Algebraic Geometry for Splines

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PREFACE

The thesis is structured as follows:

Chapter 1: We present a brief background and historical introduction to the topics of the thesis. We give a description of each chapter with the main contributions in each of them.

Chapter 2–5: Each chapter corresponds to a research paper, one recently accepted for publication in the *Journal of Symbolic Computation*, and the other three in preparation.

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This thesis wouldn't have been concluded without the help of many persons. The purpose of these lines it to try to express my deep gratitude to all of them.

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Chapter 1

INTRODUCTION

This chapter contains the general framework in which this work has been developed. The first section consists of an introduction to the SAGA project and the principal objectives of this thesis. The main topics of study are introduced in the next two sections, by giving a brief historical account together with some basic definitions and prerequisites. We conclude with an overview of the thesis and future directions of research.

1 Background

This thesis is part of the EU–FP7 Initial Training Network SAGA: ShApes, Geometry and Algebra (2008-2012). The SAGA project stems from the shared belief between the academia and industrial partners, that Computer Aided Design and Manufacturing (CAD/CAM) will be greatly enhanced by exploiting mathematical results and techniques covering the full spectrum from Algebraic Geometry and Computer Algebra, to Computer Aided Geometric Design (CAGD).

Geometric Modeling, the study of methods and algorithms for the description of shapes, plays a central role in CAGD. It usually involves piecewise algebraic representations of shapes, and their effective treatment leads to the resolution of polynomial systems of equations, which requires the use of stable and efficient tools. Within the numerical analysis community, the use of high-order polynomial representations has been conceived as a new way to break the complexity barrier caused by piecewise linear representations, and to deal efficiently with free-form geometry.

When modeling curves and surfaces algebraically, using just one polynomial does not give much flexibility. Instead, one should use piecewise polynomials to approximate larger regions of a CAD-model. This keeps the polynomial degree lower and allows for more flexible approximations. We aim to establish a better basis for the application of multivariate algebraic splines to problems in geometric modeling.

All the current approximations to CAD from algebraic geometry rely on mathematical foundations based on the use of complex numbers and projective geometry while, in practice, all the considered practical questions are presented over the real numbers and in an affine setting. This obvious remark implies the need of analyzing, from the point of view of *real algebraic geometry*, the consideration of curves, surfaces and solids as semi-algebraic sets (i.e., sets defined by means of polynomial equalities and inequalities).

This thesis addresses, from an algebraic geometry perspective, two relevant problems in CAGD: the problem of constructing and analyzing piecewise polynomial or spline functions on polyhedral subdivisions in $\mathbb{R}^d$, and how toric degenerations of *real toric varieties* are related to the regular control surfaces of *toric Bézier patches*. The forthcoming sections of this chapter are devoted to a detailed description of these problems.

2 Triangular spline spaces

The original interest in spline functions arose from the solution of partial differential equations by the finite element method. Nowadays, splines are important not only in numerical analysis, but are also a widely recognized tool in approximation theory, image analysis and CAGD.

A *spline* or *piecewise polynomial* function is a function defined on some partition Δ of a region in $\mathbb{R}^d$ with a specified degree of global smoothness.

We focus on the case where Δ is a triangulation or a tetrahedral partition of a region in the plane or in $\mathbb{R}^3$, respectively. However, most of the constructions can be adapted to rectilinear partitions and generalized to any dimension $d > 3$.

Perhaps the most remarkable aspect of these objects is the interplay between the underlying combinatorics and geometry of the subdivision and the algebraic properties of the resulting set of functions.

The simplest form of this idea are univariate spline functions, defined over intervals of the real line [13]. Let us consider the interval $[a, b] \subset \mathbb{R}$ and the subdivision $[a, c] \cup [c, b]$, $a < c < b$. A spline function takes the form

$$f(x) = \begin{cases} f_1(x) & \text{if } x \in [a, c] \\ f_2(x) & \text{if } x \in [c, b] \end{cases} \tag{2.1}$$

with polynomials $f_1, f_2 \in \mathbb{R}[x]$. We can always make trivial spline functions using the same polynomial $f_1 = f_2$; in general the previous formula gives us a well–defined function f on $[a, b]$ if and only if $f_1(c) = f_2(c)$. In such case, f is continuous on $[a, b]$. Since the polynomials f_1 and f_2 are C^∞ functions and their derivatives are also polynomials, we can consider the piecewise polynomial derivative functions

$$\begin{cases} f_1^{(r)}(x) & \text{if } x \in [a, c] \\ f_2^{(r)}(x) & \text{if } x \in [c, b] \end{cases}$$

for $r \geq 0$. As above, f is a C^r function on $[a, b]$ (that is, f is r-times differentiable and its rth derivative, $f^{(r)}$, is continuous) if and only if $f_1^{(s)}(c) = f_2^{(s)}(c)$ for each s, $0 \leq s \leq r$.

We can represent a spline function as in (2.1) by the ordered pair $(f_1, f_2) \in \mathbb{R}[x]^2$, and the C^r splines form a vector subspace of $\mathbb{R}[x]^2$, under the usual componentwise addition and scalar multiplication. In practice, it is more common to consider spline functions where the degree of each component is bounded by some fixed integer k, that space is denote by C_k^r and it is also a vector subspace of $\mathbb{R}[x]^2$.

In correspondence to subdivisions of intervals in $\mathbb{R}$, for the multivariate case we consider subdivisions of polyhedral regions in $\mathbb{R}^d$. The major new feature in $\mathbb{R}^d$, $d \geq 2$ is the bigger geometric freedom possible in constructing such subdivisions.

The case of general continuous piecewise polynomials over higher-dimension simplicial subdivisions can be successfully treated using a variety of methods. However, serious difficulties already begin to arise in the case of planar simplicial subdivisions.

For a polyhedral complex Δ in $\mathbb{R}^d$, generalizing the univariate splines above, for each $r \geq 0$ we denote by $C^r(\Delta)$ the collection of C^r functions f on Δ such that for every cell $\delta \in \Delta$ the restriction $f|_\delta$ is a polynomial function $f_\delta \in \mathbb{R}[x_1, ..., x_d]$. For $k \geq r$, $C_k^r(\Delta)$ is the subset of $C^r(\Delta)$ such that the restriction of f to each cell in Δ is a polynomial function of degree $\leq k$.

Two fundamental problems in this area are: to determine the dimension of $C_k^r(\Delta)$ as a vector space, in function of known information about the subdivision Δ, and the associated problem of determining a basis for this space. The first of these problems is the main topic in this thesis.

In [51], the problem of finding the dimension of bivariate spline spaces was explicitly formulated for the first time. This conjecture was initially for a square mesh with all diagonals drawn in, as in Figure 1.1. The work of Strang on this subject started with an idea of approximation by piecewise polynomial functions described at the end of a lecture by Courant. That idea was to triangulate the domain and to introduce the space of continuous piecewise linear functions, instead of sines and cosines, Bessel functions, or Legendre polynomials. For a regular domain the latter mentioned basis are still adequate but on an irregular domain the situation is completely different, as these functions are virtually useless. In one variable the question is comparatively simple. The term “multivariate splines” we use follows Schoenberg [16, 47]; these papers gave birth to the theory of splines (see also [34], for a discussion of these early developments).

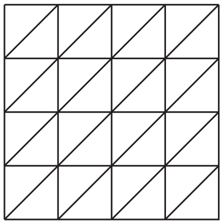


Figure 1.1. Square mesh.

A heuristic calculation was suggested to compute the dimension of the spline space, leading to a more general conjecture; Strang generalized its conjecture to any triangulated domain in [52]. In [37] Morgan and Scott computed the dimension of $C_k^r(\Delta)$ for any triangulation and any $k \geq 5$, by constructing a nodal basis. Their result took into account singular vertices, i.e., vertices in the interior of a quadrilateral that is triangulated by its two diagonals. Triangulating a rectangle with two crossing diagonals results in the dimension of $C_2^1(\Delta)$ being one higher than the combinatorially identical triangulation in which the central vertex is not the intersection of the diagonals.

With this latter result, the authors showed that Strang's conjecture was not valid for general triangulations. They extended its result to fourth–degree piecewise polynomials for quite general meshes, in their unpublished work [38]. In that paper appears the first indication that it might be very difficult to give closed formulas for dimensions of spline spaces, namely, the discovery of the example in Figure 1.2, known as the Morgan–Scott triangulation. This example shows that taking account of the slopes of the edges is not enough to describe the dimension of spline spaces in general, and has been intensively studied in the literature to determine exactly when the dimension of $C_2^1(\triangle_{\mathrm{MS}})$ is six, and when it changes to seven. It turns out that the dimension is only seven for very special choices of the interior vertices, included the symmetric configuration shown on the left in Figure 1.2 [17].

The first paper that gives formulas explaining in detail how the dimension

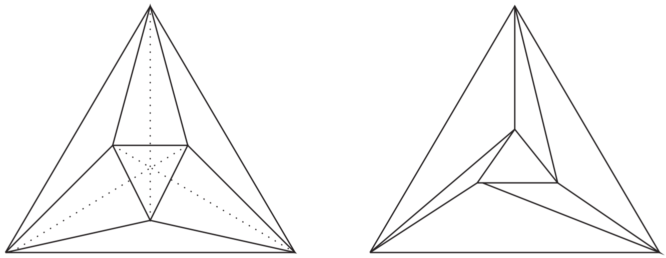


Figure 1.2. The Morgan–Scott triangulations $\triangle_{\rm MS}$.

depends on the geometry of the triangulations, and in particular, on the slopes of the edges surrounding each interior vertex was [48]. In this article, Schumaker presented a lower bound; an upper bound was reported in [49]. In [1], the dimension of $C^r_k(\Delta)$ for $k \geq 4r+1$ was obtained by using Bernstein–Bézier methods. The result was extended to $k \geq 3r+2$ in [26]. There is a wide literature on bounds, that we do not mention here but for further information we refer to the book [32].

There has not been much success in understanding the dimension of spline spaces $C^r_k(\Delta)$ of degrees $d < 3r+2$ for general triangulations. Billera in [7] developed a homological approach to the problem of finding the dimension of a spline space $C^r_k(\Delta)$ defined on triangulated d-dimensional regions in $\mathbb{R}^d$. By applying the approach to triangulated manifolds Δ in the plane, he proved the generic dimension of $C^1_k(\Delta)$, by combining his construction with results of Whiteley [55–57] on the so–called spline matrices. The importance of homological methods is that they provide a unified approach to many problems in this context. It gives a way of doing a lot of complicated linear algebra in a very organized way. This thesis follows this homological approach and some related posterior developments.

What Billera shows, is that there is a great deal of invariance if we consider the dimensions of all the $C^r_k(\Delta)$'s as a whole. This yields, for instance, a proof that for large k, $\dim C^r_k(\Delta)$ is given by a polynomial function of k [9].

Using the fact that $C^r(\Delta)$ has the structure of a commutative ring under pointwise multiplication of functions, in [8], it was proved that $C^0(\Delta)$, on a d-dimensional simplicial complex Δ, is a quotient of the Stanley–Reisner ring A_Δ of Δ, and as a consequence he derived the dimensions (as vector spaces over $\mathbb{R}$) of the subspaces $C^0_k(\Delta)$.

In [46], Schenck and Stillman introduced a chain complex different from the one used by Billera. The lower homology modules of the chain complex in this construction differ from the ones introduced in [7]. With this construction, in the planar case, [42] gives necessary and sufficient conditions for freeness of $C^r(\Delta)$ and shows that the first three terms of its Hilbert polynomial can be determined from the combinatorics and local geometry of Δ.

However, there is no corresponding statement for $d \geq 3$; Bernstein–Bézier methods have been used in [4, 5] for tetrahedral partitions. It is also shown, in [5] that the dimension problem for the trivariate case cannot be settled until we fully understand the dimension problem for bivariate splines. We give an approach to this problem by using the result on ideals of fat points.

Schenck also considered the connection between the $\mathbb{R}$-algebra on a d-dimensional simplicial complex Δ embedded in $\mathbb{R}^d$, and the Stanley–Reisner ring of Δ. He introduced a criterion to determine which elements of the Stanley–Reisner ring correspond to splines of high-order smoothness. These ideas, and in particular the work of Schenck, Stillman and Geramita are the main references of the present work.

3 Toric Varieties

Parametric Bézier curves and surfaces are widely used to represent geometric objects in CAGD; they are used in animation software such as Adobe flash, as well as for design, testing and manufacture of airplane wings.

Current CAD technology is essentially based on NURBS (Non-uniform rational basis spline). From the Algebraic Geometry point of view NURBS are rectangular patchworks composed of the simplest rational surface pieces, parameterized by the product of two projective lines. The natural modeling process of smooth surfaces with complicated topology usually generates NURBS with n-sided ($n \neq 4$) holes. Toric Bézier patches were proposed to solve the hole filling problem.

Toric varieties were introduced in the early 1970's in algebraic geometry. The close relation of this theory with combinatorics of convex polytopes, makes it very attractive for applications. In CAGD, Bézier surfaces play a central role. Tensor product Bézier surfaces and Bézier triangles are projections of Segre and Veronese surface patches, which are the two simplest cases of real projective toric surfaces. But, they are not the unique real toric surfaces that can be used in CAGD; by considering a rational Bézier triangular surface with zero weights at appropriate control points one can obtain a hexagonal patch [54] (see also [12, 50] for an introduction to toric varieties for geometric modeling).

In [31], Krasauskas introduced toric Bézier patches, as a generalization of the classical Bézier triangular and tensor product patches to arbitrary polygons whose vertices have integer coordinates. They are based upon toric varieties and naturally associated to lattice polytopes.

We introduce some definitions and notations, the main references are [15] and [22], which is also the notation which we adopt in the related chapter of the thesis. For a positive integer d, and $i = 0, \dots, d$, the Bernstein polynomial $\beta_{i;d}(x)$ is defined by

$$\beta_{i;d}(x) := x^i(d-x)^{d-i}\,.$$

Given weights $w_0, \dots, w_d \in \mathbb{R}_>$ (positive real numbers), and control points $\mathbf{b}_0, \dots, \mathbf{b}_d \in \mathbb{R}^n$, the parametrized rational Bézier curve is defined by

$$F(x) := \frac{\sum_{i=0}^{d} w_i \mathbf{b}_i \beta_{i;d}(x)}{\sum_{i=0}^{d} w_i \beta_{i;d}(x)} : [0,d] \longrightarrow \mathbb{R}^n\,,$$

the domain $[0,d]$ rather than $[0,1]$ is the natural convention for toric patches.

The *control polygon* of the curve is the union of segments $\overline{\mathbf{b}_0,\mathbf{b}_1}$,..., $\overline{\mathbf{b}_{d-1},\mathbf{b}_d}$, as in Figure 1.3 from [22].

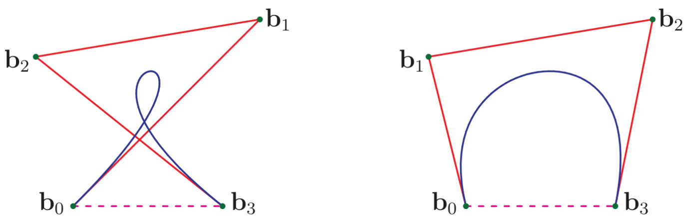


Figure 1.3. Rational Bézier curves with their control polygons.

There are two ways to extend this to surfaces. The most straightforward gives rational tensor product patches, the other yielding triangular Bézier patches.

A rational tensor product patch associated to a set of weights $w_{(i,j)} \in \mathbb{R}_>$ and control points $\mathbf{b}_{i,j} \in \mathbb{R}^n$ for $i = 0, \dots, c$ and $j = 0, \dots, d$ is given by the map $F : [0,c] \times [0,d] \to \mathbb{R}^n$ defined similarly as above, for Bernstein polynomials $\beta_{i;c}(x)$ and $\beta_{j;d}(y)$.

For the triangular Bézier patches, we consider the bivariate Bernstein polynomials for the lattice points in the triangle

$$d\triangle := \{(x,y) \in \mathbb{R}^2 : 0 \le x, y \ \text{ and } \ x+y \le d\}.$$

Krasauskas's toric patches are a natural extension of the previous two. We start by a finite set $\mathcal{A} \subset \mathbb{Z}^d$ of integer lattice points. We denote by

Conv($\mathcal{A}$), the convex hull of $\mathcal{A}$. For each edge e of Conv($\mathcal{A}$) there is a valid inequality $h_e(\mathbf{x}) \geq 0$ on Conv($\mathcal{A}$), where $h_e(\mathbf{x})$ is a linear polynomial with integer coefficients having no common integer factors that vanishes on the edge e. If E is the set of edges, for each lattice point $\mathbf{a} \in \mathcal{A}$ the *toric basis function* $\beta_{\mathbf{a},\mathcal{A}} : \mathrm{Conv}(\mathcal{A}) \to \mathbb{R}$ is defined by

$$\beta_{\mathbf{a},\mathcal{A}}(\mathbf{x}) := \prod_{e \in E} h_e(\mathbf{x})^{h_e(\mathbf{a})}.$$

Thus, a toric Bézier patch of shape $\mathcal{A}$ is given by a collection of positive weights $w = (w_{\mathbf{a}} : \mathbf{a} \in \mathcal{A}) \in \mathbb{R}^{\mathcal{A}}_{>}$ and control points $\mathcal{B} = \{\mathbf{b}_{\mathbf{a}} : \mathbf{a} \in \mathcal{A}\} \subset \mathbb{R}^n$, defining a map

$$F_{\mathcal{A},w,\mathcal{B}}(x,y) := \frac{\sum_{\mathbf{a}\in\mathcal{A}} w_{\mathbf{a}} \mathbf{b}_{\mathbf{a}} \beta_{\mathbf{a},\mathcal{A}}(\mathbf{x})}{\sum_{\mathbf{a}\in\mathcal{A}} w_{\mathbf{a}} \beta_{\mathbf{a},\mathcal{A}}(\mathbf{x})} : \mathrm{Conv}(\mathcal{A}) \longrightarrow \mathbb{R}^n \,. \tag{3.1}$$

The image of Conv($\mathcal{A}$) under the map F is a *toric Bézier patch* of shape $\mathcal{A}$.

The control points of a Bézier curve are naturally connected in sequence to give the control polygon, which is a piecewise caricature of the curve. For a surface patch there are however, many ways to interpolate the control points by edges to form a control net. There also may not be a way to fill in these edges with polygons to form a polytope. Even when this is possible, the significance of this structure for the shape of the patch is not evident.

Carl de Boor and Ron Goldman proposed to explore the significance for modeling of such control structures, i.e., of the control points plus the edges. Craciun, García–Puente and Sottile, considered such control structures and limiting patches in [15] but the results were restricted to triangulations. By working on the generality of Krasauskas' toric patches, García–Puente, Sottile and Zhu provided an answer to that question [22]: these control structures encode limiting positions of the patch when the weights assume extreme values. They proved that regular control surfaces are limits of toric Bézier patches and that if a patch is sufficiently close to a control surface, then that control surface must be regular; see in Figure 1.4 two examples of rational bicubic patches with the control points and extreme weights from [22].

The control structure in these examples is a regular decomposition of the 3×3 grid. It is regular as it is induced from the upper convex hull of the graph, see Figure 1.5. In Figure 1.6, there is an irregular decomposition of the same configuration of points; if it were the upper convex hull of the graph of the function on the grid points, then we may assume that the central square is flat and then, the value of the function at the vertex is lower than the values at the clockwise neighbor, which is impossible [22].

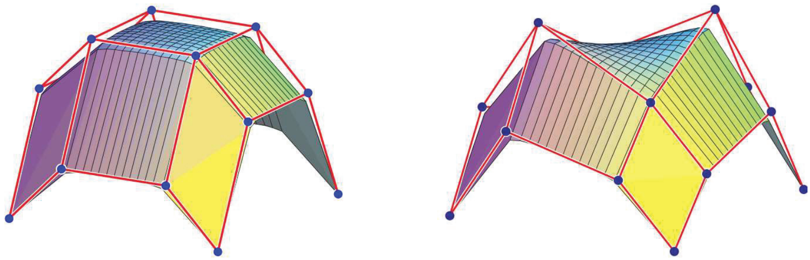
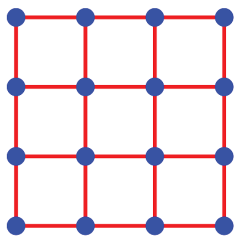


Figure 1.4. Rational patches with extreme weights.



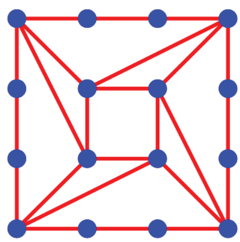


Figure 1.5. Regular decomposition. **Figure 1.6.** Irregular decomposition.

The proof in [22] is based on the fact that the map F in (3.1) admits a factorization

$$F : \mathrm{Conv}(\mathcal{A}) \xrightarrow{\beta_{\mathcal{A}}} \triangle^{\mathcal{A}} \xrightarrow{w\cdot} \triangle^{A} \xrightarrow{\pi_{\mathcal{B}}} \mathbb{R}^n,$$

where $\triangle^{\mathcal{A}} \subset \mathbb{R}^{\mathcal{A}}$ is the standard simplex of dimension $\#\mathcal{A} - 1$, the map $\beta_{\mathcal{A}}$ is given by the toric basis functions $\beta_{\mathbf{a},\mathcal{A}}$, the map $w\cdot$ is coordinatewise multiplication by the weights w, and $\pi_{\mathcal{B}}$ is a projection given by the control points $\mathcal{B}$. This factorization clarifies the role of the weights in a toric patch. The image $\beta_{\mathcal{A}}(\mathrm{Conv}(\mathcal{A})) \subset \triangle^{\mathcal{A}}$ is a standard toric variety $X_{\mathcal{A}}$. Acting on this by $w\cdot$, gives a translated toric variety $X_{\mathcal{A},w}$. The authors use results on the limiting position of the translates $X_{\mathcal{A},w}$ as the weights are allowed to vary; these are the so-called toric degenerations.

Krasauskas's definition of toric Bézier patches still makes sense if the set of points $\mathcal{A}$ consists of real (not-necessarily rational) vectors. This leads to the notion of *irrational toric patch*, as the blending functions are no longer rational functions.

The extension of the result in [22] includes a study of the translates of the irrational projective toric varieties parametrized by any finite configuration of points $\mathcal{A} \subset \mathbb{R}^d$, and this leads to a stronger analogous result and moreover, to a new and elementary interpretation of the secondary polytope of $\mathcal{A}$ as the natural space of toric degenerations of the toric variety under the Hausdorff metric.

4 Overview of the thesis

The work and results in this thesis are presented in the next four chapters, which are developed as follows.

In Chapter 2, we address the problem of determining the dimension of the space of bivariate splines $C_k^r(\Delta)$ for a triangulated region Δ in the plane. Using the homological approach introduced by Billera in [7], we recall some properties of the homology modules and reproduce in detail the construction of the chain complex presented by Schenck and Stillman in [46]; this chain complex agrees with the complex studied by Billera except at the vertices, having different lower homology modules which have nicer properties. With this approach, and by numbering the vertices, we establish formulas for lower and upper bounds on the dimension of the spline space.

The main contribution of the paper is the new formula for an upper bound. The formula applies to any ordering established on the interior vertices of the partition, contrarily to the upper bound formulas in [49], [32]. Having no restriction on the ordering makes it possible to obtain accurate approximation to the dimension and even exact value in many cases. As a consequence, we also give a short proof for the dimension formula when $k \geq 4r + 1$, this latter result and some other examples that we present illustrate the interest of the homology construction for proving exact dimension formulas.

In Chapter 3, we consider the spline dimension problem in $\mathbb{R}^3$. This problem has been studied using Bernstein–Bézier methods in a series of papers by Alfeld, Schumaker, Sirvent and Whiteley [3–5]. The results in these papers do not take into account the geometry of the faces surrounding the interior edges or interior vertices. A variant of that approach by Lau [33], gives a lower bound for simply connected tetrahedral partitions. The formula, although it contains a term which takes into account the geometry of faces surrounding interior edges, is missing the term involving the number of interior vertices. This often makes the lower bound much smaller than the one presented in [5].

In this chapter, Δ is a connected 3-dimensional simplicial complex supported on a ball, and we explore the homological approach, analogous the used in Chapter 2, to find the dimension of $C_k^r(\Delta)$. We prove lower and upper bounds by applying homological techniques and exploring connections of splines with ideals generated by powers of linear forms, ideals of fat points, Fröberg's conjecture, and the weak Lefschetz property. The formulas we present, apply to any degree k and order of smoothness r, and include terms that explicitly depend on the number of different planes surrounding the edges and vertices in the interior of Δ. In some cases, they give better approximations to the exact dimension, and more importantly perhaps, the construction gives an insight into ways of approaching this problem.

In Chapter 4, we study the the ring structure of the space of C^1 spline functions on a planar domain. Besides the interest that the vector subspaces of splines on a finite d-dimensional simplicial complex Δ in $\mathbb{R}^d$, have for practical applications, $C^r(\Delta)$ forms a ring under pointwise multiplication, which is also interesting to study as an algebraic object.

It was proved by Billera in [8], that, as a ring, $C^0(\Delta)$ is a quotient of the Stanley–Reisner ring A_Δ of Δ. Since $C^{r+1}(\Delta) \subset C^r(\Delta)$, Billera's result implies that there is a descending chain of subrings contained in A_Δ. The homological approach in [7] can be related to a homology where A_Δ appears. By this argument, Schenck [43] obtained a local characterization of those elements of A_Δ corresponding to elements of $C^r(\Delta)$.

We consider the planar case, i.e., Δ is a 2-dimensional simplicial complex. By using Schenck's characterization, we study the ring structure of those elements of the Stanley–Reisner ring which correspond to C^1 splines. We present some examples, results and conjectures about the generators of $C^1(\Delta)$ as a ring, when the triangulation is generic. Our study is presented following the complexity of the triangulation.

We state a series of conjectures, where the most remarkable result to which they would lead to, would be the strong relation between the module of syzygies of the set of linear forms vanishing on the interior edges of the triangulation Δ, and elements that generate $C^1(\Delta)$ as a subring of the Stanley–Reisner ring A_Δ. This work also would give a way of exploring the structure as a subring of $C^r(\Delta)$ for higher order of smoothness r.

In Chapter 5, we recall the definitions of irrational toric varieties, the notions of secondary polytope and regular subdivision of a point set $\mathcal{A}$, and of toric degenerations of $X_\mathcal{A} \subset \mathbb{P}^\mathcal{A}_\mathbb{R}$.

We study translates of the irrational projective toric variety $X_\mathcal{A} \subset \mathbb{P}^n_\mathbb{R}$ parametrized by any configuration $\mathcal{A}$ of $n+1$ points in $\mathbb{R}^d$. We show that any sequence of translates of irrational toric varieties yields to a regular subdivision $\mathcal{S}$ of $\mathcal{A}$ and a weight vector $w^\infty \in \mathbb{R}^\mathcal{A}_>$, the sequence admits a convergent subsequence which coincides with the toric degeneration associated to $\mathcal{S}$ and w^∞. This leads to the main (still conjectural) result in this paper, which is that all Hausdorff limits of translates of irrational toric varieties are in fact toric degenerations. This leads to a new and elementary interpretation of the secondary polytope of $\mathcal{A}$ as the natural space of toric degenerations of $X_{\mathcal{A}}$ under the Hausdorff metric.

In [22], García-Puente, Sottile and Zhu, proved that the union of toric varieties corresponding to all the facets of a decomposition $\mathcal{S}$ of $\mathcal{A}$, is the limit of the corresponding sequence of translates of the toric variety $X_{\mathcal{A},w_{(t)}}$ under a toric degeneration, for a family of weights $\{w_{(t)}\}_t$. The result was proved for $\mathcal{A} \subset \mathbb{Z}^d$ but it is also valid for irrational toric varieties, namely for the case $\mathcal{A} \subset \mathbb{R}^d$.

A weak converse of this result was also proved in [22]: if a sequence of translates of $X_{\mathcal{A}}$ has a limit in the Hausdorff topology, then this limit is a union of toric varieties for some regular subdivision $\mathcal{S}$ of $\mathcal{A}$ and a weight vector $w \in \mathbb{R}^{\mathcal{A}}_{>}$. We prove a stronger result, namely that every sequence of translates admits a convergent subsequence, in the Hausdorff topology, to some toric degeneration. Hence, the set of translates of $X_{\mathcal{A}}$ is naturally compactified by the set of toric degenerations of $X_{\mathcal{A}}$.

5 Future directions of research

This thesis is devoted to the study of spline spaces. As we can see from the results developed throughout this work, there is an interesting interplay between the underlying combinatorics and geometry of the subdivision and the algebraic properties of the resulting functions. It is along these lines direction that we foresee the following directions for future research:

- The homological structure makes possible to consider mixed splines, this is, splines where the order of smoothness may vary along the connections of the faces of the subdivisions. This would lead to a homological construction for the so-called *supersplines*.

- The further study of a valid framework for Fröberg's conjecture, possibly by relating that problem to results about the Weak Lefschetz Property. Eventually, this could allow having resolutions of ideals of powers of linear forms in several variables, yielding better bounds on the dimension of the spline space in a more general setting.

- The scheme $\operatorname{Proj} C^0(\hat{\Delta}) = \operatorname{Proj} A_\Delta$ is "equal" to Δ, where Δ is viewed as a union, in projective space of dimension equal to the number of vertices of Δ minus 1, of n-dimensional linear spaces, each corresponding to a n-dimensional face, intersecting transversally. It would be interesting to understand the geometric interpretation of the Proj of the generalized Stanley–Reisner rings, $\operatorname{Proj} C^r(\hat{\Delta})$. The conjecture is that these are twisted versions of the geometric objects appearing in the $r = 0$ case, where each face corresponds to a (singular) higher degree variety in a higher dimensional space.

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