Monotone multilevel for FOSLS linear elastic contact

G. Rovi, B. Kober, G. Starke, R. Krause

Universität Duisburg - Essen, Germany
Università della Svizzera italiana, Switzerland

February 4, 2019

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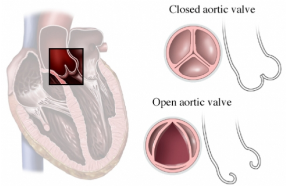
Offen im Denken

Examples of contact problems



- Contact problems with incompressible materials.
- Quantities of interest: the forces generated by the contact.

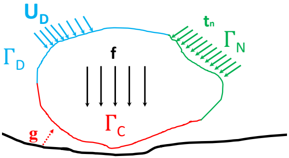
Signorini's problem: strong formulation

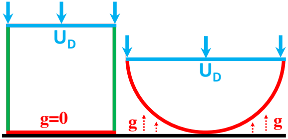
- **Linear Elasticity:**

$$
\begin{cases}
\operatorname{div}\boldsymbol{\sigma} + \mathbf{f} = 0 & \Omega \quad \text{momentum balance equation} \\
\mathcal{A}\boldsymbol{\sigma} - \boldsymbol{\varepsilon}(\mathbf{u}) = 0 & \Omega \quad \text{constitutive law} \\
\mathbf{u} = \mathbf{u}_D & \Gamma_D \quad \text{Dirichlet BC} \\
\boldsymbol{\sigma}\mathbf{n} = \mathbf{t}_N & \Gamma_N \quad \text{Neumann BC}
\end{cases}
$$

- **Contact Constraints:**
  $\partial\Omega = \Gamma_C \cup \Gamma_D \cup \Gamma_N$, $\Gamma_i \cap \Gamma_j = \emptyset$ for $i, j = D, N, C$, $i \neq j$

$$
\begin{cases}
\mathbf{u}\cdot\mathbf{n} - g \leq 0 & \Gamma_C \quad \text{impenetrability} \\
(\boldsymbol{\sigma}\mathbf{n})\cdot\mathbf{n} \leq 0 & \Gamma_C \quad \text{direction of the surface pressure} \\
(\mathbf{u}\cdot\mathbf{n} - g)\left((\boldsymbol{\sigma}\mathbf{n})\cdot\mathbf{n}\right) = 0 & \Gamma_C \quad \text{complementarity condition} \\
\mathbf{t}_j^T(\boldsymbol{\sigma}\mathbf{n}_j) = 0 & \Gamma_C \quad \text{frictionless condition}
\end{cases}
$$





- **First Order System Least-Squares (FOSLS) Functional**

$$C_1,\ C_2,\ C_3 > 0$$

$$\mathcal{J}(\mathbf{u},\boldsymbol{\sigma}) = C_1 \left\| \mathrm{div}\boldsymbol{\sigma} + \mathbf{f} \right\|^2_{L^2(\Omega)^d} + C_2 \left\| \mathcal{A}\boldsymbol{\sigma} - \boldsymbol{\varepsilon}(\mathbf{u}) \right\|^2_{L^2(\Omega)^d} + C_3 \langle \mathbf{u}\cdot\mathbf{n} - g, (\boldsymbol{\sigma}\mathbf{n})\cdot\mathbf{n} \rangle_{\Gamma_C}$$

- **Convex Set K**

$$K = \{(\mathbf{u},\boldsymbol{\sigma}) \in \left[H^1_{\Gamma_d}(\Omega)\right]^d \times \left[H_{\mathrm{div},\Gamma_N}(\Omega)\right]^d : \ \mathbf{u}\cdot\mathbf{n} - g \leq 0,\ (\boldsymbol{\sigma}\mathbf{n})\cdot\mathbf{n} \leq 0,\ \mathbf{t}_j^T(\boldsymbol{\sigma}\mathbf{n}_j) = 0 \quad \Gamma_C\}$$

- Find $(\mathbf{u},\boldsymbol{\sigma}) \in K$, such that:

•**Minimization problem:** $\mathcal{J}(\mathbf{u},\boldsymbol{\sigma}) \leq \mathcal{J}(\mathbf{v},\boldsymbol{\tau}) \qquad \forall (\mathbf{v},\boldsymbol{\tau}) \in K$

$$\Longleftrightarrow$$

•**Variational Inequality:**

$$\begin{cases} \left\langle \dfrac{\partial \mathcal{J}(\mathbf{u},\boldsymbol{\sigma};\mathbf{f},g)}{\partial \mathbf{u}}, \mathbf{v}-\mathbf{u} \right\rangle \geq 0 \\ \left\langle \dfrac{\partial \mathcal{J}(\mathbf{u},\boldsymbol{\sigma};\mathbf{f},g)}{\partial \boldsymbol{\sigma}}, \boldsymbol{\tau}-\boldsymbol{\sigma} \right\rangle \geq 0 \end{cases} \qquad \forall (\mathbf{v},\boldsymbol{\tau}) \in K$$

Rolf Krause, Benjamin Müller, and Gerhard Starke. An adaptive least-squares mixed finite element method for the Signorini problem. Numerical Methods for Partial Differential Equations, 33(1):276-289, 2017.

Advantages of the Least-Squares Approach

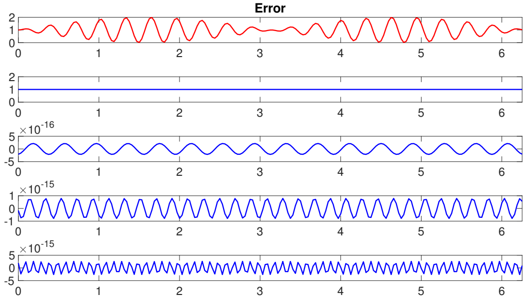
- Direct access to stress $\boldsymbol{\sigma}$ (friction, plasticity...)
- Dealing with incompressible materials ($\lambda \to \infty$)
- FOSLS functional as an a posteriori error estimator
- Flexible choice of finite element spaces (low order: $\mathbf{u}_h \in P^1$, $\boldsymbol{\sigma}_h \in \mathcal{RT}_0$)
- Symmetric positive definite system

Disadvantages of the Least-Squares Approach

- The functional is fictitious, not physical
- The asymmetry of the stress tensor
- Find proper weights C_1, C_2, C_3
- Large condition number: need for a **multilevel method**

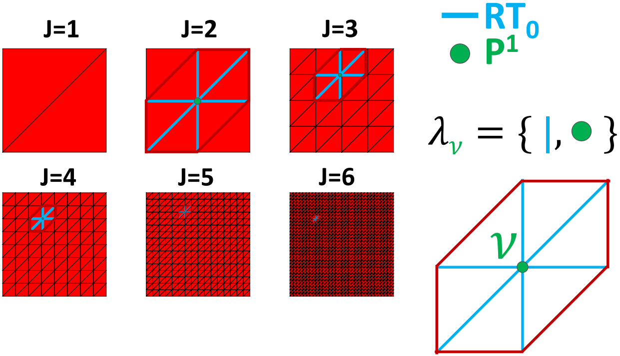
Attia, Frank S., Zhiqiang Cai, and Gerhard Starke. "First-order system least squares for the Signorini contact problem in linear elasticity". SIAM Journal on Numerical Analysis 47.4 (2009): 3027-3043.

- A proper smoother must smooth error components related to the large eigenvalues of the system operator;



- Gauß-Seidel smooths H^1, **but not** H_{div}, high frequencies of the error;
- The kernel $\text{Ker}(\text{div}) = \{\boldsymbol{\tau} \in H_{\text{div}}, \text{div}\,\boldsymbol{\tau} = 0\}$ is too large;
- Patch-smoother for divergence-free components of the error;

Linear Multilevel: nested meshes and patches



- Minimization of the functional $\mathcal{J}(\mathbf{u}, \boldsymbol{\sigma})$ on $\mathrm{span}(\lambda_{j,\nu})$
- Exploit ν-patches to smooth the error in H^1 and H_{div} simultaneously

Ralf Hiptmair. Multigrid method for H(div) in three dimensions. Electron. Trans. Numer. Anal, 6(1):133-152, 1997.

Douglas N Arnold, Richard S Falk, and Ragnar Winther. Multigrid in H(div) and H(curl). Numerische Mathe- matik, 85(2):197-217, 2000.

Gerhard Starke. Gauss-Newton multilevel methods for least-squares finite element computations of variably saturated subsurface flow. Computing, 64(4):323-338, 2000.

Define:

- $\mathbf{x}_J^k = (\mathbf{u}_J^k, \boldsymbol{\sigma}_J^k) \in K_J$ k-th iterate
- $\mathbf{x}_{J,0} = \mathbf{x}_J^k$
- $\mathbf{x}_{j,0} = \mathbf{x}_{j+1,N_{j+1}}$, for $j = J-1, \ldots, 1$

Compute a sequence of intermediate iterates $\mathbf{x}_{j,\nu} = \mathbf{x}_{j,\nu-1} + \mathbf{c}_{j,\nu}$:

$$\begin{aligned}
&\mathcal{J} \le \mathcal{J}(\mathbf{x}_{j,\nu} + \mathbf{y}) \quad \forall \mathbf{y} \in K_{j,\nu}^* \qquad j = J, \ldots, 2, \quad \nu = 1, \ldots, N_j \\
&\mathcal{J}(\mathbf{x}_{2,N_2} + \mathbf{c}_1) \le \mathcal{J}(\mathbf{x}_{2,N_2} + \mathbf{y}) \quad \forall \mathbf{y} \in K_1^* \qquad j = 1
\end{aligned}$$

with the **exact** local closed convex sets $K_{j,\nu}^*$ and K_1^*:

$$\begin{aligned}
K_{j,\nu}^*(\mathbf{x}_{j,\nu}) &= \left\{\mathbf{y} \in \operatorname{span}\{\boldsymbol{\lambda}_{j,\nu}\} : \quad \mathbf{y} + \mathbf{x}_{j,\nu} \in K_J\right\} \\
K_1^*(\mathbf{x}_{2,N_2}) &= \left\{\mathbf{y} \in \operatorname{span}\{\boldsymbol{\lambda}_1\} : \quad \mathbf{y} + \mathbf{x}_{2,N_2} \in K_J\right\}
\end{aligned}$$

Ralf Kornhuber. Monotone multigrid methods for elliptic variational inequalities I. Numerische Mathematik, 69(2):167-184, 1994.

Ralf Kornhuber and Rolf Krause. Adaptive multigrid methods for Signorini's problem in linear elasticity. Computing and Visualization in Science, 4(1):9-20, 2001.

Define:

- $\mathbf{c}_{j,\nu} = (\tilde{\mathbf{u}}_{j,\nu}, \tilde{\boldsymbol{\sigma}}_{j,\nu})$ correction at level j, patch ν
- $\mathbf{c}_{J,0} = \mathbf{x}_J^k$, $\mathbf{c}_{j,0} = \mathbf{0}$ for $j = J-1, \ldots, 1$
- $\mathbf{w}_{j,\nu} = \sum_{\mu=0}^{\nu} \mathbf{c}_{j,\mu}$

Compute a sequence of intermediate corrections $\mathbf{c}_{j,\nu} \in K_{j,\nu}(\mathbf{w}_{j,\nu-1})$ and $\mathbf{c}_1 \in K_1$:

$$\begin{aligned}
&\mathcal{J}(\mathbf{w}_{j,\nu-1} + \mathbf{c}_{j,\nu}) \le \mathcal{J}(\mathbf{w}_{j,\nu-1} + \mathbf{y}) && \forall\, \mathbf{y} \in K_{j,\nu} && j = J, \ldots, 2, \;\; \nu = 1, \ldots, N_j \\
&\mathcal{J}(\mathbf{c}_1) \le \mathcal{J}(\mathbf{y}) && \forall\, \mathbf{y} \in K_1 && j = 1
\end{aligned}$$

with the **coarse convex sets** K_j and the **approximate** local closed convex sets $K_{j,\nu}$:

$$\begin{aligned}
&K_{j,\nu}(\mathbf{w}_{j,\nu-1}) = \left\{ \mathbf{y} \in \operatorname{span}\{\lambda_{j,\nu}\} : \; \mathbf{y} + \mathbf{w}_{j,\nu-1} \in K_j \right\} \\
&K_1 \subset K_2 \subset \ldots \subset K_{J-1} \subset K_J
\end{aligned}$$

Coarse Convex Sets:

$$K_j = \Big\{ \mathbf{x}_j = (\mathbf{u}_j, \boldsymbol{\sigma}_j) \in X_j : \ \mathbf{u}_j|_{\Gamma_D} = \mathbf{u}_D, \ \boldsymbol{\sigma}_j|_{\Gamma_N} = \mathbf{t}_N, \\ \mathbf{u}_j \cdot \mathbf{n}_j|_{\Gamma_C} \le g_{j,u_n}, \ \mathbf{n}^T(\boldsymbol{\sigma}_j \mathbf{n}) \le g_{j,\sigma_n}, \ \mathbf{t}_j^T(\boldsymbol{\sigma}\mathbf{n}_j) = 0 \Big\} \qquad j = J$$

$$K_j = \Big\{ \mathbf{x}_j = (\mathbf{u}_j, \boldsymbol{\sigma}_j) \in X_j : \ \mathbf{u}_j|_{\Gamma_D} = \mathbf{0}, \ \boldsymbol{\sigma}_j|_{\Gamma_N} = \mathbf{0}, \\ \mathbf{u}_J \cdot \mathbf{n}_j|_{\Gamma_C} \le g_{j,u_n}, \ \mathbf{n}^T(\boldsymbol{\sigma}_j \mathbf{n}) \le g_{j,\sigma_n}, \ \mathbf{t}_j^T(\boldsymbol{\sigma}\mathbf{n}_j) = 0 \Big\} \qquad j = J-1, \dots, 1$$

Coarse Constraints:

- $\tilde{\mathbf{u}}_{j,\nu}$ and $\tilde{\boldsymbol{\sigma}}_{j,\nu}$ are the components of the correction $\mathbf{c}_{j,\nu}$.

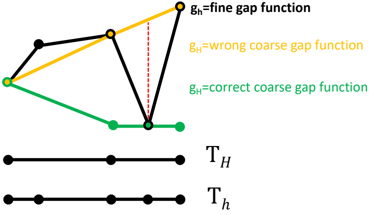
$$g_{j,u_n} = \begin{cases} g & j = J \\ I^j_{j+1,u_n}\left(g_{j+1,u_n} - \sum_{\nu=1}^{N_{j+1}} \left[\tilde{\mathbf{u}}_{j+1,\nu}|_{\Gamma_C} \right]_n \right) & j = J-1, \dots, 1 \end{cases}$$

$$g_{j,\sigma_n} = \begin{cases} 0 & j = J \\ I^j_{j+1,\sigma_n}\left(g_{j+1,\sigma_n} - \sum_{\nu=1}^{N_{j+1}} \left[\tilde{\boldsymbol{\sigma}}_{j+1,\nu}|_{\Gamma_C} \right]_n \right) & j = J-1, \dots, 1 \end{cases}$$

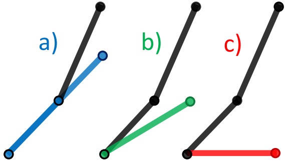
Non-Linear Projection Operators:

I^j_{j+1,u_n}, I^j_{j+1,σ_n} chosen so that $K_1 \subset K_2 \subset \dots \subset K_{J-1} \subset K_J$.

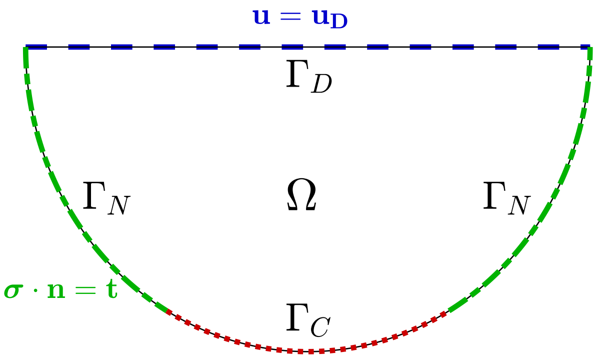
- Wrong and correct coarse constraints



- Different consistent coarse constraints



Hertzian Contact - Setting



$\mathbf{u}_D = (0, -0.01)^T$, $\mathbf{t} = \mathbf{0}$, $\mathbf{f} = \mathbf{0}$, $\mu = 1$, $\lambda = 1, \infty$ (compressible and **incompressible**)

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Hertzian Contact, two-level problem

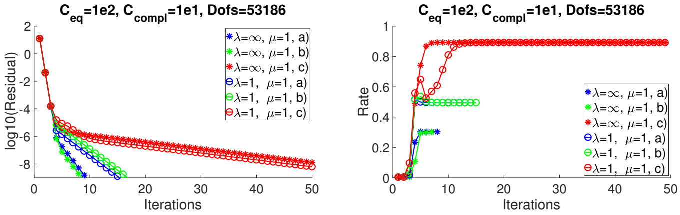

Figure: Mesh with $h_{max}/h_{min} = 7.0567$

- **First phase**: non-linear, capturing high frequencies
- **Second phase**: linear, known active set (blue, green), and not already known active set(red)
- **Similar behaviour** of compressible and incompressible cases

Signorini's problem, square mesh

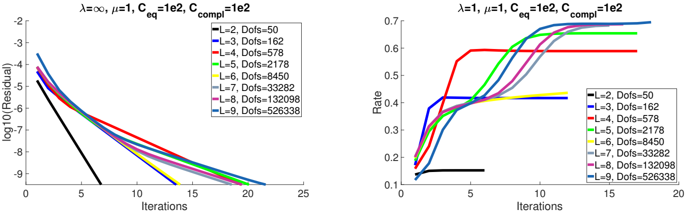

Figure: Square mesh. Compressible material.

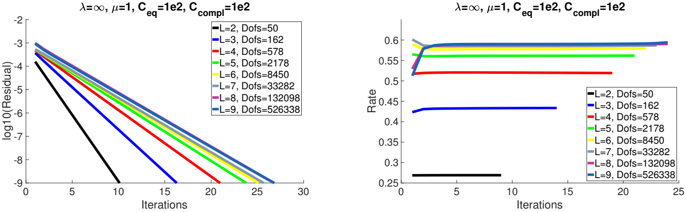

Figure: Square mesh. Incompressible material.

- Purely linear problem: $h-$ and $J-$ independency