Performance Evaluation of Networks

Sara Alouf

Objectives of Course

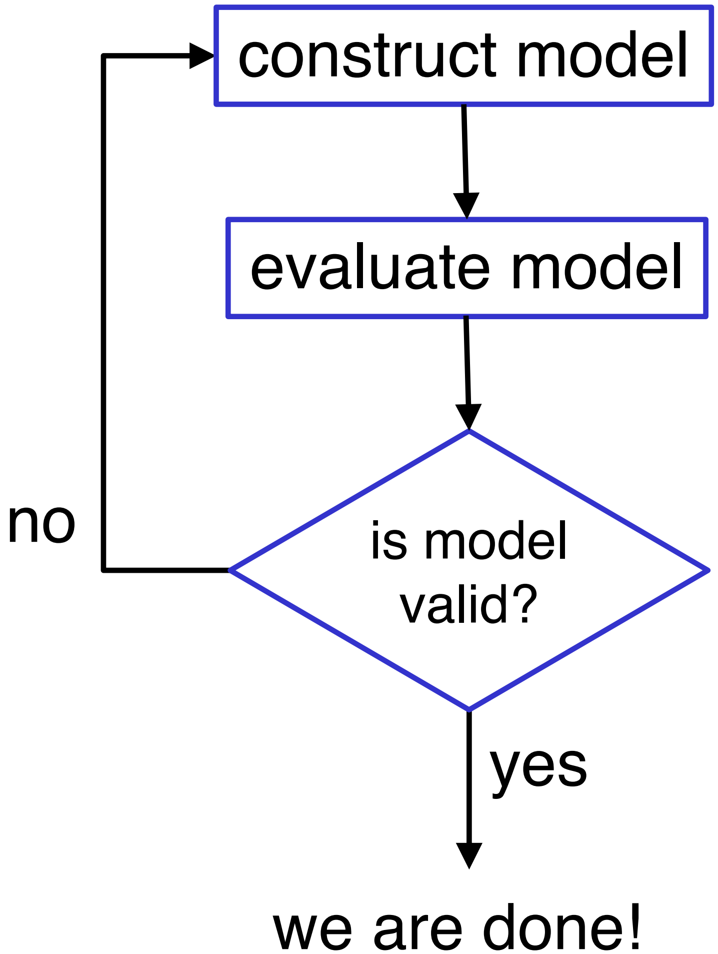
- Introduce analytical tools

- Answer questions like
  - Throughput of WiFi
  - If arrival/service rates double will response time stay same?
  - « 1 machine speed *s* » or « *n* machines speed *s/n* » ?
  - How many staff in call center to keep call rejection low?
  - and many others …

Performance Evaluation

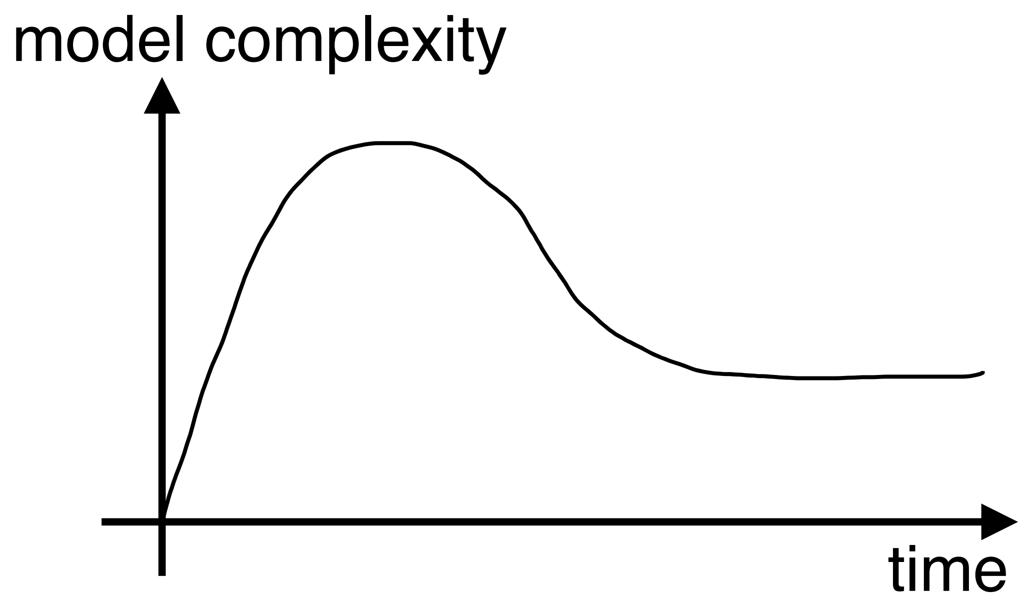
	Measurement	Simulation	Analysis
When	Prototyping Monitoring Tuning	Anytime	Anytime
Cost	High	Moderate	Low
Accuracy	Varies	Moderate	Low
Scalability	Low	Medium	High
Salability	High	Medium	Low
Tools	Instrumentation	Languages	Mathematics

Modeling Cycle



- abstract essential features
- ignore non-essential ones

- measurement
- simulations
- analysis



About the Course

- Website
  - https://lms.univ-cotedazur.fr/course/view.php?id=14278

    EIIN925 - ECUE Perform.Evaluation of Networks

    8 participants

    Lecture notes, slides, homeworks
  - http://www-sop.inria.fr/members/Sara.Alouf/PEN/

    Homeworks of past years
- Schedule: every Tuesday for 8 weeks
  - Markov Chains (3 lectures), Queues (3 lectures), Use cases (1 lecture), Exam (last session)
- Grade
  - 6 homeworks (60%), 1 exam (40%)

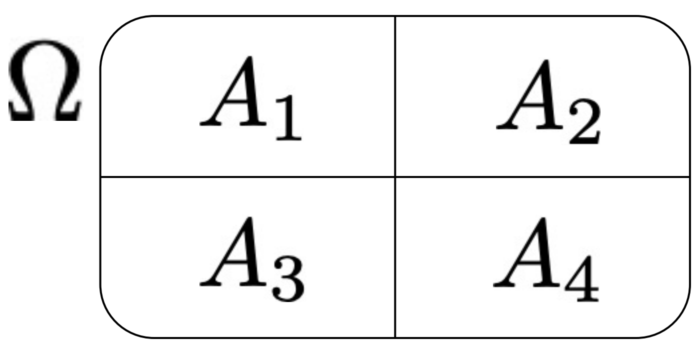
Brief Refresher

- Bayes formula

$$\boxed{P(A \mid B) = \frac{P(A \cap B)}{P(B)}}$$

- Law of total probability (use a partition)

$$\boxed{P(A) = \sum_{i=1}^{n} P(A \cap A_i) = \sum_{i=1}^{n} P(A \mid A_i)\, P(A_i)}$$



$$A_i \cap A_j = \emptyset$$

$$A_1 \cup A_2 \cup \cdots \cup A_n = \Omega$$

Brief Refresher

- Stochastic process : collection of random variables (rvs)

$$\mathbf{X} = \{X(t),\, t \in T\}$$

$X(t)$ is a rv mapping from Ω into some set $\mathcal{E} \subset \mathbb{R}$

- Poisson process : counting process rate λ

$$\{N(t), t \in T\} \qquad \boxed{\mathrm{E}[N(t)] = \lambda t}$$

  - start at 0
  - independent increments
  - count of events in $t-$long interval is Poisson variable

$$\boxed{P\big(N(t+s) - N(s) = k\big) = \frac{e^{-\lambda t}(\lambda t)^k}{k!} \qquad k = 0, 1, \ldots}$$

Part 1 Markov Chains

- Definition:

  A Markov process is a stochastic process that verifies the Markov property

$$P(X(t) \leq x \,|\, X(t_1) = x_1, \ldots, X(t_n) = x_n)$$
$$= P(X(t) \leq x \,|\, X(t_n) = x_n)$$
$$x_1, \ldots, x_n, x \in \mathcal{E}$$
$$t_1, \ldots, t_n, t \in T$$
$$t_1 < t_2 < \cdots < t_n < t$$

- Discrete space ➜ Markov chain

Ch 1 - Discrete-Time Markov Chain (DTMC)

- Discrete-time version of Markov property

$$P(X(n+1) = j \,|\, X(0) = i_0, \ldots, X(n) = i)$$
$$= P(X(n+1) = j \,|\, X(n) = i)$$

$$i_0, i_1, \ldots, i_{n-1}, i, j \in \mathcal{E}$$

- DTMC finite : state-space is finite

- DTMC is homogeneous : transition independent of step

$$p_{i,j} = P(X(n+1) = j \,|\, X(n) = i) \quad \forall i, j \in \mathcal{E}$$

One-step transition probability from state i to state j

Transition Matrix

- Square matrix containing all one-step transition prob.

$$\mathbf{P} = \begin{pmatrix} p_{0,0} & p_{0,1} & \cdots & p_{0,j} & \cdots \\ p_{1,0} & p_{1,1} & \cdots & p_{1,j} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ p_{i,0} & p_{i,1} & \cdots & p_{i,j} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \qquad \sum = 1$$

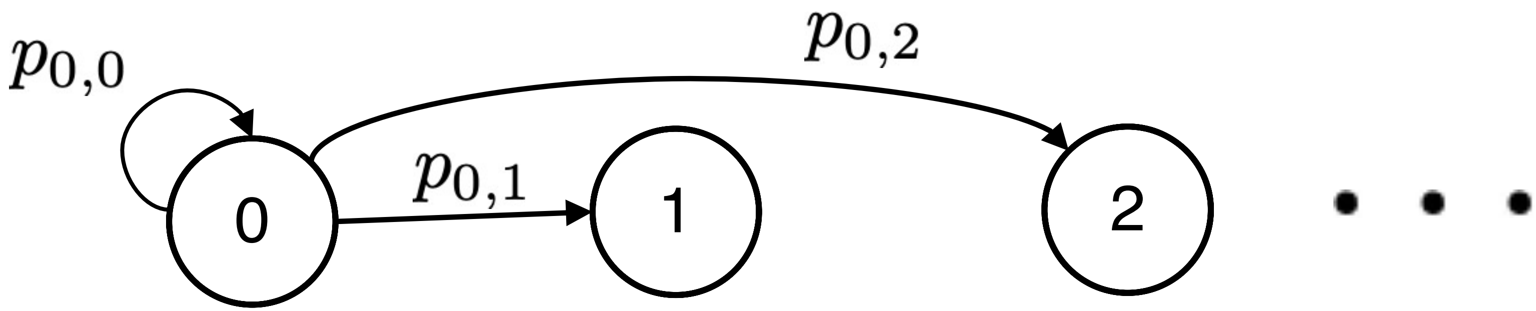
$$p_{i,j} \geq 0 \quad \forall i, j \in \mathcal{E}$$

$$\boxed{\sum_{j \in \mathcal{E}} p_{i,j} = 1 \quad \forall i \in \mathcal{E}}$$

➜ stochastic matrix

normalizing equation

Transition Diagram



Sum of arrows out of a state is 1

n-Step Transition Probability / Matrix

- n-step transition probability

$$\boxed{p_{i,j}^{(n)} = P(X(n) = j \,|\, X(0) = i)}$$

$$= P(X(n+1) = j \,|\, X(1) = i)$$

what counts is the difference between the two time steps

- n-step transition matrix

$$\mathbf{P}^{(n)} := \left[p_{i,j}^{(n)}\right] = \begin{pmatrix} p_{0,0}^{(n)} & p_{0,1}^{(n)} & \cdots \\ p_{1,0}^{(n)} & p_{1,1}^{(n)} & \cdots \\ \vdots & \vdots & \vdots \end{pmatrix} \quad \sum = 1$$

Chapman-Kolmogorov Equation

- Proposition 1: For all $n \geq 0, m \geq 0, i, j \in \mathcal{E}$

$$p_{i,j}^{(n+m)} = \sum_{k \in \mathcal{E}} p_{i,k}^{(n)} \, p_{k,j}^{(m)}$$

$$\mathbf{P}^{(n+m)} = \mathbf{P}^{(n)} \, \mathbf{P}^{(m)}$$

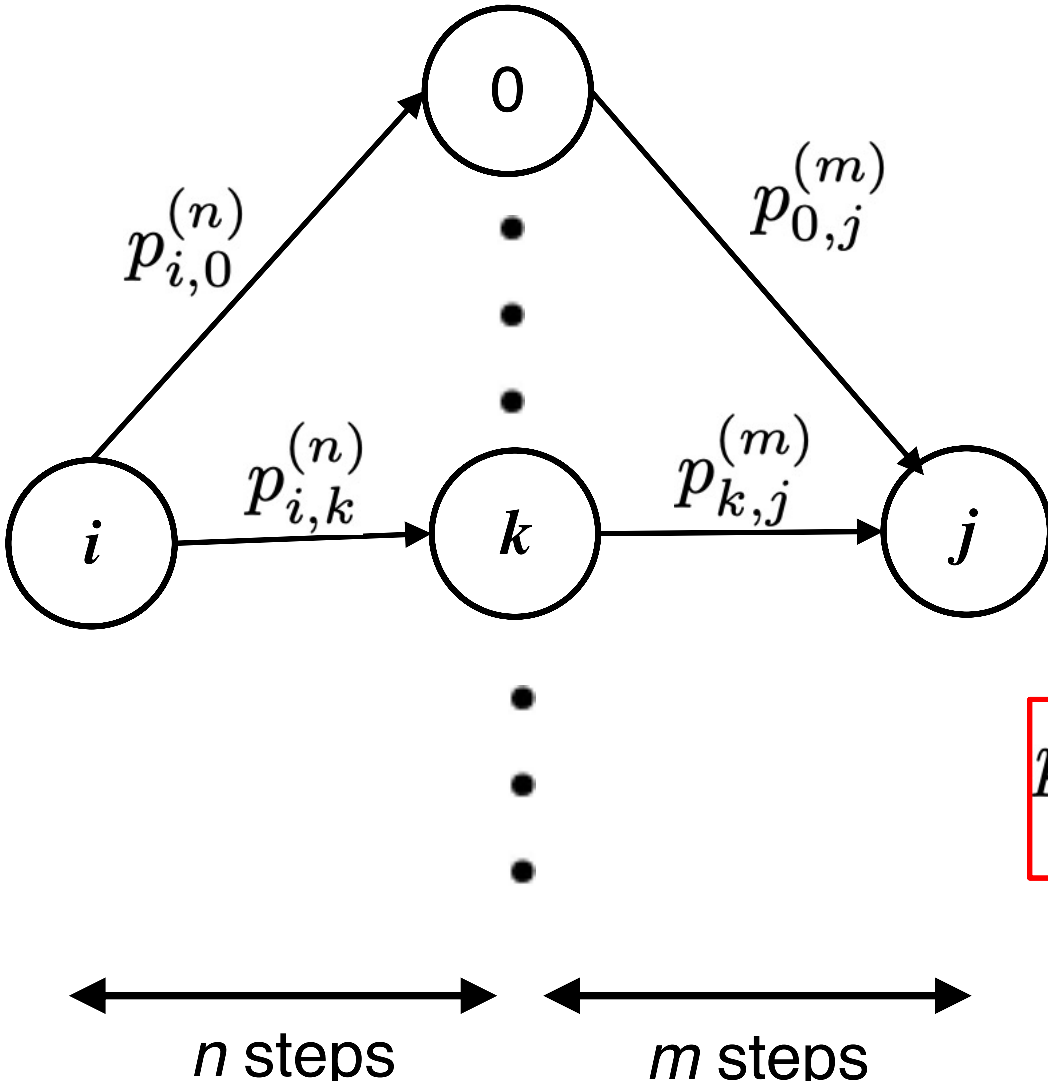
- Therefore $$\mathbf{P}^{(n)} = \mathbf{P}^{n}$$

  n-step transition matrix = n-th power of transition matrix

- Proof: use law of total probability and Markov property

  (see lecture notes page 8)

Chapman-Kolmogorov Equation



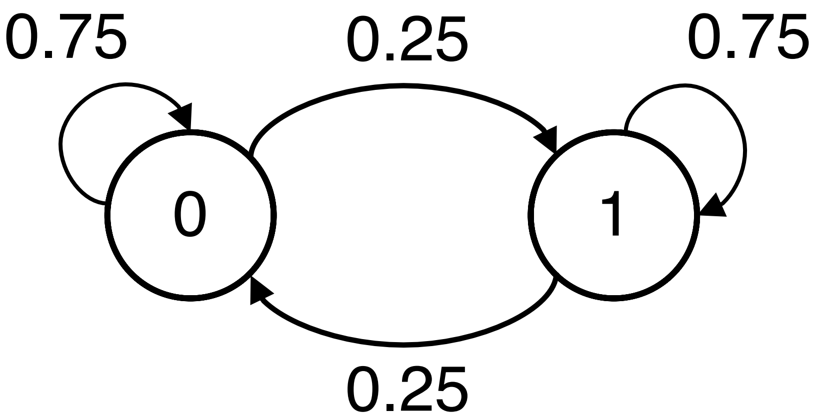
$$\boxed{p_{i,j}^{(n+m)} = \sum_{k \in \mathcal{E}} p_{i,k}^{(n)} \, p_{k,j}^{(m)}}$$

Example 2 page 8

- Communication channel transmits 0/1 through several stages
- From one stage to another, digit is unchanged with probability 0.75
- Question: Giving a 0 to stage 1, what is the probability that it is received as a 0 after stage 5?
- $X(n)$ state of system at step *n* is digit value after stage *n*
- $X(0)$ is value entered to stage 1
- State space = {0, 1}
- Markov property is verified ➜ $\{X(n), n \geq 0\}$ is a DTMC

- We are looking for $p_{0,0}^{(5)}$ the first element of matrix $\mathbf{P}^5$

Example 2 page 8



$$\mathbf{P} = \begin{bmatrix} 0.75 & 0.25 \\ 0.25 & 0.75 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

We need to compute the 5th power of the transition matrix

$$\mathbf{P}^2 = \frac{1}{8}\begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$$

$$\mathbf{P}^5 = \frac{1}{64}\begin{bmatrix} 33 & 31 \\ 31 & 33 \end{bmatrix}$$

$$\mathbf{P}^4 = \frac{1}{32}\begin{bmatrix} 17 & 15 \\ 15 & 17 \end{bmatrix}$$

$$\boxed{p_{0,0}^{(5)} = 33/64 = 0.515625}$$

15 minutes break

Transient State Distribution

- We want probability that the system is in state *i* at time *n*

$$\pi_i(n) := P(X(n) = i)$$

- Assume initial distribution is known

$$\pi_i(0) = P(X(0) = i), \quad \forall i \in \mathcal{E}$$

normalization $\displaystyle\sum_{i \in \mathcal{E}} \pi_i(0) = 1$

- Law of total probability

$$P(X(n) = j) = \pi_j(n) = \sum_{i \in \mathcal{E}} \pi_i(0)\, p_{i,j}^{(n)}$$

- In matrix notation $\boxed{\pi(n) = \pi(0)\,\mathbf{P}^n}$

Limiting State Distribution

- From equation giving transient distribution

$$\pi(n) = \pi(n-1)\,\mathbf{P}$$

- If limit exists $\quad \pi = \lim_{n\to\infty} \pi(n) = (\pi_i, i \in \mathcal{E})$

$$\boxed{\pi = \pi\,\mathbf{P}}$$

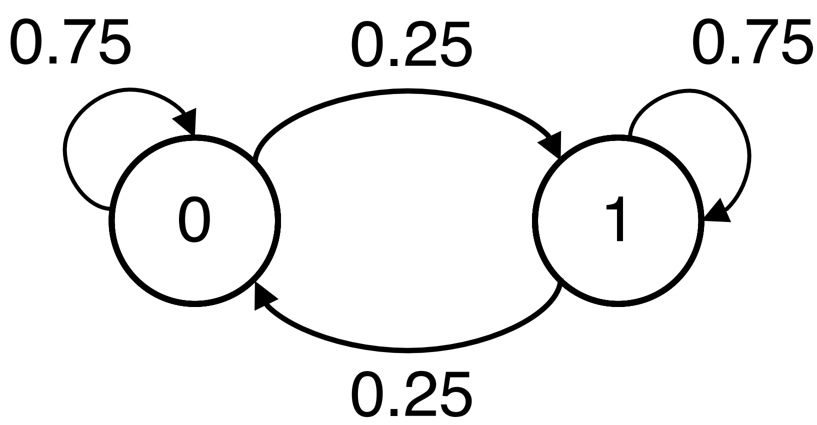
$$\rightarrow \qquad \pi_i = \sum_{j\in\mathcal{E}} \pi_j p_{j,i} \ , \quad i \in \mathcal{E}$$

- Normalization $\quad \sum_{i\in\mathcal{E}} \pi_i = 1$

$$\boxed{\pi\,\mathbf{1} = 1}$$

DTMC Property 1: Aperiodicity?

- For state *i* define $d(i) = \gcd\{n | p_{i,i}^{(n)} > 0\}$
- If 1 then state is aperiodic, otherwise state is periodic
- If all states are aperiodic ➜ DTMC is aperiodic



$$d(0) = \gcd\{1, 2, 3, 4, \ldots\} = 1$$
$$d(1) = \gcd\{1, 2, 3, 4, \ldots\} = 1$$

this DTMC is aperiodic

DTMC Property 2: Irreducibility?

- State *j* is reachable from state *i* if for some *n* $p_{i,j}^{(n)} > 0$
- Two states communicate if each can reach the other
- DTMC is irreducible if any pair of states communicate
- Checking irreducibility on transition diagram
  - If there is a path going through all states then DTMC is irreducible

DTMC Property 3: Positive Recurrence?

- Does the DTMC return to a given state?

Let f_i be probability to return to state i if starting there

- $f_i < 1$ ➜ state i is transient
- $f_i = 1$ ➜ state i is recurrent
  - Mean time between visits is finite

    ➜ state i is positive recurrent
  - Mean time between visits is infinite

    ➜ state i is null recurrent

- If DTMC irreducible, all states are the same
- DTMC is positive recurrent if all its states are

DTMC Property 4: Ergodicity?

- A DTMC is ergodic if it is aperiodic, irreducible and positive recurrent
- Limiting distribution $\lim_{n\to\infty} \pi(n) = \pi(0) \lim_{n\to\infty} \mathbf{P}^n$
- Invariant measure is solution of system (if it exists)

$$\pi = \pi\mathbf{P}$$

$$\pi\,\mathbf{1} = 1$$

- Long-run distribution $\lim_{n\to\infty} \frac{S_j(n)}{n}$

DTMC Property 4: Ergodicity?

- If DTMC ergodic

Long-run distribution = invariant measure

= limiting distribution

Existence of Limiting Distribution

- If homogeneous DTMC is aperiodic and irreducible
- If system of equations

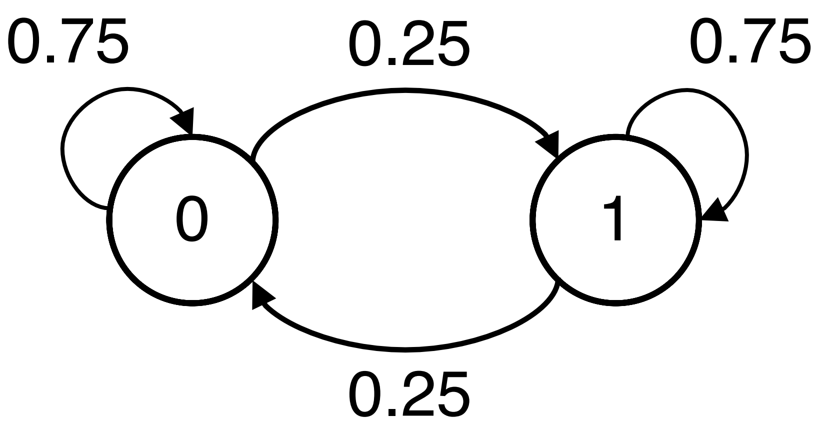
$$\pi = \pi \mathbf{P}$$
$$\pi \, \mathbf{1} = 1$$

has unique strictly positive solution

$\rightarrow$ $$\lim_{n \to \infty} \Big(P(X(n) = i), i \in \mathcal{E} \Big) = \pi$$

The limiting distribution exists and it is the invariant measure

Example 2 page 8



$$\mathbf{P} = \begin{bmatrix} 0.75 & 0.25 \\ 0.25 & 0.75 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

- DTMC is aperiodic and irreducible
- $\pi = \pi\mathbf{P} \Leftrightarrow \pi_0 = 0.75\pi_0 + 0.25\pi_1 \qquad \Rightarrow \pi_0 = \pi_1$

  $\pi\mathbf{1} = 1 \Leftrightarrow \pi_0 + \pi_1 = 1$

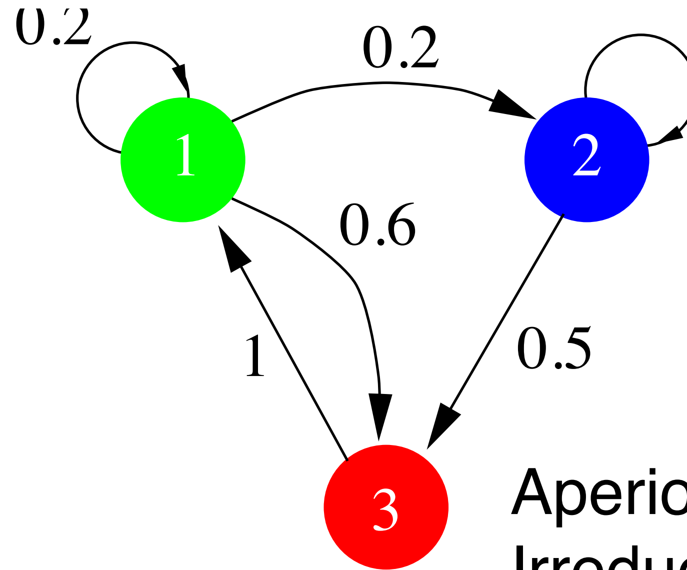
- The solution is unique and strictly positive $\quad \pi = \frac{1}{2}(1,1)$

➜ this is the limiting distribution

Example on page 12

- DTMC with $\mathcal{E} = \{1, 2, 3\}$

$$\mathbf{P} = \begin{pmatrix} 0.2 & 0.2 & 0.6 \\ 0 & 0.5 & 0.5 \\ 1 & 0 & 0 \end{pmatrix}$$



Aperiodic? ✔
Irreducible? ✔

- From transition diagram

$$\left\{\begin{array}{lcl} \pi_1 & = & 0.2\pi_1 + \pi_3 \\ \pi_2 & = & 0.2\pi_1 + 0.5\pi_2 \\ \pi_3 & = & 0.6\pi_1 + 0.5\pi_2 \\ 1 & = & \pi_1 + \pi_2 + \pi_3 \end{array}\right. \Rightarrow \left\{\begin{array}{lcl} \pi_3 & = & \frac{4}{5}\pi_1 \\ \pi_2 & = & \frac{2}{5}\pi_1 \\ & & \\ 1 & = & \pi_1\left(1 + \frac{2}{5} + \frac{4}{5}\right) \end{array}\right.$$

$$\Rightarrow \boxed{\pi_1 = \frac{5}{11}, \pi_2 = \frac{2}{11}, \pi_3 = \frac{4}{11}}$$

For next week

- Lesson 1 to revise
- Homework 1 to return on Tuesday 11 January before 9.30 am
- Lesson 2 to read before Lecture 2

Performance Evaluation of Networks

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Ch 1 - Discrete-Time Markov Chain (DTMC)

- Properties of DTMC:
  - aperiodicity, irreducibility, positive recurrence, ergodicity
- *n*-step transition matrix $\mathbf{P}^n$
- Transient distribution

$$\pi(n) = \pi(0)\,\mathbf{P}^n$$

- Limiting distribution

$$\lim_{n\to\infty} \pi(n) = \pi(0) \lim_{n\to\infty} \mathbf{P}^n$$

- Stationary distribution

$$\pi = \pi\mathbf{P}$$

$$\pi\,\mathbf{1} = 1$$

Ch 2 - Continuous-Time Markov Chain (CTMC)

- Stochastic process $\{X(t), t \geq 0\}$ with discrete state-space and verifying Markov property

$$P(X(t) = j \,|\, X(s_1) = i_1, \ldots, X(s_{n-1}) = i_{n-1}, X(s) = i)$$
$$= P(X(t) = j \,|\, X(s) = i)$$
$$i_1, \ldots, i_{n-1}, i, j \in \mathcal{E}$$
$$0 \leq s_1 < \ldots < s_{n-1} < s < t$$

is a continuous-time Markov chain

- CTMC is homogeneous if

$$P(X(t) = j \,|\, X(s) = i) = p_{i,j}(t - s)$$
$$\forall i, j \in \mathcal{E}, 0 \leq s < t$$

Chapman-Kolmogorov Equation

- Proposition 4: For all $t > 0, s > 0, i, j \in \mathcal{E}$

$$\boxed{p_{i,j}(t+s) = \sum_{k \in \mathcal{E}} p_{i,k}(t)\, p_{k,j}(s)} \qquad \boxed{\mathbf{P}(t+s) = \mathbf{P}(t) \cdot \mathbf{P}(s)}$$

DTMC:

$$p_{i,j}^{(n+m)} = \sum_{k \in \mathcal{E}} p_{i,k}^{(n)}\, p_{k,j}^{(m)} \qquad \mathbf{P}^{(n+m)} = \mathbf{P}^{(n)}\, \mathbf{P}^{(m)}$$

$$\mathbf{P}^{(n)} = \mathbf{P}^{n}$$

- Proof: use law of total probability and Markov property
- Matrix notation

$$\mathbf{P}(t) = \left[p_{i,j}(t)\right]_{i,j \in \mathcal{E}}$$

- CTMC: real steps vs. DTMC: integer steps

Infinitesimal Generator

- Define

$$q_{i,i} := \lim_{h\to 0} \frac{p_{i,i}(h)-1}{h} \le 0$$

$$q_{i,j} := \lim_{h\to 0} \frac{p_{i,j}(h)}{h} \ge 0$$

$$q_{i,i} = -\sum_{j\neq i} q_{i,j}$$

- Matrix $\boxed{\mathbf{Q} = [q_{i,j}]} = \lim_{h\to 0} \frac{\mathbf{P}(h)-\mathbf{I}}{h}$ $\quad \mathbf{I} = \begin{bmatrix} 1 & 0 & \cdots \\ 0 & 1 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$

- If CTMC can take any integer value

$$\mathbf{Q} = \begin{pmatrix} -\sum_{j\neq 0} q_{0,j} & q_{0,1} & q_{0,2} & \cdots \\ q_{1,0} & -\sum_{j\neq 1} q_{1,j} & q_{1,2} & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} \quad \sum = 0$$

Interpretation

- $q_{i,j}$ is the transition rate to state j when in state i
- $-q_{i,i}$ is transition rate out of state i

- Sojourn time in state i is Exp($-q_{i,i}$) $P(S(i) > x) = \mathrm{e}^{q_{i,i}x}$

Proof: assume $X(0) = i$ and sojourn time is $S(i)$

$$P(S(i) > x + h) = P\big(S(i) > x \textbf{ and } X(t) = i, x < t \leq x + h\big)$$
$$= P(S(i) > x)\, P\big(X(t) = i, x < t \leq x + h\big)$$

When $h \to 0, \quad P\big(X(t) = i, x < t \leq x + h\big) \approx p_{i,i}(h)$

$$q_{i,i} = \lim_{h \to 0} \frac{p_{i,i}(h) - 1}{h} \quad \Rightarrow \quad p_{i,i}(h) = 1 + hq_{i,i} + o(h)$$

Proof of Exp Sojourn Time (continued)

$$P(S(i) > x + h) - P(S(i) > x) = P(S_i > x)(hq_{i,i} + o(h))$$

Divide by h and take limit as $h \to 0$

$$\frac{dP(S(i) > x)}{dx} = q_{i,i}\, P(S(i) > x)$$

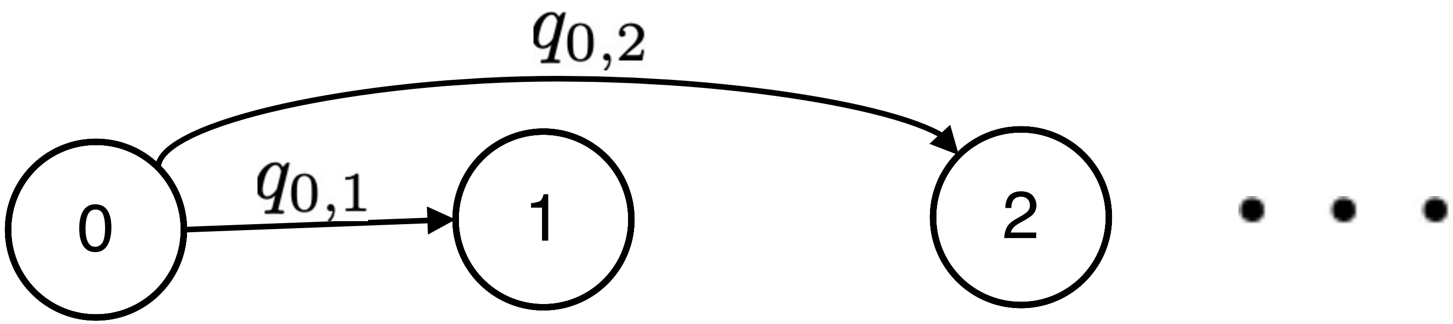
Initial condition is $P(S(i) > 0) = 1$

Solution of differential equation is

$$P(S(i) > x) = \exp(q_{i,i}x)$$

➜ Sojourn time is exponentially distributed

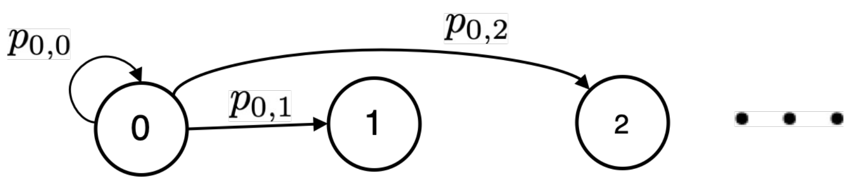
Transition Rate Diagram



- Sum of arrows out of a state i is $-q_{i,i}$

- For DTMC

Transition Diagram



Sum of arrows out of a state is 1

- Probability to go to state j when in i is

$$p(i,j) = \frac{q_{i,j}}{\sum_{j \neq i} q_{i,j}} = \frac{q_{i,j}}{-q_{i,i}} \qquad \boxed{\Rightarrow \;\; q_{i,j} = -q_{i,i}\, p(i,j)}$$

Transient State Distribution

- We want probability that the system is in state *i* at time *t*

$$\pi_i(t) := P(X(t) = i)$$

- Law of total probabilities: for any j

$$\pi_j(t+h) = \sum_{i \in \mathcal{E}} p_{i,j}(h)\,\pi_i(t) = \sum_{i \neq j} p_{i,j}(h)\pi_i(t) + p_{j,j}(h)\pi_j(t)$$

- Subtract $\pi_j(t)$ from both sides then divide by h

$$\boxed{\frac{\pi_j(t+h) - \pi_j(t)}{h}} = \sum_{\substack{i \in \mathcal{E} \\ i \neq j}} \boxed{\frac{p_{i,j}(h)}{h}}\pi_i(t) + \boxed{\frac{p_{j,j}(h) - 1}{h}}\pi_j(t)$$

- $h \to 0:$ $$\frac{d\pi_j(t)}{dt} = \sum_{\substack{i \in \mathcal{E} \\ i \neq j}} q_{i,j}\pi_i(t) + q_{j,j}\pi_j(t) = \sum_{i \in \mathcal{E}} q_{i,j}\pi_i(t)$$

Transient State Distribution

- For any j, $$\frac{d\pi_j(t)}{dt} = \sum_{i\in\mathcal{E}} q_{i,j}\pi_i(t)$$
- Row vector $$\pi(t) = (\pi_i(t), i \in \mathcal{E})$$
- In matrix notation $$\frac{d}{dt}\pi(t) = \pi(t)\mathbf{Q}, \qquad t \geq 0$$
- Solution $$\boxed{\pi(t) = \pi(0)\, e^{\mathbf{Q}t}}, \qquad t \geq 0$$
- By definition $$e^{\mathbf{Q}t} = \sum_{k=0}^{\infty} \frac{(\mathbf{Q}t)^k}{k!}$$
- For DTMC $$\boxed{\pi(n) = \pi(0)\,\mathbf{P}^n}$$

Limiting State Distribution

- From equation giving transient distribution
$$\pi(t) = \pi(0)\, e^{\mathbf{Q}(t-h)} e^{\mathbf{Q}h} = \pi(t-h) e^{\mathbf{Q}h}$$
- For DTMC $\pi(n) = \pi(n-m)\mathbf{P}^m$
- If limit exists $\pi = \lim_{t\to\infty} \pi(t) = (\pi_i, i \in \mathcal{E})$
- $h \to 0: \quad \pi(t) = \pi(t)(\mathbf{I} + \mathbf{Q}h + o(h))$
- $t \to \infty: \quad \pi = \pi + \pi\mathbf{Q}h + \pi o(h)$
- Divide by h and take limit as $h \to 0$ $\boxed{\pi\mathbf{Q} = 0}$
- Normalization $\boxed{\pi\mathbf{1} = 1}$ For DTMC $\pi\mathbf{P} = \pi$

CTMC Properties

- A CTMC is irreducible if all pairs of states communicate
- A CTMC is positive recurrent if all states are positive recurrent
- A CTMC is ergodic if irreducible and positive recurrent
- Irreducible + finite state-space ➜ positive recurrent
  ➜ ergodic

➜ long-run distribution = invariant measure
= limiting distribution

Existence of Limiting Distribution

- If homogeneous CTMC is irreducible
- If system of equations

$$\pi \mathbf{Q} = 0$$

$$\pi \mathbf{1} = 1$$

has unique strictly positive solution

➜ $$\lim_{t \to \infty} \Big(P(X(t) = i), i \in \mathcal{E} \Big) = \pi$$

The limiting distribution exists and it is the stationary distribution

15 minutes break

Balance Equations

- Develop $\pi \mathbf{Q} = 0$

- For all $i \in \mathcal{E}$ $\quad \sum_{j \in \mathcal{E}} \pi_j \, q_{j,i} = 0$

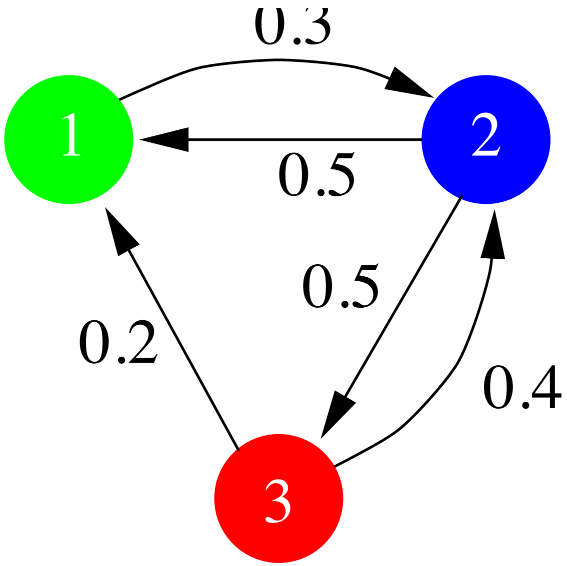
$$-\pi_i \, q_{i,i} = \sum_{j \neq i} \pi_j \, q_{j,i}$$

$$\boxed{\left(\sum_{j \neq i} q_{i,j} \right) \pi_i = \sum_{j \neq i} q_{j,i} \, \pi_j}$$

probability flow rate out of state i = probability flow rate into state i

- $q_{j,i}$: transition rate to state i when in state j
- $q_{j,i}\pi_j$: probability flow rate from state j to state i

Example



$$\mathcal{E} = \{1, 2, 3\}$$

CTMC irreducible? ✔

$$\mathbf{Q} = \begin{pmatrix} -0.3 & 0.3 & 0 \\ 0.5 & -1 & 0.5 \\ 0.2 & 0.4 & -0.6 \end{pmatrix}$$

flow out = flow in

$$\left\{ \begin{array}{rcl} 0.3\pi_1 & = & 0.5\pi_2 + 0.2\pi_3 \\ (0.5+0.5)\pi_2 & = & 0.3\pi_1 + 0.4\pi_3 \\ (0.2+0.4)\pi_3 & = & 0.5\pi_2 \\ \pi_1 + \pi_2 + \pi_3 & = & 1 \end{array} \right. \Rightarrow \left\{ \begin{array}{l} \pi_2 = 0.3\pi_1 + 0.4\pi_3 \\ 1.2\pi_3 = \pi_2 \\ \pi_1 + \pi_2 + \pi_3 = 1 \end{array} \right.$$

Example

$$\begin{cases} \pi_2 = 0.3\pi_1 + 0.4\pi_3 \\ 1.2\pi_3 = \pi_2 \quad ✔ \\ \pi_1 + \pi_2 + \pi_3 = 1 \end{cases} \qquad \Rightarrow 1.2\pi_3 = 0.3\pi_1 + 0.4\pi_3$$

$$\Rightarrow \pi_1 = \frac{8}{3}\pi_3 \quad ✔$$

- Normalization

$$\pi_3\left(\frac{8}{3} + \frac{6}{5} + 1\right) = 1 \quad \Leftrightarrow \quad \pi_3\left(\frac{40+18+15}{15}\right) = 1 \quad \Leftrightarrow \quad \pi_3 = \frac{15}{73}$$

$$\boxed{\Rightarrow \quad \pi = (\pi_1, \pi_2, \pi_3) = \frac{1}{73}\Big(40, 18, 15\Big)}$$

sanity check
$\Sigma = 1$

- Solution is unique and strictly positive

  ➜ this is the limiting distribution

Recap

	DTMC	CTMC
Chapman-Kolmogorov	$\mathbf{P}^{(n+m)} = \mathbf{P}^{(n)}\,\mathbf{P}^{(m)}$	$\mathbf{P}(t+s) = \mathbf{P}(t)\cdot\mathbf{P}(s)$
Main matrix	Transition matrix P	Infinitesimal generator Q
with characteristics	Row sum is 1	Row sum is 0
Diagram	Transition probabilities	Transition rates
with characteristics	Arrows out sum to 1	No loops
Transient distribution	$\pi(n) = \pi(0)\,\mathbf{P}^n$	$\pi(t) = \pi(0)\,e^{\mathbf{Q}t}$
Stationary distribution	$\pi\mathbf{P} = \pi$ $\quad \pi\mathbf{1} = 1$	$\pi\mathbf{Q} = 0$ $\quad \pi\mathbf{1} = 1$
Properties to check	Aperiodicity, irreducibility	Irreducibility
Sojourn time	Geometric	$\boxed{\text{Exp}(-q_{i,i})}$
Markov property	Easy to check	????

$$P(X(t)=j \,|\, X(s_1)=i_1,\ldots,X(s_{n-1})=i_{n-1},X(s)=i)$$

$$= P(X(t)=j \,|\, X(s)=i) = p_{i,j}(t-s)$$

$$i_1,\ldots,i_{n-1},i,j \in \mathcal{E}$$

$$0 \le s_1 < \ldots < s_{n-1} < s < t$$

Impossible to check for CTMC

Construction Rule #1

- Continuous-time stochastic process $\{X(t), t \geq 0\}$
- If for each state i
  - ▶ Process stays in i for a time that is Exp(τ_i)
  - ▶ Once sojourn time is over, process jumps to state j with prob a_{ij} (a_{ii} = 0, their sum for all j is 1)

➜ $\{X(t), t \geq 0\}$ is a CTMC

- Infinitesimal generator

$$\mathbf{Q} = \left[q_{i,j}\right]_{i,j\in\mathcal{E}}, \quad q_{i,j} = \begin{cases} \tau_i a_{ij} & i \neq j \\ -\tau_i & i = j \end{cases}$$

Construction Rule #2

- Continuous-time stochastic process $\{X(t), t \geq 0\}$
- For each state i, as soon as process enters state i
  - For each other state j generate sample for $Y_{i,j}$ from Exp($\mu_{i,j}$)

    (if no transition from i to j, then $\mu_{i,j}$ is 0 and sample is ∞)
  - The first sample to expire makes process jump to corresponding state

➜ $\{X(t), t \geq 0\}$ is a CTMC

- Infinitesimal generator

$$\mathbf{Q} = \left[q_{i,j}\right]_{i,j \in \mathcal{E}}, \quad q_{i,j} = \begin{cases} \mu_{i,j} & j \neq i \\ -\sum_{k \neq i} \mu_{i,k} & j = i \end{cases}$$

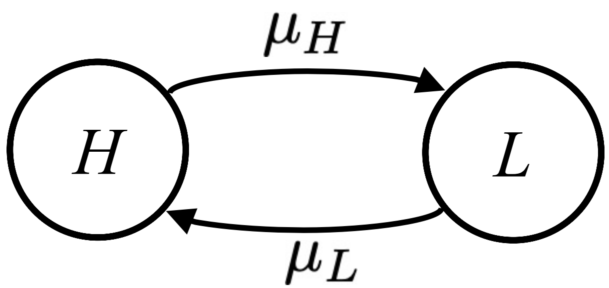
Construction Rule #2

- In practice we use this construction when at each state i
  - Several processes or events can cause a state change
  - Each lasts for a time Exp()
- Example:
  - Jobs are submitted to a server according to Poisson process rate λ ➜ time to submit job is Exp(λ)
  - A job has 1 task with prob ½ and 2 tasks with prob ½
  - Server processes tasks one at a time, service time Exp(μ)
  - State is number of tasks, state-space is set of integers

$$\left.\begin{array}{lll} i \to i+1 & \mathrm{Exp}(\lambda/2) & \\ i \to i+2 & \mathrm{Exp}(\lambda/2) & \\ i \to i-1 & \mathrm{Exp}(\mu) & i > 0 \end{array}\right\} \text{ we have a CTMC}$$

Example 2.1 Page 19

- Stochastic process $\mathbf{Y} = \{Y(t), t \geq 0\}$ alternates between states *H* and *L*
- Sojourn time in *H* is Exp(μ_H) with mean $1/\mu_H$
- Sojourn time in *L* is Exp(μ_L) with mean $1/\mu_L$
- $\mathcal{E} = \{H, L\}$ prob 1 to switch state
- Construction rule #1 ➜ $\mathbf{Y}$ is a CTMC
- Infinitesimal generator $\mathbf{Q} = \begin{pmatrix} -\mu_H & \mu_H \\ \mu_L & -\mu_L \end{pmatrix}$
- Transition diagram



Global Balance Equations

- Balance equations: for any state i

$$\left(\sum_{j \neq i} q_{i,j}\right) \pi_i = \sum_{j \neq i} q_{j,i}\, \pi_j$$

probability flow rate out of state i

= probability flow rate into state i

- Global balance equations: S subset of $\mathcal{E}$

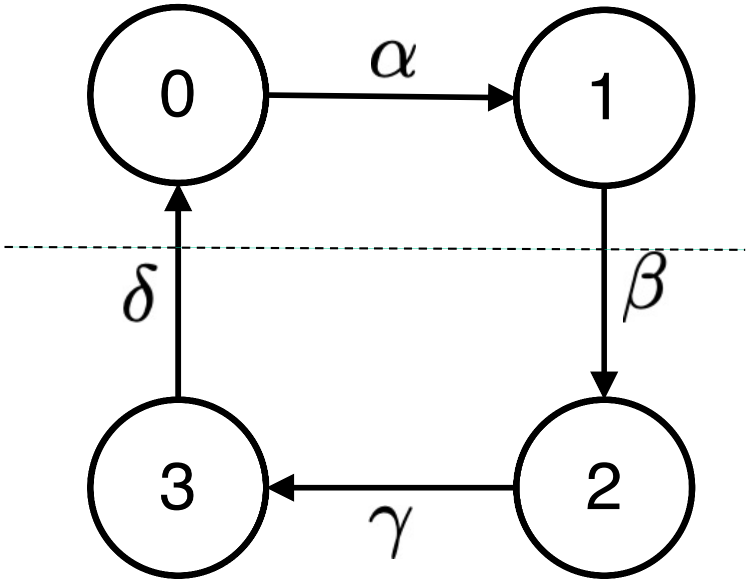
probability flow rate out of S = probability flow rate into S

$$\boxed{\sum_{i \in \mathcal{S}} \sum_{j \in \overline{\mathcal{S}}} \pi_i q_{i,j} \;=\; \sum_{i \in \overline{\mathcal{S}}} \sum_{j \in \mathcal{S}} \pi_i q_{i,j}}$$

  - If $\mathcal{S} = \{i\}$ ➜ balance equation for state i

Global Balance Equations

- Proof: use $\pi \mathbf{Q} = 0$ and $\mathbf{Q1} = 0$ and find that diff is 0
- Example $\mathcal{S} = \{0, 1\}$



global balance equation

$$\beta \pi_1 = \delta \pi_3$$

balance equations

$$\boxed{\begin{aligned} \alpha \pi_0 &= \delta \pi_3 \\ \beta \pi_1 &= \alpha \pi_0 \end{aligned}}$$

$$\gamma \pi_2 = \beta \pi_1$$

$$\delta \pi_3 = \gamma \pi_2$$

- By summing first two expressions we find the result

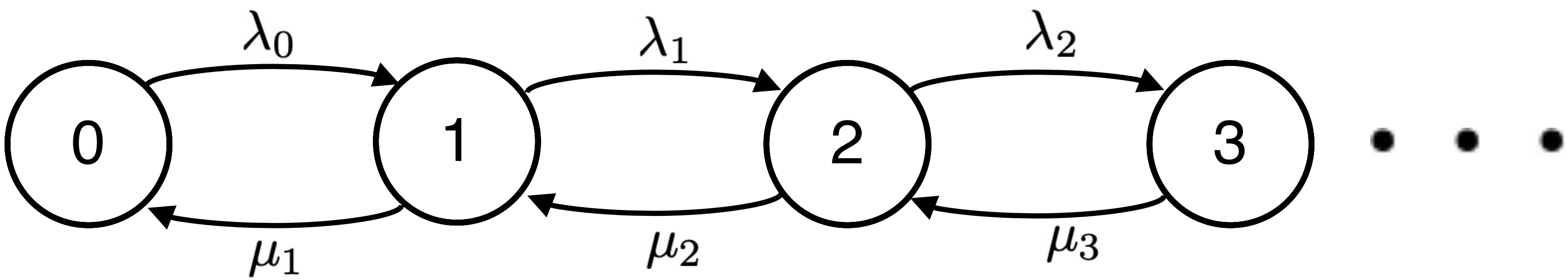
Birth and Death Process

- Particular CTMC over set of integers (or a subset)
  - New state is a neighbor of the previous state
- Birth and Death process seen as population size
- Birth rate when state is $i \geq 0, \quad \lambda_i = q_{i,i+1}$
- Death rate when state is $i \geq 1, \quad \mu_i = q_{i,i-1}$
- Infinitesimal generator is tri-diagonal matrix

$$\mathbf{Q} = \begin{pmatrix} -\lambda_0 & \lambda_0 & 0 & \cdots & & \\ \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 & 0 & \cdots & \\ 0 & \mu_2 & -(\lambda_2+\mu_2) & \lambda_2 & 0 & \cdots \\ \vdots & 0 & \mu_3 & -(\lambda_3+\mu_3) & \lambda_3 & \ddots \\ & \vdots & 0 & \mu_4 & \ddots & \ddots \\ & & \vdots & \ddots & \ddots & \end{pmatrix}$$

Birth and Death Process

- Transition rate diagram $\mathcal{E} = \mathbb{N}$



- Balance equations

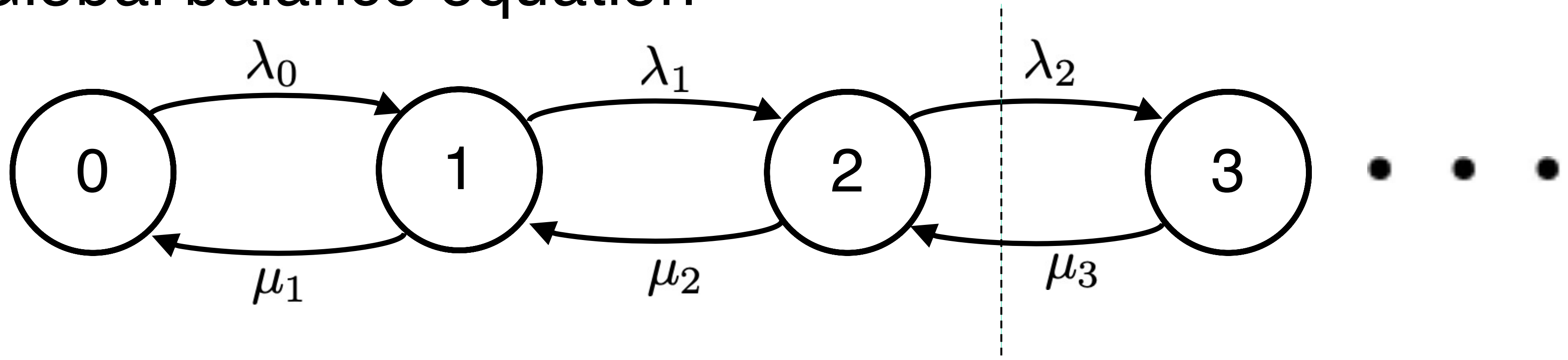
$$
\begin{aligned}
\cancel{\lambda_0\pi_0} &= \cancel{\mu_1\pi_1} \\
(\cancel{\lambda_1} + \cancel{\mu_1})\pi_1 &= \cancel{\lambda_0\pi_0} + \cancel{\mu_2\pi_2} \\
&\vdots \\
(\cancel{\lambda_{i-1}} + \cancel{\mu_{i-1}})\pi_{i-1} &= \cancel{\lambda_{i-2}\pi_{i-2}} + \cancel{\mu_i\pi_i} \\
(\lambda_i + \cancel{\mu_i})\pi_i &= \cancel{\lambda_{i-1}\pi_{i-1}} + \mu_{i+1}\pi_{i+1}
\end{aligned}
$$

$$\lambda_i\pi_i = \mu_{i+1}\pi_{i+1}$$

$$\boxed{\Rightarrow \quad \pi_{i+1} = \frac{\lambda_i}{\mu_{i+1}}\pi_i, \quad i = 0, 1, \ldots}$$

Birth and Death Process

- Global balance equation



$$\lambda_2\pi_2 = \mu_3\pi_3$$

- For any state $i > 0$ $\quad \lambda_{i-1}\pi_{i-1} = \mu_i\pi_i$

Birth and Death Process

- By recurrence $$\boxed{\pi_i = \frac{\lambda_0\lambda_1\cdots\lambda_{i-1}}{\mu_1\mu_2\cdots\mu_i}\pi_0, \quad i = 1, 2, \ldots}$$
- Normalization $$\sum_{i=0}^{\infty}\pi_i = 1$$

$$\pi_0\left(1 + \frac{\lambda_0}{\mu_1} + \frac{\lambda_0\lambda_1}{\mu_1\mu_2} + \cdots + \frac{\lambda_0\lambda_1\cdots\lambda_{i-1}}{\mu_1\mu_2\cdots\mu_i} + \cdots\right) = 1$$

- Define $$C := \sum_{i\geq 0}\prod_{j=1}^{i}\frac{\lambda_{j-1}}{\mu_j}$$
- If $C < \infty$ (stability condition) then $\boxed{\pi_0 = \frac{1}{C} > 0}$
- Irreducible ➜ limiting distribution exists ($C < \infty$)

Time-Reversibility

- A CTMC is time-reversible if for any two states $i, j \in \mathcal{E}$

$$\pi_i q_{i,j} = \pi_j q_{j,i}$$

- A birth and death process is time-reversible
- Consequence:

  forwards chain and reverse chain are statistically identical and described by same transition diagram

- Example

  Forward trajectory …0121232101232101010…

  Backward trajectory …0101012321012321210…

  Probability flow rates between any two states are same in either direction (e.g. #23 = #32)

For next week

- Lesson 2 to revise
- Homework 2 to return on Tuesday 24 September before 9.00 am
- Lesson 3 to read before Lecture 3

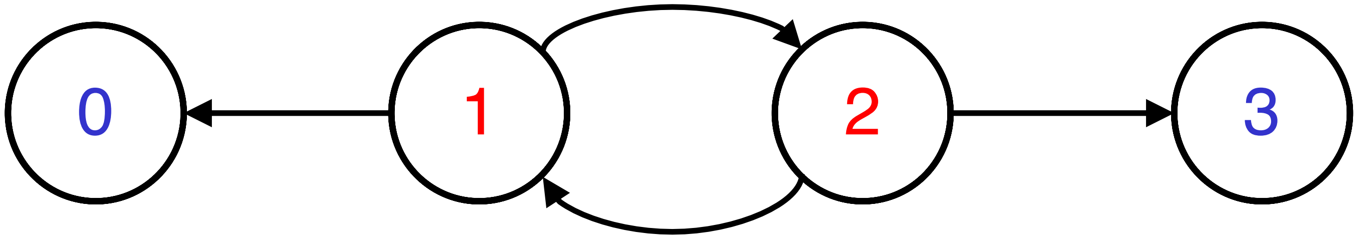
Performance Evaluation of Networks

Sara Alouf

Ch 3 – Absorbing Markov Chains

- Chapters 1 and 2: Irreducible Markov chains
  - Transient distribution
  - Steady-state / limiting distribution
- Absorbing Markov Chains

  ➜ some states are a dead end



0, 3: absorbent

1, 2: transient

- Transient distribution
- Time until absorption
- Probability to be absorbed in a given absorbing state

Discrete-Time Absorbing Markov Chain

- Homogeneous DTMC $\{X(n), n \geq 0\}$
- State space $\mathcal{E} := \{\boxed{1, 2, \ldots, N}, \boxed{1^*, 2^*, \ldots, M^*}\}$

  N transient states M absorbing states

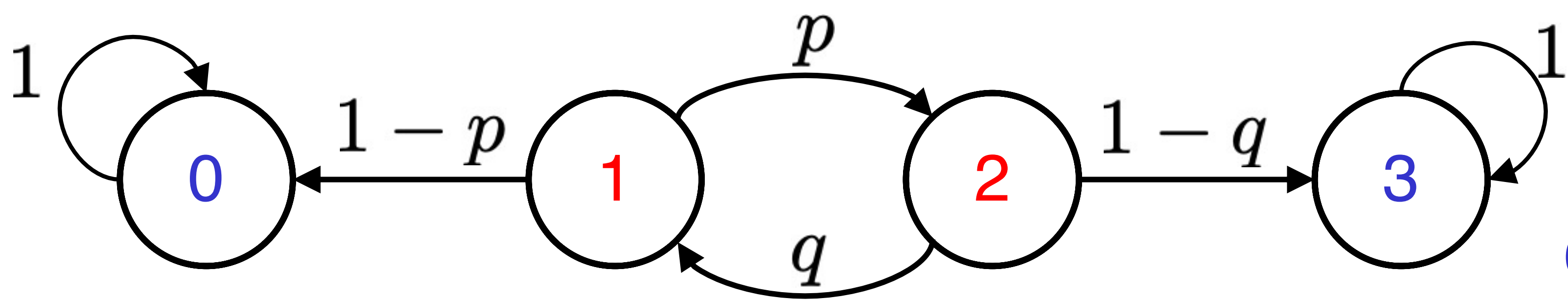
- Transition matrix $\mathbf{P} = [p_{i,j}]_{i,j \in \mathcal{E}}$

$$\mathbf{P} = \begin{pmatrix} a_{1,1} & \cdots & a_{1,N} & r_{1,1*} & \cdots & r_{1,M*} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{N,1} & \cdots & a_{N,N} & r_{N,1*} & \cdots & r_{N,M*} \\ 0 & \cdots & 0 & 1 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 1 \end{pmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{R} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}$$

$\mathbf{A} = [a_{i,j}]_{1 \leq i,j \leq N}$ $\mathbf{R} = [r_{i,j}]_{1 \leq i \leq N, 1^* \leq j \leq M^*}$

Example

- DTMC with transition diagram



1, 2: transient

0, 3: absorbent

- State-space $\mathcal{E} = \{1, 2, 0, 3\}$ ➜ order is important!

- Transition matrix (follow order)

$$\mathbf{P} = \begin{array}{c} \\ 1 \\ 2 \\ 0 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 0 & 3 \end{array} \\ \left[\begin{array}{cc|cc} 0 & p & 1-p & 0 \\ q & 0 & 0 & 1-q \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \end{array} = \begin{bmatrix} \mathbf{A} & \mathbf{R} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} = \begin{bmatrix} 0 & p \\ q & 0 \end{bmatrix}$$

$$\mathbf{R} = \begin{bmatrix} r_{1,0} & r_{1,3} \\ r_{2,0} & r_{2,3} \end{bmatrix} = \begin{bmatrix} 1-p & 0 \\ 0 & 1-q \end{bmatrix}$$

n-step Transition Matrix

- Transition matrix $\mathbf{P} = \begin{bmatrix} \mathbf{A} & \mathbf{R} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}$

- From Chapman-Kolmogorov equation $\mathbf{P}^{(n)} = \mathbf{P}^n$

$$\mathbf{P}^2 = \begin{bmatrix} \mathbf{A} & \mathbf{R} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{A} & \mathbf{R} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} = \begin{bmatrix} \mathbf{A}^2 & \mathbf{AR} + \mathbf{R} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}$$

$$\mathbf{P}^3 = \begin{bmatrix} \mathbf{A}^2 & \mathbf{AR} + \mathbf{R} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{A} & \mathbf{R} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} = \begin{bmatrix} \mathbf{A}^3 & \mathbf{A}^2\mathbf{R} + \mathbf{AR} + \mathbf{R} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}$$

$$\boxed{\mathbf{P}^n = \begin{bmatrix} \mathbf{A}^n & \sum_{k=0}^{n-1} \mathbf{A}^k\mathbf{R} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}}$$

- Transient distribution $\pi(n) = \pi(0)\,\mathbf{P}^n$

- Limiting distribution $\lim_{n\to\infty} \pi(n) = \pi(0) \boxed{\lim_{n\to\infty} \mathbf{P}^n}$

Fundamental Matrix

- Define $N \times N$ matrix $\mathbf{N} = [n_{i,j}]_{1 \leq i,j \leq N}$

  $n_{i,j}$ expected number of visits to state j if initially in i

- Define $X_j^{(n)} = \mathbb{1}(X(n) = j) = \begin{cases} 1 & X(n) = j \\ 0 & \text{otherwise} \end{cases}$

- We have $n_{i,j} = E\left[\sum_{n \geq 0} X_j^{(n)} \,|\, X(0) = i\right]$

- Proposition 8: the fundamental matrix is

$$\boxed{\mathbf{N} = (\mathbf{I} - \mathbf{A})^{-1}}$$

Nota bene:
elements in $\mathbf{N}$ are
positive or null

Proof of $\mathbf{N} = (\mathbf{I} - \mathbf{A})^{-1}$

$$E\left[\sum_{k=0}^{n} X_j^{(k)} \,\middle|\, X(0) = i\right] = \sum_{k=0}^{n} E\left[X_j^{(k)} \,\middle|\, X(0) = i\right]$$

$$= \sum_{k=0}^{n} E\left[\mathbb{1}(X(k) = j) \,|\, X(0) = i\right]$$

expected number of visits

in $n + 1$ steps

$$= \sum_{k=0}^{n} P\big(X(k) = j \,|\, X(0) = i\big)$$

$$= \sum_{k=0}^{n} \boxed{a_{i,j}^{(k)}}$$

(i, j) element in $\mathbf{A}^k$

$$E\left[\lim_{n\to\infty} \sum_{k=0}^{n} X_j^{(k)} \,\middle|\, X(0) = i\right] = \lim_{n\to\infty} E\left[\sum_{k=0}^{n} X_j^{(k)} \,\middle|\, X(0) = i\right]$$

$$n_{i,j} = \sum_{k\geq 0} a_{i,j}^{(k)}$$

Proof of $\mathbf{N} = (\mathbf{I} - \mathbf{A})^{-1}$

- For any $i, j = 1, \ldots N$ $\qquad n_{i,j} = \sum_{k \geq 0} a_{i,j}^{(k)}$

$$\Rightarrow \quad \boxed{\mathbf{N} = \sum_{k \geq 0} \mathbf{A}^k}$$

- Consider finite sum

$$(\mathbf{I} - \mathbf{A}) \sum_{k=0}^{n} \mathbf{A}^k = \sum_{k=0}^{n} \mathbf{A}^k - \sum_{k=1}^{n+1} \mathbf{A}^k = \mathbf{I} - \mathbf{A}^{n+1}$$

- Let $n \to \infty$ $\qquad a_{i,j}^{(n+1)} \to 0 \qquad \Rightarrow \qquad \mathbf{A}^{n+1} \to 0$

$$\rightarrow \quad (\mathbf{I} - \mathbf{A})\mathbf{N} = \mathbf{I}$$

- If inverse exists ➜ $\boxed{\mathbf{N} = (\mathbf{I} - \mathbf{A})^{-1}}$

Absorption Probabilities

- Define $N \times M^*$ stochastic matrix $\mathbf{B} = \left[b_{i,j}\right]_{\substack{i\in\{1,\ldots,N\} \\ j\in\{1^*,\ldots,M^*\}}}$

  $b_{i,j}$ probability to be absorbed in j if initially in i

- Proposition 9: $\boxed{\mathbf{B} = \mathbf{NR}}$ recall $\mathbf{P} = \begin{bmatrix} \mathbf{A} & \mathbf{R} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}$

- For $i \in \{1,\ldots,N\}$, $j \in \{1^*,\ldots,M^*\}$

$$b_{i,j} = \sum_{n\geq 0} P(X(n+1) = j \,|\, X(0) = i)$$

use law of tot. prob.

k last transient state

$$= \sum_{n\geq 0} \sum_{k=1}^{N} \overbrace{\boxed{P(X(n+1) = j \,|\, X(n) = k,}}^{r_{k,j}} \cancel{X(0) = i})$$

$$\times \underbrace{\boxed{P(X(n) = k \,|\, X(0) = i)}}_{a_{i,k}^{(n)}}$$

use Markov property

Absorption Probabilities

- For $i \in \{1, \ldots, N\},\ j \in \{1^*, \ldots, M^*\}$

$$\begin{aligned} b_{i,j} &= \sum_{n \geq 0} \sum_{k=1}^{N} r_{k,j} a_{i,k}^{(n)} \\ &= \sum_{k=1}^{N} \left(\sum_{n \geq 0} a_{i,k}^{(n)} \right) r_{k,j} \\ &= \sum_{k=1}^{N} n_{i,k} r_{k,j} \end{aligned}$$

- In matrix notation

$$\mathbf{B} = \mathbf{NR} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{R}$$

Nota bene:
elements in $\mathbf{B}$ are probabilities
$\mathbf{B}$ is stochastic matrix

Limit of $\mathbf{P}^n$

- *n*-step transition matrix

$$\mathbf{P}^n = \begin{bmatrix} \mathbf{A}^n & \sum_{k=0}^{n-1} \mathbf{A}^k \mathbf{R} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}$$

- As $n \to \infty$ $\qquad \mathbf{A}^n \to 0$

$$\sum_{k=0}^{n-1} \mathbf{A}^k \mathbf{R} \to \mathbf{NR}$$

$$\mathbf{P}^n \to \begin{bmatrix} \mathbf{0} & \mathbf{B} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}$$

stochastic matrix

Expected Absorption Time

- Define column vector $\mathbf{T} = (T(i), i = 1, \ldots, N)^T$

  $T(i)$ expected time until absorption if initially in state i

  time = number of steps

- For any absorbing state $j \in \{1^*, \ldots, M^*\}, \quad T(j) = 0$
- Proposition 10: $\boxed{\mathbf{T} = \mathbf{N.1}}$
- $n_{i,j}$ expected number of visits to state j if initially in i

  ➜ $T(i) = \sum_{j=1}^{N} n_{i,j}$ for $i \in \{1, 2, \ldots, N\}$

Expected Absorption Time

- Corollary 1: Column vector $\mathbf{T} = (T(i), i = 1, \ldots, N)^T$ is solution of $\boxed{\mathbf{T} = \mathbf{1} + \mathbf{A}\mathbf{T}}$
- Fundamental matrix $\mathbf{N} = (\mathbf{I} - \mathbf{A})^{-1}$
- $\mathbf{T} = \mathbf{N}.\mathbf{1} \quad \Leftrightarrow \quad \mathbf{T} = (\mathbf{I} - \mathbf{A})^{-1}.\mathbf{1}$

$$\Leftrightarrow \quad (\mathbf{I} - \mathbf{A})\mathbf{T} = \mathbf{1}$$

$$\Leftrightarrow \quad \mathbf{T} - \mathbf{A}\mathbf{T} = \mathbf{1}$$

$$\Leftrightarrow \quad \mathbf{T} = \mathbf{1} + \mathbf{A}\mathbf{T}$$

Nota bene:
elements in $\mathbf{T}$ are
strictly positive

Example

- Document in P2P system is replicated over K peers
- Retrieve requests occur at beginning of every minute
- $X(n)$ number of replicas available just before minute n
- Peers connect/disconnect from system ➜ $X(n)$ stochastic
- If at n no copy is available ➜ request fails $X(n) = F$
- State-space $\mathcal{E} = \{1, \ldots, K, F\}$
- Transition matrix

$$\mathbf{P} = \left[\begin{array}{cccc|c} \frac{1}{K+1} & \frac{1}{K+1} & \cdots & \frac{1}{K+1} & \frac{1}{K+1} \\ \vdots & \vdots & & \vdots & \vdots \\ \frac{1}{K+1} & \frac{1}{K+1} & \cdots & \frac{1}{K+1} & \frac{1}{K+1} \\ \hline 0 & 0 & \cdots & 0 & 1 \end{array}\right]$$

Example

- Describe $\{X(n), n \geq 0\}$
  - ▶ Process observed every minute when requests arrive

    ➜ discrete-time process
  - ▶ Transition probabilities independent of step

    ➜ homogeneous
  - ▶ States 1 to K are transient
  - ▶ State F is absorbent

  $\{X(n), n \geq 0\}$ is absorbing homogeneous DTMC
- Limiting probability of a failure?
  - ▶ Regardless of initial distribution the chain will be absorbed in its unique absorbing state

    ➜ limiting probability of failure is 1

Example

- Compute expected absorption times when $K = 3$

$$\mathbf{P} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{R} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \quad \Rightarrow \quad \mathbf{A} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

- $T(F) = 0$
- $\mathbf{T} = \mathbf{1} + \mathbf{A}\mathbf{T}$

$$\Rightarrow \begin{bmatrix} T(1) \\ T(2) \\ T(3) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} T(1) \\ T(2) \\ T(3) \end{bmatrix}$$

$$\Rightarrow T(1) = T(2) = T(3) = 1 + \frac{1}{4}\Big(T(1) + T(2) + T(3)\Big)$$

$$\Rightarrow 4T(1) = 4 + 3T(1) \qquad \boxed{\Rightarrow T(1) = T(2) = T(3) = 4}$$

- $\mathbf{N} = (\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

$\sum = T(1) = 4$

$\sum = T(2) = 4$

$\sum = T(3) = 4$

$\mathbf{T} = \mathbf{N}.\mathbf{1}$

15 minutes break

Continuous-Time Absorbing Markov Chain

- Homogeneous CTMC $\{X(t), t \geq 0\}$
- State space $\mathcal{E} := \{\underbrace{1, 2, \ldots, N}_{N \text{ transient states}}, \underbrace{1^*, 2^*, \ldots, M^*}_{M \text{ absorbing states}}\}$

N transient states M absorbing states

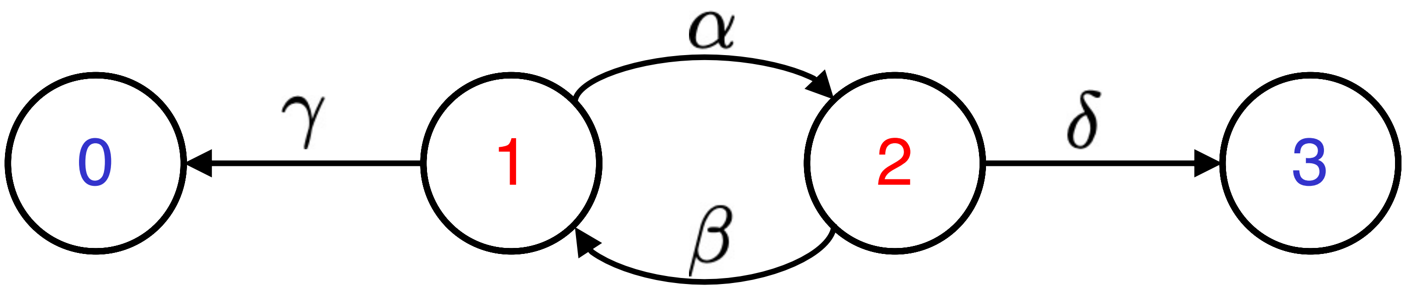
- Infinitesimal generator $\mathbf{Q} = \left[q_{i,j}\right]_{i,j\in\mathcal{E}}$

$$\mathbf{Q} = \begin{pmatrix} q_{1,1} & \cdots & q_{1,N} & \tilde{r}_{1,1*} & \cdots & \tilde{r}_{1,M*} \\ \vdots & & \vdots & \vdots & & \vdots \\ q_{N,1} & \cdots & q_{N,N} & \tilde{r}_{N,1*} & \cdots & \tilde{r}_{N,M*} \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix} = \begin{bmatrix} \tilde{\mathbf{Q}} & \tilde{\mathbf{R}} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

$\tilde{\mathbf{Q}} = [q_{i,j}]_{1\leq i,j\leq N}$ $\tilde{\mathbf{R}} = [\tilde{r}_{i,j}]_{1\leq i\leq N, 1^*\leq j\leq M^*}$

Example

- CTMC with transition rate diagram



1, 2: transient

0, 3: absorbent

- State-space $\mathcal{E} = \{1, 2, 0, 3\}$ ➜ order is important!
- Infinitesimal generator (follow order)

$$\mathbf{Q} = \begin{array}{c} \\ 1 \\ 2 \\ 0 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 0 & 3 \end{array} \\ \left[\begin{array}{cc|cc} -(\alpha+\gamma) & \alpha & \gamma & 0 \\ \beta & -(\beta+\delta) & 0 & \delta \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

$$= \begin{bmatrix} \tilde{\mathbf{Q}} & \tilde{\mathbf{R}} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

$$\tilde{\mathbf{Q}} = \begin{bmatrix} -(\alpha+\gamma) & \alpha \\ \beta & -(\beta+\delta) \end{bmatrix}$$

$$\tilde{\mathbf{R}} = \begin{bmatrix} \tilde{r}_{1,0} & \tilde{r}_{1,3} \\ \tilde{r}_{2,0} & \tilde{r}_{2,3} \end{bmatrix} = \begin{bmatrix} \gamma & 0 \\ 0 & \delta \end{bmatrix}$$

Embedded Markov Chain at Jump Times

- Construct DTMC from CTMC
- Observe $\{X(t), t \geq 0\}$ at jump times ➜ $\{X(n), n \geq 0\}$
- Transition probabilities at jump times

$$p(i,j) = \begin{cases} \frac{q_{i,j}}{-q_{i,i}} & i \in \{1, \ldots, N\}, j \in \mathcal{E}, j \neq i \\ 1 & j = i \in \{1^*, \ldots, M^*\} \text{ (by convention)} \\ 0 & \text{otherwise} \end{cases}$$

- Transition matrix (stochastic) $\mathbf{P} = \Big[p(i,j)\Big]_{i,j,\in\mathcal{E}} = \begin{bmatrix} \mathbf{A} & \mathbf{R} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}$

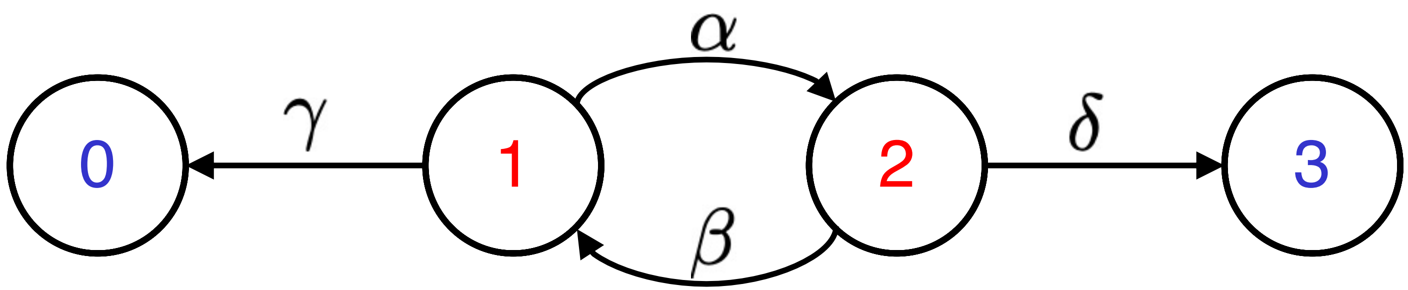
$$\mathbf{A} = [a_{i,j}]_{1 \leq i,j \leq N}$$

$$a_{i,j} = \begin{cases} \frac{q_{i,j}}{-q_{i,i}} & j \neq i \\ 0 & j = i \end{cases}$$

$$\mathbf{R} = [r_{i,j}]_{1 \leq i \leq N, 1^* \leq j \leq M^*}$$

$$r_{i,j} = \frac{q_{i,j}}{-q_{i,i}}$$

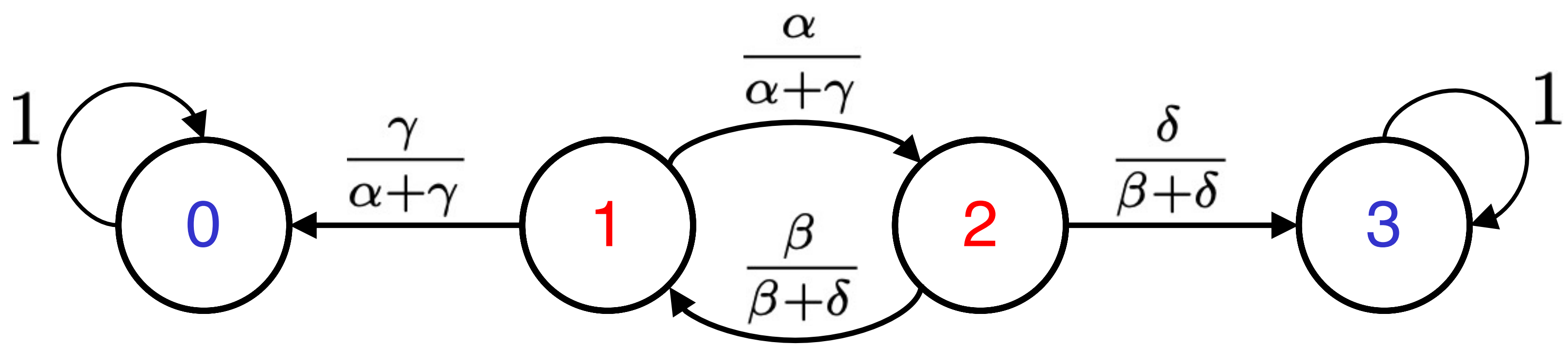
Example

- 

1, 2: transient

0, 3: absorbent

- Embedded Markov chain at jump times



$$\mathbf{P} = \begin{array}{c} \\ 1 \\ 2 \\ 0 \\ 3 \end{array} \!\!\begin{array}{c} \begin{array}{cccc} 1 & 2 & 0 & 3 \end{array} \\ \left[\begin{array}{cc|cc} 0 & \frac{\alpha}{\alpha+\gamma} & \frac{\gamma}{\alpha+\gamma} & 0 \\ \frac{\beta}{\beta+\delta} & 0 & 0 & \frac{\delta}{\beta+\delta} \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \end{array}$$

$$\mathbf{A} = \begin{bmatrix} 0 & \frac{\alpha}{\alpha+\gamma} \\ \frac{\beta}{\beta+\delta} & 0 \end{bmatrix}$$

$$\mathbf{R} = \begin{bmatrix} \frac{\gamma}{\alpha+\gamma} & 0 \\ 0 & \frac{\delta}{\beta+\delta} \end{bmatrix}$$

Mean Number of Visits

- Use embedded Markov chain and results of absorbing DTMC

$$\mathbf{N} = [n_{i,j}]_{1 \le i,j \le N} \qquad \boxed{\mathbf{N} = (\mathbf{I} - \mathbf{A})^{-1}}$$

$n_{i,j}$ expected number of visits to state j if initially in i

- Visits durations are different!

- In absorbing DTMC

➜ a visit lasts for a constant step time

- In Markov chain embedded at jump times of absorbing CTMC

➜ a visit in transient i lasts for a random time that is Exp($-q_{i,i}$)

Absorption Probabilities

- Use embedded Markov chain and results of absorbing DTMC

$$\mathbf{B} = \left[b_{i,j}\right]_{\substack{i\in\{1,\ldots,N\}\\ j\in\{1^*,\ldots,M^*\}}} \qquad \mathbf{B} = \mathbf{NR} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{R}$$

$b_{i,j}$ probability to be absorbed in j if initially in i

- Matrix is stochastic

Expected Absorption Time

- In absorbing CTMC

  number of visits to transient state $\neq$ expected time spent in that state

- Define column vector $\mathbf{T} = (T(i), i = 1, \ldots, N)^T$

  $T(i)$ expected time until absorption if initially in state i

  time $\neq$ number of steps

- Define $N \times N$ matrix $\tilde{\mathbf{T}} = [t_{i,j}]_{1 \leq i,j \leq N}$

  $t_{i,j}$ expected time spent in transient j if initially in state i

- $\boxed{T(i) = \sum_{j=1}^{N} t_{i,j}}$ $\quad t_{i,j} = n_{i,j} \times \frac{1}{-q_{j,j}}, \quad i, j = 1, \ldots, N$

Expected Absorption Time

- Proposition 11: $\tilde{\mathbf{T}} = -\tilde{\mathbf{Q}}^{-1}$ $\boxed{\mathbf{T} = -\tilde{\mathbf{Q}}^{-1}.\mathbf{1}}$
- Define $\mathbf{D} = \mathbf{diag}\left(\frac{1}{q_{1,1}}, \ldots, \frac{1}{q_{N,N}}\right)$

$$\mathbf{ND} = \begin{bmatrix} n_{1,1} & \cdots & n_{1,N} \\ \vdots & & \vdots \\ n_{N,1} & \cdots & n_{N,N} \end{bmatrix} \begin{bmatrix} \frac{1}{q_{1,1}} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{q_{N,N}} \end{bmatrix} = \begin{bmatrix} \frac{n_{1,1}}{q_{1,1}} & \cdots & \frac{n_{1,N}}{q_{N,N}} \\ \vdots & & \vdots \\ \frac{n_{N,1}}{q_{1,1}} & \cdots & \frac{n_{N,N}}{q_{N,N}} \end{bmatrix}$$

$$\Rightarrow \tilde{\mathbf{T}} = -\mathbf{N}\,\mathbf{D} = -(\mathbf{I}-\mathbf{A})^{-1}\mathbf{D} = -\left(\mathbf{D}^{-1}(\mathbf{I}-\mathbf{A})\right)^{-1}$$

$$\mathbf{D}^{-1}(\mathbf{I}-\mathbf{A}) = \begin{bmatrix} q_{1,1} & & 0 \\ & \ddots & \\ 0 & & q_{N,N} \end{bmatrix} \begin{bmatrix} 1 & \cdots & \frac{q_{1,N}}{q_{1,1}} \\ \vdots & & \vdots \\ \frac{q_{N,1}}{q_{N,N}} & \cdots & 1 \end{bmatrix}$$

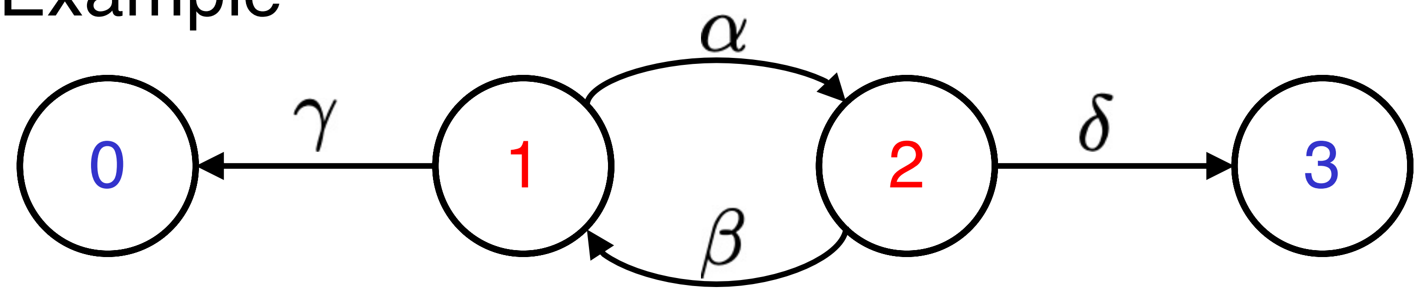
Nota bene: elements in $\mathbf{T}$ are strictly positive

$$= \begin{bmatrix} q_{1,1} & \cdots & q_{1,N} \\ \vdots & & \vdots \\ q_{N,1} & \cdots & q_{N,N} \end{bmatrix} = \tilde{\mathbf{Q}} \qquad \Rightarrow \boxed{\tilde{\mathbf{T}} = -\tilde{\mathbf{Q}}^{-1}}$$

Expected Absorption Time

- Corollary 2: Column vector $\mathbf{T} = (T(i), i = 1, \ldots, N)^T$ is solution of $\boxed{\tilde{\mathbf{Q}}\mathbf{T} = -\mathbf{1}}$

- Example



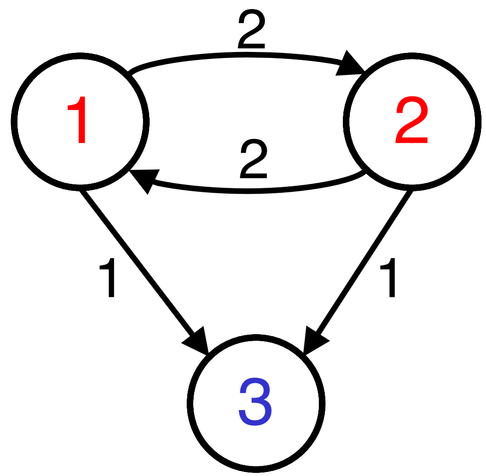
$$\tilde{\mathbf{Q}}\mathbf{T} = \begin{bmatrix} -(\alpha+\gamma) & \alpha \\ \beta & -(\beta+\delta) \end{bmatrix} \begin{bmatrix} T(1) \\ T(2) \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

$$\Rightarrow \begin{cases} -(\alpha+\gamma)\,T(1) + \alpha\,T(2) & = & -1 \\ \beta\,T(1) - (\beta+\delta)\,T(2) & = & -1 \end{cases} \Rightarrow \begin{cases} T(1) = \frac{\alpha+\beta+\delta}{\beta\gamma+\delta\alpha+\delta\gamma} \\ T(2) = \frac{\alpha+\beta+\gamma}{\beta\gamma+\delta\alpha+\delta\gamma} \end{cases}$$

Example 4 page 30

- Absorbing homogeneous CTMC

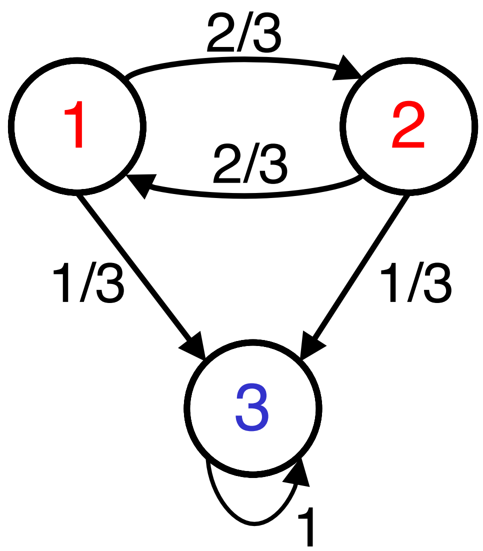
$$\mathbf{Q} = \left[\begin{array}{cc|c} -3 & 2 & 1 \\ 2 & -3 & 1 \\ \hline 0 & 0 & 0 \end{array}\right]$$



- Expected absorption time

$$\begin{aligned} -3\,T(1) + 2\,T(2) &= -1 \\ 2\,T(1) - 3\,T(2) &= -1 \end{aligned} \quad \Rightarrow T(1) = T(2) = 1$$

- Embedded Markov chain



$$\mathbf{P} = \left[\begin{array}{cc|c} 0 & \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & 0 & \frac{1}{3} \\ \hline 0 & 0 & 1 \end{array}\right] \Rightarrow \mathbf{A} = \begin{bmatrix} 0 & \frac{2}{3} \\ \frac{2}{3} & 0 \end{bmatrix} \quad \mathbf{R} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}$$

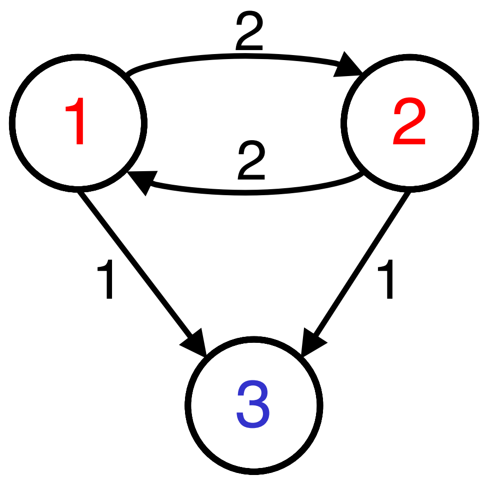
Example 4 page 30

- Mean number of visits in transient states

$$\mathbf{I}-\mathbf{A}=\begin{bmatrix}1 & -\frac{2}{3}\\ -\frac{2}{3} & 1\end{bmatrix}=\frac{1}{3}\begin{bmatrix}3 & -2\\ -2 & 3\end{bmatrix}$$

$$(\mathbf{I}-\mathbf{A})^{-1}=3\begin{bmatrix}3 & -2\\ -2 & 3\end{bmatrix}^{-1}=\frac{3}{5}\begin{bmatrix}3 & 2\\ 2 & 3\end{bmatrix}=\mathbf{N}$$

- Absorbing probabilities



$$\mathbf{B}=\mathbf{NR}=\frac{3}{5}\begin{bmatrix}3 & 2\\ 2 & 3\end{bmatrix}\frac{1}{3}\begin{bmatrix}1\\ 1\end{bmatrix}=\begin{bmatrix}1\\ 1\end{bmatrix}=\begin{bmatrix}b_{1,3}\\ b_{2.3}\end{bmatrix}$$

For next week

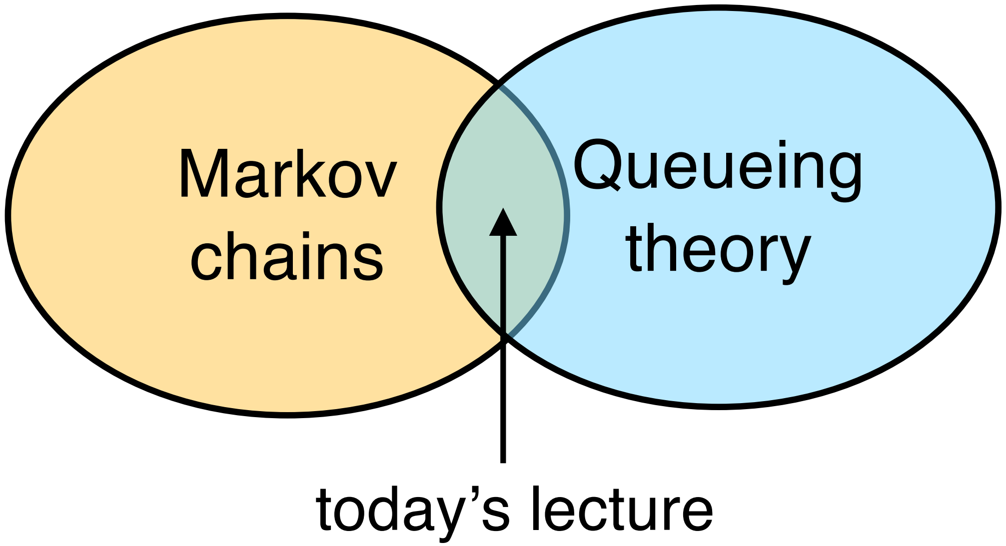
- Lesson 3 to revise
- Homework 3 to return on Tuesday 1 October before 9 am
- Lesson 4 to read before Lecture 4

Performance Evaluation of Networks

Sara Alouf

Part II – Queueing Theory

- Part I – Markov chains
  - Irreducible
    - Discrete-time Markov chains (Chapter 1)
    - Continuous-time Markov chains (Chapter 2)
  - Absorbing
    - Discrete-time and continuous-time (Chapter 3)
- Part II – Queueing Theory



Part II – Queueing Theory

- What is a queue?
  - Supermarket, bank, postoffice, administrations, etc.
  - CPU, servers, clusters, etc.
  - Manufacturing, product lines, etc.
  - …
- System with
  - at least one service facility
  - potentially a waiting room (finite or infinite)
  - customers
- Representation

Kendall's Notation

- Describe a queueing system

$$A \,/\, B \,/\, c \,/\, K$$

A ➜ distribution of interarrivals

B ➜ distribution of service times

c ➜ number of servers

K ➜ number of customers in system (omitted if infinite)

- Distributions often used
  - Exponential ➜ M
  - Deterministic ➜ D
  - General ➜ G
  - Erlang ➜ E
  - Phase-type ➜ PH

What is Not in Kendall's Notation

- Service discipline (scheduling)
  - First-in-first-out (First-come-first-served)
  - Last-in-first-out (last-come-first-served)
  - Processor sharing
  - Random
  - Shortest-job-first
  - Shortest-processing-time-first
- Multiple waiting rooms / queues: Join discipline
  - Random
  - Join-shortest-queue
  - Best-out-of-*d*

$M\,/\,M\,/\,1$ Queue

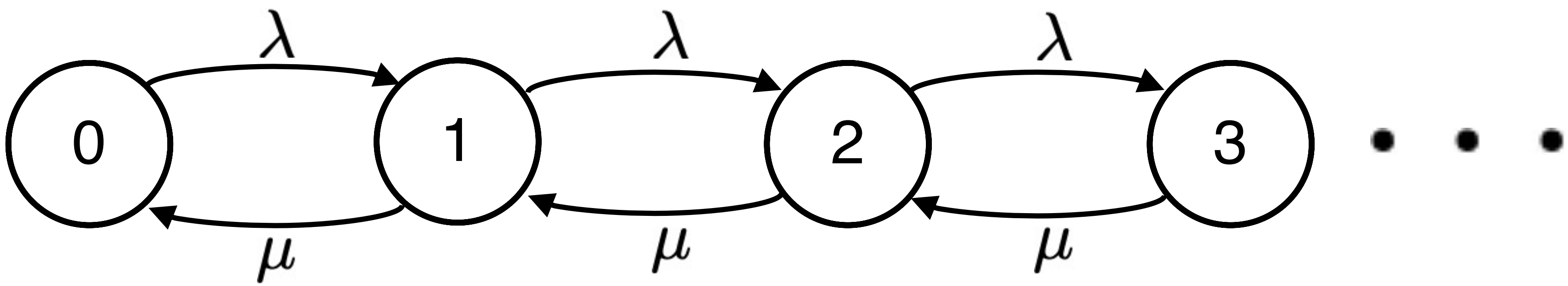
- Arrivals: Poisson rate λ ➜ interarrival time Exp(λ)
- Service time Exp(μ)
- Independence between arrivals and service times
- Service discipline ➜ not relevant (memoryless property)
- $X(t)$ number of customers in system at time t (queue size)
- Proposition 12: $\{X(t), t \geq 0\}$ is a birth-and-death process, birth rate λ, death rate μ
- Proof: construction rule #2

$$i \to i+1 \qquad \mathrm{Exp}(\lambda)$$

$$i \to i-1 \qquad \mathrm{Exp}(\mu) \quad i > 0$$

$M / M / 1$ Queue

- Transition diagram, $\mathcal{E} = \mathbb{N}$



- Global balance equations

$$\lambda \pi_{i-1} = \mu \pi_i, \qquad i \geq 1$$

- System utilization $\rho = \dfrac{\lambda}{\mu}$

- Proposition 13: If $\rho < 1$ limiting/stationary distribution is

$$\pi_i = (1 - \rho)\rho^i, \quad i \geq 0$$

$M / M / 1$ Queue

- Proof
$$\begin{aligned}\pi_i &= \rho\pi_{i-1} \\ &= \rho^i \pi_0\end{aligned}$$

- Normalization $\sum_{i\geq 0} \pi_i = 1$

$$\Leftrightarrow \quad \pi_0 \sum_{i\geq 0} \rho^i = 1$$ sum of terms in geometric progression

- If $\boxed{\rho < 1} \Rightarrow \quad \pi_0 \frac{1}{1-\rho} = 1$

stability condition

$$\Rightarrow \quad \pi_0 = 1 - \rho$$ (ρ system utilization)

$$\Rightarrow \quad \pi_i = (1-\rho)\rho^i, \quad i \geq 0$$

$M / M / 1$ Queue

- Let X stationary version of queue size ➜ $X \sim \mathrm{Geom}(1-\rho)$

  $1-\rho$ probability to find system empty

  X number of « failed trials » before finding system empty

- Expected queue size

$$E[X] = \sum_{i \geq 0} i\,\pi_i = (1-\rho) \sum_{i \geq 0} i\,\rho^i$$

$$= \rho(1-\rho) \sum_{i \geq 1} i\,\rho^{i-1} \quad = \rho(1-\rho) \left(\sum_{i \geq 1} \rho^i \right)'$$

$$\Rightarrow \boxed{E[X] = \frac{\rho}{1-\rho}}$$

- Throughput (= rate of everything that goes through the system)

$$\mathrm{Thpt} = \mu\,(1-\pi_0) = \mu\,\rho = \lambda$$

Burke's Theorem

- Suppose $M/M/1$ starts in steady-state
  - Departure process is Poisson with rate λ
  - Queue size at t independent of departures before time t

- Proof: use time-reversibility of $M/M/1$

  Forward chain identical to Backwards chain

  ✔ Arrival process is Poisson with rate λ

  ✔ Queue size at t independent of arrivals after time t

$M / M / 1 / K$ Queue

- $M / M / 1$ finite waiting room ➜ at most K customers in system
- Queue size is a finite birth-and-death process

$$\pi_i = \rho^i \pi_0, \quad 0 \le i \le K$$

- Normalization $$\sum_{i=0}^{K} \pi_i = 1 \quad \Leftrightarrow \quad \pi_0 \sum_{i=0}^{K} \rho^i = 1$$

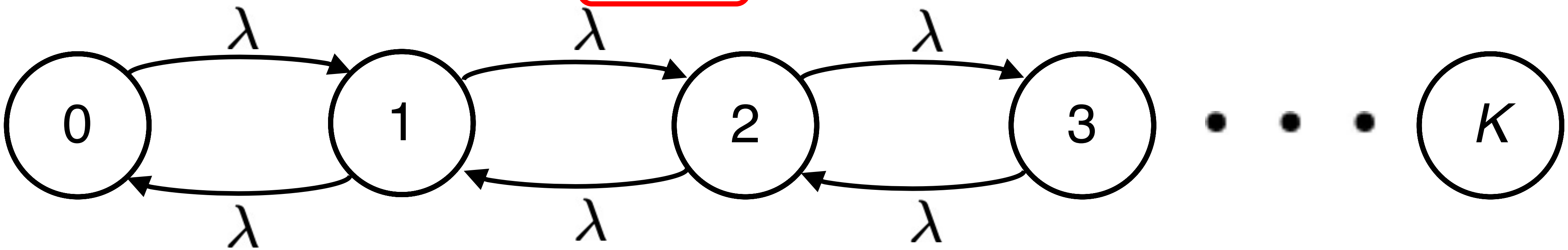
- If $\rho \neq 1 \Rightarrow$ $$\pi_0 \frac{1-\rho^{K+1}}{1-\rho} = 1 \quad \Leftrightarrow \quad \pi_0 = \frac{1-\rho}{1-\rho^{K+1}}$$

$$\Rightarrow \quad \boxed{\pi_i = \frac{(1-\rho)\,\rho^i}{1-\rho^{K+1}}, \quad i = 0, 1, \ldots, K}$$

- If $\rho = 1 \Rightarrow$ $$\pi_0 = \boxed{\frac{1}{K+1} = \pi_i, \quad i = 0, 1, \ldots, K}$$

$M\,/\,M\,/\,1\,/\,K$ Queue

- Transition diagram, $\boxed{\rho = 1}$, $\mathcal{E} = \{0, 1, \ldots, K\}$



- We are equally likely to go left or right
- Stationary process X is Uniform between 0 and K

$$\pi_i = \frac{1}{K+1}, \quad i = 0, 1, \ldots, K$$

- Expected queue size

$$E[X] = \sum_{i=0}^{K} i\,\pi_i = \frac{1}{K+1} \sum_{i=0}^{K} i = \frac{1}{K+1} \frac{K(K+1)}{2} = \frac{K}{2}$$

$M / M / 1 / K$ Queue

- Stationary distribution always exists!

  no stability condition (finite system)

- Finite system ➜ customers may find system full!

  loss probability ?

- PASTA : Poisson Arrivals See Time Averages
- Birth-and-death process is ergodic
  - Time averages = stationary distribution
- Loss probability = prob customer arrives and sees full queue

$$P_{\text{loss}} = \pi_K = \frac{(1-\rho)\,\rho^K}{1-\rho^{K+1}}$$

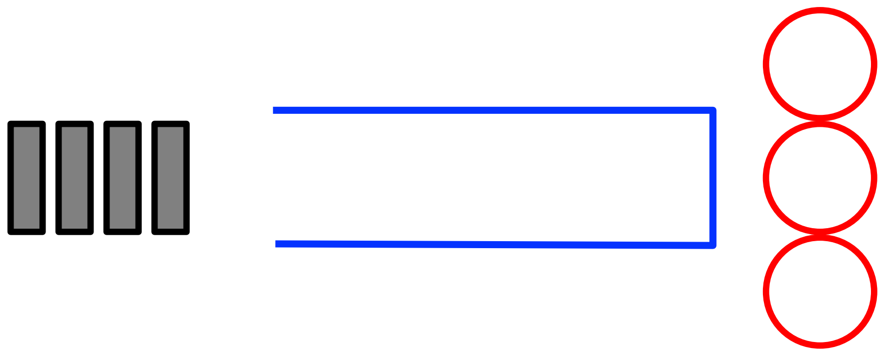
- Throughput $\quad \text{Thpt} = \mu\,(1-\pi_0) = \lambda\,(1-\pi_K)$

$M / M / c$ Queue

- Poisson arrivals rate λ ➜ interarrival time Exp(λ)
- Service time Exp(μ)
- Multiple servers
- Infinite waiting room
- Queue size is a birth-and-death process
  - $i \to i+1$ birth rate λ
  - $i \to i-1$ death rate $\begin{aligned} \mu_i &= i\mu, \quad i = 1, 2, \ldots, c-1 \\ &= c\mu, \quad i \geq c \end{aligned}$
- System utilization $\rho = \dfrac{\lambda}{c\mu}$
- Infinite system ➜ stability condition might be needed!

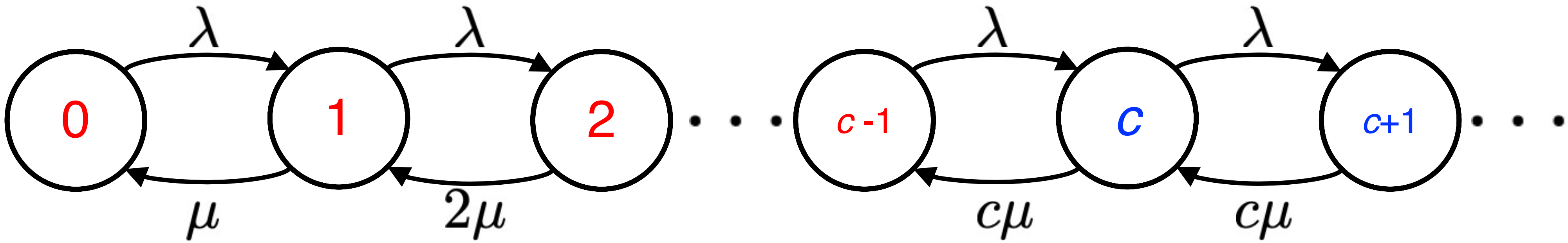
$M / M / c$ Queue

- Representation



Example:
call center

- Transition diagram, $\mathcal{E} = \mathbb{N}$



Customers served upon arrival

new customer needs to wait

$M\,/\,M\,/\,c$ Queue

- Stationary distribution: for $i = 1, 2, \ldots$

$$\pi_i = \frac{\lambda_0 \lambda_1 \cdots \lambda_{i-1}}{\mu_1 \mu_2 \cdots \mu_i} \pi_0 = \begin{cases} \left(\frac{\lambda}{\mu}\right)^i \frac{1}{i!} \pi_0 & i = 0, 1, \ldots, c \\ \left(\frac{\lambda}{\mu}\right)^i \frac{1}{c!} \frac{1}{c^{i-c}} \pi_0 & i \geq c \end{cases}$$

- Normalization: if $\boxed{\rho < 1}$ (stability condition)

$$\pi_0 = \left[\sum_{i=0}^{c-1} \left(\frac{\lambda}{\mu}\right)^i \frac{1}{i!} + \left(\frac{\lambda}{\mu}\right)^c \frac{1}{c!} \left(\frac{1}{1-\rho}\right) \right]^{-1}$$

$$\rho = \frac{\lambda}{c\mu}$$

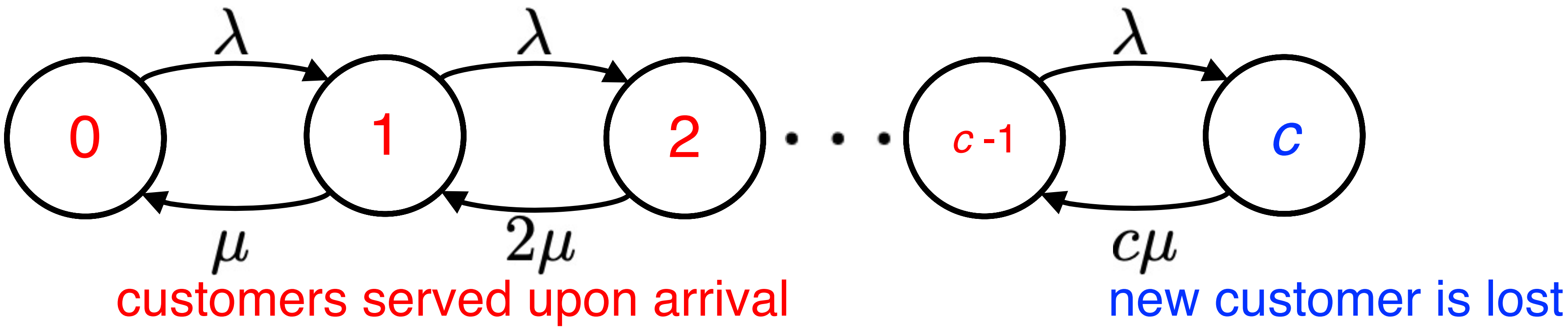
- Probability of waiting (PASTA)

$$P_{\text{wait}} = \sum_{i \geq c} \pi_i = \frac{\pi_0 \left(\lambda/\mu\right)^c}{c!} \sum_{i \geq 0} \left(\frac{\lambda}{c\mu}\right)^i = \frac{\pi_0 \left(c\rho\right)^c}{c!(1-\rho)}$$

15 minutes break

$M\,/\,M\,/\,c\,/\,c$ Queue

- Multi-server queue with no waiting room
- Pure loss system (call center without music)
- Finite system ➜ no stability condition
- Queue size is a birth-and-death process
  - $i \to i+1$ birth rate $\lambda_i = \lambda \quad i = 0, 1, \ldots, c-1$
  - $i \to i-1$ death rate $\mu_i = i\mu, \quad i = 1, 2, \ldots, c$
- Transition diagram $\mathcal{E} = \{0, 1, \ldots, c\}$



$M/M/c/c$ Queue

- Stationary distribution

$$\pi_i = \pi_0 \left(\frac{\lambda}{\mu}\right)^i \frac{1}{i!} \quad i = 0, 1, \ldots, c$$

$$\pi_0 = \left[\sum_{i=0}^{c} \left(\frac{\lambda}{\mu}\right)^i \frac{1}{i!}\right]^{-1}$$

- Loss probability

$$\boxed{P_{\text{loss}} = \pi_c = \frac{\left(\frac{\lambda}{\mu}\right)^c \frac{1}{c!}}{\sum_{i=0}^{c} \left(\frac{\lambda}{\mu}\right)^i \frac{1}{i!}}}$$

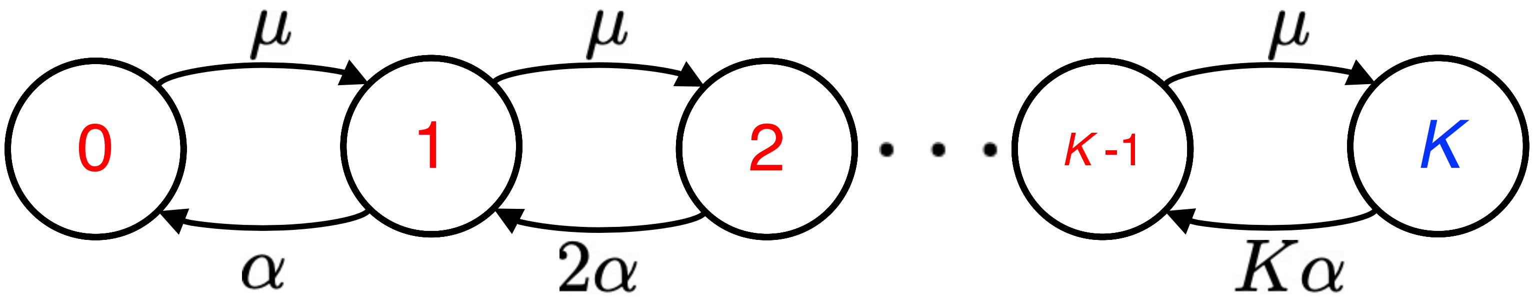
- Erlang's loss formula (1917)
- Major historical role in dimensioning phone systems
- Insensitivity property: holds for any service time distribution

Example: Repair Person Model

- K machines
- One repair person
- Each machine breaks after time Exp(α) (break rate is α)
- Upon a breakdown repair request is sent
- Repair person spends time Exp(μ) to repair one machine
- Questions
  - Probability that i machines are working normally
  - Overall failure rate?
- Define $X(t)$ number of functional machines at time t
- State space $\mathcal{E} = \{0, 1, \ldots, K\}$

Example: Repair Person Model

- Possible transitions?
  - $i \to i+1$ repair is over, time Exp(μ)
  - $i \to i-1$ break occurs, time Exp($i\alpha$)
- Using construction rule #2, process is a (homogeneous) CTMC
- It is also a birth-and-death-process
  - Birth rate (of functional machines) $\lambda_i = \mu$
  - Death rate $\mu_i = i\,\alpha$
- Transition diagram



Example: Repair Person Model

- Number of functional machines

  = Queue size of $M / M / K / K$

- Probability that i machines are working normally

$$\pi_i = \frac{\left(\frac{\mu}{\alpha}\right)^i \frac{1}{i!}}{\sum_{j=0}^{K} \left(\frac{\mu}{\alpha}\right)^j \frac{1}{j!}}$$

- Overall failure rate $$\sum_{i=1}^{K} (i\,\alpha)\,\pi_i = \alpha E[X]$$

- Overall repair rate $$\sum_{i=0}^{K-1} \mu\,\pi_i = \mu\,(1 - \pi_K)$$

- In steady-state overall failure rate = overall repair rate

Little's Formula

- Relates three quantities in steady-state
  - Occupancy in a system $\overline{N}$
  - Entrance rate to the system λ
  - Sojourn time in system $\overline{T}$

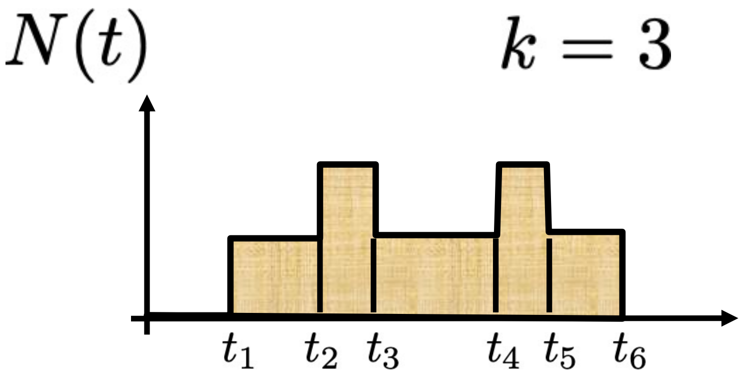
$$\boxed{\overline{N} = \lambda \overline{T}}$$

- Valid for work-conserving systems

  system may not be idle if customers waiting
- No assumption on any distribution
- No assumption on service discipline
- Independence assumption

Proof of Little's Formula

- Steady-state ➜ system empties infinitely often
- Let 0 and C be two times when system is empty
- Let k be number of customers served in $(0, C)$
- $\{a_i\}_{i=1,\ldots,k}$ arrival instants
- $\{d_i\}_{i=1,\ldots,k}$ departure instants
- $\{t_n\}_{n=1,\ldots,2k}$ all instants
- Number of customers over time $N(t)$

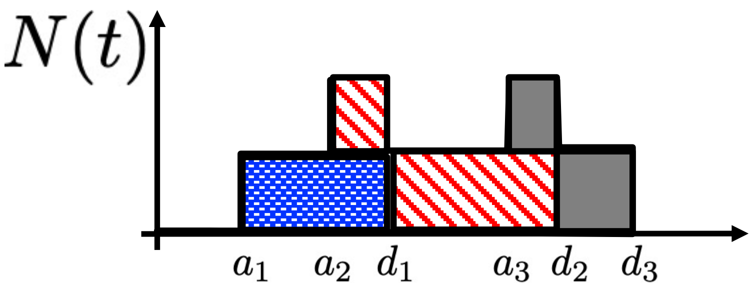


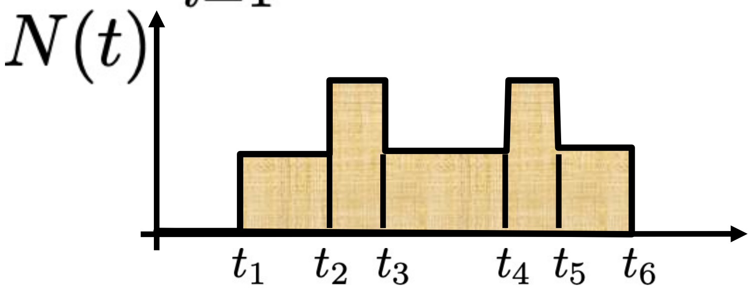
$$\overline{N} = \frac{1}{C}\int_0^C N(t)\,dt$$

$$= \frac{1}{C}\sum_{i=1}^{2k-1} N(t_i^+)\,(t_{i+1} - t_i)$$

Proof of Little's Formula

- Mean sojourn time $$\overline{T} = \frac{1}{k}\sum_{i=1}^{k}(d_i - a_i)$$





sum of horizontal boxes = sum of vertical boxes

$$\sum_{i=1}^{k}(d_i - a_i) = \sum_{i=1}^{2k-1} N(t_i^+)\,(t_{i+1} - t_i)$$

$$\Rightarrow \quad \overline{T}\,k = \overline{N}\,C$$

- When $C \to \infty,\ \frac{k}{C} \to \lambda \quad \Rightarrow \quad \boxed{\overline{T}\lambda = \overline{N}}$

Example 5 Page 39

- Consider $M / M / 1$, arrival rate λ, service rate μ
- If $\rho = \dfrac{\lambda}{\mu} < 1$ (queue is stable)
- Mean number of customers $\overline{N} = E[X] = \dfrac{\rho}{1-\rho} > 0$
- Entrance rate = arrival rate (no losses)
- Expected sojourn time

  By Little's formula $\overline{T} = \dfrac{\overline{N}}{\lambda} = \dfrac{\rho}{\lambda(1-\rho)} \Rightarrow \boxed{\overline{T} = \dfrac{1}{\mu - \lambda}}$
- Expected waiting time

$$\overline{W} = \overline{T} - \frac{1}{\mu} = \frac{\rho}{\mu(1-\rho)}$$

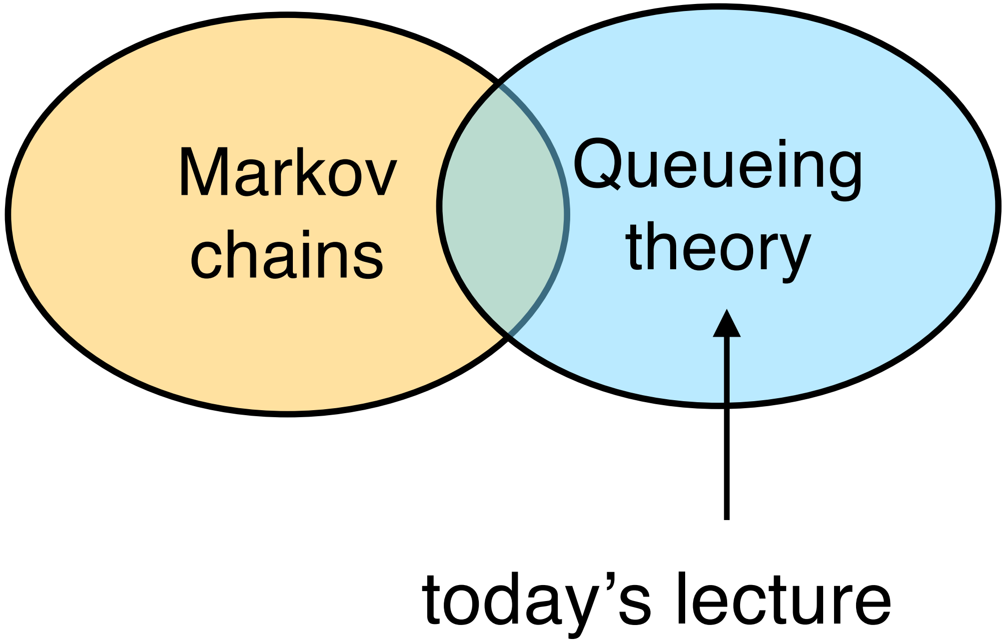
For next week

- Lesson 4 to revise
- Homework 4 to return on Tuesday 8 October before 9 am
- Lesson 5 to read before Lecture 5

Performance Evaluation of Networks

Sara Alouf

Ch 5 – The General Service Time Queue



- $M / G / 1$ FIFO queue
- $M / G / 1$ FIFO queue with vacations

$M / G / 1$ FIFO Queue

- Arrivals Poisson process rate λ
- Service time ➜ General distribution and independence
  - Service times are independent identically distributed
  - σ generic service time

$$G(x) = P(\sigma \leq x), \quad x \geq 0$$

$$E[\sigma] = \int_0^\infty (1 - G(x))\, dx = \frac{1}{\mu}$$

$$E[\sigma^2] = \int_0^\infty x^2\, dG(x)$$

$$\mathrm{Var}[\sigma] = E[\sigma^2] - (E[\sigma])^2$$

- First-in-first-out service discipline

$M / G / 1$ FIFO Queue

- Load $\rho = \dfrac{\lambda}{\mu}$

- Queue size $N(t)$ ➜ not Markov chain

  sojourn time in a state is not Exp()

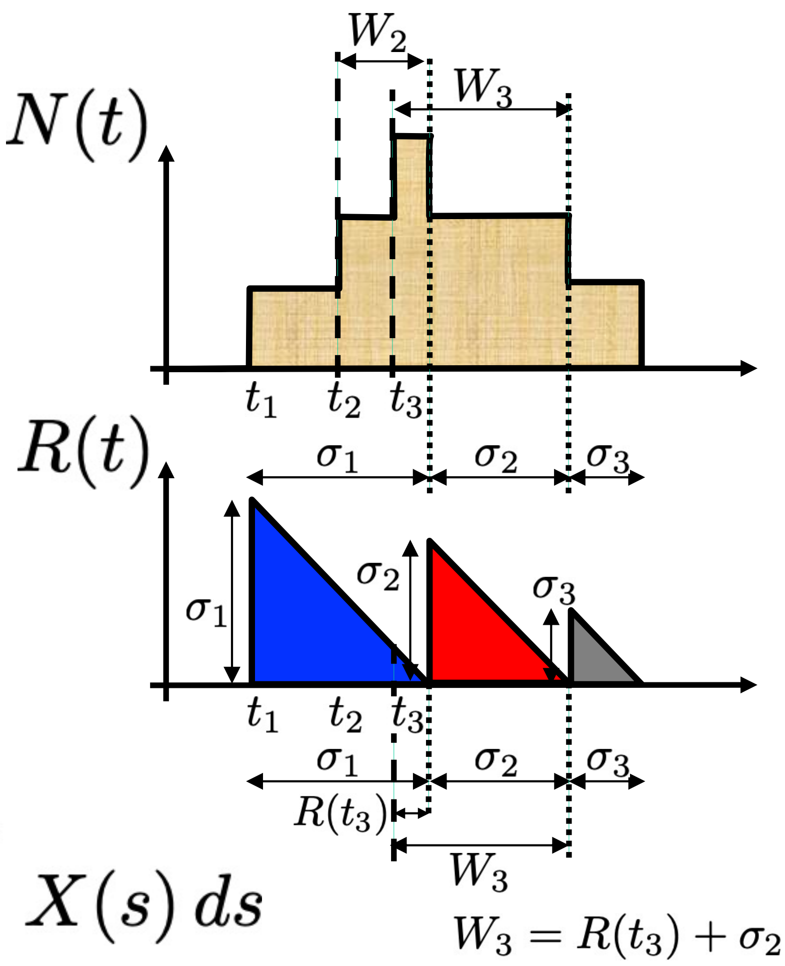
- Expected waiting time in steady-state $\overline{W}$

- Pollaczek-Khinchin formula

$$\boxed{\overline{W} = \frac{\lambda E[\sigma^2]}{2(1-\rho)}} = \frac{\rho}{2(1-\rho)} \cdot \frac{\mathrm{Var}(\sigma) + E[\sigma]^2}{E[\sigma]}$$

- Higher service time variability ➜ longer waiting times

Proof of Pollaczek-Khinchin Formula



- For customer i
  - Arrival time t_i
  - Service time σ_i
  - Waiting time W_i
- Number of customers waiting
  - at time t ➜ $X(t)$
  - at time t_i ➜ $X(t_i) = X(t_i^-)$
  - expectation $\overline{X} = \lim_{t\to\infty} \frac{1}{t} \int_0^t X(s)\,ds$
- Residual service time
  - at time t ➜ $R(t)$
  - expectation $\overline{R} = \lim_{t\to\infty} \frac{1}{t} \int_0^t R(s)\,ds$

Proof of Pollaczek-Khinchin Formula

- Customer i sees $X(t_i)$ customers waiting

$$W_i = R(t_i) + \sigma_{i-1} + \sigma_{i-2} + \ldots + \sigma_{i-X(t_i)}$$

$$E[W_i] = E[R(t_i)] + E\left[\sum_{j=1}^{X(t_i)} \sigma_{i-j}\right]$$

- $X(t_i)$ consists of customers $i-1$, …, $i - X(t_i)$

  they have not been served yet

  ➜ $X(t_i)$ independent of their service time

- Use Wald's formula

$$\underbrace{\boxed{E[W_i]}}_{\overline{W}} = \underbrace{\boxed{E[R(t_i)]}}_{??} + \underbrace{\boxed{E[X(t_i)]}}_{??} E[\sigma]$$

Proof of Pollaczek-Khinchin Formula

- Take limit $i \to \infty$ and use PASTA property
- We have

$$\lim_{i\to\infty} E[R(t_i)] = \lim_{t\to\infty} \frac{1}{t}\int_0^t R(s)\,ds = \overline{R}$$

expected residual service time at arrival epochs in steady state

= time average of residual service time

- Similarly

$$\lim_{i\to\infty} E[X(t_i)] = \lim_{t\to\infty} \frac{1}{t}\int_0^t X(s)\,ds = \overline{X}$$

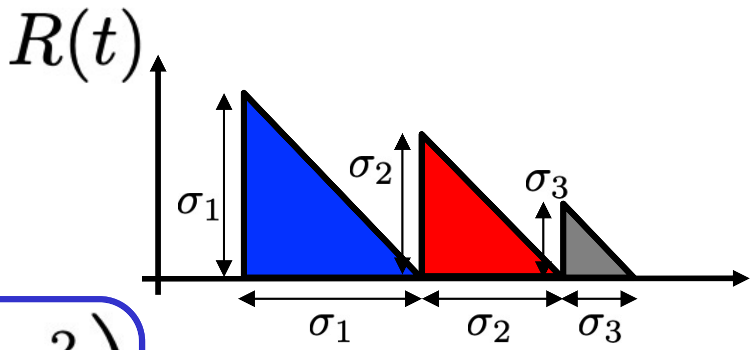
- Therefore

$$E[W_i] = E[R(t_i)] + E[X(t_i)]E[\sigma] \quad \rightarrow \quad \overline{W} = \overline{R} + \frac{\overline{X}}{\mu}$$

Proof of Pollaczek-Khinchin Formula

- If $\rho < 1$ (stability condition of $M / G / 1$)

  ➜ queue empties infinitely often
- Let 0 and C be two times when system is empty
- Let k be number of customers served in (0, C)
- Expected residual service time



$$\overline{R} = \lim_{C\to\infty} \frac{1}{C} \sum_{i=1}^{k} \frac{\sigma_i^2}{2}$$

$$= \boxed{\lim_{\substack{C\to\infty \\ k\to\infty}} \left(\frac{k}{C}\right)} \boxed{\lim_{\substack{C\to\infty \\ k\to\infty}} \left(\frac{1}{k} \sum_{i=1}^{k} \frac{\sigma_i^2}{2}\right)}$$

$$= \lambda \frac{E[\sigma^2]}{2}$$

Proof of Pollaczek-Khinchin Formula

- Apply Little's formula on waiting room

$$\overline{X} = \lambda \overline{W}$$

- Recall $\overline{W} = \overline{R} + \frac{\overline{X}}{\mu} = \overline{R} + \frac{\lambda}{\mu} \overline{W}$

$$\Rightarrow \quad \overline{W}(1-\rho) = \overline{R}$$

$$\Leftrightarrow \quad \overline{W} = \frac{\overline{R}}{1-\rho}$$

$$\Leftrightarrow \quad \boxed{\overline{W} = \frac{\lambda E[\sigma^2]}{2(1-\rho)}}$$

$M / G / 1$ FIFO Queue

- Expected waiting time

$$\overline{W} = \frac{\lambda\, E[\sigma^2]}{2\,(1-\rho)}$$

- Expected sojourn time

$$\overline{T} = \overline{W} + \frac{1}{\mu} = \frac{1}{\mu} + \frac{\lambda\, E[\sigma^2]}{2\,(1-\rho)}$$

- Expected number of customers waiting

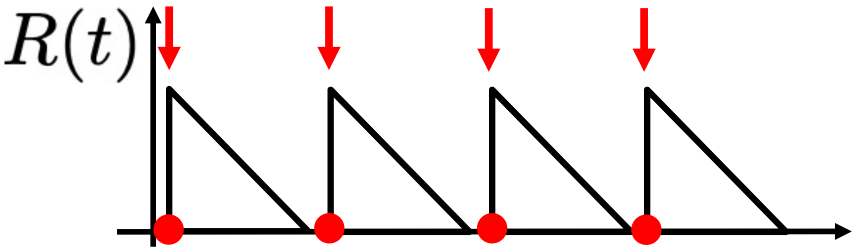
$$\overline{X} = \lambda\,\overline{W} = \frac{\lambda^2\, E[\sigma^2]}{2\,(1-\rho)}$$

- Expected queue size

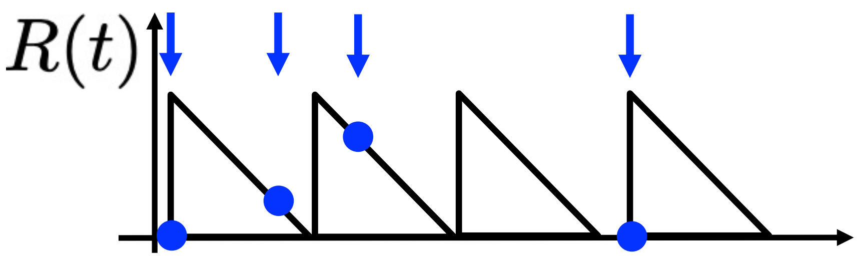
$$\overline{N} = \overline{X} + \rho = \lambda\overline{T} = \rho + \frac{\lambda^2\, E[\sigma^2]}{2\,(1-\rho)}$$

Example When PASTA Not True

- Consider $D\,/\,D\,/\,1$ FIFO queue
- Arrivals every second ➜ $\lambda = 1$ s^{-1}
- Service time 0.9 second ➜ $\mu = 1/0.9$ s^{-1}
- Load is very high ➜ $\rho = 0.9$



if Poisson arrivals



- Time average

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t R(s)\,ds = \lambda\frac{E[\sigma^2]}{2} = \frac{(0.9)^2}{2} = 0.405$$

- Average at arrival epochs $\displaystyle\lim_{i\to\infty} E[R(t_i)] = 0$

$M/M/1$ Versus $M/D/1$ FIFO

- Consider two queues
  - Poisson arrival rate λ
  - One server
  - Infinite waiting room
  - FIFO service discipline
- Different service time distribution but same average $1/\mu$
  - $M/M/1$ queue : Exp(μ) $\quad E[\sigma^2] = \dfrac{2}{\mu^2}$
  - $M/D/1$ queue : $\sigma = 1/\mu$ $\quad E[\sigma^2] = \dfrac{1}{\mu^2}$
- Expected waiting time

$$\overline{W}_{M/M/1} = \frac{\lambda E[\sigma^2]}{2(1-\rho)} = \frac{\lambda}{\mu^2(1-\rho)} \qquad \boxed{\overline{W}_{M/D/1} = \frac{\overline{W}_{M/M/1}}{2}}$$

$M / G / 1$ FIFO Queue With Vacations

- $M / G / 1$ FIFO but if queue empty: server ➜ vacation
  - Maintenance, background task, sleep mode, power off
- First-in-first-out service discipline
- Arrivals Poisson process rate λ
- Service time ➜ General distribution and independence
  - Service times are independent identically distributed
  - σ generic service time

$$G(x) = P(\sigma \leq x), \quad x \geq 0$$

$$E[\sigma] = \int_0^\infty (1 - G(x))\, dx = \frac{1}{\mu}$$

$$E[\sigma^2] = \int_0^\infty x^2\, dG(x)$$

$M/G/1$ FIFO Queue With Vacations

- Vacation time ➜ General distribution and independence
  - vacation durations are independent identically distributed
  - V generic vacation duration

$$F(x) = P(V \leq x), \quad x \geq 0$$

$$E[V] = \int_0^\infty (1 - F(x))\, dx$$

$$E[V^2] = \int_0^\infty x^2\, dF(x)$$

- Load $\rho = \dfrac{\lambda}{\mu}$

- Queue size not Markov chain (sojourn time not Exp())

$M / G / 1$ FIFO Queue With Vacations

- If $\rho < 1$(stability condition) expected waiting time steady-state

$$\boxed{\overline{W} = \frac{\lambda E[\sigma^2]}{2(1-\rho)} + \frac{E[V^2]}{2E[V]}}$$

$$= \overline{W}_{M/G/1} + \boxed{\frac{E[V^2]}{2E[V]}}$$

- Higher vacations variability ➜ longer waiting times

$$\frac{E[V^2]}{2E[V]} = \frac{\mathrm{Var}[V]}{2E[V]} + \frac{E[V]}{2}$$

- To lessen impact of vacations ➜ V deterministic, small
- If cost to go on vacation ➜ tradeoff to be found

15 minutes break

$M / G / 1$ FIFO Queue With Vacations

- Expected waiting time in steady-state $\overline{W}$

$$\boxed{\overline{W} = \frac{\lambda E[\sigma^2]}{2(1-\rho)} + \frac{E[V^2]}{2E[V]}}$$

Proof

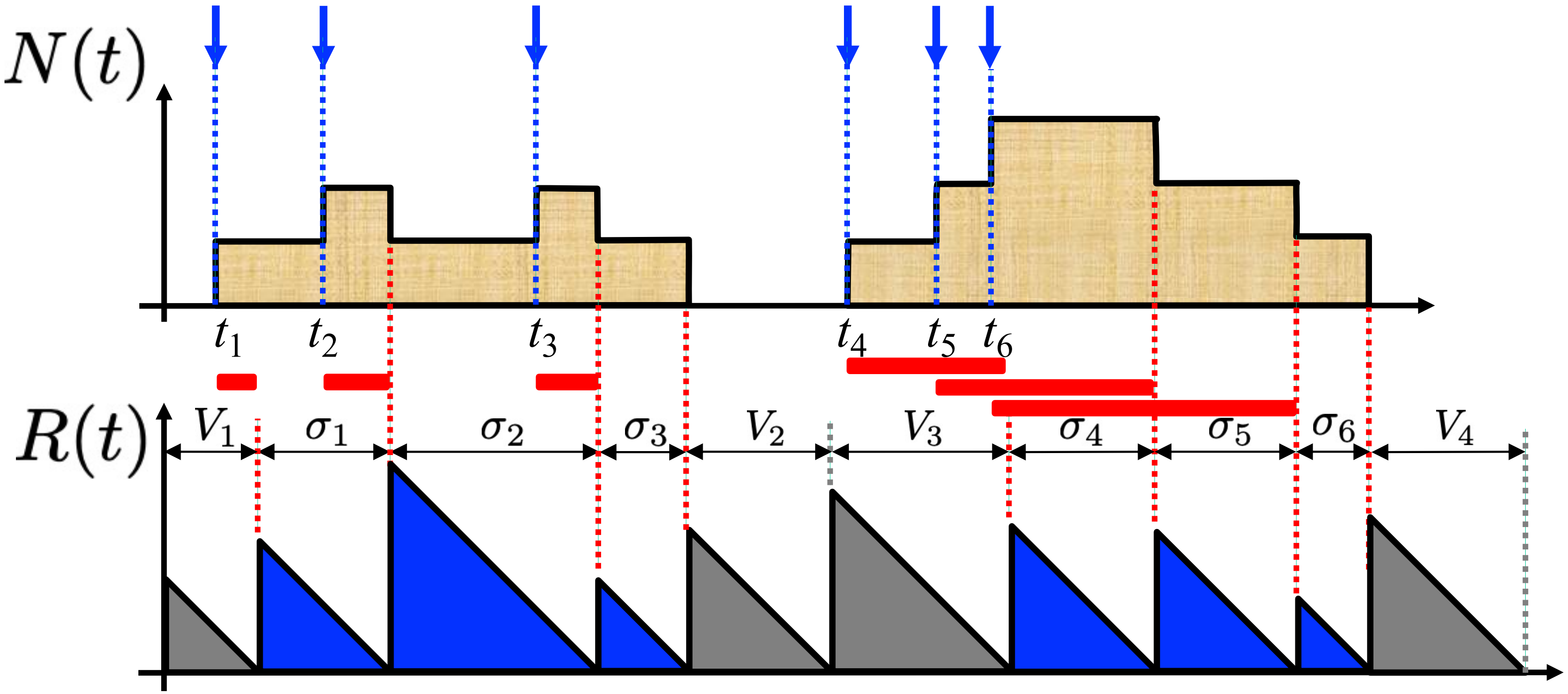
- For customer i
  - Arrival time t_i
  - Service time σ_i
  - Waiting time W_i

- Number of customers waiting
  - at time t ➜ $X(t)$
  - at time t_i ➜ $X(t_i) = X(t_i^-)$
  - expectation $\overline{X} = \lim_{t \to \infty} \frac{1}{t} \int_0^t X(s)\, ds$

- kth server vacation time V_k

Proof

- Residual time at server at time t ➜ $R(t)$
  - if server busy ➜ residual service time
  - if server in vacation ➜ residual vacation time



$$W_6 = R(t_6) + \sigma_4 + \sigma_5$$

Proof

- Customer i sees $X(t_i)$ customers waiting

$$W_i = R(t_i) + \sigma_{i-1} + \sigma_{i-2} + \ldots + \sigma_{i-X(t_i)}$$

$$E[W_i] = E[R(t_i)] + E\left[\sum_{j=1}^{X(t_i)} \sigma_{i-j}\right]$$

- Use Wald's formula ($X(t_i)$ independent of all σ_{i-j})

$$\boxed{E[W_i]} = \boxed{E[R(t_i)]} + \boxed{E[X(t_i)]} E[\sigma]$$

- Take limit $i \to \infty$ and use PASTA property

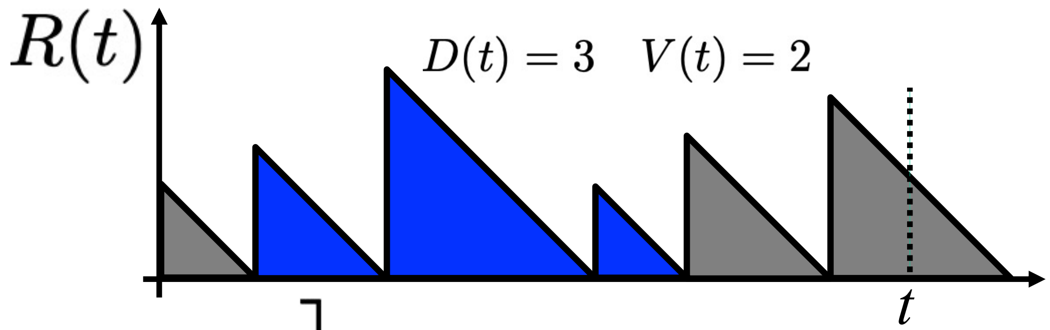
$$\left.\begin{aligned} \overline{W} &= \overline{R} + \frac{\overline{X}}{\mu} \\ \overline{X} &= \lambda \overline{W} \end{aligned}\right\} \Rightarrow \boxed{\overline{W} = \frac{\overline{R}}{1-\rho}}$$

- By Little's formula $\overline{X} = \lambda \overline{W}$

same expression
different
$\overline{R}$

Proof: Find $\overline{R}$

- $\rho < 1$ ➜ queue will empty infinitely often
- $D(t)$ ➜ number of customers fully served in (0, t)
- $V(t)$ ➜ number of complete vacations in (0, t)
- Expected residual time



$$\overline{R} = \lim_{t\to\infty} \frac{1}{t} \int_0^t R(u)du$$

$$= \lim_{t\to\infty} \frac{1}{t} \left[\sum_{i=1}^{D(t)} \frac{\sigma_i^2}{2} + \sum_{k=1}^{V(t)} \frac{V_k^2}{2} + \text{trapezoid} \right]$$

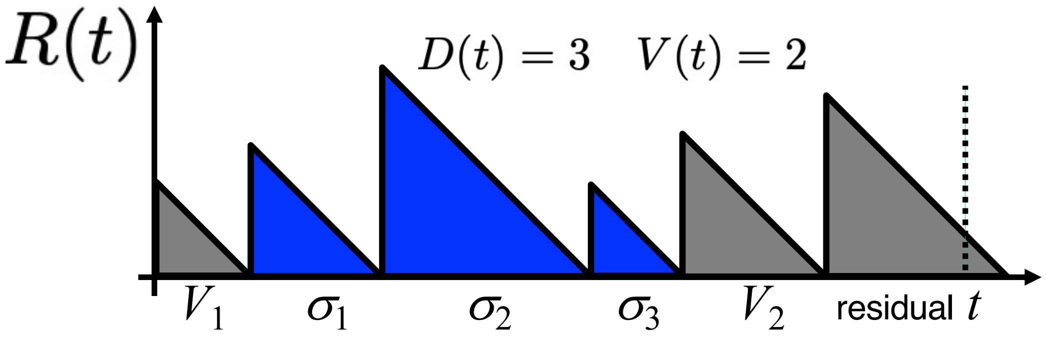
$$= \lim_{t\to\infty} \left[\boxed{\frac{D(t)}{t}} \boxed{\frac{1}{D(t)} \sum_{i=1}^{D(t)} \frac{\sigma_i^2}{2}} + \frac{V(t)}{t} \boxed{\frac{1}{V(t)} \sum_{k=1}^{V(t)} \frac{V_k^2}{2}} + \frac{\cancel{\text{trapezoid}}}{t} \right]$$

$$= \lambda \frac{E[\sigma^2]}{2} + \boxed{\left(\lim_{t\to\infty} \frac{V(t)}{t} \right)} \frac{E[V^2]}{2}$$

?

Proof: Find $\lim_{t\to\infty} \frac{V(t)}{t}$

- We have



$$t = \sum_{i=1}^{D(t)} \sigma_i + \sum_{k=1}^{V(t)} V_k + \text{residual}$$

$$1 = \boxed{\frac{D(t)}{t}} \boxed{\frac{1}{D(t)} \sum_{i=1}^{D(t)} \sigma_i} + \frac{V(t)}{t} \boxed{\frac{1}{V(t)} \sum_{k=1}^{V(t)} V_k} + \cancel{\frac{\text{residual}}{t}}$$

take limit $t \to \infty$

$$1 = \lambda E[\sigma] + \boxed{\left(\lim_{t\to\infty} \frac{V(t)}{t} \right)} E[V]$$

$$\Rightarrow \quad \lim_{t\to\infty} \frac{V(t)}{t} = \frac{1-\rho}{E[V]}$$

Proof: Recap

- $$\lim_{t\to\infty} \frac{V(t)}{t} = \frac{1-\rho}{E[V]}$$

- $$\overline{R} = \lambda \frac{E[\sigma^2]}{2} + \left(\lim_{t\to\infty} \frac{V(t)}{t}\right) \frac{E[V^2]}{2}$$
$$= \lambda \frac{E[\sigma^2]}{2} + \left(\frac{1-\rho}{E[V]}\right) \frac{E[V^2]}{2}$$

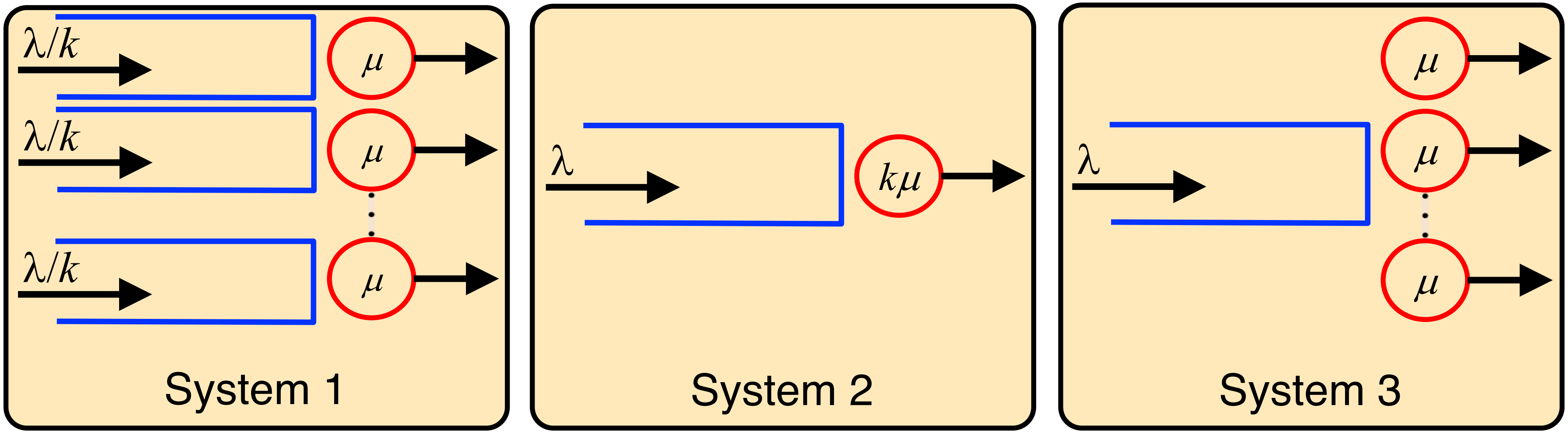
- $$\overline{W} = \frac{\overline{R}}{1-\rho}$$
$$= \frac{\lambda E[\sigma^2]}{2(1-\rho)} + \frac{E[V^2]}{2E[V]} \quad ✔$$
$= \overline{W}_{M/G/1} +$ effect of vacations

Exercize: Compare Different Organizations

- Global traffic Poisson rate λ
- Total service rate $k\mu$

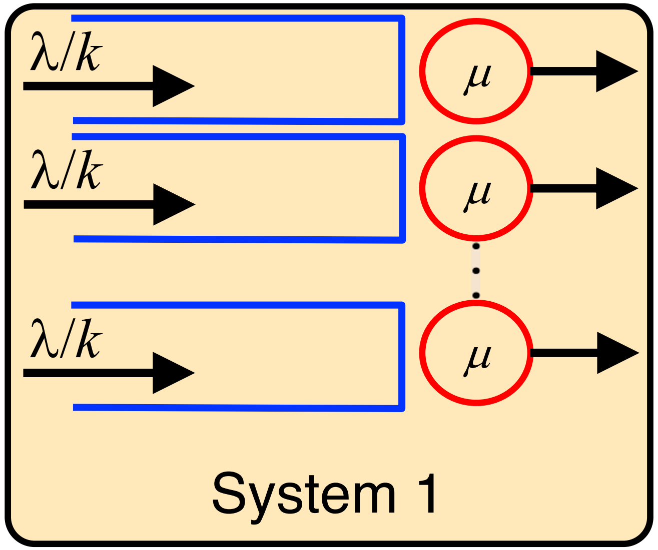
$$\rho = \frac{\lambda}{k\mu}$$

- Service time is exponentially distributed
- Objective: compare three systems organizations



- Order systems according to expected sojourn time

System 1

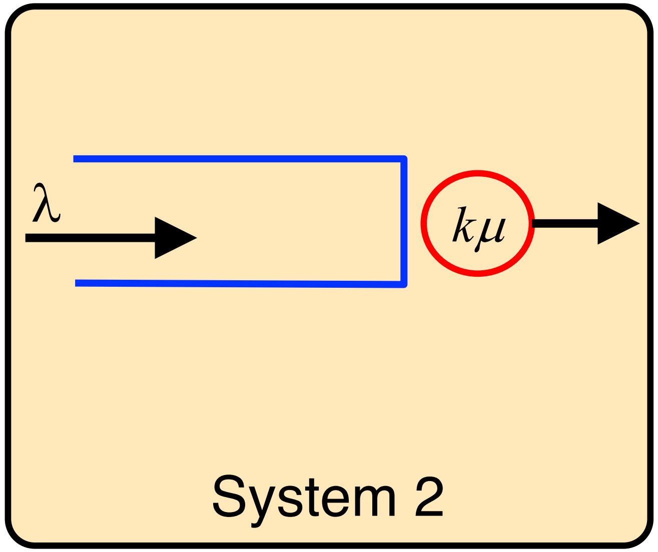


- Each queue is $M/M/1$ queue
- Arrival rate λ/k
- Service rate μ
- Infinite queue: stability condition $\lambda/k < \mu \quad (\rho < 1)$
- Expected queue size in one queue $\dfrac{\lambda/k}{\mu - \lambda/k}$
- Expected number of customers in System 1

$$\overline{N}_1 = \frac{\lambda}{\mu - \lambda/k} = \frac{k\rho}{1-\rho}$$

- By Little's fomula $\overline{T}_1 = \dfrac{\overline{N}_1}{\lambda} = \dfrac{1}{\mu - \lambda/k} = \dfrac{k}{k\mu - \lambda}$

System 2



- Queue is $M/M/1$ queue
- Arrival rate λ
- Service rate $k\mu$
- Infinite queue: stability condition $\lambda < k\mu \;\; (\rho < 1)$

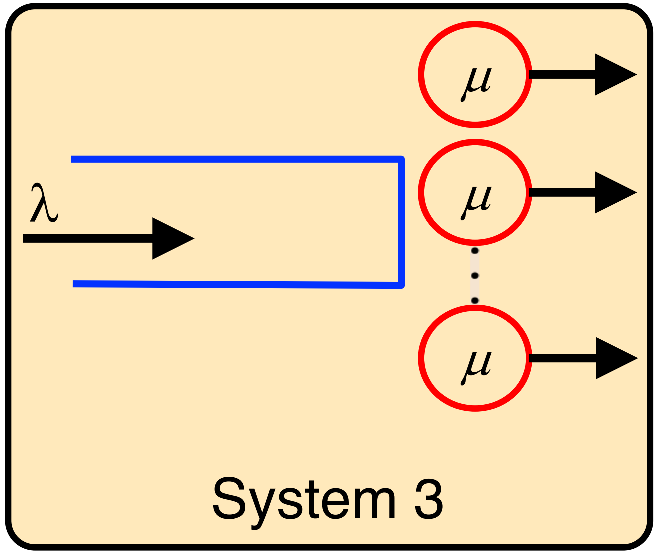
- Expected number of customers in System 2

$$\overline{N}_2 = \frac{\lambda}{k\mu - \lambda} = \frac{\rho}{1-\rho}$$

- By Little's fomula

$$\overline{T}_2 = \frac{\overline{N}_2}{\lambda} = \frac{1}{k\mu - \lambda}$$

System 3



- Queue is $M/M/k$ queue
- Arrival rate λ
- Service rate μ
- Infinite queue: stability condition $\lambda < k\mu \;\; (\rho < 1)$
- Expected number of customers in System 3

$$\overline{N}_3 = \overline{N}_{\text{wait}} + \overline{N}_{\text{servers}}$$

- By Little's formula on all servers

$$\overline{N}_{\text{servers}} = \lambda \frac{1}{\mu}$$

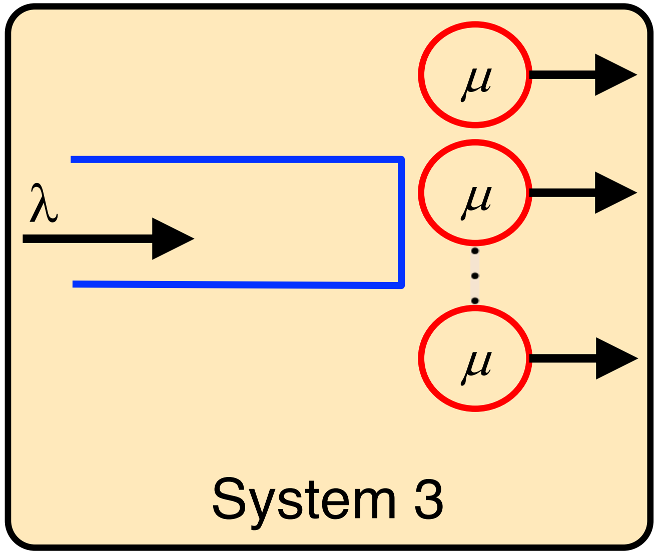
- Let X_{wait} number of customers waiting in queue

$$\overline{N}_{\text{wait}} = E[X_{\text{wait}}]$$

law of total probabilities

$$= E[X_{\text{wait}}|\text{wait}]P_{\text{wait}} + \cancel{E[X_{\text{wait}}|\text{no wait}]}(1 - P_{\text{wait}})$$

System 3



Conditioning on fact all servers busy X_{wait} same as queue size in $M/M/1$ queue with arrival rate λ service rate $k\mu$

$$E[X_{\text{wait}}|\text{wait}] = \overline{N}_2$$

- All servers are busy with probability (see lecture 4)

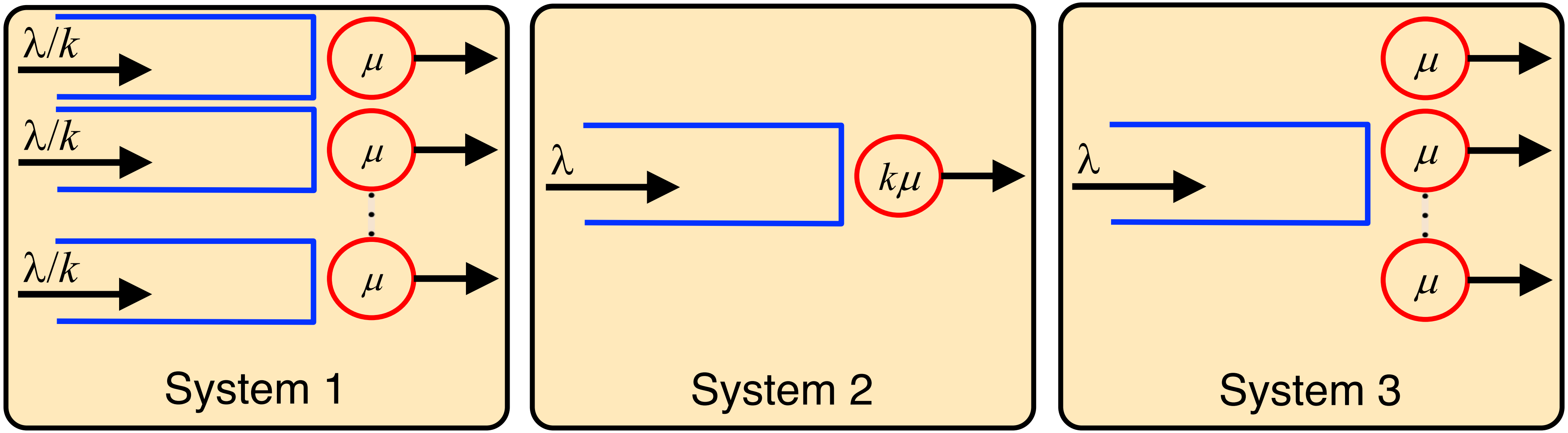
$$P_{\text{wait}} = \frac{\pi_0 (k\rho)^k}{k!(1-\rho)} = \frac{\frac{(k\rho)^k}{k!(1-\rho)}}{\sum_{i=0}^{k-1} \frac{(k\rho)^i}{i!} + \frac{(k\rho)^k}{k!(1-\rho)}}$$

- Expected number of customers in System 3

$$\overline{N}_3 = \frac{\lambda}{k\mu - \lambda} P_{\text{wait}} + \frac{\lambda}{\mu}$$

- By Little's fomula $\overline{T}_3 = \frac{1}{k\mu - \lambda} P_{\text{wait}} + \frac{1}{\mu}$

Exercize: Compare Different Organizations

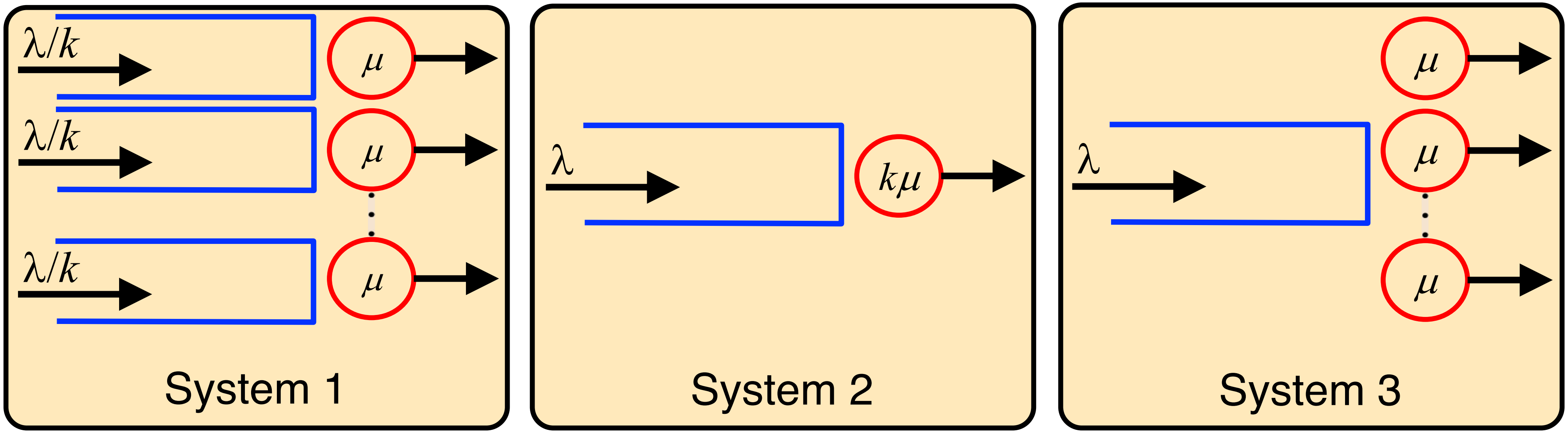


$$\overline{N}_1 = \frac{k\rho}{1-\rho} \qquad \overline{N}_2 = \frac{\rho}{1-\rho} \qquad \overline{N}_3 = \overline{N}_2 P_{\text{wait}} + \frac{\lambda}{\mu}$$

$$\overline{T}_1 = \frac{k}{k\mu - \lambda} \qquad \overline{T}_2 = \frac{1}{k\mu - \lambda} \qquad \overline{T}_3 = \overline{T}_2 P_{\text{wait}} + \frac{1}{\mu}$$

- System 2 is k times better than System 1
- What about System 3? $\dfrac{\overline{T}_3}{\overline{T}_2} = P_{\text{wait}} + k(1-\rho) > 1$

Exercize: Compare Different Organizations



- What about System 3? $$\frac{\overline{T}_3}{\overline{T}_2} = P_{\text{wait}} + k(1-\rho) > 1$$
- If very low utilization ➜ ratio close to k

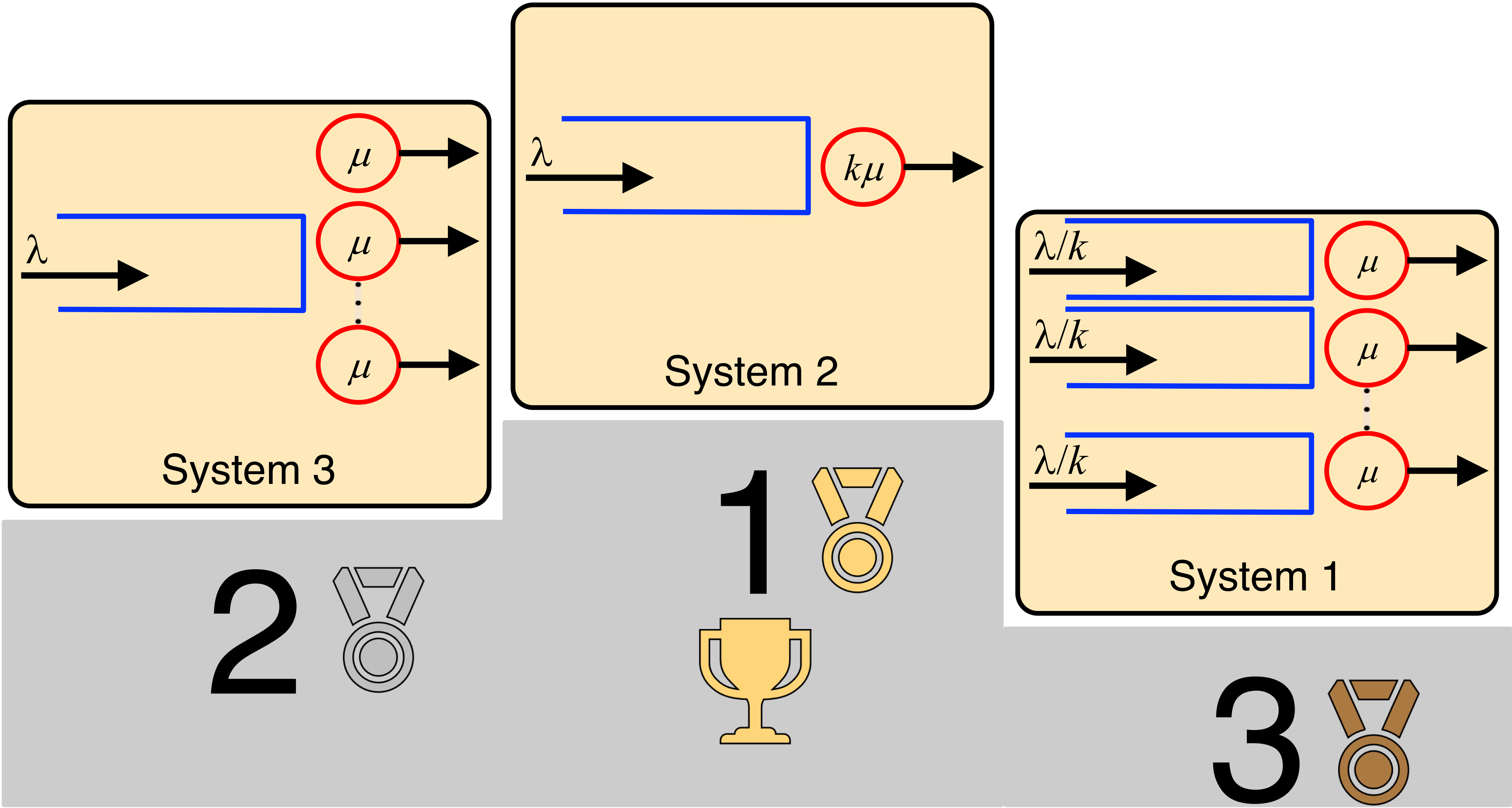
  System 3 almost k times worse than System 2

  ➜ System 3 slightly better than System 1
- If very high utilization ➜ ratio close to 1

  System 3 almost same (slightly worse) as System 2

Conclusion $\overline{T}_2 < \overline{T}_3 < \overline{T}_1$

- … and the winner is: System 2!



For next week

- Lesson 5 to revise
- Homework 5 to return on Tuesday 15 October before 9 am
- Lesson 6 to read before Lecture 6

Performance Evaluation of Networks

Sara Alouf

Ch 6 – Queueing Networks

- Until now only one queue (one service)
  - Chapter 4: $M/M/1$, $M/M/1/K$, $M/M/c$, $M/M/c/c$
  - Chapter 5: $M/G/1$ FIFO, $M/G/1$ FIFO with vacations
- Multiple queues ➜ network of queues
  - Open versus closed network

    Open ➜ customers enter and leave network of queues

    Closed ➜ fixed number of customers in network of queues
  - Single class versus multi-class

    Single ➜ all customers are identical

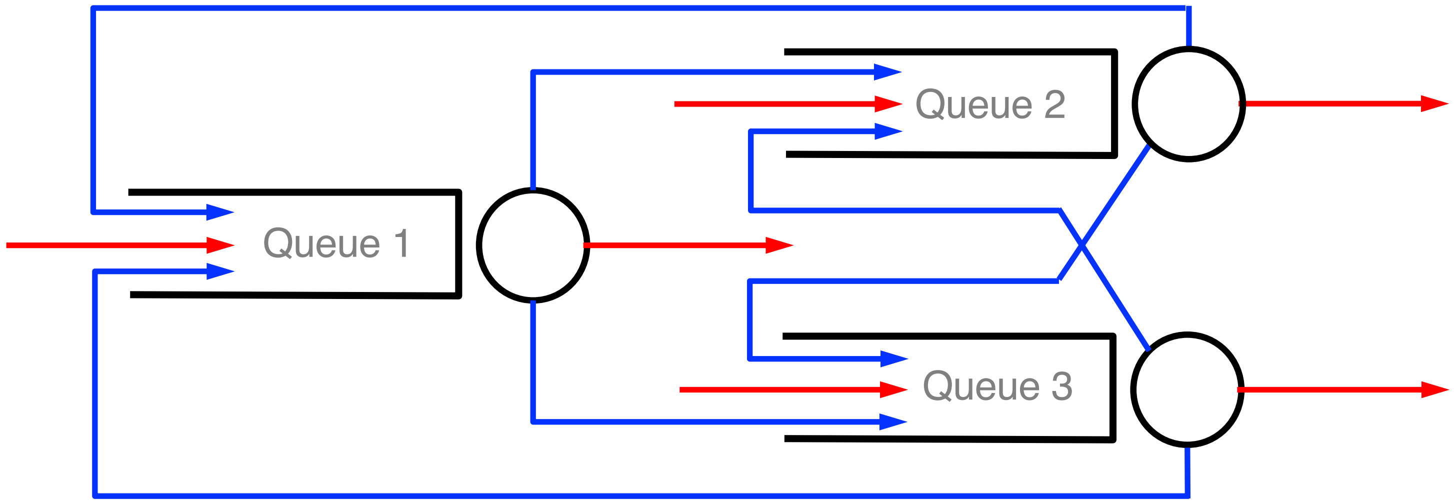
    ➜ same routing rules among queues for all

    Multi-class ➜ customers are identical within each class

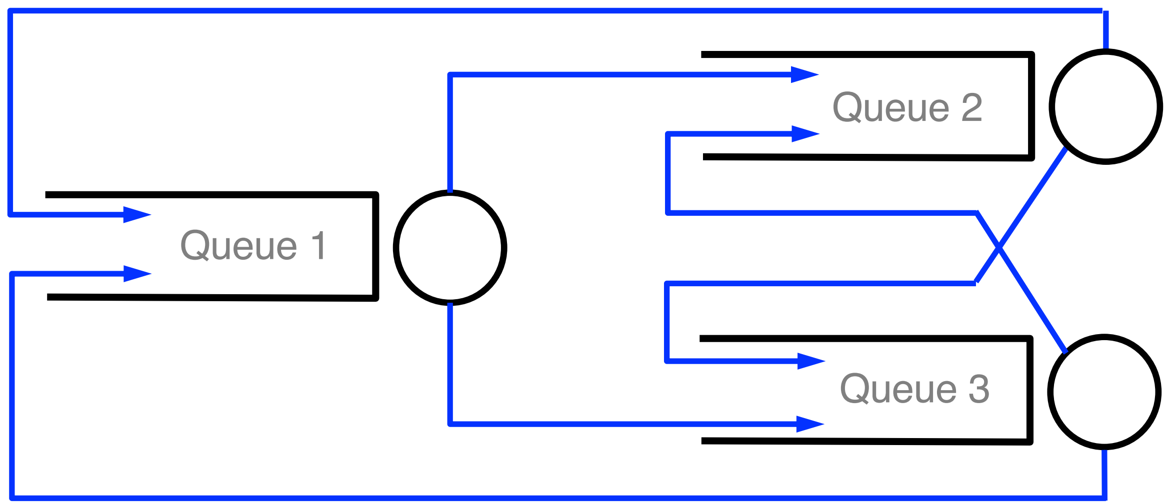
    ➜ routing rules among queues depend on class

Open Versus Closed Network

- Open network

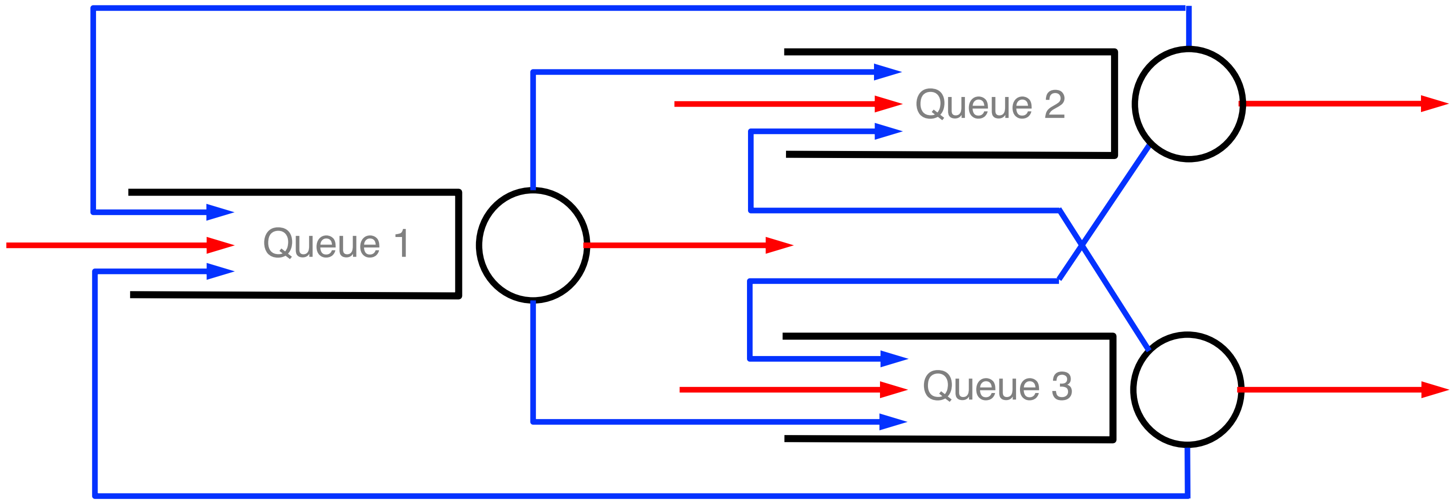


- Closed network

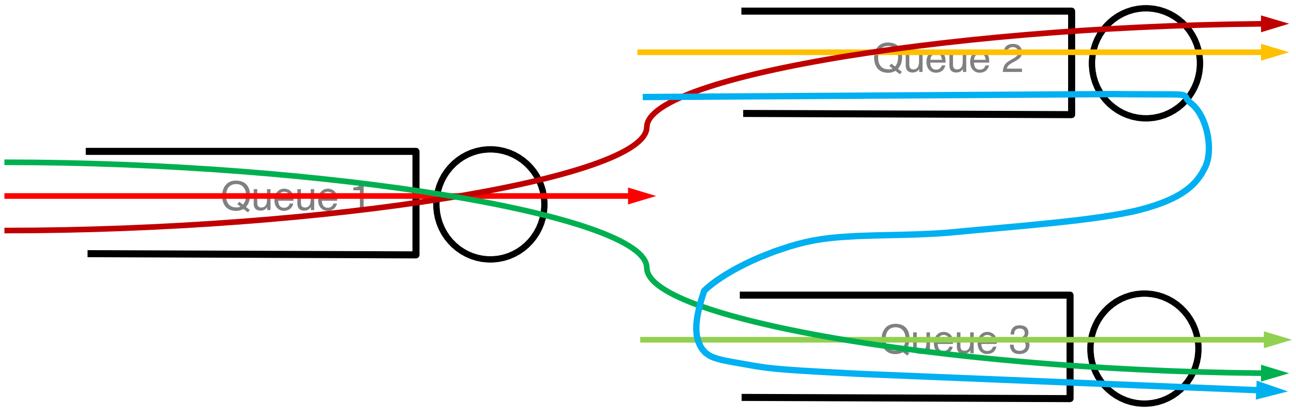


Single Class Versus Multi-Class

- Single class



- Multi-class ➜ specific routes



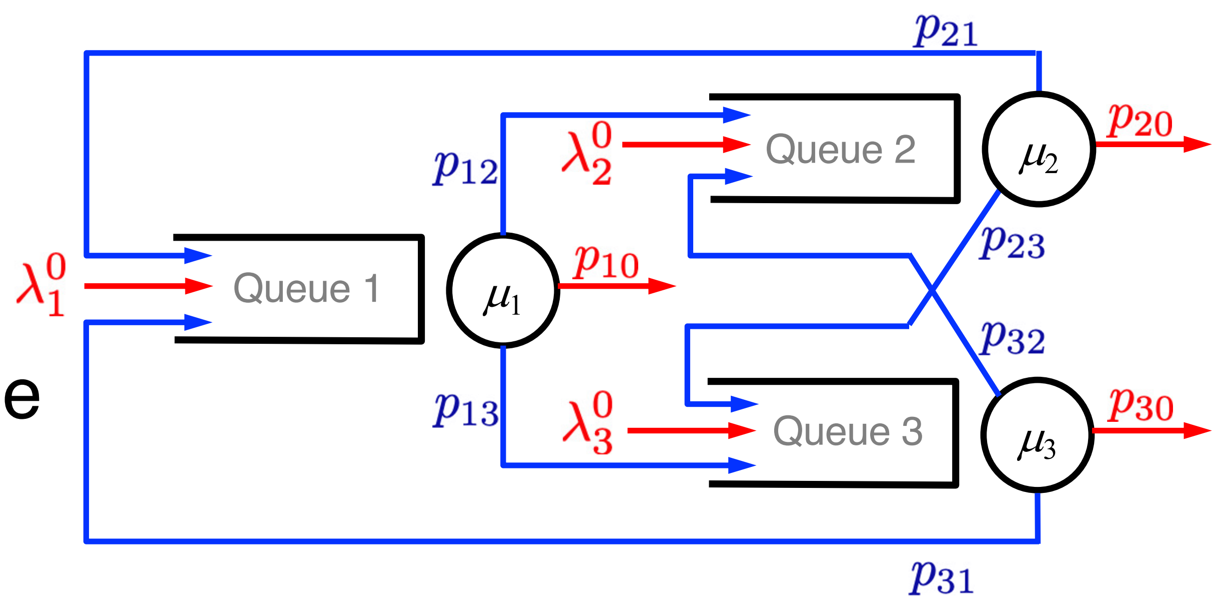
Open Jackson Network

- Open network
- Single class
- K queues
- One server per queue
- Infinite capacity
- In queue i
  - External customers Poisson process rate λ_i^0
  - Service time is Exp(μ_i)
  - Served customer
    - leaves system with probability p_{i0}
    - goes to queue j with probability p_{ij}

$$\Rightarrow \sum_{j=0}^{K} \boxed{p_{ij}} = 1$$

routing probabilities

  - Arrivals and service independent



Open Jackson Network

- How to study this system?
- Can we study each queue separately?

No!

inter-arrivals in one queue depend on events in other queues

- State of system = queue size in each queue
- $X_i(t)$ queue size at time t in queue i
- Define vector $\mathbf{X}(t) = \big(X_1(t), \ldots, X_K(t)\big) \quad \forall t \geq 0$
- State space $\mathcal{E} = \mathbb{N}^K$
- Is $\{\mathbf{X}(t),\, t \geq 0\}$ a Markov process?

Yes!

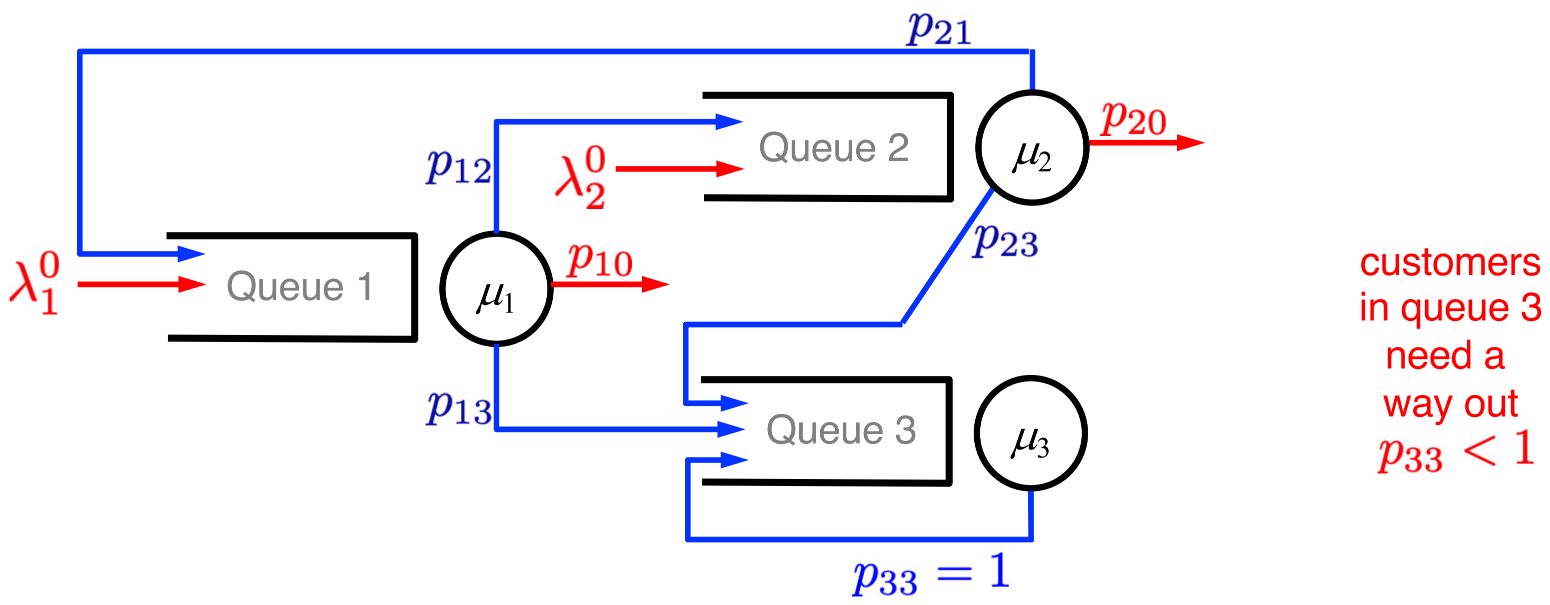
use construction rule #2

Using Construction Rule #2

- Consider state $\mathbf{n} = (n_1, \ldots, n_K) \in \mathbb{N}^K$
- Define $\mathbf{e}_i = (0, \ldots, 0, \underset{\substack{\uparrow \\ \text{position } i}}{1}, 0, \ldots, 0) \in \mathbb{N}^K$
- Possible transitions?
  - $\mathbf{n} \to \mathbf{n} + \mathbf{e}_i$ external arrival in queue i, time Exp(λ_i^0)
  - $\mathbf{n} \to \mathbf{n} - \mathbf{e}_i$ service end in queue i + leave system
    $n_i > 0$ time Exp($\mu_i p_{i0}$)
  - $\mathbf{n} \to \mathbf{n} - \mathbf{e}_i + \mathbf{e}_j$ service end in queue i + move to queue j
    $n_i > 0$ $j \neq i$ time Exp($\mu_i p_{ij}$)
- $\{\mathbf{X}(t),\, t \geq 0\}$ is a homogeneous CTMC ✔

Irreducibility?

- Need to check that any two states communicate



- Can state $\mathbf{n} = (n_1, n_2, n_3)$ with $n_3 > 0$ reach $\mathbf{0} = (0, 0, 0)$?

No!

- Markov chain of shown network is not irreducible

Routing Matrix

- Define matrix with internal routing probabilities

$$\mathbf{P} = \begin{pmatrix} \boxed{p_{11} & p_{12} & \cdots & p_{1K}} \\ p_{21} & p_{22} & \cdots & p_{2K} \\ \vdots & \vdots & \ddots & \vdots \\ p_{K1} & p_{K2} & \cdots & p_{KK} \end{pmatrix} \qquad \sum = 1 - p_{10}$$

- Definitions
  - Queue is open if its customers are certain to leave system
  - If all queues open ➜ network of queues is completely open

sufficient condition

one queue with $p_{i0} > 0$ and paths from all other queues to it

Irreducibility Condition

- Matrix $\mathbf{I} - \mathbf{P}$ is invertible

$$\mathbf{I} - \mathbf{P} = \begin{pmatrix} \boxed{1 - p_{11} & -p_{12} & \cdots & -p_{1K}} \\ -p_{21} & 1 - p_{22} & \cdots & -p_{2K} \\ \vdots & \vdots & \ddots & \vdots \\ -p_{K1} & -p_{K2} & \cdots & 1 - p_{KK} \end{pmatrix} \quad \sum = p_{10}$$

- Jackson queueing network is completely open

- Above statements are equivalent

Stationary/Limiting Distribution

- We want to compute vector $\pi = \left(\pi_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^K\right)$

$$\pi_{\mathbf{n}} = \lim_{t\to\infty} P\big(\mathbf{X}(t) = \mathbf{n}\big)$$
$$= \lim_{t\to\infty} P\big(X_1(t) = n_1, X_2(t) = n_2, \ldots, X_K(t) = n_K\big)$$

- Existence of Limiting Distribution
  - If homogeneous CTMC is irreducible ✔
  - If system of equations $\pi\mathbf{Q} = 0$

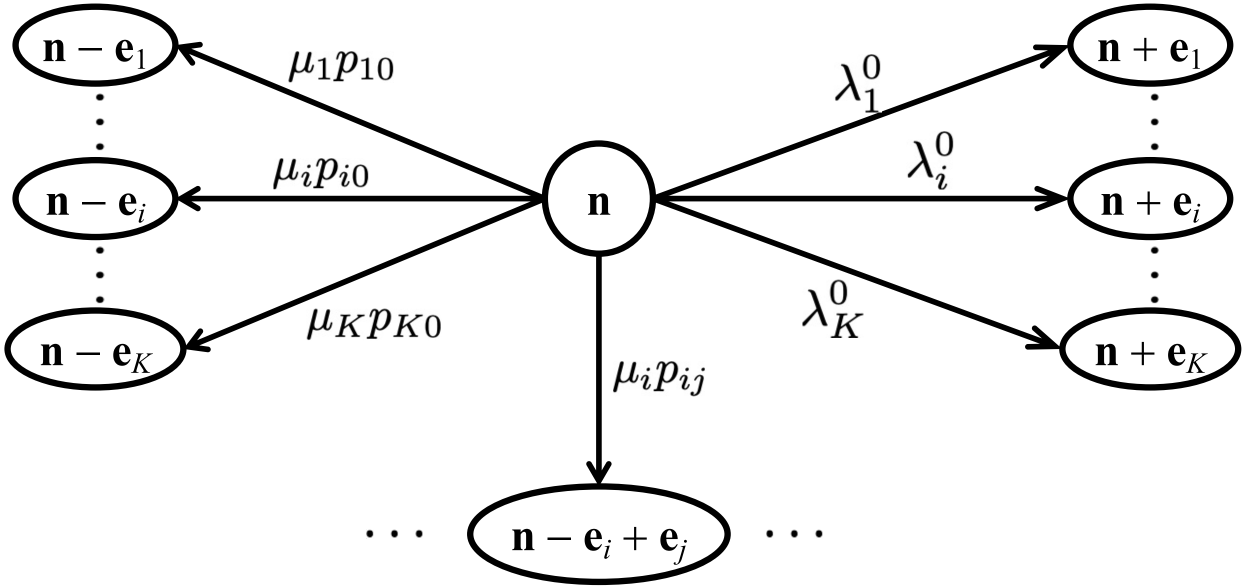
    $\pi\mathbf{1} = 1$?

    has unique strictly positive solution

    ➜ limiting distribution exists and it is the solution found

Balance Equations: Flow Out of State $\mathbf{n}$

- Showing transitions out of state $\mathbf{n}$



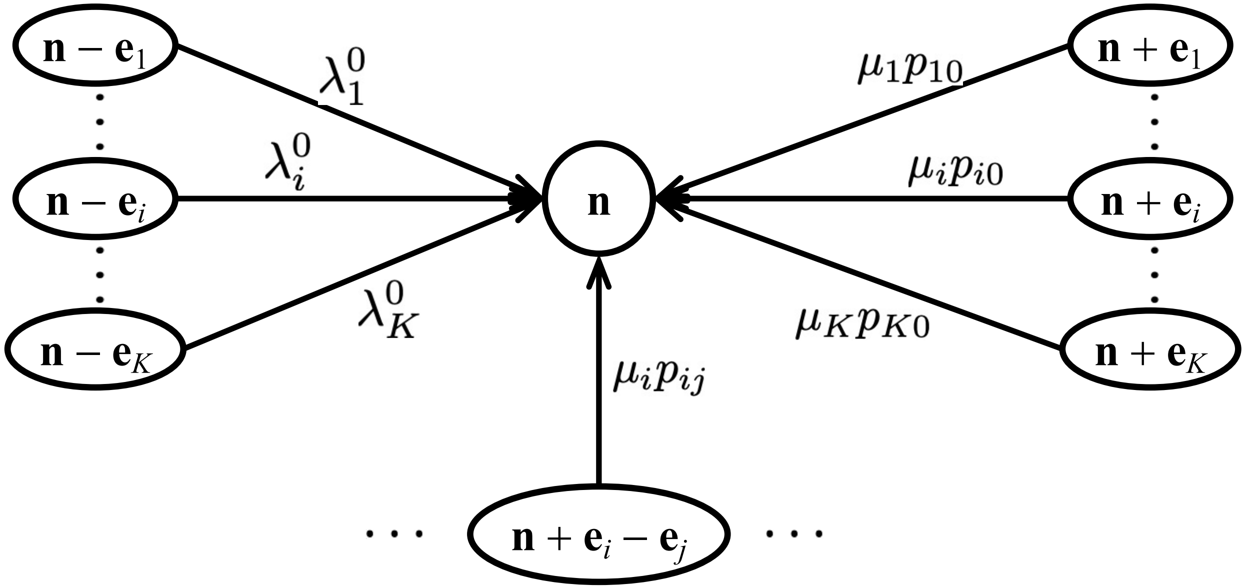
- Flow out $\pi_{\mathbf{n}}\left(\sum_{i=1}^{K}\lambda_i^0+\sum_{i=1}^{K}\boxed{\mathbb{1}_{n_i>0}}\boxed{(1-p_{ii})}\mu_i\right)$

prob to leave queue

$= 1$ if $n_i > 0$

Balance Equations: Flow In State $\mathbf{n}$

- Showing transitions into state $\mathbf{n}$



- Flow in

$$\underbrace{\boxed{\sum_{i=1}^{K} \mathbb{1}_{n_i>0}\lambda_i^0 \pi_{\mathbf{n}-\mathbf{e}_i}}}_{\text{exogenous arrivals}} + \underbrace{\boxed{\sum_{i=1}^{K} \mu_i p_{i0} \pi_{\mathbf{n}+\mathbf{e}_i}}}_{\text{departure from queue}} + \underbrace{\boxed{\sum_{i=1}^{K} \sum_{\substack{j=1 \\ j\neq i}}^{K} \mathbb{1}_{n_j>0}\, \mu_i p_{ij} \pi_{\mathbf{n}+\mathbf{e}_i-\mathbf{e}_j}}}_{\text{movements between queues}}$$

Proposition 18 (Jackson, 1957)

- Find unique nonnegative solution of system

traffic equations
$$\lambda_i = \lambda_i^0 + \sum_{j=1}^{K} p_{ji}\,\lambda_j, \qquad i = 1, 2, \ldots, K$$

- Define $\rho_i = \lambda_i / \mu_i$ for all i

- If $\lambda_i < \mu_i$ for all i

$$\boxed{\pi_{\mathbf{n}} = \prod_{i=1}^{K} (1-\rho_i)\rho_i^{n_i}, \quad \forall \mathbf{n} \in \mathbb{N}^K}$$

- Traffic equations using routing matrix $\mathbf{P}$

$$\left.\begin{aligned} \boldsymbol{\lambda} &= (\lambda_1, \ldots \lambda_K) \\ \boldsymbol{\lambda}^0 &= (\lambda_1^0, \ldots \lambda_K^0) \end{aligned}\right\} \quad \Leftrightarrow \quad \begin{aligned} \boldsymbol{\lambda} &= \boldsymbol{\lambda}^0 + \boldsymbol{\lambda}\mathbf{P} \\ \boldsymbol{\lambda} &= \boldsymbol{\lambda}^0 \underbrace{\boxed{(\mathbf{I}-\mathbf{P})^{-1}}}_{\text{irreducibility}} \end{aligned}$$

Proposition 19 (Jackson, 1957)

- If matrix $\mathbf{I} - \mathbf{P}$ invertible
- If $\boxed{\lambda_i < \mu_i \text{ for all } i}$ stability condition
- Then

$$\pi_{\mathbf{n}} = \prod_{i=1}^{K} (1 - \rho_i) \rho_i^{n_i}, \quad \forall \mathbf{n} \in \mathbb{N}^K$$

where $\boldsymbol{\lambda} = \boldsymbol{\lambda}^0 (\mathbf{I} - \mathbf{P})^{-1}$

- What is the meaning of λ_i?

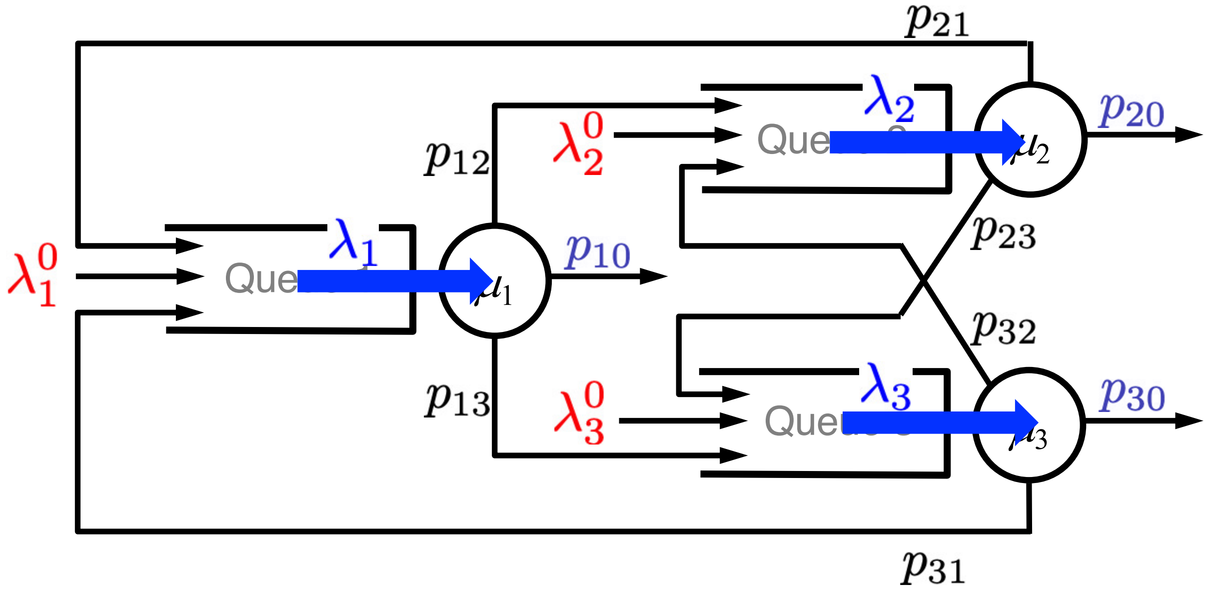
➜ total arrival rate to queue i

also throughput of queue i

Example

- In steady-state

$$\lambda_1 = \lambda_1^0 + p_{21}\lambda_2 + p_{31}\lambda_3$$
$$\lambda_2 = \lambda_2^0 + p_{12}\lambda_1 + p_{32}\lambda_3$$
$$\lambda_3 = \lambda_3^0 + p_{13}\lambda_1 + p_{23}\lambda_2$$



- By summing traffic equations

$$p_{10}\lambda_1 + p_{20}\lambda_2 + p_{30}\lambda_3 = \lambda_1^0 + \lambda_2^0 + \lambda_3^0$$

system output rate = system input rate

- In general $$\sum_{i=1}^{K} p_{i0}\,\lambda_i = \sum_{i=1}^{K} \lambda_i^0$$

Stationary/Limiting Distribution

- Let us check that

$$\pi_{\mathbf{n}} = \prod_{i=1}^{K} (1-\rho_i)\rho_i^{n_i}, \quad \forall \mathbf{n} \in \mathbb{N}^K$$

is solution of the system of equation

$$\pi \mathbf{Q} = 0$$

$$\pi \mathbf{1} = 1$$

Checking $\pi \mathbf{1} = 1$

- We have $\displaystyle\sum_{\mathbf{n}\in\mathbb{N}^K} \pi_{\mathbf{n}} = \sum_{n_1\in\mathbb{N},\ldots,n_K\in\mathbb{N}} \prod_{i=1}^{K} (1-\rho_i)\rho_i^{n_i}$

$$= \sum_{n_1\in\mathbb{N}} \cdots \sum_{n_K\in\mathbb{N}} \prod_{i=1}^{K} (1-\rho_i) \prod_{i=1}^{K} \rho_i^{n_i}$$

$$= \prod_{i=1}^{K} (1-\rho_i) \sum_{n_1\in\mathbb{N}} \rho_1^{n_1} \cdot \ldots \cdot \sum_{n_K\in\mathbb{N}} \rho_K^{n_K}$$

If $\rho_i < 1$ for all i $\displaystyle = \prod_{i=1}^{K} (1-\rho_i) \prod_{i=1}^{K} \frac{1}{1-\rho_i}$

$$= 1 \checkmark$$

Checking $\pi \mathbf{Q} = 0$

- From $\quad \pi_{\mathbf{n}} = \prod_{i=1}^{K} (1-\rho_i)\rho_i^{n_i}, \quad \forall \mathbf{n} \in \mathbb{N}^K$

  we find following relations

$$\pi_{\mathbf{n}-\mathbf{e}_i} = \frac{\pi_{\mathbf{n}}}{\rho_i} \qquad\qquad \pi_{\mathbf{n}+\mathbf{e}_i} = \rho_i \pi_{\mathbf{n}} \qquad\qquad \pi_{\mathbf{n}+\mathbf{e}_i-\mathbf{e}_j} = \frac{\rho_i \pi_{\mathbf{n}}}{\rho_j}$$

- Balance equation for state $\mathbf{n}$

$$\cancel{\pi_{\mathbf{n}}} \left(\sum_{i=1}^{K} \lambda_i^0 + \sum_{i=1}^{K} \mathbb{1}_{n_i>0} (1-p_{ii})\mu_i \right)$$

$$= \sum_{i=1}^{K} \mathbb{1}_{n_i>0} \lambda_i^0 \boxed{\pi_{\mathbf{n}-\mathbf{e}_i}} + \sum_{i=1}^{K} \mu_i p_{i0} \boxed{\pi_{\mathbf{n}+\mathbf{e}_i}} + \sum_{i=1}^{K} \sum_{\substack{j=1 \\ j\neq i}}^{K} \mathbb{1}_{n_j>0}\, \mu_i p_{ij} \boxed{\pi_{\mathbf{n}+\mathbf{e}_i-\mathbf{e}_j}}$$

$\rho_i = \lambda_i / \mu_i$

$$= \sum_{i=1}^{K} \mathbb{1}_{n_i>0} \lambda_i^0 \frac{\cancel{\pi_{\mathbf{n}}}}{\rho_i} + \sum_{i=1}^{K} \mu_i p_{i0} \rho_i \cancel{\pi_{\mathbf{n}}} + \sum_{i=1}^{K} \sum_{\substack{j=1 \\ j\neq i}}^{K} \mathbb{1}_{n_j>0}\, \mu_i p_{ij} \frac{\rho_i \cancel{\pi_{\mathbf{n}}}}{\rho_j}$$

$$\sum_{i=1}^{K} \lambda_i^0 + \sum_{i=1}^{K} \mathbb{1}_{n_i>0} (1-p_{ii})\mu_i = \sum_{i=1}^{K} \mathbb{1}_{n_i>0} \frac{\lambda_i^0 \mu_i}{\lambda_i} + \sum_{i=1}^{K} p_{i0}\, \lambda_i + \sum_{i=1}^{K} \sum_{\substack{j=1 \\ j\neq i}}^{K} \mathbb{1}_{n_j>0} p_{ij} \frac{\lambda_i \mu_j}{\lambda_j}$$

Checking $\pi \mathbf{Q} = 0$

$$\sum_{i=1}^{K} \lambda_i^0 + \boxed{\sum_{i=1}^{K} \mathbb{1}_{n_i>0} (1 - \cancel{p_{ii}})\mu_i} = \sum_{i=1}^{K} \mathbb{1}_{n_i>0} \frac{\lambda_i^0 \mu_i}{\lambda_i} + \boxed{\sum_{i=1}^{K} p_{i0}\, \lambda_i} + \sum_{i=1}^{K} \sum_{\substack{j=1 \\ \cancel{j \neq i}}}^{K} \mathbb{1}_{n_j>0} p_{ij} \frac{\lambda_i \mu_j}{\lambda_j}$$

$$\sum_{i=1}^{K} \lambda_i^0 - \sum_{i=1}^{K} p_{i0}\, \lambda_i = -\sum_{i=1}^{K} \mathbb{1}_{n_i>0}\, \mu_i + \sum_{i=1}^{K} \mathbb{1}_{n_i>0} \frac{\lambda_i^0 \mu_i}{\lambda_i} + \sum_{i=1}^{K} \sum_{j=1}^{K} \mathbb{1}_{n_j>0} p_{ij} \frac{\lambda_i \mu_j}{\lambda_j}$$

$$= -\sum_{i=1}^{K} \mathbb{1}_{n_i>0}\, \mu_i + \sum_{i=1}^{K} \mathbb{1}_{n_i>0} \frac{\lambda_i^0 \mu_i}{\lambda_i} + \sum_{j=1}^{K} \sum_{i=1}^{K} \mathbb{1}_{n_i>0} p_{ji} \frac{\lambda_j \mu_i}{\lambda_i}$$

$$= -\sum_{i=1}^{K} \mathbb{1}_{n_i>0} \frac{\mu_i}{\lambda_i} \left(\boxed{\lambda_i - \lambda_i^0 - \sum_{j=1}^{K} p_{ji} \lambda_j} \right)$$

traffic equations

$0 = 0$ ✔

Conclusion: limiting distribution exists and is

$$\pi_{\mathbf{n}} = \prod_{i=1}^{K} (1 - \rho_i) \rho_i^{n_i}, \quad \forall \mathbf{n} \in \mathbb{N}^K$$

Product-Form Solution $\pi_{\mathbf{n}} = \prod_{i=1}^{K}(1-\rho_i)\rho_i^{n_i}$

- Queue size of $M/M/1$ with arrival rate λ_i and service rate μ_i has limiting distribution

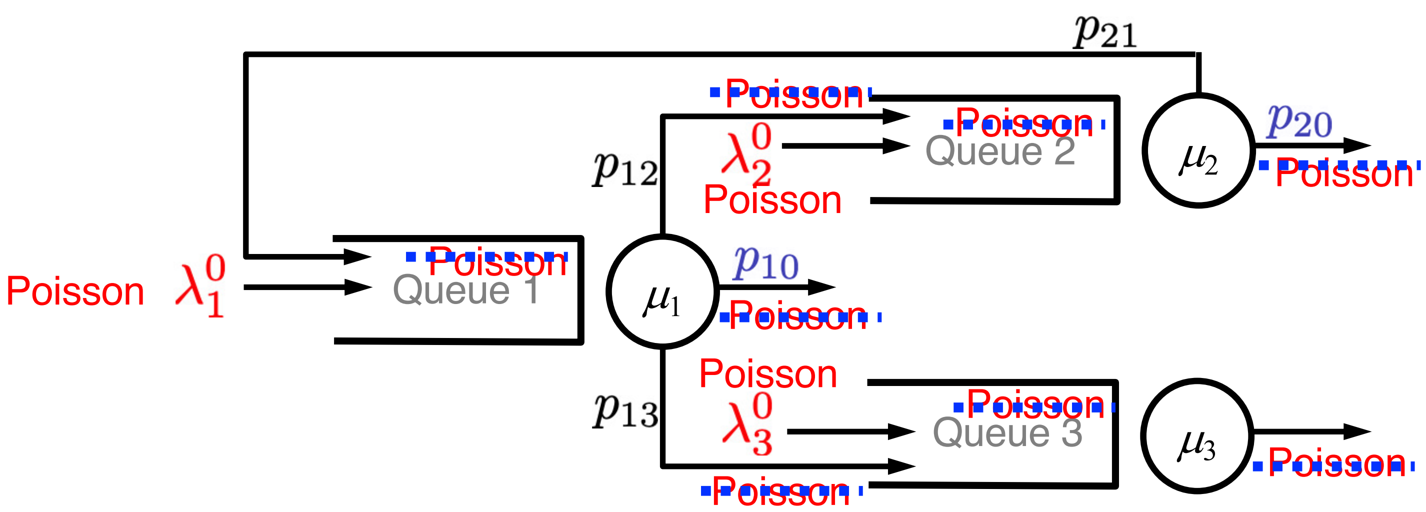
$$\pi_{n_i} = \lim_{t\to\infty} P(X_i(t) = n_i) = (1-\rho_i)\rho_i^{n_i}, \quad \forall n_i \in \mathbb{N}$$

- Solution is product of solution of each queue taken alone
- But
  - Queue sizes are correlated between queues
  - Arrivals are not Poisson in each queue in general
  - Each queue is not $M/M/1$
- Yet
  - Queue size of each queue same distribution as $M/M/1$

Arrivals are Not Poisson in Each Queue

- Burke's theorem
  - ▶Departure process of stationary $M / M / 1$ is Poisson
- Thinning of Poisson process ➜ Poisson process
- Aggregation of independent Poisson process

  ➜ Poisson process



15 minutes break

Jackson Network of Multi-Servers Queues

- Open Jackson network where queue i has c_i servers
- Find unique nonnegative solution of system

traffic equations

$$\lambda_i = \lambda_i^0 + \sum_{j=1}^{K} p_{ji}\,\lambda_j, \qquad i = 1, 2, \ldots, K$$

- Define $\mu_i(r) = \mu_i \min(r, c_i)$ for any positive r and for all i

- Define $\rho_i = \lambda_i / c_i \mu_i$ for all i
- If $\lambda_i < c_i \mu_i$ for all i

$$\pi_{\mathbf{n}} = \prod_{i=1}^{K} C_i \left(\frac{\lambda_i^{n_i}}{\prod_{r=1}^{n_i} \mu_i(r)} \right), \quad \forall \mathbf{n} \in \mathbb{N}^K$$

$$C_i = \left[\sum_{r=0}^{c_i - 1} \left(\frac{\lambda_i}{\mu_i} \right)^r \frac{1}{r!} + \left(\frac{\lambda_i}{\mu_i} \right)^{c_i} \frac{1}{c_i!} \left(\frac{1}{1 - \rho_i} \right) \right]^{-1}$$

Jackson Network of Multi-Servers Queues

- Product-form solution

- Each term in product relates to distribution of specific queue

- Queue i is not $M/M/c_i$ but distribution of queue size is same

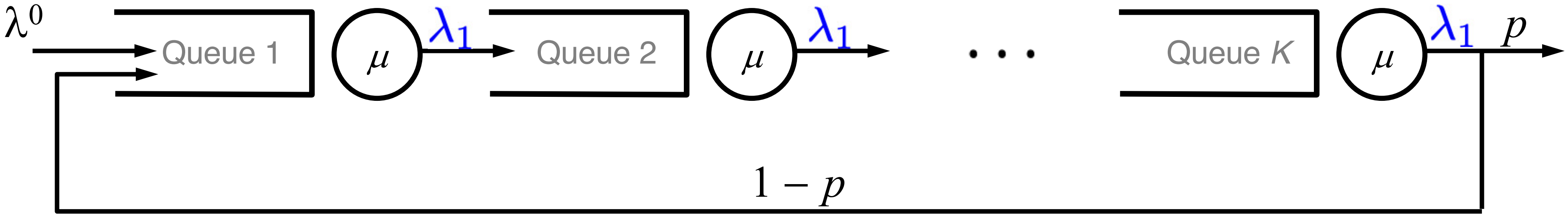
Example 10 Page 52

- Packets travel through K identical nodes to reach destination
- Destination cannot decode a ratio of $1-p$ packets due to transmission errors along the path

  ➜ a negative acknowledgement (NACK) is sent to source
- Source resends packet upon receiving a NACK
- Source generates packets according to Poisson rate λ^0
- NACK travel time is negligible
- Each node has service time Exp(μ)
- Questions:
  - Mean number of packets in network?
  - Expected sojourn time in network?

Example 10 Page 52

- We have a Jackson network



- Routing probabilities

$$p_{i,i+1} = 1 \quad i = 1, \ldots, K-1 \qquad p_{K,0} = p \qquad p_{K,1} = 1-p$$

- All nodes are open ➜ network completely open

$p > 0$

- Traffic equations

$$\left.\begin{array}{l} \lambda_1 = \lambda^0 + (1-p)\lambda_K \\ \lambda_i = \lambda_{i-1} \quad i = 2, \ldots, K \end{array}\right\} \Rightarrow \quad \lambda_i = \frac{\lambda^0}{p} \quad i = 1, \ldots, K$$

Example 10 Page 52

- Irreducibility condition: $p > 0$
- Stability condition: $\lambda^0 < p\mu$
- Joint distribution of number of customers in system

$$\pi_{\mathbf{n}} = \prod_{i=1}^{K} (1-\rho)\rho^{n_i}, \quad \forall \mathbf{n} \in \mathbb{N}^K$$

$$= \left(\frac{p\mu - \lambda^0}{p\mu}\right)^K \left(\frac{\lambda^0}{p\mu}\right)^{n_1+\cdots+n_K}$$

- Queue size in queue i similar to $M / M / 1$ with arrival rate λ^0/p and service rate μ

$$\overline{X_i} = \frac{\lambda^0}{p\mu - \lambda^0}$$

Example 10 Page 52

- Expected number of packets in system

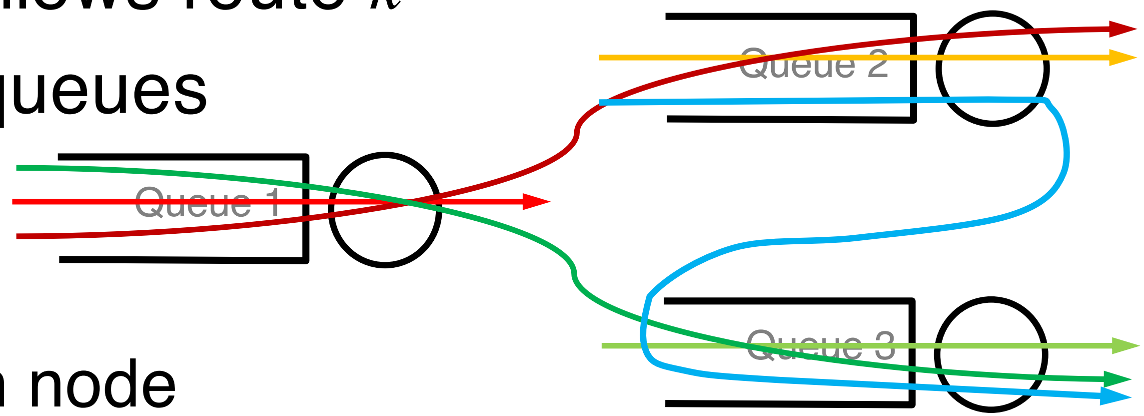
$$\overline{N} = \sum_{i=1}^{K} \overline{X_i} = \frac{K\lambda^0}{p\mu - \lambda^0}$$

- By Little: expected sojourn time in system is

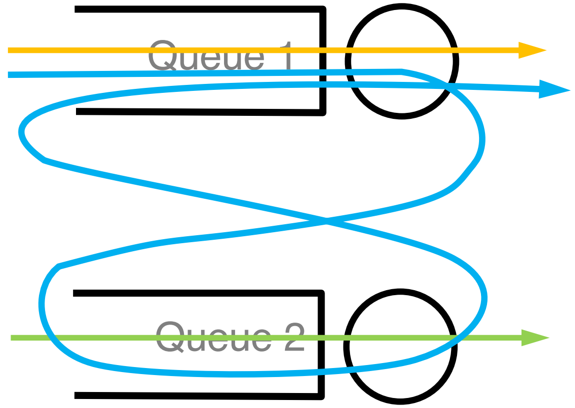
$$\overline{T} = \frac{\overline{N}}{\lambda^0} = \frac{K}{p\mu - \lambda^0}$$

Multiclass Kelly Network

- Objective: study systems where paths through network are deterministic
- Class k ➜ route $r_k = (r_k^1, \ldots, r_k^{n_k})$
  - repetitions are possible within a route
- Customer in class k follows route k
- System = network of queues
  - K nodes/queues
  - R routes
  - Single server at each node
  - Infinite waiting room
  - Poisson arrivals in class k rate λ_k
  - Service time Exp(μ_i) at node i



Multiclass Kelly Network



- How to study this system?
- Can we study each queue separately?

No!

arrivals in one queue depend on events in other queues

- State = queue size in each queue and each class
- $X_{ik}(t)$ number of class k customers at time t in queue i
- Define matrix $\mathbf{X}(t) = [X_{ik}(t)]_{\substack{1 \le i \le K \\ 1 \le k \le R}} \quad \forall t \ge 0$
- State space $\mathcal{E}_{KR}$: set of K–by–R matrices w/ entries in $\mathbb{N}$
- Is $\{\mathbf{X}(t),\, t \ge 0\}$ a Markov process?

No!

need to track number of visits to same node

Stationary/Limiting distribution

- For $\mathbf{N} = \left[n_{ik} \right]_{\substack{1 \le i \le K \\ 1 \le k \le R}} \in \mathcal{E}_{KR}$ define

$$\pi_{\mathbf{N}} = \lim_{t \to \infty} P(\mathbf{X}(t) = \mathbf{N}) = \lim_{t \to \infty} P\big(X_{ik}(t) = n_{ik}; 1 \le i \le K, 1 \le k \le R\big)$$

- Kelly (1975)
  - Compute global arrival rate of class k customers in node i

$$\hat{\lambda}_{ik} = \lambda_k \sum_{j=1}^{n_k} \mathbb{1}_{r_k^j = i} = \begin{cases} 0 & \text{if node } i \text{ not in route } r_k \\ \ell \lambda_k & \text{if node } i \text{ appears } \ell \text{ times in route } r_k \end{cases}$$

  - Compute global arrival rate in node i $\quad \hat{\lambda}_i = \sum_{k=1}^{R} \hat{\lambda}_{ik}$
  - If $\hat{\lambda}_i < \mu_i$ (stability condition) for each node

$$\pi_{\mathbf{N}} = \prod_{i=1}^{K} \left(1 - \frac{\hat{\lambda}_i}{\mu_i}\right) \boxed{\binom{\sum_{k=1}^{R} n_{ik}}{n_{i1}, n_{i2}, \ldots, n_{iR}}} \prod_{k=1}^{R} \left(\frac{\hat{\lambda}_{ik}}{\mu_i}\right)^{n_{ik}}$$

multinomial coefficient

$$\frac{\left(\sum_{k=1}^{R} n_{ik}\right)!}{\prod_{k=1}^{R} n_{ik}!}$$

Stationary/Limiting distribution

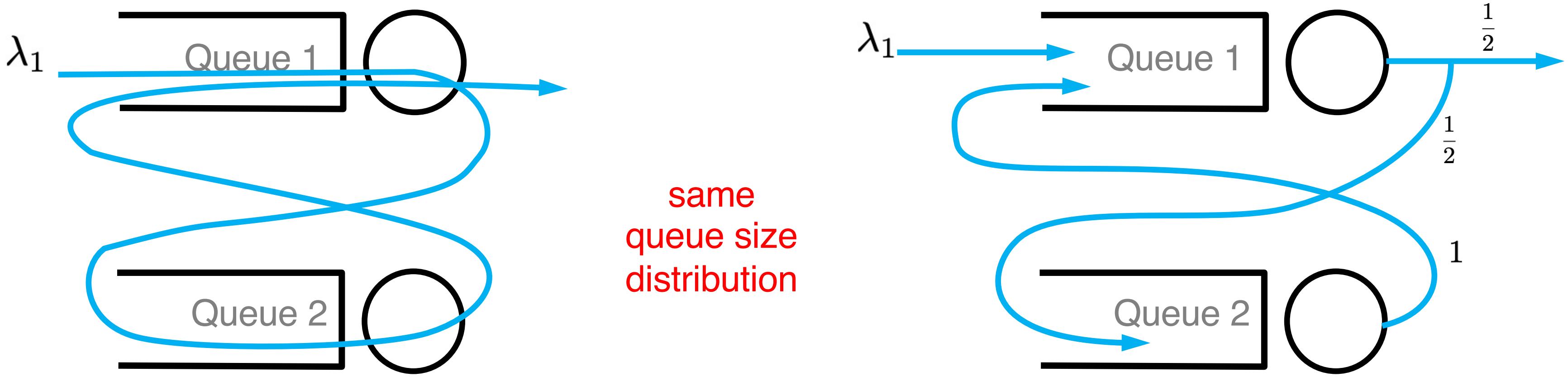
- Product-form solution

$$\pi_{\mathbf{N}} = \prod_{i=1}^{K} \boxed{\left(1 - \frac{\hat{\lambda}_i}{\mu_i}\right) \binom{\sum_{k=1}^{R} n_{ik}}{n_{i1}, n_{i2}, \ldots, n_{iR}} \prod_{k=1}^{R} \left(\frac{\hat{\lambda}_{ik}}{\mu_i}\right)^{n_{ik}}}$$

distribution of queue size in queue i

- If single class ($R = 1$) ➜ same as in Jackson network

$$\pi_{\mathbf{N}} = \prod_{i=1}^{K} \left(1 - \frac{\hat{\lambda}_i}{\mu_i}\right) \left(\frac{\hat{\lambda}_i}{\mu_i}\right)^{n_i}$$



Expected Number of Customers

- Expected number of class k customers in queue i

$$\overline{N}_{ik} = E[X_{ik}] = \frac{\hat{\lambda}_{ik}}{\mu_i - \hat{\lambda}_i}$$

- Expected number of customers in queue i

$$\overline{N}_i = \sum_{k=1}^{R} \overline{N}_{ik} = \frac{\sum_{k=1}^{R} \hat{\lambda}_{ik}}{\mu_i - \hat{\lambda}_i} = \frac{\hat{\lambda}_i}{\mu_i - \hat{\lambda}_i}$$

- Expected number of class k customers in network

$$\overline{N}^{(k)} = \sum_{i=1}^{K} \overline{N}_{ik}$$

Expected Sojourn Time

- Use Little's formula
- Expected sojourn time of class k customers

$$\overline{T}_k = \frac{\overline{N}^{(k)}}{\lambda_k} = \frac{1}{\lambda_k} \sum_{i=1}^{K} \frac{\hat{\lambda}_{ik}}{\mu_i - \hat{\lambda}_i}$$

$$\hat{\lambda}_{ik} = \begin{cases} 0 \\ \ell\lambda_k \end{cases}$$

- Expected sojourn time of arbitrary customer

$$\overline{T} = \frac{\sum_{i=1}^{K} \overline{N}_i}{\sum_{k=1}^{R} \lambda_k} = \frac{1}{\sum_{k=1}^{R} \lambda_k} \sum_{i=1}^{K} \frac{\hat{\lambda}_i}{\mu_i - \hat{\lambda}_i}$$

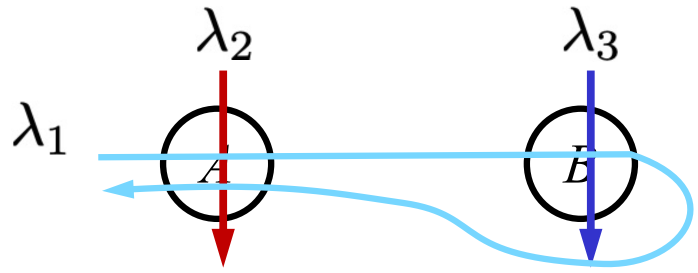
- We have $$\overline{T} = \sum_{k=1}^{R} \boxed{\frac{\lambda_k}{\sum_{l=1}^{R} \lambda_l}} \overline{T}_k$$

probability customer is from class k

Example

- Kelly network 2 nodes 3 routes

  $r_1 = (A, B, A) \quad r_2 = (A) \quad r_3 = (B)$



- Service rates μ_A and μ_B
- We can compute

$$\hat{\lambda}_{A1} = 2\lambda_1 \qquad \hat{\lambda}_{A2} = \lambda_2 \qquad \hat{\lambda}_{A3} = 0 \qquad \hat{\lambda}_A = 2\lambda_1 + \lambda_2$$

$$\hat{\lambda}_{B1} = \lambda_1 \qquad \hat{\lambda}_{B2} = 0 \qquad \hat{\lambda}_{B3} = \lambda_3 \qquad \hat{\lambda}_B = \lambda_1 + \lambda_3$$

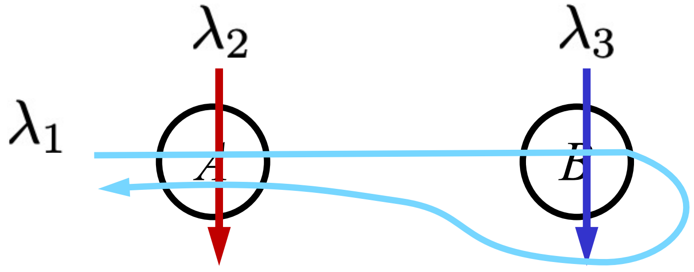
- Expected number of customers

$$\boxed{\overline{N}_{A1} = \frac{2\lambda_1}{\mu_A - 2\lambda_1 - \lambda_2}} \quad \boxed{\overline{N}_{A2} = \frac{\lambda_2}{\mu_A - 2\lambda_1 - \lambda_2}} \quad \boxed{\overline{N}_{A3} = 0} \quad \boxed{\overline{N}_A = \frac{2\lambda_1 + \lambda_2}{\mu_A - 2\lambda_1 - \lambda_2}}$$

$$\boxed{\overline{N}_{B1} = \frac{\lambda_1}{\mu_B - \lambda_1 - \lambda_3}} \quad \boxed{\overline{N}_{B2} = 0} \quad \boxed{\overline{N}_{B3} = \frac{\lambda_3}{\mu_B - \lambda_1 - \lambda_3}} \quad \boxed{\overline{N}_B = \frac{\lambda_1 + \lambda_3}{\mu_B - \lambda_1 - \lambda_3}}$$

$$\boxed{\overline{N}^{(1)} = \overline{N}_{A1} + \overline{N}_{B1}} \quad \boxed{\overline{N}^{(2)} = \overline{N}_{A2} + \overline{N}_{B2}} \quad \boxed{\overline{N}^{(3)} = \overline{N}_{A3} + \overline{N}_{B3}} \quad \boxed{\overline{N}_A + \overline{N}_B}$$

Example



- Expected number of customers per class / in network

$$\boxed{\overline{N}_{A1} = \frac{2\lambda_1}{\mu_A - 2\lambda_1 - \lambda_2}} \quad \boxed{\overline{N}_{A2} = \frac{\lambda_2}{\mu_A - 2\lambda_1 - \lambda_2}} \quad \boxed{\overline{N}_{A3} = 0} \quad \boxed{\overline{N}_A = \frac{2\lambda_1 + \lambda_2}{\mu_A - 2\lambda_1 - \lambda_2}}$$

$$\boxed{\overline{N}_{B1} = \frac{\lambda_1}{\mu_B - \lambda_1 - \lambda_3}} \quad \boxed{\overline{N}_{B2} = 0} \quad \boxed{\overline{N}_{B3} = \frac{\lambda_3}{\mu_B - \lambda_1 - \lambda_3}} \quad \boxed{\overline{N}_B = \frac{\lambda_1 + \lambda_3}{\mu_B - \lambda_1 - \lambda_3}}$$

$$\boxed{\overline{N}^{(1)} = \overline{N}_{A1} + \overline{N}_{B1}} \quad \boxed{\overline{N}^{(2)} = \overline{N}_{A2} + \overline{N}_{B2}} \quad \boxed{\overline{N}^{(3)} = \overline{N}_{A3} + \overline{N}_{B3}} \quad \boxed{\overline{N}_A + \overline{N}_B}$$

- Expected sojourn time per class / in network

$$\overline{T}_1 = \frac{\overline{N}^{(1)}}{\lambda_1} \boxed{= \frac{2}{\mu_A - 2\lambda_1 - \lambda_2} + \frac{1}{\mu_B - \lambda_1 - \lambda_3}}$$

$$\overline{T}_2 = \frac{\overline{N}^{(2)}}{\lambda_2} \boxed{= \frac{1}{\mu_A - 2\lambda_1 - \lambda_2}}$$

$$\overline{T}_3 = \frac{\overline{N}^{(3)}}{\lambda_3} \boxed{= \frac{1}{\mu_B - \lambda_1 - \lambda_3}}$$

$$\boxed{\overline{T} = \frac{\overline{N}_A + \overline{N}_B}{\lambda_1 + \lambda_2 + \lambda_3}}$$

$$= \frac{\lambda_1 \overline{T}_1 + \lambda_2 \overline{T}_2 + \lambda_3 \overline{T}_3}{\lambda_1 + \lambda_2 + \lambda_3}$$

$$= \frac{\overline{N}^{(1)} + \overline{N}^{(2)} + \overline{N}^{(3)}}{\lambda_1 + \lambda_2 + \lambda_3}$$

For next time (in two weeks)

- Lesson 6 to revise

- Homework 6 to return on Tuesday 5 November before 9 am

- Next lecture given by Alain Jean-Marie
  - Use case
  - pyMarmote tool

- Instructions to install Marmote before next lesson
  - https://marmote.gitlabpages.inria.fr/marmote/