Monotone multilevel for FOSLS linear elastic contact

G. Rovi$^{\triangle}$, B. Kober$^{\circ}$, G. Starke$^{\circ}$, R. Krause$^{\triangle}$

$\circ$: Università Duisburg - Essen, Germany
$\triangle$: Università della Svizzera italiana, Switzerland

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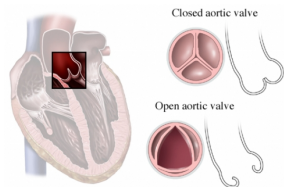
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Examples of contact problems

Monotone
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- Contact problems with incompressible materials.
- Quantities of interest: the forces generated by the contact.

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Signorini's problem: strong formulation

- **First Order System Linear Elasticity**
  Find displacement **u**, stress $\boldsymbol{\sigma}$ of the body Ω:

$$\begin{cases} \text{div}\boldsymbol{\sigma} + \mathbf{f} = 0 & \Omega & \text{momentum balance equation} \\ \mathcal{A}\boldsymbol{\sigma} - \boldsymbol{\varepsilon}(\mathbf{u}) = 0 & \Omega & \text{constitutive law} \\ \mathbf{u} = \mathbf{u}_D & \Gamma_D & \text{Dirichlet BC} \\ \boldsymbol{\sigma}\mathbf{n} = \mathbf{t}_N & \Gamma_N & \text{Neumann BC} \end{cases}$$

where $\boldsymbol{\varepsilon}(\mathbf{u}) = \dfrac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T)$, $\mathcal{A}\boldsymbol{\sigma} = \dfrac{1}{2\mu}\left(\boldsymbol{\sigma} - \dfrac{\lambda}{d\lambda + 2\mu}\text{tr}\boldsymbol{\sigma}\mathbf{I}\right)$ and μ, λ are the Lamé parameters

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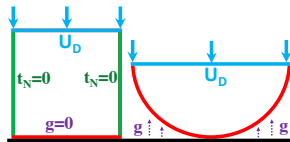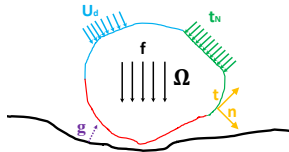
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- **Contact Constraints**
  Given the gap function $g \geq 0$, the normal and tangent vectors $\mathbf{n}$ and $\mathbf{t}$:

$$\begin{cases} \mathbf{u} \cdot \mathbf{n} - g \leq 0 & \Gamma_C & \text{impenetrability} \\ (\boldsymbol{\sigma}\mathbf{n}) \cdot \mathbf{n} \leq 0 & \Gamma_C & \text{direction of the surface pressure} \\ (\mathbf{u} \cdot \mathbf{n} - g)\left((\boldsymbol{\sigma}\mathbf{n}) \cdot \mathbf{n}\right) = 0 & \Gamma_C & \text{complementarity condition} \\ \mathbf{t}_i^T(\boldsymbol{\sigma}\mathbf{n}) = 0 & \Gamma_C & \text{frictionless condition} \end{cases}$$

Portion of Γ_C actually in contact **not known a priori $\Rightarrow$ non-linearity**

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The Least-Squares Formulation

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- **First Order System Least-Squares (FOSLS) Functional**

  $C_1, \ C_2, \ C_3 > 0$

  $$\mathcal{J}(\mathbf{u}, \boldsymbol{\sigma}) = C_1 \left\| \mathrm{div}\boldsymbol{\sigma} + \mathbf{f} \right\|^2_{L^2(\Omega)^d} + C_2 \left\| \mathcal{A}\boldsymbol{\sigma} - \boldsymbol{\varepsilon}(\mathbf{u}) \right\|^2_{L^2(\Omega)^d} + C_3 \langle \mathbf{u} \cdot \mathbf{n} - g, (\boldsymbol{\sigma}\mathbf{n}) \cdot \mathbf{n} \rangle_{\Gamma_c}$$

● Rolf Krause, Benjamin Müller, and Gerhard Starke. An adaptive least-squares mixed finite element method for the Signorini problem.
Numerical Methods for Partial Differential Equations, 33(1):276-289, 2017.

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- **Convex Set K**

  $K = \{(\mathbf{u}, \boldsymbol{\sigma}) \in \left[H^1_{\Gamma_D}(\Omega) \right]^d \times \left[H_{\operatorname{div}, \Gamma_N}(\Omega) \right]^d : \ \mathbf{u} \cdot \mathbf{n} - g \leq 0, \ (\boldsymbol{\sigma}\mathbf{n}) \cdot \mathbf{n} \leq 0, \ \mathbf{t}_i^T(\boldsymbol{\sigma}\mathbf{n}) = 0 \quad \Gamma_C \}$

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- **First Order System Least-Squares (FOSLS) Functional**

  $C_1,\ C_2,\ C_3 > 0$

  $\mathcal{J}(\mathbf{u}, \boldsymbol{\sigma}) = C_1 \|\mathrm{div}\boldsymbol{\sigma} + \mathbf{f}\|_{L^2(\Omega)^d}^2 + C_2 \|\mathcal{A}\boldsymbol{\sigma} - \boldsymbol{\varepsilon}(\mathbf{u})\|_{L^2(\Omega)^d}^2 + C_3 \langle \mathbf{u} \cdot \mathbf{n} - g, (\boldsymbol{\sigma}\mathbf{n}) \cdot \mathbf{n} \rangle_{\Gamma_c}$

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- Find $(\mathbf{u}, \boldsymbol{\sigma}) \in K$, such that:

  •**Minimization problem:** $\quad \mathcal{J}(\mathbf{u}, \boldsymbol{\sigma}) \le \mathcal{J}(\mathbf{v}, \boldsymbol{\tau}) \quad \forall (\mathbf{v}, \boldsymbol{\tau}) \in K$

  $$\Longleftrightarrow$$

  •**Variational Inequality:** $\quad \begin{cases} \left\langle \dfrac{\partial \mathcal{J}(\mathbf{u}, \boldsymbol{\sigma}; \mathbf{f}, g)}{\partial \mathbf{u}}, \mathbf{v} - \mathbf{u} \right\rangle \ge 0 \\[3mm] \left\langle \dfrac{\partial \mathcal{J}(\mathbf{u}, \boldsymbol{\sigma}; \mathbf{f}, g)}{\partial \boldsymbol{\sigma}}, \boldsymbol{\tau} - \boldsymbol{\sigma} \right\rangle \ge 0 \end{cases} \quad \forall (\mathbf{v}, \boldsymbol{\tau}) \in K$

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Discretization

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- **Discretized domain** Ω_L

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- Discretized domain Ω_L
- **FE space** $X_L = P^1_{\Gamma_D}(\Omega_L) \times \mathcal{RT}_{0,\Gamma_N}(\Omega_L)$ with $\mathbf{x}_L = (\mathbf{u}_L, \boldsymbol{\sigma}_L) \in X_L$

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- $\mathbf{f}_L$, $\mathbf{u}_{D,L}$, $\mathbf{t}_{N,L}$, g_L FE representations of $\mathbf{f}$, $\mathbf{u}_D$, $\mathbf{t}_N$, g

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- Discrete FOSLS Functional

$$\mathcal{J}(\mathbf{x}_L) = \frac{1}{2}\mathbf{x}_L^T \mathbf{A}_L \mathbf{x}_L - \mathbf{x}_L^T \mathbf{f}_L$$

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Discretization

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- Convex Set K_L (in general $K_L \not\subseteq K$)

$$\mathbf{x}_L \in K_L \qquad \Longleftrightarrow \qquad \mathbf{B}_L \mathbf{x}_L \leq \mathbf{g}_L$$

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- **Minimization problem:**
  Find $\mathbf{x}_L \in K_L$

$$\arg\min \mathcal{J}(\mathbf{x}_L) = \frac{1}{2}\mathbf{x}_L^T \mathbf{A}_L \mathbf{x}_L - \mathbf{x}_L^T \mathbf{f}_L$$
$$\mathbf{B}_L \mathbf{x}_L \leq \mathbf{g}_L$$

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Disadvantages and Advantages of the FOSLS

Pros

- Direct access to stress $\boldsymbol{\sigma}$ (friction, plasticity...)
- Dealing with incompressible materials ($\lambda \to \infty$)
- FOSLS functional as an a posteriori error estimator
- Flexible choice of finite element spaces (**low order**: $\mathbf{u}_L \in P^1$, $\boldsymbol{\sigma}_L \in \mathcal{RT}_0$)
- Symmetric positive definite system

• Attia, Frank S., Zhiqiang Cai, and Gerhard Starke. "First-order system least squares for the Signorini contact problem in linear elasticity". SIAM Journal on Numerical Analysis 47.4 (2009): 3027-3043.

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Disadvantages and Advantages of the FOSLS

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Cons

- The functional is fictitious, not physical
- The asymmetry of the stress tensor
- Find proper weights C_1, C_2, C_3
- Large condition number: **need for a preconditioner**

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Disadvantages and Advantages of the FOSLS

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$$\left\{\begin{array}{l} \textbf{Functional to be minimized} \\ \textbf{Local constraints} \\ \textbf{Need for a preconditioner} \end{array}\right. \quad \Rightarrow \quad \textbf{Monotone Multilevel}$$

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Monotone Multilevel strategy

**Monotone
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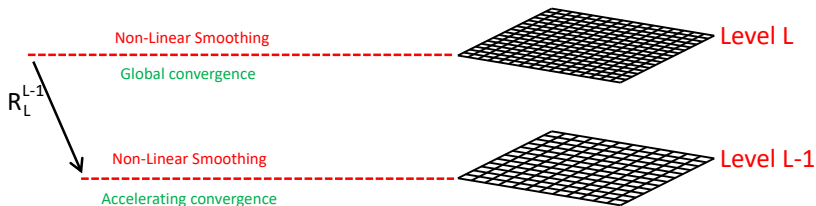contact**

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- Successive energy minimization by means of local corrections

- No correction can increase energy

- Fine space corrections on fine grid (non-linear Gauß-Seidel) $\Rightarrow$ global convergence

- Coarse space corrections $\Rightarrow$ accelerating convergence

Monotone Multilevel by energy minimization

Monotone multilevel for FOSLS linear elastic contact

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Non-Linear Smoothing

Global convergence

Level L

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Monotone Multilevel by energy minimization

R_i^{i-1} restriction operator ($i = L, ..., 2$)

Non-Linear Smoothing

Global convergence

R_L^{L-1}

Level L

Non-Linear Smoothing

Accelerating convergence

Level L-1

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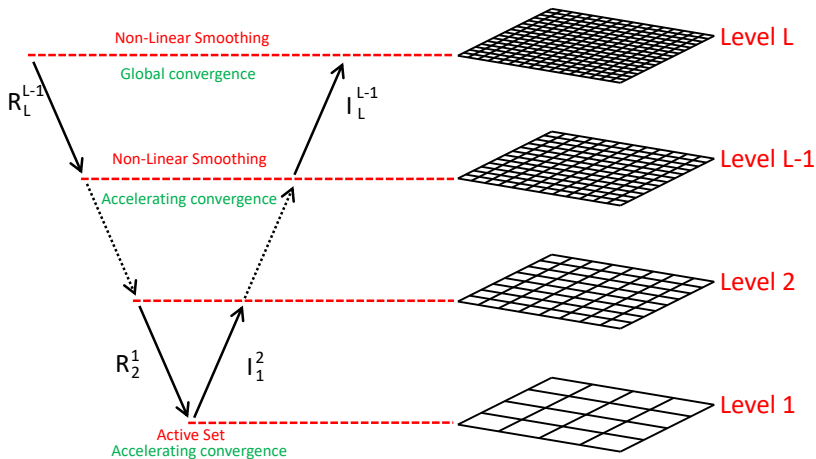# Monotone Multilevel by energy minimization

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R_i^{i-1} restriction operator ($i = L, ..., 2$)

R_L^{L-1}

Non-Linear Smoothing
Global convergence
Level L

Non-Linear Smoothing
Accelerating convergence
Level L-1

Level 2

R_2^1

Active Set
Accelerating convergence
Level 1

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Monotone Multilevel by energy minimization

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R_i^{i-1} restriction operator, I_{i-1}^i interpolation operator ($i = L, ..., 2$)

Level L

Non-Linear Smoothing

Global convergence

R_L^{L-1}

I_L^{L-1}

Non-Linear Smoothing

Level L-1

Accelerating convergence

Level 2

R_2^1

I_1^2

Level 1

Active Set

Accelerating convergence

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Multilevel ingredients

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Smoother

- Standard non-linear Gauß-Seidel smooths H^1, **but not** H_{div}
- The kernel $\text{Ker}(\text{div}) = \{\boldsymbol{\tau} \in H_{\text{div}}, \text{div}\,\boldsymbol{\tau} = 0\}$ is too large
- Patch-smoother for divergence-free components of the error

Interpolations and restrictions

- Standard P^1 and RT_0 interpolations and restrictions for primal and dual variables
- Non-linear projections for constraint representation on coarser levels

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LS Patch smoother

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- Mesh level $j = 1, ..., L$, vertex $\nu = 1, ..., N_j$
- Patch$_{j,\nu}$= dofs of node ν and surrounding edges/faces (2D/3D)
- $K_{j,\nu}$= local closed convex set spanned by basis functions in Patch$_{j,\nu}$
- Minimization of $\mathcal{J}$ on $K_{j,\nu}$
- Error smoothed in H^1 and H_{div} simultaneously

$K_{j,1}$ $K_{j,2}$ $K_{j,3}$

● RT_0
● P^1

$K_{j,4}$ $K_{j,5}$ $K_{j,6}$

$K_{j,\nu}$=span$\{●,●\}$
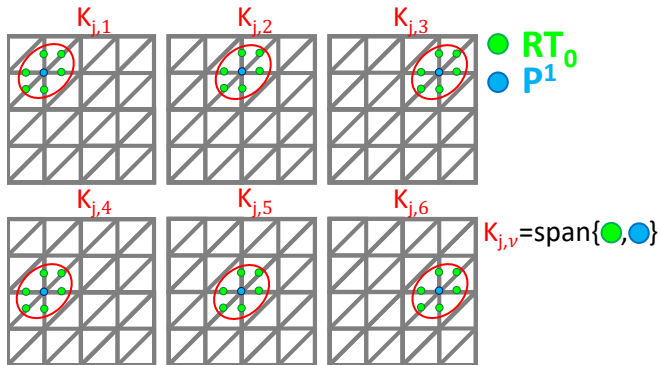
● Ralf Hiptmair. Multigrid method for H(div) in three dimensions. Electron. Trans. Numer. Anal, 6(1):133-152, 1997.
● Douglas N Arnold, Richard S Falk, and Ragnar Winther. Multigrid in H(div) and H(curl). Numerische Mathe- matik, 85(2):197-217, 2000.
● Gerhard Starke. Gauss-Newton multilevel methods for least-squares finite element computations of variably saturated subsurface flow. Computing, 64(4):323-338, 2000.

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Non-Linear Projections for Coarse Constraints

- **Exact monotone multilevel**
  Comparing coarse corrections $\mathbf{c}_j$ with **fine constraint** $\Rightarrow$ **suboptimal complexity**

- **Approximate monotone multilevel**
  Comparing of coarse corrections $\mathbf{c}_j$ with **coarse constraint** $\Rightarrow$ **optimal complexity**
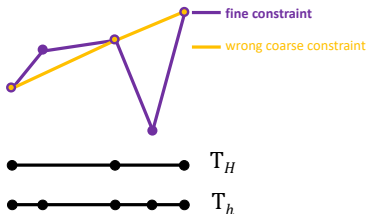
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- **Approximate monotone multilevel**
  Comparing of coarse corrections $\mathbf{c}_j$ with **coarse constraint** $\Rightarrow$ **optimal complexity**

fine constraint

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Non-Linear Projections for Coarse Constraints

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- **Exact monotone multilevel**
  Comparing coarse corrections $\mathbf{c}_j$ with **fine constraint** $\Rightarrow$ **suboptimal complexity**

- **Approximate monotone multilevel**
  Comparing of coarse corrections $\mathbf{c}_j$ with **coarse constraint** $\Rightarrow$ **optimal complexity**

T_H

T_h

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Non-Linear Projections for Coarse Constraints

Monotone
multilevel
for FOSLS
linear
elastic
contact

G.
Rovi[△],
B.
Kober[○],
G.
Starke[○],
R.
Krause[△]

- **Exact monotone multilevel**
  Comparing coarse corrections c_j with **fine constraint** $\Rightarrow$ **suboptimal complexity**

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fine constraint
wrong coarse constraint
constraint inconsistency

T_H

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Non-Linear Projections for Coarse Constraints

- **Exact monotone multilevel**
  Comparing coarse corrections c_j with **fine constraint** $\Rightarrow$ **suboptimal complexity**

- **Approximate monotone multilevel**
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— fine constraint
— wrong coarse constraint
------ constraint inconsistency
— correct coarse constraint

T_H

T_h

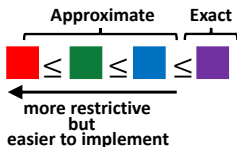# Non-Linear Projections for Coarse Constraints

Monotone multilevel for FOSLS linear elastic contact

G. Rovi[△],
B. Kober[○],
G. Starke[○],
R. Krause[△]

- **Exact monotone multilevel**
  Comparing coarse corrections c_j with **fine constraint** $\Rightarrow$ **suboptimal complexity**

- **Approximate monotone multilevel**
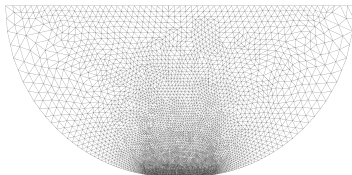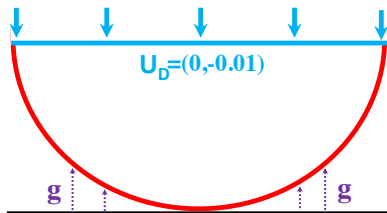 Comparing of coarse corrections c_j with **coarse constraint** $\Rightarrow$ **optimal complexity**

- Different consistent coarse constraints

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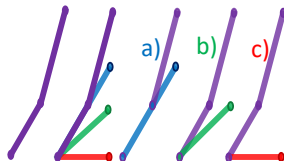
Hertzian Contact - Setting

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G.
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$\mu = 1$, $\lambda = 1, \infty$ (compressible and **incompressible**)

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Hertzian Contact, two-level problem

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Starke[○],
R.
Krause[△]

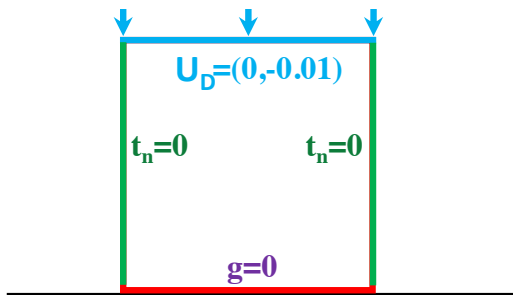Figure: Mesh with $h_{max}/h_{min} = 7.0567$
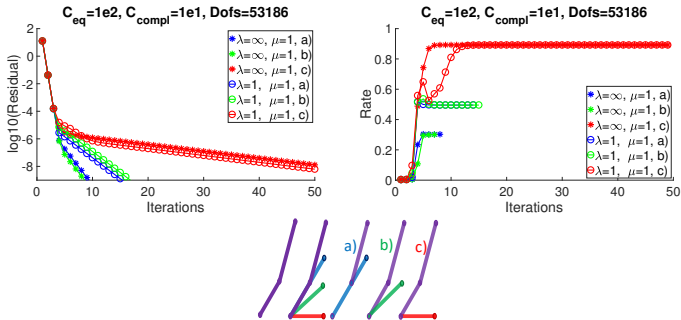
- **First phase**: non-linear, capturing high frequencies
- **Second phase**: linear
  - green, blue: **known active set** $\Rightarrow$ faster
  - red: **not already known active set** $\Rightarrow$ slower
- green, blue similar behaviour: pick green $\Rightarrow$ easier to implement
- **Incompressibility** easily solvable

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Hertzian Contact - Setting

$\mu = 1,\ \lambda = 1, \infty$ (compressible and **incompressible**)

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Signorini's problem, square mesh

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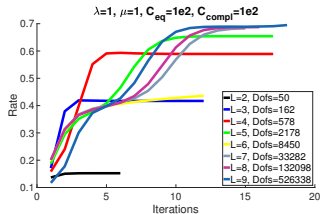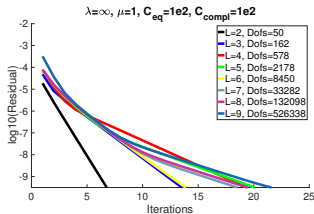Rovi△,
B.
Kober°,
G.
Starke°,
R.
Krause△

Figure: Square mesh. Compressible material.
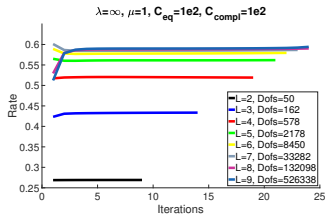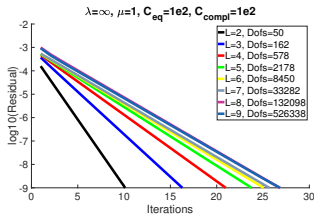
Figure: Square mesh. Incompressible material.

- Purely linear problem: $h-$ and $L-$ independency

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Conclusions

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- Monotone multilevel for FOSLS linear elastic contact

- Limit case: h– and L– independency

- Similar behaviour of compressible and incompressible cases

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Thank you for your attention!

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Exact Monotone Multilevel

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Starke[○],
R.
Krause[△]

Define:

- $\mathbf{x}_j^k = (\mathbf{u}_j^k, \boldsymbol{\sigma}_j^k) \in K_J$ k-th iterate
- $\mathbf{x}_{J,0} = \mathbf{x}_J^k$
- $\mathbf{x}_{j,0} = \mathbf{x}_{j+1, N_{j+1}}$, for $j = J - 1, ..., 1$

Compute a sequence of intermediate iterates $\mathbf{x}_{j,\nu} = \mathbf{x}_{j,\nu-1} + \mathbf{c}_{j,\nu}$:

$$\mathcal{J} \leq \mathcal{J}(\mathbf{x}_{j,\nu} + \mathbf{y}) \quad \forall \mathbf{y} \in K_{j,\nu}^*, \quad j = J, ..., 2, \quad \nu = 1, ..., N_j$$

$$\mathcal{J}(\mathbf{x}_{2,N_2} + \mathbf{c}_1) \leq \mathcal{J}(\mathbf{x}_{2,N_2} + \mathbf{y}) \quad \forall \mathbf{y} \in K_1^* \quad j = 1$$

with the **exact** local closed convex sets $K_{j,\nu}^*$ and K_1^*:

$$K_{j,\nu}^*(\mathbf{x}_{j,\nu}) = \left\{ \mathbf{y} \in \text{span}\{\boldsymbol{\lambda}_{j,\nu}\} : \quad \mathbf{y} + \mathbf{x}_{j,\nu} \in K_J \right\}$$

$$K_1^*(\mathbf{x}_{2,N_2}) = \left\{ \mathbf{y} \in \text{span}\{\boldsymbol{\lambda}_1\} : \quad \mathbf{y} + \mathbf{x}_{2,N_2} \in K_J \right\}$$

Ralf Kornhuber. Monotone multigrid methods for elliptic variational inequalities I. Numerische Mathematik, 69(2):167-184, 1994.

Ralf Kornhuber and Rolf Krause. Adaptive multigrid methods for Signorini's problem in linear elasticity. Computing and Visualization in Science, 4(1):9-20, 2001.

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Approximate Monotone Multilevel

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Define:

- $\mathbf{c}_{j,\nu} = (\tilde{\mathbf{u}}_{j,\nu}, \tilde{\boldsymbol{\sigma}}_{j,\nu})$ correction at level j, patch ν
- $\mathbf{c}_{J,0} = \mathbf{x}_J^k$, $\mathbf{c}_{j,0} = \mathbf{0}$ for $j = J-1, ..., 1$
- $\mathbf{w}_{j,\nu} = \sum_{\mu=0}^{\nu} \mathbf{c}_{j,\mu}$

Compute a sequence of intermediate corrections $\mathbf{c}_{j,\nu} \in K_{j,\nu}(\mathbf{w}_{j,\nu-1})$ and $\mathbf{c}_1 \in K_1$:

$$\mathcal{J}(\mathbf{w}_{j,\nu-1} + \mathbf{c}_{j,\nu}) \leq \mathcal{J}(\mathbf{w}_{j,\nu-1} + \mathbf{y}) \quad \forall \, \mathbf{y} \in K_{j,\nu} \qquad j = J, ..., 2, \ \nu = 1, ..., N_j$$
$$\mathcal{J}(\mathbf{c}_1) \leq \mathcal{J}(\mathbf{y}) \qquad\qquad\qquad \forall \, \mathbf{y} \in K_1 \qquad j = 1$$

with the **coarse convex sets** K_j and the **approximate** local closed convex sets $K_{j,\nu}$:

$$K_{j,\nu}(\mathbf{w}_{j,\nu-1}) = \left\{ \mathbf{y} \in \text{span}\{\lambda_{j,\nu}\} : \ \mathbf{y} + \mathbf{w}_{j,\nu-1} \in K_j \right\}$$
$$K_1 \subset K_2 \subset ... \subset K_{J-1} \subset K_J$$

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Coarse Convex Sets and Constraints

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Coarse Convex Sets:

$$K_j = \Big\{ \mathbf{x}_j = (\mathbf{u}_j, \boldsymbol{\sigma}_j) \in X_j : \; \mathbf{u}_j|_{\Gamma_D} = \mathbf{u}_D, \; \boldsymbol{\sigma}_j|_{\Gamma_N} = \mathbf{t}_N,$$

$$\mathbf{u}_j \cdot \mathbf{n}_j|_{\Gamma_C} \le g_{j,u_n}, \; \mathbf{n}^T(\boldsymbol{\sigma}_j \mathbf{n}) \le g_{j,\sigma_n}, \; \mathbf{t}_j^T(\boldsymbol{\sigma} \mathbf{n}_j) = 0 \Big\} \qquad j = J$$

$$K_j = \Big\{ \mathbf{x}_j = (\mathbf{u}_j, \boldsymbol{\sigma}_j) \in X_j : \; \mathbf{u}_j|_{\Gamma_D} = \mathbf{0}, \; \boldsymbol{\sigma}_j|_{\Gamma_N} = \mathbf{0},$$

$$\mathbf{u}_J \cdot \mathbf{n}_j|_{\Gamma_C} \le g_{j,u_n}, \; \mathbf{n}^T(\boldsymbol{\sigma}_j \mathbf{n}) \le g_{j,\sigma_n}, \; \mathbf{t}_j^T(\boldsymbol{\sigma} \mathbf{n}_j) = 0 \Big\} \qquad j = J-1, ..., 1$$

Coarse Constraints:

- $\tilde{\mathbf{u}}_{j,\nu}$ and $\tilde{\boldsymbol{\sigma}}_{j,\nu}$ are the components of the correction $\mathbf{c}_{j,\nu}$.

$$g_{j,u_n} = \begin{cases} g & j = J \\ \mu_{j+1,u_n}^j \left(g_{j+1,u_n} - \sum_{\nu=1}^{N_{j+1}} \left[\tilde{\mathbf{u}}_{j+1,\nu}|_{\Gamma_C} \right]_n \right) & j = J-1, ..., 1 \end{cases}$$

$$g_{j,\sigma_n} = \begin{cases} 0 & j = J \\ \mu_{j+1,\sigma_n}^j \left(g_{j+1,\sigma_n} - \sum_{\nu=1}^{N_{j+1}} \left[\tilde{\boldsymbol{\sigma}}_{j+1,\nu}|_{\Gamma_C} \right]_n \right) & j = J-1, ..., 1 \end{cases}$$

Non-Linear Projection Operators:
μ_{j+1,u_n}^j, μ_{j+1,σ_n}^j chosen so that $K_1 \subset K_2 \subset ... \subset K_{J-1} \subset K_J$.

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Normal displacement Non-Linear Projection

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$$v_H(\nu_{H,1}) \leq v_h(\nu_{H,1})$$
$$v_H(\nu_{H,2}) \leq v_h(\nu_{H,2})$$
$$\frac{1}{2}(v_H(\nu_{H,1}) + v_H(\nu_{H,2})) \leq v_h(\nu_h)$$

$$\forall \varepsilon_H \in \mathcal{E}_H \cap \Gamma_{C,H}$$

It is easy to see that, on e_H, the following values satisfy the three conditions above:

a) $\begin{cases} \tilde{v}_H(\nu_{H,1}) = \min(v_h(\nu_{H,1}), \max(v_h(\nu_h), 2v_h(\nu_h) - v_h(\nu_{H,2}))) \\ \tilde{v}_H(\nu_{H,2}) = \min(v_h(\nu_{H,2}), \max(v_h(\nu_h), 2v_h(\nu_h) - v_h(\nu_{H,1}))) \end{cases}$ $\forall \varepsilon_H \in \mathcal{E}_H \cap \Gamma_C$

b) $\begin{cases} \tilde{v}_H(\nu_{H,1}) = \min(v_h(\nu_{H,1}), v_h(\nu_h)) \\ \tilde{v}_H(\nu_{H,2}) = \min(v_h(\nu_{H,2}), v_h(\nu_h)) \end{cases}$ $\forall \varepsilon_H \in \mathcal{E}_H \cap \Gamma_C$

c) $\begin{cases} \tilde{v}_H(\nu_{H,1}) = \min(v_h(\nu_{H,1}), v_h(\nu_h), v_h(\nu_{H,2})) \\ \tilde{v}_H(\nu_{H,2}) = \min(v_h(\nu_{H,1}), v_h(\nu_h), v_h(\nu_{H,2})) \end{cases}$ $\forall \varepsilon_H \in \mathcal{E}_H \cap \Gamma_C$

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Pressure Non-Linear Projection

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$$s_H(\phi_H) \leq s_h(\phi_h) \quad \forall \phi_h \in P_{\phi_H}^{\phi_h}$$

Thus:

$$s_H = I_{h,\sigma_n}^H s_h = \sum_{\phi_{H_i} \in T_H} \left[\lambda_{\Sigma_H, H_i} \right]_n s_H(\phi_{H_i}) \qquad \text{with} \qquad s_H(\phi_{H_i}) = \min_{\phi_h \in P_{\phi_{H_i}}^{\phi_h}} s_h(\phi_h)$$

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Truncated Basis

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Krause$^{\triangle}$

$$\left[\tilde{\boldsymbol{\lambda}}_{U_j,\nu}\right]_i = \begin{cases} \left[\boldsymbol{\lambda}_{U_j,\nu}\right]_i & \nu \in \mathcal{N}_j \setminus \mathcal{N}_j^{\bullet}, \; i = n, t \\ 0 & \nu \in \mathcal{N}_j^{\bullet}, \qquad i = n \\ \left[\boldsymbol{\lambda}_{U_j,\nu}\right]_i & \nu \in \mathcal{N}_j^{\bullet}, \qquad i = t \end{cases}$$

$$\left[\tilde{\boldsymbol{\lambda}}_{\Sigma_j,\phi}\right]_i = \begin{cases} \left[\boldsymbol{\lambda}_{\Sigma_j,\phi}\right]_i & \phi \in \mathcal{F}_j \setminus \mathcal{F}_j^{\bullet}, \; i = n, t \\ 0 & \phi \in \mathcal{F}_j^{\bullet} \qquad i = n \\ \left[\boldsymbol{\lambda}_{\Sigma_j,\phi}\right]_i & \phi \in \mathcal{F}_j^{\bullet}, \qquad i = t \end{cases}$$