DAG
Sheet 4

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1. Definition 9.2 in Ziegler's *Lectures on Polytopes* constructs the linear map

$$P \xrightarrow{\pi^c} Q^c := \left\{ \begin{pmatrix} \pi(x) \\ cx \end{pmatrix} : x \in P \right\} \subset \mathbb{R}^{q+1}$$

from a projection $\pi : P \subset \mathbb{R}^p \to Q \subset \mathbb{R}^q$ and a linear function $c \in (\mathbb{R}^q)^\star$. Is it possible to give an algorithm to determine the set of lower faces $\mathcal{L}^\downarrow(Q^c)$ of Q^c from just the set of facet normals of Q, the projection π, and the linear function c, without running a convex hull algorithm on Q^c?

2. Show that

$$\int_P f(x)\,\mathrm{d}x = \mathrm{vol}(P) \cdot f(p_0)$$

for any polytope P, where $p_0 = \frac{1}{\mathrm{vol}(P)} \int_P x\,\mathrm{d}x$ denotes the barycenter of P.

3. Complete the proof of Theorem 9.6 in Ziegler's *Lectures on Polytopes*.

1. It is not possible, since we cannot obtain Q^c (or $\mathcal{L}^\downarrow(Q^c)$) just from Q, π and c.
For example, take $q = 1$, Q any 1-polytope (any interval), $\pi(x, y) = x$ and $c = (0, 1)$, then π^c is the identity in P, so $Q^c = P$, but we are not given P, so it's impossible to get Q^c or just $\mathcal{L}^\downarrow(Q^c)$.

2. Directly from the linearity of f and the linearity property of the integral:

$$\mathrm{vol}(P) f(p_0) = \mathrm{vol}(P) f\left(\frac{1}{\mathrm{vol}(P)} \int_P x\,\mathrm{d}x\right) = f\left(\int_P x\,\mathrm{d}x\right) = \int_P f(x)\,\mathrm{d}x$$

3. I will enumerate the parts of the sketch of Ziegler's proof which are not clear and I'll explain them:

   (i) $\Sigma(P, Q)$ is convex.

Proof. Given two points $a, b \in \Sigma(P, Q)$ and their corresponding sections γ_a, γ_b. Then $(ta + (1-t)b) = \int t\gamma_a + (1-t)\gamma_b = \int \gamma dx$ since the convex combination of 2 sections is a section (by the linearity of the integral and π). □

(ii) $\dim(\Sigma(P, Q)) \leq p - q$ with $\dim(P) = p$, $\dim(Q) = q$.

Proof. We have to see that $\pi^{(-1)}(r_0)$ has dimension p-q. Observing that, since we have supposed that the dimensions of P and Q are maximal, $\ker(\pi) = p - q$ and so we get the result. □

(iii) "Every piecewise linear section that is not tight can be changed locally in two directions; thus it can be written as a convex combination of two other sections".

Proof. □

(iv) "To detect the vertices of $\Sigma(P, Q)$, we use that they arise as the unique maxima for generic linear functions $c \in (\mathbb{R}^p)^*$. However, if c is generic, then every fiber $\pi^{-1}(r)$ for $r \in Q$ has a unique maximal element with respect to c".

Proof. The problem of this part is the definition of *generic* linear functions. In this case there's only one possibility for the definition of generic:

Definition 0.1. *A linear function c will be generic with respect to $\Gamma(P, Q)$ if c is generic with respect to P and with respect to π, that is that c is not parallel to $\pi^{(-1)}(r)$ for $r \in Q$.*

Now with this definition of generic, it is clear that every fiber $\pi^{-1}(r)$ has a unique maximal element with respect to c. It is "generic" since the set of forbidden c has area 0. □

(v) "A simple computation shows that for a continuous section $\gamma : Q \to P$, the point $\frac{1}{\mathrm{vol}(Q)} \int \gamma dx \in \Sigma(P, Q)$ lies in the face defined by c if and only if the image of the section is entirely contained in the collection of faces $F^c \subseteq L(P)$".

Proof. □