SUBDIVISION AND SPLINE SPACES

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ABSTRACT. A standard construction in approximation theory is mesh refinement. For a simplicial or polyhedral mesh $\Delta \subseteq \mathbb{R}^k$, we study the subdivision Δ' obtained by subdividing a maximal cell of Δ . We give sufficient conditions for the module of splines on Δ' to split as the direct sum of splines on Δ and splines on the subdivided cell. As a consequence, we obtain dimension formulas and explicit bases for several commonly used subdivisions and their multivariate generalizations.

1. Introduction

Splines are fundamental objects in approximation theory, computer aided geometric design and modeling, and the finite element method for solving PDEs. Starting with a simplicial or polyhedral complex Δ (or mesh) which partitions a region in \mathbb{R}^k , it may be the case that the mesh is too coarse for the specific application. So a natural approach is to refine the mesh via subdivision.

We use Δ to denote a k-dimensional simplicial complex in \mathbb{R}^k , Δ_i the set of i-dimensional faces, and Δ_i^0 the set of interior i-dimensional faces; all k-dimensional faces are considered interior so $\Delta_k = \Delta_k^0$. Although we work in the simplicial setting, all our results generalize easily to the polyhedral case. We analyze a special type of subdivision, where the original mesh Δ is modified by subdividing a single maximal cell $\sigma \in \Delta_k$. For the resulting object Δ' to be a complex, it is necessary that any modifications made to the boundary of σ occur only on $\sigma \cap \partial(\Delta)$.

The principal technique we use is the homological approach introduced by Billera in [2], combined with the observation that if $\widehat{\Delta}$ is the cone over Δ , then $S^r(\widehat{\Delta})$ is a graded module over the polynomial ring $R = \mathbb{R}[x_0, \dots, x_k]$, and the dimension of $S_d^r(\Delta)$ is the dimension of the d^{th} graded piece of $S^r(\widehat{\Delta})$.

Our main result is a sufficient condition for the set of splines on $\widehat{\Delta}'$ to split as the direct sum of splines on $\widehat{\Delta}$ and splines on the subdivided cell $\widehat{\Delta}''$:

$$S^r(\widehat{\Delta}') \simeq S^r(\widehat{\Delta}) \bigoplus \left(S^r(\widehat{\Delta}'')/\mathbb{R}[x_0,\ldots,x_k]\right),$$

where quotienting of the second summand by $\mathbb{R}[x_0, \ldots, x_k]$ corresponds to eliminating the splines defined by the same polynomial on all maximal cells of Δ'' . As a consequence, we obtain dimension formulas and explicit bases for several commonly used subdivisions, their multivariate generalizations, as well as on various intermediate subdivisions. For these subdivisions, $S^r(\Delta')$ is free, and a generalization [7] of Schumaker's lower bound for the planar case [10] gives the correct dimension.

²⁰⁰⁰ Mathematics Subject Classification. Primary 41A15, Secondary 13D40, 52C99. Key words and phrases. spline, dimension formula.

H. Schenck is partially supported by NSF grant #1312071.

T. Sorokina is partially supported by a grant from the Simons Foundation #235411.

2. Homology and subdivisions

We work with the modification of Billera's complex introduced in [9]. Throughout this paper, our basic references are [5] for splines and [4] for algebra.

Definition 2.1. For a full-dimensional simplicial complex $\Delta \subseteq \mathbb{R}^k$, let $\mathcal{R}/\mathcal{J}(\Delta)$ be the complex of $R = \mathbb{R}[x_0, \dots, x_k]$ modules, with differential ∂_i the usual boundary operator in relative (modulo boundary) homology.

$$0 \longrightarrow \bigoplus_{\sigma \in \Delta_k} R \xrightarrow{\partial_k} \bigoplus_{\tau \in \Delta_{k-1}^0} R/J_{\tau} \xrightarrow{\partial_{k-1}} \bigoplus_{\psi \in \Delta_{k-2}^0} R/J_{\psi} \xrightarrow{\partial_{k-2}} \dots \xrightarrow{\partial_1} \bigoplus_{v \in \Delta_0^0} R/J_v \longrightarrow 0,$$

where for an interior i-face $\gamma \in \Delta_i^0$, we define

$$J_{\gamma} = \langle l_{\widehat{\tau}}^{r+1} \mid \gamma \subseteq \tau \in \Delta_{k-1} \rangle.$$

The ideal J_{γ} is generated by $r+1^{st}$ powers of homogenizations $l_{\widehat{\tau}}$ of linear forms l_{τ} whose vanishing defines the affine span of faces τ containing γ . The top homology module of $\mathcal{R}/\mathcal{J}(\Delta)$ computes splines of smoothness r on $\widehat{\Delta}$.

Theorem 2.2. [7] If Δ is a topological k-ball, then the module $S^r(\widehat{\Delta})$ is free iff $H_i(\mathcal{R}/\mathcal{J}(\Delta)) = 0$ for all i < k. In this case,

$$\dim S^r(\widehat{\Delta})_d = \sum_{i=0}^k (-1)^i \dim(\mathcal{R}/\mathcal{J}_{k-i})_d,$$

where $\mathcal{R}/\mathcal{J}_{k-i} = \bigoplus_{\psi \in \Delta_k^0} R/J_{\psi}$.

2.1. **Split subdivisions.** Our strategy is to relate splines on a simplicial complex Δ to splines on a complex Δ' obtained by subdividing some $\sigma \in \Delta_k$.

Definition 2.3. Let $\Delta \subseteq \mathbb{R}^k$ be a k-dimensional simplicial complex, $\sigma \in \Delta_k$, and Δ'' a subdivision of σ , such that $\partial(\sigma) = \partial(\Delta'')$ on Δ^0 . Then the resulting subdivision Δ' is again a simplicial complex, and we call the subdivision a **simple** subdivision. For each i-face $\gamma \in \Delta'_i$, let $J(\Delta')_{\gamma}$ denote the ideal in Definition 2.1. We call a simple subdivision Δ' split if for every $\gamma \in \partial(\Delta'')_i$ but not in $\partial(\Delta')$,

$$J(\Delta')_{\gamma} = J(\Delta)_{\gamma}$$
.

Note that Definition 2.3 imposes no conditions on faces of $\Delta'' \cap \partial(\Delta')$. The following example illustrates simple and split subdivisions.

Example 2.4. We start with Δ depicted in Figure 1, and subdivide the interior triangle into three subtriangles as in Figure 2. Both subdivision Δ' in Figure 3 and $\widetilde{\Delta}'$ in Figure 4 are simple. Moreover, when r=1, both Δ' and $\widetilde{\Delta}'$ are split, because when r=1 as soon as there are three distinct slopes at a vertex, J(v) is the square of the ideal of the vertex. However, when $r\geq 2$, only Δ' in Figure 3, where the new edge has the same slope as an existing edge, is a split subdivision.

For any subdivision Δ' of Δ , there is a tautological map of chain complexes

$$\mathcal{R}/\mathcal{J}(\Delta) \xrightarrow{v} \mathcal{R}/\mathcal{J}(\Delta'),$$

where if $\gamma \in \Delta_k$ is subdivided into $\gamma'_1, \ldots, \gamma'_m \in \Delta'_k$, v is induced by the map

$$\bigoplus_{\gamma \in \Delta_k} R \stackrel{v}{\longrightarrow} \bigoplus_{\gamma' \in \Delta'_k} R, \text{ which sends } 1_{\gamma} \text{ to } \sum_{i=1}^m 1_{\gamma'_i}.$$

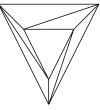


Figure 1. Δ







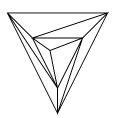


Figure 3. Δ'

Figure 4. $\widetilde{\Delta}'$

In general v will have both a kernel and a cokernel.

Lemma 2.5. For a simple subdivision Δ' of Δ , $\ker(v)$ is supported on $\partial(\Delta'')$, and $\operatorname{coker}(v)$ is supported on $(\Delta'')^0$. If Δ' is split then $\operatorname{ker}(v) = 0$.

Proof. The faces of $\Delta \setminus \sigma$ and $\Delta' \setminus \Delta''$ are identical, so for $\gamma \in \Delta \setminus \sigma = \Delta' \setminus \Delta''$ the ideal $J(\gamma)$ is also the same. In particular, both $\ker(v)$ and $\operatorname{coker}(v)$ are nonzero only on Δ'' . No face of Δ meets $(\Delta'')^0$ save σ itself, which v maps to the sum of k-faces of Δ'' . Thus

$$\operatorname{coker}(v) = \left\{ \begin{array}{ll} R^{|\Delta_k''|}/R, & \text{in degree } k, \\ \bigoplus_{\gamma \in \Delta_i''} R/J(\Delta')_{\gamma}, & \text{for all } i < k, \end{array} \right.$$

and $\operatorname{coker}(v)$ is nonzero only on the interior of Δ'' . For the kernel, if $\gamma \in \Delta \setminus \sigma$, then

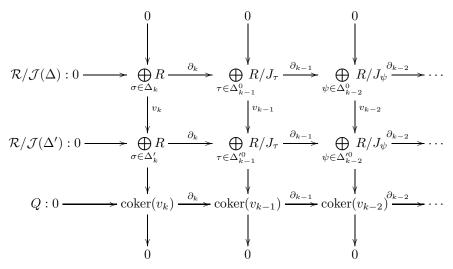
$$J(\Delta)_{\gamma} = J(\Delta')_{\gamma},$$

so the kernel can only be nonzero on $\partial(\sigma) = \partial(\Delta'')$. If Δ' is split, then again

$$J(\Delta)_{\gamma} = J(\Delta')_{\gamma}$$

for
$$\gamma \in \partial(\sigma)$$
, hence $\ker(v) = 0$.

Proposition 2.6. A split subdivision Δ' of Δ gives rise to a short exact sequence of complexes: a commuting diagram where the columns are exact, and the rows form complexes:



Proof. Follows from Lemma 2.5 and Definition 2.3.

Corollary 2.7. If Δ' is a split subdivision of Δ and $H_{k-1}(\mathcal{R}/\mathcal{J}(\Delta)) = 0$, then

$$0 \longrightarrow S^r(\widehat{\Delta}) \longrightarrow S^r(\widehat{\Delta}') \longrightarrow H_k(Q) \longrightarrow 0$$

is an exact sequence, so

$$\dim S^r(\widehat{\Delta}')_d = \dim S^r(\widehat{\Delta})_d + \dim H_k(Q)_d.$$

Proof. Recall that a short exact sequence of complexes yields a long exact sequence in homology

$$\to H_{i+1}(Q) \to H_i(\mathcal{R}/\mathcal{J}(\Delta)) \to H_i(\mathcal{R}/\mathcal{J}(\Delta')) \to H_i(Q) \to H_{i-1}(\mathcal{R}/\mathcal{J}(\Delta)) \to$$
 and the result follows.

2.2. Main result. We are now ready to explore why split subdivisions are special.

Theorem 2.8. If Δ' is a split subdivision of Δ , then for i < k,

$$\operatorname{coker}(v_i) \simeq \mathcal{R}/\mathcal{J}(\Delta'')_i$$

and in degree k

$$\operatorname{coker}(v_k) \simeq (\bigoplus_{\sigma \in \Delta_k^{"}} R)/R,$$

with the quotient map is via the diagonal.

Proof. Since the subdivision is split, the complexes $\mathcal{R}/\mathcal{J}(\Delta)$ and $\mathcal{R}/\mathcal{J}(\Delta')$ agree on the common faces, which are all faces save those in the interior of Δ'' . This means that the vertical maps in the double complex in Proposition 2.6 are either the identity or zero on each individual term of the direct sums. Hence the cokernel of v_i is simply $\mathcal{R}/\mathcal{J}(\Delta'')_i$. The exception to this is on σ , which maps via the identity to each k-face of Δ'' .

Theorem 2.9. If Δ' is a split subdivision of Δ and both $S^r(\widehat{\Delta})$ and $S^r(\widehat{\Delta}'')$ are free, then $S^r(\widehat{\Delta}')$ is free.

Proof. By Theorem 2.8, $H_k(Q) \simeq S^r(\widehat{\Delta}'')$, modulo constant splines. The result follows from the long exact sequence in homology associated with the short exact sequence of complexes in Proposition 2.6 coupled with Theorem 4.10 of [7].

3. Applications and computations

In this section, we apply the results of § 2 to various subdivisions Δ'' of σ such that $S^r(\widehat{\Delta}'')$ is free. Our starting point is recent progress in obtaining the dimension of multivariate splines of arbitrary degree and smoothness on the so-called Alfeld split $A(T_k)$ of an k-dimensional simplex T_k in \mathbb{R}^k , which is a higher dimensional analog of the Clough-Tocher split of a triangle, see [1] or Sections 18.3, 18.7 of [5]. The split $A(T_k)$ is obtained from a single simplex T_k by adding a single interior vertex u, and then coning over the boundary of T_k . A formula for the dimension of the space $S^r_d(A(T_k))$ of splines of smoothness r and polynomial degree at most d on $A(T_k)$ was conjectured in [6], and proved in [8], where it was also shown that the module is free.

Theorem 3.1. [8] Let $A(T_k)$ be the Alfeld split of an k-simplex T_k in \mathbb{R}^k . Then

$$\dim S_d^r(A(T_k)) = \binom{d+k}{k} + A(k,d,r),$$

where

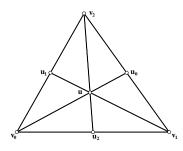
$$A(k,d,r) := \begin{cases} k \binom{d+k - \frac{(r+1)(k+1)}{2}}{k}, & \text{if } r \text{ is odd,} \\ \binom{d+k - 1 - \frac{r(k+1)}{2}}{k} + \dots + \binom{d - \frac{r(k+1)}{2}}{k}, & \text{if } r \text{ is even.} \end{cases}$$

Moreover, the associated module of splines $S^r(\widehat{A}(T_k))$ is free for any r.

The Alfeld split can be further refined to obtain other splits useful in applications, in particular for constructing macro-element spaces. Such constructions are not possible unless the exact dimension of the spline space of interest is known. In this section we concentrate on k-dimensional analogs of two known refinements of Alfeld splits in \mathbb{R}^2 and \mathbb{R}^3 . The first one is known in \mathbb{R}^2 as the Powell-Sabin split of a triangle, see Figure 5, and Sections 6.3, 7.3, 8.4 of [5], and the references therein. Its analog in \mathbb{R}^3 has been called both Worsey-Farin and Clough-Tocher, see e.g. Section 18.4 and 18.8 of [5] with the references therein. In order to eliminate any ambiguity, we introduce the following definition in \mathbb{R}^k .

Definition 3.2. For a full-dimensional k-simplex $T_k := [v_0, v_1, \ldots, v_k] \subseteq \mathbb{R}^k$, let $A(T_k)$ be the Alfeld split with the interior vertex u. The facet split $F(T_k)$ is obtained by further subdividing $A(T_k)$ as follows. For each $i = 0, \ldots, k$, let F_i be the facet of T_k opposite vertex v_i . Let u_i be the point strictly interior to F_i and collinear with v_i and u. Each u_i induces a (k-1)-dimensional Alfeld split $A(F_i)$ of F_i . Finally, cone u over $A(F_i)$ forming a pyramid P_i in \mathbb{R}^k . The collection of k+1 pyramids P_i is the facet split $F(T_k)$.

Note that if u is the barycenter of T_k , and each u_i is the barycenter of F_i , then the collinearity condition is satisfied. $F(T_k)$ consists of $k^2 + k$ simplices, has one interior vertex u, and 2k + 2 boundary vertices. Figure 7 shows an unfinished (for clarity) $F(T_3)$, where Definition 3.2 was carried out for i = 0 only, thus splitting the facet F_0 only and forming a pyramid P_0 while keeping the remaining three subtetrahedra of $A(T_3)$ intact.



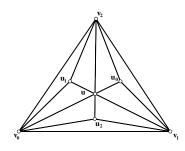


FIGURE 5. $F(T_2)$

FIGURE 6. $AA(T_2)$

Proposition 3.3. Let T_{k-1} be a (k-1)-simplex in \mathbb{R}^k , and let P_k be a pyramid in \mathbb{R}^k obtained by forming a cone with the base $A(T_{k-1})$ and an arbitrary vertex not coplanar with aff (T_{k-1}) . Then

$$\dim S_d^r(P_k) = \binom{d+k}{k} + P(k, d, r),$$

where

$$P(k,d,r) := \begin{cases} (k-1)\binom{d+k-\frac{(r+1)k}{2}}{k}, & \text{if } r \text{ is odd} \\ \binom{d+k-1-\frac{rk}{2}}{k} + \dots + \binom{d+1-\frac{rk}{2}}{k}, & \text{if } r \text{ is even.} \end{cases}$$

Moreover, the associated module of splines $S^r(\widehat{P}_k)$ is free for any r.

Proof. Since a pyramid $P_k \subseteq \mathbb{R}^k$ is a cone over the Alfeld split $A_{k-1} \subseteq \mathbb{R}^{k-1}$ of a tetrahedron T_{k-1} , Theorem 3.1 yields

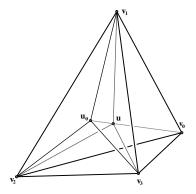
$$\dim S_d^r(P_k) = \sum_{i=0}^d \left[\binom{i+k-1}{k-1} + A(k-1,i,r) \right] = \binom{d+k}{k} + P(k,d,r),$$

and the proof is complete.

The second refinement of interest is the k-dimensional analog of the so-called double Clough-Tocher split in \mathbb{R}^2 , see Figure 6, and Section 7.5 of [5] along with the references therein. We shall call the new refinement the *double Alfeld* split to emphasize the multivariate nature.

Definition 3.4. For an k-simplex $T_k := [v_0, v_1, \ldots, v_k] \subseteq \mathbb{R}^k$, let $A(T_k)$ be the Alfeld split with the interior vertex u. The double Alfeld split $AA(T_k)$ is obtained by further subdividing $A(T_k)$ as follows for each $i = 0, \ldots, k$. Let F_i be the facet of T_k opposite vertex v_i . Let u_i be a point strictly interior to the simplex $T_k^i := [u, F_i]$ and collinear with v_i and u. Each u_i induces an Alfeld split $A(T_k^i)$ of T_k^i . The collection of k+1 Alfeld splits $A(T_k^i)$ is the double Alfeld split $AA(T_k)$.

Note that if u is the barycenter of T_k , and each u_i is the barycenter of T_k^i , then the collinearity condition is satisfied. $AA(T_k)$ consists of $(k+1)^2$ simplices, has k+2 interior vertices, and k+1 boundary vertices. Figure 8 shows an unfinished (for clarity) $AA(T_3)$, where Definition 3.4 was carried out for i=0 only, thus splitting



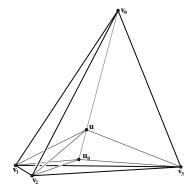


FIGURE 7. A part of $F(T_3)$

FIGURE 8. A part of $AA(T_3)$

the subterahedron T_3^0 only while keeping the remaining three subtetrahedra of $A(T_3)$ intact. We are now ready to prove the main result of this section.

Theorem 3.5. For an k-simplex T_k in \mathbb{R}^k let $F(T_k)$ and $AA(T_k)$ be the associated facet and double Alfeld splits as in Definition 3.2 and 3.4. Then

$$\dim S_d^r(F(T_k)) = \binom{d+k}{k} + A(k,d,r) + (k+1)P(k,d,r),$$

$$\dim S_d^r(AA(T_k)) = \binom{d+k}{k} + (k+2)A(k,d,r),$$

where A(k,d,r) and P(k,d,r) are as in Theorem 3.1 and Proposition 3.3, respectively. Moreover, the associated modules of splines $S^r(\widehat{F}(T_k))$ and $S^r(\widehat{AA}(T_k))$ are free for any r.

Proof. We start by subdividing the single simplex $\sigma := [u, v_1, v_2 \dots, v_k]$ in $A(T_k)$, as in Definition 2.3. In the case of the facet split, Δ_F'' is the pyramid P_k described in Theorem 3.3. Figure 7 demonstrates the 3D case, where the point u_0 is placed in the face $[v_1, v_2, v_3]$. For the double Alfeld split, Δ_{AA}'' is the Alfeld split of σ . Figure 8 depicts the 3D case, where the point u_0 is placed in the interior of the tetrahedron $[u, v_1, v_2, v_3]$. In either case, due to the collinearity conditions on u, v_0 and u_0 , the resulting subdivisions Δ_F' and Δ_{AA}' are simple and split as in Definition 2.3. Then by Corollary 2.7 and Theorem 2.8, we obtain

$$\dim S_d^r(\Delta_F') = \dim S_d^r(A(T_k)) + \dim S_d^r(P_k) - \binom{d+k}{k},$$

$$\dim S_d^r(\Delta_{AA}') = \dim S_d^r(A(T_k)) + \dim S_d^r(A(T_k)) - \binom{d+k}{k}.$$

Moreover, since by Theorem 2.9 the associated modules of splines $S^r(\widehat{\Delta}'_F)$ and $S^r(\widehat{\Delta}'_{AA})$ are free, we can apply the same technique to the next simplex $[u, v_0, v_2, v_3, \ldots, v_k]$ in the intermediate subdivision Δ'_F or Δ'_{AA} , and so on. Using the dimension formulae in Theorem 3.1 and Proposition 3.3 completes the proof. \square

4. Remarks

Remark 4.1. As a consequence of the split of the module of splines on Δ' , we also obtain explicit bases for $S^r(\Delta')$ essentially as a union of generators for $S^r(\Delta)$ and $S^r(\Delta'')$. The generators are not as useful in applications as more traditional B-spline or Bernstein-Bézier bases, and an efficient conversion algorithm is an open computational problem.

Remark 4.2. The proof of Theorem 3.5 holds for partial facet and double Alfeld splits, i.e. for the case where not every tetrahedron in $A(T_k)$ is subdivided. Such partial subdivisions are useful in the context of boundary finite elements.

Remark 4.3. As Example 2.4 demonstrates, the requirement of the collinearity in both Definition 3.2 and 3.4 can be omitted for r = 1.

Remark 4.4. The splitting method of \S 2 can be applied to more subdivisions, including those of a simplex in \mathbb{R}^k . We focused on two well-known splits that do not require consideration of multiple cases stemming from exact geometry. We also note that the freeness of the modules of splines involved in the splitting method is a sufficient but not a necessary condition. We are investigating extensions of the results here to other situations.

Acknowledgements Computations in the Macaulay2 package of Grayson and Stillman (available at http://www.math.uiuc.edu/Macaulay2) and in Alfeld's spline software (available at http://www.math.utah.edu/~pa) were essential to this work. We also thank the Mathematische Forschungsinstitut Oberwolfach, where our collaboration began.

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