Sheet 4

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Solution 1

There is one thing which we do not find entirely clear in this problem: the question whether we are given P or not.

If P is not given, we have discussed with other colleagues the problem, and we have come to the conclusion that the lower faces of Q^c are not determined from only the linear map $\pi : \mathbb{R}^p \longrightarrow \mathbb{R}^{q+1}$, and so it is not possible to give an algorithm to find such faces. As an example, we could consider the projection to the first coordinate, $\pi\colon \mathbb{R}^2 \to Q \subset \mathbb{R}$, and the vector $c = (0, 1)$. Then, for $x = (x_1, x_2) \in \mathbb{R}^2$, it is $c \cdot x = x_2$, and so $\pi^c = \mathrm{id}$. Hence, for any polytope $P \subset \mathbb{R}^2$, we have that $Q^c = \mathrm{id}(P) = P$, and so for different choices of P we get different sets of lower faces.

Nevertheless, we think that since π is defined on P, then P is given along with π, and so the lower faces are determined. We have found an algorithm which determines the set of lower faces of Q^c but, since we cannot distinguish the interior points of Q^c from those in the boundary, we use at some point a convex hull algorithm to solve this.

Recall that the lower faces of Q^c are those whose valid inequality is given by a hyperplane h with $h_{d+1} < 0$ and a vector h_0, with $x \cdot h \leq h_0$ for all $x \in Q^c$.

With "convex hull algorithm" we refer to any algorithm that can distinguish points in the boundary from points in the interior of a given polytope.

Our algorithmn does the following:

1. For each vertex $x_i \in P$, compute $\pi^c(x_i)$, so that it is $Q^c = \mathrm{conv}\{\pi^c(x_i)\}$.
2. Use a convex hull algorithm to find the incidence graph of Q^c.

3. For all subsets of $d+1$ vertices:

   (a) Check if they are in the same hyperplane.
   If they are, compute the hyperplane (h, h_0) with $h_{d+1} < 0$.

   (b) For all other points v, check if $v \cdot h \leq h_0$:

      i. If not, then it cannot be a lower facet, so we stop.

      ii. If $v \cdot h = h_0$, then v is in the hyperplane, so we add v to the subset, and compute the convex hull of this new subset to find the real facet.

   If at the end, all vertices v verify $v \cdot h \leq h_0$, then we have a valid inequality, and thus a facet, and this facet will be a lower facet by definition. We can find lower dimensional faces by intersecting lower facets.

We have been searching for ways to avoid this convex hull (or any equivalent idea) from the algorithm, but we cannot find a way to do this with the information that we have. We have noticed that by having the set of facet normals of Q, we have the first q coordinates of the normal vectors of all facets in Q^c, so we only need to determine this last coordinate, which will depend on c and P. Sadly we did not find a way to fulfill this purpose, and neither a way to prove that such algorithm does not exist.

Solution 2

Since f is linear, we can write $f(x) = Ax$ for some matrix A. Then,

$$f(x) = Ax = \left(\sum_{i=1}^{n} a_{1i}x_i, \dots, \sum_{i=1}^{n} a_{ni}x_i \right) = (f_1(x), \dots, f_n(x)).$$

For each coordinate $j = 1, \dots, n$, using the linearity of the integral,

$$\int_Q f_j(x)\ dx = \int_Q \sum_{i=1}^{n} a_{ji}x_i\ dx = \sum_{i=1}^{n} a_{ji} \int_Q x_i dx = \sum_{i=1}^{n} a_{ji} \cdot r_{0i} = \mathrm{vol}(Q) A_j \cdot r_0,$$

where A_j is the jth row of A. Then,

$$\int_Q f(x)\ dx = (A_1 \cdot r_0 \mathrm{vol}(Q), \dots, A_n \cdot r_0 \mathrm{vol}(Q)) = A \cdot r_0 \cdot \mathrm{vol}(Q) = f(r_0) \cdot \mathrm{vol}(Q).$$

Solution 3

We follow the proof formally proving Ziegler's claims.

First of all, any convex combination of two sections is a section again. If γ_1, γ_2 are both sections of π, then for any $t \in [0,1]$, since π is linear,

$$\pi \circ (t\gamma_1 + (1-t)\gamma_2) = t(\pi \circ \gamma_1) + (1-t)(\pi \circ \gamma_2) = t \cdot \mathrm{id} + (1-t) \cdot \mathrm{id} = \mathrm{id}.$$

Now, by the linearity of the integral, $\Sigma(P,Q)$ is convex. If $p, q \in \Sigma(P,Q)$, then we can write

$$p = \tfrac{1}{\mathrm{vol}(Q)} \int_Q \gamma_p(x)dx, \quad q = \tfrac{1}{\mathrm{vol}(Q)} \int_Q \gamma_q(x)dx$$

and for any $t \in [0,1]$,

$$tp + (1-t)q = \frac{1}{\mathrm{vol}(Q)} \int_Q (t\gamma_p + (1-t)\gamma_q)(x)dx \in \Sigma(P,Q)$$

Now, the dimension of $\Sigma(P,Q)$ cannot be larger than $\dim P - \dim Q$, since $\Sigma(P,Q) \subset \pi^{-1}(r_0)$, which has this dimension. Here there are two unproven assertions. First of all, that $\Sigma(P,Q) \subset \pi^{-1}(r_0)$, and secondly that $\dim \pi^{-1}(r_0) = \dim P - \dim Q$. The second assertion is obtained by noticing that, by the first isomorphism theorem, π induces an isomorphism $\mathbb{R}^p / \ker \pi \simeq \mathbb{R}^q$, so assuming P and Q are full dimensional, it must be $\dim P - \dim Q = \dim \ker \pi = \dim \pi^{-1}(r_0)$. For the first assertion, take $p \in \Sigma(P,Q)$, along with its associated section γ. Then,

$$\pi(p) = \frac{1}{\mathrm{vol} Q} \int_Q (\pi \circ \gamma)(x)dx = \frac{1}{\mathrm{vol} Q} \int_Q x dx = r_0.$$

We must also check that $p \in P$. If γ is piecewise linear, and $\{p_1, \dots, p_n\}$ are the vertices of P, we can express $\gamma(x) = \sum_{i=1}^n t_i(x) p_i$, where $\sum t_i(x) = 1$ for all $x \in Q$. Then,

$$\frac{1}{\mathrm{vol} Q} \int_Q \gamma(x)dx = \frac{1}{\mathrm{vol} Q} \sum p_i \int_Q t_i(x) = \frac{1}{\mathrm{vol} Q} \sum p_i t_i(r_0) \mathrm{vol} Q = \sum p_i t_i(r_0),$$

and since $\sum t_i(r_0) = 1$, this belongs to P.

At this point, Ziegler claims: "Every piecewise linear *non tight* section can be changed locally in two opposite directions; thus it can be written as a convex combination of other two sections that have a different integral. Thus we get that the set $\Sigma(P,Q)$ is the convex hull of the integrals $\frac{1}{\mathrm{vol} Q} \int_Q \gamma(x)dx$ for which γ is a tight (piecewise linear, continuous) section. From this we conclude that $\Sigma(P,Q)$ is a polytope."

First thing to notice here is that Ziegler does not give a definition of tight section. We have searched for possible definitions and, taking into account the statement of the theorem, we have come up with the following one:

We may assume all sections γ are continuous and take values on the boundary of P. If not, we can express, for each $x \in Q$, $\gamma(x)$ as a convex combination of two continuous sections taking values in $\pi^{-1}(x) \cap \partial P$. Now, for any continuous section γ taking values on the boundary, we can find a hyperplane c such that $\gamma(x)$ maximizes c for each $x \in Q$. This c determines a coherent subdivision of Q by the map π^c. We will say that a section is tight if this induced subdivision is tight. Conversely, for any tight subdivision defined by a generic $c \in (\mathbb{R}^p)^*$, we can define a section $\gamma(x)$ as the point in $\pi^{-1}(x)$ maximizing c. This way we have a bijection between tight subdivisions and tight sections. We say that a linear form c is *generic* if the corresponding hyperplane associated to c is not parallel to any of the facets in P (i.e. c is in general position with respect to all facet defining hyperplanes).

Now, since we only have finitely many tight subdivisions (since P and Q have a finite number of faces), we only have finitely many tight sections. If a section is not tight, there is a face F in P, in the corresponding subdivision, such that $\dim \pi(F) < \dim F$. For each x such that $\gamma(x) \in F$, we can move inside $\pi^{-1}(x)$ in opposite directions towards the boundary of F, so that the corresponding subdivision uses faces of lower dimension, instead of F. We can keep doing this until reaching a section which is tight.

Now, to find the vertices of $\Sigma(P, Q)$ we use the fact that each of them corresponds to a valid inequality where equality is reached by a single point, and so they maximize a generic linear form $c \in (\mathbb{R}^p)^*$ when restricted to P. This is due to the fact that a generic linear form cannot be maximized simultaneously in two or more points, and so it will correspond to a vertex.

Ziegler also claims that if c is generic, then every fiber $\pi^{-1}(r)$ for $r \in Q$ has a unique maximal element with respect to c, so we can then define the section γ^c as before, being $\gamma^c(r)$ the point maximizing c in $\pi^{-1}(r)$ for each $r \in Q$. Here Ziegler claims that this section is unique (clear, by construction, and because the maximum is obtained in a unique point), coherent (because it induces a coherent subdivision) and tight.

This section is tight because, since c is generic, on each face the maximum is obtained in the boundary of the face, so for each face F of the subdivision in Q, $r \in F$, and $p \in \pi^{-1}(x)$ we always move (to maximize c) towards the boundary of each face of P whose intersection with $\pi^{-1}(F)$ is non empty, and the final face defined by $\gamma(F)$ ends up having the same dimension as F, so the induced subdivision is tight and thus the section.

Finally, Ziegler claims that the point corresponding to this section in $\Sigma(P, Q)$ is a vertex because it maximizes c. To see this, notice that by construction, $c \cdot \gamma^c(x) \geq c \cdot \gamma(x)$ for any section γ. Then,

$$c \cdot \left(\frac{1}{\mathrm{vol} Q} \int_Q \gamma^c(x) dx \right) = \frac{1}{\mathrm{vol} Q} \int_Q c \cdot \gamma^c(x) dx \geq \frac{1}{\mathrm{vol} Q} \int_Q c \cdot \gamma(x) dx \geq$$
$$\geq c \cdot \left(\frac{1}{\mathrm{vol} Q} \int_Q \gamma(x) dx \right)$$

So the point corresponding to γ^c is the only point reaching equality in a valid inequality, and thus it is a vertex.

Hence if we join everything, we have that each tight section defines a generic hyperplane which defines a vertex in $\Sigma(P, Q)$. Conversely, any vertex in $\Sigma(P, Q)$ is of the form $\frac{1}{\mathrm{vol} Q} \int_Q \gamma^c(x) dx$, for some generic $c \in (\mathbb{R}^p)^*$, and so γ^c is a tight section and thus defines a tight subdivision. Thus, we get a bijection between the vertices of $\Sigma(P, Q)$ and the tight subdivisions of $\pi \colon P \to Q$.

To finish the proof, we extend this correspondence between subdivisions and vertices to higher dimensional faces. The idea is that, for a given face, now the defining hyperplane c is not generic, and so more than one point could maximize c in a given fiber. Nevertheless, these points will belong to the same face, so we can define the section by choosing one of them, and induce a subdivision by including the whole face. Ziegler does this formally:

Each face has a defining hyperplane c, and so c defines a coherent subdivision of Q by the mapping π^c. Ziegler claims then that a point $\frac{1}{\mathrm{vol} Q} \int_Q \gamma$ is in the face defined by c if and only if $\gamma(Q)$ is contained in the collection of lower faces defining the subdivision. Notice that the point defined by γ is in the face defined by c iff it maximizes c, and $\gamma(Q) \subset \mathcal{F}^c$ iff $\gamma(x)$ maximizes c in each $\pi^{-1}(x)$

For the right implication, assume $\frac{1}{\mathrm{vol} Q} \int_Q \gamma$ maximizes c, and let $y = \gamma(x)$ for a given $x \in Q$. Then, y maximizes γ in $\pi^{-1}(x)$ (otherwise we could get a section γ' such that $c \cdot \int_Q \gamma' \geq c \cdot \int_Q \gamma$). Then,

$$\pi^c(y) = \left(\begin{array}{c} \pi(y) \\ cy \end{array} \right) = \left(\begin{array}{c} x \\ cy \end{array} \right),$$

so since y maximizes c, $\pi^c(y)$ is in a lower face.

For the left implication, assume $\gamma(x)$ in a lower face for each $x \in Q$. Then, $\gamma(x)$ maximizes c in each $\pi^{-1}(x)$. Hence,

$$c \cdot \frac{1}{\mathrm{vol} Q} \int_Q \gamma = \frac{1}{\mathrm{vol} Q} \int_Q c \cdot \gamma$$

is maximal, and so the point $\frac{1}{\mathrm{vol} Q} \int_Q \gamma$ is in the face defined by c.