# CK0048 - Métodos Numéricos II

Trabalho 6

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# 1 Quadraturas de Gauss-Hermite, Gauss-Laguerre e Gauss-Chebyshev, para n = 4

## 1.1 Gauss-Hermite

$$H_{\mathrm{n}}(x) = (-1)^n \, e^{x^2} \frac{d^n}{dx^n} e^{-x^2} \tag{1}$$

Iniciamos com o desenvolvimento do polinômio para $n = 4$:

$$\begin{aligned} H_4(x) &= (-1)^4 \, e^{x^2} \tfrac{d^4}{dx^4} e^{-x^2} \\ &= -2\left(-8e^{-x^2}x^4 + 24e^{-x^2}x^2 - 6e^{-x^2}\right)e^{x^2} \\ &= 16x^4 - 48x^2 + 12 \end{aligned} \tag{2}$$

Após o desenvolvimento, calculamos as raízes e os pesos respectivos.

$$H_4(x) = 0$$
$$16x^4 - 48x^2 + 12 = 0$$

$$x_1 = \sqrt{\frac{3+\sqrt{6}}{2}},\ x_2 = \sqrt{\frac{3-\sqrt{6}}{2}},$$
$$x_3 = -\sqrt{\frac{3-\sqrt{6}}{2}},\ x_4 = -\sqrt{\frac{3+\sqrt{6}}{2}} \tag{3}$$

Com os dados das raízes, podemos calcular os pesos $w_1$, $w_2$, $w_3$ e $w_4$ com a fórmula

$$w_k = \frac{2^{n-1} n! \sqrt{\pi}}{n^2 \left[H_{n-1}\left(x_i\right)\right]^2} \tag{4}$$

obtendo então:

$$\begin{aligned} w_1 = w_4 &= \frac{2^{4-1} \cdot 4! \cdot \sqrt{\pi}}{4^2 \cdot \left[H_3\left(\pm\sqrt{\frac{3+\sqrt{6}}{2}}\right)\right]^2} \\ &= \frac{2^{4-1} \cdot 4! \cdot \sqrt{\pi}}{4^2 \cdot \left[4\sqrt{3\left(3+\sqrt{6}\right)}\right]^2} \\ &= \frac{\sqrt{\pi}\left(3-\sqrt{6}\right)}{12} \\ w_2 = w_3 &= \frac{2^{4-1} \cdot 4! \cdot \sqrt{\pi}}{4^2 \cdot \left[H_3\left(\pm\sqrt{\frac{3-\sqrt{6}}{2}}\right)\right]^2} \\ &= \frac{\sqrt{\pi}\left(3+\sqrt{6}\right)}{12} \end{aligned} \tag{5}$$

Temos por fim os seguintes valores para $x_k$ e $w_k$:

$$x_1 = \sqrt{\frac{3+\sqrt{6}}{2}},\ x_2 = \sqrt{\frac{3-\sqrt{6}}{2}},$$

$$x_3 = -\sqrt{\frac{3-\sqrt{6}}{2}},\ x_4 = -\sqrt{\frac{3+\sqrt{6}}{2}}$$

$$w_1 = w_4 = \frac{\sqrt{\pi}\left(3-\sqrt{6}\right)}{12}$$

$$w_2 = w_3 = \frac{\sqrt{\pi}\left(3+\sqrt{6}\right)}{12}$$

## 1.2 Gauss-Laguerre

$$L_n(x) = \frac{e^x}{n!}\frac{d^n}{dx^n}\left(e^{-x}x^n\right) \tag{6}$$

Iniciamos com o desenvolvimento do polinômio para $n = 4$:

$$\begin{aligned} L_4(x) &= \tfrac{e^x}{n!}\tfrac{d^n}{dx^n}\left(e^{-x}x^n\right) \\ &= \tfrac{e^x}{4!}\cdot e^{-x}\left(x^4 - 16x^3 + 72x^2 - 96x + 24\right) \\ &= \tfrac{x^4}{24} - \tfrac{2x^3}{3} + 3x^2 - 4x + 1 \end{aligned} \tag{7}$$

Após o desenvolvimento, calculamos as raízes e os pesos respectivos.

$$\begin{aligned} L_4(x) &= 0 \\ \frac{x^4}{24} - \frac{2x^3}{3} + 3x^2 - 4x + 1 &= 0 \\ \\ x_1 &\approx 0.32254, \\ x_2 &\approx 1.74576, \\ x_3 &\approx 4.53662, \\ x_4 &\approx 9.39507 \end{aligned} \tag{8}$$

Com os dados das raízes, podemos calcular os pesos $w_1$, $w_2$, $w_3$ e $w_4$ com a fórmula

$$w_k = \frac{x_i}{(n+1)^2 \cdot \left[L_{n+1}(x_i)\right]^2} \tag{9}$$

Porém, para $n = 4$, precisamos de $L_{n+1}(x_i)$. Vamos calcular $L_5(x)$ a seguir:

$$\begin{aligned}
L_5(x) &= \tfrac{e^x}{n!}\tfrac{d^n}{dx^n}\left(e^{-x}x^n\right) \\
&= \tfrac{e^x}{5!}\cdot\left(e^{-x}\left(-x^5+25x^4-200x^3+600x^2-600x+120\right)\right) \\
&= \tfrac{-x^5+25x^4-200x^3+600x^2-600x+120}{120} \\
&= -\tfrac{x^5}{120}+\tfrac{5x^4}{24}-\tfrac{5x^3}{3}+5x^2-5x+1
\end{aligned} \tag{10}$$

Utilizando para os pesos o polinômio $L_5(x_i)$ calculado, obtemos então:

$$\begin{aligned}
w_i &= \tfrac{x_i}{(5)^2\cdot[L_5(x_i)]^2} \\
w_1 &\approx \tfrac{0.32254}{(5)^2\cdot[-0.14623]^2} \approx 0.60335 \\
w_2 &\approx \tfrac{1.74576}{(5)^2\cdot[0.44201]^2} \approx 0.35742 \\
w_3 &\approx \tfrac{4.53662}{(5)^2\cdot[-2.16017]^2} \approx 0.03888 \\
w_4 &\approx \tfrac{9.39507}{(5)^2\cdot[26.39773]^2} \approx 0.00053
\end{aligned} \tag{11}$$

Temos por fim os seguintes valores para $x_k$ e $w_k$:

$$x_1 \approx 0.32254, x_2 \approx 1.74576$$
$$x_3 \approx 4.53662, x_4 \approx 9.39507$$

$$w_1 \approx 0.60335, w_2 \approx 0.35742$$
$$w_3 \approx 0.03888, w_4 \approx 0.00053$$

### 1.3 Gauss-Chebyshev

$$T_{n(x) = \frac{(-2)^n \cdot n!}{(2n)!}} \cdot \sqrt{1-x^2} \cdot \frac{d^n}{dx^n}\left(1-x^2\right)^{n-\frac{1}{2}} \tag{12}$$

Iniciamos com o desenvolvimento do polinômio para $n = 4$:

$$\begin{aligned} T_4(x) &= \frac{(-2)^4 \cdot 4!}{(2\cdot 4)!} \cdot \sqrt{1-x^2} \cdot \frac{d^4}{dx^4}\left(1-x^2\right)^{4-\frac{1}{2}} \\ &= \frac{(-2)^4 \cdot\ 4!}{(2\cdot\ 4)!} \cdot \sqrt{1-x^2} \cdot \left(-\frac{7\left(-120x^4+120x^2-15\right)}{\sqrt{1-x^2}}\right) \\ &= 8x^4 - 8x^2 + 1 \end{aligned} \tag{13}$$

Após o desenvolvimento, calculamos as raízes e os pesos respectivos.

$$T_4(x) = 0$$
$$8x^4 - 8x^2 + 1 = 0$$

$$\begin{gathered} x_1 = \frac{\sqrt{2+\sqrt{2}}}{2}, x_2 = \frac{\sqrt{2-\sqrt{2}}}{2}, \\ x_3 = -\frac{\sqrt{2+\sqrt{2}}}{2}, x_4 = -\frac{\sqrt{2-\sqrt{2}}}{2}, \end{gathered} \tag{14}$$

No caso da quadratura de Gauss-Chebyshev, não precisamos das raízes para calcular os pesos (que são idênticos) utilizamos portanto a fórmula

$$w_k = \frac{\pi}{n} \tag{15}$$

obtendo então

$$w_1 = w_2 = w_3 = w_4 = \frac{\pi}{4} \tag{16}$$

Temos por fim os seguintes valores para $x_k$ e $w_k$:

$$x = \pm\frac{\sqrt{2\pm\sqrt{2}}}{2}$$
$$w_{1..4} = \frac{\pi}{4}$$

## 1.4 Quadro-Resumo

| n | $H_n(x)$ | | $L_n(x)$ | | $T_n(x)$ | |
|---|---|---|---|---|---|---|
| | $x_k$ | $w_k = \frac{2^{n-1} n! \sqrt{\pi}}{n^2 [H_{n-1}(x_i)]^2}$ | $x_k$ | $w_k = \frac{x_i}{(n+1)^2 \cdot [L_{n+1}(x_i)]^2}$ | $x_k$ | $w_k = \frac{\pi}{n}$ |
| 2 | $\begin{cases} x_1 = -\frac{1}{\sqrt{2}} \\ x_2 = \frac{1}{\sqrt{2}} \end{cases}$ | $w_1 = w_2 = \frac{\sqrt{\pi}}{2}$ | $\begin{cases} x_1 = 2 - \sqrt{2} \\ x_2 = 2 + \sqrt{2} \end{cases}$ | $\begin{cases} x_1 = \frac{1}{4} \cdot (2 + \sqrt{2}) \\ x_2 = \frac{1}{4} \cdot (2 - \sqrt{2}) \end{cases}$ | $\begin{cases} x_1 = -\frac{1}{\sqrt{2}} \\ x_2 = \frac{1}{\sqrt{2}} \end{cases}$ | $w_{1..2} = \frac{\pi}{2}$ |
| 3 | $\begin{cases} x_1 = -\sqrt{\frac{3}{2}} \\ x_2 = 0 \\ x_3 = \sqrt{\frac{3}{2}} \end{cases}$ | $w_1 = w_3 = \frac{\sqrt{\pi}}{6}$ <br> $w_2 = \frac{2 \cdot \sqrt{\pi}}{3}$ | $\begin{cases} x_1 = 0.4157745568 \\ x_2 = 2.2942893603 \\ x_3 = 6.2899459829 \end{cases}$ | $\begin{cases} w_1 = 0.7110930099 \\ w_2 = 0.2785177336 \\ w_3 = 0.0103892565 \end{cases}$ | $\begin{cases} x_1 = -\frac{\sqrt{3}}{2} \\ x_2 = 0 \\ x_3 = \frac{\sqrt{3}}{2} \end{cases}$ | $w_{1..3} = \frac{\pi}{3}$ |
| 4 | $\begin{cases} x_1 = \sqrt{\frac{3+\sqrt{6}}{2}} \\ x_2 = \sqrt{\frac{3-\sqrt{6}}{2}} \\ x_3 = -\sqrt{\frac{3-\sqrt{6}}{2}} \\ x_4 = -\sqrt{\frac{3+\sqrt{6}}{2}} \end{cases}$ | $w_1 = w_4 = \frac{\sqrt{\pi}(3-\sqrt{6})}{12}$ <br> $w_2 = w_3 = \frac{\sqrt{\pi}(3+\sqrt{6})}{12}$ | $\begin{cases} x_1 = 0.32254 \\ x_2 = 1.74576 \\ x_3 = 4.53662 \\ x_4 = 9.39507 \end{cases}$ | $\begin{cases} w_1 = 0.60335 \\ w_2 = 0.35742 \\ w_3 = 0.03888 \\ w_4 = 0.00053 \end{cases}$ | $\begin{cases} x_1 = \frac{\sqrt{2+\sqrt{2}}}{2} \\ x_2 = \frac{\sqrt{2-\sqrt{2}}}{2} \\ x_3 = -\frac{\sqrt{2+\sqrt{2}}}{2} \\ x_4 = -\frac{\sqrt{2-\sqrt{2}}}{2} \end{cases}$ | $w_{1..4} = \frac{\pi}{4}$ |