Monotone multilevel for FOSLS linear elastic contact

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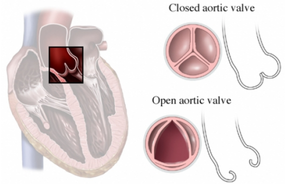
Offen im Denken

Examples of contact problems



- Contact problems with incompressible materials.
- Quantities of interest: the forces generated by the contact.

- **First Order System Linear Elasticity**
  Given displacement $\mathbf{u}$, stress $\boldsymbol{\sigma}$, body Ω:

$$\begin{cases} \mathrm{div}\boldsymbol{\sigma} + \mathbf{f} = 0 & \Omega \quad \text{momentum balance equation} \\ \mathcal{A}\boldsymbol{\sigma} - \boldsymbol{\varepsilon}(\mathbf{u}) = 0 & \Omega \quad \text{constitutive law} \\ \mathbf{u} = \mathbf{u}_D & \Gamma_D \quad \text{Dirichlet BC} \\ \boldsymbol{\sigma}\mathbf{n} = \mathbf{t}_N & \Gamma_N \quad \text{Neumann BC} \end{cases}$$

where $\boldsymbol{\varepsilon}(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T)$, $\boldsymbol{\mathcal{A}}\boldsymbol{\sigma} = \frac{1}{2\mu}\left(\boldsymbol{\sigma} - \frac{\lambda}{d\lambda + 2\mu}\mathrm{tr}\boldsymbol{\sigma}\mathbf{I}\right)$ and μ, λ are the Lamé parameters

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Signorini's problem: strong formulation

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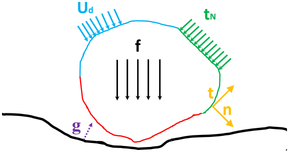
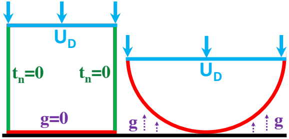
- **Contact Constraints**
  Given the gap function g, the normal and tangent vectors $\mathbf{n}$ and $\mathbf{t}$:

$$
\begin{cases}
\mathbf{u}\cdot\mathbf{n} - g \leq 0 & \Gamma_C & \text{impenetrability} \\
(\boldsymbol{\sigma}\mathbf{n})\cdot\mathbf{n} \leq 0 & \Gamma_C & \text{direction of the surface pressure} \\
(\mathbf{u}\cdot\mathbf{n} - g)\left((\boldsymbol{\sigma}\mathbf{n})\cdot\mathbf{n}\right) = 0 & \Gamma_C & \text{complementarity condition} \\
\mathbf{t}_i^T(\boldsymbol{\sigma}\mathbf{n}) = 0 & \Gamma_C & \text{frictionless condition}
\end{cases}
$$





- **First Order System Least-Squares (FOSLS) Functional**

$$C_1,\ C_2,\ C_3 > 0$$

$$\mathcal{J}(\mathbf{u},\boldsymbol{\sigma}) = C_1 \left\|\mathrm{div}\boldsymbol{\sigma} + \mathbf{f}\right\|^2_{L^2(\Omega)^d} + C_2 \left\|\mathcal{A}\boldsymbol{\sigma} - \boldsymbol{\varepsilon}(\mathbf{u})\right\|^2_{L^2(\Omega)^d} + C_3 \langle \mathbf{u}\cdot\mathbf{n} - g, (\boldsymbol{\sigma}\mathbf{n})\cdot\mathbf{n}\rangle_{\Gamma_C}$$

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- **Convex Set K**

$$K = \{(\mathbf{u},\boldsymbol{\sigma}) \in \left[H^1_{\Gamma_d}(\Omega)\right]^d \times \left[H_{\mathrm{div},\Gamma_N}(\Omega)\right]^d : \mathbf{u}\cdot\mathbf{n} - g \leq 0,\ (\boldsymbol{\sigma}\mathbf{n})\cdot\mathbf{n} \leq 0,\ \mathbf{t}_i^T(\boldsymbol{\sigma}\mathbf{n}) = 0 \quad \Gamma_C\}$$

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The Least-Squares Formulation

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- Find $(\mathbf{u},\boldsymbol{\sigma}) \in K$, such that:
  - **Minimization problem:** $\mathcal{J}(\mathbf{u},\boldsymbol{\sigma}) \le \mathcal{J}(\mathbf{v},\boldsymbol{\tau}) \qquad \forall(\mathbf{v},\boldsymbol{\tau}) \in K$

$$\Longleftrightarrow$$

  - **Variational Inequality:**

$$\begin{cases} \left\langle \dfrac{\partial\mathcal{J}(\mathbf{u},\boldsymbol{\sigma};\mathbf{f},g)}{\partial\mathbf{u}}, \mathbf{v}-\mathbf{u}\right\rangle \ge 0 \\[2ex] \left\langle \dfrac{\partial\mathcal{J}(\mathbf{u},\boldsymbol{\sigma};\mathbf{f},g)}{\partial\boldsymbol{\sigma}}, \boldsymbol{\tau}-\boldsymbol{\sigma}\right\rangle \ge 0 \end{cases} \qquad \forall(\mathbf{v},\boldsymbol{\tau}) \in K$$

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- **FE space** $X_L = P^1_{\Gamma_D}(\Omega_L) \times \mathcal{RT}_{0,\Gamma_N}(\Omega_L)$ with $\mathbf{x}_L = (\mathbf{u}_L, \boldsymbol{\sigma}_L) \in X_L$

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- **Minimization problem:**
  Find $\mathbf{x}_L \in K_L$

$$\operatorname{argmin} \mathcal{J}(\mathbf{x}_L; \mathbf{f}_L) = \frac{1}{2}\mathbf{x}_L^T \mathbf{A}_L \mathbf{x}_L - \mathbf{x}_L^T \mathbf{f}_L$$
$$\mathbf{B}_L \mathbf{x}_L \leq \mathbf{g}_L$$

Disadvantages and Advantages of the FOSLS

Pros

- Direct access to stress $\boldsymbol{\sigma}$ (friction, plasticity...)
- Dealing with incompressible materials ($\lambda \to \infty$)
- FOSLS functional as an a posteriori error estimator
- Flexible choice of finite element spaces (**low order**: $\mathbf{u}_L \in P^1$, $\boldsymbol{\sigma}_L \in \mathcal{RT}_0$)
- Symmetric positive definite system

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- The asymmetry of the stress tensor
- Find proper weights C_1, C_2, C_3
- Large condition number: **need for a preconditioner**

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$$\left\{\begin{array}{l}\textbf{Functional to be minimized}\\ \textbf{Local constraints}\\ \textbf{Need for a preconditioner}\end{array}\right. \quad \Rightarrow \quad \textbf{Monotone Multilevel}$$

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- Successive energy minimization by means of local corrections
- No correction can increase energy
- Fine space corrections on fine grid (non-linear Gauß-Seidel) $\Rightarrow$ global convergence
- Coarse space corrections $\Rightarrow$ accelerating convergence

Monotone Multilevel by energy minimization

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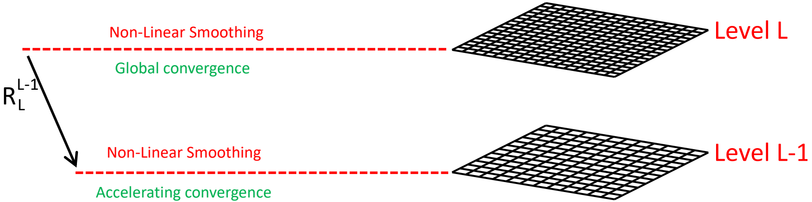
R_i^{i-1} restriction operator $(i = L, ..., 2)$



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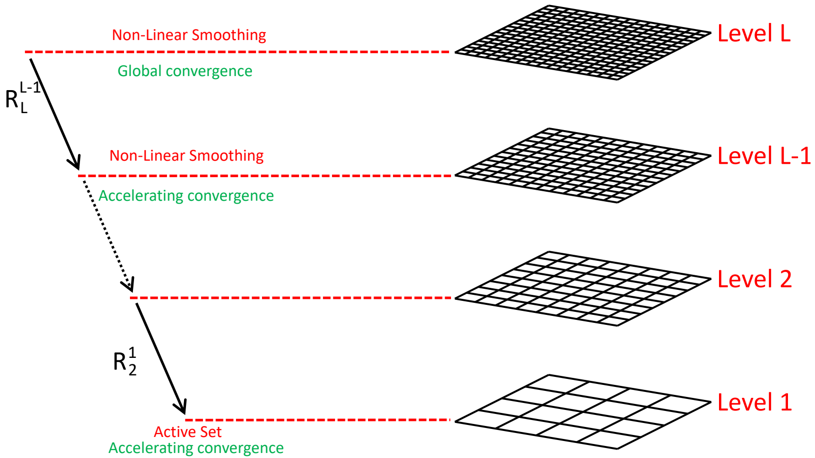
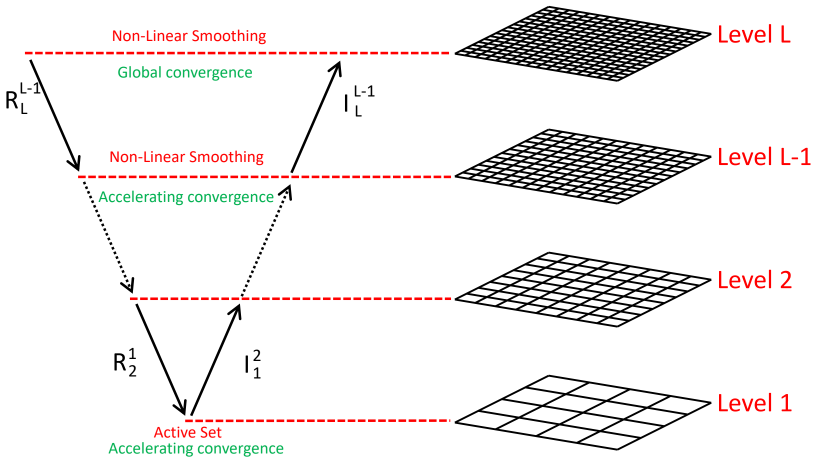


R_i^{i-1} restriction operator, I_{i-1}^{i} interpolation operator ($i = L, ..., 2$)



Smoother

- Standard non-linear Gauß-Seidel smooths H^1, **but not** H_{div}
- The kernel $\text{Ker}(\text{div}) = \{\boldsymbol{\tau} \in H_{\text{div}}, \text{div}\,\boldsymbol{\tau} = 0\}$ is too large
- Patch-smoother for divergence-free components of the error

Interpolations and restrictions

- Standard P^1 and RT_0 interpolations and restrictions for primal and dual variables
- Non-linear projections for constraint representation on coarser levels

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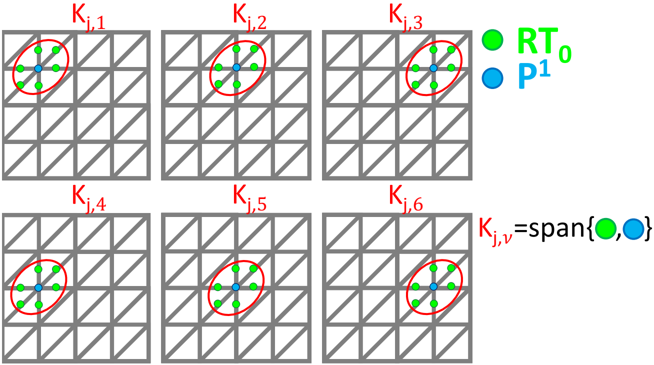
LS Patch smoother

- Mesh level $j = 1, \ldots, L$, vertex $\nu = 1, \ldots, N_j$
- Patch$_{j,\nu}$= dofs of node ν and surrounding edges/faces (2D/3D)
- $K_{j,\nu}$= local closed convex set spanned by basis functions in Patch$_{j,\nu}$
- Minimization of $\mathcal{J}$ on $K_{j,\nu}$
- Error smoothed in H^1 and H_{div} simultaneously



• Ralf Hiptmair. Multigrid method for H(div) in three dimensions. Electron. Trans. Numer. Anal, 6(1):133-152, 1997.
• Douglas N Arnold, Richard S Falk, and Ragnar Winther. Multigrid in H(div) and H(curl). Numerische Mathe- matik, 85(2):197-217, 2000.
• Gerhard Starke. Gauss-Newton multilevel methods for least-squares finite element computations of variably saturated subsurface flow. Computing, 64(4):323-338, 2000.

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- **Exact monotone multilevel**
  Comparison of coarse function $\mathbf{c}_j$ with **fine constraint** $\Rightarrow$ **suboptimal complexity**
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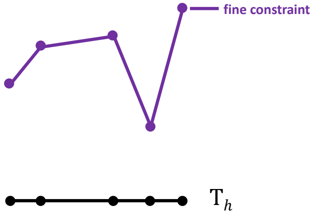

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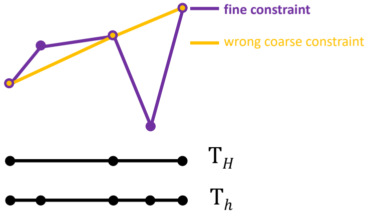

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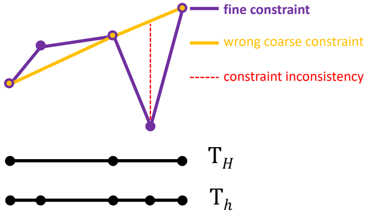

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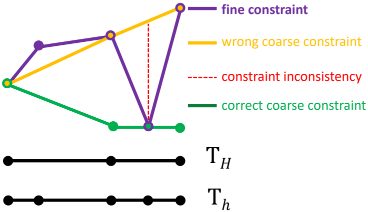

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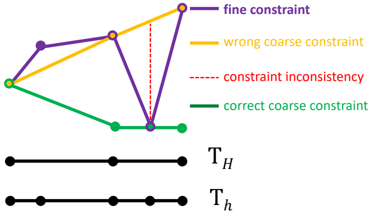
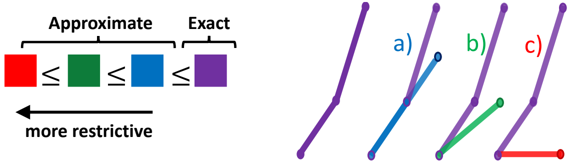
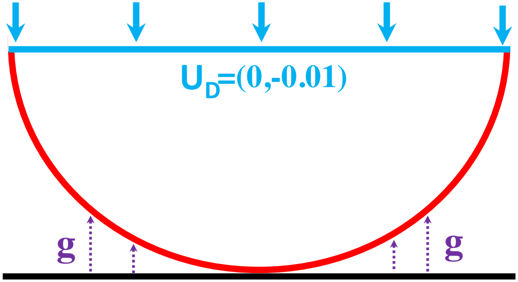


- Different consistent coarse constraints



Hertzian Contact - Setting



$\mu = 1,\ \lambda = 1, \infty$ (compressible and **incompressible**)

Hertzian Contact, two-level problem

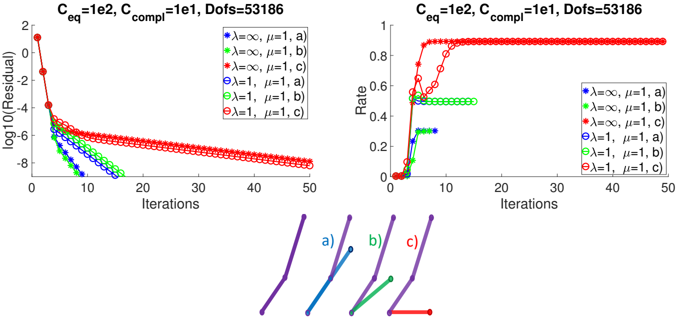
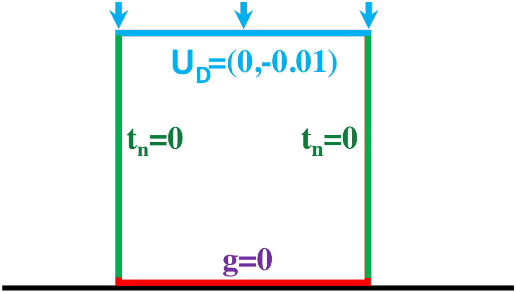


Figure: Mesh with $h_{max}/h_{min} = 7.0567$

- **First phase**: non-linear, capturing high frequencies
- **Second phase**: linear, known active set (blue, green), and not already known active set(red)
- **Similar behaviour** of compressible and incompressible cases

Hertzian Contact - Setting



$\mu = 1,\ \lambda = 1, \infty$ (compressible and **incompressible**)

Signorini's problem, square mesh

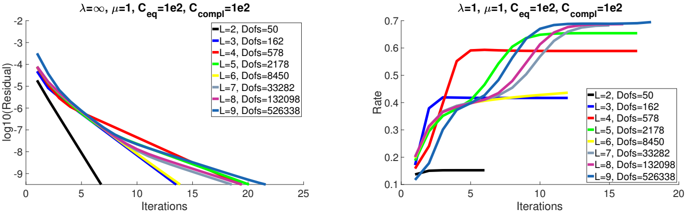


Figure: Square mesh. Compressible material.

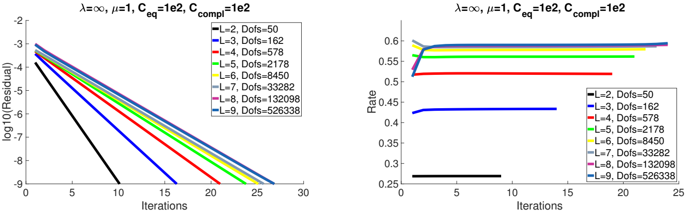


Figure: Square mesh. Incompressible material.

- Purely linear problem: $h-$ and $L-$ independency

Conclusions

- Monotone multilevel for FOSLS linear elastic contact
- Limite case: $h-$ and $L-$ independency
- Similar behaviour of compressible and incompressible cases

Thank you for your attention!

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Define:

- $\mathbf{x}_J^k = (\mathbf{u}_J^k, \boldsymbol{\sigma}_J^k) \in K_J$ k-th iterate
- $\mathbf{x}_{J,0} = \mathbf{x}_J^k$
- $\mathbf{x}_{j,0} = \mathbf{x}_{j+1,N_{j+1}}$, for $j = J-1, ..., 1$

Compute a sequence of intermediate iterates $\mathbf{x}_{j,\nu} = \mathbf{x}_{j,\nu-1} + \mathbf{c}_{j,\nu}$:

$$\mathcal{J} \leq \mathcal{J}(\mathbf{x}_{j,\nu} + \mathbf{y}) \quad \forall \mathbf{y} \in K_{j,\nu}^* \qquad j = J, ..., 2, \quad \nu = 1, ..., N_j$$
$$\mathcal{J}(\mathbf{x}_{2,N_2} + \mathbf{c}_1) \leq \mathcal{J}(\mathbf{x}_{2,N_2} + \mathbf{y}) \quad \forall \mathbf{y} \in K_1^* \qquad j = 1$$

with the **exact** local closed convex sets $K_{j,\nu}^*$ and K_1^*:

$$K_{j,\nu}^*(\mathbf{x}_{j,\nu}) = \left\{\mathbf{y} \in \operatorname{span}\{\boldsymbol{\lambda}_{j,\nu}\} : \quad \mathbf{y} + \mathbf{x}_{j,\nu} \in K_J\right\}$$
$$K_1^*(\mathbf{x}_{2,N_2}) = \left\{\mathbf{y} \in \operatorname{span}\{\boldsymbol{\lambda}_1\} : \quad \mathbf{y} + \mathbf{x}_{2,N_2} \in K_J\right\}$$

Ralf Kornhuber. Monotone multigrid methods for elliptic variational inequalities I. Numerische Mathematik, 69(2):167-184, 1994.

Ralf Kornhuber and Rolf Krause. Adaptive multigrid methods for Signorini's problem in linear elasticity. Computing and Visualization in Science, 4(1):9-20, 2001.

Approximate Monotone Multilevel

Define:

- $\mathbf{c}_{j,\nu} = (\tilde{\mathbf{u}}_{j,\nu}, \tilde{\boldsymbol{\sigma}}_{j,\nu})$ correction at level j, patch ν
- $\mathbf{c}_{J,0} = \mathbf{x}_J^k$, $\mathbf{c}_{j,0} = \mathbf{0}$ for $j = J-1, ..., 1$
- $\mathbf{w}_{j,\nu} = \sum_{\mu=0}^{\nu} \mathbf{c}_{j,\mu}$

Compute a sequence of intermediate corrections $\mathbf{c}_{j,\nu} \in K_{j,\nu}(\mathbf{w}_{j,\nu-1})$ and $\mathbf{c}_1 \in K_1$:

$$\begin{aligned} &\mathcal{J}(\mathbf{w}_{j,\nu-1} + \mathbf{c}_{j,\nu}) \leq \mathcal{J}(\mathbf{w}_{j,\nu-1} + \mathbf{y}) \quad \forall\, \mathbf{y} \in K_{j,\nu} \qquad j = J, ..., 2, \;\; \nu = 1, ..., N_j \\ &\mathcal{J}(\mathbf{c}_1) \leq \mathcal{J}(\mathbf{y}) \qquad\qquad\qquad\qquad\quad \forall\, \mathbf{y} \in K_1 \qquad\quad j = 1 \end{aligned}$$

with the **coarse convex sets** K_j and the **approximate** local closed convex sets $K_{j,\nu}$:

$$\begin{aligned} &K_{j,\nu}(\mathbf{w}_{j,\nu-1}) = \left\{\mathbf{y} \in \mathrm{span}\{\lambda_{j,\nu}\} : \; \mathbf{y} + \mathbf{w}_{j,\nu-1} \in K_j\right\} \\ &K_1 \subset K_2 \subset ... \subset K_{J-1} \subset K_J \end{aligned}$$

Coarse Convex Sets:

$$K_j = \Big\{ \mathbf{x}_j = (\mathbf{u}_j, \boldsymbol{\sigma}_j) \in X_j : \ \mathbf{u}_j|_{\Gamma_D} = \mathbf{u}_D, \ \boldsymbol{\sigma}_j|_{\Gamma_N} = \mathbf{t}_N,$$
$$\mathbf{u}_j \cdot \mathbf{n}_j|_{\Gamma_C} \le g_{j,u_n}, \ \mathbf{n}^T(\boldsymbol{\sigma}_j \mathbf{n}) \le g_{j,\sigma_n}, \ \mathbf{t}_j^T(\boldsymbol{\sigma}\mathbf{n}_j) = 0 \Big\} \qquad j = J$$

$$K_j = \Big\{ \mathbf{x}_j = (\mathbf{u}_j, \boldsymbol{\sigma}_j) \in X_j : \ \mathbf{u}_j|_{\Gamma_D} = \mathbf{0}, \ \boldsymbol{\sigma}_j|_{\Gamma_N} = \mathbf{0},$$
$$\mathbf{u}_J \cdot \mathbf{n}_j|_{\Gamma_C} \le g_{j,u_n}, \ \mathbf{n}^T(\boldsymbol{\sigma}_j \mathbf{n}) \le g_{j,\sigma_n}, \ \mathbf{t}_j^T(\boldsymbol{\sigma}\mathbf{n}_j) = 0 \Big\} \qquad j = J-1, \dots, 1$$

Coarse Constraints:

- $\tilde{\mathbf{u}}_{j,\nu}$ and $\tilde{\boldsymbol{\sigma}}_{j,\nu}$ are the components of the correction $\mathbf{c}_{j,\nu}$.

$$g_{j,u_n} = \begin{cases} g & j = J \\ I^j_{j+1,u_n}\left(g_{j+1,u_n} - \sum_{\nu=1}^{N_{j+1}} \left[\tilde{\mathbf{u}}_{j+1,\nu}|_{\Gamma_C} \right]_n \right) & j = J-1, \dots, 1 \end{cases}$$

$$g_{j,\sigma_n} = \begin{cases} 0 & j = J \\ I^j_{j+1,\sigma_n}\left(g_{j+1,\sigma_n} - \sum_{\nu=1}^{N_{j+1}} \left[\tilde{\boldsymbol{\sigma}}_{j+1,\nu}|_{\Gamma_C} \right]_n \right) & j = J-1, \dots, 1 \end{cases}$$

Non-Linear Projection Operators:

I^j_{j+1,u_n}, I^j_{j+1,σ_n} chosen so that $K_1 \subset K_2 \subset \dots \subset K_{J-1} \subset K_J$.

Normal displacement Non-Linear Projection

$$
\left.\begin{aligned}
&v_H(\nu_{H,1}) \leq v_h(\nu_{H,1})\\
&v_H(\nu_{H,2}) \leq v_h(\nu_{H,2})\\
&\frac{1}{2}(v_H(\nu_{H,1}) + v_H(\nu_{H,2})) \leq v_h(\nu_h)
\end{aligned}\right. \qquad \forall \varepsilon_H \in \mathcal{E}_H \cap \Gamma_{C,H}
$$

It is easy to see that, on e_H, the following values satisfy the three conditions above:

$$
a) \quad \begin{cases} \tilde{v}_H(\nu_{H,1}) = \min(v_h(\nu_{H,1}), \max(v_h(\nu_h), 2v_h(\nu_h) - v_h(\nu_{H,2}))) \\ \tilde{v}_H(\nu_{H,2}) = \min(v_h(\nu_{H,2}), \max(v_h(\nu_h), 2v_h(\nu_h) - v_h(\nu_{H,1}))) \end{cases} \qquad \forall \varepsilon_H \in \mathcal{E}_H \cap \Gamma_C
$$

$$
b) \quad \begin{cases} \tilde{v}_H(\nu_{H,1}) = \min(v_h(\nu_{H,1}), v_h(\nu_h)) \\ \tilde{v}_H(\nu_{H,2}) = \min(v_h(\nu_{H,2}), v_h(\nu_h)) \end{cases} \qquad \forall \varepsilon_H \in \mathcal{E}_H \cap \Gamma_C
$$

$$
c) \quad \begin{cases} \tilde{v}_H(\nu_{H,1}) = \min(v_h(\nu_{H,1}), v_h(\nu_h), v_h(\nu_{H,2})) \\ \tilde{v}_H(\nu_{H,2}) = \min(v_h(\nu_{H,1}), v_h(\nu_h), v_h(\nu_{H,2})) \end{cases} \qquad \forall \varepsilon_H \in \mathcal{E}_H \cap \Gamma_C
$$

$$s_H(\phi_H) \leq s_h(\phi_h) \quad \forall \phi_h \in P_{\phi_H}^{\phi_h}$$

Thus:

$$s_H = I_{h,\sigma_n}^H s_h = \sum_{\phi_{H_i} \in T_H} \left[\lambda_{\Sigma_H, H_i}\right]_n s_H(\phi_{H_i}) \qquad \text{with} \qquad s_H(\phi_{H_i}) = \min_{\phi_h \in P_{\phi_{H_i}}^{\phi_h}} s_h(\phi_h)$$

Truncated Basis

$$\left[\tilde{\boldsymbol{\lambda}}_{U_j,\nu}\right]_i = \begin{cases} \left[\boldsymbol{\lambda}_{U_j,\nu}\right]_i & \nu \in \mathcal{N}_j \setminus \mathcal{N}_j^\bullet, \quad i = n, t \\ 0 & \nu \in \mathcal{N}_j^\bullet, \qquad i = n \\ \left[\boldsymbol{\lambda}_{U_j,\nu}\right]_i & \nu \in \mathcal{N}_j^\bullet, \qquad i = t \end{cases}$$

$$\left[\tilde{\boldsymbol{\lambda}}_{\Sigma_j,\phi}\right]_i = \begin{cases} \left[\boldsymbol{\lambda}_{\Sigma_j,\phi}\right]_i & \phi \in \mathcal{F}_j \setminus \mathcal{F}_j^\bullet, \quad i = n, t \\ 0 & \phi \in \mathcal{F}_j^\bullet \qquad i = n \\ \left[\boldsymbol{\lambda}_{\Sigma_j,\phi}\right]_i & \phi \in \mathcal{F}_j^\bullet, \qquad i = t \end{cases}$$