Discrete and Algorithmic Geometry: Sheet 4

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1. Definition 9.2 in Ziegler's *Lectures on Polytopes* constructs the linear map

$$P \xrightarrow{\pi^c} Q^c := \left\{ \begin{pmatrix} \pi(x) \\ cx \end{pmatrix} : x \in P \right\} \subset \mathbb{R}^{q+1}$$

from a projection $\pi : P \subset \mathbb{R}^p \to Q \subset \mathbb{R}^q$ and a linear function $c \in (\mathbb{R}^p)^\star$. Is it possible to give an algorithm to determine the set of lower faces $\mathcal{L}^\downarrow(Q^c)$ of Q^c from just the set of facet normals of Q, the projection π, and the linear function c, without running a convex hull algorithm on Q^c?

2. Show that

$$\int_P f(x)\,\mathrm{d}x \;=\; \mathrm{vol}(P) \cdot f(p_0)$$

for any polytope P and linear function f, where $p_0 = \frac{1}{\mathrm{vol}(P)} \int_P x\,\mathrm{d}x$ denotes the barycenter of P.

3. Complete the proof of Theorem 9.6 in Ziegler's *Lectures on Polytopes*, possibly referring to [1].

1.

It is not possible to give such an algorithm.

This is because π, c and the set of facets of Q do not determine the lower faces of Q^c. Consider $\pi : \mathbb{R}^2 \to \mathbb{R}$ that deletes the last coordinate. Consdier then $c = (0,1)$. In this case, π^c is the identity in $\mathbb{R}^2$. Since in this case $q = 1$, the interval is the only polytope the set of facet normals of q is always the same, so $q = 1$, the only relevant information is π and c, but in this case, π^c is the identity. Therefore, it such algorithm existed, the set of lower faces would be the same for all polygons, which is not true.

2.

Using the fact that f is linear and linearity of the integral:

$$\mathrm{vol}(P) f(p_0) = \mathrm{vol}(P) f\left(\frac{1}{\mathrm{vol}(P)} \int_P x\,\mathrm{d}x\right) = f\left(\int_P x\,\mathrm{d}x\right) = \int_P f(x)\,\mathrm{d}x \tag{1}$$

3.

Claim 1. **$\boldsymbol{\Sigma(P,Q)}$ is a convex set**

Proof. Consider two points $y_1, y_2 \in \Sigma$, and a convex combination of them $y = q_1 y_1 + q_2 y_2$. Then $y_1 = \int_Q \gamma_1$, $y_2 = \int_Q \gamma_2$ for some γ_1, γ_2 sections.

Then, by linearity of the integral: $y = \int_Q q_1\gamma_1 + q_2\gamma_2$. So we have to see that the convex combination of sections is a section. Let $\gamma \overset{\text{def}}{=} q_1\gamma_1 + q_2\gamma_2$. Indeed, by linearity of π:

$$\pi(\gamma(x)) = \pi(q_1\gamma_1(x) + q_2\gamma_2(x)) = q_1\pi(\gamma_1(x)) + q_2\pi(\gamma_2(x)) = (q_1+q_2)x = x \tag{2}$$

□

Claim 2. **$\boldsymbol{\Sigma(P,Q) \subseteq \pi^{-1}(r_0)}$**

Proof. We want to see that $y \in \Sigma(P,Q) \implies \pi(y) = \boldsymbol{r_0}$. Consider $y \in \Sigma(P,Q)$. Then there exists $\gamma : Q \to P$ section, such that $y = \frac{1}{\mathrm{vol}(Q)} \int_Q \gamma(x)dx$ Then:

$$\pi(y) = \pi\left(\frac{1}{\mathrm{vol}(Q)} \int_Q \gamma(x)dx\right) \tag{3}$$

Using linearity of π and of the integral, this is equal to [1]

$$\frac{1}{\mathrm{vol}(Q)} \int_Q \pi(\gamma(x))dx = \frac{1}{\mathrm{vol}(Q)} \int_Q x dx = \boldsymbol{r_0} \tag{4}$$

□

[1] This step probably needs some more explanation. Or maybe not. It's just putting coordinates.

Claim 3. $\mathbf{dim(\Sigma(\boldsymbol{P},\boldsymbol{Q})) \leq \boldsymbol{p-q}}$

Proof. To prove this, we only need to prove that $\dim \pi^{-1}(\boldsymbol{r_0}) = p-q$. For a section to be tight, its corresponding subdivision must be tight as well, because a non-tight subdivision cannot be the image of a section.

Indeed, since π is a linear function, $\dim(\pi^{-1}(\boldsymbol{r_0})) = \dim\ker(\pi) = p - \dim \mathrm{Im}\pi$. Since we are assuming that P and Q have full dimension, $\dim \mathrm{Im}\pi = q$. □

Remark 1. A section $\gamma : Q \to P$ is uniquely defined by its image $\gamma(Q)$.

Proof. Given $x \in Q$, $\gamma(x)$ will be the only element in $\pi^{-1}(x) \cap \gamma(Q)$. This set has exactly one element because $\pi \circ \gamma = \mathrm{id}_Q$. □

Definition 1. A section $\gamma : Q \to P$ is *tight* [2] if:

$$\gamma(Q) = \bigcup_{F \in \mathcal{F}} F \qquad \text{for } \mathcal{F} \subseteq L(P) \text{ a subset of faces of } P \tag{5}$$

Remark 2. For a section to be tight, its corresponding subset of faces $\mathcal{F}$ must define a π-induced subdivision of Q, that is also tight.

Proof. First observe that a section is an homeomorphism when restricted to its image, because it is a continuous function, and its inverse (the restriction of π) is a linear (and thus continuous) map. This means, in particular, that γ has to respect dimensions of faces.

For a subset of faces $\mathcal{F} \in L(P)$ to define a π-induced subdivision, it must satisfy condition (ii) in [2, Def 9.1]. Since γ maintains dimensions and π is a linear projection, for all $F \in \mathcal{F}$, $\pi^{-1}(\pi(F)) = F$, so condition (ii) is always satisfied.

By the same dimensional argument, the π-induced subdivision of Q defined by γ must also be tight. □

Note that given $\mathcal{F} \in L(P)$, the only issue for $\mathcal{F}$ to define a tight section is the part of defining a section, because if it does, then it is trivially tight [3].

With this definition, there is trivially a finite number of tight sections, since each section is defined by its image, which is determined by a subset of $L(P)$, and there are a finite number of them.

Remark 3. The partial order on $\omega(P, Q)$ defined in [2, Sec. 9.1] is indeed a partial order. In particular, each π-induced subdivision $\mathcal{F}$ is determined by the union of its faces in P, $\bigcup_{F \in \mathcal{F}} F$.

Proof. Let $\mathcal{F}$ be a π-induced subdivision and let $X = \bigcup_{F \in \mathcal{F}} F$. Let $\mathcal{G}$ be an arbitrary π-induced subdivision satisfying $\bigcup_{G \in \mathcal{G}} G = X$. Let $H_1, H_2, ..., H_l \subseteq X$ be the maximal elements from $L(P)$ contained in X. Then all the H_i must be in $\mathcal{G}$. It is clear that $\pi(H_i)$ must be the maximal faces in $\pi(\mathcal{G})$ and thus $\pi(\mathcal{G}) = L(\pi(G_1)) \cup \cdots \cup L(\pi(G_l))$, as $\pi(\mathcal{G})$ must be a polytopal complex. Finally, condition (ii) in [2, Def. 9.1] implies that the faces in $\mathcal{G}$ must be the ones satisfying $G = \pi^{-1}(J) \cup X$ for some face J of $\pi(\mathcal{G})$, so $\mathcal{G}$ is unequivocally determined by X. □

Claim 4. A π-section γ is not tight if and only if there is a face $F \in L(P)$ such that $F \nsubseteq \gamma(Q)$ and $\mathrm{relint}\ F \cap \gamma(Q) \neq \emptyset$.

Proof. We will prove the contrapositive statement, i.e. γ is tight if and only if for every face $F \in L(P)$ such that $\mathrm{relint}\ F \cap \gamma(Q) \neq \emptyset$ then $F \subseteq \gamma(Q)$:

$\boxed{\Rightarrow}$ Suppose that for some $r \in Q$ and $F \in L(P)$, $\gamma(r) \in \mathrm{relint}\ F$. Then any face $G \in L(P)$ contains $\gamma(r)$ if and only if $F \leq G$. If γ is tight, then $\gamma(Q)$ is an union of faces from P, so $\gamma(Q)$ must contain a face greater than F and in consequence it contains F itself.

$\boxed{\Leftarrow}$ Suppose that $\gamma(Q)$ contains all the faces of P whose relative interior intersect. Note that for every $r \in Q$, $\gamma(r)$ belongs to the relative interior of exactly one face of $L(P)$, namely the minimal face containing $\gamma(r)$. Let us denote by $F(r)$ to such face. Then, clearly $\gamma(Q) = \bigcup_{r \in Q} F(r)$ and γ is tight. □

We will denote by $L_n(P)$ to the set of faces

Claim 5. Let γ be a π-section, $r \in Q$ and $F \in L(P)$ such that $\gamma(r) \in \mathrm{relint}\ F$. If every open set (relative to Q) $B \subseteq Q$ satisfying $r \in B$ verifies $\gamma(B) \nsubseteq F$, then there exists a face $G > F$ such that $\gamma(Q) \cap \mathrm{relint}\ G \neq \emptyset$.

Proof. By hypothesis $\gamma(r) \in \gamma(Q)\backslash$. □

Claim 6. If a section γ is not tight, there exist two sections γ_1, γ_2 such that γ is a convex combination of γ_1 and γ_2, and the three points of $\Sigma(P, Q)$ defined by them are different.

Proof. □ Coming soon.

Claim 7. $\Sigma(P, Q)$ is a polytope.

Proof. We know by claim 1 that it is convex. By claim 6 and the number of tight sections being finite, we know that only a finite number of points cannot be expressed as a convex combination of different elements in $\Sigma(P, Q)$. Therefore, it is the convex hull of a finite number of points. □

Remark 4. Now we can say that we can restrict to sections that are piece-wise linear over a subdivision of Q, because all points of Σ are convex combinations of points defined by tight sections, that are piece-wise linear over their subdivision of Q.

Definition 2. Given $S \subset \mathbb{R}^p$ we will call the *direction* $\overrightarrow{S}$ of S to the vector subspace of $\mathbb{R}^p$ spanned by all vectors of the form $x - y$ for some $x, y \in S$.

We will use the following results:

Theorem 1. *Let $P \subset \mathbb{R}^p$ be a polytope and $c \in (\mathbb{R}^p)^\star$ a linear function. Then c reaches its maximum over P in a non-empty face of P. In other words:* $\arg\max(c_{|_P}) \in L(P)$.

Remark 5. In the previous theorem, it is direct that if $F = \arg\max(c_{|_P})$ then $\overrightarrow{F} \subseteq \ker c$.

Definition 3. Given a polytope $P \in \mathbb{R}^P$, we will say that a linear function $c \in (\mathbb{R}^p)^\star$ is *generic* with respect to P if it reaches its maximum exactly in one vertex of P, i.e $\arg\max(c_{|_P}) \in V(P)$.

Corollary 1. *Let $P \in \mathbb{R}^P$ be a polytope, and $c \in (\mathbb{R}^p)^\star$ be a linear function such that for any non-vertex face $F \in L(P)$ it is satisfied $\overrightarrow{F} \nsubseteq \ker c$. Then c is generic respect to P.*

Lemma 1. *Let $P \subset \mathbb{R}^p$ be a polytope and let $A \subset \mathbb{R}^p$ be an affine set. Then the intersection $P \cap A$ is also a polytope and its non-empty faces are of the form $F \cap A$ for some $F \in L(P)$.*

Lemma 2. *Given $s \in P$ and $v \in \mathbb{R}^p$ such that $s + v \in P$. There exists $U \subseteq P$, an open set (with the topology of P), $s \in U$, and $\varepsilon > 0$ such that $U + \varepsilon v \subseteq P$.*

Proof. □ Coming soon.

Definition 4. Given a two polytopes $P \subset \mathbb{R}^p$, $Q \subset \mathbb{R}^q$ and a projection between them $\pi : \mathbb{R}^p \to \mathbb{R}^q$, $\pi(P) = Q$, we will say that a linear function $c \in (\mathbb{R}^p)^\star$ is **generic** with respect to π over P if every face $F \in L(P)$ such that $\overrightarrow{F} \cap \overrightarrow{\ker \pi} \neq \{0\}$ satisfies $\overrightarrow{F} \cap \overrightarrow{\ker \pi} \nsubseteq \ker c$.

From now on we will keep the notation used in last definition.

Corollary 2. *If a linear function $c \in (\mathbb{R}^p)^\star$ is generic with respect to π over P then it is also generic with respect to each fiber $\pi^{-1}(r)$, $r \in Q$.*

As its name suggests, "genericness" is an "almost-sure" property:

Lemma 3. *Under the canonical identification $(\mathbb{R}^p)^\star \simeq \mathbb{R}^p$ the following sets of linear functions in $(\mathbb{R}^p)^\star$ are closed with empty interior:*

(1) The set of non-generic functions with respect to P.

(2) The set of non-generic functions with respect to π over P.

Proof. We will prove the statement for case (2). To prove it for case (1) one can proceed analogously. Note that given a set $S \subset \mathbb{R}^p$ and a function $c \in (\mathbb{R}^p)^\star$, $S \subseteq \ker c$ is equivalent to $c \in S^\perp$. Now, note that there are finitely many linear subspaces of the form $G = \overrightarrow{F} \cap \overrightarrow{\ker \pi}$ with $G \neq \{0\}$. Finally, for any of such G's, $G^\perp$ is trivially closed and it also has empty interior, as $\dim G < p$. □

This way, given $c \in (\mathbb{R}^p)^\star$ generic with respect to π over P, we can define a section γ^c as

$$\gamma^c(\boldsymbol{r}) = \underset{y \in \pi^{-1}(\boldsymbol{r})}{\arg\max}\{c(y)\} \tag{6}$$

Claim 8. The map γ^c is indeed a section.

Proof. □ Coming soon.

[2]Because working with tight sections without defining them seems to be too *Zieglery*.

[3]In fact, this observation suggests that the term *tight* is not well suited for this kind of sections, but for the sake of clarity, we wanted to use the same naming as in [2].

Claim 9. The section previously defined γ^c is tight, and its corresponding subdivision of Q is π-coherent.

Proof. Just note that $S = \{(x, y) \in \mathbb{R}^{q+1} \,|\, x \in Q\,, y = c(\gamma^c(x))\}$ is the union of the lower faces of Q^c. This implies that $(\pi^c)^{-1}(S) = \gamma^c(Q)$ is the union of faces of a coherent subdivision. $\square$

Now we know that tight coherent subdivisions of Q correspond to vertices of Σ. We still have to prove the correspondence of the poset ω_{coh} with the face lattice of $\Sigma(P, Q)$.

To do that, let us first note that every element $c \in (\mathbb{R}^p)^\star$ defines both a face in Σ and a coherent subdivision in Q^c:

1. The face it defines is ϕ^c, given by the valid inequality $c(s) \geq \min_{y \in \Sigma} c(y)$.
2. The coherent subdivision it defines is the one given by $\mathcal{F}^c$, as in [2, Def. 9.2].

We will see that we can find a bijection between faces of Σ and coherent subdivisions of Q through the elements $c \in (\mathbb{R}^p)^\star$.

Definition 5. Let $s \in \Sigma$, then

$$\Gamma(s) \stackrel{\text{def}}{=} \left\{ \gamma : Q \to P \text{ section} : \ s = \frac{1}{\text{vol}(Q)} \int_Q \gamma(x)\, x \right\} \tag{7}$$

Definition 6. Given $c \in (\mathbb{R}^p)^\star$, let us define the following sets of sections:

$$\Gamma(\mathcal{F}^c) \stackrel{\text{def}}{=} \left\{ \gamma \ : \ \gamma(Q) \subseteq \bigcup_{F \in \mathcal{F}^c} F \right\}, \qquad \Gamma(\phi^c) \stackrel{\text{def}}{=} \bigcup_{s \in \phi^c} \Gamma(s) \tag{8}$$

Definition 7. Given $c \in (\mathbb{R}^p)^\star$, we define the functional $\mathcal{A}^c : \{\text{sections of } \pi\} \to \mathbb{R}$ as:

$$\gamma \mapsto \mathcal{A}^c(\gamma) \stackrel{\text{def}}{=} \frac{1}{\text{vol}(Q)} \int_Q c(\gamma(x))\, \mathrm{d}x \tag{9}$$

Theorem 2. *Given γ_1, γ_2 sections,*

1. $\gamma_1, \gamma_2 \in \Gamma(\mathcal{F}^c) \implies \mathcal{A}(\gamma_1) = \mathcal{A}(\gamma_2)$.

2. $\gamma_1 \in \Gamma(\mathcal{F}^c), \gamma_2 \notin \Gamma(\mathcal{F}^c) \implies \mathcal{A}(\gamma_1) < \mathcal{A}(\gamma_2)$.

Proof. $\boxed{1.}$
$\mathcal{A}^c(\gamma_1) - \mathcal{A}^c(\gamma_2) = \frac{1}{\text{vol}(Q)} \int_Q [c(\gamma_1(x)) - c(\gamma_2(x))]\, \mathrm{d}x$.

Since $\gamma_1, \gamma_2 \in \Gamma(\mathcal{F}^c)$, we know that $\forall x \in Q$, the points $\binom{x}{c(\gamma_1(x))}$, $\binom{x}{c(\gamma_2(x))}$ are in a lower face of Q^c. Since their first q coordinates are equal, and the face is a lower face, it must happen that the last coordinate is also the same. This implies that $[c(\gamma_1(x)) - c(\gamma_2(x))] = 0$. Hence $\mathcal{A}^c(\gamma_1) - \mathcal{A}^c(\gamma_2) = 0$.

$\boxed{2.}$ Since $\gamma_2 \notin \Gamma(\mathcal{F}^c)$, there exists $r \in Q$ such that $\gamma(r)$ does not minimize c in $\pi^{-1}(r)$. Let $s \in \arg\min_{y \in \pi^{-1}(r)} \{c(y)\}$,

Then we can define $v \stackrel{\text{def}}{=} s - \gamma(r)$. By construction, we are in the hypothesis of lemma 2, so there exists $B \subseteq P$ open, and $\varepsilon > 0$, $\gamma_2(r) \in B$ and $B + \varepsilon v \subseteq P$. $\square$

Completio of proof coming soon

This last theorem implies that, indeed $\Gamma(\mathcal{F}^c) = \Gamma(\phi^c)$, which gives the desired bijection between the face lattice of Σ and ω_{coh}.

References

[1] Louis J. Billera and Bernd Sturmfels, *Fiber polytopes.*, Ann. Math. (2) **135** (1992), no. 3, 527–549 (English).

[2] Günter M Ziegler, *Lectures on polytopes*, vol. 152, Springer Science & Business Media, 2012.