EHRHART POLYNOMIALS AND SPLITS OF COXETER MATROIDS

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1. COXETER MATROIDS

Matroids are a combinatorial structure that generalizes, for instance, the concept of families of subspaces of a vector space. One way among many to associate a matroid M to a configuration X of n vectors in a vector space V is to specify all subsets B of $[n] = \{1,2,\dots,n\}$ that index bases of V among X. Abstractly, a matroid on $[n]$ can be characterized as a system M of subsets of $[n]$ that satisfies *Steinitz' basis exchange axiom*:

If $A \neq B \in \mathsf{M}$ and $a \in A \setminus B$, there exists some $b \in B \setminus A$ such that $A - a + b \in \mathsf{M}$.

One way to recover geometry from this combinatorial abstraction is to work with *characteristic vectors*, by assigning to each basis B the $0/1$-vector $\chi(B)$ of length n that has a '1' precisely in the coordinates indexed by B. The convex hull

$$\mathsf{MBP}(\mathsf{M}) \;=\; \operatorname{conv}\{\chi(B) : B \in \mathsf{M}\}$$

of these points is called the *matroid base polytope* of M.

Obviously, one can study the polytope of characteristic vectors associated to any set system, but it is less than clear what, if anything, one might learn from it. However, in the case of matroids these polytopes are quite well-behaved:

- Since all bases have the same cardinality, all characteristic vectors $\chi(B)$ lie on a sphere of radius $\sqrt{|B|}$, and therefore all of them are vertices of $\mathsf{MBP}(\mathsf{M})$.
- Edges reflect basis exchange: Two vertices $\chi(A)$, $\chi(B)$ span an edge in $\mathsf{MBP}(\mathsf{M})$ iff A, B satisfy Steinitz' axiom.

Hidden just beneath the surface of Steinitz' axiom we find the action of the *symmetric group S_n* on $[n]$: we can regard a Steinitz interchange $a \leftrightarrow b$ as the transposition (a,b), and such transpositions generate S_n. Geometrically, each edge of $\mathsf{MBP}(\mathsf{M})$ materializes the orthogonal reflection of its vertices across a hyperplane of equation $x_i = x_j$, say, and all of *those* form a very classical object: the hyperplane arrangement associated to the root system A_{n-1}.

There are various more or less abstract definitions to generalize these concepts from A_{n-1} and its associated regular polytope (the simplex) to the other classical root systems: BC_n (cubes), D_n, H_3 (dodeca/icosahedron), H_4 (120-cell and 600-cell), F_4 (24-cell), E_6, E_7, E_8, but the hands-down winner is the following charming theorem by Israel Gelfand and Vera Serganova:

Theorem-Definition (Gelfand–Serganova, 1987; cf. [1])
Let Q be a convex polytope. Consider, for each edge e of Q, the hyperplane H_e orthogonal to e that passes through its midpoint. Let W be the group generated by the reflections in all H_e. Then W is finite iff Q is a Coxeter matroid (polytope).

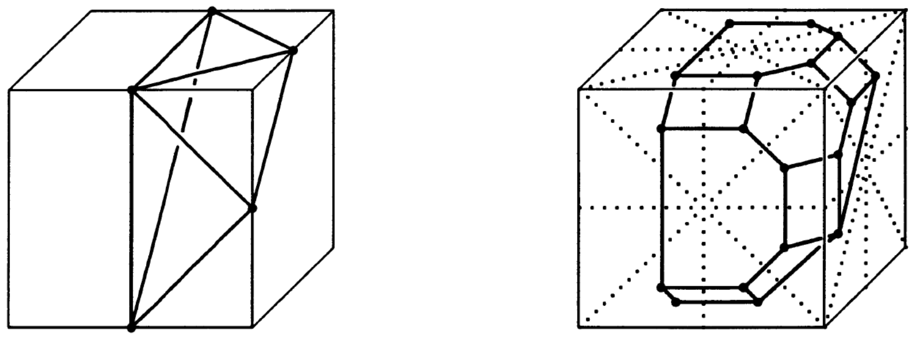

FIGURE 1. An illustration in [1] of two BC_3 Coxeter matroid polytopes

2. UNIFORM COXETER MATROIDS AND THE WYTHOFF CONSTRUCTION

The set of bases of the *uniform matroid* $U(k,n)$ is the entire set $\binom{[n]}{k}$, so that the corresponding matroid base polytope is the hypersimplex $\Delta(k,n)$. For fixed n, this hypersimplex may be generated by reflecting the point $(1,\dots,1,0,\dots,0)$ with k entries equal to 1 in all the hyperplanes $H_{i,j} = \{x \in \mathbb{R}^n : x_i = x_j\}$, $i < j$, of the fixed root system A_{n-1}. The matroid base polytope of any matroid of rank k on n elements is then a subpolytope of $\Delta(k,n)$.

Similarly, suppose that τ is a classical root system of rank n, with some numbering[1] of its n simple roots. For any subset $R \subseteq [n]$, the *uniform Coxeter matroid polytope $M_{\tau,R}$ of type τ and rings R* is the convex hull of the orbit of a point *not* on the hyperplanes with normal vectors among the simple roots indexed by R, and *on* the hyperplanes with normal vectors *not* indexed by R. This is called the *Wythoff construction*, and it is traditional to represent R by drawing rings around the dots corresponding to the hyperplanes indexed by R in a Coxeter diagram of τ. Thus, $\Delta(k,n) = M_{\mathsf{A}_{n-1},R}$ for any $R \in \binom{[n]}{k}$, but for other root systems distinct choices of R generally yield distinct uniform Coxeter matroid polytopes.

Example 2.1. Figure 2 shows the 8 classes of Coxeter matroids of type B_2 with $R = \{0,1\}$. Figures 3, 4 and 5 show the classes of Coxeter matroids of type D_3 with $R = \{0,1\}$, $R = \{1,2\}$ and $R = \{0,1,2\}$, respectively.

3. CHALLENGES

3.1. **Enumeration.** Complete enumeration of all matroids or Coxeter matroids doesn't make much sense, given the large symmetry groups that operate on these objects. Therefore, we will always enumerate only pairwise non-isomorphic Coxeter matroids, where we consider two Coxeter matroid polytopes $M_{\tau,R}$, $M'_{\tau,R}$ to be isomorphic if they are related by a symmetry induced by the root system τ.

The number of isomorphism classes of A_{n-1}-matroids of rank r on n elements is known for $r \le 3$ and $n \le 12$, for $r = 4$ and $n \le 10$, and for $r = 5$ and $n \le 9$. The largest computation among these is that of Matsumoto et al [4], who determined the number of isomorphism classes for $(r,n) = (4,10)$ to be 4 886 380 924.

Tables 1 and 2 summarize the results of our enumeration isomorphism classes of some Coxeter matroids using `polymake` [2].

Challenge 3.1. Confirm or correct, and extend if possible, the data in Tables 1 and 2.

[1] For example, you could use the convention in https://en.wikipedia.org/wiki/Root_system#Explicit_construction_of_the_irreducible_root_systems

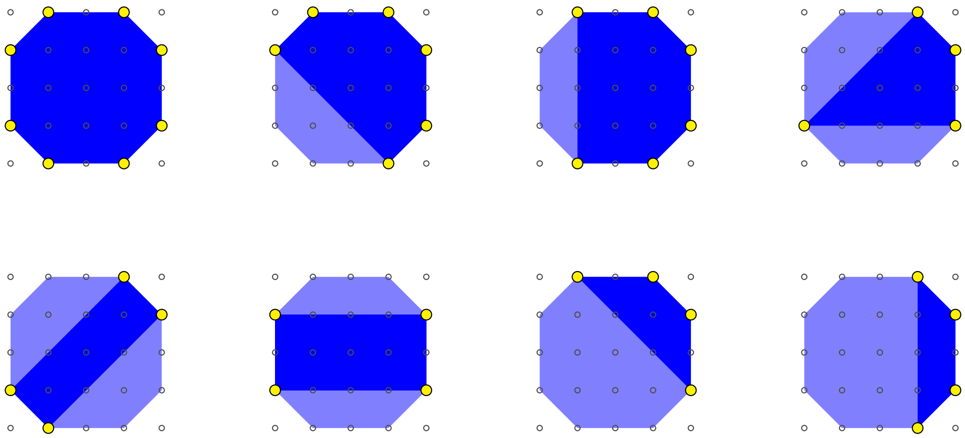

FIGURE 2. The 8 isomorphism classes of Coxeter matroids of type B_2 with $R = \{0, 1\}$

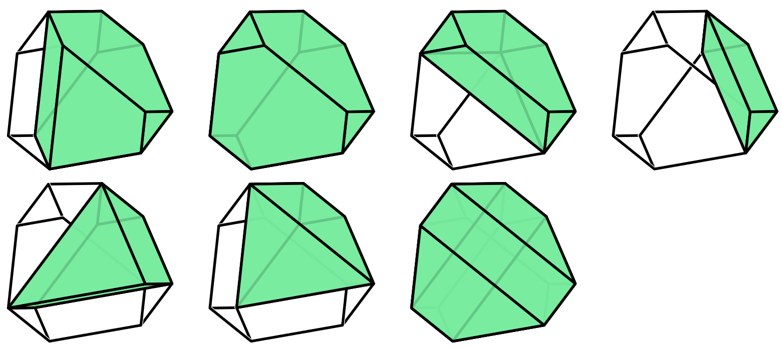

FIGURE 3. The 7 classes of Coxeter matroids of type D_3 with set of rings $R = \{0, 1\}$

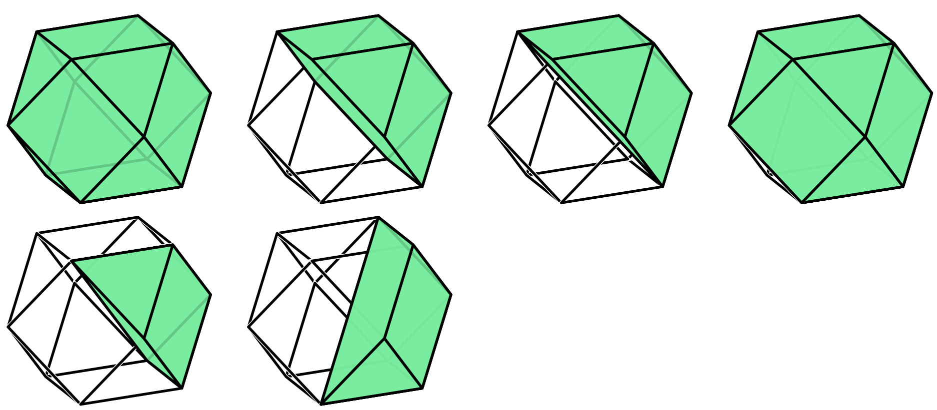

FIGURE 4. The 6 classes of Coxeter matroids of type D_3 with set of rings $R = \{1, 2\}$

3.2. **Facets of Coxeter matroids.**

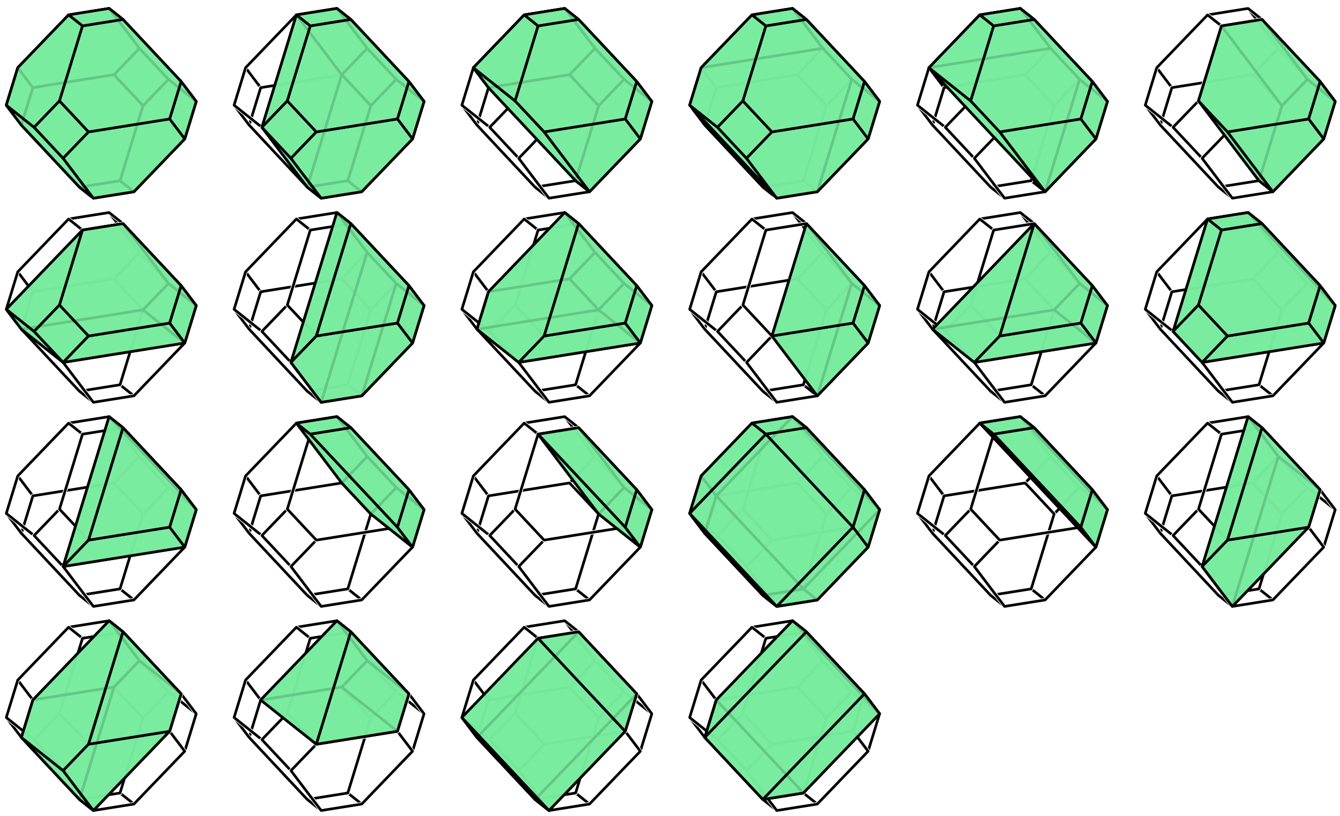

FIGURE 5. The 22 classes of Coxeter matroids of type D_3 with set of rings $R = \{0, 1, 2\}$

Root system τ \ Rings R	0	1	2	01	02	12	012
$\mathsf{B}_3, \mathsf{C}_3$	**3** 6	**22** 12	**9** 8	**83** 24	**79** 24	**109** 24	48
D_3	**2** 6	**1** 4	**1** 4	**7** 12	**7** 12	**6** 12	**22** 24
H_3	**1089** 20	**9701** 30	**57** 12	60	60	60	120

TABLE 1. In boldface, the known numbers of isomorphism classes of 3-dimensional Coxeter matroids. Smaller, lowered and gray, the number of vertices of the uniform Coxeter matroid $M_{\tau,R}$ of which the isomorphism classes are subpolytopes. Note that the following choices of rings R yield combinatorially isomorphic uniform Coxeter matroids: For D_3, $R \in \{1, 2\}$, $R \in \{01, 02\}$

Challenge 3.2. What can you say about the combinatorial types of faces of Coxeter matroid polytopes? For A_{n-1}, the only possible 2-faces are triangles and squares [1, Theorem 1.12.8], but for the other types even this very basic question seems to be open (though easy).

Challenge 3.3. Let τ be a root system of dimension d. Describe and count the facets of $M_{\tau,[d]}$.

Challenge 3.4. Given a subset $F \subset [d]$, is there always a facet of $M_{\tau,[d]}$ whose edges are parallel to the vectors in F?

Root system τ \ Rings R	0	1	2	3	01	02	03	12	13	23
$\mathsf{B}_4, \mathsf{C}_4$	**4** 8	**133** 24	**873** 32	**67** 16	48	96	64	96	96	64
D_4	**3** 8	**67** 24	**3** 8	**3** 8	48	32	32	48	48	32
F_4	**2345** 24	96	96	**2345** 24	192	288	144	288	288	192
H_4	600	1200	720	120	2400	3600	2400	3600	3600	1440

TABLE 2. In boldface, the known numbers of isomorphism classes of 4-dimensional Coxeter matroids. Smaller, lowered and gray, the number of vertices of the uniform Coxeter matroid $M_{\tau,R}$ of which the isomorphism classes are subpolytopes. Note that the following choices of rings R yield combinatorially isomorphic uniform Coxeter matroids: For D_4, $R \in \{0,2,3\}$, $R \in \{01,12,13\}$, $R \in \{02,03,23\}$; For F_4, $R \in \{0,3\}$, $R \in \{1,2\}$; etc

Challenge 3.5. Classify the facets of $M_{\tau,[d]}$ that are orthogonal to a root vector.

Challenge 3.6. Using the result of Challenge 3.5, determine if there are Coxeter matroids all of whose facets are orthogonal to a root.

Challenge 3.7. Determine the symmetry classes of facets of $M_{\tau,[d]}$ under the action of τ.

Challenge 3.8. The hyperplanes defining facets of $M_{\mathsf{B}_3,02}$ (which are squares, rectangles and triangles) are parallel to the hyperplanes defining facets of $M_{\mathsf{B}_3,012}$ (which are octagons —from cutting off the vertices of a square— , hexagons —from cutting off the vertices of a triangle— , and rectangles). The same appears to happen for $M_{\mathsf{D}_3,12}$.

For which other types does this happen, in general dimension d?

Conjecture 3.9. Let τ be a root system of dimension d, and let $Q \subset M_{\tau,R}$ be a Coxeter matroid. Then each facet of Q is parallel to a facet of $M_{\tau,[d]}$.

Challenge 3.10. Let τ be a root system of dimension d. Describe the orbits of the vectors in τ under τ, and thus determine the equivalence classes of edges of Coxeter matroids.

3.3. **Concordant Coxeter Matroids.**

Definition 3.11. A Coxeter matroid $Q \subseteq M_{\tau,R}$ is *concordant* if each facet of Q is parallel to a facet of $M_{\tau,R}$ (and not just of $M_{\tau,[d]}$).

Example 3.12. Our partial enumeration yields the results depicted in `concordant-examples.pdf`.

Observation 3.13. By Challenge 3.8, all Coxeter matroids belonging to $M_{\mathsf{B}_3,02}$ or $M_{\mathsf{D}_3,12}$ are concordant.

Definition 3.14. A *picky* concordant matroid is a concordant matroid all of whose facets belong to the same symmetry type.

Challenge 3.15. Which types have picky concordant matroids, and how many of them? This is what we know so far:

type \ rings	0	1	2	3	01	02	12	012
B_3	1	0	1		1	1	2	
D_3	0				0		0	0
B_4	1	0	0	1				
D_4	0	0	0	0				
F_4	12							

Explain this table! Can you extend it to B_n, D_n for $n \geq 5$?

3.4. **Lattice Coxeter matroids.** The classical root systems are all *crystallographic*, with the exception of H_3 and H_4. This means that the root vectors generate a lattice.

Challenge 3.16. For all crystallographic root systems, find out if there is a canonical realization of each uniform Coxeter matroid as a lattice polytope in the root lattice

Challenge 3.17. If the answer to Challenge 3.16 is positive, find out if there is any relation between the Ehrhart polynomials of the Coxeter matroid polytopes of a fixed uniform Coxeter matroid.

3.5. **Splits of Coxeter matroids.** A very important construction in toric and tropical geometry is the subdivision, in the case A_{n-1}, of matroid polytopes into smaller matroid polytopes. This was first considered by Lafforgue [3]. Speyer [5] conjectured that the subdivision corresponding to *series-parallel* matroids have the largest number of faces, and he proved this in the case of tropical linear spaces.

Challenge 3.18. What can you say about subdivisions of Coxeter matroid polytopes into smaller Coxeter matroid polytopes?

References

[1] Alexandre V. Borovik, I. M. Gelfand, and Neil White. *Coxeter matroids. With illustrations by Anna Borovik.* Boston, MA: Birkhäuser, 2003.

[2] Ewgenij Gawrilow and Michael Joswig. `polymake`: a framework for analyzing convex polytopes. In *Polytopes—combinatorics and computation (Oberwolfach, 1997)*, volume 29 of *DMV Sem.*, pages 43–73. Birkhäuser, Basel, 2000.

[3] L. Lafforgue. *Chirurgie des grassmanniennes.* Providence, RI: American Mathematical Society (AMS), 2003.

[4] Yoshitake Matsumoto, Sonoko Moriyama, Hiroshi Imai, and David Bremner. Matroid enumeration for incidence geometry. *Discrete Comput. Geom.*, 47(1):17–43, 2012.

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