Sparse Polynomial Systems with many Positive Solutions from Bipartite Simplicial Complexes

Frédéric Bihan and Pierre-Jean Spaenlehauer

Abstract

Consider a regular triangulation of the convex-hull P of a set $\mathcal{A}$ of n points in $\mathbb{R}^d$, and a real matrix C of size $d \times n$. A version of Viro's method allows to construct from these data an unmixed polynomial system with support $\mathcal{A}$ and coefficient matrix C whose number of positive solutions is bounded from below by the number of d-simplices which are *positively decorated* by C. We show that all the d-simplices of a triangulation can be positively decorated if and only if the triangulation is *balanced*, which in turn is equivalent to the fact that its dual graph is *bipartite*. This allows us to identify, among classical families, monomial supports which admit *maximally positive systems*, *i.e.* systems all toric complex solutions of which are real and positive. These families give some evidence in favor of a conjecture due to Bihan. We also use this technique in order to construct fewnomial systems with many positive solutions. This is done by considering a simplicial complex with bipartite dual graph included in a regular triangulation of the cyclic polytope.

1 Introduction

Real solutions of multivariate polynomial systems are central objects in many areas of mathematics. Positive solutions (*i.e.* solutions all coordinates of which are real and positive) are of special interest as they contain meaningful information in several applications, *e.g.* robotics, optimization, algebraic statistics, etc. In the 70s, foundational results by Kouchnirenko [26], Khovanskii [27] and Bernshtein [2] have laid theoretical ground for the study of the algebraic structure of sparse polynomial equations in strong connection with the development of toric and tropical geometry. These breakthroughs opened the door to computational techniques for sparse elimination [13, 14, 19, 41, 43].

Let $\mathcal{A} \subset \mathbb{Z}^d$ be a finite point configuration. We consider unmixed sparse polynomial systems $f_1(X_1, \ldots, X_d) = \cdots = f_d(X_1, \ldots, X_d) = 0$ with support $\mathcal{A}$. This means that $\mathcal{A}$ coincides with the set of exponent vectors $\mathbf{a}$ of the monomials $X^{\mathbf{a}}$ appearing in each equation. Kouchnirenko's theorem [26] states that the number of toric complex solutions (no coordinate is zero) which are non-degenerate (the Jacobian matrix of the system is invertible at the solution) is bounded by the normalized volume of the convex hull of $\mathcal{A}$.

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Viro's method [44] (see also [3, 33, 42] for instance) is one of the roots of tropical geometry and has been used with great success for constructing real algebraic varieties with interesting topological types. It allows to recover under certain conditions the topological type for t close to 0 of a real algebraic variety defined by a system whose coefficients depend polynomially on a positive parameter t. Here we apply a version of Viro's method which has already been used in [40]. Given any finite configuration $\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\} \subset \mathbb{Z}^d$, where $n = |\mathcal{A}|$, an unmixed real polynomial system with support $\mathcal{A}$ can be written as $C \cdot (X^{\mathbf{a}_1}, \ldots, X^{\mathbf{a}_n})^T = 0$, where C is a real matrix of size $d \times n$ called *coefficient matrix*. Given a regular triangulation of the convex-hull of $\mathcal{A}$ associated with a height function $h : \mathcal{A} \mapsto \mathbb{R}$, we look at the deformed system $C \cdot (t^{h(\mathbf{a}_1)} x^{\mathbf{a}_1}, \ldots, t^{h(\mathbf{a}_n)} x^{\mathbf{a}_n})^T = 0$. For $t > 0$ sufficiently small, the number of complex (resp., real, positive) toric solutions of this deformed system is at least the total number of complex (resp., real, positive) toric solutions of the sub-systems obtained by truncating the initial system to the d-simplices of the triangulation. We note that as far as we are only concerned with positive solutions, this construction works in the same manner if we allow real exponent vectors *i.e.* if $\mathcal{A} \subset \mathbb{R}^d$. If the triangulation is unimodular, which means that all d-simplices have normalized volume one, then all these sub-systems are linear up to monomial changes of coordinates. Generically, a real linear system has one complex toric solution, which is in fact a real solution. Since the number of d-simplices in any unimodular triangulation of the convex-hull of $\mathcal{A}$ is equal to its normalized volume, this construction produces polynomial systems whose all toric complex solutions are real [40, Corollary 2.4].

One goal of the present paper was to analyze under which conditions this construction produces polynomial systems whose all toric complex solutions are positive. Such polynomial systems are called *maximally positive* in [3], where a conjecture about their supports has been proposed [3, Conjecture 0.6].

Main results. Consider a regular full-dimensional pure simplicial complex supported on a point configuration $\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\} \subset \mathbb{R}^d$ and a coefficient matrix C of size $d \times n$ which encodes a map $\mathcal{A} \to \mathbb{R}^d$. We call a d-simplex of this simplicial complex *positively decorated* by C if the kernel of a $d \times (d+1)$ submatrix of C corresponding to this simplex contains vectors all coordinates of which are positive. This condition can be read off from the signs of maximal minors of C. Said otherwise, the simplices that are positively decorated can be identified on the oriented matroid associated to C. Moreover, our construction produces a polynomial system whose number of positive solutions is at least the number of d-simplices of a regular triangulation of the convex-hull of $\mathcal{A}$ which are positively decorated by C. Our first observation is that any simplicial complex supported on $\mathcal{A}$ whose all d-simplices are positively decorated has a dual graph which is bipartite. This leads us to investigate three types of full-dimensional pure simplicial complexes: *balanced* simplicial complexes have the property that their set of vertices is $(d+1)$-coloriable (two adjacent vertices have different colors) [39, Section III.4]; *positively decorated* simplicial complexes are characterized by the existence of a coefficient matrix which positively decorates all their d-simplices; finally, *bipartite* simplicial complexes are characterized by the property that their dual graphs are bipartite. We shall see that balancedness implies that the complex can be positively decorated, which in turn implies that the complex is bipartite. For triangulations,

these three properties are equivalent. Consequently, if a point configuration $\mathcal{A}$ admits a regular, unimodular and balanced triangulation, then our construction produces maximally positive polynomial systems with support $\mathcal{A}$.

In order to illustrate this result, we check that some classical families of polytopes, namely order polytopes, hypersimplices, cross polytopes and alcoved polytopes, admit regular unimodular balanced triangulations and thus provide point configurations supporting maximally positive systems. As a by-product, we verify that these classes of point configurations have a basis of affine relations with coefficients in $\{-2, -1, 1, 2\}$: this gives evidence in favor of Bihan's conjecture [3, Conjecture 0.6].

Interesting computational problems arise from this analysis: if Γ is a full-dimensional pure simplicial complex in $\mathbb{R}^d$ whose dual graph is connected, deciding if it is balanced or if its dual graph is bipartite is computationally easy. However, deciding if it is positively decorable seems to be a nontrivial problem, which can be restated as deciding the existence of a realizable oriented matroid verifying conditions given by the combinatorial structure of the complex. We also show that the problem of decorating a simplicial complex can be recast as a low-rank matrix completion problem with positivity constraints.

We apply our results to the problem of constructing fewnomial systems with many positive solutions comparatively to their number of monomials. Let d be the number of variables (and equations) and $d + k + 1$ the total number of monomials of a polynomial system. As a particular case of more general bounds, Khovanskii [27] obtained an upper bound on the number of (non-degenerate) positive solutions of such a system which depends only on d and k. This bound was later improved by Bihan and Sottile [6] to some constant times $2^{\binom{k}{2}} d^k$. When $k \gg d$, taking d univariate polynomials with distinct variables provides a construction with many positive solutions, while for $d \gg k$ the record construction is due to Bihan, Rojas and Sottile [7]. We focus here in the case $k = d$, where the best construction so far is to consider a system with d quadratic univariate equations with distinct variables, yielding 2^d positive solutions. By considering a subsimplicial complex of a regular triangulation of the cyclic polytope, we construct a pure simplicial complex of dimension d on $2d + 1$ vertices whose dual graph is bipartite. Its number of d-simplices grows as $O((1 + \sqrt{2})^d / \sqrt{d})$. Consequently, if this simplicial complex is positively decorable, then there exists a system with at least that many positive solutions. For $d = 1, 3, 5$, we compute explicitly such decorations and we ask the question whether they exist for any d. In particular, for $d = 5$ this yields a system with 11 monomials and 38 positive solutions.

Related works. Configurations of points $\mathcal{A}$ that support maximally positive systems (systems such that all toric complex solutions are positive) have been characterized when $\mathcal{A}$ is the set of vertices of a simplex (see *e.g.* [4]) or when $\mathcal{A}$ is a circuit, see [3]. When $\mathcal{A}$ is any finite subset of $\mathbb{Z}$, it follows from Descartes' rule of signs that $\mathcal{A}$ should coincide with the intersection of its convex-hull with $\mathbb{Z}$. Based on these characterizations, Bihan conjectured that if $\mathcal{A} \subset \mathbb{Z}^d$ is the support of a maximally positive polynomial system, then there is a basis of affine relations for $\mathcal{A}$ with coefficients at most 2 in absolute value [3]. By [29, Lemma 7.6], this is equivalent to saying that the homogeneous toric ideal $I_{\mathcal{A}}$ associated to $\mathcal{A}$ can be written $I_{\mathcal{A}} = J : \langle X_1 \cdots X_n \rangle^{\infty}$, where J is generated by binomials with exponents at

most 2. Balanced regular triangulations have also been used in [35] in order to get lower bounds for the number of real solutions of Wronski polynomial systems. Namely, given a regular balanced triangulation of a convex polytope in $\mathbb{R}^d$, a Wronski polynomial system is an unmixed polynomial system where all polynomials have the form $\sum_{i=0}^{d} c_i \, \varphi_i(X)$ with $c_0, \ldots, c_d \in \mathbb{R}$ and $\varphi_i(X)$ is a linear combination with positive coefficients of monomials with given color i. Since the triangulation is bipartite, all its d-simplices get a sign $\pm$ so that two adjacent d-simplices have opposite signs. Soprunova and Sottile showed in [35] that under certain conditions on the polytope the absolute value of the difference between the number of positive and negative odd normalized volume d-simplices of the triangulation provides a lower bound on the number of real solutions of any Wronski polynomial system associated to this triangulation. In fact, Sottile informed us that the existence of maximally positive Wronski polynomial systems when the support admits a regular balanced unimodular triangulation follows from [35, Lemma 3.9]. Notice that our construction can be used to produce maximally positive systems which are not Wronski polynomial systems. In general, triangulations need not be balanced, but under some conditions, they admit a minimal branched balanced covering. This has been investigated by Izmestiev and Joswig [21, 22] for combinatorial 3-manifolds. The connection between the oriented matroid defined by the matrix of coefficients and the number of positive solutions has been investigated by Müller et al. in [30, Theorem 1.5], where they give a sufficient condition on this oriented matroid for a sparse system to have at most one positive solution.

Organization of the paper. Section 2 focuses on simplices and describes the construction of Viro's system and its relation with the oriented matroid associated to a given coefficient matrix C. In Section 3, the relationship between balanced, positively decorated and bipartite simplicial complexes is investigated. Section 4 focuses on unimodular and regular triangulations of classical polytopes, and we identify classes of polytopes for which there exists such a triangulation which is balanced, yielding construction of maximally positive polynomial systems. In Section 5, we focus on the cyclic polytope and we propose a construction of a bipartite subsimplicial complex with many simplices. We finish in Section 6 by relating the problem of positive decorability of a simplicial complex with two computational problems: the existence of realizable oriented matroids satisfying a specific condition, and the low-rank matrix completion problem with extra positivity constraints.

2 Positively decorated simplices

We start by focusing on systems of d equations in d variables involving $d+1$ monomials. This case corresponds to simplices and they shall serve as building blocks which will be glued to form simplicial complexes. Such systems are equivalent to linear systems up to a monomial

map and the positivity of their solution can be read off from the signs of the maximal minors of the matrix recording the coefficients of the system.

Definition 2.1. *A $d \times (d+1)$ matrix M with real entries is called* oriented *if all the values $(-1)^i \mathrm{minor}(M, i)$ are nonzero and have the same sign, where $\mathrm{minor}(M, i)$ is the determinant of the square matrix obtained by removing the i-th column.*

The terminology "oriented" follows from the fact that for $d = 2$, the signs of the minors determine orientations of the edges of a 2-dimensional simplex compatible with an orientation of the plane (see Figure 1). The following proposition is elementary, but it plays a central role in the sequel of the paper.

Proposition 2.2. *Let M be a full rank $d \times (d+1)$ matrix with real entries. Then the following statements are equivalent:*

1. *the matrix M is oriented;*
2. *for any $A \in \mathrm{GL}_n(\mathbb{R})$, $A \cdot M$ is an oriented matrix;*
3. *for any permutation matrix $P \in \mathfrak{S}_{d+1}$, $M \cdot P$ is an oriented matrix;*
4. *all the coordinates of any non-zero vector in the kernel of the matrix are non-zero and share the same sign;*
5. *there exists $i \in \{1, \ldots, d+1\}$ such that the i-th column vector of M belongs to the interior of the negative cone generated by the other column vectors of M;*
6. *for any $i \in \{1, \ldots, d+1\}$, the i-th column vector of M belongs to the interior of the negative cone generated by the other column vectors of M.*

Proof. The equivalence (1) $\Leftrightarrow$ (4) follows from Cramer's rule. (2) $\Rightarrow$ (1) and (3) $\Rightarrow$ (1) are proved directly by instanciating A and P to the identity matrix. (1) $\Rightarrow$ (2) follows from

$$\mathrm{sign}((-1)^i \mathrm{minor}(A \cdot M, i)) = \mathrm{sign}(\det(A)) \cdot \mathrm{sign}((-1)^i \mathrm{minor}(M, i)).$$

(3) $\Leftrightarrow$ (4) is a consequence of the fact that permuting the columns of M is equivalent to permuting the coordinates of the kernel vectors. Finally, the equivalence between (4), (5) and (6) is obvious once we have noticed that a positive vector $(x_1, \ldots, x_{d+1})$ belongs to the kernel of M if and only if $\mathbf{m}_i = \sum_{j \neq i} -\frac{x_j}{x_i} \mathbf{m}_j$ assuming $x_i \neq 0$, where $\mathbf{m}_1, \ldots, \mathbf{m}_{d+1}$ are the column vectors of M. $\square$

We let X denote the set of variables $\{X_1, \ldots, X_d\}$. A solution $\mathbf{v} = (v_1, \ldots, v_d)$ of a system $f_1(X) = \cdots = f_d(X) = 0$ is called *non-degenerate* if all the functions f_i are C^1 at $\mathbf{v}$ and the Jacobian matrix of $(f_1, \ldots, f_d)$ is invertible at $\mathbf{v}$. Throughout the paper, $\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\} \subset \mathbb{Z}^d$ denotes a finite point configuration, and the coordinates of these points are recorded in a $d \times n$ matrix A (we assume that an ordering of the points has been arbitrarily fixed). We let $\widetilde{A}$ denote the matrix obtained by adding a first row whose entries are all 1. For $\mathbf{a} = (a_1, \ldots, a_d) \in \mathbb{Z}^d$, the shorthand $X^{\mathbf{a}}$ stands for the monomial $X_1^{a_1} \ldots X_d^{a_d}$.

Proposition 2.3. *Let $\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_{d+1}\} \subset \mathbb{Z}^d$ and*

$$f_i(X) = \sum_{j=1}^{d+1} C_{i,j} X^{\mathbf{a}_j}, \quad 1 \le i \le d$$

be a system of d Laurent polynomials with real coefficients involving $d+1$ monomials. If $\widetilde{A}$ is invertible, then the system $f_1(X) = \cdots = f_d(X) = 0$ has one non-degenerate solution in the positive orthant if and only if the $d \times (d+1)$ matrix C recording the coefficients of the system is oriented.

Proof. To any invertible $(d+1) \times (d+1)$ real matrix S with columns $(\mathbf{s}_1, \ldots, \mathbf{s}_{d+1})$, we associate the bijection of the positive orthant

$$\mu_S : \begin{array}{ccc} \mathbb{R}_+^{d+1} & \to & \mathbb{R}_+^{d+1} \\ \mathbf{x} & \mapsto & (\mathbf{x}^{\mathbf{s}_1}, \ldots, \mathbf{x}^{\mathbf{s}_{d+1}}). \end{array}$$

Its inverse map is $\mu_{S^{-1}}$. Let $\ell_1(X_0, \ldots, X_d) = \cdots = \ell_d(X_0, \ldots, X_d) = 0$ be the linear system defined by

$$\ell_i(X_0, \ldots, X_d) = \sum_{j=0}^{d} C_{i,j+1} X_j.$$

By Cramer's rule, this system has a solution in the positive orthant if and only if the matrix C is oriented. Since $(\ell_i \circ \mu_{\widetilde{A}})(1, X_1, \ldots, X_d) = f_i(X)$, the positive solutions of $f_1(X) = \cdots = f_d(X) = 0$ are in bijection with those of $\ell_1(1, X_1, \ldots, X_d) = \cdots = \ell_d(1, X_1, \ldots, X_d) = 0$. $\quad\square$

Let $\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\} \subset \mathbb{R}^d$ be a finite point configuration, and assume that the convex-hull of $\mathcal{A}$ is a full-dimensional polytope Q. Let (Γ, ν) be a regular triangulation of the convex hull of $\mathcal{A}$ *i.e.* Γ is a triangulation and ν is a convex function, linear on each simplex of Γ, but not linear on the union of two different maximal simplices of Γ. Regular triangulations are sometimes called coherent or convex in the literature. Let C be a $d \times n$ matrix with real entries. We say that C *positively decorates* a d-simplex $\Delta = \mathrm{conv}(\mathbf{a}_{i_1}, \ldots, \mathbf{a}_{i_{d+1}}) \in \Gamma$ if the $d \times (d+1)$ submatrix of C given by its columns $\{i_1, \ldots, i_{d+1}\}$ is oriented.

Consider the following family of polynomial systems parametrized by a positive real number t:

$$f_{1,t}(X) = \cdots = f_{d,t}(X) = 0, \tag{2.1}$$

where

$$f_{i,t}(X) = \sum_{j=1}^{n} C_{ij} t^{\nu(\mathbf{a}_j)} X^{\mathbf{a}_j} \in \mathbb{R}[X_1, \ldots, X_d], \quad i = 1, \ldots, d, \quad t > 0.$$

For each positive real value of t, this system has support included in $\mathcal{A}$.

Theorem 2.4. *There exists $t_0 \in \mathbb{R}_+$ such that for all $0 < t < t_0$ the number of non-degenerate solutions of (2.1) contained in the positive orthant is bounded from below by the number of maximal simplices in Γ which are positively decorated by C.*

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Proof. Let $\Gamma_1, \ldots, \Gamma_k$ be the maximal simplices in Γ which are positively decorated by C. Let ν_ℓ be the restriction of ν to Γ_ℓ for $\ell = 1, \ldots, k$. The function ν_ℓ is affine hence there exist $\mathbf{a}_\ell = (a_{1\ell}, \ldots, a_{d\ell}) \in \mathbb{R}^d$ and $b_\ell \in \mathbb{R}$ such that $\nu_\ell(x) = \langle a_\ell, x \rangle + b_\ell$ for any $x = (x_1, \ldots, x_d) \in \Gamma_\ell$. Moreover, since ν is convex and not affine on the union of two distinct maximal simplices of Γ, setting $Y = (Y_1, \ldots, Y_d)$ and $Yt^{-\mathbf{a}_\ell} = (Y_1 t^{-a_{1\ell}}, \ldots, Y_d t^{-a_{d\ell}})$ we get

$$\frac{f_{i,t}(Yt^{-\mathbf{a}_\ell})}{t^{b_\ell}} = f_i^{(\ell)}(Y) + r_{i,t}(Y), \quad i = 1, \ldots, d,$$

where $f_i^{(\ell)}(Y) = \sum_{w_j \in \Gamma_\ell} C_{ij} Y^{w_j}$ and $r_{i,t}(Y)$ is a polynomial in $\mathbb{R}[Y_1, \ldots, Y_d]$ whose coefficients are products of real numbers by positive powers of t. Since Γ_ℓ is positively decorated by C, by Proposition 2.3 the system $f_1^{(\ell)}(Y) = \cdots = f_d^{(\ell)}(Y) = 0$ has one non-degenerate positive solution. Let K be a compact set in the positive orthant which contains all the non-degenerate positive solutions of the systems $f_1^{(\ell)}(Y) = \cdots = f_d^{(\ell)}(Y) = 0$ for $\ell = 1, \ldots, k$. If $t > 0$ is small enough, the sets $t^{-\mathbf{a}_\ell} \cdot K = \{(y_1 t^{-a_{1\ell}}, \ldots, y_d t^{-a_{n\ell}}) \, | \, (y_1, \ldots, y_d) \in K\}$, $\ell = 1, \ldots, k$, are pairwise disjoint and each one contains at least one non-degenerate positive solution of the system (2.1). $\square$

Recall that Q is the convex-hull of $\mathcal{A}$ and that $\mathrm{Vol}(\cdot)$ stands for the Euclidean volume of $\mathbb{R}^d$ multiplied by $d!$. The triangulation Γ is called *unimodular* if A has integer entries and any maximal simplex $\Gamma_i \in \Gamma$ verifies $\mathrm{Vol}(\Gamma_i) = 1$. Sturmfels [40] showed that if A has integer entries and Γ is unimodular then for $t > 0$ small enough the system (2.1) has exactly $\mathrm{Vol}(Q)$ non-degenerate solutions with non-zero real coordinates, and no other solution with non-zero complex coordinates. The following proposition shows that if moreover the triangulation is positively decorated, then all these solutions are positive. We stress that the existence of maximally positive systems for monomial supports admitting a regular, unimodular and regular triangulation was already proved in [35, Lemma 3.9] using Wronski systems.

Proposition 2.5. *If A is a matrix with integer entries, Γ is a unimodular regular triangulation and all its d-simplices are positively decorated by C, then for $t > 0$ small enough the system (2.1) has exactly $\mathrm{Vol}(Q)$ non-degenerate positive solutions, and no other solution with non-zero complex coordinates.*

Proof. By Theorem 2.4, the system (2.1) has at least $\mathrm{Vol}(Q)$ non-degenerate solutions in the positive orthant for $t > 0$ small enough. On the other hand, the system (2.1) has at most $\mathrm{Vol}(Q)$ non-degenerate solutions with non-zero complex coordinates by Kouchnirenko Theorem [26]. $\square$

Polynomial systems whose all non-degenerate solutions with non-zero complex coordinates are contained in the positive orthant are called *maximally positive* in [3]. We shall put a special focus on classical polytopes admitting maximally positive polynomial systems in Section 4.

3 Bipartite dual graphs and balanced triangulations

The aim of this section is to show how Theorem 2.4 may be used to construct systems of polynomials with prescribed support and many positive real solutions. Throughout this paper, by simplicial complex, we always mean a full-dimensionsal pure geometric simplicial complex embedded in $\mathbb{R}^d$, see [28, Definition 2.3.5].

Definition 3.1. *Let $\mathcal{A} \subset \mathbb{R}^d$ be a finite configuration of points, represented by a $d \times n$ matrix A. A* positively decorated simplicial complex *supported on $\mathcal{A}$ is a pair (Γ, C), where Γ is a simplicial complex whose vertex set is a subset of $\mathcal{A}$ and C is a $d \times n$ matrix such that every submatrix of size $d \times (d+1)$ corresponding to a d-simplex in Γ is an oriented matrix.*

Throughout this paper, we represent a pure d-dimensional simplicial complex as a finite set $\{\tau_1, \ldots, \tau_k\}$, where $\tau_i \subset \{1, \ldots, n\}$ has cardinality $d+1$. For $\tau \in \Gamma$ a d-simplex and C a coefficient matrix associated to the point configuration $\mathcal{A}$, we let C_τ denote the $d \times (d+1)$ submatrix of C whose columns correspond to the $d+1$ vertices in τ. Let τ be a d-simplex in $\mathbb{R}^d$ with vertices $\mathbf{a}_1, \ldots, \mathbf{a}_{d+1}$, and let A be the corresponding $d \times (d+1)$ matrix. Let $\mathbf{a}$ be any point in the interior of τ and let D_τ be the $d \times (d+1)$ matrix with columns $\mathbf{a}_1 - \mathbf{a}, \mathbf{a}_2 - \mathbf{a}, \ldots, \mathbf{a}_{d+1} - \mathbf{a}$. Clearly, the oriented matroid defined by D_τ does not depend on the choice of $\mathbf{a}$. We say that two matrices of the same size define the same (resp., opposite) oriented matroid if they have the same bases and two bases corresponding to the same set of columns have determinant of the same (resp., opposite) sign.

Lemma 3.2. *The matrix D_τ is oriented. Therefore, a $d \times (d+1)$ matrix C_τ positively decorates τ if and only if either C_τ and D_τ define the same oriented matroid, or C_τ and D_τ define opposite oriented matroids.*

Proof. Let $A(i)$ be the matrix obtained by removing the i-th column from the matrix with columns (in this order) $\mathbf{a}_1 - \mathbf{a}_i, \mathbf{a}_2 - \mathbf{a}_i, \ldots, \mathbf{a}_{d+1} - \mathbf{a}_i$. Clearly, $\mathrm{minor}(D_\tau, i) \cdot \det A(i) > 0$. On the other hand, we compute that $(-1)^{i+1} \det A(i)$ coincide for $i = 1, \ldots, d+1$ with the determinant of the $(d+1) \times (d+1)$ matrix $\widetilde{A}$ obtained by adding to A a first row of ones. Thus, all $(-1)^i \cdot \mathrm{minor}(D_\tau, i)$ have the same sign, which means that D_τ is oriented. It follows that C_τ positively decorates τ if and only if either $\mathrm{minor}(C_\tau, i) \cdot \mathrm{minor}(D_\tau, i) > 0$ for all i, or $\mathrm{minor}(C_\tau, i) \cdot \mathrm{minor}(D_\tau, i) < 0$ for all i. $\qquad\square$

Lemma 3.3. *Let τ and τ' be two n-simplices in $\mathbb{R}^n$ with a common facet. Assume that the simplicial complex $\{\tau, \tau'\}$ is positively decorated by a matrix C. If C_τ and D_τ define the same oriented matroid, then $C_{\tau'}$ and $D_{\tau'}$ define the opposite oriented matroids.*

Proof. Due to Proposition 2.2, we may permute simultaneously the columns of A and C so that τ corresponds to the columns $1, \ldots, d, d+1$ and τ' corresponds to the columns $1, \ldots, d, d+2$. Thus, the submatrix C_τ given by the columns $1, \ldots, d, d+1$ of C and the submatrix $C_{\tau'}$ given by the columns $1, \ldots, d, d+2$ of C are oriented.

Choose two points $\mathbf{a}$ and $\mathbf{a}'$ in the interior of τ and τ' respectively which are symmetric with respect to the common facet with vertices $\mathbf{a}_1, \ldots, \mathbf{a}_d$. Then, $\det(\mathbf{a}_1 - \mathbf{a}, \mathbf{a}_2 - \mathbf{a}, \ldots, \mathbf{a}_d - \mathbf{a})$

and $\det(\mathbf{a}_1 - \mathbf{a}', \mathbf{a}_2 - \mathbf{a}', \ldots, \mathbf{a}_d - \mathbf{a}')$ have opposite signs (a hyperplane symmetry has negative determinant). This means that $\mathrm{minor}(D_\tau, d+1)$ and $\mathrm{minor}(D_{\tau'}, d+1)$ have opposite signs. On the other hand, $\mathrm{minor}(C_\tau, d+1)$ and $\mathrm{minor}(C_{\tau'}, d+1)$ are equal. It follows that if C_τ and D_τ define the same oriented matroid, then $C_{\tau'}$ and $D_{\tau'}$ define the opposite oriented matroids. $\qquad\square$

The *dual graph* of a pure simplicial complex is the adjacency graph of its simplices of maximal dimension.

Proposition 3.4. *Let Γ be a pure simplicial complex of dimension d in $\mathbb{R}^d$. If there exists a matrix C such that the pair (Γ, C) is a positively decorated simplicial complex, then the dual graph of Γ is bipartite.*

Proof. This is a straightforward consequence of Lemma 3.3. In fact, the proofs of Lemma 3.3 and Lemma 3.2 show how one can associate a sign $s(\tau)$ to each d-simplex τ of Γ so that any two adjacent (with a common facet) such simplices have opposite signs. Let $\widetilde{A}$ denote the matrix obtained by adding a first row of 1 to A, and let $\widetilde{A}_\tau$ be the square submatrix corresponding to a d-simplex τ of Γ. Let C_τ be the submatrix of C corresponding to τ. Since (Γ, C) is positively decorated, all $(-1)^{i+1}\mathrm{minor}(C_\tau, i)$ have the same sign $s(C_\tau) \in \{\pm 1\}$. We define then $s(\tau)$ as the sign of the product $\det(\widetilde{A}_\tau) \cdot s(C_\tau)$. If we denote by $\widetilde{C}_\tau$ the matrix obtained by adding a first row of 1 to C_τ, we immediately obtain that $s(\tau)$ is the sign of $\det(\widetilde{A}_\tau) \cdot \det(\widetilde{C}_\tau)$. $\qquad\square$

A large class of simplicial complexes with bipartite dual graphs is provided by *balanced* simplicial complexes.

Definition 3.5. *[39, Section III.4] A $(d+1)$-coloration of a simplicial complex Γ supported on $\mathcal{A}$ is a map $\gamma : \mathcal{A} \to \{1, \ldots, d+1\}$ (or more generally a map from $\mathcal{A}$ to a set with $d+1$ elements) such that $\gamma(\mathbf{a}_1) \neq \gamma(\mathbf{a}_2)$ for any edge (a_1, a_2) of γ. A d-dimensional simplicial complex Γ supported on $\mathcal{A}$ is called balanced if there exists such a coloration.*

Such colorations are sometimes called *foldings* since they can extended to a map from Γ to the d-dimensional standard simplex which is linear on each d-simplex of Γ. Similarly, balanced triangulations are sometimes referred to as *foldable* triangulations, see *e.g.* [25] and references within.

Proposition 3.6. *Let $\mathbf{e}_i$ be the i-th canonical basis vector of $\mathbb{R}^d$ and $\mathbf{e}_{d+1}$ be the vector $(-1, \ldots, -1)$. Let Γ be a simplicial complex supported on $\mathcal{A} = \{\mathbf{a}_i\}_{1 \leq i \leq |\mathcal{A}|}$. If Γ is balanced and $\gamma : \mathcal{A} \to \{1, \ldots, d+1\}$ is a $(d+1)$-coloring of Γ, then the matrix C with columns $(\mathbf{e}_{\gamma(\mathbf{a}_i)})_{1 \leq i \leq |\mathcal{A}|}$ positively decorates Γ.*

Proof. By construction, every $d \times (d+1)$ submatrix of C corresponding to a d-simplex of Γ is a column permutation of the $d \times (d+1)$ matrix with columns $(\mathbf{e}_1, \ldots, \mathbf{e}_{d+1})$. This latter matrix is oriented and hence the statement follows from Proposition 2.2. $\qquad\square$
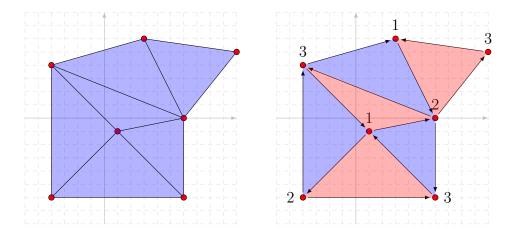
Figure 1: The balanced simplicial complex from Example 3.8, a 3-coloration of its 1-skeleton, and the induced orientations of its edges.

A pure simplicial complex Γ is called *locally strongly connected* if the dual graph of the star of any vertex is connected. By [23, Proposition 6] and [23, Corollary 11], a locally strongly connected and simply-connected complex Γ on a finite set $\mathcal{A}$ is balanced if and only if its dual graph is bipartite, see also [21, Theorem 5]. It is worth noting that any triangulation of $\mathcal{A}$ (*i.e.* a triangulation of the convex-hull of $\mathcal{A}$ with vertices in $\mathcal{A}$) is locally strongly connected and simply connected.

Theorem 3.7. *Assume that a finite full-dimensional point configuration $\mathcal{A}$ in $\mathbb{R}^d$ admits a regular triangulation Γ which is balanced, or equivalently, whose dual graph is bipartite. Then there exists a polynomial system with support $\mathcal{A}$ whose number of positive solutions is at least the number of d-simplices of Γ. If furthermore $\mathcal{A} \subset \mathbb{Z}^d$ and the triangulation Γ is unimodular, then there exists a polynomial system with support $\mathcal{A}$ which is maximally positive.*

Proof. This is a direct consequence of Theorem 2.4, Proposition 3.6 and Kouchnirenko's theorem [26]. $\qquad\square$

We finish this section by an explicit example of a polynomial system with prescribed number of positive solutions obtained using this construction.

Example 3.8. *Let $d = 2$, $\mathcal{A} = (XY^{-1}, X^{-4}Y^{-6}, X^{-4}Y^4, X^6, X^3Y^6, X^{10}Y^5, X^6Y^{-6})$, $\Gamma = \{\{1,2,3\}, \{1,3,4\}, \{3,4,5\}, \{4,5,6\}, \{1,2,7\}, \{1,4,7\}\}$. Choosing heights*

$$
\begin{array}{lll}
\nu(1,-1) = 0 & \nu(-4,4) = 0 & \nu(10,5) = 10 \\
\nu(-4,-6) = 0 & \nu(6,0) = 3 & \nu(6,-6) = 2 \\
& \nu(3,6) = 5 &
\end{array}
$$

provides us with a regular triangulation of $\mathcal{A}$ which has the balanced subsimplicial complex described in Figure 1. Applying the construction of Theorem 2.4 and Proposition 3.6, we obtain the following system, depending on a parameter t:
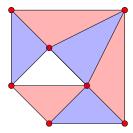
10

Figure 2: A two-dimensional simplicial complex whose dual graph is bipartite but which is not balanced. The white triangle is not a 2-simplex of the complex.

$$
\begin{aligned}
f_1 &= XY^{-1} - X^{-4}Y^4 + t^5 X^3 Y^6 - t^{10} X^{10} Y^5 - t^2 X^6 Y^{-6} \\
f_2 &= X^{-4}Y^{-6} - X^{-4}Y^4 + t^3 X^6 - t^{10} X^{10} Y^5 - t^2 X^6 Y^{-6}
\end{aligned}
$$

which has at least six solutions in the positive orthant for t sufficiently small. Setting $t = 1/1000$ and using Gröbner bases library FGb [16] and the real solver rs_isolate_sys [34] in Maple confirms that this system has indeed six positive solutions.

At this point, we would like to recapitulate the relationship between the properties of simplicial complexes studied in this section. As discussed above, balanced simplicial complexes are always positively decorable (Proposition 3.6) and the dual graph of positively decorable simplicial complexes is necessarily bipartite (Proposition 3.4). In summary:

$$\text{balanced} \implies \text{positively decorable} \implies \text{bipartite}.$$

By results of Joswig [23], for locally strongly connected simplicial complexes which are simply connected, these three properties are equivalent. However, it is not the case for simplicial complexes which do not verify these assumptions as Figure 2 gives an example of a simplicial complex whose dual graph is bipartite but which is not balanced. The reader can verify that it is positively decorable, hence positive decorability is not equivalent to balancedness.

We do not know an example of a pure simplicial complex of dimension d embedded in $\mathbb{R}^d$ whose dual graph is bipartite but which is not positively decorable, and we therefore ask the following question.

Question 3.9. *Is a pure full-dimensional simplicial complex positively decorable if and only if its dual graph is bipartite? If not, construct a counterexample.*

4   Polytopes and maximally positive systems

Next, we turn our attention to finite sets which are supports of maximally positive systems. We recall that maximally positive systems have all their toric complex solutions in the positive orthant.

4.1   Order polytopes

Here we recall the description given in [35], which is itself based on [37]. Let P be any finite partially ordered set, a *poset* for short, on n elements. A *chain* of P is a subset which is totally ordered. Two subsets A, B of P are called incomparable if for all $(a, b) \in A \times B$, the elements a and b are incomparable. The *order polytope* $\mathcal{O}(P)$ of P is the set of points $y \in [0, 1]^P$ such that $y_a \leq y_b$ whenever $a \leq b$ in P.

Example 4.1.   *1. If all elements of P are incomparable, then $\mathcal{O}(P) = [0, 1]^P$.*

    2. If P is a totally ordered set with d elements, then $\mathcal{O}(P)$ is a primitive d-simplex. For instance, if $P = \{1, 2, \ldots, n\}$ with usual increasing order, then $\mathcal{O}(P)$ is the unit simplex with vertices the origin and $\sum_{i=k}^d \mathbf{e}_i$ for $k = 1, \ldots, d$, where $\mathbf{e}_i$ stands for the i-th vector of the canonical basis.

    3. If $P = \{1, 2, 3\}$ with order given by $1 \leq 2$ and $1 \leq 3$, then $\mathcal{O}(P)$ is the convex-hull of the set of points $(0, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, $(0, 1, 1)$, $(1, 1, 1)$.

    4. If P is a disjoint union of incomparable chains of lenghts $d_1, \ldots, d_k$, then the order polytope is isomorphic to a Cartesian product of unit simplices of dimensions $d_1, \ldots, d_k$.

    5. More generally, if P is a disjoint union of incomparable sub-posets $P_1, \ldots, P_k$, then $\mathcal{O}(P) \simeq \prod_{i=1}^k \mathcal{O}(P)_i$.

A *linear extension* of P is an order-preserving bijection from P to $[d] = \{1, 2, \ldots, d\}$. To each linear extension λ of P, we associate an unimodular d-dimensional simplex $\Gamma_\lambda \subset \mathcal{O}(P)$ defined by $0 \leq y_{a_1} \leq \cdots \leq y_{a_d} \leq 1$, where $a_i = \lambda^{-1}(i)$. The vertices of Γ_λ are $(0, \ldots, 0)$ and $\sum_{i=k}^d \mathbf{e}_{\lambda^{-1}(i)}$ for $k = 1, \ldots, d$, where $(\mathbf{e}_a)_{a \in P}$ stands for the canonical basis of $\mathbb{R}^P$. The simplices Γ_λ are the d-dimensional simplices of a triangulation $\Gamma(P)$ of $\mathcal{O}(P)$ called *canonical triangulation*. Two d-dimensional simplices Γ_λ and $\Gamma_{\lambda'}$ in $\Gamma(P)$ have a common facet if and only if there is transposition τ of $[d]$ such that $\lambda' = \tau \circ \lambda$. Fixing a linear extension of P identifies each linear extension of P with a permutation of $[d]$, where the fixed linear extension is identified with the identity permutation. The *sign* of a simplex Γ_λ is then defined as the sign of the corresponding permutation. Note that this is defined up to the choice of a fixed linear extension, another choice eventually changes simultaneously all signs. Thus, the adjacency graph of d-dimensional simplices of $\Gamma(P)$ is bipartite. It follows then from [23, Corollary 11] that $\Gamma(P)$ is balanced. In fact, this can be shown directly by noting that the map $y \mapsto |y|$, (where $|y|$ is the number of non-zero coordinates of y) restricts to a $(d+1)$-coloration (with values in $\{0, 1, \ldots, d\}$) on $\{0, 1\}^P$ giving different values to a pair of adjacent vertices of $\Gamma(P)$. It turns out that the canonical triangulation $\Gamma(P)$ is also regular. A convex function certifying the convexity of $\Gamma(P)$ is given by $\nu(y) = |y|^2$ [35, Lemma 4.6]. We now summarize these properties of $\Gamma(P)$.

Proposition 4.2. *[35] The canonical triangulation of $\mathcal{O}(P)$ is regular, unimodular, and balanced.*

Theorem 4.3. *For any poset P, there exists a real polynomial system with Newton polytope $\mathcal{O}(P)$ which is maximally positive.*

Proof. This follows readily from Theorem 3.7 and Proposition 4.2. $\square$

We can be more explicit. As we already saw, the map $(y_a)_{a \in P} \in \mathcal{O}(P) \cap \mathbb{Z}^P \mapsto |y|$ is a $(d + 1)$-coloring map onto $\{0, 1, \ldots, d\}$ giving distinct values to adjacent vertices of $\Gamma(P)$. Then, as shown in the proof of Proposition 3.6, the triangulation $\Gamma(P)$ is decorated by the matrix C with column vector $-(e_1 + \cdots + e_d)$ corresponding to $(0, \ldots, 0)$ and column vector $e_{|w|}$ for any other vertex w of $\Gamma(P)$. We now use the convex function $y \mapsto |y|^2$, which shows that $\Gamma(P)$ is regular, to get the following Viro polynomial system.

$$\sum_{w \in \mathrm{Vert}(\Gamma) \setminus \{(0,\ldots,0)\}} t^{|w|^2} e_{|w|} X^w - (e_1 + \cdots + e_d) = 0. \tag{4.1}$$

By Proposition 2.5, for $t > 0$ small enough the system (4.1) is maximally positive. For instance, if $P = \{1, 2, 3\}$ with partial order defined by $1 \leq 2$, then $\mathcal{O}(P)$ is the convex-hull of the points $(0, 0, 0)$, $(0, 1, 0)$, $(1, 1, 0)$, $(0, 0, 1)$, $(0, 1, 1)$, $(1, 1, 1)$ (a prism) and (4.1) may be written as $t(x_2 + x_3) = t^4(x_1 x_2 + x_2 x_3) = t^9 x_1 x_2 x_3 = 1$.

Systems of multidegree $(1, \ldots, 1)$ are a special case of Theorem 4.3: their support is the order polytope of a disjoint union of incomparable chains (see Example 4.1, item 4). This implies the following statement.

Corollary 4.4. *Let d be any positive integer and $(d_1, \ldots, d_k)$ be any partition of d into positive integers. Set $X = (X_1, \ldots, X_k)$, where $X_i = (X_{ij})_{1 \leq j \leq d_i}$ for $i = 1, \ldots, k$. Among multilinear (in other words multidegree $(1, \ldots, 1)$) polynomial systems in $X = (X_1, \ldots, X_k)$, there exists a maximally polynomial system.*

There is another possible construction of maximally positive multilinear systems, which uses strictly totally positive matrices (*i.e.* matrices whose all minors of any size is positive). Note that such matrices exist [1, Thm 2.7]. The interest of the following proposition is that it gives a direct construction of maximally positive multilinear systems which does not depend on a parameter t. However, it does not seem to generalize easily to multi-homogeneous systems, while the construction above does (we give a proof of this fact at the end of this section).

Proposition 4.5. *Let d be any positive integer and $d_1 + \cdots + d_k = d$ be a partition of d into positive integers. Then there exist strictly totally positive matrices $(T^{(1)}, \ldots, T^{(k)})$ of respective dimensions $(d_1 + 1) \times d, \ldots, (d_k + 1) \times d$ such that all toric complex solutions of the system $f_1(X) = \cdots = f_d(X) = 0$ lie in the positive orthant, where*

$$f_i = \left((-1)^{d_1+1} T^{(1)}_{d_1+1,i} + \sum_{j=1}^{d_1} (-1)^j T^{(1)}_{j,i} X_{1j} \right) \times \cdots \times \left((-1)^{d_k+1} T^{(k)}_{d_k+1,i} + \sum_{j=1}^{d_k} (-1)^j T^{(k)}_{j,i} X_{kj} \right).$$

Proof. By Kouchnirenko's theorem, the number of complex toric non-degenerate solutions is bounded by the multinomial coefficient $\binom{d}{d_1,\ldots,d_k}$. For any partition of the set $\{1,\ldots,d\}$ into k parts $E_1 \cup \cdots \cup E_k$ of respective sizes $n_1,\ldots,n_k$, we associate the linear system $\ell_1(X) = \cdots = \ell_d(X) = 0$, where

$$\ell_i(X) = (-1)^{d_u+1}T^{(u)}_{d_u+1,i} + \sum_{j=1}^{d_u}(-1)^j T^{(u)}_{j,i}X_{1j}, \text{ where } u \in \mathbb{N} \text{ is such that } i \in E_u.$$

By construction, $\ell_i(X)$ divides $f_i(X)$, hence a solution of the linear system is a solution of $f_1(X) = \cdots = f_d(X) = 0$. Next, note that the condition that the matrices T are totally positive imply that the linear system $\ell_1(X) = \cdots = \ell_d(X) = 0$ has a unique solution in the positive orthant. There are $\binom{d}{d_1,\ldots,d_k}$ possible partitions of $\{1,\ldots,d\}$ into k parts $E_1\cup\cdots\cup E_k$ of respective sizes $d_1,\ldots,d_k$. Each of them yields one solution in the positive orthant. To finish the proof, we prove that there exist $(T^{(1)},\ldots,T^{(k)})$ such that all these solutions are distinct. This is done by noticing that the set of t-uples of strictly totally positive matrices is a non-empty open subset for the Euclidean topology of t-uples of matrices with real entries, while the set of such tuples of matrices leading to coalescing solutions is Zariski closed and hence has Lebesgue measure 0. $\square$

The vertices of $\mathcal{O}(P)$ are characteristic functions of upper order ideals of P. The set of such upper order ideals ordered by inclusion is a distributive lattice. The toric ideal of the set of integer points of the order polytope $\mathcal{O}(P)$ is generated by binomials $x_J x_K - x_{J\wedge K}x_{J\vee K}$ over all incomparable upper order ideals J, K of P, where $J \wedge K = J\cap K$ and $J \vee K = J\cup K$ [18]. These binomials are homogeneous binomials with exponents at most 2 (in fact at most 1). Thus Bihan's conjecture (see the introduction) holds true for order polytopes.

We finish this section by reporting on other classical families of polytopes which admit unimodular balanced regular triangulations. We stress that in all the following cases, Bihan's conjecture [3] holds.

Cross polytopes. The d-dimensional cross polytope is the subset of $\mathbb{R}^d$ defined by points verifying $|x_1| + \cdots + |x_d| \leq 1$. It has normalized volume 2^d and has a regular unimodular triangulation obtained by slicing it along the coordinate hyperplanes $x_i = 0$ for $i \in \{1,\ldots,d\}$. Associating a sign to each of the simplex of the triangulation counting the parity of the number of negative coordinates provides a 2-coloring of its dual graph. By Proposition 3.6 this triangulation is balanced.

Products of balanced triangulations. If $\mathcal{A}_1$, $\mathcal{A}_2$ are two point configurations which admit regular balanced unimodular triangulations, then the product of these triangulations yields a regular balanced subdivision of the convex hull of $\mathcal{A}_1 \times \mathcal{A}_2$ into products of simplices. Joswig and Witte show in [24] that this subdivision can be refined into a regular balanced unimodular triangulation. Consequently, if $\mathcal{A}_1$ and $\mathcal{A}_2$ admit regular balanced and unimodular triangulation, then so does $\mathcal{A}_1 \times \mathcal{A}_2$.

Joins of balanced triangulations. If $P_1 \subset \mathbb{R}^{d_1}$, $P_2 \subset \mathbb{R}^{d_2}$ are two full-dimensional polytopes admitting regular unimodular balanced triangulations, then the natural triangulation of the join $P_1 \star P_2 \subset \mathbb{R}^{d_1+d_2+1}$ by joins $\sigma_1 \star \sigma_2$ of full-dimensional simplices in the triangulations of P_1 and P_2 is regular and unimodular (see [17, Section 2.3.2]). The fact that this triangulation is balanced can be seen by coloring the vertices of the triangulation of P_1 by $\{0, \ldots, d_1\}$ and the vertices of the triangulation of P_2 by $\{d_1 + 1, \ldots, d_1 + d_2 + 2\}$.

Alcoved polytopes. Alcoved polytopes are polytopes whose codimension 1 faces lie on hyperplanes of the form $x_i - x_j = \ell$, with $\ell \in \mathbb{N}$. Lam and Postnikov [31] showed that such polytopes admit a natural unimodular triangulation compatible with the subdivision of $\mathbb{R}^d$ given by the complements of all the hyperplanes of the form $x_i - x_j = \ell$, for $\ell \in \mathbb{N}$. The fact that this triangulation of alcoved polytopes is regular and balanced is a special case of Lemma 4.6 below.

Let $\{b_1, \ldots, b_s\} \subset \mathbb{Z}^d$ be a collection of vectors. This induces an infinite arrangement $\mathcal{H}$ of hyperplanes $\{x \in \mathbb{R}^d \mid \langle b_i, x \rangle = k\}$ for all $i = 1, \ldots, s$ and $k \in \mathbb{Z}$. Let Q be a d-dimensional lattice polytope in $\mathbb{R}^d$. Assume that each $(d - 1)$-dimensional face of Q is contained in some hyperplane of $\mathcal{H}$. Consider the subdivision Γ of Q obtained by slicing Q along the hyperplanes of $\mathcal{H}$.

Lemma 4.6. *The subdivision Γ is regular and its dual graph is bipartite.*

Proof. The regularity of Γ is obtained by considering the restriction to the vertices of Γ of $f(x) = \sum_{i=1}^{s} \langle b_i, x \rangle^2$ (see [17, Theorem 2.4]). It remains to show that the dual graph of Γ is bipartite. In fact we show that the dual graph of the subdivision of $\mathbb{R}^n$ obtained by slicing along the hyperplanes of $\mathcal{H}$ is bipartite. For any given $i = 1, \ldots, s$, associate a sign to each connected component of

$$\mathbb{R}^d \setminus \cup_{k \in \mathbb{Z}}\{x \in \mathbb{R}^d \mid \langle b_i, x \rangle = k\}$$

so that adjacent components get opposite signs. Any connected component of $\mathbb{R}^d \setminus \mathcal{H}$ is a common intersection of such connected components for $i = 1, \ldots, s$, and we equip it with the product of the corresponding signs. Clearly, two adjacents components of $\mathbb{R}^d \setminus \mathcal{H}$ have opposite signs. $\square$

The hypersimplex. For $d, k \in \mathbb{N}$, $k \leq d$, the hypersimplex $\Delta_{d,k}$ is the convex hull of all vectors in $\mathbb{R}^d$ whose coordinates are k ones and $d-k$ zeros. Several unimodular triangulations of the hypersimplex have been proposed. One is given by Sturmfels in [41]. Another one is described by Stanley in [38]. As explained in [31, Section 2.3], the hypersimplex is linearly equivalent to an alcoved polytope. Consequently, it admits a unimodular balanced and regular triangulation.

Multi-homogeneous systems. A multi-homogeneous system with respect to a partition $d_1 + \cdots + d_k = d$ and with multi-degrees $(\ell_1, \ldots, \ell_k)$ is a system whose monomial support correspond to the set of lattice points in the polytope $\ell_1 \Delta_{d_1} \times \cdots \times \ell_k \Delta_{d_k}$, where Δ_{d_i} is the convex hull of $\{\mathbf{0}, \mathbf{e}_1, \ldots, \mathbf{e}_{d_i}\}$ in $\mathbb{R}^{d_i}$. Using the construction by Joswig and Witte [24], it is therefore sufficient to show that $\ell \Delta_d \subset \mathbb{R}^d$ admits a regular balanced unimodular
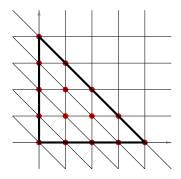
Figure 3: A regular balanced unimodular triangulation of $4\,\Delta_2$

triangulation for any $d, \ell \in \mathbb{N}$. Under the linear change of variables $z_i = x_1 + \cdots + x_i$, we see that $\ell\,\Delta_d \subset \mathbb{R}^d$ is defined by inequalities $z_1 - z_0 \geq 0$, $z_{i+1} - z_i \geq 0$ for $i = 1, \ldots, d - 1$, $z_d - z_0 \leq \ell$ and $z_0 = 0$. Thus as explained in [31, Section 2.3], slicing $\mathbb{R}^{d+1}$ by hyperplanes $z_j - z_i = k$ for $1 \leq i < j \leq d$ and $k \in \mathbb{Z}$ (together with $z_0 = 0$) gives a unimodular triangulation of $\ell\,\Delta_d \subset \mathbb{R}^d$. This triangulation is regular and balanced by Lemma 4.6. Itenberg and Viro constructed another interesting regular unimodular triangulation of $\ell\,\Delta_d$ by induction on the dimension d [20, Section 5]. This triangulation is obtained by taking joins of dilates of the simplex Δ_{d-1}. Using the property that the natural triangulation of a join of polytopes equipped with regular, unimodular and balanced triangulations is also regular, unimodular and balanced, one can easily show inductively that the triangulation used in [20] is balanced.

5   Fewnomial systems with many positive solutions

In this section, we consider the following classical problem.

Problem 5.1. *Let $d, k \in \mathbb{N}$ be integers. Let $\Xi_{d,k}$ be the maximum number of non-degenerate solutions in the positive orthant over all polynomial systems of d equations in d variables with real coefficients involving at most $d + k + 1$ monomials. Find lower and upper bounds on $\Xi_{d,k}$.*

Khovanskii [27] proved that $\Xi_{d,k}$ is bounded from above by a function of d and k, and he proved that $\Xi_{d,k} \leq 2^{\binom{d+k}{2}}(d+1)^{d+k}$ [27]. This bound was later improved by Bihan and Sottile [6] to

$$\Xi_{d,k} \leq \frac{e^2 + 3}{4} 2^{\binom{k}{2}} d^k.$$

This is currently the best known upper bound. Bihan, Rojas and Sottile [7] proved the following lower bound:

$$(\lfloor d/k \rfloor + 1)^k \leq \Xi_{d,k},$$

or more generally, if $d_1 + \cdots + d_k = d$ is a partition of d, then $\prod_{1 \leq i \leq k}(d_i + 1) \leq \Xi_{d,k}$.

A system consisting of d dense univariate polynomials of degree $\lfloor k/d \rfloor + 1$ proves the inequality:

$$(\lfloor k/d \rfloor + 1)^d \leq \Xi_{d,k}.$$

More generally, if $k_1 + \cdots + k_d = k$ is a partition of k, then $\prod_{1 \leq i \leq \ell}(k_i + 1) \leq \Xi_{d,k}$.

It is natural to study the sharpness of these two last bounds. The first one is sharp when $k = 1$ by [3]. The second bound is sharp when $d = 1$ by Descartes' rule of signs. When $k = d$, these two bounds give $2^d \leq \Xi_{d,d}$.

In order to construct systems with many positive solutions, we now turn our attention to subsimplicial complexes of triangulations of cyclic polytopes. These triangulations are good candidates since they enjoy a large number of simplices compared to their number of vertices.

Definition 5.2. *Let $a_1 < a_2 < \cdots < a_n$ be n real numbers. For $d, n \in \mathbb{N}$, the cyclic polytope $C(n, d)$ associated to $(a_1, \ldots, a_n)$ is the convex hull in $\mathbb{R}^d$ of the points $\{(a_i, a_i^2, \ldots, a_i^d)\}_{1 \leq i \leq n}$. The combinatorial structure of $C(n, d)$ is independent of the choice of the real numbers $a_1, \ldots, a_n$.*

Projecting the lower part of $C(n, d + 1)$ with respect to the last coordinate onto the first d coordinates yields a regular triangulation of $C(n, d)$. This triangulation denoted by $\hat{\mathbf{0}}_{n,d}$ is the minimum element for the *higher Stasheff-Tamari orders* on $C(n, d)$ [12]. We now recall the description of $\hat{\mathbf{0}}_{n,d}$ given in [12, Lemma 2.3] and provide a proof for the reader's convenience. In what follows, we represent a d-simplex of $\hat{\mathbf{0}}_{n,d}$ by the increasing sequence of $d + 1$ integer numbers between 1 and n which labels its vertices.

Proposition 5.3. *If d is odd, the d-simplices of $\hat{\mathbf{0}}_{n,d}$ are*

$$1 \leq i_1 < i_1 + 1 < i_2 < i_2 + 1 < \cdots < i_{\frac{d+1}{2}} < i_{\frac{d+1}{2}} + 1 \leq n,$$

while for even d they are the simplices

$$1 = i_1 < i_2 < i_2 + 1 < \cdots < i_{\frac{d}{2}} < i_{\frac{d}{2}} + 1 \leq n.$$

The triangulation $\hat{\mathbf{0}}_{n,d}$ is regular, and its number of d-simplices is

$$\begin{cases} \binom{n-(d+1)/2}{(d+1)/2} & \text{if d is odd} \\\\ \binom{n-1-d/2}{d/2} & \text{if d is even.} \end{cases}$$

Proof. In the following, we show that $\hat{\mathbf{0}}_{n,d}$ is the regular triangulation of $C(n, d)$ associated with the height function $(a_i, a_i^2, \ldots, a_i^d) \mapsto P(a_i)$, where $P(x) = x^{d+1}$. All monic univariate polynomials P of degree $d + 1$ give rise to height functions defining the same triangulation $\hat{\mathbf{0}}_{n,d}$. Indeed, if P_1 and P_2 are two such polynomials, then $P_1 - P_2$ is a polynomial of degree d and can therefore be expressed as an affine function on the rational normal curve of degree d.

Let $1 \leq j_1 < \cdots < j_{d+1} \leq n$ be any maximal simplex of $\hat{\mathbf{0}}_{n,d}$. The intervals of $\mathbb{R}$ where the univariate polynomial $P(T) = (T - a_{j_1}) \cdots (T - a_{j_{d+1}})$ is nonnegative is the union

$$\begin{cases} [a_{j_1}, a_{j_2}] \cup [a_{j_3}, a_{j_4}] \cup \cdots \cup [a_{j_d}, a_{j_{d+1}}] & \text{if } d \text{ is odd, or} \\ ]-\infty, a_{j_1}] \cup [a_{j_2}, a_{j_3}] \cup \cdots \cup [a_{j_d}, a_{j_{d+1}}] & \text{if } d \text{ is even.} \end{cases}$$

The condition that P is nonnegative on $\{a_1, \ldots, a_n\}$ is equivalent to saying that none of the a_i belongs to these intervals. This implies that every simplex has the form claimed in Proposition 5.3. For instance, if d is even, it implies that a_1 is a vertex of all the d-simplices. Since P is positive on $\{a_1, \ldots, a_n\} \setminus \{a_{j_1}, \ldots, a_{j_{d+1}}\}$, the simplex $1 \leq j_1 < \ldots < j_{d+1} \leq n$ belongs to the regular triangulation induced by the polynomial $P(T) = (T - a_{j_1}) \cdots (T - a_{j_{d+1}})$. But we showed that this triangulation does not depend on the choice of the monic polynomial P and is therefore $\hat{\mathbf{0}}_{n,d}$. $\qquad \square$

Unfortunately, the triangulation $\hat{\mathbf{0}}_{n,d}$ cannot be positively decorated (except in trivial cases) since its dual graph is not bipartite.

Example 5.4. *The triangulation $\hat{\mathbf{0}}_{6,3}$ of the cyclic polytope $C(6,3)$ consists of the following tetrahedra (together with their faces): $A = 1 < 2 < 3 < 4$, $B = 1 < 2 < 4 < 5$, $C = 1 < 2 < 5 < 6$, $D = 2 < 3 < 4 < 5$ $E = 2 < 3 < 5 < 6$, $F = 3 < 4 < 5 < 6$. Its dual graph is shown in Figure 4. It is not bipartite since it contains cycles of length 3.*
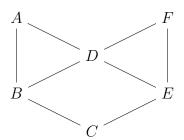

Figure 4: Dual graph of $\hat{\mathbf{0}}_{6,3}$.

We now restrict our attention to the case where d is odd. We represent any d-simplex $1 \leq i_1 < i_1 + 1 < i_2 < i_2 + 1 < \cdots < i_{\frac{d+1}{2}} < i_{\frac{d+1}{2}} + 1 \leq n$ of $\hat{\mathbf{0}}_{n,d}$ by the sequence $1 \leq i_1 < \cdots < i_{\frac{d+1}{2}} \leq n - 1$. Note that $i_j - i_{j+1} \geq 2$ for $j = 1, \ldots, \frac{d+1}{2}$. Consider the subsimplicial d-dimensional complex $\mathbf{S}_{n,d}$ of $\hat{\mathbf{0}}_{n,d}$ whose maximal simplices are the d-simplices $1 \leq i_1 < \cdots < i_{\frac{d+1}{2}} \leq n - 1$ such that for all j, either i_j is odd, or $i_{j+1} - i_j > 2$.

Proposition 5.5. *For d odd, the dual graph of the simplicial complex $\mathbf{S}_{n,d}$ is bipartite and the number of d-simplices of $\mathbf{S}_{n,d}$ equals the coefficient of $X^n Y^{(d+1)/2}$ in the series expansion of the rational function*

$$\frac{1 + X + X^3 Y}{1 - X^2 - X^2 Y - X^4 Y}.$$

Proof. Two simplices $1 \leq i_1 < \cdots < i_{(d+1)/2} \leq n$ and $1 \leq i'_1 < \cdots < i'_{\frac{d+1}{2}} \leq n$ of $\mathbf{S}_{n,d}$ are adjacent if and only if the two sets $\{i_1, i_1 + 1, i_2, i_2 + 1, \ldots, i_{\frac{d+1}{2}}, i_{\frac{d+1}{2}} + 1\}$ and $\{i'_1, i'_1 + 1, i'_2, i'_2 + 1, \ldots, i'_{\frac{d+1}{2}}, i'_{\frac{d+1}{2}} + 1\}$ share d elements in common. This implies that the symmetric difference of the sets $\{i_1, \ldots, i_{\frac{d+1}{2}}\}$ and $\{i'_1, \ldots, i'_{\frac{d+1}{2}}\}$ is an interval $I = \{k + 1, \ldots, k + 2\ell\}$ for some integers k and ℓ. Since i_j is odd, or $i_{j+1} - i_j > 2$, the cardinality of I must equal two. Hence, in $\mathbf{S}_{n,d}$, the parity of $i_1 + \cdots + i_{\frac{d+1}{2}}$ alternates between adjacent d-simplices. Consequently, the dual graph is bipartite.

Let $c_{n,d}$ denote the number of d-simplices in $\mathbf{S}_{n,d}$. It is easily verified that $c_{n,d}$ verifies the recurrence relation $c_{n,d} = c_{n-2,d} + c_{n-2,d-2} + c_{n-4,d-2}$. Let $G(X,Y) = \sum_{i,j \geq 0} c_{n,d} X^n Y^d$ be the associated generating series. The linear recurrence and the initial conditions imply that $(1 - X^2 - X^2Y^2 - X^4Y^2)G(X,Y) = (1 + X + X^3Y^3)$. Consequently, $c_{n,d}$ equals the coefficient of $X^n Y^d$ in the series expansion of

$$G(X,Y) = \frac{Y + XY + X^3Y^3}{1 - X^2 - X^2Y^2 - X^4Y^2}.$$

$\square$

The simplicial complex $\mathbf{S}_{n,d}$ is bipartite but it is not balanced in general, as demonstrated by the following example.

Example 5.6. *The simplicial complex $\mathbf{S}_{6,3}$ is obtained by removing D from $\hat{\mathbf{0}}_{6,3}$, see Example 5.4. It is easily seen that this simplicial complex is not balanced although its dual graph is bipartite. In fact the star of the vertex 4 contains the tetrahedra A, B, and F and its star is thus not connected. Consequently, the simplicial complex is not locally strongly connected.*

Corollary 5.7. *For integers $i, j \in \mathbb{N}$ and a rational function $S \in \mathbb{Z}(X,Y)$, we let $[X^iY^j]S(X,Y)$ denote the coefficient of X^iY^j in the formal series expansion of S. If the simplicial complex $\mathbf{S}_{2d+1,d}$ is positively decorable, then for d odd,*

$$[X^{2d+1}Y^{(d+1)/2}]\frac{1 + X + X^3Y}{1 - X^2 - X^2Y - X^4Y} \leq \Xi_{d,d}.$$

The best known lower bound on $\Xi_{d,d}$ is $2^d \leq \Xi_{d,d}$. The following proposition shows that if $\mathbf{S}_{2d+1,d}$ is positively decorable, then we would obtain the following sharper lower bound on $\Xi_{d,d}$.

Proposition 5.8. *For d odd, let c_d denote the number of d-simplices in $\mathbf{S}_{2d+1,d}$. As d grows,*

$$c_d \sim \frac{(\sqrt{2} + 1)^d}{\sqrt{d}} \frac{2^{1/4}(1 + \alpha)}{4\alpha\sqrt{\pi}},$$

where $\alpha = 3 - 2\sqrt{2}$. Consequently, $\lim c_d^{1/d} = \sqrt{2} + 1$.

Proof. For a bivariate series $G(X, Y)$, let $H(X, Y)$ and $U(X, Y)$ be the series such that $H(X^2, Y) = (XG(X) - XG(-X))/2$ and $U(X^2, Y) = (H(X, Y) + H(-X, Y))/2$. Note that, if we set $k = (d+1)/2$, then for d odd, we have

$$[X^k Y^k] U(X, Y) = [X^{2d+1} Y^{(d+1)/2}] G(X, Y).$$

If G is the series in Corollary 5.7, then $U(X, Y)$ equals

$$U(X, Y) = \frac{X(XY^2 - 2Y - 1)}{1 + X^2 Y^2 + (-Y^2 - 4Y - 1)X}.$$

The diagonal series $D(Z) = \sum_{i \geq 0}([X^i Y^i] U(X, Y)) Z^i$ is obtained by considering $U(X/Y, Y)/Y$ as a univariate function with coefficients in $\mathbb{R}(X)$ and by examinating its poles which tend to zero as X goes to zero, see *e.g.* [10, Section 4] and references therein. In this setting, $U(X/Y, Y)/Y$ has three poles:

$$0, \quad \frac{X^2 - 4X + 1 - \sqrt{(X-1)^2(X^2 - 6X + 1)}}{2X}, \quad \frac{X^2 - 4X + 1 + \sqrt{(X-1)^2(X^2 - 6X + 1)}}{2X}.$$

Only the first two poles tend to zero as X goes to zero, and the sum of their residues gives the desired algebraic series

$$D(X) = \frac{1}{2}\left(\frac{1+X}{\sqrt{X^2 - 6X + 1}} - 1\right).$$

The series $D(X)$ is analytic at 0. By [15, Theorem IV.7], the asymptotic exponential behavior of the coefficients of the series expansion is then dictated by its complex singularity nearest to the origin. In our case, the singularity of $D(X)$ with smallest complex module is $\alpha = 3 - 2\sqrt{2}$. Consequently, we obtain that

$$\limsup_{k \to \infty} \left([X^k Y^k] U(X, Y)\right)^{1/k} = \frac{1}{3 - 2\sqrt{2}} = \left(\sqrt{2} + 1\right)^2.$$

Using the relation $k = (n+1)/2$ proves the second statement of the proposition. In order the prove the first statement, we need to look more precisely at the nature of the singularity of $D(X)$ at $X = 3 - 2\sqrt{2}$. First, we note that $\sqrt{\alpha - X}\, D(X)$ is analytic at α, hence

$$D(X) = \frac{1}{\sqrt{\alpha - X}} \sum_{i \in \mathbb{N}} a_i (\alpha - X)^i,$$

for some real values $a_i \in \mathbb{R}$. Next, using a Taylor expansion around 0, we get for all $i \in \mathbb{N}$:

$$[X^k]\frac{(\alpha - X)^i}{\sqrt{\alpha - X}} = \frac{(2k - 2i - 1)!!}{2^k k!} \underset{k \to \infty}{\sim} \frac{1}{\alpha^k (2k)^i \sqrt{\alpha \pi k}}.$$

Evaluating $(\alpha - X) D(X)$ at 0 provides us with the value of a_0. Finally we get

$$[X^k] D(X) \underset{k \to \infty}{\sim} \frac{1 + \alpha}{4 \, 2^{1/4}} \frac{1}{\alpha^k \sqrt{\alpha \pi k}}$$

and using $c_d = [X^{(d+1)/2}] D(X)$ concludes the proof. $\qquad\square$

d	2^d	c_d
1	2	2
3	8	8
5	32	38
7	128	192
9	512	1002
11	2048	5336
13	8192	28814
15	32768	157184
17	131072	864146
19	524288	4780008
21	2097152	26572086

Table 1: For d odd, comparison between 2^d, the best known lower bound on $\Xi_{d,d}$ with the number of simplices c_d in the simplicial complex $\mathbf{S}_{2d+1,d}$.

We report in Table 1 the number of simplices in $\mathbf{S}_{2d+1,d}$ for the first odd values of d.

Example 5.9. *The simplicial complex for the cyclic polytope $C(6,3)$ is positively decorable. Ordering the vertices with respect to their relative position on the rational normal curve of degree 3, a coefficient matrix which decorates the simplicial complex is*

$$\begin{bmatrix} 1 & 0 & 3 & -4 & 0 & -1 \\ -2 & 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 3 & -3 & 1 & -3 \end{bmatrix}$$

By Theorem 2.4, this implies that for $t > 0$ sufficiently small, the polynomial system

$$
\begin{aligned}
1 + 3t^{2^4}X^2Y^{2^2}Z^{2^3} - 4t^{3^4}X^3Y^{3^2}Z^{3^3} - t^{5^4}X^5Y^{5^2}Z^{5^3} &= 0 \\
-2 + tXYZ + t^{2^4}X^2Y^{2^2}Z^{2^3} - t^{5^4}X^5Y^{5^2}Z^{5^3} &= 0 \\
3t^{2^4}X^2Y^{2^2}Z^{2^3} - 3t^{3^4}X^3Y^{3^2}Z^{3^3} + t^{4^4}X^4Y^{4^2}Z^{4^3} - 3t^{5^4}X^5Y^{5^2}Z^{5^3} &= 0
\end{aligned}
$$

has at least 5 real solutions in the positive orthant. Consequently, $5 \leq \Xi_{3,2}$.

We end this section by a special case of Question 3.9:

Question 5.10. *For d odd, is the bipartite simplicial complex $\mathbf{S}_{2d+1,d}$ always positively decorable?*

We verified that it is the case for $d = 1, 3, 5$. A general positive answer to this question would imply that $\limsup\limits_{d \to \infty} (\Xi_{d,d})^{1/d} \geq \sqrt{2} + 1$.

6 Realizable oriented matroids and positive matrix completion

In this section, we study the problem of decorating positively a simplicial complex from a computational viewpoint and we exhibit a connection to the problems of the realizability of oriented matroids and low-rank matrix completion problem. We start by the following characterization of oriented matrices (compare with Proposition 2.2).

Proposition 6.1. *Let M be a full rank $d \times (d+1)$ matrix with real entries. The following statements are equivalent:*

1. *M is an oriented matrix;*

2. *If $v_1, \ldots, v_{d+1}$ are the columns vectors of M, then the common intersection of half spaces $H_i^+ = \{x \in \mathbb{R}^d , \langle v_i, x \rangle \geq 0\}$ for $i = 1, \ldots, d+1$ is reduced to the zero vector.*

Proof. We use Proposition 2.2. Assume M is oriented. Then by Proposition 2.2 there exist positive real numbers x_i such that $\sum_{i=1}^{d+1} x_i v_i = 0$. Thus if $\delta \in \cap_{i=1}^{d+1} H_i^+$, then $\langle v_i, \delta \rangle = 0$ for $i = 1, \ldots, d+1$, which gives $\delta = 0$ since $v_1, \ldots, v_{d+1}$ generate $\mathbb{R}^d$. Conversely, assume that $\cap_{i=1}^{d+1} H_i^+ = \{0\}$. Since M has maximal rank, we may assume that $v_1, \ldots, v_d$ form a basis of $\mathbb{R}^d$ permuting the columns of M if necessary. Take a dual basis, that is, vectors $u_1, \ldots, u_d \in \mathbb{R}^d$ such that $\langle u_i, v_j \rangle = 1$ if $i = j$ and 0 otherwise. Note that $u_1, \ldots, u_d \in \cap_{i=1}^d H_i^+$. Write $v_{d+1} = \sum_{i=1}^d \lambda_i v_i$ with $\lambda_1, \ldots, \lambda_d \in \mathbb{R}$. If some coefficient λ_i is nonnegative, then $\langle u_i, v_{d+1} \rangle \geq 0$, and thus u_i is a non-zero vector in $\cap_{i=1}^{d+1} H_i^+$. It follows that $\lambda_1, \ldots, \lambda_d < 0$, which implies that M is oriented by Proposition 2.2. $\square$

A $d \times n$ matrix C with columns vectors $v_1, \ldots, v_n \in \mathbb{R}^d$ determines an oriented matroid on $\{1, \ldots, n\}$ with chirotope given by the signs of the maximal minors of C. Consider the hyperplane arrangement $\mathcal{H} = \{H_1, \ldots, H_n\}$, where $H_i = \{x \in \mathbb{R}^d , \langle v_i, x \rangle = 0\}$. Each connected component of the complementary part (chamber for short) gets a *sign vector* $s = (s_1, \ldots, s_n) \in \{\pm 1\}^n$ recording the signs of the linear forms $\langle v_i, \cdot \rangle$ on that chamber (where as usual $s_i = 1$ means that $\langle v_i, \cdot \rangle$ is nonnegative on the considered chamber). It turns out that these sign vectors are precisely the covectors of the oriented matroid determined by C, see [8, p. 11].

Proposition 6.2. *A simplicial complex of dimension d with vertices indexed by $\{1, \ldots, n\}$ can be positively decorated if and only if there exists a realizable oriented matroid of rank d on $\{1, \ldots, n\}$ such that for any d-simplex $j_1 < \cdots < j_{d+1}$ of the simplicial complex, there is no covector $s \in \{\pm 1\}^n$ of the oriented matroid which satisfies $s_{i_1} = \cdots = s_{i_{d+1}} = 1$.*

Proof. This follows from Proposition 6.1. $\square$

Proposition 6.2 shows that computational techniques for classifying oriented matroids may yield a solution to the problem of decorating simplicial complexes. Another option is to rely on techniques for low-rank matrix completion with positivity constraints. The following positive variant of low-rank matrix completion appears in several applicative problems in compressed sensing, see *e.g.* [9].

Problem 6.3 (Positive matrix completion problem). *Let $p, q, r \in \mathbb{N}$ be three integers with $r \leq \min(p, q)$, and M be a $p \times q$ non-negative real matrix with missing entries. Complete the matrix with* positive *real numbers, such that the completed matrix has rank r.*

The next proposition shows that positively decorating a simplicial complex can be done by solving a positive matrix completion problem.

Proposition 6.4. *Let Γ be a pure simplicial complex of dimension d on n vertices, with d-simplices $\tau_1, \ldots, \tau_\ell$. Let M be a $n \times \ell$ nonnegative matrix such that $M_{i,j} > 0$ if and only if j is a vertex of τ_i. If M has rank $n - d$, then any full rank $d \times n$ matrix C such that $C \cdot M = \mathbf{0}$ positively decorates Γ.*

Proof. By construction, each $d \times (d+1)$ submatrix of C corresponding to a d-simplex of Γ has a positive vector in its kernel and is therefore oriented. $\qquad\square$

Example 6.5. *We continue our running example 5.9, which corresponds to a bipartite simplicial complex on 6 vertices. It has 5 d-simplices, hence decorating this complex is equivalent to finding a rank 3 matrix of size 6×5 which has nonzero entries at positions prescribed by the complex. A solution to this problem is*

$$
\begin{array}{c}
\begin{array}{ccccc} A & B & C & E & F \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}
\left[
\begin{array}{ccccc}
1 & 4 & 1 & 0 & 0 \\
1 & 8 & 3 & 2 & 0 \\
3 & 0 & 0 & 3 & 6 \\
4 & 4 & 0 & 0 & 4 \\
0 & 3 & 3 & 6 & 3 \\
0 & 0 & 1 & 3 & 2
\end{array}
\right]
\end{array}
\qquad
\begin{array}{rcl}
A & = & [1234] \\
B & = & [1245] \\
C & = & [1256] \\
E & = & [2356] \\
F & = & [3456]
\end{array}
$$

The matrix in Example 5.9 is a basis of the left kernel of this matrix.

Using this reformulation of the initial problem, we used the software `NewtonSLRA` [36] to solve the matrix completion problem and to compute a decoration of $\mathbf{S}_{11,5}$. This shows that $\Xi_{5,5} \geq 38$, see Appendix A for the description of the Viro system that was obtained. `NewtonSLRA` is an iterative numerical algorithm with local quadratic convergence which can solve low-rank matrix completion problems. However, this software is not designed to handle positivity constraints so we randomized the starting point of the iteration and ran it until it converged to a nonnegative matrix. More specific computational methods would be needed to solve larger problems.

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A   Polynomial system with 5 variables, 5 equations, 11 monomials and 38 positive solutions

Let C be the coefficient matrix

$$C = \begin{bmatrix} \frac{14036}{26031} & \frac{-29047}{45845} & \frac{22485}{134218} & \frac{-20647}{80496} & \frac{14312}{69515} & \frac{-39015}{127243} & \frac{-6739}{42098} & \frac{19359}{360623} & \frac{16000}{83529} & \frac{1804}{131469} & \frac{4862}{44061} \\[2mm] \frac{19937}{61149} & \frac{-8379}{77942} & \frac{-2105}{18949} & \frac{5635}{122379} & \frac{9229}{59989} & \frac{5391}{113671} & \frac{17593}{33547} & \frac{-50525}{112808} & \frac{-13843}{33458} & \frac{18357}{116882} & \frac{-54686}{132521} \\[2mm] \frac{6391}{94296} & \frac{-3329}{144100} & \frac{7957}{156078} & \frac{-5685}{48451} & \frac{-14459}{74653} & \frac{30218}{245615} & \frac{-12227}{25927} & \frac{49127}{145204} & \frac{-14117}{47609} & \frac{29515}{59658} & \frac{-42328}{83609} \\[2mm] \frac{-12249}{145219} & \frac{-13663}{97873} & \frac{-25831}{90582} & \frac{26287}{33739} & \frac{6818}{23407} & \frac{-14579}{44765} & \frac{-11126}{58889} & \frac{2247}{122770} & \frac{11139}{100537} & \frac{14421}{74818} & \frac{-60016}{644607} \\[2mm] \frac{15984}{47945} & \frac{-22523}{72834} & \frac{-10734}{41165} & \frac{8531}{24837} & \frac{-21257}{47591} & \frac{22017}{37075} & \frac{5346}{284353} & \frac{19757}{194173} & \frac{5740}{83029} & \frac{-62271}{466111} & \frac{5591}{37902} \end{bmatrix}.$$

Then for $t > 0$ sufficiently small, the system

$$C \cdot \begin{bmatrix} 1 \\ tX_1X_2X_3X_4X_5 \\ t^{2^6}X_1^2X_2^{2^2}X_3^{2^3}X_4^{2^4}X_5^{2^5} \\ t^{3^6}X_1^3X_2^{3^2}X_3^{3^3}X_4^{3^4}X_5^{3^5} \\ \vdots \\ t^{10^6}X_1^{10}X_2^{10^2}X_3^{10^3}X_4^{10^4}X_5^{10^5} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

of 5 polynomial equations in $\mathbb{R}[X_1, \ldots, X_5]$ has at least 38 positive solutions.

Authors' addresses:

Frédéric Bihan, Laboratoire de Mathématiques, Université Savoie Mont Blanc, Campus Scientifique, 73376 Le Bourget-du-Lac Cedex, France, `frederic.bihan@univ-savoie.fr`

Pierre-Jean Spaenlehauer, CARAMEL project, INRIA Grand-Est; Université de Lorraine; CNRS, UMR 7503; LORIA, Nancy, France, `pierre-jean.spaenlehauer@inria.fr`