Discrete and Algorithmic Geometry: Sheet 4

Ander Elkoroaristizabal Peleteiro & Filip Cano Córdoba & Alberto Larrauri Borroto

1. Definition 9.2 in Ziegler's *Lectures on Polytopes* constructs the linear map

$$P \xrightarrow{\pi^c} Q^c := \left\{ \begin{pmatrix} \pi(x) \\ cx \end{pmatrix} : x \in P \right\} \subset \mathbb{R}^{q+1}$$

from a projection $\pi : P \subset \mathbb{R}^p \to Q \subset \mathbb{R}^q$ and a linear function $c \in (\mathbb{R}^p)^\star$. Is it possible to give an algorithm to determine the set of lower faces $\mathcal{L}^\downarrow(Q^c)$ of Q^c from just the set of facet normals of Q, the projection π, and the linear function c, without running a convex hull algorithm on Q^c?

2. Show that

$$\int_P f(x)\,\mathrm{d}x \;=\; \mathrm{vol}(P) \cdot f(p_0)$$

for any polytope P and linear function f, where $p_0 = \frac{1}{\mathrm{vol}(P)} \int_P x\,\mathrm{d}x$ denotes the barycenter of P.

3. Complete the proof of Theorem 9.6 in Ziegler's *Lectures on Polytopes*, possibly referring to [1].

1.

It is not possible to give such an algorithm.

This is because π, c and the set of facets of Q do not determine the lower faces of Q^c. Consider $\pi : \mathbb{R}^2 \to \mathbb{R}$ that deletes the last coordinate. Consdier then $c = (0,1)$. In this case, π^c is the identity in $\mathbb{R}^2$. Since in this case $q = 1$, the interval is the only polytope the set of facet normals of q is always the same, so $q = 1$, the only relevant information is π and c, but in this case, π^c is the identity. Therefore, it such algorithm existed, the set of lower faces would be the same for all polygons, which is not true.

2.

Using the fact that f is linear and linearity of the integral:

$$\mathrm{vol}(P) f(p_0) = \mathrm{vol}(P) f\left(\frac{1}{\mathrm{vol}(P)} \int_P x\,\mathrm{d}x\right) = f\left(\int_P x\,\mathrm{d}x\right) = \int_P f(x)\,\mathrm{d}x \tag{1}$$

3.

Claim 1. $\Sigma(P,Q)$ is a convex set.

Proof. Consider two points $y_1, y_2 \in \Sigma$, and a convex combination of them $y = q_1 y_1 + q_2 y_2$. Then $y_1 = \int_Q \gamma_1$, $y_2 = \int_Q \gamma_2$ for some γ_1, γ_2 sections.

Then, by linearity of the integral: $y = \int_Q q_1\gamma_1 + q_2\gamma_2$. So we have to see that the convex combination of sections is a section. Let $\gamma \overset{\text{def}}{=} q_1\gamma_1 + q_2\gamma_2$. Indeed, by linearity of π:

$$\pi(\gamma(x)) = \pi(q_1\gamma_1(x) + q_2\gamma_2(x)) = q_1\pi(\gamma_1(x)) + q_2\pi(\gamma_2(x)) = (q_1+q_2)x = x \tag{2}$$

□

Claim 2. $\Sigma(P,Q) \subseteq \pi^{-1}(r_0)$.

Proof. We want to see that $y \in \Sigma(P,Q) \implies \pi(y) = \boldsymbol{r_0}$. Consider $y \in \Sigma(P,Q)$. Then there exists $\gamma : Q \to P$ section, such that $y = \frac{1}{\mathrm{vol}(Q)} \int_Q \gamma(x)dx$ Then:

$$\pi(y) = \pi\left(\frac{1}{\mathrm{vol}(Q)} \int_Q \gamma(x)dx\right) \tag{3}$$

Using linearity of π and of the integral, this is equal to [1]

$$\frac{1}{\mathrm{vol}(Q)} \int_Q \pi(\gamma(x))dx = \frac{1}{\mathrm{vol}(Q)} \int_Q x dx = \boldsymbol{r_0} \tag{4}$$

□

[1] This step probably needs some more explanation. Or maybe not. It's just putting coordinates.

Claim 3. $\dim(\Sigma(P,Q)) \leq p-q$.

Proof. To prove this, we only need to prove that $\dim \pi^{-1}(\boldsymbol{r_0}) = p-q$. For a section to be tight, its corresponding subdivision must be tight as well, because a non-tight subdivision cannot be the image of a section.

Indeed, since π is a linear function, $\dim(\pi^{-1}(\boldsymbol{r_0})) = \dim\ker(\pi) = p - \dim \mathrm{Im}\pi$. Since we are assuming that P and Q have full dimension, $\dim \mathrm{Im}\pi = q$. $\square$

Remark 1. A section $\gamma : Q \to P$ is uniquely defined by its image $\gamma(Q)$.

Proof. Given $x \in Q$, $\gamma(x)$ will be the only element in $\pi^{-1}(x) \cap \gamma(Q)$. This set has exactly one element because $\pi \circ \gamma = \mathrm{id}_Q$. $\square$

Definition 1. Given $S \subset \mathbb{R}^p$ we will call the *direction* $\overrightarrow{S}$ of S to the vector subspace of $\mathbb{R}^p$ spanned by all vectors of the form $x-y$ for some $x, y \in S$.

Lemma 1. *Let $v \in \mathbb{R}^p$. Then the function $f_v : P \leftarrow \mathbb{R}$ defined by $s \mapsto \max_{s+tv\in P} t$ is continuous.*

Proof. Note that f_v is indeed well defined: for any $s \in P$ the maximum t does exist.

Let $c^1x \leq c^1_0, \dots, c^kx \leq c^k_0$ be the facet-defining inequalities for P. Then P is the region of points satisfying them all.

Without loss of generality we may assume that the c^i's such that $c^iv > 0$ are $c^1, \dots, c^j$. Given $s \in P, t \in \mathbb{R}$ satisfying $s+tv \in P$, there exists an $\varepsilon > 0$ such that $s+(t+\varepsilon)v \in P$ if and only if $c^i(s+tv) < c^i_0$ for $i \in \{1, \dots, j\}$. Then, by necessity, $\max\{t \in \mathbb{R} \,|\, s+tv \in P\} = \max\{t \in \mathbb{R} \,|\, c^i(s+tv) \leq c^i_0 \;\; \forall i \in \{1,\dots,j\} = min_{1\leq i\leq j}\{\max\{t \in \mathbb{R} \,|\, c^i(s+tv) \leq c^i_0\}\}$. It is satisfied that $\max\{t \in \mathbb{R} \,|\, c^i(s+tv) \leq c^i_0\} = \frac{c^i_0 - c^is}{c^iv}$ for $1 \leq i \leq j$. In consequence $f_v(s) = \min_{1\leq i\leq j} \frac{c^i_0-c^is}{c^iv}$, and f_v is the minimum of some finitely many continuous functions. Thus, f_v is continuous. $\square$

We have the following corollary.

Corollary 1. *Given $s \in P$ and $v \in \mathbb{R}^p$ such that $s+v \in P$. There exists $U \subseteq P$, an open set (with the topology of P), $s \in U$, and $\varepsilon > 0$ such that $U + \varepsilon v \subseteq P$.*

Lemma 2. $\dim \Sigma(P,Q) \geq \dim P - \dim Q$

Proof. Still not written. This is the only lemma not yet written.

$\square$

We still don't know that $\Sigma(P,Q)$ is a polytope. Since we need to talk of its faces, we will make the following auxiliary definition.

Definition 2. Given a convex set $C \subseteq \mathbb{R}^p$, a linear function $c \in (\mathbb{R}^p)^\star$ and a scalar c_0, we say that $c(y) \leq c_0$ is a *valid inequality* of C it is satisfied in all $y \in C$.

The equality region of a valid inequality of C is a *face*.

Note that this definition is the same as the one for faces of polytopes, but in this case a face may be empty or not a polytope. With the inclusion defined partial order, the faces of a convex set have also poset structure. This way if the underlying convex set is a polytope, we recover the definition of faces for a polytope.

Let us first note that every element $c \in (\mathbb{R}^p)^\star$ defines both a face in Σ and a coherent subdivision in Q^c:

1. The face it defines is ϕ^c, given by the valid inequality $c(s) \geq \min_{y\in\Sigma} c(y)$.

2. The coherent subdivision it defines is the one given by $\mathcal{F}^c$, as in [2, Def. 9.2].

We will see that we can find a bijection between faces of Σ and coherent subdivisions of Q through the elements $c \in (\mathbb{R}^p)^\star$.

Definition 3. Let $s \in \Sigma$, then

$$\Gamma(s) \stackrel{\text{def}}{=} \left\{\gamma : Q \to P \text{ section} : \; s = \frac{1}{\mathrm{vol}(Q)} \int_Q \gamma(x)\, x\right\} \tag{5}$$

Definition 4. Given $c \in (\mathbb{R}^p)^\star$, let us define the following sets of sections:

$$\Gamma(\mathcal{F}^c) \stackrel{\text{def}}{=} \left\{\gamma \,:\, \gamma(Q) \subseteq \bigcup_{F\in\mathcal{F}^c} F\right\}, \qquad \Gamma(\phi^c) \stackrel{\text{def}}{=} \bigcup_{s\in\phi^c} \Gamma(s) \tag{6}$$

Definition 5. Given $c \in (\mathbb{R}^p)^\star$, we define the functional $\mathcal{A}^c : \{\text{sections of } \pi\} \to \mathbb{R}$ as:

$$\gamma \mapsto \mathcal{A}^c(\gamma) \stackrel{\text{def}}{=} \frac{1}{\operatorname{vol}(Q)} \int_Q c(\gamma(x))\, \mathrm{d}x \tag{7}$$

Theorem 1. *Given γ a section:*

1. $\gamma \in \Gamma(\mathcal{F}^c) \implies \mathcal{A}^c(\gamma) = \min_{\sigma \text{ section}} \mathcal{A}^c(\sigma)$ *(i.e., γ minimizes $\mathcal{A}^c$).*

2. $\gamma \notin \Gamma(\mathcal{F}^c) \implies \exists \gamma' : \mathcal{A}^c(\gamma') < \mathcal{A}^c(\gamma)$.

Proof.
$\boxed{1.}$
First let us note that for all $r \in Q$, the function c has a minimum in $\pi^{-1}(r)$. [2] Also, it is clear that for any section σ

$$c(\sigma(r)) \geq c_0(r) \stackrel{\text{def}}{=} \min_{y \in \pi^{-1}(r)} c(y), \tag{8}$$

and therefore $\mathcal{A}^c(\sigma) \geq \frac{1}{\operatorname{vol}(Q)} \int_Q c_0(x) \mathrm{d}x$. If $\gamma \in \Gamma(\mathcal{F}^c)$, we know that $\forall r \in Q$, the point $\binom{r}{c(\gamma(r))}$ is in a lower face of Q^c. If $c(\gamma(r))$ was not minimal in $\pi^{-1}(r)$, we could find $y \in \pi^{-1}(r)$ such that $c(y) < c(\gamma(r))$. But since $y \in \pi^{-1}(r)$, this would produce a point $\binom{r}{c(y)}$. This is not possible because being the point in a lower face of Q^c and having the same in the first q coordinates, it cannot be that the last coordinate is decreased.
$\boxed{2.}$
Since $\gamma \notin \Gamma(\mathcal{F}^c)$, there exists $r \in Q$ such that $\gamma(r)$ does not minimize c in $\pi^{-1}(r)$. Let $s \in \arg\min_{y \in \pi^{-1}(r)} \{c(y)\}$,
Then we can define $v \stackrel{\text{def}}{=} s - \gamma(r)$. By construction, we are in the hypothesis of corollary 1, so there exist an open neighbourhood $B \subseteq P$ of $\gamma(r)$ and $\varepsilon > 0$, such that $\gamma(r) \in B$ and $B + \varepsilon v \subseteq P$. Let us define $U \stackrel{\text{def}}{=} \gamma^{-1}(B)$. This set is open because γ is continuous and B is open. Then we define the function $f \colon \mathbb{R}^q \to \mathbb{R}$ as a continuous function with the following properties: [3]

1. For all $x \notin U$, satisfies $f(x) = 0$.
2. For all $x \in U$, satisfies $0 \leq f(x) \leq \varepsilon$.
3. Has positive integral: $\int_Q f(x) dx > 0$.

Now let us define $\gamma' \colon Q \to P$ as $\gamma'(x) \stackrel{\text{def}}{=} \gamma(x) + v f(x)$. This is indeed a section because:

1. It is continuous, because γ and f are continuous.
2. Since $\gamma(r), s \in \pi^{-1}(r)$, $v \in \ker \pi$, and therefore $\pi(\gamma'(x)) = \pi(\gamma(x)) + \pi(v f(x)) = x + f(x)\pi(v) = x$.

Let us show now that $\mathcal{A}^c(\gamma') < \mathcal{A}^c(\gamma)$.

$$\mathcal{A}^c(\gamma') = \frac{1}{\operatorname{vol}(Q)} \int_Q \left[c(\gamma(x)) + c(v f(x))\right] \mathrm{d}x = \mathcal{A}^c(\gamma') + \frac{c(v)}{\operatorname{vol}(Q)} \int_Q f(x) \mathrm{d}x \tag{9}$$

Here using that $c(v) < 0$ and property 3 in the definition of $f(x)$, we get the desired inequality. ☐

Corollary 2. *For all $c \in (\mathbb{R}^p)^\star$, the identity $\Gamma(\mathcal{F}^c) = \Gamma(\phi^c)$ is satisfied.*

Proof. From the previous theorem, we know that sections γ that minimize $\mathcal{A}^c$ are precisely those satisfying $\gamma \in \mathcal{F}^c$. For a section γ, let $s_\gamma \in \Sigma$ be the point it defines, then $\gamma \in \Gamma(\phi^c) \iff c(s_\gamma) = \min_{y \in \Sigma} c(y) \iff \gamma$ minimizes $\mathcal{A}^c$. ☐

Corollary 3. *We have a poset isomorphism between $\omega_{coh} \cup \{\emptyset\}$ and the faces of Σ* [4]*.*

Proof. The bijection is clear with corollary 2. Now let's see that it respects the partial order. Given $\mathcal{F}_1 \leq \mathcal{F}_2$. By definition of the partial order, this means that $\bigcup_{F \in \mathcal{F}_1} F \subset \bigcup_{F \in \mathcal{F}_2} F$. Therefore

$$\Gamma(\mathcal{F}_1) = \left\{ \gamma : \gamma(Q) \subseteq \bigcup_{F \in \mathcal{F}_1} F \subset \bigcup_{F \in \mathcal{F}_2} F \right\} \subset \Gamma(\mathcal{F}_2) \tag{10}$$

By corollary 2, we have $\Gamma(\phi_1) \subset \Gamma(\phi_2) \iff \phi_1 \subset \phi_2$.

☐

[2] This is because $\pi^{-1}(r)$ is a polytope and c is a linear function.
[3] This function can be constructed even to be infinitely differentiable with bump functions.
[4] As a convex set, we still have not proved that Σ is a polytope.

Notice that because of this isomorphism we know that the face-lattice of Σ has a finite number of elements. Because of definition 2 this implies that there is also a finite number of inequalities that these faces satisfy as equalities, and hence Σ is a polytope. The identification of vertices with the tight π-coherent subdivisions of Q is made through [2, Lemma 9.5], which concludes the proof.

References

[1] Louis J. Billera and Bernd Sturmfels, *Fiber polytopes.*, Ann. Math. (2) **135** (1992), no. 3, 527–549 (English).

[2] Günter M Ziegler, *Lectures on polytopes*, vol. 152, Springer Science & Business Media, 2012.