

# Monotone multilevel for FOSLS linear elastic contact

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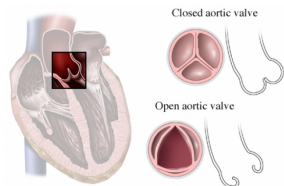
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# Examples of contact problems

Monotone  
multilevel  
for FOSLS  
linear  
elastic  
contact

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- Contact problems with incompressible materials.
- Quantities of interest: the forces generated by the contact.

# Signorini's problem: strong formulation

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- **First Order System Linear Elasticity**

Given displacement  $\mathbf{u}$ , stress  $\boldsymbol{\sigma}$ , body  $\Omega$ :

$$\begin{cases} \operatorname{div} \boldsymbol{\sigma} + \mathbf{f} = 0 & \Omega & \text{momentum balance equation} \\ \mathcal{A} \boldsymbol{\sigma} - \boldsymbol{\varepsilon}(\mathbf{u}) = 0 & \Omega & \text{constitutive law} \\ \mathbf{u} = \mathbf{u}_D & \Gamma_D & \text{Dirichlet BC} \\ \boldsymbol{\sigma} \mathbf{n} = \mathbf{t}_N & \Gamma_N & \text{Neumann BC} \end{cases}$$

where  $\boldsymbol{\varepsilon}(\mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T)$ ,  $\mathcal{A} \boldsymbol{\sigma} = \frac{1}{2\mu} \left( \boldsymbol{\sigma} - \frac{\lambda}{d\lambda + 2\mu} \operatorname{tr} \boldsymbol{\sigma} \mathbf{I} \right)$  and  $\mu, \lambda$  are the Lamé parameters

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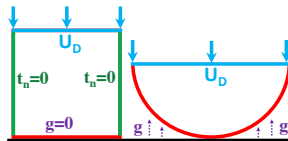
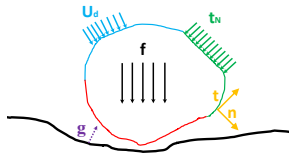
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## Contact Constraints

Given the gap function  $g$ , the normal and tangent vectors  $\mathbf{n}$  and  $\mathbf{t}$ :

$$\begin{cases} \mathbf{u} \cdot \mathbf{n} - g \leq 0 & \Gamma_C \text{ impenetrability} \\ (\boldsymbol{\sigma} \mathbf{n}) \cdot \mathbf{n} \leq 0 & \Gamma_C \text{ direction of the surface pressure} \\ (\mathbf{u} \cdot \mathbf{n} - g) ((\boldsymbol{\sigma} \mathbf{n}) \cdot \mathbf{n}) = 0 & \Gamma_C \text{ complementarity condition} \\ \mathbf{t}_i^T (\boldsymbol{\sigma} \mathbf{n}) = 0 & \Gamma_C \text{ frictionless condition} \end{cases}$$



# The Least-Squares Formulation

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- **First Order System Least-Squares (FOSLS) Functional**

$$C_1, C_2, C_3 > 0$$

$$\mathcal{J}(\mathbf{u}, \boldsymbol{\sigma}) = C_1 \|\operatorname{div} \boldsymbol{\sigma} + \mathbf{f}\|_{L^2(\Omega)^d}^2 + C_2 \|\mathcal{A} \boldsymbol{\sigma} - \boldsymbol{\varepsilon}(\mathbf{u})\|_{L^2(\Omega)^d}^2 + C_3 \langle \mathbf{u} \cdot \mathbf{n} - g, (\boldsymbol{\sigma} \mathbf{n}) \cdot \mathbf{n} \rangle_{\Gamma_c}$$

- Rolf Krause, Benjamin Müller, and Gerhard Starke. An adaptive least-squares mixed finite element method for the Signorini problem. Numerical Methods for Partial Differential Equations, 33(1):276-289, 2017.

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- Convex Set  $K$

$$K = \{(\mathbf{u}, \boldsymbol{\sigma}) \in [H_{\Gamma_d}^1(\Omega)]^d \times [H_{\operatorname{div}, \Gamma_N}(\Omega)]^d : \mathbf{u} \cdot \mathbf{n} - g \leq 0, (\boldsymbol{\sigma} \mathbf{n}) \cdot \mathbf{n} \leq 0, \mathbf{t}_i^T(\boldsymbol{\sigma} \mathbf{n}) = 0 \quad \Gamma_C\}$$

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- Find  $(\mathbf{u}, \boldsymbol{\sigma}) \in K$ , such that:

- **Minimization problem:**  $\mathcal{J}(\mathbf{u}, \boldsymbol{\sigma}) \leq \mathcal{J}(\mathbf{v}, \boldsymbol{\tau}) \quad \forall (\mathbf{v}, \boldsymbol{\tau}) \in K$

$$\Longleftrightarrow$$

- **Variational Inequality:** 
$$\left\{ \begin{array}{l} \left\langle \frac{\partial \mathcal{J}(\mathbf{u}, \boldsymbol{\sigma}; \mathbf{f}, g)}{\partial \mathbf{u}}, \mathbf{v} - \mathbf{u} \right\rangle \geq 0 \\ \left\langle \frac{\partial \mathcal{J}(\mathbf{u}, \boldsymbol{\sigma}; \mathbf{f}, g)}{\partial \boldsymbol{\sigma}}, \boldsymbol{\tau} - \boldsymbol{\sigma} \right\rangle \geq 0 \end{array} \right. \quad \forall (\mathbf{v}, \boldsymbol{\tau}) \in K$$

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- Discretized domain  $\Omega_L$



# Discretization

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- **Discretized domain**  $\Omega_L$
- **FE space**  $X_L = P_{\Gamma_D}^1(\Omega_L) \times \mathcal{RT}_{0,\Gamma_N}(\Omega_L)$  with  $\mathbf{x}_L = (\mathbf{u}_L, \boldsymbol{\sigma}_L) \in X_L$

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- $\mathbf{f}_L, \mathbf{u}_{D,L}, \mathbf{t}_{N,L}, g_L$  FE representations of  $\mathbf{f}, \mathbf{u}_D, \mathbf{t}_N, g$

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- **Discrete FOSLS Functional**

$$\mathcal{J}(\mathbf{x}_L; \mathbf{f}_L) = \frac{1}{2} \mathbf{x}_L^T \mathbf{A}_L \mathbf{x}_L - \mathbf{x}_L^T \mathbf{f}_L$$

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- **Convex Set**  $K_L$  (in general  $K_L \not\subseteq K$ )

$$\mathbf{x}_L \in K_L \quad \Longleftrightarrow \quad \mathbf{B}_L \mathbf{x}_L \leq \mathbf{g}_L$$

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- Minimization problem:

Find  $\mathbf{x}_L \in K_L$

$$\begin{aligned} \operatorname{argmin} \mathcal{J}(\mathbf{x}_L; \mathbf{f}_L) &= \frac{1}{2} \mathbf{x}_L^T \mathbf{A}_L \mathbf{x}_L - \mathbf{x}_L^T \mathbf{f}_L \\ &\mathbf{B}_L \mathbf{x}_L \leq \mathbf{g}_L \end{aligned}$$

# Disadvantages and Advantages of the FOSLS

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## Pros

- Direct access to stress  $\sigma$  (friction, plasticity...)
- Dealing with incompressible materials ( $\lambda \rightarrow \infty$ )
- FOSLS functional as an a posteriori error estimator
- Flexible choice of finite element spaces (**low order**:  $\mathbf{u}_L \in P^1$ ,  $\sigma_L \in \mathcal{RT}_0$ )
- Symmetric positive definite system

• Attia, Frank S., Zhiqiang Cai, and Gerhard Starke. "First-order system least squares for the Signorini contact problem in linear elasticity". SIAM Journal on Numerical Analysis 47.4 (2009): 3027-3043.

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- The functional is fictitious, not physical
- The asymmetry of the stress tensor
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**Functional to be minimized**  
**Local constraints**  
**Need for a preconditioner**  $\Rightarrow$  **Monotone Multilevel**

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# Monotone Multilevel strategy

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- Successive energy minimization by means of local corrections
- No correction can increase energy
- Fine space corrections on fine grid (non-linear Gauß-Seidel)  $\Rightarrow$  global convergence
- Coarse space corrections  $\Rightarrow$  accelerating convergence

# Monotone Multilevel by energy minimization

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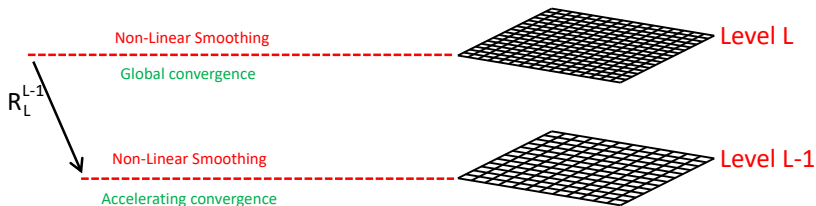


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$R_i^{i-1}$  restriction operator ( $i = L, \dots, 2$ )

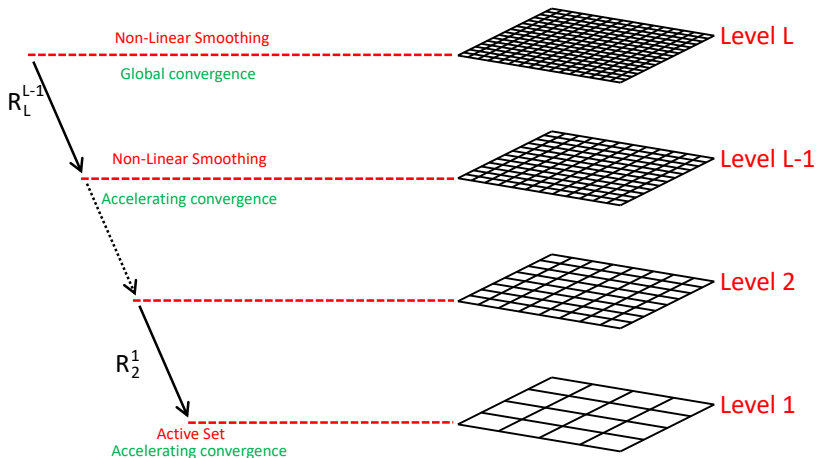


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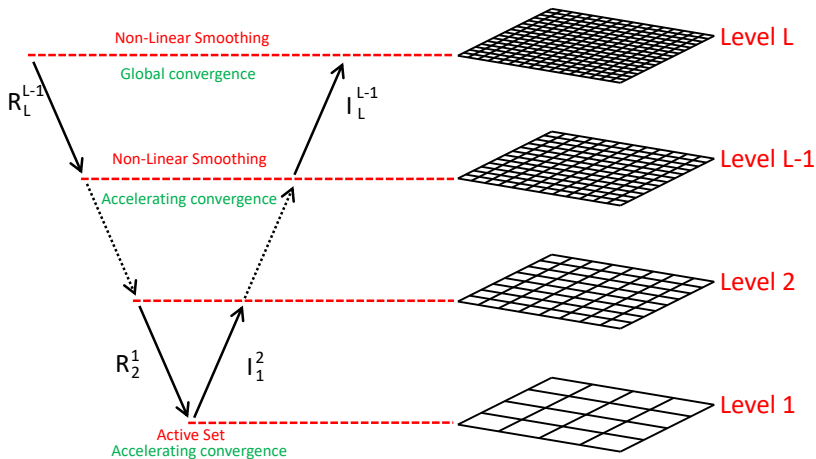


# Monotone Multilevel by energy minimization

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$R_i^{i-1}$  restriction operator,  $I_{i-1}^i$  interpolation operator ( $i = L, \dots, 2$ )



## Smoother

- Standard non-linear Gauß-Seidel smooths  $H^1$ , **but not**  $H_{\text{div}}$
- The kernel  $\text{Ker}(\text{div}) = \{\boldsymbol{\tau} \in H_{\text{div}}, \text{div } \boldsymbol{\tau} = 0\}$  is too large
- Patch-smoother for divergence-free components of the error

## Interpolations and restrictions

- Standard  $P^1$  and  $RT_0$  interpolations and restrictions for primal and dual variables
- Non-linear projections for constraint representation on coarser levels

# Multilevel ingredients

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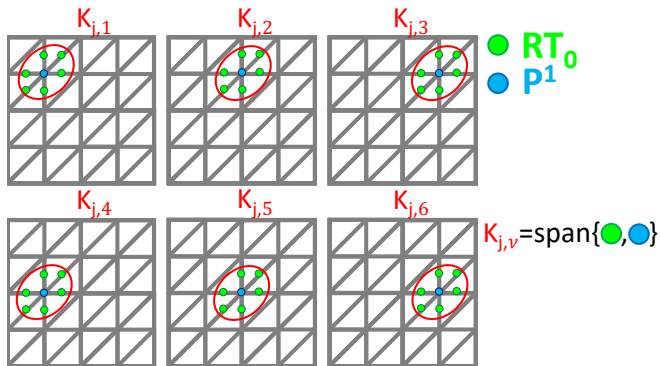
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# LS Patch smoother

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- Mesh level  $j = 1, \dots, L$ , vertex  $\nu = 1, \dots, N_j$
- $\text{Patch}_{j,\nu}$  = dofs of node  $\nu$  and surrounding edges/faces (2D/3D)
- $K_{j,\nu}$  = local closed convex set spanned by basis functions in  $\text{Patch}_{j,\nu}$
- Minimization of  $\mathcal{J}$  on  $K_{j,\nu}$
- Error smoothed in  $H^1$  and  $H_{\text{div}}$  simultaneously



- Ralf Hiptmair. Multigrid method for  $H(\text{div})$  in three dimensions. Electron. Trans. Numer. Anal, 6(1):133-152, 1997.
- Douglas N Arnold, Richard S Falk, and Ragnar Winther. Multigrid in  $H(\text{div})$  and  $H(\text{curl})$ . Numerische Mathematik, 85(2):197-217, 2000.
- Gerhard Starke. Gauss-Newton multilevel methods for least-squares finite element computations of variably saturated subsurface flow. Computing, 64(4):323-338, 2000.



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# Non-Linear Projections for Coarse Constraints

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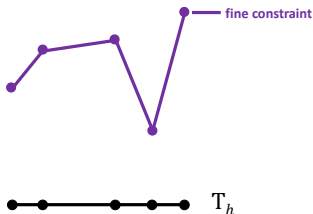
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Comparison of coarse function  $\mathbf{c}_j$  with **fine constraint**  $\Rightarrow$  **suboptimal complexity**
- **Approximate monotone multilevel**  
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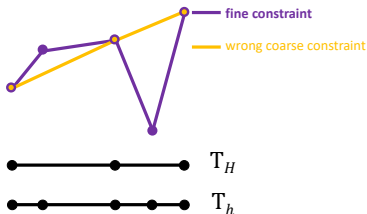


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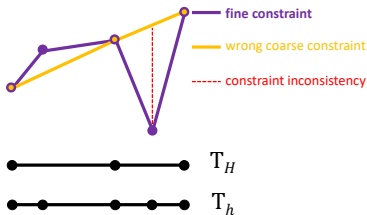


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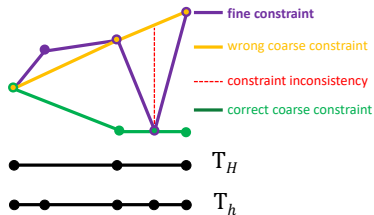


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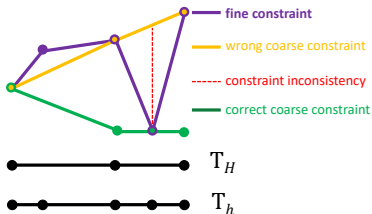


# Non-Linear Projections for Coarse Constraints

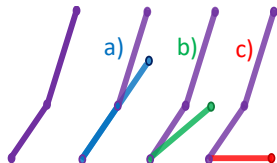
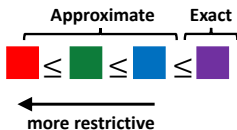
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for FOSLS  
linear  
elastic  
contact

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G. Starke,  
R. Krause

- **Exact monotone multilevel**  
Comparison of coarse function  $c_j$  with **fine constraint**  $\Rightarrow$  **suboptimal complexity**
- **Approximate monotone multilevel**  
Comparison of coarse function  $c_j$  with **coarse constraint**  $\Rightarrow$  **optimal complexity**

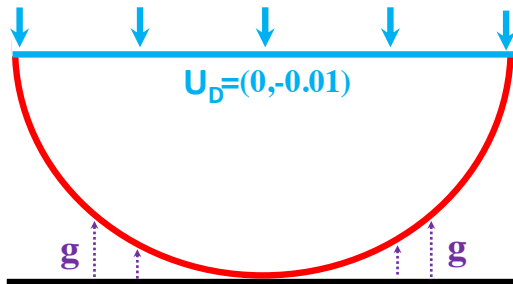


- Different consistent coarse constraints



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$\mu = 1$ ,  $\lambda = 1, \infty$  (compressible and **incompressible**)



# Hertzian Contact, two-level problem

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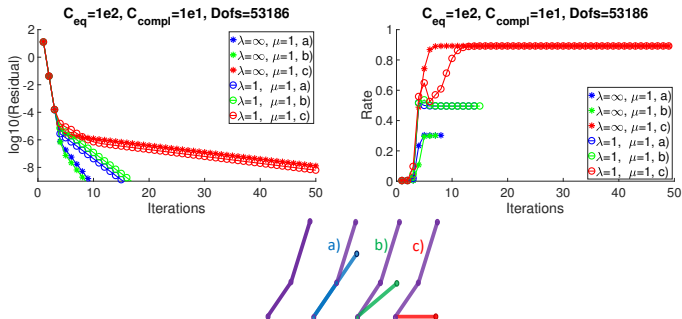


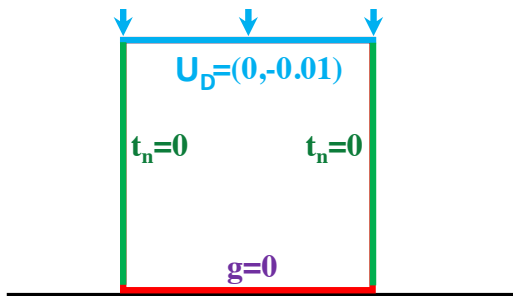
Figure: Mesh with  $h_{\max}/h_{\min} = 7.0567$

- **First phase:** non-linear, capturing high frequencies
- **Second phase:** linear, known active set (blue, green), and not already known active set (red)
- **Similar behaviour** of compressible and incompressible cases

# Hertzian Contact - Setting

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$\mu = 1$ ,  $\lambda = 1, \infty$  (compressible and **incompressible**)

# Signorini's problem, square mesh

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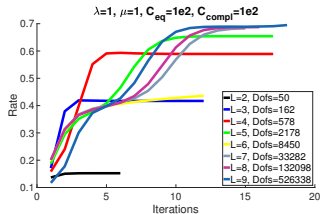
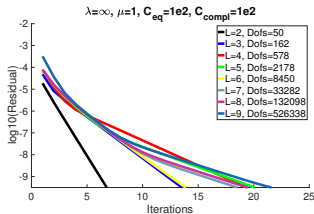


Figure: Square mesh. Compressible material.

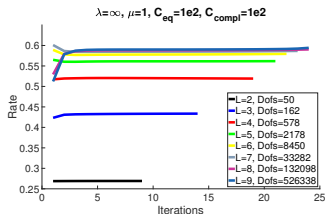
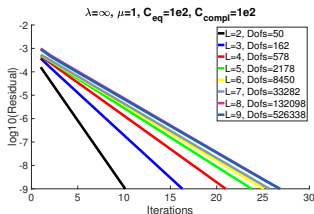


Figure: Square mesh. Incompressible material.

- Purely linear problem:  $h$ - and  $L$ - independency

# Conclusions

Monotone  
multilevel  
for FOSLS  
linear  
elastic  
contact

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R. Krause

- Monotone multilevel for FOSLS linear elastic contact
- Limite case:  $h$ - and  $L$ - independency
- Similar behaviour of compressible and incompressible cases

Monotone  
multilevel  
for FOSLS  
linear  
elastic  
contact

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Thank you for your attention!

**Monotone  
multilevel  
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linear  
elastic  
contact**

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G. Starke,  
R. Krause**

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contact**

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# Exact Monotone Multilevel

Monotone  
multilevel  
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linear  
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contact

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Define:

- $\mathbf{x}_j^k = (\mathbf{u}_j^k, \boldsymbol{\sigma}_j^k) \in K_J$   $k$ -th iterate
- $\mathbf{x}_{J,0} = \mathbf{x}_J^k$
- $\mathbf{x}_{j,0} = \mathbf{x}_{j+1,N_{j+1}}$ , for  $j = J-1, \dots, 1$

Compute a sequence of intermediate iterates  $\mathbf{x}_{j,\nu} = \mathbf{x}_{j,\nu-1} + \mathbf{c}_{j,\nu}$ :

$$\mathcal{J} \leq \mathcal{J}(\mathbf{x}_{j,\nu} + \mathbf{y}) \quad \forall \mathbf{y} \in K_{j,\nu}^* \quad j = J, \dots, 2, \quad \nu = 1, \dots, N_j$$

$$\mathcal{J}(\mathbf{x}_{2,N_2} + \mathbf{c}_1) \leq \mathcal{J}(\mathbf{x}_{2,N_2} + \mathbf{y}) \quad \forall \mathbf{y} \in K_1^* \quad j = 1$$

with the **exact** local closed convex sets  $K_{j,\nu}^*$  and  $K_1^*$ :

$$K_{j,\nu}^*(\mathbf{x}_{j,\nu}) = \{\mathbf{y} \in \text{span}\{\boldsymbol{\lambda}_{j,\nu}\} : \mathbf{y} + \mathbf{x}_{j,\nu} \in K_J\}$$

$$K_1^*(\mathbf{x}_{2,N_2}) = \{\mathbf{y} \in \text{span}\{\boldsymbol{\lambda}_1\} : \mathbf{y} + \mathbf{x}_{2,N_2} \in K_J\}$$

Ralf Kornhuber. Monotone multigrid methods for elliptic variational inequalities I. Numerische Mathematik, 69(2):167-184, 1994.

Ralf Kornhuber and Rolf Krause. Adaptive multigrid methods for Signorini's problem in linear elasticity. Computing and Visualization in Science, 4(1):9-20, 2001.



# Approximate Monotone Multilevel

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multilevel  
for FOSLS  
linear  
elastic  
contact

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R. Krause

Define:

- $\mathbf{c}_{j,\nu} = (\tilde{\mathbf{u}}_{j,\nu}, \tilde{\boldsymbol{\sigma}}_{j,\nu})$  correction at level  $j$ , patch  $\nu$
- $\mathbf{c}_{J,0} = \mathbf{x}_J^k$ ,  $\mathbf{c}_{j,0} = \mathbf{0}$  for  $j = J - 1, \dots, 1$
- $\mathbf{w}_{j,\nu} = \sum_{\mu=0}^{\nu} \mathbf{c}_{j,\mu}$

Compute a sequence of intermediate corrections  $\mathbf{c}_{j,\nu} \in K_{j,\nu}(\mathbf{w}_{j,\nu-1})$  and  $\mathbf{c}_1 \in K_1$ :

$$\begin{aligned} \mathcal{J}(\mathbf{w}_{j,\nu-1} + \mathbf{c}_{j,\nu}) &\leq \mathcal{J}(\mathbf{w}_{j,\nu-1} + \mathbf{y}) \quad \forall \mathbf{y} \in K_{j,\nu} & j = J, \dots, 2, \quad \nu = 1, \dots, N_j \\ \mathcal{J}(\mathbf{c}_1) &\leq \mathcal{J}(\mathbf{y}) & \forall \mathbf{y} \in K_1 & j = 1 \end{aligned}$$

with the **coarse convex sets**  $K_j$  and the **approximate** local closed convex sets  $K_{j,\nu}$ :

$$\begin{aligned} K_{j,\nu}(\mathbf{w}_{j,\nu-1}) &= \{\mathbf{y} \in \text{span}\{\lambda_{j,\nu}\} : \mathbf{y} + \mathbf{w}_{j,\nu-1} \in K_j\} \\ K_1 &\subset K_2 \subset \dots \subset K_{J-1} \subset K_J \end{aligned}$$

# Coarse Convex Sets and Constraints

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## Coarse Convex Sets:

$$\begin{aligned}
 K_j &= \left\{ \mathbf{x}_j = (\mathbf{u}_j, \boldsymbol{\sigma}_j) \in X_j : \mathbf{u}_j|_{\Gamma_D} = \mathbf{u}_D, \boldsymbol{\sigma}_j|_{\Gamma_N} = \mathbf{t}_N, \right. \\
 &\quad \left. \mathbf{u}_j \cdot \mathbf{n}_j|_{\Gamma_C} \leq g_{j,u_n}, \mathbf{n}^T(\boldsymbol{\sigma}_j \mathbf{n}) \leq g_{j,\sigma_n}, \mathbf{t}_j^T(\boldsymbol{\sigma}_j \mathbf{n}_j) = 0 \right\} \quad j = J \\
 K_j &= \left\{ \mathbf{x}_j = (\mathbf{u}_j, \boldsymbol{\sigma}_j) \in X_j : \mathbf{u}_j|_{\Gamma_D} = \mathbf{0}, \boldsymbol{\sigma}_j|_{\Gamma_N} = \mathbf{0}, \right. \\
 &\quad \left. \mathbf{u}_j \cdot \mathbf{n}_j|_{\Gamma_C} \leq g_{j,u_n}, \mathbf{n}^T(\boldsymbol{\sigma}_j \mathbf{n}) \leq g_{j,\sigma_n}, \mathbf{t}_j^T(\boldsymbol{\sigma}_j \mathbf{n}_j) = 0 \right\} \quad j = J-1, \dots, 1
 \end{aligned}$$

## Coarse Constraints:

- $\tilde{\mathbf{u}}_{j,\nu}$  and  $\tilde{\boldsymbol{\sigma}}_{j,\nu}$  are the components of the correction  $\mathbf{c}_{j,\nu}$ .

$$\begin{aligned}
 g_{j,u_n} &= \begin{cases} g & j = J \\ l_{j+1,u_n}^j \left( g_{j+1,u_n} - \sum_{\nu=1}^{N_{j+1}} [\tilde{\mathbf{u}}_{j+1,\nu}|_{\Gamma_C}]_n \right) & j = J-1, \dots, 1 \end{cases} \\
 g_{j,\sigma_n} &= \begin{cases} 0 & j = J \\ l_{j+1,\sigma_n}^j \left( g_{j+1,\sigma_n} - \sum_{\nu=1}^{N_{j+1}} [\tilde{\boldsymbol{\sigma}}_{j+1,\nu}|_{\Gamma_C}]_n \right) & j = J-1, \dots, 1 \end{cases}
 \end{aligned}$$

## Non-Linear Projection Operators:

$l_{j+1,u_n}^j, l_{j+1,\sigma_n}^j$  chosen so that  $K_1 \subset K_2 \subset \dots \subset K_{J-1} \subset K_J$ .

# Normal displacement Non-Linear Projection

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$$v_H(\nu_{H,1}) \leq v_h(\nu_{H,1})$$

$$v_H(\nu_{H,2}) \leq v_h(\nu_{H,2})$$

$$\forall \varepsilon_H \in \mathcal{E}_H \cap \Gamma_{C,H}$$

$$\frac{1}{2}(v_H(\nu_{H,1}) + v_H(\nu_{H,2})) \leq v_h(\nu_h)$$

It is easy to see that, on  $\varepsilon_H$ , the following values satisfy the three conditions above:

$$a) \quad \begin{cases} \tilde{v}_H(\nu_{H,1}) = \min(v_h(\nu_{H,1}), \max(v_h(\nu_h), 2v_h(\nu_h) - v_h(\nu_{H,2}))) \\ \tilde{v}_H(\nu_{H,2}) = \min(v_h(\nu_{H,2}), \max(v_h(\nu_h), 2v_h(\nu_h) - v_h(\nu_{H,1}))) \end{cases} \quad \forall \varepsilon_H \in \mathcal{E}_H \cap \Gamma_C$$

$$b) \quad \begin{cases} \tilde{v}_H(\nu_{H,1}) = \min(v_h(\nu_{H,1}), v_h(\nu_h)) \\ \tilde{v}_H(\nu_{H,2}) = \min(v_h(\nu_{H,2}), v_h(\nu_h)) \end{cases} \quad \forall \varepsilon_H \in \mathcal{E}_H \cap \Gamma_C$$

$$c) \quad \begin{cases} \tilde{v}_H(\nu_{H,1}) = \min(v_h(\nu_{H,1}), v_h(\nu_h), v_h(\nu_{H,2})) \\ \tilde{v}_H(\nu_{H,2}) = \min(v_h(\nu_{H,1}), v_h(\nu_h), v_h(\nu_{H,2})) \end{cases} \quad \forall \varepsilon_H \in \mathcal{E}_H \cap \Gamma_C$$

# Pressure Non-Linear Projection

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linear  
elastic  
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$$s_H(\phi_H) \leq s_h(\phi_h) \quad \forall \phi_h \in P_{\phi_H}^{\phi_h}$$

Thus:

$$s_H = I_{h,\sigma_n}^H s_h = \sum_{\phi_{H_i} \in T_H} [\lambda_{\Sigma_H, H_i}]_n s_H(\phi_{H_i}) \quad \text{with} \quad s_H(\phi_{H_i}) = \min_{\phi_h \in P_{\phi_{H_i}}^{\phi_h}} s_h(\phi_h)$$

# Truncated Basis

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linear  
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R. Krause

$$[\tilde{\lambda}_{U_j, \nu}]_i = \begin{cases} [\lambda_{U_j, \nu}]_i & \nu \in \mathcal{N}_j \setminus \mathcal{N}_j^\bullet, \quad i = n, t \\ 0 & \nu \in \mathcal{N}_j^\bullet, \quad i = n \\ [\lambda_{U_j, \nu}]_i & \nu \in \mathcal{N}_j^\bullet, \quad i = t \end{cases}$$

$$[\tilde{\lambda}_{\Sigma_j, \phi}]_i = \begin{cases} [\lambda_{\Sigma_j, \phi}]_i & \phi \in \mathcal{F}_j \setminus \mathcal{F}_j^\bullet, \quad i = n, t \\ 0 & \phi \in \mathcal{F}_j^\bullet, \quad i = n \\ [\lambda_{\Sigma_j, \phi}]_i & \phi \in \mathcal{F}_j^\bullet, \quad i = t \end{cases}$$