

Sovereign Partial Default in Continuous Time

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- Continuous time analysis and solution methods for the *partial default* quantitative theory of Arellano, Mateos-Planas, and Ríos-Rull (2023, JPE)
- Partial default theory consistent with...
 - endogenous length of default crisis,
 - exiting the crisis with a high debt level compared to outset, arrears,
 - implicit seniority among creditors, Schleg, Trebesch, and Wright (2019)
- Computation with
 - Upwind finite difference scheme, Achdou et al. (2022, ReStud)
 - Deep neural network, Maliar, Maliar, and Winant (2021, JME)

Partial Default

- Sovereign chooses with discretion what share of the *due debt service payment* to make. Default on flow, not on stock.
- The share of payment not made accumulates as *arrears*, extra debt.
- No “market exclusion,” sovereign can issue new bond units at all times, at prices at which lenders break even in expectation.
- Convex penalty function of default, with discontinuity at 0. *Inaction region*.

In discrete time:

$$\begin{aligned}c_t &= \phi(d_t)z_t - (1 - d_t)b_t + q_t\ell_t \\ b_{t+1} &= (1 - \delta)b_t + \kappa d_t b_t + \ell_t\end{aligned}$$

$$\begin{aligned} V(B_0, z_0) = & \max_{\{c_t, d_t\}_{t \in [0, \infty]}} \mathbb{E}_0 \left\{ \int_0^\infty e^{-\rho t} u(c_t) dt \right\} \\ \text{s.t. } & c_t = \phi(d_t, z_t) e^{z_t} - (1 - d_t) (\delta + \lambda) B_t + q_t \ell_t \\ & \frac{dB_t}{dt} = -\delta B_t + \kappa (\delta + \lambda) d_t B_t + \ell_t \end{aligned}$$

The Sovereign

$$V(B_0, z_0) = \max_{\{c_t, d_t\}_{t \in [0, \infty]}} \mathbb{E}_0 \left\{ \int_0^\infty e^{-\rho t} u(c_t) dt \right\}$$

s.t.

$$c_t = \phi(d_t, z_t) e^{z_t} - (1 - d_t) (\delta + \lambda) B_t + q_t \ell_t$$
$$\frac{dB_t}{dt} = -\delta B_t + \kappa (\delta + \lambda) d_t B_t + \ell_t$$

Consumption c_t :

- GDP: $\phi(d_t, z_t) e^{z_t}$
- Debt service payment (minus): $(1 - d_t) (\delta + \lambda) B_t$
- New issuance proceeds: $q_t \ell_t$

The Sovereign

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s.t. $c_t = \phi(d_t, z_t) e^{z_t} - (1 - d_t) (\delta + \lambda) B_t + q_t \ell_t$

$$\frac{dB_t}{dt} = -\delta B_t + \kappa (\delta + \lambda) d_t B_t + \ell_t$$

Drift of debt dB_t/dt :

- Maturing debt (minus): δB_t
- Arrears: $\kappa (\delta + \lambda) d_t B_t$
- New issuance: ℓ_t

The Sovereign: HJB and FOCs

$$\rho V(B, z) = \max_{c, d \in [0, 1]} \left\{ u(c) + S(B, z, c, d, q) V_B(B, z) - \mu z V_z(B, z) + \frac{\sigma^2}{2} V_{zz}(B, z) \right\} \quad (1)$$

$$S(B, z, c, d, q) \equiv \frac{c - \phi(d, z) e^z}{q(B, z)} + \left[\left(\frac{1}{q(B, z)} + \left(\kappa - \frac{1}{q(B, z)} \right) d \right) (\delta + \lambda) - \delta \right] B$$

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FOCs:

$$c^*(B, z) = u_c^{-1} \left(-\frac{V_B(B, z)}{q(B, z)} \right) \quad (2)$$

$$d_{\text{int}}^*(B, z) = \min \left\{ 1, \phi_d^{-1} \left((1 - \kappa q(B, z)) (\delta + \lambda) \frac{B}{e^z}, z \right) \right\} \quad (3)$$

$$q_t = \mathbb{E}_t \int_t^\infty e^{-(r+\delta)(s-t) + \int_t^s \kappa(\delta+\lambda) d_\tau d\tau} (\delta + \lambda)(1 - d_s) d_s$$

$$\zeta(d^*(B, z))q(B, z) = (1 - d^*(B, z))(\lambda + \delta) + \tilde{S}(B, z)q_B(B, z) - \mu z q_z(B, z) + \frac{\sigma^2}{2} q_{zz}(B, z) \quad (4)$$

With,

- Effective discount rate, inclusive of arrears: $\zeta(d) \equiv r + \delta - \kappa(\delta + \lambda)d$
- Equilibrium drift of debt: $\tilde{S}(B, z)$

A *Markov Perfect Equilibrium* consist of

- the sovereign's value function $V(B, z)$,
- policy functions for consumption and default, $c^*(B, z)$ and $d^*(B, z)$, and
- the bond price function $q(B, z)$,

such that

- given q and V , the policies satisfy FOCs (2) and (3),
- given q and policies, the sovereign's value satisfies the HJB equation (1), and
- given policy functions, the bond price satisfies equation (4).

Parameter	Value	Comment
<i>Preferences</i>		
ν	2.0	AMPRR
ρ	0.047	Implied discount rate
<i>Debt</i>		
δ	0.13	Macauley duration
κ	0.7	Partial default haircut
r	0.039	Ct. compounding rate
λ	r	Normalization
<i>Endowment Process</i>		
μ	0.221	Time aggregation of AMPRR AR(1)
σ	0.062	
<i>Default Penalty</i>		
γ_0	0.0476	AMPRR
γ_1	2.0	AMPRR
γ_2	0.12	AMPRR
\tilde{z}	-0.062	AMPRR

$$u(c_t) = \begin{cases} \frac{c_t^{1-\nu}}{1-\nu} & \text{if } \nu \neq 1 \\ \log c_t & \text{if } \nu = 1 \end{cases}$$

$$\phi(d_t, z_t) = (1 - \gamma_0 d_t^{\gamma_1}) \times [1 - (z_t - \tilde{z}) \gamma_2 \mathbb{1}_{\{d_t > 0 \text{ and } z_t > \tilde{z}\}}]$$

$$dz_t = -\mu z_t dt + \sigma dW_t$$

(Ornstein-Uhlenbeck with $[\underline{z}, \bar{z}]$ barriers)

Computation: Two Methods

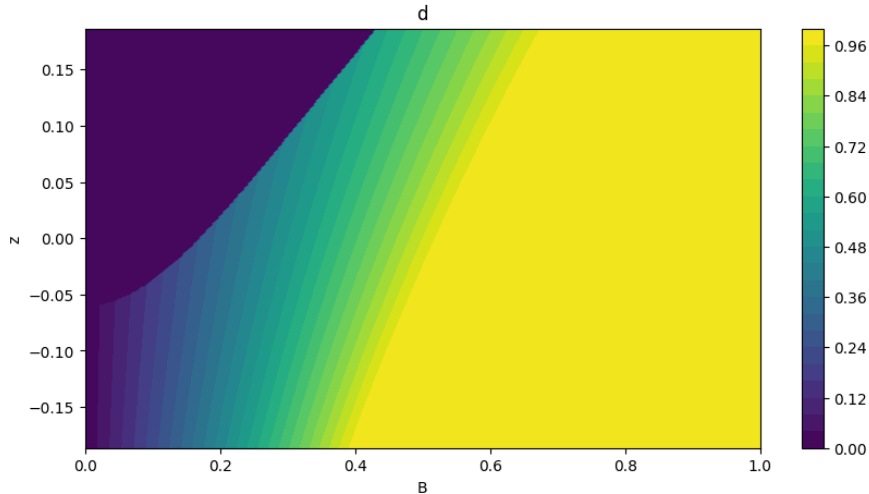
Today, *deep neural network*:

- Neural network inputs $\langle B, z \rangle$, outputs $\langle V, q, c \rangle$
- Minimize minibatch residuals of HJB for V , HJB for q , and FOC for c
- Stochastic gradient descent
- Method amenable to extensions with many state variables

Work in progress, *upwind finite difference scheme*:

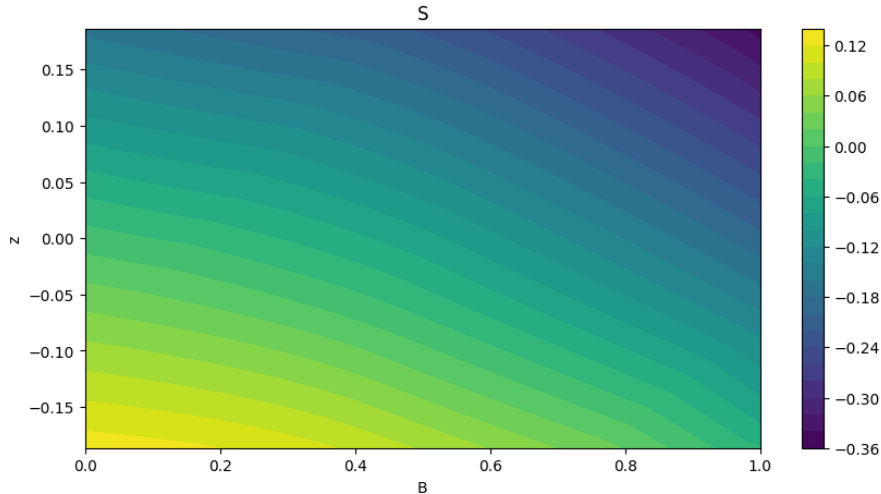
- Fast and precise
- Use in literature on traditional default models in continuous time, Hurtado, Nuño, and Thomas (2023, JEEA) and Borstein (2020, JEDC)

Preliminary Results: Default Policy



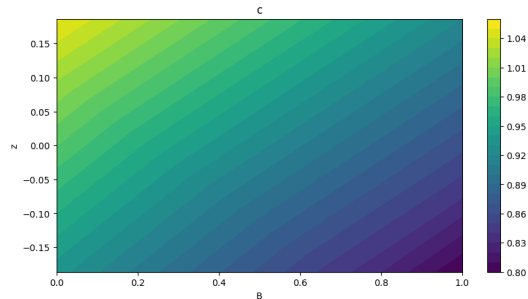
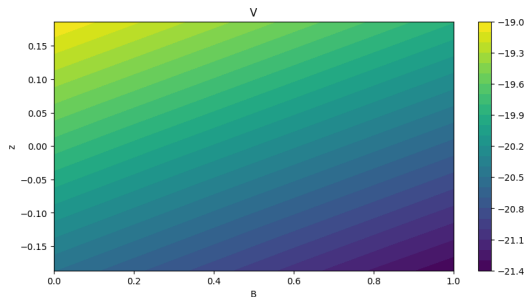
- Inaction zone ($d = 0$) dark blue, no debt service ($d = 1$) in yellow. Intermediate intensities in blue-green range.

Preliminary Results: Drift of B



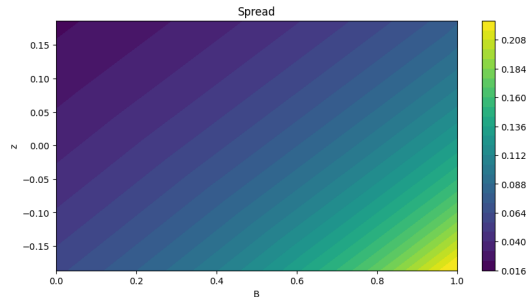
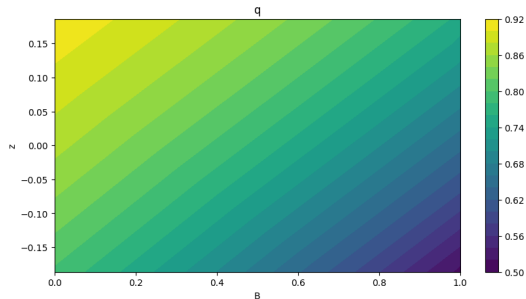
- Borrow aggressively when debt is low and endowment is low, reduce debt at high debt and/or high endowment. Keep B in place at $S = 0$, mid-green.

Preliminary Results: Value Function and Consumption



- Value and consumption increasing in endowment (z) and decreasing in debt (B)

Preliminary Results: Bond Price and Spreads



- Spreads decreasing in endowment (z) and increasing in debt (B)