Solving Default Models*

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Abstract

This article surveys issues and methods for the computation of sovereign default and related models. It lays out algorithms for the solution and simulation of such models, with long-term debt, using discrete choice methods. It concludes by briefly considering the impact of extensions common in quantitative work, such as physical capital or self-fulfilling crises, on the solution method.

Keywords: sovereign default, numerical solution, simulation, discrete choice methods
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1 Introduction

This article lays out solution and simulation algorithms for sovereign default models with long-term debt, in the tradition of Eaton and Gersovitz (1981), Arellano (2008), Hatchondo and Martinez (2009), and Chatterjee and Eyigungor (2012). Some of these modeling and numerical techniques are also relevant for the study of consumer unsecured credit (Chatterjee et al. 2023) or corporate bankruptcy (Hennessy and Whited 2005; Corbae and D'Erasmo 2021).

These incomplete markets models feature default on the equilibrium path and bond yields that endogenously compensate lenders for the risk that the country might default on its obligations. Such models are challenging to compute for at least two reasons. First, the presence of the discrete default choice induces substantial nonlinearities in policies and bond prices in areas of the state space with a non-trivial risk of default. Second, with long-term debt and a lack of commitment over future bond issuance behavior, debt dilution is a major driver of bond yields. The importance of debt dilution for matching key features of yields data and the disconnect between short-term default rates and yield spreads is well understood (Hatchondo, Martinez, and Sosa-Padilla 2016; Hatchondo, Martinez, and Roch 2020) but so is the role played by debt dilution in frustrating convergence of standard algorithms (Chatterjee and Eyigungor 2012). Informally, debt dilution is the incentive of lenders to offer the sovereign depressed bond prices, for a given level of borrowing today, because they correctly forecast that in the future the sovereign will borrow more, which in turn makes the likelihood of eventual default higher. Even if the sovereign is unlikely to default next period, because debt is long term, lenders are concerned about all future borrowing and default opportunities, and offer bond prices consistent with discretionary equilibrium behavior.

With short-term debt, the model has an unique equilibrium which can be shown to be the fixed point of a contraction (Auclert and Rognlie 2016; Aguiar and Amador 2019) and many standard numerical methods can deliver convergence to a relatively precise solution. On the other hand, with long-term debt, multiplicity of equilibria is pervasive (Aguiar and Amador 2020; Stangebye 2023) and the source of numerical instability for common methods (Chatterjee and Eyigungor 2012). How to identify and select among equilibria is an ongoing area of research.

The literature has pursued several solution methods for long-term debt models: papers like Hatchondo and Martinez (2009) and Hatchondo, Martinez, and Sosa-Padilla (2016) employ cubic spline approximation, numerical constrained maximization routines for the borrowing choice, and take the limit of the finite horizon economy. Chatterjee and Eyigungor (2012) restrict attention to discrete state spaces and perturb borrowing with an iid, continuously distributed extra endowment shock. Also based on the idea of small perturbations to the borrowing policy, which weakens the tight destabilizing interdependence between borrowing policies and bond prices, Dvorkin et al. (2021) introduce taste shocks, as commonly employed in the structural estimation of discrete choice models. This latter approach will be the one described in detail in this article. Finally, more recently, Bai et al. (2023) employ adaptive sparse grids together with adjustment costs, while Gu and Stangebye (2023) develop an approach based on Gaussian Process Dynamic Programming.

One advantage of discrete choice methods over other approaches is that they eliminate the need for the use of computationally costly constrained maximization routines, as with the use of interpolation, or the algorithmic partition of a continuous shock support, as done by Chatterjee and Eyigungor (2012). The resulting simplicity of the method enables its use in models with richer features, including news shock (Dvorkin et al. 2020), monetary policy and pricing frictions (Arellano, Bai, and Mihalache 2020), currency unions (Wolf and Zessner-Spitzenberg 2021), maturity choice in restructuring (Mihalache 2020), or partial default and debt relief policies (Arellano, Bai, and Mihalache 2023), to name but a few. The closed-form expression characterizing the choice probabilities and expected value mean that the algorithms presented here are particularly well suited for computation on the GPU and relatively easy to parallelize over multiple computer nodes, in a cluster setting.

A potential concern raised by the discrete choice methods employed here is whether the addition of such Extreme Value taste shocks alters the incentives faced by the sovereign. Briglia et al. (2022) argue that they do, in the form of additional incentives for precautionary savings, in a model with incomplete markets but without default. They suggest ways to quantify and correct for the additional mechanism induced by agents being aware that their future choices will be perturbed by taste shocks. Briglia et al. (2022) also discuss the issue of grid sensitivity and argue that the appropriate correction needs to be based on the measure of the feasible budget set rather than the number of grid points.

To fix ideas, we start in Section 2 by laying out a canonical sovereign default model with long-term debt. In Section 3, we describe the alterations required by the use of discrete choice methods, namely augmenting the model with Extreme Value taste shocks, and detail the solution and simulation algorithms in turn. Finally, Section 4 briefly comments on how our algorithms are impacted by several salient extensions common in applied, quantitative work.

2 The Canonical Sovereign Default Model

Consider the following canonical sovereign default model with long-term debt, a real, endowment model, as in Hatchondo and Martinez (2009) and Chatterjee and Eyigungor (2012). We focus on the case of long-term debt because of the special role and quantitative importance of debt dilution for the level and volatility of spreads and default risk, as discussed by Chatterjee and Eyigungor (2015) and Hatchondo, Martinez, and Sosa-Padilla (2016).

This is an infinite horizon, discrete time setting. The model consists of the sovereign of a small open economy and a unit measure of competitive, risk-neutral international lenders. No distinction is drawn between the sovereign and its domestic private sector, a "centralized borrowing, centralized default" assumption. We discuss the agents' problems in turn and define the equilibrium of this economy. In Section 4 we briefly consider extensions which relax some of the assumptions made, including recovery and lender risk aversion.

2.1 The Sovereign's Problem

In any one period, the sovereign can either have good credit standing, in which case it has access to international financial markets, or be in a state of default. While in good credit standing, the sovereign has the option to declare default, stop debt service payments, and transition to the default regime. With a constant probability, default is resolved and the sovereign reenters international markets without outstanding debt and good credit standing.[1]

The sovereign receives an endowment y governed by a discrete Markov chain with support $\mathbb{Y}$ and transition matrix Π_y. While in good credit standing, the sovereign starts each period with endowment realization y and outstanding debt level B. It then chooses whether to default on its debt or not, by comparing the value it achieves by servicing the debt and retaining market access, $\tilde{V}^r(y, B)$, and the value it can achieve by entering default, $\tilde{V}^d(y)$. The value of default is independent of the debt level upon default, B, because of our assumption of no recovery, full debt repudiation. The policy function $\mathcal{D}(y, B) \in \{0, 1\}$ encodes the sovereign's default decision, and the value function of starting the period with states $\{y, B\}$ is given by $\tilde{V}(y, B)$, with

$$\tilde{V}(y, B) = \max_{d \in \{0,1\}} \left\{ (1-d)\tilde{V}^r(y, B) + d\tilde{V}^d(y) \right\} \tag{1}$$

and the corresponding arg max encoded in $\mathcal{D}(y, B)$.

If the sovereign chooses to not default, it makes a debt service payment proportional to the stock of outstanding debt, κB, and has the option to issues or buy back units of its long-term bond in international markets.[2] Lenders face the sovereign with a bond price schedule $\tilde{q}(y, B')$ which depends on the level of debt the sovereign will carry into next period B' and the current endowment y, as they are predictive of future default and borrowing behavior. The sovereign's problem with market access is given by

$$\begin{aligned} \tilde{V}^r(y, B) = \max_{B'} \; & u(c) + \beta \mathbb{E}_{y'|y} \tilde{V}(y', B') \\ \text{s.t. } & c = y - \kappa B + \left[B' - (1-\delta)B\right] \tilde{q}(y, B') \end{aligned} \tag{2}$$

and we denote the arg max by $\mathcal{B}(y, B)$.

The parameter δ controls the duration of the debt. A share δ of the outstanding debt matures while the rest, $(1-\delta)B$, remains outstanding. As a result, if the sovereign is to have outstanding debt level B' next period, it can only issue $B' - (1-\delta)B$ additional units this period. These are the units that it can sell to lenders at price $\tilde{q}(y, B')$.

If the sovereign is excluded from international markets, in a state of default, it makes no choices and consumes its endowment subject to a penalty, $c = h(y) \leq y$. The sovereign reenters markets next period, with a good credit standing, with constant probability χ. The value associated with

1. Section 4.1 addresses the case of endogenous, positive recovery rates and references works with an endogenous length of market exclusion.

2. Note that markets are incomplete because the only financial asset traded is a defaultable, long-term bond, even though the endowment process y induces potentially many states of the world next period.

the default regime is

$$\tilde{V}^d(y) = u(h(y)) + \beta \mathbb{E}_{y'|y}\left[\chi \tilde{V}(y',0) + (1-\chi)\tilde{V}^d(y')\right]. \tag{3}$$

2.2 International Lenders

International lenders are competitive and risk-neutral. Their opportunity cost of funds is given by the international risk-free rate r. A representative lender from the unit mass of lenders faces the following problem over the choice of bonds it purchases a,

$$\max_a \left\{ -\tilde{q}(y,B')a + \frac{1}{1+r}\mathbb{E}_{y'|y}(1-\mathcal{D}(y',B'))\left[\kappa + (1-\delta)\tilde{q}(y',\mathcal{B}(y',B'))\right] a \right\}, \tag{4}$$

which is linear in a. For an interior optimum, the bond price must satisfy

$$\tilde{q}(y,B') = \frac{1}{1+r}\mathbb{E}_{y'|y}(1-\mathcal{D}(y',B'))\left[\kappa + (1-\delta)\tilde{q}(y',\mathcal{B}(y',B'))\right]. \tag{5}$$

If the sovereign defaults next period, the lenders receive no payment and the bonds are worthless. If instead the sovereign does not default, each unit of debt makes service payment κ and a share $1-\delta$ will remain outstanding, with next period market value of $\tilde{q}(y',\mathcal{B}(y',B'))$ per unit. In expectations, lenders break even.

It is straightforward to show that the risk-free bond price, that is if the sovereign does not default in any state, is $q^{\mathrm{rf}} = \frac{\kappa}{\delta + r}$, and the yield-to-maturity of the bond is given by $\frac{\kappa}{\tilde{q}(y,B')} - \delta$. The spread is, then, the difference between the bond's yield-to-maturity and the risk-free rate r.[3]

Sections 4.1 and 4.2 concern the cases of positive recovery and lender risk aversion, respectively.

2.3 Equilibrium

We restrict attention to Markov Perfect equilibria, where only the payoff-relevant variables y and B are states. Our notation throughout reflects this restriction. We can now define the model's equilibrium.

A *Recursive Markov Perfect Equilibrium* consists of

1. the sovereign's value functions $\tilde{V}(y,B)$, $\tilde{V}^r(y,B)$, and $\tilde{V}^d(y)$,
2. the sovereign's default and issuance policies $\mathcal{D}(y,B)$ and $\mathcal{B}(y,B)$, and
3. the bond price schedule $\tilde{q}(y,B')$,

such that

a. given the sovereign's policies, the bond price schedule satisfies equation (5), so that lenders break even in expectation,

3. It is often convenient to normalize $\kappa = \delta + r$ so that the risk-free bond price is $q^{\mathrm{rf}} = 1$. This has the useful side-effect of making the units of B be those of the endowment.

b. the sovereign's policies solve to the maximization problems in equations (1) and (2), and

c. the value functions satisfy equations (1), (2), and (3).

The restriction to Markov Perfect equilibria is not innocuous, although a standard way to introduce equilibrium default. It rules out mechanisms based on reputation—although see the related work by Amador and Phelan (2021, 2023)—and commitment over future default or borrowing policies. Recent work by Hatchondo, Martinez, and Roch (2020) and Mateos-Planas et al. (2023) revisits the implications of commitment in default models.

3 Computation with Taste Shocks

We follow Dvorkin et al. (2021) and Gordon (2019) in augmenting the model with taste shock and computing it with discrete choice methods.[4] This approach greatly alleviates the convergence problems discussed by Chatterjee and Eyigungor (2012). Chatterjee and Eyigungor (2012) restrict attention to B values in a discrete set $\mathbb{B}$ and introduce a second continuous, iid endowment shock, which acts akin to a randomization device across B' options which deliver comparable values to the sovereign. Our approach here also requires that B be in a discrete set but it does not rely on changes to the endowment process. Instead, we induce choice probabilities over B' via Extreme Value Type I shocks associated with each B' option, with the computational advantage that such a functional form assumption delivers closed-form expressions for choice probabilities and expected values, a multinominal logit structure (Train 2009). These methods are widespread in structural applied work and have proven critical for the study of consumer credit by Chatterjee et al. (2023).[5]

3.1 Augmenting the Model with Taste Shocks

We add choice-specific taste shocks ε_d and ε_r to the two options in equation (1), to obtain

$$V(y,B) = \mathbb{E}_{\varepsilon_d,\varepsilon_r} \max_{d\in\{0,1\}} \left\{ (1-d)\left[V^r(y,B) + \eta\varepsilon_r\right] + d\left[V^d(y) + \eta\varepsilon_d\right] \right\} - \eta\gamma, \tag{6}$$

where η is a scaling parameter which governs the relative importance of taste shocks in shaping choices[6] and $\gamma = 0.57721566\ldots$ is the Euler-Mascheroni constant, which is subtracted in order to correct the effect on the value from the non-zero mean of the $\varepsilon_\cdot$ shocks. Note that now $V(y,B)$ is the *ex-ante* value to the sovereign, before observing the $\varepsilon_\cdot$ shocks, which is why we do not need to treat the shocks as extra state variables. We drop the tilde for the bond price schedule

4. For completeness, we will introduce taste shocks to both the default $\mathcal{D}$ and borrowing decisions $\mathcal{B}$, although key to the method's success are the perturbations to the B' choice.

5. For Chatterjee et al. (2023), taste shocks are helpful in preventing lenders from perfectly inferring heterogeneous borrowers' private type by observing their choices, as taste shocks imply that all feasible action are taken with positive probability.

6. As we take $\eta \to 0$, taste shocks are eliminated and the sovereign picks the best option based on the comparison of V^r and V^d alone. If, instead, we take $\eta \to \infty$, all feasible choices are picked with equal probability, in particular here independently of the values of V^r and V^d.

and the value functions incorporating taste shocks, to highlight that they can be understood as approximations to the corresponding objects of the model in Section 2.

Given our distributional assumption, standard arguments imply that the *ex-ante* probability of choosing default is given by

$$\Pr(d=1|y,B) = \frac{\exp\frac{V^d(y)}{\eta}}{\exp\frac{V^d(y)}{\eta} + \exp\frac{V^r(y,B)}{\eta}} \tag{7}$$

and the corresponding expected value is

$$V(y,B) = \eta \log\left[\exp\frac{V^d(y)}{\eta} + \exp\frac{V^r(y,B)}{\eta}\right]. \tag{8}$$

We proceed in a similar manner in introducing taste shocks in the borrowing problem (2).[7] First, to ease exposition, let us introduce an auxiliary function

$$W(y,B,B') = u(y - \kappa B + [B' - (1-\delta)B]q(y,B')) + \beta \mathbb{E}_{y'|y} V(y',B'), \tag{9}$$

the net-of-taste-shocks value of choosing B' in state $\{y,B\}$, and set $W(y,B,B') = -\infty$ whenever the associate consumption level is not strictly positive. Then, (2) becomes

$$V^r(y,B) = \mathbb{E}_{\{\varepsilon_{B'}\}} \max_{B'} \left\{W(y,B,B') + \rho\varepsilon_{B'}\right\} - \rho\gamma \tag{10}$$

and the resulting choice probabilities and ex-ante value are

$$\Pr(B'=i|y,B) = \frac{\exp\frac{W(y,B,i)}{\rho}}{\sum_j \exp\frac{W(y,B,j)}{\rho}} \tag{11}$$

and

$$V^r(y,B) = \rho\log\left[\sum_j \exp\frac{W(y,B,j)}{\rho}\right], \tag{12}$$

respectively. The parameter ρ controls the magnitude of taste shocks for borrowing. It does not need to be equal to η, which plays an analogous role for the default decision.

Finally, the bond price schedule consistent with these choice probabilities is given by

$$q(y,B') = \frac{1}{1+r}\mathbb{E}_{y'|y}(1-\Pr(d'=1|y',B'))\left[\kappa + (1-\delta)\sum_j \Pr(B''=j|y',B')q(y',j)\right]. \tag{13}$$

We are now ready to turn to this augmented model's solution algorithm and numerical

7. We will implicitly restrict attention to the quantitatively-relevant case of debt, $B' \geq 0$. If the sovereign could hold assets, start the period with $B < 0$, we would have to confront the possibility of "default" on an asset position, which would happen with positive probability due to the taste shocks.

implementation.

3.2 Solution Algorithm

The logarithmic and exponential functions in equations (11) and (12) are prone to numerical issues related to underflow or overflow, depending on the sign and magnitude of $W(y, B, B')$. The standard approach to sidestep these problems is to rewrite these expressions as

$$\Pr(B' = i|y, B) = \frac{\exp \frac{W(y,B,i) - \overline{W}(y,B)}{\rho}}{\sum_j \exp \frac{W(y,B,j) - \overline{W}(y,B)}{\rho}} \tag{14}$$

and

$$V^r(y, B) = \overline{W}(y, B) + \rho \log \left[\sum_j \exp \frac{W(y, B, j) - \overline{W}(y, B)}{\rho} \right], \tag{15}$$

where $\overline{W}(y, B) = \max_{B'} W(y, B, B')$, the maximum over the choices, without considering taste shocks. This formulation is conceptually equivalent but numerically much better behaved, as all the $W - \overline{W}$ terms are weakly negative and therefore the exponents are bounded above by 1.

Analogously, for the default decision, defining $\overline{V}(y, B) = \max\{V^d(y),\ V^r(y, B)\}$, we use

$$\Pr(d = 1|y, B) = \frac{\exp \frac{V^d(y) - \overline{V}(y,B)}{\eta}}{\exp \frac{V^d(y) - \overline{V}(y,B)}{\eta} + \exp \frac{V^r(y,B) - \overline{V}(y,B)}{\eta}} \tag{16}$$

and

$$V(y, B) = \overline{V}(y, B) + \eta \log \left[\exp \frac{V^d(y) - \overline{V}(y, B)}{\eta} + \exp \frac{V^r(y, B) - \overline{V}(y, B)}{\eta} \right]. \tag{17}$$

We are now ready to lay out the main algorithm. We will use an "one loop"[8] algorithm, where, in each iteration, we use the bond price schedule q_0 and value functions V_0 and V_0^d from the previous iteration to construct this iteration's corresponding objects.

Step 1. **Initialization.** One-time initial setup:

a) Fix parameter values and functional forms for $u(c)$ and $h(y)$.

b) Construct grids for the y and B states, $\mathbb{Y}$ and $\mathbb{B}$, and the transition matrix Π_y.

c) Set initial guesses: $q_0(y, B') = q^{\text{rf}}$, $V_0(y, B) = \frac{u(y - \kappa B)}{1 - \beta}$, and $V_0^d(y) = \frac{u(h(y))}{1 - \beta}$.

Step 2. **Update.** Construct new values, policies, and prices:

a) Use V_0 and V_0^d on the RHS of equation (3) to construct V_1^d.

b) Use q_0 and V_0 to construct W, using (9).

8. Arellano (2008) describes a "two loops" algorithm: given a bond price schedule, an inner loop iterates on value functions to convergence, while an outer loop iterates to convergence on the bond price schedule. Such an approach is generally slower and not necessarily more likely to converge than our "one loop" algorithm.

c) Construct V_1^r and choice probabilities $\Pr(B' = i|y, B)$ using equations (14) and (15).

d) Use V_1^r and V_1^d to construct V_1 and $\Pr(d = 1|y, B)$ using (16) and (17).

e) Use borrowing $\Pr(B' = i|y, B)$ and default $\Pr(d = 1|y, B)$ choice probabilities, together with q_0, to construct q_1 using (13).

Step 3. **Check for Convergence.** Compare $||V_1 - V_0||$, $||V_1^d - V_0^d||$, and $||q_1 - q_0||$ against convergence thresholds. If all three norms are smaller than their respective thresholds, stop, and use the objects constructed in the latest iteration as the equilibrium policies, values, and prices. Otherwise, update the guesses, $q_0 \leftarrow q_1$, $V_0^d \leftarrow V_1^d$, $V_0 \leftarrow V_1$ and return to Step 2.

3.3 Simulating the Model

With the model's solution in hand, we can use equilibrium bond prices, together with borrowing and default policies, to simulate time paths for our model. The simulation algorithm is as follows:

Step 1. **Initialization.**

a) Fix the length of the simulation, T periods.

b) Simulate the entire time path for the exogenous shock $\{y_t\}_{t=2}^T$ using its transition matrix and an initial value y_1 (e.g., the unconditional mean).

c) Set the initial debt levels $B_0 = B_1 = 0$ and assume that the country is in good credit standing in period 1, with $d_1 = 0$.

Step 2. **Simulate.** For each period $t \in \{2, 3, \dots, T\}$:

a) If the country was in default in period $t-1$, with $d_{t-1} = 1$, then return them to market with probability χ. If the draw is such that the country returns, set $d_t = 0$ and $B_t = 0$. If the draw is unsuccessful, set $d_t = 1$ and $B_t = B_{t-1}$, as the country must remain excluded.

b) If $d_t = 0$, the country may choose to default this period with probability $\Pr(d_t = 1|y_t, B_t)$. Draw from the uniform distribution to simulate this event. If the country does default, set $d_t = 1$, otherwise keep $d_t = 0$.

c) If $d_t = 0$, simulate B_{t+1} from the pmf induced by the choice probabilities $\Pr(B_{t+1}|y_t, B_t)$ and using it to compute the yield-to-maturity spread sp_t implied by $q(y_t, B_{t+1})$. Use the expression for consumption in (2) to construct c_t and the trade balance to GDP ratio, $tby_t = 1 - c_t/y_t$. If instead we have $d_t = 1$, set $B_{t+1} = B_t$, $sp_t = \infty$, $c_t = h(y_t)$, and $tby_t = 0$.

Step 3. **Sample Selection.** Mark as invalid the first $K \geq 2$ periods, to avoid any dependency on the simulation's initial conditions, and any period in default, with $d_t = 1$. Since our model lacks recovery, the trade balance is abnormally negative in the first few periods following the return to market, so we might want to also mark an invalid $N \geq 0$ periods after the return to market, in which case we mark period t as invalid whenever $\sum_{\tau=0}^{N} d_{t-\tau} \neq 0$. We compute statistics over valid periods only, with care taken to only include valid periods following valid periods in the computation of autocorrelations.

3.4 A Sample Parameterization

A full calibration of the model to a set of target data moments is beyond the scope of this article. Instead, we briefly describe a sample quarterly parameterization, based on common functional forms and parameter values which deliver quantitative features roughly in line with the literature, to illustrate a typical result.

We assume that the utility function exhibits constant relative risk aversion, with a risk aversion coefficient $\sigma = 2$ and set $u(c) = (1-\beta)\frac{c^{1-\sigma}-1}{1-\sigma} = (1-\beta)(1-c^{-1})$. The penalty function $h(y)$ is shaped by two parameters, as in Chatterjee and Eyigungor (2012), $h(y) = y - \max\{0, \lambda_0 y + \lambda_1 y^2\}$, with $\lambda_0 \leq 0 \leq \lambda_1$. The endowment process y is governed by an AR(1) process with unconditional mean 1, autocorrelation ρ_y and standard deviation of innovations σ_y, which we discretize into a Markov chain using standard quadrature methods.

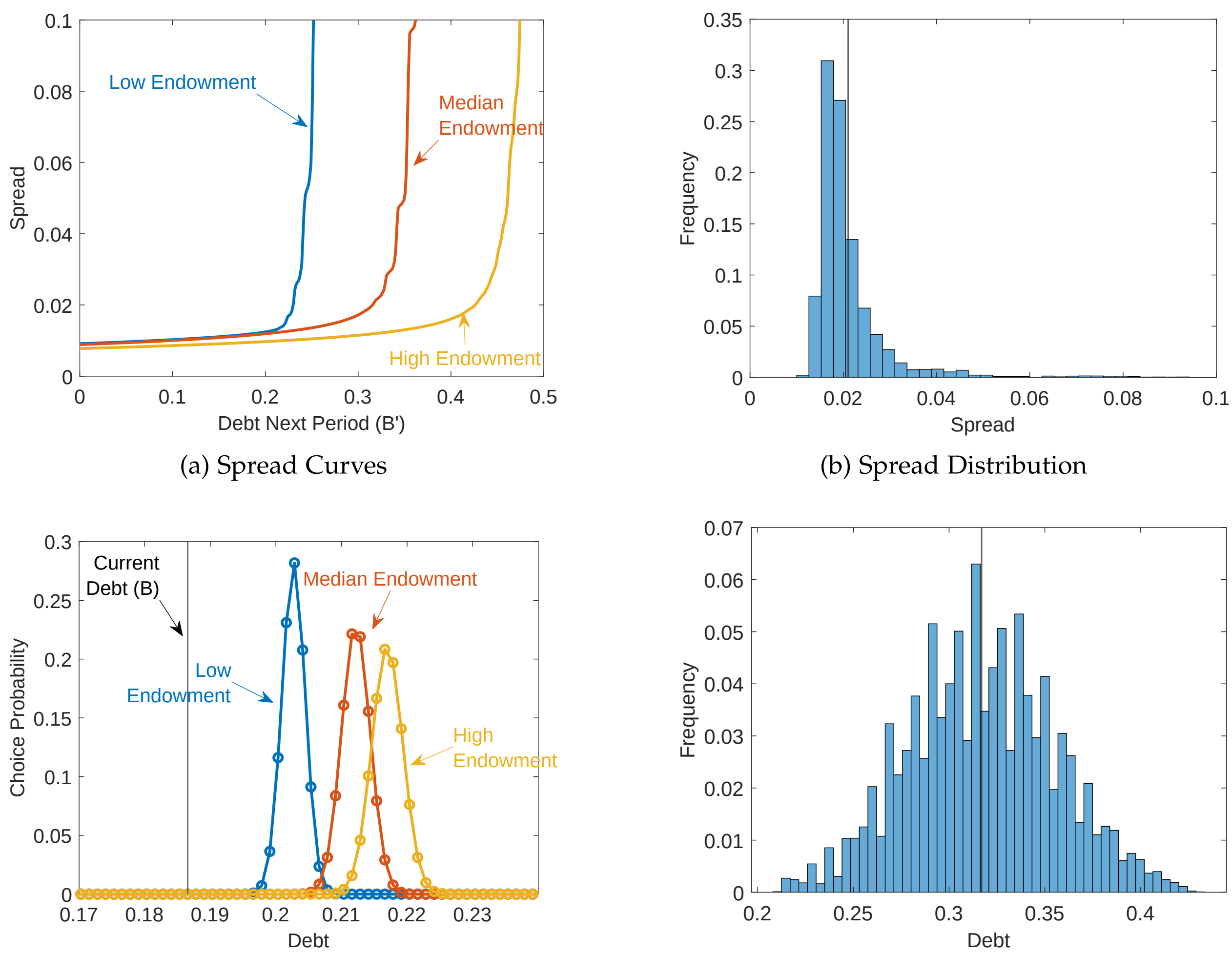
Table 1 collects the parameter values while Table 2 and Figure 1 display key moments and select policies, respectively. The solution uses $\#\mathbb{Y} = 31$ grid points for y and $\#\mathbb{B} = 600$ grid points for B.

	Value	Description
Preferences and endowment		
σ	2.0	CRRA
β	0.9775	Sovereign discounting
ρ_y	0.95	Endowment autocorrelation
σ_y	0.005	Endowment innovation
International lending		
r	0.01	Risk-free rate
δ	0.04	5 years debt duration
κ	$r+\delta$	Normalization, $q^{\text{rf}} = 1$
Default		
λ_0	-0.48	Penalty linear term
λ_1	0.525	Penalty quadratic term
χ	0.125	2 years average market exclusion
Taste shocks		
η	$5e^{-4}$	Default choice
ρ	$1e^{-5}$	Borrowing choice

Table 1: Parameter Values

Moment	Value
Mean	
Debt to GDP	7.9
Spread	2.1
Standard Deviation	
Spread	0.9
GDP	1.5
Consumption	1.7
Correlation with GDP	
Spread	−44.7
Trade Balance to GDP	−29.4

Table 2: Moments (%)



(a) Spread Curves

(b) Spread Distribution

(c) Choice Probabilities over B'

(d) Debt Distribution

Figure 1: Sample Plots

4 Extensions

In Section 2, we restricted attention to a canonical, baseline environment, to highlight the default and borrowing choices, and abstracted from many arguably important features of the data. We now consider briefly several extensions which mitigate some of these limitations of the textbook model and address whether and how our algorithms must be adapted.

4.1 Haircuts and Recovery

In the data, debt is seldom repudiate wholesale. Instead, as the evidence collected by Cruces and Trebesch (2013) shows, lenders enjoy eventually at least a partial recovery. Sovereigns swap the old, defaulted instruments for new debt, often involving an extension of the maturity structure (Mihalache 2020), and resume debt service payments, under the new terms. Two major branches of the literature concern recovery. First, following Yue (2010), several papers employ generalized Nash bargaining to determine the level of recovery following default, but maintain the assumption of a market exclusion spell of exogenous, random duration. The second set of papers focuses, instead, on the endogenous determination of the length of market exclusion, via non-cooperative, alternating offers bargaining, so that both the eventual haircut and the length of time negotiations take are endogenous. Pitchford and Wright (2012) and Dvorkin et al. (2021) are salient examples.

Finally, it is common in applied work for authors to assume a constant recovery rate, for example Hatchondo, Martinez, and Sosa-Padilla (2016). Such as assumption makes it more likely that the model exhibit what Chatterjee and Eyigungor (2015) call "maximum dilution before default," for the sovereign to borrow an arbitrarily large amount in the current period, $B' \to \infty$, default next period with certainty, but receive today from its international lenders the net present value of the eventual recovery. A common solution, to rule out this behavior in equilibrium, is to invoke underwriting standards and impose that bond prices cannot fall below a certain exogenous threshold, e.g. $q(y, B') \geq q_{\min}$. Equivalently, an upper bound on spreads can be imposed.

The presence of recovery does not alter substantively the numerical strategy of Section 3.2, especially in the case of the extension with Nash bargaining over recovery rates (Yue 2010).

4.2 Lender Risk Aversion

It is relatively straightforward to depart from the stark assumption of risk-neutral, deep-pocketed, competitive international lenders, either because one might want to induce a risk premium component to the spread or in order to study the consequences for emerging markets of shocks to the lenders' wealth or home market conditions.

Lizarazo (2013) and Aguiar et al. (2016) consider risk-averse lenders, who allocate an endogenous share of their wealth to buying the sovereign's risky debt, with the rest saved using risk-free investment options in the financial center. Related, Morelli, Ottonello, and Perez (2022) consider collateral-constrained global financial intermediaries, to study the cross-country comovements caused by the tightness of their financial constraint. Hatchondo, Martinez, and Sosa-Padilla (2016) employ lenders with Epstein-Zin preferences and a random trend endowment, while Bianchi, Hatchondo, and Martinez (2018) rely on a reduced-form, exponential-affine pricing kernel, a one-factor SDF as in Vasicek (1977).

Independent of the structure driving the lenders' discount factor, equation (5) becomes

$$q(y, s, B') = \mathbb{E}_{y',s'|y,s} \left\{ m(s, s') \left(1 - \mathcal{D}(y', s', B')\right) \left[\kappa + (1 - \delta) q(y', \mathcal{B}(y', s', B'))\right] \right\}, \tag{18}$$

where s is a shock driving the lenders' pricing kernel m, be it the lenders' wealth level, endowment growth rate, or a reduced-form factor. All expectations must now be taken over not only y' but also s'. Moreover, all value functions require s as an additional state variable. Our algorithm remains otherwise unchanged.

4.3 Self-fulfilling Crises

So far we have followed the Eaton and Gersovitz (1981) timing assumption. First, the sovereign decides whether to default or not and then, conditional on not defaulting, it can auction off more bond units. Implicit in this timing is the assumption that the sovereign cannot decide to default after dealing with its lenders, depending on the outcome of the bond auction. In contrast, an extensive literature on self-fulfilling crises studies defaults caused by coordination failures in sovereign debt markets, based on the timing assumption of Cole and Kehoe (2000): the sovereign attempts to auction off additional bond units and then, after observing the outcome of the auction, can decide to default. This opens up the possibility for the following notion of self-fulfilling crisis. Each lender might worry that not enough lenders will participate in the auction, in which case the sovereign cannot issue enough bond units to roll over its debt and it will choose to default once the auction concludes on unfavorable terms. In some states of the world, two equilibria are possible: 1) all lenders participate in the auction, the sovereign is able to roll over its debt, and it chooses not to default, confirming the lenders' belief that the auction will be successful, the "good equilibrium", and 2) no lenders participate in the auction, the sovereign cannot roll over its debt and it chooses to default, confirming the lenders' belief that joining the auction is undesirable, the "bad equilibrium." It is standard to assume that lenders coordinate on one of the two equilibria based on the realization of a public sunspot variable.

To fix ideas, consider the following structure, loosely based on the recursive formulation in the Online Appendix of Bocola and Dovis (2019), who study the interactions between maturity choice and the risk of self-fulfilling default during the European Debt Crisis of the early 2010s. We introduce a sunspot variable $\xi \in \{0, 1\}$ and assume it is iid with $\Pr(\xi = 1) = p$. When $\xi = 0$, self-fulfilling crises are not possible and lenders always coordinate on the "good equilibrium." Instead, when $\xi = 1$, if a failed auction would trigger a default, lenders will indeed lock out the sovereign out of the primary market and indeed force a default.

If the sovereign were to be locked out of markets but continue to service the outstanding debt, it would achieve a value given by

$$V^{\ell}(y, B) = u(y - \kappa B) + \beta \mathbb{E}_{y',\xi'|y} V(y', \xi', (1-\delta)B). \tag{19}$$

Then, abstracting from taste shocks on the default decision, the default policy is given by

$$\mathcal{D}(y, \xi, B) = \begin{cases} 1, & \text{if } V^r(y, B) < V^d(y) \\ 1, & \text{if } V^{\ell}(y, B) < V^d(y) \text{ and } \xi = 1 \\ 0, & \text{otherwise.} \end{cases} \tag{20}$$

The first branch of equation (20) reflects fundamental default, where the sovereign is better off defaulting even if it could enjoy normal market access. The second is the self-fulfilling default case, the sunspot variable in unfavorable, $\xi = 1$, and default is preferable to servicing the debt without rolling over.

Using the previous characterization of default behavior, we can write the start-of-period value function as

$$V(y, \xi, B) = \mathcal{D}(y, \xi, B) V^d(y) + (1 - \mathcal{D}(y, \xi, B)) V^r(y, B), \tag{21}$$

and stress that V^{ℓ} is never reached in equilibrium, the country is never forced to service its debt without rolling over if doing so would not trigger a default.

Finally, the bond price schedule is now given by

$$q(y, B') = \frac{1}{1+r} \mathbb{E}_{y',\xi'|y} (1 - \mathcal{D}(y', \xi', B')) \left[\kappa + (1-\delta) \sum_j \Pr(B'' = j | y', B') q(y', j) \right], \tag{22}$$

and it is not a function of the current ξ, because it is not predictive of future ξ' or y' realization.

Note that this extension did not require any changes to the sovereign's problem under market access (10), but it does call for a small update to the continuation value of the W function, to account for the sunspot shock,

$$W(y, B, B') = u(y - \kappa B + [B' - (1-\delta)B] q(y, B')) + \beta \mathbb{E}_{y',\xi'|y} V(y', \xi', B'). \tag{23}$$

The taste shock structure in (11-12) is otherwise unaffected. The algorithm in Section 3.2 remains largely unchanged, once the ξ shock is incorporated as a state variable for V.

4.4 Multiple Endogenous States, Nested Choices

Our baseline model features one exogenous state variable, the endowment y, and one endogenous state variable, the debt level B, while the sovereign chooses the debt level for next period B' conditional on not defaulting this period. We now turn briefly to the question of how one can extend the method laid out in Section 3 to models with multiple endogenous state variables. Examples include settings with multiple debt maturities (Arellano and Ramanarayanan 2012;

Hatchondo, Martinez, and Sosa-Padilla 2016; Mihalache 2020), maturity choice (Bocola and Dovis 2019; Dvorkin et al. 2021), or capital and investment (Gordon and Guerron-Quintana 2018; Asonuma and Joo 2022).

To fix ideas, consider a model with debt and physical capital, as in Gordon and Guerron-Quintana (2018), and maintain for tractability the assumption that the sovereign makes all domestic decisions, including the level of investment. Now, the state variables are the productivity level y, the debt level B, and the capital stock K, while the sovereign's choices include the debt level B' and the capital stock K' for next period.

Equation (9) becomes

$$\begin{aligned} W(y,B,K,B',K') &= u(c) + \beta \mathbb{E}_{y'|y} V(y',B',K') \\ &\text{with } c = yK^{\alpha} - \kappa B + [B' - (1-\delta)B]q(y,B',K') - K' + (1-\Delta)K - \Phi(K,K'), \end{aligned} \tag{24}$$

where Δ is the depreciation rate for capital, $\Phi(K,K')$ is an adjustment cost, and output is given by yK^{α}, as we abstract from labor supply for simplicity.

A straightforward way to proceed is to treat each $\{B',K'\}$ pair as a discrete choice and assign an Extreme Value Type I shock to each of them. In this case, choice probabilities and the expected value function are

$$\Pr(B' = b_i, K' = k_i | y, B, K) = \frac{\exp \frac{W(y,B,K,b_i,k_i)}{\rho}}{\sum_j \exp \frac{W(y,B,K,b_j,k_j)}{\rho}} \tag{25}$$

and

$$V^r(y,B,K) = \rho \log \left[\sum_j \exp \frac{W(y,B,K,b_j,k_j)}{\rho} \right], \tag{26}$$

similar to before, in Section 3.

While straightforward, treating each $\{B',K'\}$ pair as a choice means that we cannot control the quantitative importance of taste shocks for the B' choice separately from that for K', the parameter ρ plays both roles by necessity. A tractable way to circumvent this limitation is to use a nested discrete choice structure. One interpretation is that we are allowing for correlations between the state shocks across the B' or K' dimensions of choice (Dvorkin et al. 2021). Another, more informal, way to think of this structure is that the sovereign chooses B' conditional on having chosen a particular K', subject to taste shocks, and that K' is chosen, subject to taste shocks, for a fixed B'.

Conditional on an arbitrary B', choice probabilities over K' are given by

$$\Pr(K' = k_i | y, B, K, B') = \frac{\exp \frac{W(y,B,K,B',k_i)}{\rho_K}}{\sum_j \exp \frac{W(y,B,K,B',k_j)}{\rho_K}} \tag{27}$$

with associated ex-ante value

$$W_K(y,B,K,B') = \rho_K \log \left[\sum_j \exp \frac{W(y,B,K,B',k_j)}{\rho_K} \right]. \tag{28}$$

Then, the outer choice over B' satisfies

$$\Pr(B' = b_i | y, B, K) = \frac{\exp \frac{W_K(y,B,K,b_i)}{\rho_B}}{\sum_j \exp \frac{W_K(y,B,K,b_j)}{\rho_B}} \tag{29}$$

and the ex-ante value of repayment becomes

$$V^r(y, B, K) = \rho_B \log \left[\sum_j \exp \frac{W_K(y, B, K, b_j)}{\rho_B} \right]. \tag{30}$$

In the simulation, we would first draw B' from the pmf $\Pr(B'|y, B, K)$ and then K' from $\Pr(K'|y, B, K, B')$. Finally, note that this formulation also supports the limit case under which the choice of K' is deterministic given B', under $\rho_K \to 0$.

While apparently more cumbersome than treating each $\{B', K'\}$ pair as its own choice, symmetrically, this nested approach allows us to use the two parameters ρ_K and ρ_B to separately control the magnitude of taste shocks over the two dimensions of choice.

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