Discrete and Algorithmic Geometry: Sheet 4

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1. Definition 9.2 in Ziegler's *Lectures on Polytopes* constructs the linear map

$$P \xrightarrow{\pi^c} Q^c := \left\{ \begin{pmatrix} \pi(x) \\ cx \end{pmatrix} : x \in P \right\} \subset \mathbb{R}^{q+1}$$

from a projection $\pi : P \subset \mathbb{R}^p \to Q \subset \mathbb{R}^q$ and a linear function $c \in (\mathbb{R}^p)^\star$. Is it possible to give an algorithm to determine the set of lower faces $\mathcal{L}^\downarrow(Q^c)$ of Q^c from just the set of facet normals of Q, the projection π, and the linear function c, without running a convex hull algorithm on Q^c?

2. Show that

$$\int_P f(x)\,\mathrm{d}x = \mathrm{vol}(P) \cdot f(p_0)$$

for any polytope P and linear function f, where $p_0 = \frac{1}{\mathrm{vol}(P)} \int_P x\,\mathrm{d}x$ denotes the barycenter of P.

3. Complete the proof of Theorem 9.6 in Ziegler's *Lectures on Polytopes*, possibly referring to [1].

1.

It is not possible to give such an algorithm.

This is because π, c and the set of facets of Q do not determine the lower faces of Q^c. Consider $\pi : \mathbb{R}^2 \to \mathbb{R}$ that deletes the last coordinate. Consdier then $c = (0, 1)$. In this case, π^c is the identity in $\mathbb{R}^2$. Since in this case $q = 1$, the interval is the only polytope the set of facet normals of q is always the same, so $q = 1$, the only relevant information is π and c, but in this case, π^c is the identity. Therefore, it such algorithm existed, the set of lower faces would be the same for all polygons, which is not true.

2.

Using the fact that f is linear and linearity of the integral:

$$\mathrm{vol}(P) f(p_0) = \mathrm{vol}(P) f\left(\frac{1}{\mathrm{vol}(P)} \int_P x\,\mathrm{d}x\right) = f\left(\int_P x\,\mathrm{d}x\right) = \int_P f(x)\,\mathrm{d}x \tag{1}$$

3.

Unless stated otherwise, when we consider sets of the form $\pi^{-1}(U)$, for some $U \subseteq Q$, they will be intersected with P. That is, we will be considering $\pi \colon P \to Q$, and not its equivalent map $\pi \colon \mathbb{R}^p \to \mathbb{R}^q$.

Lemma 1. *$\Sigma(P, Q)$ is a convex set.*

Proof. Consider two points $y_1, y_2 \in \Sigma(P, Q)$, and a convex combination of them $y = q_1 y_1 + q_2 y_2$. Then $y_1 = \int_Q \gamma_1$, $y_2 = \int_Q \gamma_2$ for some γ_1, γ_2 sections.

Then, by linearity of the integral: $y = \int_Q q_1\gamma_1 + q_2\gamma_2$. So we have to see that the convex combination of sections is a section. Let $\gamma \overset{\text{def}}{=} q_1\gamma_1 + q_2\gamma_2$. Indeed, since π is affine and $q_1 + q_2 = 1$:

$$\pi(\gamma(x)) = \pi(q_1\gamma_1(x) + q_2\gamma_2(x)) = q_1\pi(\gamma_1(x)) + q_2\pi(\gamma_2(x)) = (q_1 + q_2)x = x \tag{2}$$

$\square$

Lemma 2. $\dim(\Sigma(P, Q)) \leq p - q$.

Proof. First we will prove that $\Sigma(P, Q) \subseteq \pi^{-1}(r_0)$, and then we will prove that the dimension of the fiber $\pi^{-1}(r_0)$ is $p - q$.

We want to see that $y \in \Sigma(P, Q) \implies \pi(y) = \boldsymbol{r_0}$. Consider $y \in \Sigma(P, Q)$. Then there exists $\gamma : Q \to P$ section, such that $y = \frac{1}{\mathrm{vol}(Q)} \int_Q \gamma(x)dx$. Then

$$\pi(y) = \pi\left(\frac{1}{\mathrm{vol}(Q)} \int_Q \gamma(x)dx\right) \tag{3}$$

Using linearity of π and of the integral, this is equal to

$$\frac{1}{\mathrm{vol}(Q)}\int_Q \pi(\gamma(x))dx = \frac{1}{\mathrm{vol}(Q)}\int_Q x dx = \boldsymbol{r_0} \tag{4}$$

Hence $\Sigma(P,Q) \subseteq \pi^{-1}(r_0)$.

Now we need to prove that $\dim \pi^{-1}(\boldsymbol{r_0}) = p - q$. For the sake of simplicity, we will consider now $\pi^{-1}(\boldsymbol{r_0})$ as an affine space in $\mathbb{R}^p$ (i.e., considering $\pi \colon \mathbb{R}^p \to \mathbb{R}^q$.) Indeed, since π is an affine function, it has an associated linear function $\overline{\pi}$, and $\dim(\pi^{-1}(\boldsymbol{r_0})) = \dim\ker(\overline{\pi}) = p - \dim \mathrm{Im}\overline{\pi}$. Since we are assuming that P and Q have full dimension, $\dim \mathrm{Im}\overline{\pi} = q$. $\square$

Lemma 3. *Let $v \in \mathbb{R}^p$. Then the function $f_v : P \to \mathbb{R}$ defined by $s \mapsto \max\limits_{s+tv\in P}\{t\}$ is continuous.*

Proof. Note that f_v is indeed well defined: for any $s \in P$ the maximum t does exist.

Let $\boldsymbol{c^1}(x) \le c_0^1, \dots, \boldsymbol{c^k}(x) \le c_0^k$ be the facet-defining inequalities for P, for $\boldsymbol{c^i} \in (\mathbb{R}^p)^\star$. Then P is the region of points satisfying them all.

Without loss of generality we may assume that the c^i's such that $\boldsymbol{c^i}(v) > 0$ are $\boldsymbol{c^1}, \dots, \boldsymbol{c^j}$. Given $s \in P, t \in \mathbb{R}$ satisfying $s+tv \in P$, there exists an $\varepsilon > 0$ such that $s+(t+\varepsilon)v \in P$ if and only if $\boldsymbol{c^i}(s+tv) < c_0^i$ for $i \in \{1, \dots, j\}$.

Then max in the definition of f_v can be computed with only j inequalities:

$$\max\{t \in \mathbb{R} \,:\, s + tv \in P\} = \max\left\{t \in \mathbb{R} \,:\, \boldsymbol{c^i}(s+tv) \le c_0^i \quad \forall i \in \{1,\dots,j\}\right\}$$

Since we are computing a maximum restricted to inequalities, it can be computed as

$$f_v(s) = \min_{1\le i\le j}\left\{\max\{t\in\mathbb{R} \,:\, \boldsymbol{c^i}(s+tv) \le c_0^i\}\right\}.$$

For all $i \in \{1,\dots,j\}$ it is satisfied that $\max\{t\in\mathbb{R} \,:\, \boldsymbol{c^i}(s+tv) \le c_0^i\} = \frac{c_0^i - \boldsymbol{c^i}(s)}{\boldsymbol{c^i}(v)}$ for $1 \le i \le j$. In consequence $f_v(s) = \min_{1\le i\le j}\frac{c_0^i - \boldsymbol{c^i}(s)}{\boldsymbol{c^i}(v)}$. Then f_v is continuous since it is the minimum of finitely many continuous functions. $\square$

We have the following corollary.

Corollary 1. *Given $s \in P$ and $v \in \mathbb{R}^p$ such that $s+v \in P$. There exists $U \subseteq P$, an open set (with the topology of P), $s \in U$, and $\varepsilon > 0$ such that $U + \varepsilon v \subseteq P$.*

Lemma 4. $\dim \Sigma(P,Q) \ge \dim P - \dim Q$

Proof. Notice that the interior of P is non-empty, as $\dim P = p$. Let $u \in \mathit{int}\, P$ and $r = \pi(u)$. Then the fiber $\hat{P} \stackrel{\text{def}}{=} \pi^{-1}(r)$ is a polytope of dimension $p-q$. Indeed, $\hat{P} = P \cap (u + \ker\overline{\pi}) \supset B \cap (u + \ker\overline{\pi})$ for some open ball $B \subset P$ centered in u. Trivially $\dim(B \cap (u+\ker\overline{\pi})) = \dim\ker\overline{\pi} = p-q$.

Now take γ a section. By definition $\gamma(r) \in \hat{P}$. As $\hat{P}$ is a polytope with direction $\ker\overline{\pi}$, there exist linearly independent vectors $v_1, \dots, v_{p-q} \in \ker\overline{\pi}$ such that $\gamma(r) + v_i \in \hat{P}$ and in particular $\gamma(r) + v_i \in P$ for every i. Let us denote $s \stackrel{\text{def}}{=} \frac{1}{\mathrm{vol}(Q)}\int_Q \gamma(x)\, x$. By definition $s \in \Sigma(P,Q)$. We will show that there exist $\delta_1, \dots, \delta_{p-q} > 0$ such that $s + \delta_i v_i \in \Sigma(P,Q)$ for each i.

Fix $1 \le i \le p-q$. Using *lemma* 8 we conclude that there exists an open (relative to P) set U_i containing s and $\varepsilon_i > 0$ such that $z + \varepsilon_i v_i \in P$ for every $z \in U_i$. Let us define $V_i \stackrel{\text{def}}{=} \gamma^{-1}(U_i)$. This set is open because γ is continuous and U_i is open. Then we define the function $f_i \colon \mathbb{R}^q \to \mathbb{R}$ as a continuous function with the following properties: [1]

1. For all $x \notin V_i$, satisfies $f_i(x) = 0$.

2. For all $x \in V_i$, satisfies $0 \le f_i(x) \le \varepsilon$.

3. Has positive integral: $\delta_i \stackrel{\text{def}}{=} \frac{1}{\mathrm{vol}\, Q}\int_Q f_i(x)dx > 0$.

Now let us define $\gamma_i \colon Q \to P$ as $\gamma_i(x) \stackrel{\text{def}}{=} \gamma(x) + v_i f_i(x)$. This is indeed a section because:

1. It is continuous, because γ and f_i are continuous.

2. Since $v_i \in \ker\overline{\pi}$, we compute $\pi(\gamma_i(x)) = \pi(\gamma(x)) + \overline{\pi}(v_i f_i(x)) = x + f_i(x)\pi(v_i) = x$.

Finally, we have $\frac{1}{\mathrm{vol}\, Q}\int_Q [\gamma(x) + f_i(x)v_i]\mathrm{d}x = s + \delta_i v_i \in \Sigma(P,Q)$. $\square$

[1] This function can be constructed even to be infinitely differentiable with bump functions.

Now we need to prove that $\Sigma(P,Q)$ is a compact set. In this step we will follow a different direction from the one proposed in the sketch of the proof in [2, Thm 9.6], where it is shown that $\Sigma(P,Q)$ is a polytope by means of the introduction of tight sections. Instead, we will make use of a strong result from functional analysis:

Theorem 1 (Open mapping theorem)**.** *Let X and Y be Banach spaces and $T : X \to Y$ a surjective continuous linear map. Then T is an open map.*

Consider the vector space $\mathcal{G}$ of continuous maps $\gamma : Q \to \mathbb{R}^p$, equipped with the norm $||\gamma|| = \max_{r\in Q} ||\gamma(r)||$, where $\|\cdot\|$ is the usual euclidean norm in $\mathbb{R}^p$. It is easily shown that $\mathcal{G}$ is a Banach space, i.e it is complete. Also, $\mathbb{R}^p$ equipped with its usual norm is a Banach space as well. We will define the following linear operators:

Definition 1. Let $r \in Q$. Then the evaluation operator $\mathrm{Ev}_r : \mathcal{G} \to \mathbb{R}^p$ is the one defined by $\mathrm{Ev}_r(\gamma) = \gamma(r)$.

Remark 1. The operator Ev_r is continuous. Given $\gamma_1, \gamma_2 \in \mathcal{G}$ such that $||\gamma_1 - \gamma_2|| < \varepsilon$, then by definition $||\gamma_1(r) - \gamma_2(r)|| < \varepsilon$.

Lemma 5. *The set Γ of sections is closed in $\mathcal{G}$.*

Proof. By definition, the sections are precisely the maps $\gamma \in \mathcal{G}$ such that $\gamma(r) \in \pi^{-1}(r)$ for every $r \in Q$. Let $r \in Q$. Then the set of maps $\gamma \in \mathcal{G}$ satisfying $\gamma(r) \in \pi^{-1}(r)$ is $\mathrm{Ev}_r^{-1}(\pi^{-1}(r))$. As $\pi^{-1}(r)$ is closed and Ev_r is continuous, this last set is closed in $\mathcal{G}$. The set of sections can be written as $\bigcap_{r\in Q} \mathrm{Ev}_r^{-1}(\pi^{-1}(r))$, an intersection of closed sets in $\mathcal{G}$, therefore it is closed itself. ☐

Now, let $S : \mathcal{G} \to \mathbb{R}^p$ be the linear operator defined by

$$S(\gamma) = \frac{1}{\mathrm{vol}(Q)} \int_Q \gamma(x)x$$

This new operator is continuous as well. Indeed, if $||\gamma_1 - \gamma_2|| < \varepsilon$ it takes a simple computation to show $||S(\gamma_1 - \gamma_2)|| < \varepsilon$. Our operator S is also surjective: for any $s \in \mathbb{R}^p$ take $\gamma \equiv s$ (the constant function with value s). Then $S(\gamma) = s$. Thus, using *theorem* 1 we conclude that S is an open map.

With this we have essentially proven the following.

Corollary 2. *The set $\Sigma(P,Q)$ is compact*

Proof. Notice that $\Sigma(P,Q) = S(\Gamma)$. As S is an open map and Γ is closed, $\Sigma(P,Q)$ must be closed as well.

To see that $\Sigma(P,Q)$ is bounded take $s_1, s_2 \in \Sigma(P,Q)$ and $\gamma_1, \gamma_2 \in \Gamma$ such that $S(\gamma_1) = s_1$ and $S(\gamma_2) = s_2$. Then $s_1 - s_2 = \frac{1}{\mathrm{vol}(Q)} \int_Q (\gamma_1(x) - \gamma_2(x)x \leq \mathrm{diam}(P)$, where $\mathrm{diam}(P)$ denotes the diameter of P.

Now the result holds, as $\Sigma(P,Q)$ is a closed bounded set in $\mathbb{R}^p$. ☐

We still don't know that $\Sigma(P,Q)$ is a polytope. Since we need to talk of its faces, we will make the following auxiliary definition.

Definition 2. Given a convex set $C \subseteq \mathbb{R}^p$, a linear function $\boldsymbol{c} \in (\mathbb{R}^p)^\star$ and a scalar c_0, we say that $\boldsymbol{c}(y) \leq c_0$ is a ***valid inequality*** of C it is satisfied in all $y \in C$.

The equality region of a valid inequality of C is a ***face***.

Note that this definition is the same as the one for faces of polytopes, but in this case a face may be empty or not a polytope. With the inclusion defined partial order, the faces of a convex set have also poset structure. This way if the underlying convex set is a polytope, we recover the definition of faces for a polytope.

Let us first note that every element $\boldsymbol{c} \in (\mathbb{R}^p)^\star$ defines both a face in $\Sigma(P,Q)$ and a coherent subdivision in Q^c:

1. The face it defines is ϕ^c, given by the valid inequality $\boldsymbol{c}(s) \geq \min_{y\in\Sigma(P,Q)} \boldsymbol{c}(y)$.
2. The coherent subdivision it defines is the one given by $\mathcal{F}^c$, as in [2, Def. 9.2].

We will see that we can find a bijection between faces of $\Sigma(P,Q)$ and coherent subdivisions of Q through the elements $\boldsymbol{c} \in (\mathbb{R}^p)^\star$.

Definition 3. Let $s \in \Sigma(P,Q)$, then

$$\Gamma(s) \stackrel{\text{def}}{=} \left\{ \gamma : Q \to P \text{ section} : \; s = \frac{1}{\mathrm{vol}(Q)} \int_Q \gamma(x)\,x \right\} \tag{5}$$

Definition 4. Given $\boldsymbol{c} \in (\mathbb{R}^p)^\star$, let us define the following sets of sections:

$$\Gamma(\mathcal{F}^c) \stackrel{\text{def}}{=} \left\{\gamma \,:\, \gamma(Q) \subseteq \bigcup_{F\in\mathcal{F}^c} F\right\}, \qquad \Gamma(\phi^c) \stackrel{\text{def}}{=} \bigcup_{s\in\phi^c} \Gamma(s) \tag{6}$$

Remark 2. Notice that $\Gamma(\phi^c)$ is not empty as we have proven that $\Sigma(P,Q)$ is a compact set, but $\Gamma(\mathcal{F}^c)$ may be.

Also, $\gamma \in \Gamma(\phi^c) \iff \boldsymbol{c}(s_\gamma) = \min_{y\in\Sigma(P,Q)} c(y) \iff \gamma$ minimizes $\mathcal{A}^c$. In particular, the minimum of $\mathcal{A}^c$ does exist.

Definition 5. Given $c \in (\mathbb{R}^p)^\star$, we define the functional $\mathcal{A}^c : \{\text{sections of } \pi\} \to \mathbb{R}$ as:

$$\gamma \mapsto \mathcal{A}^c(\gamma) \stackrel{\text{def}}{=} \frac{1}{\operatorname{vol}(Q)} \int_Q \boldsymbol{c}(\gamma(x))\,\mathrm{d}x \tag{7}$$

Theorem 2. *Given γ a section:*

1. $\gamma \notin \Gamma(\mathcal{F}^c) \implies \exists\gamma' : \mathcal{A}^c(\gamma') < \mathcal{A}^c(\gamma)$.
2. $\gamma \in \Gamma(\mathcal{F}^c) \implies \mathcal{A}^c(\gamma) = \min_{\sigma \text{ section}} \mathcal{A}^c(\sigma)$ *(i.e., γ minimizes $\mathcal{A}^c$).*

Proof.

$\boxed{1.}$

Since $\gamma \notin \Gamma(\mathcal{F}^c)$, there exists $r \in Q$ such that $\gamma(r)$ does not minimize $\boldsymbol{c}$ in $\pi^{-1}(r)$. Let $s \in \underset{y\in\pi^{-1}(r)}{\arg\min}\{\boldsymbol{c}(y)\}$, Then we can define $v \stackrel{\text{def}}{=} s-\gamma(r)$. Clearly $\gamma(r)+v \in P$. Thus we are in conditions to apply lemma 8. Analogously to the proof of lemma 4 we can construct a section $\gamma'(x) = \gamma(x) + f(x)v$, where $f : Q \to \mathbb{R}$ is a continuous function satisfying:

1. For some open set $V \subset Q$ such that $r \in V$, $f|_{Q\setminus V} = 0$ and $f|_V > 0$.
2. Has positive integral: $\frac{1}{\operatorname{vol} Q}\int_Q f_i(x)dx > 0$.

Let us show now that $\mathcal{A}^c(\gamma') < \mathcal{A}^c(\gamma)$.

$$\mathcal{A}^c(\gamma') = \frac{1}{\operatorname{vol}(Q)}\int_Q [\boldsymbol{c}(\gamma(x)) + \boldsymbol{c}(vf(x))]\,\mathrm{d}x = \mathcal{A}^c(\gamma) + \frac{\boldsymbol{c}(v)}{\operatorname{vol}(Q)}\int_Q f(x)\mathrm{d}x > \mathcal{A}^c(\gamma) \tag{8}$$

Here using that $\boldsymbol{c}(v) < 0$ and property 2 in the definition of $f(x)$, we get the desired inequality.

This implies that $\Gamma(\phi^c) \subseteq \Gamma(\mathcal{F}^c)$. Thus, $\Gamma(\mathcal{F}^c)$ is not empty.

$\boxed{2.}$

First let us note that for all $r \in Q$, the function c has a minimum in $\pi^{-1}(r)$. [2] Also, it is clear that for any section σ

$$\boldsymbol{c}(\sigma(r)) \geq c_0(r) \stackrel{\text{def}}{=} \min_{y\in\pi^{-1}(r)} c(y), \tag{9}$$

and therefore $\mathcal{A}^c(\sigma) \geq \frac{1}{\operatorname{vol}(Q)}\int_Q c_0(x)\mathrm{d}x$. If $\gamma \in \Gamma(\mathcal{F}^c)$, we know that $\forall r \in Q$, the point $\binom{r}{\boldsymbol{c}(\gamma(r))}$ is in a lower face of Q^c. If $\boldsymbol{c}(\gamma(r))$ was not minimal in $\pi^{-1}(r)$, we could find $y \in \pi^{-1}(r)$ such that $\boldsymbol{c}(y) < \boldsymbol{c}(\gamma(r))$. But since $y \in \pi^{-1}(r)$, this would produce a point $\binom{r}{\boldsymbol{c}(y)}$. This is not possible because being the point in a lower face of Q^c and having the same in the first q coordinates, it cannot be that the last coordinate is decreased. $\square$

Corollary 3. *For all $\boldsymbol{c} \in (\mathbb{R}^p)^\star$, the identity $\Gamma(\mathcal{F}^c) = \Gamma(\phi^c)$ is satisfied (see remark 2).*

Lemma 6. *Let $\mathcal{F}_1, \mathcal{F}_2$ be coherent subdivisions. Then $\Gamma(\mathcal{F}_1) = \Gamma(\mathcal{F}_2)$ if and only if $\mathcal{F}_1 = \mathcal{F}_2$.*

Proof. If $\mathcal{F}_1 \neq \mathcal{F}_2$, we may assume without loss of generality that there exists a maximal face $F \in \mathcal{F}_1$, such that $F \not\subseteq \bigcup_{G\in\mathcal{F}_2} G$. Let $H \stackrel{\text{def}}{=} \pi(F)$. Notice that H is a polytope. Also, being F maximal in subdivision $\mathcal{F}_1$ implies that $\dim H = q$ and in consequence $\pi(\operatorname{relint} F) = \operatorname{relint} H = \operatorname{int} H$.

The union of faces in $\mathcal{F}_2$ is a closed set, so there is a point $s \in \mathit{int}\, F$ such that $s \notin \bigcup_{G\in\mathcal{F}_2} G$. Take $\gamma \in \Gamma(\mathcal{F})$. Then $\gamma(\pi(F)) \subseteq F$. Let $r \stackrel{\text{def}}{=} \pi(s)$, $v \stackrel{\text{def}}{=} s - \gamma(r)$ and $U \subset H$ be an open set of $\mathbb{R}^q$ satisfying $s \in U$. Then by virtue of lemma 3 and the might of bump functions we can define a section $\gamma'(x) = \gamma(x) + f(x)v$, where f is a continuous function such that $f(r) = 1$ and $f(x) = 0$ in $Q \setminus U$, satisfying $\gamma'(H) \subset F$. Finally, we have $\gamma' \in \Gamma(\mathcal{F}_1) \setminus \Gamma(\mathcal{F}_2)$ and in consequence $\Gamma(\mathcal{F}_1) \neq \Gamma(\mathcal{F}_2)$. $\square$

[2] This is because $\pi^{-1}(r)$ is a polytope and $\boldsymbol{c}$ is a linear function.

Corollary 4. *We have a poset isomorphism between $\omega_{coh} \cup \{\emptyset\}$ and the faces of $\Sigma(P,Q)$* [3].

Proof. Given a π-induced coherent subdivision $\mathcal{F}$, by definition it is $\mathcal{F}^c$ for some $c \in (\mathbb{R}^p)^\star$. So we can map it to the face of $\Sigma(P,Q)$ defined by that c, namely ϕ^c. Shortly $\mathcal{F}^c \mapsto \phi^c$ and $\emptyset \mapsto \emptyset$. We will prove that this map is indeed an isomorphism.

This map is well defined (i.e., does no depend on the choice of c) because in the definition of $\Gamma(\mathcal{F}^c)$ there is no explicit use of the particular c, but only of $\mathcal{F}^c$, so $\Gamma(\mathcal{F}^c)$ is independent of the choice of c. By corollary 3, it is equal to $\Gamma(\phi^c)$, and by definition of the aforementioned set, $\phi^{c_1} = \phi^{c_2} \iff \Gamma(\phi^{c_1}) = \Gamma(\phi^{c_2})$.

It is surjective because all faces are defined by an element $c \in (\mathbb{R}^p)^\star$, except the empty face. This is why we add $\{\emptyset\}$ to the poset ω_{coh}. It is injective because of lemma 6.

Now let's see that it respects the partial order. Given $\mathcal{F}_1 \leq \mathcal{F}_2$. By definition of the partial order, this means that $\bigcup_{F\in\mathcal{F}_1} F \subseteq \bigcup_{F\in\mathcal{F}_2} F$. Therefore

$$\Gamma(\mathcal{F}_1) = \left\{ \gamma : \gamma(Q) \subseteq \bigcup_{F\in\mathcal{F}_1} F \subseteq \bigcup_{F\in\mathcal{F}_2} F \right\} \subseteq \Gamma(\mathcal{F}_2) \tag{10}$$

By corollary 3, we have $\Gamma(\phi_1) \subseteq \Gamma(\phi_2) \iff \phi_1 \subseteq \phi_2$.

□

Notice that because of this isomorphism we know that the face-lattice of $\Sigma(P,Q)$ has a finite number of elements. Because of definition 2 this implies that there is also a finite number of inequalities that these faces satisfy as equalities, and hence $\Sigma(P,Q)$ is a polytope. The identification of vertices with the tight π-coherent subdivisions of Q is made through [2, Lemma 9.5], which concludes the proof.

Additional Material

We have followed a different path than Ziegler in our proof, because we think needs less technical lemmas and details than the original one. However, in this *additional material* we want to sketch how the proof in [2, Thm. 9.4] would be filled with details. We will provide all the definitions and intermediate results needed. Some of the proofs are also completed, while some others are left incomplete.

Remark 3. A section $\gamma : Q \to P$ is uniquely defined by its image $\gamma(Q)$.

Proof. Given $x \in Q$, $\gamma(x)$ will be the only element in $\pi^{-1}(x) \cap \gamma(Q)$. This set has exactly one element because $\pi \circ \gamma = \mathrm{id}_Q$. □

Definition 6. A section $\gamma : Q \to P$ is ***tight*** [4] if:

$$\gamma(Q) = \bigcup_{F\in\mathcal{F}} F \qquad \text{for } \mathcal{F} \subseteq L(P) \text{ a subset of faces of } P \tag{11}$$

Remark 4. For a section to be tight, its corresponding subset of faces $\mathcal{F}$ must define a π-induced subdivision of Q, that is also tight.

Proof. First observe that a section is an homeomorphism when restricted to its image, because it is a continuous function, and its inverse (the restriction of π) is a linear (and thus continuous) map. This means, in particular, that γ has to respect dimensions of faces.

For a subset of faces $\mathcal{F} \in L(P)$ to define a π-induced subdivision, it must satisfy condition (ii) in [2, Def 9.1]. Since γ maintains dimensions and π is a linear projection, for all $F \in \mathcal{F}$, $\pi^{-1}(\pi(F)) = F$, so condition (ii) is always satisfied.

By the same dimensional argument, the π-induced subdivision of Q defined by γ must also be tight. □

Note that given $\mathcal{F} \in L(P)$, the only issue for $\mathcal{F}$ to define a tight section is the part of defining a section, because if it does, then it is trivially tight [5].

With this definition, there is trivially a finite number of tight sections, since each section is defined by its image, which is determined by a subset of $L(P)$, and there are a finite number of them.

Remark 5. The partial order on $\omega(P,Q)$ defined in [2, Sec. 9.1] is indeed a partial order. In particular, each π-induced subdivision $\mathcal{F}$ is determined by the union of its faces in P, $\bigcup_{F\in\mathcal{F}} F$.

[3] As a convex set, we still have not proved that $\Sigma(P,Q)$ is a polytope.

[4] Because working with tight sections without defining them seems to be too *Zieglery*.

[5] In fact, this observation suggests that the term *tight* is not well suited for this kind of sections, but for the sake of clarity, we wanted to use the same naming as in [2].

Proof. Let $\mathcal{F}$ be a π-induced subdivision and let $X = \bigcup_{F\in\mathcal{F}} F$. Let $\mathcal{G}$ be an arbitrary π-induced subdivision satisfying $\bigcup_{G\in\mathcal{G}} G = X$. Let $H_1, H_2, ..., H_l \subseteq X$ be the maximal elements from $L(P)$ contained in X. Then all the H_i must be in $\mathcal{G}$. It is clear that $\pi(H_i)$ must be the maximal faces in $\pi(\mathcal{G})$ and thus $\pi(\mathcal{G}) = L(\pi(G_1)) \cup \cdots \cup L(\pi(G_l))$, as $\pi(\mathcal{G})$ must be a polytopal complex. Finally, condition (ii) in [2, Def. 9.1] implies that the faces in $\mathcal{G}$ must be the ones satisfying $G = \pi^{-1}(J) \cup X$ for some face J of $\pi(\mathcal{G})$, so $\mathcal{G}$ is unequivocally determined by X. $\square$

Claim 1. A section γ is not tight if and only if there is a face $F \in L(P)$ such that $F \nsubseteq \gamma(Q)$ and relint $F \cap \gamma(Q) \neq \emptyset$.

Proof. We will prove the contrapositive statement, i.e. γ is tight if and only if for every face $F \in L(P)$ such that relint $F \cap \gamma(Q) \neq \emptyset$ then $F \subseteq \gamma(Q)$:

$\boxed{\Rightarrow}$ Suppose that for some $r \in Q$ and $F \in L(P)$, $\gamma(r) \in$ relint F. Then any face $G \in L(P)$ contains $\gamma(r)$ if and only if $F \leq G$. If γ is tight, then $\gamma(Q)$ is an union of faces from P, so $\gamma(Q)$ must contain a face greater than F and in consequence it contains F itself.

$\boxed{\Leftarrow}$ Suppose that $\gamma(Q)$ contains all the faces of P whose relative interior intersect. Note that for every $r \in Q$, $\gamma(r)$ belongs to the relative interior of exactly one face of $L(P)$, namely the minimal face containing $\gamma(r)$. Let us denote by $F(r)$ to such face. Then, clearly $\gamma(Q) = \bigcup_{r\in Q} F(r)$ and γ is tight. $\square$

We will denote the set of faces by $L_n(P)$.

Claim 2. Let γ be a section, $r \in Q$ and $F \in L(P)$ such that $\gamma(r) \in$ relint F. If every open set (relative to Q) $B \subseteq Q$ satisfying $r \in B$ verifies $\gamma(B) \nsubseteq F$, then there exists a face $G > F$ such that $\gamma(Q) \cap$ relint $G \neq \emptyset$.

Proof. Left incomplete. $\square$

Claim 3. If a section γ is not tight, there exist two sections γ_1, γ_2 such that γ is a convex combination of γ_1 and γ_2, and the three points of $\Sigma(P, Q)$ defined by them are different.

Proof. Left incomplete. $\square$

Claim 4. $\Sigma(P, Q)$ is a polytope.

Proof. We know by lemma 1 that it is convex. By claim 3 and the number of tight sections being finite, we know that only a finite number of points cannot be expressed as a convex combination of different elements in $\Sigma(P, Q)$. Therefore, it is the convex hull of a finite number of points. $\square$

Remark 6. Now we can say that we can restrict to sections that are piece-wise linear over a subdivision of Q, because all points of $\Sigma(P, Q)$ are convex combinations of points defined by tight sections, that are piece-wise linear over their subdivision of Q.

Definition 7. Given $S \subset \mathbb{R}^p$ we will call the ***direction*** $\overrightarrow{S}$ of S to the vector subspace of $\mathbb{R}^p$ spanned by all vectors of the form $x - y$ for some $x, y \in S$.

We will use the following results:

Theorem 3. *Let $P \subset \mathbb{R}^p$ be a polytope and $c \in (\mathbb{R}^p)^\star$ a linear function. Then c reaches its maximum over P in a non-empty face of P. In other words:* $\arg\max(c_{|_P}) \in L(P)$.

Remark 7. In the previous theorem, it is direct that if $F = \arg\max(c_{|_P})$ then $\overrightarrow{F} \subseteq \ker c$.

Definition 8. Given a polytope $P \in \mathbb{R}^P$, we will say that a linear function $c \in (\mathbb{R}^p)^\star$ is ***generic*** with respect to P if it reaches its maximum exactly in one vertex of P, i.e $\arg\max(c_{|_P}) \in V(P)$.

Corollary 5. *Let $P \in \mathbb{R}^P$ be a polytope, and $c \in (\mathbb{R}^p)^\star$ be a linear function such that for any non-vertex face $F \in L(P)$ it is satisfied $\overrightarrow{F} \nsubseteq \ker c$. Then c is generic respect to P.*

Lemma 7. *Let $P \subset \mathbb{R}^p$ be a polytope and let $A \subset \mathbb{R}^p$ be an affine set. Then the intersection $P \cap A$ is also a polytope and its non-empty faces are of the form $F \cap A$ for some $F \in L(P)$.*

Lemma 8. *Given $s \in P$ and $v \in \mathbb{R}^p$ such that $s + v \in P$. There exists $U \subseteq P$, an open set (with the topology of P), $s \in U$, and $\varepsilon > 0$ such that $U + \varepsilon v \subseteq P$.*

Proof. Left incomplete. $\square$

Definition 9. Given a two polytopes $P \subset \mathbb{R}^p$, $Q \subset \mathbb{R}^q$ and a projection between them $\pi : \mathbb{R}^p \to \mathbb{R}^q$, $\pi(P) = Q$, we will say that a linear function $c \in (\mathbb{R}^p)^\star$ is ***generic*** with respect to π over P if every face $F \in L(P)$ such that $\overrightarrow{F} \cap \overrightarrow{\ker \pi} \neq \{0\}$ satisfies $\overrightarrow{F} \cap \overrightarrow{\ker \pi} \nsubseteq \ker c$.

From now on we will keep the notation used in last definition.

Corollary 6. *If a linear function $c \in (\mathbb{R}^p)^\star$ is generic with respect to π over P then it is also generic with respect to each fiber $\pi^{-1}(r)$, $r \in Q$.*

As its name suggests, "genericness" is an "almost-sure" property:

Lemma 9. *Under the canonical identification $(\mathbb{R}^p)^\star \simeq \mathbb{R}^p$ the following sets of linear functions in $(\mathbb{R}^p)^\star$ are closed with empty interior:*

(1) The set of non-generic functions with respect to P.

(2) The set of non-generic functions with respect to π over P.

Proof. We will prove the statement for case (2). To prove it for case (1) one can proceed analogously. Note that given a set $S \subset \mathbb{R}^p$ and a function $c \in (\mathbb{R}^p)^\star$, $S \subseteq \ker c$ is equivalent to $c \in S^\perp$. Now, note that there are finitely many linear subspaces of the form $G = \overrightarrow{F} \cap \overrightarrow{\ker \pi}$ with $G \neq \{0\}$. Finally, for any of such G's, $G^\perp$ is trivially closed and it also has empty interior, as $\dim G < p$. □

This way, given $c \in (\mathbb{R}^p)^\star$ generic with respect to π over P, we can define a section γ^c as

$$\gamma^c(\boldsymbol{r}) = \underset{y \in \pi^{-1}(\boldsymbol{r})}{\arg\max}\{c(y)\} \tag{12}$$

Claim 5. The map γ^c is indeed a section.

Proof. Left incomplete. □

Claim 6. The section previously defined γ^c is tight, and its corresponding subdivision of Q is π-coherent.

Proof. Just note that $S = \{(x, y) \in \mathbb{R}^{q+1} \,|\, x \in Q\,, y = c(\gamma^c(x))\}$ is the union of the lower faces of Q^c. This implies that $(\pi^c)^{-1}(S) = \gamma^c(Q)$ is the union of faces of a coherent subdivision. □

Now with all this, we could apply theorem 2 and its corollaries to get the isomorphism. At this point, there is no need to worry about technical details of any of the Γ's being empty, since we have already proved that $\Sigma(P, Q)$ is a polytope via tight sections.

References

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