Monotone multilevel for FOSLS linear elastic contact

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February 6, 2019

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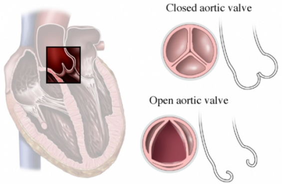
Offen im Denken

Examples of contact problems



- Contact problems with incompressible materials.
- Quantities of interest: the forces generated by the contact.

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Signorini's problem: strong formulation

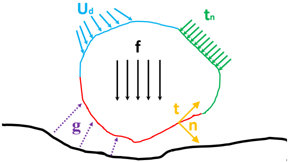
- **First Order System Linear Elasticity:**

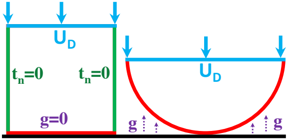
$$\begin{cases} \operatorname{div}\boldsymbol{\sigma} + \mathbf{f} = 0 & \Omega & \text{momentum balance equation} \\ \mathcal{A}\boldsymbol{\sigma} - \boldsymbol{\varepsilon}(\mathbf{u}) = 0 & \Omega & \text{constitutive law} \\ \mathbf{u} = \mathbf{u}_D & \Gamma_D & \text{Dirichlet BC} \\ \boldsymbol{\sigma}\mathbf{n} = \mathbf{t}_N & \Gamma_N & \text{Neumann BC} \end{cases}$$

where $\boldsymbol{\varepsilon}(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T)$, $\boldsymbol{\mathcal{A}} = \frac{1}{2\mu}\left(\boldsymbol{\sigma} - \frac{\lambda}{d\lambda + 2\mu}\operatorname{tr}\boldsymbol{\sigma}\mathbf{I}\right)$ and μ, λ are the Lamé parameters

- **Contact Constraints:**
$\partial\Omega = \Gamma_C \cup \Gamma_D \cup \Gamma_N$, $\Gamma_i \cap \Gamma_j = \emptyset$ for $i, j = D, N, C$, $i \neq j$

$$\begin{cases} \mathbf{u}\cdot\mathbf{n} - g \le 0 & \Gamma_C & \text{impenetrability} \\ (\boldsymbol{\sigma}\mathbf{n})\cdot\mathbf{n} \le 0 & \Gamma_C & \text{direction of the surface pressure} \\ (\mathbf{u}\cdot\mathbf{n} - g)\,((\boldsymbol{\sigma}\mathbf{n})\cdot\mathbf{n}) = 0 & \Gamma_C & \text{complementarity condition} \\ \mathbf{t}_i^T(\boldsymbol{\sigma}\mathbf{n}) = 0 & \Gamma_C & \text{frictionless condition} \end{cases}$$





The Least-Squares Formulation

- **First Order System Least-Squares (FOSLS) Functional**

$$C_1,\ C_2,\ C_3 > 0$$

$$\mathcal{J}(\mathbf{u},\boldsymbol{\sigma}) = C_1 \left\|\mathrm{div}\boldsymbol{\sigma} + \mathbf{f}\right\|^2_{L^2(\Omega)^d} + C_2 \left\|\mathcal{A}\boldsymbol{\sigma} - \boldsymbol{\varepsilon}(\mathbf{u})\right\|^2_{L^2(\Omega)^d} + C_3 \langle \mathbf{u}\cdot\mathbf{n} - g, (\boldsymbol{\sigma}\mathbf{n})\cdot\mathbf{n}\rangle_{\Gamma_C}$$

- **Convex Set K**

$$K = \{(\mathbf{u},\boldsymbol{\sigma}) \in \left[H^1_{\Gamma_d}(\Omega)\right]^d \times \left[H_{\mathrm{div},\Gamma_N}(\Omega)\right]^d :\ \mathbf{u}\cdot\mathbf{n} - g \le 0,\ (\boldsymbol{\sigma}\mathbf{n})\cdot\mathbf{n} \le 0,\ \mathbf{t}_i^T(\boldsymbol{\sigma}\mathbf{n}) = 0 \quad \Gamma_C\}$$

- Find $(\mathbf{u},\boldsymbol{\sigma}) \in K$, such that:
  - **Minimization problem:** $\mathcal{J}(\mathbf{u},\boldsymbol{\sigma}) \le \mathcal{J}(\mathbf{v},\boldsymbol{\tau}) \qquad \forall (\mathbf{v},\boldsymbol{\tau}) \in K$

$$\Longleftrightarrow$$

  - **Variational Inequality:**

$$\begin{cases} \left\langle \dfrac{\partial \mathcal{J}(\mathbf{u},\boldsymbol{\sigma};\mathbf{f},g)}{\partial \mathbf{u}}, \mathbf{v}-\mathbf{u} \right\rangle \ge 0 \\[2ex] \left\langle \dfrac{\partial \mathcal{J}(\mathbf{u},\boldsymbol{\sigma};\mathbf{f},g)}{\partial \boldsymbol{\sigma}}, \boldsymbol{\tau}-\boldsymbol{\sigma} \right\rangle \ge 0 \end{cases} \qquad \forall (\mathbf{v},\boldsymbol{\tau}) \in K$$

Rolf Krause, Benjamin Müller, and Gerhard Starke. An adaptive least-squares mixed finite element method for the Signorini problem. Numerical Methods for Partial Differential Equations, 33(1):276-289, 2017.

Discretization

- **Discretized domain** Ω_J
- **FEM space** $X_J = P^1_{\Gamma_D}(\Omega_J) \times \mathcal{RT}_{0,\Gamma_N}(\Omega_J)$ with $\mathbf{x}_J = (\mathbf{u}_J, \boldsymbol{\sigma}_J) \in X_J$
- $\mathbf{f}_J$, $\mathbf{u}_{D,J}$, $\mathbf{t}_{N,J}$, g_J FE representations of $\mathbf{f}$, $\mathbf{u}_D$, $\mathbf{t}_N$, g
- **Discrete FOSLS Functional**

$$\mathcal{J}(\mathbf{x}_J; \mathbf{f}_J) = \frac{1}{2}\mathbf{x}_J^T \mathbf{A}_J \mathbf{x}_J - \mathbf{x}_J^T \mathbf{f}_J$$

- **Convex Set** K_J (in general $K_J \nsubseteq K$)

$$K_J = \{\mathbf{x} \in X : \ \mathbf{u}_J \cdot \mathbf{n}_J - g_J \leq 0, \ (\boldsymbol{\sigma}_J \mathbf{n}_J) \cdot \mathbf{n}_J \leq 0, \ \mathbf{t}_i^T (\boldsymbol{\sigma}\mathbf{n}) = 0 \quad \Gamma_C\}$$

- **Minimization problem:**
  Find $\mathbf{x}_J \in K_J$, such that $\mathcal{J}(\mathbf{x}_J; \mathbf{f}_J) \leq \mathcal{J}(\mathbf{y}_J; \mathbf{f}_J) \qquad \forall \mathbf{y}_J \in K_J$

Disadvantages and Advantages of the FOSLS

Pros

- Direct access to stress $\boldsymbol{\sigma}$ (friction, plasticity...)
- Dealing with incompressible materials ($\lambda \to \infty$)
- FOSLS functional as an a posteriori error estimator
- Flexible choice of finite element spaces (**low order**: $\mathbf{u}_J \in P^1$, $\boldsymbol{\sigma}_J \in \mathcal{RT}_0$)
- Symmetric positive definite system

Cons

- The functional is fictitious, not physical
- The asymmetry of the stress tensor
- Find proper weights C_1, C_2, C_3
- Large condition number: **need for a preconditioner**

$$\left\{\begin{array}{l}\textbf{Functional to be minimized}\\ \textbf{Local constraints}\\ \textbf{Need for a preconditioner}\end{array}\right. \quad \Rightarrow \quad \textbf{Monotone Multilevel}$$

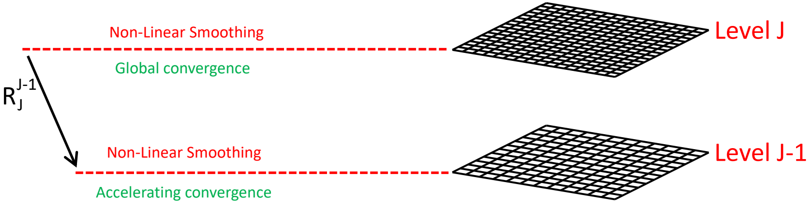
Attia, Frank S., Zhiqiang Cai, and Gerhard Starke. "First-order system least squares for the Signorini contact problem in linear elasticity". SIAM Journal on Numerical Analysis 47.4 (2009): 3027-3043.

Monotone Multilevel strategy

- Successive energy minimization by means of local corrections
- Fine space corrections on fine grid (non-linear Gauß-Seidel) $\Rightarrow$ global convergence
- Coarse space corrections $\Rightarrow$ accelerating convergence


Non-Linear Smoothing
Global convergence
Level J

R_i^{i-1} restriction operator $(i = J, ..., 2)$



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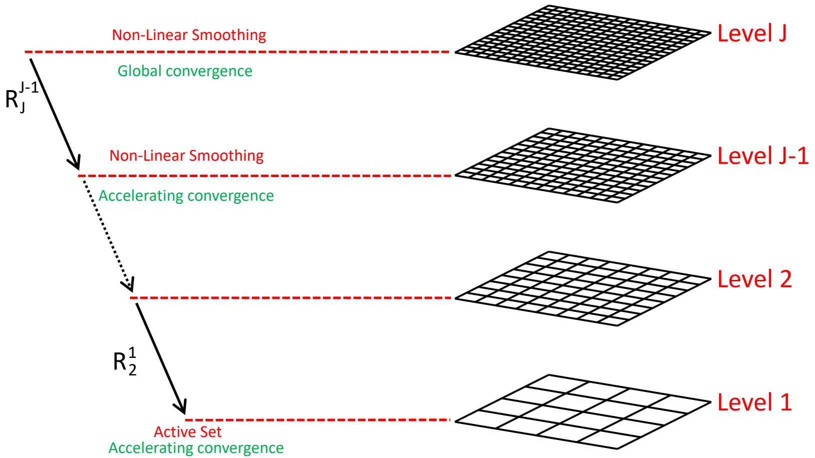
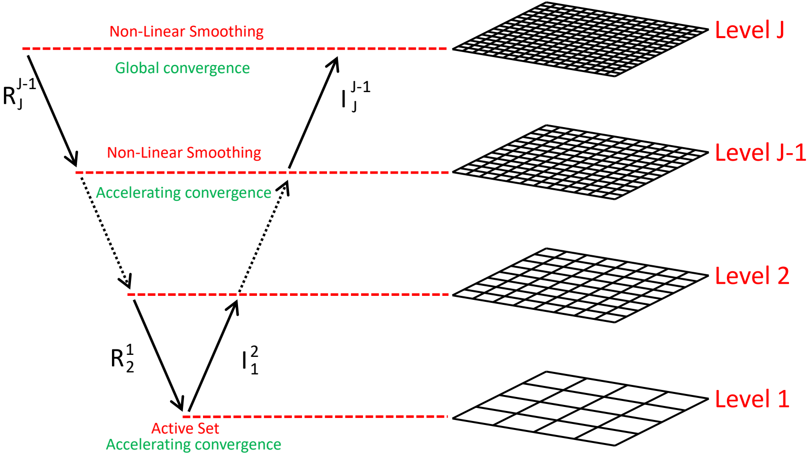
Monotone Multilevel by energy minimization

R_i^{i-1} restriction operator $(i = J, ..., 2)$



Monotone Multilevel by energy minimization

R_i^{i-1} restriction operator, I_{i-1}^{i} interpolation operator $(i = J, ..., 2)$



Multilevel ingredients

Smoother

- Standard non-linear Gauß-Seidel smooths H^1, **but not** H_{div}
- The kernel $\text{Ker}(\text{div}) = \{\boldsymbol{\tau} \in H_{\text{div}}, \text{div}\, \boldsymbol{\tau} = 0\}$ is too large
- Patch-smoother for divergence-free components of the error

Interpolations and restrictions

- Standard P^1 and RT_0 interpolations and restrictions for primal and dual variables
- Non-linear projections for constraint representation on coarser levels

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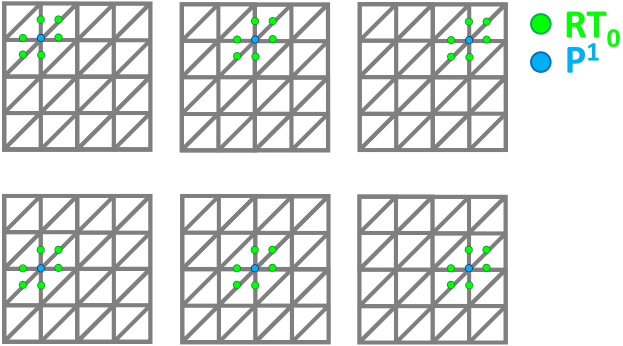
LS Patch smoother



- Minimization of $\mathcal{J}(\mathbf{u}, \boldsymbol{\sigma})$ on span$(\lambda_{j,\nu})$
- Exploit ν-patches to smooth the error in H^1 and H_{div} simultaneously

Ralf Hiptmair. Multigrid method for H(div) in three dimensions. Electron. Trans. Numer. Anal, 6(1):133-152, 1997.

Douglas N Arnold, Richard S Falk, and Ragnar Winther. Multigrid in H(div) and H(curl). Numerische Mathe- matik, 85(2):197-217, 2000.

Gerhard Starke. Gauss-Newton multilevel methods for least-squares finite element computations of variably saturated subsurface flow. Computing, 64(4):323-338, 2000.

Multilevel ingredients

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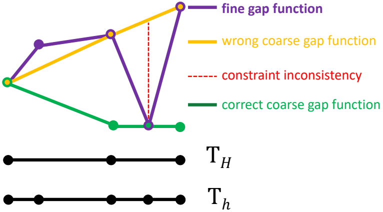
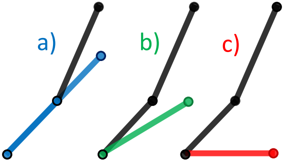
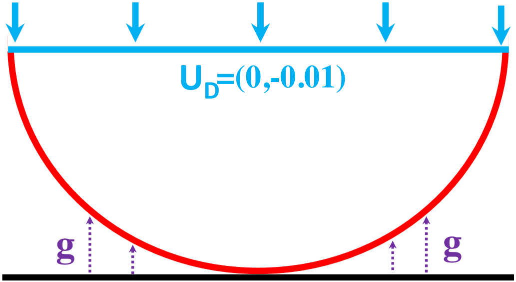
- Wrong and correct coarse constraints



- Different consistent coarse constraints



Hertzian Contact - Setting



$\mu = 1,\ \lambda = 1, \infty$ (compressible and **incompressible**)

Hertzian Contact, two-level problem

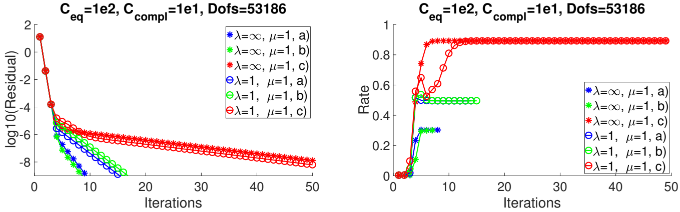


Figure: Mesh with $h_{max}/h_{min} = 7.0567$

- **First phase**: non-linear, capturing high frequencies
- **Second phase**: linear, known active set (blue, green), and not already known active set(red)
- **Similar behaviour** of compressible and incompressible cases

Hertzian Contact - Setting

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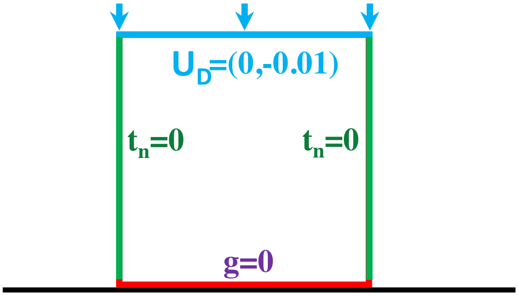

$\mu = 1,\ \lambda = 1, \infty$ (compressible and **incompressible**)

Signorini's problem, square mesh

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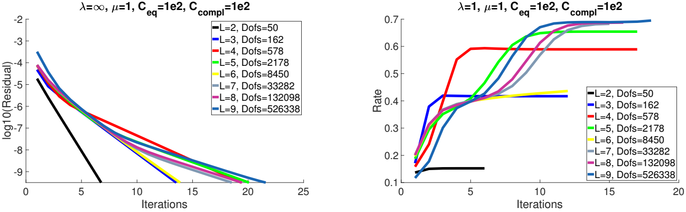

Figure: Square mesh. Compressible material.

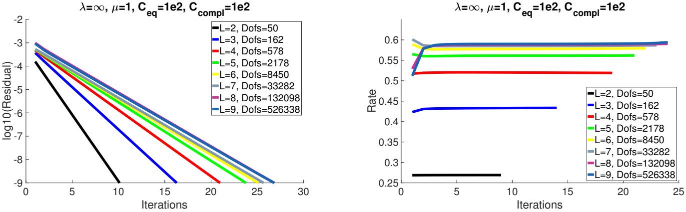

Figure: Square mesh. Incompressible material.

- Purely linear problem: $h-$ and $J-$ independency

Conclusions

- Monotone multilevel for FOSLS linear elastic contact
- Limite case: h– and J– independency
- Similar behaviour of compressible and incompressible cases

Thank you for your attention!

Define:

- $\mathbf{x}_J^k = (\mathbf{u}_J^k, \boldsymbol{\sigma}_J^k) \in K_J$ k-th iterate
- $\mathbf{x}_{J,0} = \mathbf{x}_J^k$
- $\mathbf{x}_{j,0} = \mathbf{x}_{j+1,N_{j+1}}$, for $j = J-1, ..., 1$

Compute a sequence of intermediate iterates $\mathbf{x}_{j,\nu} = \mathbf{x}_{j,\nu-1} + \mathbf{c}_{j,\nu}$:

$$
\begin{aligned}
&\mathcal{J} \leq \mathcal{J}(\mathbf{x}_{j,\nu} + \mathbf{y}) \quad \forall \mathbf{y} \in K^*_{j,\nu} \qquad j = J, ..., 2, \quad \nu = 1, ..., N_j \\
&\mathcal{J}(\mathbf{x}_{2,N_2} + \mathbf{c}_1) \leq \mathcal{J}(\mathbf{x}_{2,N_2} + \mathbf{y}) \quad \forall \mathbf{y} \in K^*_1 \qquad j = 1
\end{aligned}
$$

with the **exact** local closed convex sets $K^*_{j,\nu}$ and K^*_1:

$$
\begin{aligned}
K^*_{j,\nu}(\mathbf{x}_{j,\nu}) &= \{\mathbf{y} \in \mathrm{span}\{\boldsymbol{\lambda}_{j,\nu}\} : \quad \mathbf{y} + \mathbf{x}_{j,\nu} \in K_J\} \\
K^*_1(\mathbf{x}_{2,N_2}) &= \{\mathbf{y} \in \mathrm{span}\{\boldsymbol{\lambda}_1\} : \quad \mathbf{y} + \mathbf{x}_{2,N_2} \in K_J\}
\end{aligned}
$$

Ralf Kornhuber. Monotone multigrid methods for elliptic variational inequalities I. Numerische Mathematik, 69(2):167-184, 1994.

Ralf Kornhuber and Rolf Krause. Adaptive multigrid methods for Signorini's problem in linear elasticity. Computing and Visualization in Science, 4(1):9-20, 2001.

Approximate Monotone Multilevel

Define:

- $\mathbf{c}_{j,\nu} = (\tilde{\mathbf{u}}_{j,\nu}, \tilde{\boldsymbol{\sigma}}_{j,\nu})$ correction at level j, patch ν
- $\mathbf{c}_{J,0} = \mathbf{x}_J^k$, $\mathbf{c}_{j,0} = \mathbf{0}$ for $j = J-1, \dots, 1$
- $\mathbf{w}_{j,\nu} = \sum_{\mu=0}^{\nu} \mathbf{c}_{j,\mu}$

Compute a sequence of intermediate corrections $\mathbf{c}_{j,\nu} \in K_{j,\nu}(\mathbf{w}_{j,\nu-1})$ and $\mathbf{c}_1 \in K_1$:

$$
\begin{array}{lll}
\mathcal{J}(\mathbf{w}_{j,\nu-1} + \mathbf{c}_{j,\nu}) \le \mathcal{J}(\mathbf{w}_{j,\nu-1} + \mathbf{y}) & \forall\, \mathbf{y} \in K_{j,\nu} & j = J, \dots, 2, \;\; \nu = 1, \dots, N_j \\
\mathcal{J}(\mathbf{c}_1) \le \mathcal{J}(\mathbf{y}) & \forall\, \mathbf{y} \in K_1 & j = 1
\end{array}
$$

with the **coarse convex sets** K_j and the **approximate** local closed convex sets $K_{j,\nu}$:

$$
\begin{aligned}
&K_{j,\nu}(\mathbf{w}_{j,\nu-1}) = \left\{ \mathbf{y} \in \mathrm{span}\{\lambda_{j,\nu}\} : \; \mathbf{y} + \mathbf{w}_{j,\nu-1} \in K_j \right\} \\
&K_1 \subset K_2 \subset \dots \subset K_{J-1} \subset K_J
\end{aligned}
$$

Coarse Convex Sets:

$$K_j = \Big\{ \mathbf{x}_j = (\mathbf{u}_j, \boldsymbol{\sigma}_j) \in X_j : \mathbf{u}_j|_{\Gamma_D} = \mathbf{u}_D,\ \boldsymbol{\sigma}_j|_{\Gamma_N} = \mathbf{t}_N, \\ \mathbf{u}_j \cdot \mathbf{n}_j|_{\Gamma_C} \le g_{j,u_n},\ \mathbf{n}^T(\boldsymbol{\sigma}_j \mathbf{n}) \le g_{j,\sigma_n},\ \mathbf{t}_j^T(\boldsymbol{\sigma}\mathbf{n}_j) = 0 \Big\} \qquad j = J$$

$$K_j = \Big\{ \mathbf{x}_j = (\mathbf{u}_j, \boldsymbol{\sigma}_j) \in X_j : \mathbf{u}_j|_{\Gamma_D} = \mathbf{0},\ \boldsymbol{\sigma}_j|_{\Gamma_N} = \mathbf{0}, \\ \mathbf{u}_J \cdot \mathbf{n}_j|_{\Gamma_C} \le g_{j,u_n},\ \mathbf{n}^T(\boldsymbol{\sigma}_j \mathbf{n}) \le g_{j,\sigma_n},\ \mathbf{t}_j^T(\boldsymbol{\sigma}\mathbf{n}_j) = 0 \Big\} \qquad j = J-1, \ldots, 1$$

Coarse Constraints:

- $\tilde{\mathbf{u}}_{j,\nu}$ and $\tilde{\boldsymbol{\sigma}}_{j,\nu}$ are the components of the correction $\mathbf{c}_{j,\nu}$.

$$g_{j,u_n} = \begin{cases} g & j = J \\ I^j_{j+1,u_n}\left(g_{j+1,u_n} - \sum_{\nu=1}^{N_{j+1}} \left[\tilde{\mathbf{u}}_{j+1,\nu}|_{\Gamma_C} \right]_n \right) & j = J-1, \ldots, 1 \end{cases}$$

$$g_{j,\sigma_n} = \begin{cases} 0 & j = J \\ I^j_{j+1,\sigma_n}\left(g_{j+1,\sigma_n} - \sum_{\nu=1}^{N_{j+1}} \left[\tilde{\boldsymbol{\sigma}}_{j+1,\nu}|_{\Gamma_C} \right]_n \right) & j = J-1, \ldots, 1 \end{cases}$$

Non-Linear Projection Operators:

I^j_{j+1,u_n}, I^j_{j+1,σ_n} chosen so that $K_1 \subset K_2 \subset \ldots \subset K_{J-1} \subset K_J$.

$$\begin{aligned} v_H(\nu_{H,1}) &\leq v_h(\nu_{H,1}) \\ v_H(\nu_{H,2}) &\leq v_h(\nu_{H,2}) \qquad \forall \varepsilon_H \in \mathcal{E}_H \cap \Gamma_{C,H} \\ \frac{1}{2}(v_H(\nu_{H,1}) + v_H(\nu_{H,2})) &\leq v_h(\nu_h) \end{aligned}$$

It is easy to see that, on e_H, the following values satisfy the three conditions above:

$$a) \quad \begin{cases} \tilde{v}_H(\nu_{H,1}) = \min(v_h(\nu_{H,1}), \max(v_h(\nu_h), 2v_h(\nu_h) - v_h(\nu_{H,2}))) \\ \tilde{v}_H(\nu_{H,2}) = \min(v_h(\nu_{H,2}), \max(v_h(\nu_h), 2v_h(\nu_h) - v_h(\nu_{H,1}))) \end{cases} \qquad \forall \varepsilon_H \in \mathcal{E}_H \cap \Gamma_C$$

$$b) \quad \begin{cases} \tilde{v}_H(\nu_{H,1}) = \min(v_h(\nu_{H,1}), v_h(\nu_h)) \\ \tilde{v}_H(\nu_{H,2}) = \min(v_h(\nu_{H,2}), v_h(\nu_h)) \end{cases} \qquad \forall \varepsilon_H \in \mathcal{E}_H \cap \Gamma_C$$

$$c) \quad \begin{cases} \tilde{v}_H(\nu_{H,1}) = \min(v_h(\nu_{H,1}), v_h(\nu_h), v_h(\nu_{H,2})) \\ \tilde{v}_H(\nu_{H,2}) = \min(v_h(\nu_{H,1}), v_h(\nu_h), v_h(\nu_{H,2})) \end{cases} \qquad \forall \varepsilon_H \in \mathcal{E}_H \cap \Gamma_C$$

Pressure Non-Linear Projection

$$s_H(\phi_H) \leq s_h(\phi_h) \quad \forall \phi_h \in P_{\phi_H}^{\phi_h}$$

Thus:

$$s_H = I_{h,\sigma_n}^{H} s_h = \sum_{\phi_{H_i} \in T_H} \left[\lambda_{\Sigma_H, H_i}\right]_n s_H(\phi_{H_i}) \qquad \text{with} \qquad s_H(\phi_{H_i}) = \min_{\phi_h \in P_{\phi_{H_i}}^{\phi_h}} s_h(\phi_h)$$

Truncated Basis

$$\left[\tilde{\boldsymbol{\lambda}}_{U_j,\nu}\right]_i = \begin{cases} \left[\boldsymbol{\lambda}_{U_j,\nu}\right]_i & \nu \in \mathcal{N}_j \setminus \mathcal{N}_j^{\bullet}, \quad i = n, t \\ 0 & \nu \in \mathcal{N}_j^{\bullet}, \qquad i = n \\ \left[\boldsymbol{\lambda}_{U_j,\nu}\right]_i & \nu \in \mathcal{N}_j^{\bullet}, \qquad i = t \end{cases}$$

$$\left[\tilde{\boldsymbol{\lambda}}_{\Sigma_j,\phi}\right]_i = \begin{cases} \left[\boldsymbol{\lambda}_{\Sigma_j,\phi}\right]_i & \phi \in \mathcal{F}_j \setminus \mathcal{F}_j^{\bullet}, \quad i = n, t \\ 0 & \phi \in \mathcal{F}_j^{\bullet} \qquad i = n \\ \left[\boldsymbol{\lambda}_{\Sigma_j,\phi}\right]_i & \phi \in \mathcal{F}_j^{\bullet}, \qquad i = t \end{cases}$$