaving state $i \neq j$? From the interpretation of $q_{i,j}$ given above it should be clear that

$$p(i,j) = \frac{q_{i,j}}{-q_{i,i}}, \quad i,j \in \mathcal{E},\, i \neq j. \tag{22}$$

What is the sojourn time distribution of the CTMC in state i? It can be shown that it is exponentially distributed with rate $-q_{i,i}$ [5, Section 2.4]. In particular, the expected sojourn time in state i is $-1/q_{i,i}$ (inverse of rate).

Proof. Assume that the CTMC enters state i at $t = 0$. Let $S(i)$ be the sojourn time in state i, meaning that at instant $S(i)$ the CTMC will leave state i. Let us show that $S(i)$ is exponentially distributed with rate $-q_{i,i}$, namely,

$$P(S(i) \leq x) = 1 - \exp(q_{i,i}x) \quad \text{or equivalently} \quad P(S(i) > x) = \exp(q_{i,i}x).$$

We have

$$\begin{aligned}
P(S(i) > x+h) &= P(S(i) > x,\ \text{CTMC state is } i \text{ in } (x, x+h)) \\
&= P(S(i) > x \,|\ \text{CTMC state is } i \text{ in } (x, x+h)) \\
&\quad \times P(\text{CTMC state is } i \text{ in } (x, x+h)) \qquad \text{(from Bayes formula)} \\
&= P(S(i) > x)\, P(\text{CTMC state is } i \text{ in } (x, x+h))
\end{aligned} \tag{23}$$

since in a Markov chain what occurs after time x does not depend on what occurred before time x (meaning that the event $\{S(i) > x\}$ is independent of whether or not an event will occur in the interval $(x, x+h)$). Let us focus on $P(\text{CTMC state is } i \text{ in } (x, x+h))$, the probability that the chain will stay in state i during $(x, x+h)$. When $h \to 0$, this probability is given by $p_{i,i}(h)$. From (19), we can write $p_{i,i}(h) = 1 + hq_{i,i} + o(h)$. Introducing this value in (23) gives

$$\begin{aligned}
P(S(i) > x+h) &= P(S(i) > x) \cdot (1 + hq_{i,i} + o(h)) \\
\Rightarrow \quad P(S(i) > x+h) - P(S(i) > x) &= P(S_i > x)(hq_{i,i} + o(h)).
\end{aligned}$$

Dividing by h and letting $h \to 0$ (recall that $o(h)/h \to 0$ as $h \to 0$) gives the following first order differential equation for $P(S(i) > x)$

$$\frac{dP(S(i) > x)}{dx} = q_{i,i}\, P(S(i) > x).$$

As $P(S(i) > 0) = 1$, the unique solution is $P(S(i) > x) = \exp(q_{i,i}x)$. This concludes the proof. ★

In summary, a homogeneous CTMC on $\mathcal{E}$ is a stochastic process which sojourns in each state it visits for an exponentially distributed amount of time (with rate $-q_{i,i}$ for state i) independent of the past evolution of the process, and upon leaving state i instantaneously joins state $j \neq i$ with the probability $p(i,j)$ given in (22). These features of a homogeneous CTMC will be used to establish construction rules #1 and #2 (see below) that we will use to check if a stochastic process is a CTMC.

Observe that the infinitesimal generator $\mathbf{Q}$ fully characterises the behavior of a homogeneous continuous-time Markov chain, alike the behavior of a discrete-time Markov chain that is fully characterised by the transition matrix $\mathbf{P}$ (see Section 1).

Transient probability distribution

Let us now focus on the transient probability distribution of a CTMC. Define the row vector $\pi(t) = (\pi_i(t), i \in \mathcal{E})$, where $\pi_i(t) := P(X(t) = i)$ is the probability that the CTMC is in state i at time t. For all $j \in \mathcal{E}$ we have (Hint: use law of total probabilities)

$$\begin{aligned}
\pi_j(t+h) &= \sum_{i \in \mathcal{E}} P(X(t+h) = j \,|\, X(t) = i)\pi_i(t) \\
&= P(X(t+h) = j \,|\, X(t) = j)\pi_j(t) + \sum_{\substack{i \in \mathcal{E} \\ i \neq j}} P(X(t+h) = j \,|\, X(t) = i)\pi_i(t) \\
&= p_{j,j}(h)\pi_j(t) + \sum_{i \neq j} p_{i,j}(h)\pi_i(t)
\end{aligned}$$

so that

$$\frac{\pi_j(t+h) - \pi_j(t)}{h} = \frac{p_{j,j}(h) - 1}{h}\pi_j(t) + \sum_{\substack{i \in \mathcal{E} \\ i \neq j}} \frac{p_{i,j}(h)}{h}\pi_i(t), \quad t \geq 0.$$

Letting[2] $h \to 0$ in the latter equation and using (19)-(20) gives

$$\frac{d\pi_j(t)}{dt} = q_{j,j}\pi_j(t) + \sum_{\substack{i \in \mathcal{E} \\ i \neq j}} q_{i,j}\pi_i(t) = \sum_{i \in \mathcal{E}} q_{i,j}\pi_i(t), \quad j \in \mathcal{E}, t \geq 0$$

or, in matrix form,

$$\frac{d}{dt}\pi(t) = \pi(t)\mathbf{Q}, \quad t \geq 0.$$

Once $\pi(0)$ is specified this first-order differential equation has a unique solution, given by[3]

$$\pi(t) = \pi(0)\, e^{\mathbf{Q}t}, \quad t \geq 0. \tag{24}$$

In practice this result is not very useful since it is in general very difficult to calculate $e^{\mathbf{Q}t}$. By definition $e^{\mathbf{Q}t} = \sum_{k=0}^{\infty} \frac{(\mathbf{Q}t)^k}{k!}$. This is especially true when $\mathbf{Q}$ has infinite dimension (infinite state-space).

[2] When the cardinality of $\mathcal{E}$ is infinite the identity $\lim_{h \downarrow 0} \sum_{\substack{i \in \mathcal{E} \\ i \neq j}} (p_{i,j}(h)/h)\pi_i(t) = \sum_{\substack{i \in \mathcal{E} \\ i \neq j}} \lim_{h \downarrow 0} (p_{i,j}(h)/h)\pi_i(t)$ is justified by the bounded convergence theorem.

[3] $e^{\mathbf{A}} = \sum_{k=0}^{\infty} \mathbf{A}^k/k!$ for any matrix $\mathbf{A}$.

Steady-state probability distribution

In the rest of this section we shall only be concerned with the steady-state probability distribution of a CTMC, namely with $\pi := \lim_{t\to\infty} \pi(t)$ whenever this limit exists and does not depend on the initial state.

In direct analogy with the definition given for a DTMC, we shall say that a homogeneous CTMC is *irreducible* if for any states $i, j \in \mathcal{E}$ there exist $s, s' > 0$ such that $p_{i,j}(s) > 0$ and $p_{j,i}(s') > 0$.

Before stating the main result of this section, let us make a guess on what the limiting distribution π, when it exits, is. Assume that $\pi := (\pi_i, i \in \mathcal{E})$ is the stationary probability distribution.

Alike for DTMCs, we first develop an argument to find π.

For any $0 < h < t$, we have from (24)

$$\pi(t) = \pi(0)\, e^{\mathbf{Q}(t-h)} e^{\mathbf{Q}h} = \pi(t-h)e^{\mathbf{Q}h} = \pi(t-h)(\mathbf{I} + \mathbf{Q}h + o(h)) \tag{25}$$

with $o(h)$ denotes any function such that $\lim_{h\to 0} o(h)/h = 0$. Letting $t \to \infty$ in (25) gives

$$\pi \mathbf{Q} h = o(h). \tag{26}$$

Dividing now both sides of (26) by h and letting $h \to 0$ gives

$$\pi \mathbf{Q} = 0. \tag{27}$$

We have such shown that if the steady-state exists (in the sense that $\pi = \lim_{t\to\infty} \pi(t)$ exists and does not depend on the initial state) then this limit satisfies the linear equation (27).

The following result gives necessary conditions for the existence of the steady-state of a CTMC.

Proposition 5 (Limiting distribution function of a CTMC)**.** *If a homogeneous CTMC with infinitesimal generator* $\mathbf{Q}$ *is irreducible, and if the system of equations*

$$\begin{aligned} \mathbf{x}\mathbf{Q} &= 0 \\ \mathbf{x}\cdot\mathbf{1} &= 1 \end{aligned}$$

has a strictly positive solution (i.e., for all $i \in \mathcal{E}$*,* $x(i)$*, the* i*-th component of the row vector* $\mathbf{x}$*, is strictly positive) then*

$$x(i) = \lim_{t\to\infty} P(X(t) = i) \tag{28}$$

for all $i \in \mathcal{E}$*, regardless of the initial state* $X(0)$*.* ∎

We shall not prove this result. We may compare this result with our earlier equation for a DTMC namely, $\pi = \pi\,\mathbf{P}$ (see Proposition 3); here $\mathbf{P}$ is the transition matrix, whereas the infinitesimal generator $\mathbf{Q}$ is a matrix of *transition rates*.

The equation $\pi\mathbf{Q} = 0$ in Proposition 5 can be rewritten as

$$\sum_{j\in\mathcal{E}} \pi_j\, q_{j,i} = 0$$

for all $i \in \mathcal{E}$, or equivalently, cf. (21),

$$- q_{i,i}\, \pi_i = \left(\sum_{j \neq i} q_{i,j} \right) \pi_i = \sum_{j \neq i} \pi_j \, q_{j,i}. \tag{29}$$

The above equation expresses the fact that for any state

the probability flow out of a state is equal to the probability flow into that state

when the system is at equilibrium. Equations (29) are called the *balance equations* and the sentence in italic below (29) is referred to as the *principle of flow conservation*.

Many examples of continuous-time Markov chains will be discussed in Sections 4 and 6.1.

Identifying continuous-time Markov chains

It remains one important question to address: how does one check that a stochastic process is a CTMC? In the discrete-time setting it is in general easy to check whether or not the Markov property (2) in Section 1 is satisfied. The one-step transition matrix $\mathbf{P}$ in (3) is usually easy to find.

In the continuous-time setting checking that the Markov property (16) holds is much more difficult since it has to be checked for a non-countable number of points in time. Below we give two *construction rules* that can be used to check whether or not a stochastic process is a CTMC and, if yes, to identify its infinitesimal generator.

Construction rule #1. Let $\{X(t), t \geq 0\}$ be a continuous-time stochastic process enjoying the following properties.

For each state i

- the process stays in state i for a random duration with an exponential distribution with parameter $\tau_i > 0$, independent of the past of the process; then
- the process instantaneously jumps into state j with probability a_{ij} (i.e. for each i there is a unique state j in which the process can jump into). We have $a_{ij} \in [0, 1]$, $a_{ii} = 0$ and

$$\sum_j a_{ij} = 1.$$

If the above holds for all states then $\{X(t), t \geq 0\}$ is a CTMC, with infinitesimal generator $\mathbf{Q} = [q_{i,j}]$ given by

$$q_{i,j} = \begin{cases} \tau_i a_{ij} & \text{if } i \neq j \\ -\tau_i & \text{if } i = j. \end{cases} \tag{30}$$

Construction rule #2. Let $\{X(t), t \geq 0\}$ be a continuous-time stochastic process with the following properties.

When the process enters state i at time t then

- for each $j \neq i$ a rv Y_{ij} with an exponential distribution with parameter μ_{ij} is generated. These rvs are mutually independent and are independent of the past of the process. One may have $\mu_{ij} = 0$ in which case $Y_{ij} = +\infty$ (corresponding to the case where state j cannot be reached from state i in one transition).
- assume that Y_{ik} is the smallest among all the rvs $\{Y_{ij}\}_j$. It is well known that Y_{ik} has an exponential distribution with parameter $\sum_j \mu_{ij}$. At time $t + Y_{ik}$ the process instantaneously jumps into state k.

If the above holds for all states then $\{X(t),\, t \geq 0\}$ is a CTMC, with infinitesimal generator $\mathbf{Q} = [q_{i,j}]$ given by

$$q_{i,j} = \begin{cases} \mu_{ij} & \text{if } i \neq j \\ -\sum_{j \neq i} \mu_{ij} & \text{if } i = j. \end{cases} \tag{31}$$

Construction #2 is useful when several phenomena try, in parallel and independently of each other, to force the process to leave state i after an "exponential time". The first which succeeds determines the next state to be visited by the process. Since the rvs $\{Y_{ij}\}_j$ are mutually independent, at each visit in state i it is not always the same which wins the competition.

2.1 A case study: Markov modulated arrivals

Consider a process $\mathbf{Y} := \{Y(t), t \geq 0\}$ that *alternates* between two states H (for High) and L (for Low), such that $Y(t) = H$ (resp. $Y(t) = L$) if the process is in state H (resp. L) at time t. The sojourn time of $\mathbf{Y}$ in state H (resp. L) is random and exponentially distributed with mean $1/\mu_H$ (resp. $1/\mu_L$). We assume that all sojourn times are mutually independent rvs and that $\mu_H > 0$ and $\mu_L > 0$.

In steady-state the probability p of finding $\mathbf{Y}$ in state H is

$$p = \frac{1/\mu_H}{1/\mu_H + 1/\mu_L} = \frac{\mu_L}{\mu_H + \mu_L} \tag{32}$$

and the probability $1 - p$ of finding it in state L is

$$1 - p = \frac{\mu_H}{\mu_H + \mu_L}.$$

Note that p is simply the ratio of time spent by $\mathbf{Y}$ on average in state H (given by $1\mu_H$) over the average duration of a cycle (given by $1/\mu_H + 1/\mu_L$), where a cycle is defined as the time that elapses before two consecutive entries of $\mathbf{Y}$ in state H or in state L.

To prove (32) rigorously, observe that under the above assumptions $\mathbf{Y}$ is an irreducible CTMC (Hint: apply rule #1) with infinitesimal generator

$$\mathbf{Q} = \begin{pmatrix} -\mu_H & \mu_H \\ \mu_L & -\mu_L \end{pmatrix}.$$

Solving for the equations (i) $\pi\,\mathbf{Q} = 0$ and (ii) $\pi \cdot \mathbf{1} = 1$ with $\pi = (\pi_H, \pi_L)$, gives

$$\pi_H = \frac{\mu_L}{\mu_H + \mu_L} \qquad \text{and} \qquad \pi_L = \frac{\mu_H}{\mu_H + \mu_L}.$$

Since $\pi = (\pi_H, \pi_L)$ is the unique strictly positive solution of eqns (i)-(ii) we conclude from Proposition 5 that π is the steady-state distribution of $\mathbf{Y}$, regardless the initial state of $Y(0)$. Hence, $p = \pi_H$ and $1 - p = \pi_L$ as announced.

Assume that during periods of type H (resp. L) objects (customers, packets, jobs, etc.) are generated according to a Poisson process with rate $\lambda_H > 0$ (resp. $\lambda_L > 0$). We further assume that the number of objects generated in different periods are independent rvs. We say that an object is of type H (resp. L) if it has been generated in a period of type H (resp. L).

Define $q_{b|a} = P(X_{n+1} = b \,|\, X_n = a)$, $a, b \in \{H, L\}$, the probability that the $(n+1)$-st object is of type b given that the n-th object is of type a. Our objectif is to compute $q_{b|a}$ for $a, b \in \{H, L\}$.

Let $\mathcal{F}_n$ be the event that the n-th and the $(n+1)$-st object arrive in the same period. We have $P(\mathcal{F}_n \,|\, X_n = a) = \lambda_a/(\lambda_a + \mu_a)$ for $a \in \{H, L\}$. Indeed, when $\mathbf{Y}$ is in state a, the next object will be generated after an exponential duration A with rate λ_a and $\mathbf{Y}$ will jump to state $b \neq a$ after an exponential duration B with rate μ_a. Hence, $P(\mathcal{F} \,|\, X_n = a) = P(A < B) = \lambda_a/(\lambda_a + \mu_a)$, since A and B are independent exponential rvs. To see this, condition on (for instance) A to get $P(A < B) = \int_0^\infty P(x < B)\lambda_a \exp(-\lambda_a x)\, dx = \int_0^\infty \exp(-\mu_a x))\lambda_a \exp(-\lambda_a x)\, dx = \lambda_a/(\lambda_a + \mu_a)$.

By the law of total probability (see Proposition 25)

$$\begin{aligned} q_{L|L} &= P(X_{n+1} = L \,|\, X_n = L, \mathcal{F}_n)P(\mathcal{F}_n \,|\, X_n = L) + P(X_{n+1} = L \,|\, X_n = L, \mathcal{F}_n^c)P(\mathcal{F}_n^c \,|\, X_n = L) \\ &= 1 \times \frac{\lambda_L}{\lambda_L + \mu_L} + P(X_{n+1} = L \,|\, X_n = L, \mathcal{F}_n^c)\, \frac{\mu_L}{\lambda_L + \mu_L} \\ &= \frac{\lambda_L}{\lambda_L + \mu_L} + q_{L|H}\, \frac{\mu_L}{\lambda_L + \mu_L}. \end{aligned} \tag{33}$$

The last equality comes from the fact that the probability $P(X_{n+1} = L \,|\, X_n = L, \mathcal{F}_n^c)$ is the same as the probability $q_{L|H}$ due to the memoryless property of the (Poisson) arrival process of objects of type L. Similarly

$$\begin{aligned} q_{L|H} &= P(X_{n+1} = L \,|\, X_n = H, \mathcal{F}_n)P(\mathcal{F}_n \,|\, X_n = H) + P(X_{n+1} = L \,|\, X_n = H, \mathcal{F}_n^c)P(\mathcal{F}_n^c \,|\, X_n = H) \\ &= 0 \times \frac{\lambda_H}{\lambda_H + \mu_H} + P(X_{n+1} = L \,|\, X_n = H, \mathcal{F}_n^c)\, \frac{\mu_H}{\lambda_H + \mu_H} \;=\; q_{L|L}\, \frac{\mu_H}{\lambda_H + \mu_H}. \end{aligned}$$

Substituting the above value of $q_{L,H}$ in (33) gives

$$q_{L|L} = \frac{\lambda_L}{\lambda_L + \mu_L} + q_{L|L}\, \frac{\mu_H}{\lambda_H + \mu_H}\, \frac{\mu_L}{\lambda_L + \mu_L}$$

so that

$$q_{L|L} = \frac{\lambda_L(\lambda_H + \mu_H)}{\lambda_L(\lambda_H + \mu_H) + \lambda_H \mu_L}, \quad \text{and therefore} \quad q_{L|H} = \frac{\lambda_L \mu_H}{\lambda_L(\lambda_H + \mu_H) + \lambda_H \mu_L}. \tag{34}$$

By symmetry

$$q_{H|H} = \frac{\lambda_H(\lambda_L + \mu_L)}{\lambda_H(\lambda_L + \mu_L) + \lambda_L \mu_H}, \quad q_{H|L} = \frac{\lambda_H \mu_L}{\lambda_H(\lambda_L + \mu_L) + \lambda_L \mu_H}. \tag{35}$$

Assume that $\lambda_H = \lambda_L = \lambda$ (all objects arrive according to a Poisson process with rate λ). Then,

$$q_{L|L} = \frac{\lambda + \mu_H}{\lambda + \mu_H + \mu_L}, \quad q_{H|L} = \frac{\mu_L}{\lambda + \mu_L + \mu_H} \quad q_{H|H} = \frac{\lambda + \mu_L}{\lambda + \mu_L + \mu_H}, \quad q_{L|H} = \frac{\mu_H}{\lambda + \mu_H + \mu_L}. \tag{36}$$

Is it possible to select the parameters in (36) so that an arriving object is equally likely to be of either type given the type of the previous arrival is known? In other words, when do we have $q_{L|L} = q_{H|L} = q_{H|H} = q_{L|H} = 1/2$? It is easy to see that this is possible if and only if $\lambda = 0$ and $\mu_H = \mu_L$, a case which is irrelevant since there is no arrival with probability 1.

On the other hand, if $\mu_H \to 0$ (resp. $\mu_L \to 0$) then $q_{L|L} \to 1$ and $q_{H|L} \to 0$ (resp. $q_{H|H} \to 1$ and $q_{L|H} \to 0$), namely all arrivals tend to be of type H (resp. type L).

2.2 Birth and death process

A *birth and death process* $\{X(t),\, t \geq 0\}$ is a continuous-time discrete-space (with state-space which is $\mathbb{N}$ or which is a subset, finite or infinite, of $\mathbb{N}$) Markov chain (or Markov process) with infinitesimal generator $\mathbf{Q} = [q_{i,j}]$ such that $q_{i,j} = 0$ for $j \notin \{i-1, i, i+1\}$ for $i \geq 1$ and $j \notin \{i, i+1\}$ for $i = 0$.

In other words, the infinitesimal generator $\mathbf{Q}$ is a tridiagonal matrix of the following form

$$\mathbf{Q} = \begin{pmatrix} -\lambda_0 & \lambda_0 & 0 & \cdots & & \\ \mu_1 & -(\lambda_1 + \mu_1) & \lambda_1 & 0 & \cdots & \\ 0 & \mu_2 & -(\lambda_2 + \mu_2) & \lambda_2 & 0 & \cdots \\ \vdots & 0 & \mu_3 & -(\lambda_3 + \mu_3) & \lambda_3 & \ddots \\ & \vdots & 0 & \mu_4 & \ddots & \ddots \\ & & \vdots & \ddots & \ddots & \end{pmatrix}$$

where we have set $q_{i,i+1} = \lambda_i$ for $i = 0, 1, \ldots$ and $q_{i,i-1} = \mu_i$ for $i = 1, 2, \ldots$.

The random variable (rv) $X(t)$ may be interpreted as the *size of the population at time t*. In that case, λ_i gives the birth-rate when the size of the population is i and μ_i gives the death-rate when the size of the population is i with $i \geq 1$ (by convention $\mu_0 = 0$).

Assume that the Markov process $\{X(t),\, t \geq 0\}$ is irreducible (if $\lambda_i, \mu_i > 0$ for all $i \in \mathbb{N}$ then it will be irreducible on $\mathbb{N}$; if $\lambda_j, \mu_j > 0$ for $j = 0, 1, \ldots, i$ and $\lambda_{i+1} = 0$ then it will be irreducible on $\{0, 1, \ldots, i\}$). Then, from Proposition 5 in Section 2 we know that if the equation $\pi \mathbf{Q} = 0$ (with $\pi = (\pi_0, \pi_1, \ldots)$) has a solution strictly positive (i.e., $\pi_i > 0$ for all $i = 0, 1, \ldots$) such that $\sum_{i=0}^{\infty} \pi_i = 1$ then $\lim_{t \to \infty} P(X(t) = i) = \pi_i$ for all $i = 0, 1, \ldots$.

Proposition 6 (Balance equations of a birth and death process)**.** *The equation $\pi \mathbf{Q} = 0$ also writes*

$$\lambda_0 \, \pi_0 = \mu_1 \, \pi_1 \tag{37}$$

$$(\lambda_i + \mu_i) \, \pi_i = \lambda_{i-1} \, \pi_{i-1} + \mu_{i+1} \, \pi_{i+1} \quad i = 1, 2, \ldots. \tag{38}$$

■

Equations (37)-(38) are the balance equations - also called the *equilibrium equations* - of a birth and death process (see Section 2 for the balance equations of an arbitrary continuous-time Markov chain). They express the property that in steady-state the flow of probability out of a state is equal to the flow of probability into that state. This observation, of a physical nature, will allow us in most cases to easily compute the equilibrium equations of a birth and death process.

From Proposition 6 we find that $\pi_{i+1} = \frac{\lambda_i}{\mu_{i+1}}\pi_i$ for $i = 0, 1, 2, \ldots$ so the following holds.

Proposition 7 (Stationary probabilities of a birth and death process)**.** *Assume that the series*

$$C := 1 + \frac{\lambda_0}{\mu_1} + \frac{\lambda_0\lambda_1}{\mu_1\mu_2} + \cdots + \frac{\lambda_0\lambda_1\cdots\lambda_{n-1}}{\mu_1\mu_2\cdots\mu_i} + \cdots = \sum_{i\geq 0}\prod_{j=1}^{i}\frac{\lambda_{j-1}}{\mu_j} < \infty \tag{39}$$

(i.e., C converges). Then, for each $n = 1, 2\ldots,$

$$\pi_n = \frac{\lambda_0\lambda_1\cdots\lambda_{n-1}}{\mu_1\mu_2\cdots\mu_n}\pi_0 \tag{40}$$

where $\pi_0 = 1/C$. ■

This result is obtained by direct substitution of (40) into (37)-(38). The computation of π_0 relies on the fact that $\sum_{n=0}^{\infty}\pi_n = 1$.

The condition (39) is called the *stability condition* of a birth and death process.

2.3 Time-reversibility

A CTMC is said to be time-reversible if for any two states $i, j \in \mathcal{E}$, with $i \neq j$, we have

$$\pi_i q_{i,j} = \pi_j q_{j,i}.$$

It is clear that the birth and death process is time-reversible.

A consequence of the time-reversibility property is that the forwards chain and the reverse chain are statistically identical and both can be described by the same transition diagram.