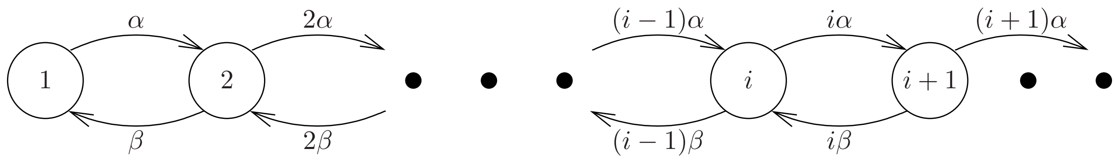
UBINET/SI5: Performance Evaluation of Networks

Correction of homework 4

4.1 Fake news

1. The number of spreaders of fake news at time t, $X(t)$, takes values in $\mathcal{E} = \mathbb{N}^* = \{1, 2, 3, \dots\}$ (there is always at least one spreader that is the source). The convincing and awakening processes compete to change the state of the system (the number of spreaders), and each duration is exponentially distributed. By construction rule 2, $\{X(t), t > 0\}$ is a CTMC. Furthermore, the convincing and awakening processes make the number of spreaders of fake news increase/decrease by one at each transition. Clearly, $\{X(t), t > 0\}$ is a birth and death process with birth rate $\lambda_i = i\alpha$ (i processes are competing) and death rate $\mu_i = (i-1)\beta$ (if i persons are spreading the fake news, only $i-1$ of them will eventually stop doing so, so there will be $i-1$ awakening processes initiated in parallel). The transition diagram is



2. We can write the global balance equations for $S = \{1, \dots, i-1\}$, and for $i \geq 2$,

$$(i-1)\alpha\pi_{i-1} = (i-1)\beta\pi_i \quad \Rightarrow \quad \pi_i = \frac{\alpha}{\beta}\pi_{i-1} = \rho\pi_{i-1} = \rho^2\pi_{i-2} = \dots = \rho^{i-1}\pi_1$$

where we have used $\rho := \alpha/\beta$.

Instead, we can start directly with the stationary distribution of a birth and death process that verifies the following equation, for $i = 2, 3, \dots$

$$\pi_i = \frac{\lambda_{i-1}\lambda_{i-2}\cdots\lambda_1}{\mu_i\mu_{i-1}\cdots\mu_2}\pi_1 = \frac{(i-1)!\alpha^{i-1}}{(i-1)!\beta^{i-1}}\pi_1 = \rho^{i-1}\pi_1.$$

This equation is also true for $i = 1$. The normalizing condition gives

$$\sum_{i=1}^{\infty}\pi_i = 1 \quad \Rightarrow \quad \pi_1 = \frac{1}{1 + \sum_{i=2}^{\infty}\rho^{i-1}} = \frac{1}{\sum_{i=0}^{\infty}\rho^i}.$$

This probability is positive if the sum in the denominator of π_1 converges. The denominator is the sum of the terms of a geometric progression, it converges for $\rho < 1$. In this case

$$\pi_1 = 1 - \rho\ , \qquad \Rightarrow \quad \pi_i = (1-\rho)\rho^{i-1}, \text{ for } i = 1, 2, \dots$$

The stability condition is $\rho < 1$ or equivalently that $\alpha < \beta$.

3. The expected number of fake news spreaders in steady-state is

$$\overline{N} = \sum_{i=1}^{\infty} i\pi_i = (1-\rho)\sum_{i=1}^{\infty} i\rho^{i-1} = (1-\rho)\cdot\frac{1}{(1-\rho)^2} = \frac{1}{1-\rho}\ .$$

4. The average convincing rate is

$$\overline{\lambda} = \sum_{i=1}^{\infty} \lambda_i\pi_i = \sum_{i=1}^{\infty} i\alpha\pi_i = \alpha\overline{N} = \frac{\alpha}{1-\rho}\ .$$

5. We will use Little's law on the set of spreaders of fake news (this is the black box), there are on average $\overline{N}$ items in the box (spreaders of fake news), and the arrival rate into the box is $\overline{\lambda}$. The average time a person acts as a spreader of fake news (expected time inside the box) is

$$\overline{T} = \frac{\overline{N}}{\overline{\lambda}} = \frac{1}{\alpha}.$$

   A spreader of fake news will spread it for an average duration of $1/\alpha$. This is also the average time a person takes to convince another one to start spreading the fake news.

   Remark: Since $\alpha < \beta$ then $1/\alpha > 1/\beta$, that is the average time a random person acts as a spreader of fake news is larger than the average time needed by a spreader other than the source (the malicious person) to awaken and stop spreading the news. This is a consequence of the fact that the source never awakens and this increases the average time spent by a random person as a fake news spreader.

6. The new death rate is $\mu_i = (i-1)\beta/(i-1) = \beta$. The stationary distribution, for $i = 2, 3, \ldots$, becomes

$$\pi_i = \frac{(i-1)!\alpha^{i-1}}{\beta^{i-1}}\pi_1 = (i-1)!\rho^{i-1}\pi_1,$$

   where the last equality holds also for $i = 1$. The normalizing condition gives

$$\sum_{i=1}^{\infty} \pi_i = 1 \quad \Rightarrow \quad \pi_1 = \frac{1}{\sum_{i=1}^{\infty}(i-1)!\rho^{i-1}}$$

   By the ratio test, the series in the denominator diverges since $\lim_{i\to\infty} \frac{i!\rho^i}{(i-1)!\rho^{i-1}} = \lim_{i\to\infty} i\rho \to \infty$. The system is always unstable regardless of the values of ρ (or α and β).

4.2 Dimensioning a server

1. The new server has twice the speed of the old server. Therefore the customers service rate is doubled, we have $\mu_{\text{new}} = 2\mu$. The utilization of the new server is $\rho_{\text{new}} = \lambda_{\text{new}}/\mu_{\text{new}} = 2\lambda/(2\mu) = \rho$. The new faster server has the same utilization as the older one.

2. The original server can be modeled as an M/M/1 queueing system with arrival rate λ and service rate μ. By Little's formula we know that the mean response time (or the expected sojourn time) is $\overline{T} = 1/(\mu - \lambda)$. After the merger, the arrival rate has doubled so $\lambda_{\text{new}} = 2\lambda$. The new server serves client with a rate $\mu_{\text{new}} = 2\mu$. The new server can be modeled as an M/M/1 queueing system with arrival rate 2λ and service rate 2μ. The expected sojourn time of the new server is $\overline{T}_{\text{new}} = 1/(\mu_{\text{new}} - \lambda_{\text{new}}) = \overline{T}/2$. The expected sojourn time is halved in the new server.

3. To maintain the same quality, we need to have $\overline{T}_{\text{new}} = \overline{T}$. Let $\mu_{\text{new}} = \alpha\mu$, with $\alpha > 1$ (new server is faster), be the new server's speed. The resulting mean response time is

$$\overline{T}_{\text{new}} = \frac{1}{\mu_{\text{new}} - \lambda_{\text{new}}} = \frac{1}{\alpha\mu - 2\lambda} = \overline{T} = \frac{1}{\mu - \lambda} \quad \Rightarrow \quad \alpha = 1 + \frac{\lambda}{\mu}.$$

   The CPU should be increased by a ratio $\rho := \lambda/\mu$ (original speed is μ, new speed is $\mu + \lambda$). Since $\lambda < \mu$, it is clear that $\alpha < 2$. We observe that the new server must be stable. We must have $2\lambda < \alpha\mu$, i.e. $\alpha > 2\rho$. Therefore $2\rho < \alpha < 2$.

   Our recommendation is to buy a server with a speed $\alpha\mu$ with $\alpha = 1 + \frac{\lambda}{\mu}$. It would be cheaper than a server with speed 2μ ($\alpha < 2$) and will achieve the same response time for customers. We note that the utilization of this suggested new server would be $\rho_{\text{new}} = \lambda_{\text{new}}/\mu_{\text{new}} = 2\lambda/(\alpha\mu) = (2/\alpha)\rho > \rho$. Our suggested faster server will be more utilized than the old one.